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# THE CENTRALIZER RESOLUTION OF THE K(2)-LOCAL SPHERE AT THE PRIME 2

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# THE CENTRALIZER RESOLUTION OF THE $K(2)$ -LOCAL SPHERE AT THE PRIME 2.

HANS-WERNER HENN

ABSTRACT. Let  $\mathbb{G}_2$  be the Morava stabilizer group at the prime 2. We construct a resolution of the  $K(2)$ -local sphere at the prime 2 in terms of certain homotopy fixed point spectra which are closely related to the spectrum of topological modular forms. This resolution is in certain ways analogous to the centralizer resolution of the  $K(n)$ -local sphere constructed in [18] if  $p$  is an odd prime and  $n = p - 1$ .

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The origin of this paper goes back to the early 2000's and was stimulated by the joint work with P. Goerss, M. Mahowald and C. Rezk [14]. Some of the results were announced in [18] but proofs of the existence of the centralizer resolutions were never published. Recent work by Beaudry [2], [3], [4] by Bobkova and Goerss [5] and a joint project with Beaudry and Goerss [6] have underlined the importance of these resolutions. The author apologizes for the delay in making these results available and he is happy to acknowledge helpful discussions with Goerss, Mahowald, Rezk, Beaudry and Bobkova which have lead to this research and to improvements of the original results and simplifications of the proofs.

## 1. INTRODUCTION

Let  $p$  be a prime, let  $n > 0$  be an integer and let  $K(n)$  be the  $n$ -th Morava  $K$ -theory at  $p$ . The category of  $K(n)$ -local spectra is a basic building block of the stable homotopy category of  $p$ -local spectra and the  $K(n)$ -localization of the sphere,  $L_{K(n)}S^0$ , plays a central role in this category. The homotopy of  $L_{K(n)}S^0$  can be studied by the Adams-Novikov spectral sequence, and by [10] this spectral sequence can be identified with the homotopy fixed point spectral sequence for the action of the extended Morava stabilizer group  $\mathbb{G}_n$  on  $E_n$ . Here  $E_n$  denotes the 2-periodic Landweber exact spectrum  $E_n$  whose coefficients in degree 0 classify deformations (in the sense of Lubin and Tate) of a suitable formal group law  $\Gamma_n$  of height  $n$  over  $\mathbb{F}_{p^n}$  and  $\mathbb{G}_n$  is the automorphism group of  $\Gamma_n$  in the category of formal group laws (cf [28]). The  $E_2$ -page of this spectral sequence is given by the continuous cohomology  $H_{cts}^*(\mathbb{G}_n, (E_n)_*)$  of  $\mathbb{G}_n$  with coefficients in  $(E_n)_*$ . It becomes therefore interesting to find resolutions of the trivial module for the group  $\mathbb{G}_n$  from which one can calculate this continuous cohomology.

If  $p$  is large with respect to  $n$  then the  $E_2$ -page satisfies  $E_2^{s,*} = 0$  for  $* > n^2$  and the spectral sequence collapses at its  $E_2$ -page. In the sequel we concentrate on the case  $n = 2$  because the case  $n = 1$  is well understood and very little is understood in explicit terms if  $n > 2$ .

For  $n = 2$  the spectral sequence collapses if and only if  $p > 3$ . In these cases the homotopy of  $L_{K(2)}S^0$  has been calculated in [29] without using the point of view of group cohomology. The results have been reinterpreted in [7] and an independant calculation for the Moore space has been carried out in [24] by using an explicit projective resolution of length 4 of the trivial  $\mathbb{G}_2$ -module  $\mathbb{Z}_p$ .

If  $n = 2$  and  $p \leq 3$  the mod- $p$  cohomological dimension of the group  $\mathbb{G}_2$  is infinite and there cannot be any projective resolution of the trivial  $\mathbb{G}_2$ -module  $\mathbb{Z}_p$  of finite length. However, for  $p = 3$  very useful resolutions of the trivial  $\mathbb{G}_2$ -module  $\mathbb{Z}_p$  of length 4 in terms of more general modules and corresponding topological resolutions of  $L_{K(2)}S^0$  exist; a “duality resolution” has been constructed in [14] and a “centralizer resolution” in [18]. These resolutions complement each other and they have been crucial in recent progress of our understanding of  $K(2)$ -local homotopy theory at the prime 3. In particular they have been used for proving the chromatic splitting conjecture for  $n = 2$  [13], for determining Hopkins’ Picard group of  $K(2)$ -local spectra [22], [15] and for identifying the Brown-Comenetz dual of the  $K(2)$ -local sphere [16].

If  $n = 2$  and  $p = 2$  our understanding is less complete although the chromatic splitting conjecture has already been successfully analyzed in [4] and [6] by heavily using the algebraic and topological duality resolution for an important subgroup  $\mathbb{S}_2^1$  of  $\mathbb{G}_2$ . The existence of an algebraic duality and an algebraic centralizer resolution of length 3 for  $\mathbb{S}_2^1$  was already announced in [18], as well as a topological centralizer resolution for the homotopy fixed point spectrum  $E_2^{h\mathbb{S}_2^1}$ , in all cases without proofs. For the algebraic duality resolution the construction was finally established in [2] and the construction of its topological counterpart was given in [5]. The latter paper relied heavily on the existence of both the algebraic and topological centralizer resolution for  $\mathbb{S}_2^1$  for which no proof has been published yet. The main purpose of this paper is to fill this gap in the literature and extend the announced results from the group  $\mathbb{S}_2^1$  to  $\mathbb{S}_2$  and even to  $\mathbb{G}_2$ . Such extensions appear to be impossible for the algebraic and topological duality resolutions.

### 1.1. Preliminaries on Morava stabilizer groups at $n = p = 2$ .

1.1.1. Let  $\Gamma$  be a formal group law of height  $n$  defined over  $\mathbb{F}_p$ , let  $q = p^n$  and assume that the automorphism group  $\mathbb{S}_n(\Gamma) := \text{Aut}_{\mathbb{F}_q}(\Gamma)$  is isomorphic to  $\mathbb{S}_n := \mathbb{S}_n(\Gamma_H)$ , the automorphism

group of the Honda formal group law.<sup>1</sup> Because the formal group law is defined over  $\mathbb{F}_p$  the Galois group  $\text{Gal}$  of the extension  $\mathbb{F}_p \subset \mathbb{F}_q$  acts on  $\mathbb{S}_n(\Gamma)$  and we get extended automorphism groups

$$\mathbb{G}_n(\Gamma) = \mathbb{S}_n(\Gamma) \rtimes \text{Gal} .$$

For  $n = p = 2$  there are two important candidates for  $\Gamma$ . In fact, there are two particularly interesting formal group laws  $\Gamma$  of height 2 over the prime field  $\mathbb{F}_2$ : the Honda formal group law  $\Gamma_H$ , i.e. the [2]-typical formal group law with [2]-series  $[2]_{\Gamma_H}(x) = x^4$ , and the formal group law  $\Gamma_E$  of the supersingular elliptic curve over  $\mathbb{F}_2$  with affine equation  $y^2 + y = x^3$ . In the remainder of this introduction  $\Gamma$  always refers to either  $\Gamma_H$  or to  $\Gamma_E$ .

If  $\overline{\mathbb{F}}_2$  denotes the algebraic closure of  $\mathbb{F}_2$  then the endomorphism rings of both formal group laws satisfy

$$\text{End}_{\mathbb{F}_4}(\Gamma) \cong \text{End}_{\overline{\mathbb{F}}_2}(\Gamma) ,$$

and because both formal group laws become isomorphic over  $\overline{\mathbb{F}}_2$  their endomorphism rings are already isomorphic over  $\mathbb{F}_4$ . Consequently the automorphisms groups  $\mathbb{S}_2(\Gamma) = \text{Aut}_{\mathbb{F}_4}(\Gamma)$  of these two formal group laws over the field  $\mathbb{F}_4$  are abstractly isomorphic. If  $\Gamma = \Gamma_H$  this group is the classical second Morava stabilizer group at  $p = 2$  and usually denoted  $\mathbb{S}_2$ , and  $\mathbb{G}_2(\Gamma)$  is usually called the extended Morava stabilizer group and denoted  $\mathbb{G}_2$ . While the groups  $\mathbb{S}_2(\Gamma)$  are abstractly isomorphic this ceases to be true for the groups  $\mathbb{G}_2(\Gamma)$  (cf. Lemma 2.2).

The endomorphism rings  $\text{End}_{\mathbb{F}_4}(\Gamma)$  contain  $\mathbb{W}$ , the ring of Witt vectors of  $\mathbb{F}_4$ . They are generated as a non-commutative  $\mathbb{W}$ -algebra by the endomorphism  $\xi_\Gamma \in \text{End}_{\mathbb{F}_4}(\Gamma)$  given by  $\xi_\Gamma(x) = x^2$ . In order to describe the endomorphism rings more explicitly we denote the image of  $w \in \mathbb{W}$  with respect to the lift of the Frobenius automorphism of  $\mathbb{F}_4$  by  ${}^\sigma w$  and we abbreviate  $\xi_\Gamma$  simply by  $\xi$ . Then the canonical algebra map from the free non-commutative  $\mathbb{W}$ -algebra  $\mathbb{W}\langle \xi \rangle$  generated by  $\xi$  to  $\text{End}_{\mathbb{F}_4}(\Gamma)$  induces an isomorphism

$$(1.1) \quad \mathbb{W}\langle \xi \rangle / (\xi w - {}^\sigma w \xi, \xi^2 - 2u) \cong \text{End}_{\mathbb{F}_4}(\Gamma)$$

where

$$(1.2) \quad u = \begin{cases} 1 & \Gamma = \Gamma_H \\ -1 & \Gamma = \Gamma_E . \end{cases}$$

An explicit isomorphism between the two rings is given by the  $\mathbb{W}$ -algebra map which sends  $\xi$  to  $\xi y$  where we can take for  $y$  any element in  $\mathbb{W}$  with the property  $yy^\sigma = -1$  (cf. [2] for an explicit choice of  $y$ ).

The ideal generated by  $\xi$  is a two-sided maximal ideal  $\mathfrak{m}$  with quotient  $\mathbb{F}_4$  and the endomorphism rings are complete with respect to the  $\mathfrak{m}$ -adic topology. This also defines a filtration on the group  $\mathbb{S}_2(\Gamma)$  indexed by half integers  $\frac{i}{2} \geq 0$  given by

$$F_{\frac{i}{2}} := F_{\frac{i}{2}}\mathbb{S}_2(\Gamma) := \{g \in \mathbb{S}_2(\Gamma) \mid g \equiv 1 \pmod{(\xi^i)}\}$$

and successive quotients

$$F_{\frac{i}{2}}/F_{\frac{i+1}{2}} \cong \begin{cases} \mathbb{F}_4^\times & i = 0 \\ \mathbb{F}_4 & i = 0 . \end{cases}$$

The group

$$\mathbb{S}_2(\Gamma) := F_{\frac{i}{2}}\mathbb{S}_2(\Gamma)$$

is a profinite 2-group, the normal 2-Sylow subgroup of  $\mathbb{S}_2(\Gamma)$ .

<sup>1</sup>This is equivalent to the endomorphism  $\xi_\Gamma$  given by  $\xi_\Gamma(x) = x^p$  satisfying  $\xi_\Gamma^n = pu$  (cf. Remark 5.2).

Inverting 2 in the endomorphism rings gives two isomorphic division algebras which we denote by  $\mathbb{D}_2(\Gamma)$ . They contain  $\mathbb{Q}_2$  as their center and are of dimension 4 over  $\mathbb{Q}_2$ . The division algebras are equipped with a valuation

$$v : \mathbb{D}_2(\Gamma)^\times \rightarrow \frac{1}{2}\mathbb{Z}$$

which extends the valuation on  $\mathbb{Q}_p$  which is normalized by  $v(p) = 1$ .

The group of units  $\mathbb{D}_2(\Gamma)^\times$  of  $\mathbb{D}_2(\Gamma)$  contains  $\mathbb{S}_2(\Gamma)$  as the group of elements of valuation 0 and from (1.1) it is clear that the action of the Galois group on  $\mathbb{S}_2(\Gamma)$  is realized by conjugation by  $\xi_\Gamma$  in  $\mathbb{D}_2(\Gamma)^\times$ . Therefore we get canonical isomorphisms

$$(1.3) \quad \mathbb{G}_2(\Gamma) \cong \mathbb{D}_2(\Gamma)^\times / \langle \xi_\Gamma^2 \rangle \cong \begin{cases} \mathbb{D}_2(\Gamma)^\times / \langle 2 \rangle & \Gamma = \Gamma_H \\ \mathbb{D}_2(\Gamma)^\times / \langle -2 \rangle & \Gamma = \Gamma_E . \end{cases}$$

The groups  $\mathbb{S}_2(\Gamma)$  and  $\mathbb{G}_2(\Gamma)$  contain  $-1$  as unique central element of order 2 and dividing out by the subgroup  $C_2$  generated by it gives us quotient groups which we will denote  $PS_2(\Gamma)$  and  $P\mathbb{G}_2(\Gamma)$ . From (1.3) it is clear we have isomorphisms

$$P\mathbb{G}_2(\Gamma_H) \cong \mathbb{D}_2(\Gamma_H)^\times / \langle 2, -1 \rangle \cong \mathbb{D}_2(\Gamma_E)^\times / \langle -2, -1 \rangle \cong P\mathbb{G}_2(\Gamma_E) .$$

1.1.2. From (1.1) we see that  $\text{End}_{\mathbb{F}_4}(\Gamma)$  is a free  $\mathbb{W}$ -module with basis 1 and  $\xi$ . Right multiplication induces  $\mathbb{W}$ -linear maps and the determinant gives a multiplicative homomorphism

$$\det : \text{End}_{\mathbb{F}_4}(\Gamma) \rightarrow \mathbb{W}$$

which, in fact takes its values in  $\mathbb{Z}_2$ . It is explicitly given as follows: if  $a, b \in \mathbb{W}$  then

$$\det(a + b\xi_\Gamma) = aa^\sigma - 2ubb^\sigma$$

with  $u = 1$  as in (1.2). This determinant induces an epimorphism

$$\det : \mathbb{S}_2(\Gamma) \rightarrow \mathbb{Z}_2^\times$$

which is often also called the reduced norm. Finally we get an epimorphism given as composition

$$\mathbb{G}_2(\Gamma) = \mathbb{S}_2(\Gamma) \rtimes \text{Gal} \xrightarrow{\det \times \text{id}} \mathbb{Z}_2^\times \times \text{Gal} \rightarrow \mathbb{Z}_2^\times \rightarrow \mathbb{Z}_2^\times / \{\pm 1\}$$

in which the second and third part are given as the obvious projections. Let  $\mathbb{G}_2^1(\Gamma)$  be the kernel of this composition and  $\mathbb{S}_2^1(\Gamma)$  resp.  $S_2^1(\Gamma)$  its intersection with  $\mathbb{S}_2(\Gamma)$  resp.  $S_2(\Gamma)$ . We observe that the action of Gal on  $\mathbb{S}_2(\Gamma)$  leaves  $\mathbb{S}_2^1(\Gamma)$  invariant and  $\mathbb{G}_2^1(\Gamma)$  is equal to the semidirect product  $\mathbb{S}_2^1(\Gamma) \rtimes \text{Gal}$ . By the definition of  $\mathbb{G}_2^1(\Gamma)$  it is clear that every finite subgroup of  $\mathbb{G}_2(\Gamma)$  is contained in  $\mathbb{G}_2^1(\Gamma)$ .

The central element  $-1 = 1 - u\xi_\Gamma^2$  (where  $u$  is as in (1.2)) is contained in  $S_2^1(\Gamma)$  and generates a central subgroup  $C_2$  of order 2. If  $H$  is any closed subgroup of  $\mathbb{G}_2(\Gamma)$  containing  $C_2$  then we will denote the quotient  $H/C_2$  by  $PH$ .

1.1.3. The groups  $\mathbb{S}_2^1(\Gamma)$ ,  $PS_2^1(\Gamma)$ ,  $\mathbb{G}_2^1(\Gamma)$ ,  $P\mathbb{G}_2^1(\Gamma)$  and  $PS_2^1(\Gamma)$  contain certain finite subgroups which figure in the statements of our main results. In all cases except that of  $\mathbb{G}_2^1(\Gamma)$  the isomorphism type of the ambient group is independent of  $\Gamma$  and only when we discuss finite subgroups of  $\mathbb{G}_2^1(\Gamma)$  the choice of  $\Gamma$  matters. In the other cases we will therefore from now on omit  $\Gamma$  from our notation.

If  $F$  is a finite subgroup of  $\mathbb{G}_2^1(\Gamma)$  which contains the central  $C_2$  and for which  $F_0 := F \cap S_2^1$  is of index 2 in  $F$  then we have a commutative diagram of groups with exact rows

$$(1.4) \quad \begin{array}{ccccccc} 1 & \longrightarrow & F_0 & \longrightarrow & F & \longrightarrow & \text{Gal} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & PF_0 & \longrightarrow & PF & \longrightarrow & \text{Gal} \longrightarrow 1 \end{array} .$$

In the following table we give a list of closed subgroups  $F \subset \mathbb{G}_2^1(\Gamma)$  and the corresponding groups  $PF \subset P\mathbb{G}_2^1$ ,  $F_0 \subset \mathbb{S}_2^1$ ,  $PF_0 \subset PS_2^1$  and  $PF_0 \cap PS_2^1 \subset PS_2^1$  which will be relevant for stating our main results. Subgroups of  $PS_2^1$  will play an important role in section 4.2.

$$(1.5) \quad \begin{array}{|c|c|c|c|c|c|c|} \hline F & \mathbb{G}_2^1(\Gamma) & G_{48}(\Gamma) & G'_{48}(\Gamma) & G_{12}(\Gamma) & C_8 & C_2 \times \text{Gal} \\ \hline PF & P\mathbb{G}_2^1 & \mathfrak{S}_4 & \mathfrak{S}'_4 & \mathfrak{S}_3 & C_4 & \text{Gal} \\ \hline F_0 & \mathbb{S}_2^1 & G_{24} & G'_{24} & C_6 & C_4 & C_2 \\ \hline PF_0 & PS_2^1 & A_4 & A'_4 & C_3 & C_2 & \{1\} \\ \hline PF_0 \cap PS_2^1 & PS_2^1 & E_2 & E'_2 & \{1\} & C_2 & \{1\} \\ \hline \end{array}$$

We refer to Section 2, in particular Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 for more details on this table. Here we are content to explain that in this table  $C_n$  denotes a cyclic group of order  $n$ , Gal is the Galois group of the extension  $\mathbb{F}_2 \subset \mathbb{F}_4$ ,  $\mathfrak{S}_n$  and  $\mathfrak{S}'_n$  denote symmetric groups on  $n$  letters,  $A_4$  and  $A'_4$  alternating groups on 4 letters and  $E_2$  and  $E'_2$  groups isomorphic to  $C_2 \times C_2$ . The groups  $G_{24}$  and  $G'_{24}$  are groups of order 24 both isomorphic to  $SL_2(\mathbb{F}_3)$ . The isomorphism type of the groups  $F = G_{48}(\Gamma)$ ,  $F = G'_{48}(\Gamma)$  and  $F = G_{12}(\Gamma)$  depends on  $\Gamma$ . The first two are maximal subgroups of  $\mathbb{G}_2^1(\Gamma)$  of order 48 which are non-conjugate in  $\mathbb{G}_2^1(\Gamma)$  but become conjugate in  $\mathbb{G}_2(\Gamma)$ . In fact, we have (cf. Lemma 2.2)

$$G_{48}(\Gamma) \cong G'_{48}(\Gamma) \cong \begin{cases} GL_2(\mathbb{F}_3) & \Gamma = \Gamma_E \\ O_{48} & \Gamma = \Gamma_H \end{cases}$$

where  $O_{48}$  denotes the binary octahedral group. For the groups  $G_{12}(\Gamma)$  we get (cf. Lemma 2.3)

$$G_{12}(\Gamma) \cong \begin{cases} C_2 \times \mathfrak{S}_3 & \Gamma = \Gamma_E \\ C_3 \rtimes C_4 & \Gamma = \Gamma_H \end{cases}$$

where  $C_3 \rtimes C_4$  denotes the semidirect of  $C_3$  with  $C_4$  acting non-trivially on  $C_3$ .

## 1.2. Main results.

Let  $G$  be a profinite group, let  $X$  be a profinite  $G$ -set such that  $X = \lim_i X_i$  with finite  $G$ -sets  $X_i$  and let  $\mathbb{W}$  be the ring of Witt vectors for a finite field  $k$  of order  $q = p^n$  for a prime  $p$  and an integer  $n > 0$ . We define

$$(1.6) \quad \mathbb{W}[[X]] = \lim_{i,k} \mathbb{W}/p^k[[X_i]] .$$

Suppose that  $G$  is equipped with a continuous homomorphism  $\phi : G \rightarrow \text{Gal}$  to the Galois group Gal of the extension  $\mathbb{F}_p \subset \mathbb{F}_q$ .

The *Galois-twisted completed group ring*  $\mathbb{W}_\phi[[G]]$  of  $G$  is the  $\mathbb{W}$ -module  $\mathbb{W}[[G]]$  with multiplication induced by  $(w_1 g_1)(w_2 g_2) = w_1 g_1 w_2 g_2$  if  $g_1, g_2 \in G$ ,  $w_1, w_2 \in \mathbb{W}$  and if  $g_1 w_2$  is the result of the Galois action of  $\phi(g_1)$  on  $w_2$ . A  $p$ -profinite  $\mathbb{W}_\phi[[G]]$ -module will also be called a *Galois-twisted  $p$ -profinite  $G$ -module*, or simply a *Galois-twisted profinite  $G$ -module* if  $p$  is understood from the context. In order to keep notation simple we will write  $\mathbb{W}[[G]]$  instead of  $\mathbb{W}_\phi[[G]]$ .

Analogous to [18] we introduce relative homological algebra in the context of Galois-twisted  $p$ -profinite  $G$ -modules. Let  $\mathcal{F}(G)$  be the set of conjugacy classes of finite subgroups of  $G$  and

assume that  $\mathcal{F}(G)$  is a finite set. A Galois-twisted  $p$ -profinite  $G$ -module  $P$  will be called  $\mathcal{F}$ -projective if it is a direct summand in a module of the form  $\bigoplus_{(F)} \mathbb{W}[[G]] \otimes_{\mathbb{W}[F]} M$  where each  $M_F$  is a  $p$ -profinite<sup>2</sup>  $\mathbb{W}[F]$ -module and the direct sum is indexed by conjugacy classes of finite subgroups of  $G$ . In the sequel we will also write  $M \uparrow_F^G$  instead of  $\mathbb{W}[[G]] \otimes_{\mathbb{W}[F]} M$ .

The class of  $\mathcal{F}$ -projective Galois-twisted  $p$ -profinite  $G$ -modules determines in the usual way a class of  $\mathcal{F}$ -exact sequences: a sequence of Galois-twisted  $p$ -profinite  $G$ -modules  $M' \rightarrow M \rightarrow M''$  is called  $\mathcal{F}$ -exact if the composition  $M' \rightarrow M''$  is trivial and

$$\mathrm{Hom}_{\mathbb{W}[[G]]}(P, M') \rightarrow \mathrm{Hom}_{\mathbb{W}[[G]]}(P, M) \rightarrow \mathrm{Hom}_{\mathbb{W}[[G]]}(P, M'')$$

is an exact sequence of abelian groups for each  $\mathcal{F}$ -projective Galois-twisted  $p$ -profinite  $G$ -module  $P$ .

An  $\mathcal{F}$ -resolution of a Galois-twisted  $p$ -profinite  $G$ -module  $M$  is a sequence of Galois-twisted  $p$ -profinite  $G$ -modules

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where each  $P_i$  is  $\mathcal{F}$ -projective and each 3-term subsequence is  $\mathcal{F}$ -exact. We note that  $\mathcal{F}$ -exactness is equivalent to the complex being split when restricted to any finite subgroup of  $G$ .

Here is the main algebraic result of this paper in which  $\mathbb{W}$  is now the ring of Witt vector of  $\mathbb{F}_4$  and the subgroups of  $P\mathbb{G}_2$  are those of table (1.5).

**Theorem 1.1.** *There exists an  $\mathcal{F}$ -resolution of the trivial Galois-twisted profinite  $P\mathbb{G}_2$ -module  $\mathbb{W}$*

$$\begin{aligned} 0 \longrightarrow \mathbb{W} \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2} \xrightarrow{\partial_4} \mathbb{W} \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2} \oplus \mathbb{W} \uparrow_{\mathrm{Gal}}^{P\mathbb{G}_2} \xrightarrow{\partial_3} \mathbb{W} \uparrow_{\mathrm{Gal}}^{P\mathbb{G}_2} \oplus \mathbb{W} \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2} \oplus \mathbb{W} \uparrow_{C_4}^{P\mathbb{G}_2} \\ \xrightarrow{\partial_2} \mathbb{W} \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2} \oplus \mathbb{W} \uparrow_{C_4}^{P\mathbb{G}_2} \oplus \mathbb{W} \uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2} \oplus \mathbb{W} \uparrow_{\mathfrak{S}'_4}^{P\mathbb{G}_2} \xrightarrow{\partial_1} \mathbb{W} \uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2} \oplus \mathbb{W} \uparrow_{\mathfrak{S}'_4}^{P\mathbb{G}_2} \xrightarrow{\varepsilon} \mathbb{W} . \end{aligned}$$

The main work towards establishing this theorem is the following result.

**Theorem 1.2.** *There exists an  $\mathcal{F}$ -resolution of the trivial Galois-twisted profinite  $P\mathbb{G}_2^1$ -module  $\mathbb{W}$*

$$0 \longrightarrow \mathbb{W} \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \xrightarrow{\partial_3} \mathbb{W} \uparrow_{\mathrm{Gal}}^{P\mathbb{G}_2^1} \xrightarrow{\partial_2} \mathbb{W} \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \oplus \mathbb{W} \uparrow_{C_4}^{P\mathbb{G}_2^1} \xrightarrow{\partial_1} \mathbb{W} \uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W} \uparrow_{\mathfrak{S}'_4}^{P\mathbb{G}_2^1} \xrightarrow{\varepsilon} \mathbb{W} .$$

**Remark 1.3.** a) The resolutions for the group  $P\mathbb{G}_2^1$  resp. for  $P\mathbb{G}_2$  can be considered as resolutions for  $\mathbb{G}_2^1(\Gamma)$  resp.  $\mathbb{G}_2(\Gamma)$  via the obvious projections  $\mathbb{G}_2^1 \rightarrow P\mathbb{G}_2^1$  resp.  $\mathbb{G}_2 \rightarrow P\mathbb{G}_2$ . In terms of the table (1.5) this has the effect of replacing a summand in the resolution of the form  $\mathbb{W} \uparrow_{PF}^{P\mathbb{G}_2^1}$  resp.  $\mathbb{W} \uparrow_{PF}^{P\mathbb{G}_2}$  by  $\mathbb{W} \uparrow_F^{\mathbb{G}_2^1}$  resp.  $\mathbb{W} \uparrow_F^{\mathbb{G}_2}$ . Unlike for  $P\mathbb{G}_2^1$  resp. for  $P\mathbb{G}_2$  the resulting resolutions for  $\mathbb{G}_2^1(\Gamma)$  resp.  $\mathbb{G}_2(\Gamma)$  will depend on the choice of  $\Gamma$ .

b) Restricted to  $\mathbb{S}_2^1(\Gamma)$  the resolution for  $\mathbb{G}_2^1(\Gamma)$  is an untwisted  $\mathcal{F}$ -resolution of  $\mathbb{W}$  which is a  $\mathbb{W}$ -linear extension of the algebraic centralizer resolution announced in [18] and used in [5]. We refer to Remark 3.3 for a justification of the terminology centralizer resolution.

Next we will describe the topological analogues of these algebraic resolutions. As in [18] we call a sequence of spectra

$$(1.7) \quad X_{\bullet} : * \rightarrow X_{-1} \xrightarrow{\alpha_0} X_0 \rightarrow X_1 \xrightarrow{\alpha_1} \dots$$

a *complex of spectra* if the composite of two consecutive maps is null-homotopic. Such a complex is called a *resolution* of  $X_{-1}$  if in addition each of the maps  $\alpha_i : X_{i-1} \rightarrow X_i$ ,  $i \geq 0$ , can be factored as  $X_{i-1} \xrightarrow{\beta_i} W_i \xrightarrow{\gamma_i} X_i$  such that  $W_{i-1} \xrightarrow{\gamma_{i-1}} X_{i-1} \xrightarrow{\beta_i} W_i$  is a cofibration for every

<sup>2</sup>The assumption that  $M$  is  $p$ -profinite was regrettably missing in [18].

$i \geq 0$  (with  $W_{-1} := *$ ). We say that the resolution is of length  $n$  if  $W_n \simeq X_n$  and  $X_i \simeq *$  if  $i > n$ .

Here are the main topological results of this paper. In their statements  $E_2$  should really read  $E_2(\Gamma)$  where  $E_2(\Gamma)$  is the 2-periodic Landweber exact spectrum whose coefficients in degree 0 classify deformations (in the sense of Lubin and Tate) of  $\Gamma$ . In order to keep notation readable we will nevertheless simply write  $E_2$  instead of  $E_2(\Gamma)$ . By the Goerss-Hopkins-Miller theorem  $\Gamma$  acts on  $E_2$ , in particular there exist homotopy fixed point spectra  $E_2^{hF}$  for all finite subgroups of  $\mathbb{G}_2(\Gamma)$  and by [10] also for all closed subgroups.

**Theorem 1.4.** *Let  $\Gamma$  be either  $\Gamma_H$  or  $\Gamma_E$ . Then there exists a resolution of  $E_2^{h\mathbb{G}_2^1(\Gamma)}$*

$$* \rightarrow E_2^{h\mathbb{G}_2^1(\Gamma)} \rightarrow E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \rightarrow E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \rightarrow E_2^{C_2 \times \text{Gal}} \rightarrow E_2^{hG_{12}(\Gamma)} \rightarrow * .$$

**Theorem 1.5.** *Let  $\Gamma$  be either  $\Gamma_H$  or  $\Gamma_E$ . Then there exist a resolution of  $L_{K(2)}S^0 \simeq E_2^{h\mathbb{G}_2(\Gamma)}$*

$$\begin{aligned} * \rightarrow L_{K(2)}S^0 \rightarrow E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \rightarrow E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \vee E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \\ \rightarrow E_2^{C_2 \times \text{Gal}} \vee E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \rightarrow E_2^{hG_{12}(\Gamma)} \vee E_2^{C_2 \times \text{Gal}} \rightarrow E_2^{hG_{12}(\Gamma)} \rightarrow * . \end{aligned}$$

**Remark 1.6.** a) Because  $G_{48}(\Gamma)$  and  $G'_{48}(\Gamma)$  are conjugate subgroups of  $\mathbb{G}_2(\Gamma)$ , the homotopy fixed point spectra  $E_2^{hG_{48}(\Gamma)}$  and  $E_2^{hG'_{48}(\Gamma)}$  have the same homotopy type.

b) There are corresponding resolutions for  $E_2^{h\mathbb{S}_2^1}$  and  $E_2^{h\mathbb{S}_2}$  which are obtained by replacing  $E_2^{hF}$  by  $E_2^{hF_0}$  where  $F$  and  $F_0$  are the finite subgroups of table (1.5).

The paper is organized as follows. In section 2 we discuss the finite subgroups of the Morava stabilizer groups at  $n = p = 2$  which figure in our main results and in section 3 we study the mod-2 cohomology algebra of  $PS_2^1$  via its restriction to the cohomology of elementary abelian 2-subgroups. Section 4 contains the construction of the algebraic centralizer resolutions and in section 5 we show how to realize the algebraic resolutions topologically.

## 2. IMPORTANT FINITE SUBGROUPS FOR MORAVA STABILIZER GROUPS AT $n = p = 2$

In this section we will elaborate on table (1.5) and describe more explicitly the relevant finite subgroups. We remark that in the general case of any prime  $p$  and any height  $n$  finite subgroups of  $\mathbb{S}_n$  have been studied by Hewett in [20] and [21] and finite subgroups of  $\mathbb{G}_n(\Gamma)$  have been studied by Bujard [9].

We will start by recalling from [2] the description of explicit maximal subgroups  $G_{24}$  and  $G'_{24}$  of  $\mathbb{S}_2$  and we prefer to work with  $\mathbb{S}_2(\Gamma_H)$  and write  $S$  instead of  $\xi_H$ .

Let  $\omega$  be a third root of unity in  $\mathbb{W}^\times$  and let

$$(2.1) \quad \pi := 1 + 2\omega .$$

By Hensel's Lemma the element  $-7 \in \mathbb{Z}_2$  has two square roots in  $\mathbb{Z}_2$ . We pick the one which satisfies  $\sqrt{-7} \equiv 1 + 4 \pmod{8}$  and let

$$(2.2) \quad \alpha := \frac{1 - 2\omega}{\sqrt{-7}} .$$

We note that  $\pi$  and  $\alpha$  both belong to  $\mathbb{S}_2$  and the reduced norm of  $\alpha$  is  $-1$  while the reduced norm of  $\pi$  is 3.

The following lemma is proved by direct calculation (cf. Lemma 2.4.3 of [2]).



**Lemma 2.1.** *Let*

$$i := \frac{1}{3}(1 + 2\omega^2)(1 - \alpha S), \quad j := \frac{1}{3}(1 + 2\omega^2)(1 - \alpha\omega^2 S), \quad k := \frac{1}{3}(1 + 2\omega^2)(1 - \alpha\omega S) .$$

*Then the elements  $\{\pm 1, \pm i, \pm j, \pm k\}$  form a subgroup of  $\mathbb{S}_2^1$  which is isomorphic to the quaternion group  $Q_8$ . This subgroup is invariant by conjugation by  $\omega$ , more precisely*

$$j = \omega i \omega^{-1}, \quad k = \omega j \omega^{-1} \quad i = \omega k \omega^{-1} .$$

*Furthermore*

$$\omega = -\frac{1}{2}(1 + i + j + k) . \quad \square$$

We let  $G_{24}$  be the subgroup generated by  $Q_8$  and  $\omega$ . It is isomorphic to the semidirect product of  $Q_8$  with  $C_3$ ,

$$(2.3) \quad G_{24} \cong Q_8 \rtimes C_3 .$$

It is easy to verify that the 16 elements of  $G_{24}$  which are not in  $Q_8$  are the elements of the form  $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$  so that

$$(2.4) \quad G_{24} = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$$

We also note that the center of  $G_{24}$  is the subgroup  $\{\pm 1\}$  and  $Q_8$  is a characteristic subgroup.

**Lemma 2.2.** *Let  $\Gamma$  be either  $\Gamma_E$  or  $\Gamma_H$ .*

*a) The subgroup of  $\mathbb{G}_2(\Gamma)$  generated by  $G_{24}$  and the image of  $1+i$  is a maximal finite subgroup  $G_{48}(\Gamma)$  of  $\mathbb{G}_2(\Gamma)$  of order 48.*

*b)  $G_{48}(\Gamma)$  is a subgroup of  $\mathbb{G}_2^1(\Gamma)$ .*

*c) The quotient  $PG_{48}(\Gamma)$  is isomorphic to  $\mathfrak{S}_4$  independent of  $\Gamma$ .*

*d) There are isomorphisms  $G_{48}(\Gamma_E) \cong GL_2(\mathbb{F}_3)$  and  $G_{48}(\Gamma_H) \cong O_{48}$ . The groups  $GL_2(\mathbb{F}_3)$  and  $O_{48}$  are not isomorphic.*

*e) The intersection  $G_{48}(\Gamma) \cap \mathbb{S}_2^1$  is  $G_{24}$ ,  $PG_{24}$  is isomorphic to  $A_4$  and  $PG_{24} \cap PS_2^1$  is the 2-Sylow subgroup of  $A_4$ , isomorphic to  $C_2 \times C_2$ .*

*Proof.* a) It is easy to see, for example from (2.4), that the element  $1+i$  normalizes the group  $G_{24}$ . The order of  $1+i$  as element of  $\mathbb{D}_2^\times$  is clearly infinite. However, because of  $(1+i)^2 = 2i$  and because of (1.3), its square in  $\mathbb{G}_2(\Gamma)$  is an element of  $\mathbb{S}_2(\Gamma)$ , equal to  $i$  if  $\Gamma = \Gamma_H$  and equal to  $-i$  if  $\Gamma = \Gamma_E$ . Because  $G_{24}$  is a maximal finite subgroup of  $\mathbb{S}_2$  of order 24 it follows that  $G_{48}(\Gamma)$  is a maximal finite subgroup of  $\mathbb{G}_2(\Gamma)$  and is of order 48.

b) Any finite subgroup of  $\mathbb{G}_2(\Gamma)$  is contained in  $\mathbb{G}_2^1(\Gamma)$ .

c) For  $F$  a subgroup of  $G$  let  $N_G(F)$  resp.  $C_G(F)$  denote the normalizer resp. centralizer of  $F$  in  $G$ . Conjugation in  $\mathbb{D}_2^\times$  induces a monomorphism from  $N_{\mathbb{D}_2^\times}(Q_8)/C_{\mathbb{D}_2^\times}(Q_8)$  to  $\text{Aut}(Q_8)$ , the group of automorphisms of  $Q_8$ . The latter group is well known to be isomorphic to  $\mathfrak{S}_4$  and the subgroup  $A_4$  of  $\mathfrak{S}_4$  is realized by conjugation in  $G_{24}/C_2 = PG_{24}$ . The element  $1+i$  belongs to  $N_{\mathbb{D}_2^\times}(Q_8)$  and it is easy to check that conjugation by it does not belong to  $A_4$ . Hence conjugation induces an epimorphism  $PG_{48}(\Gamma) \rightarrow \text{Aut}(Q_8) \cong \mathfrak{S}_4$  which for cardinality reasons has to be an isomorphism.

d) The automorphism group of the elliptic curve with equation  $y^2 + y = x^3$  over  $\mathbb{F}_4$  is isomorphic to  $G_{24}$  (cf. [30]). This group injects into the automorphism group of the formal group law over  $\mathbb{F}_4$ . Because the elliptic curve is already defined over  $\mathbb{F}_2$  we get an injection

$G_{24} \rtimes \text{Gal} \rightarrow \mathbb{G}_2(\Gamma_E)$  and the image is  $G_{48}(\Gamma_E)$ . It is elementary to verify that the group of  $\mathbb{F}_4$ -points of the elliptic curve is of order 9, isomorphic to  $\mathbb{Z}/3 \times \mathbb{Z}/3$  and that  $G_{24} \rtimes \text{Gal}$  realizes all automorphisms of  $E[3]$ . Hence  $G_{48}(\Gamma_E)$  is isomorphic to  $GL_2(\mathbb{F}_3)$ .

Next it is easy to construct an isomorphism between  $O_{48}$  and  $G_{48}(\Gamma_H)$  which restricts to the identity on  $G_{24}$ ; in fact,  $O_{48}$  can be realized within the classical unit quaternions such that  $G_{24}$  corresponds to the subgroup which contains the elements of (2.4) and the element  $(1+i) \in G_{48}(\Gamma_H)$  corresponds to the element  $\frac{1}{\sqrt{2}}(1+i) \in O_{48}$ .

In order to see that  $GL_2(\mathbb{F}_3)$  and  $O_{48}$  are not isomorphic it is enough to see that their 2-Sylow subgroups are not isomorphic. In the case of  $GL_2(\mathbb{F}_3)$  this is the semidihedral group of order 16 while in the case of  $O_{48}$  this is the generalized quaternion group of order 16 and these two groups of order 16 are not isomorphic.

e) This is now obvious. □

Then we define

$$(2.5) \quad G'_{24} := \pi G_{24} \pi^{-1}, \quad G'_{48}(\Gamma) := \pi G_{48}(\Gamma) \pi^{-1}.$$

The groups  $G_{24}$  and  $G'_{24}$  are known to be non-conjugate in  $\mathbb{S}_2^1$  and, up to conjugacy, they are the two maximal finite subgroups of  $\mathbb{S}_2^1$  (cf. [2]). Consequently  $G_{48}(\Gamma)$  and  $G'_{48}(\Gamma)$  are non-conjugate in  $\mathbb{G}_2^1(\Gamma)$  and, up to conjugacy, they are the two maximal finite subgroups of  $\mathbb{G}_2^1(\Gamma)$ . Likewise,  $\mathfrak{S}_4$  and  $\mathfrak{S}'_4$  are non-conjugate in  $P\mathbb{G}_2^1(\Gamma)$  and, up to conjugacy, they are the two maximal finite subgroups of  $P\mathbb{G}_2^1(\Gamma)$ .

**Lemma 2.3.** *Let  $\Gamma$  be either  $\Gamma_E$  or  $\Gamma_H$ .*

a) *The subgroup of  $\mathbb{G}_2(\Gamma)$  generated by  $C_6 = \langle -\omega \rangle$  and the image of  $j-k$  is a subgroup  $G_{12}(\Gamma)$  of  $\mathbb{G}_2(\Gamma)$  of order 12.*

b)  *$G_{12}(\Gamma)$  is a subgroup of  $\mathbb{G}_2^1(\Gamma)$ .*

c) *The quotient  $PG_{12}(\Gamma)$  is isomorphic to  $\mathfrak{S}_3$  independent of  $\Gamma$ .*

d) *There are isomorphisms  $G_{12}(\Gamma_E) \cong C_2 \times \mathfrak{S}_3$  and  $G_{12}(\Gamma_H) \cong C_3 \rtimes C_4$ .*

e) *The intersection  $G_{12}(\Gamma) \cap \mathbb{S}_2^1$  is  $C_6$ ,  $PC_6$  is isomorphic to  $C_3$  and  $PC_6 \cap PS_2^1$  is the trivial group.*

*Proof.* a) The element  $j-k$  normalizes the subgroup  $C_6$  generated by  $-\omega$ . In fact, a direct calculation in the division algebra using that  $\omega = -\frac{1}{2}(1+i+j+k)$  shows

$$(j-k)\omega(j-k)^{-1} = \omega^2.$$

The order of  $j-k$  as element of  $\mathbb{D}_2^\times$  is clearly infinite. However, because of  $(j-k)^2 = -2$ , its square in  $\mathbb{G}_2(\Gamma)$  is an element of  $\mathbb{S}_2(\Gamma)$ , equal to 1 if  $\Gamma = \Gamma_E$  and equal to  $-1$  if  $\Gamma = \Gamma_H$ . Then it is clear that  $G_{12}(\Gamma)$  is of order 12.

b) Any finite subgroup of  $\mathbb{G}_2(\Gamma)$  is contained in  $\mathbb{G}_2^1(\Gamma)$ .

c) This is immediate from the calculation in part a). The image of  $\omega$  in  $PG_{12}(\Gamma)$  generates a normal subgroup of order 3 and the image of  $j-k$  is of order 2 and acts non-trivially on the image of  $\omega$ .

d) This follows because the image of  $j-k$  in  $G_{12}(\Gamma)$  is of order 2 in the case of  $\Gamma = \Gamma_E$  and of order 4 in the case of  $\Gamma = \Gamma_H$ . □

The following two lemmas are elementary and their proof is left to the reader.

**Lemma 2.4.** *Let  $\Gamma$  be either  $\Gamma_E$  or  $\Gamma_H$ .*

a) *The subgroup of  $\mathbb{G}_2(\Gamma)$  generated by  $1 + i$  is a subgroup  $G_8$  of  $\mathbb{G}_2(\Gamma)$  of order 8 which contains  $C_4 = \langle i \rangle$  and is, up to isomorphism, independent of  $\Gamma$ .*

b)  *$G_8$  is a subgroup of  $\mathbb{G}_2^1(\Gamma)$ .*

c) *The quotient  $PG_8$  is isomorphic to  $C_4$ .*

d) *The intersection  $G_8 \cap \mathbb{S}_2^1$  is the subgroup  $C_4$  generated by  $i$ ,  $PC_4$  is isomorphic to  $C_2$  and  $PC_4 \cap PS_2^1 = PC_4$ .  $\square$*

**Lemma 2.5.** *Let  $\Gamma$  be either  $\Gamma_E$  or  $\Gamma_H$ .*

a) *The subgroup of  $\mathbb{G}_2(\Gamma)$  generated by  $-1$  and the Galois group is a subgroup  $G_4$  of  $\mathbb{G}_2(\Gamma)$  of order 4 which is isomorphic to  $C_2 \times \text{Gal}$  independent of  $\Gamma$ .*

b)  *$G_4$  is a subgroup of  $\mathbb{G}_2^1(\Gamma)$ .*

c) *The quotient  $PG_4$  is isomorphic to  $\text{Gal}$ .*

d) *The intersection  $G_4 \cap \mathbb{S}_2^1$  is the subgroup  $C_2$  generated by  $-1$  and  $PC_2 = PC_2 \cap PS_2^1$  is the trivial group.  $\square$*

### 3. THE MOD-2 COHOMOLOGY ALGEBRA OF $PS_2^1$

#### 3.1. Quillen's $F$ -isomorphism for the mod- $p$ cohomology of a profinite group.

Let  $G$  be a profinite group and let  $p$  be a fixed prime. The continuous cohomology  $H_c^*(G; \mathbb{F}_p)$  of  $G$  with coefficients in the trivial module  $\mathbb{F}_p$  will be abbreviated by  $H^*(G; \mathbb{F}_p)$ , or simply by  $H^*G$  if  $p$  is understood from the context. We recall that if  $G$  is the (inverse) limit of finite groups  $G_i$  then  $H^*G = \text{colim}_i H^*G_i$ .

We will assume that  $H^*G$  is finitely generated as  $\mathbb{F}_p$ -algebra. By work of Lazard [23] it is known that this holds for many interesting profinite groups, for example for profinite  $p$ -analytic groups like  $GL(n, \mathbb{Z}_p)$ , the general linear groups over the  $p$ -adic integers, or the automorphism groups of formal group laws over finite fields.

In case  $H^*G$  is finitely generated as  $\mathbb{F}_p$ -algebra Quillen has shown [26] that there are only finitely many conjugacy classes of elementary abelian  $p$ -subgroups of  $G$  (i.e. groups isomorphic to  $(\mathbb{Z}/p)^n$  for some natural number  $n$ ). In other words, the following category  $\mathcal{A}(G)$  is equivalent to a finite category: objects of  $\mathcal{A}(G)$  are all elementary abelian  $p$ -subgroups of  $G$ , and if  $E_1$  and  $E_2$  are elementary abelian  $p$ -subgroups of  $G$ , then the set of morphisms from  $E_1$  to  $E_2$  in  $\mathcal{A}(G)$  consists precisely of those homomorphisms  $\alpha : E_1 \rightarrow E_2$  of abelian groups for which there exists an element  $g \in G$  with  $\alpha(e) = geg^{-1}$  for all  $e \in E_1$ . The assignment  $E \mapsto H^*E$  determines a functor from the opposite category  $\mathcal{A}_*(G)^{op}$  to graded  $\mathbb{F}_p$ -algebras.

**Theorem 3.1. (Quillen)** [26] *Let  $G$  be a profinite group and assume  $H^*G$  is a finitely generated  $\mathbb{F}_p$ -algebra. Then the canonical map*

$$q_G : H^*G \rightarrow \lim_{\mathcal{A}(G)^{op}} H^*E$$

*is an  $F$ -isomorphism, in other words  $q$  has the following properties.*

- *If  $x \in \text{Ker}q_G$ , then  $x$  is nilpotent.*

- If  $y \in \lim_{\mathcal{A}(G)}{}^{op} H^*E$  then there exists an integer  $n$  with  $y^{p^n} \in \text{Im}q$ .

In the sequel we will call  $\mathcal{A}(G)$  the Quillen category of  $G$ .

Let  $\mathcal{A}_*(G)$  be the full subcategory of  $\mathcal{A}(G)$  whose objects are all elementary abelian  $p$ -subgroups except the trivial subgroup. The centralizer  $C_G(E)$  of an elementary abelian  $p$ -subgroup  $E$  is a closed subgroup and hence inherits a natural profinite structure from  $G$ . The assignment  $E \mapsto H^*C_G(E)$  extends to a functor from  $\mathcal{A}_*(G)$  to graded  $\mathbb{F}_p$ -algebras and the restriction homomorphisms  $H^*G \rightarrow H^*C_G(E)$  (for  $E$  running through the non-trivial elementary abelian  $p$ -subgroups of  $G$ ) induce a canonical map  $\rho : H^*G \rightarrow \lim_{\mathcal{A}_*(G)} H^*C_G(E)$ . The main result of [17] reads as follows.

**Theorem 3.2.** *Let  $G$  be a profinite group and assume  $H^*G$  is a finitely generated  $\mathbb{F}_p$ -algebra. Then the canonical map  $\rho : H^*G \rightarrow \lim_{\mathcal{A}_*(G)} H^*C_G(E)$  has finite kernel and cokernel.*

**Remark 3.3.** a) In our current approach this theorem is no longer needed. However, it played a crucial role in our initial approach to construct resolutions for  $PS_2^1$  and is ultimately the reason for naming our resolutions centralizer resolutions. Furthermore, in [18] the theorem played a crucial role for constructing algebraic centralizer resolutions at odd primes, which as the algebraic resolutions of this paper are  $\mathcal{F}$ -resolutions in the sense of Section 1.2.

b) Theorem 3.2 is not useful if  $G$  contains central elements of order  $p$ , because then  $H^*G$  appears in the limit. In these cases one can use the theorem to study the quotient of  $G$  by the maximal central elementary abelian  $p$ -subgroup of  $G$  and this was the origin for considering the groups  $PS_2^1$  and  $PS_2^1$ .

### 3.2. The Quillen category of $PS_2^1$ .

We recall from section 2 that  $S_2^1$  contains two subgroups isomorphic to  $Q_8$  and they give rise to two elementary abelian 2-subgroups  $E_2$  and  $E_2'$  in  $PS_2^1$  which are contained in the normal 2-Sylow subgroup  $PS_2^1$ .

The following result has a significant overlap with section 2.4 of [2].

**Proposition 3.4.** a) *Up to conjugacy  $PS_2^1$  contains three elementary abelian 2-subgroups of rank 1 and two of rank 2.*

b) *All automorphism groups of the category  $\mathcal{A}(PS_2^1)$  are trivial and there is exactly one morphism from each rank 1 group to each of the rank 2 groups.*

*Proof.* a) If  $E$  is an elementary abelian 2-subgroup of  $PS_2^1$  then its inverse image  $\tilde{E}$  in  $S_2^1$  is an extension of  $E$  by  $\mathbb{Z}/2$ . The structure of the possible finite 2-subgroups of the division algebra  $\mathbb{D}_2$  is explicitly known: in fact, any finite abelian subgroup must be cyclic and generates in the division algebra a cyclotomic extension the degree of which must divide 2. Hence any abelian 2-subgroup is cyclic of order 2 or 4 and this implies that any finite 2-subgroup is isomorphic to a subgroup of  $Q_8$ . In particular we see that the 2-rank of  $E$  is either 1 or 2.

Now suppose that  $F_1$  and  $F_2$  are two elementary abelian 2-subgroups of rank 1 of  $PS_2^1$ . Then  $\tilde{F}_1$  and  $\tilde{F}_2$  are two subgroups isomorphic to  $\mathbb{Z}/4$  and by the Skolem Noether theorem any isomorphism  $\varphi : \tilde{F}_1 \rightarrow \tilde{F}_2$  can be realized by conjugation by an element of  $u \in \mathbb{D}_2^\times$ , i.e.  $\varphi(x) = uxu^{-1}$  for any  $x \in \tilde{F}_1$ . If we denote a generator of  $\tilde{F}_1$  by  $i$  then  $1+i \in \mathbb{D}_2^\times$  centralizes  $F_1$ , so we can change  $u$  by any power of  $(1+i)$  and conjugation by  $u(1+i)^n$  will still give  $\varphi$ . Because the valuation of  $1+i$  is  $\frac{1}{2}$  we can choose  $n$  such that  $(1+i)^n u$  is of valuation 0. In other words, we can suppose that  $u$  is an element of  $\mathbb{S}_2$ . Furthermore, the element  $1+2i \in \mathbb{S}_2$

has reduced norm 5 and is thus a topological generator of  $\mathbb{S}_2/\mathbb{S}_2^1$ . It also centralizes  $\tilde{F}_1$  and by multiplying  $u$  by a suitable  $p$ -adic power of  $1 + 2i$  we can even assume that  $u$  is in  $\mathbb{S}_2^1$ . This implies that all rank 1 subgroups of  $PS_2^1$  are conjugate and therefore the quotient group  $\mathbb{S}_2^1/\mathbb{S}_2^1 \cong \mathbb{F}_4^\times$  which is generated by the image of  $\omega$  acts transitively on the  $PS_2^1$ -conjugacy classes of elementary abelian 2-subgroups of rank 1.

Thus there are either three or one  $PS_2^1$ -conjugacy classes of elementary abelian 2-subgroups of rank 1. If there was only one then conjugation by  $\omega$  would have to be the same as conjugation by an element in  $PS_2^1$  and this would mean that there is an element in  $\mathbb{S}_2^1$  of the form  $\omega u'$  with  $u' \in \mathbb{S}_2^1$  whose image in  $PS_2^1$  centralizes  $F_1$ , and hence  $\omega u'$  normalizes  $\tilde{F}_1$ . However,  $N_{\mathbb{S}_2^1}(\tilde{F}_1)/C_{\mathbb{S}_2^1}(\tilde{F}_1)$  is isomorphic to a subgroup of

$$N_{\mathbb{D}_2^\times}(\tilde{F}_1)/C_{\mathbb{D}_2^\times}(\tilde{F}_1) \cong \text{Aut}(\tilde{F}_1) \cong C_2$$

hence  $N_{\mathbb{S}_2}(\tilde{F}_1)$  contains the centralizer  $C_{\mathbb{S}_2}(\tilde{F}_1) \cong \mathbb{Z}_2[i]^\times$  as a subgroup of index at most 2. This implies that  $N_{\mathbb{S}_2}(\tilde{F}_1)$  is a profinite 2-group and cannot contain such an element which would have non-trivial image in  $\mathbb{F}_4^\times$ .

Next suppose  $F_1$  and  $F_2$  are two elementary abelian 2-subgroups of rank 2 of  $PS_2^1$ . Then  $\tilde{F}_1$  and  $\tilde{F}_2$  are two subgroups of  $\mathbb{S}_2^1$  isomorphic to  $Q_8$  and again by the Skolem Noether theorem any isomorphism  $\varphi : \tilde{F}_1 \rightarrow \tilde{F}_2$  can be realized by conjugation by an element of  $u \in \mathbb{D}_2^\times$ , i.e.  $\varphi(x) = uxu^{-1}$  for any  $x \in \tilde{F}_1$ . In particular, we have an isomorphism

$$N_{\mathbb{D}_2^\times}(F_1)/C_{\mathbb{D}_2^\times}(F_1) \cong \text{Aut}(Q_8) \cong \mathfrak{S}_4 .$$

In order to determine the number of conjugacy classes of elementary abelian 2-subgroups of rank 2 of  $PS_2^1$  we need to know something about the structure of the normalizer  $N_{\mathbb{S}_2}(Q_8)$ . The centralizer  $C_{\mathbb{D}_2^\times}(Q_8)$  is isomorphic to  $\mathbb{Q}_2^\times$  and the quotient  $N_{\mathbb{D}_2^\times}(Q_8)/C_{\mathbb{D}_2^\times}(Q_8)$  is generated by the image of the group  $G_{24}$  and the element  $1 + i$  (cf. the proof of part a) of Lemma 2.2). Furthermore the centralizer  $C_{\mathbb{S}_2}(Q_8)$  is isomorphic to  $\mathbb{Z}_2^\times$  and we get an isomorphism

$$(3.1) \quad N_{\mathbb{S}_2}(Q_8) \cong \mathbb{Z}_2^\times \times_{C_2} G_{24}$$

between  $N_{\mathbb{S}_2}(Q_8)$  and the central product  $\mathbb{Z}_2^\times \times_{C_2} G_{24}$  and an isomorphism

$$(3.2) \quad N_{\mathbb{S}_2}(Q_8)/C_{\mathbb{S}_2}(Q_8) \cong PG_{24} = A_4 .$$

Because the normalizer  $N_{\mathbb{D}_2^\times}(F_1)$  always contains an element  $y$  of valuation  $\frac{1}{2}$ , we can assume by changing  $u$ , if necessary, by a suitable power of  $y$  that there is an isomorphism  $\psi : \tilde{F}_1 \rightarrow \tilde{F}_2$  which is realized by conjugation in  $\mathbb{S}_2$ . In particular, in  $\mathbb{S}_2$  there is only one conjugacy class of subgroups isomorphic to  $Q_8$  and in  $PS_2$  there is only one conjugacy class of elementary abelian 2-subgroups of rank 2. This means that the group  $\mathbb{S}_2/\mathbb{S}_2^1$  acts transitively on the set of conjugacy classes of subgroups of  $\mathbb{S}_2^1$  which are isomorphic to  $Q_8$ . Because the center acts trivially on the set of conjugacy classes and the image of the center in  $\mathbb{S}_2/\mathbb{S}_2^1$  is of index 2 there are at most two conjugacy classes of  $Q_8$ 's in  $\mathbb{S}_2^1$ . We claim that there are two of them given by  $Q_8$  and  $\pi Q_8 \pi^{-1}$  where  $Q_8$  is the 2-Sylow subgroup of the group  $G_{24}$  of section 2 and  $\pi \in \mathbb{S}_2$  is the element defined in (2.1). In fact, if they were conjugate then  $\pi$  could be written as product  $xn$  with  $x \in \mathbb{S}_2^1$  and  $n \in N_{\mathbb{S}_2}(\tilde{F}_1)$ . However, from (3.1) we see that the reduced norm of such an element is always a square in  $\mathbb{Z}_2^\times$  and this contradicts the fact that the reduced norm of  $\pi$  is 3.

b) It is clear that the automorphism groups of elementary abelian 2-subgroups of rank 1 are trivial. For the automorphisms of a rank 2 subgroup we note that (3.2) implies that the automorphism group  $\text{Aut}_{\mathcal{A}(PS_2)}(PQ_8)$  is  $C_3$  because conjugation by any element of the subgroup  $Q_8$  of  $G_{24}$  induces the trivial automorphism. This in turn implies that  $\text{Aut}_{\mathcal{A}(PS_2^1)}(PQ_8)$  is trivial.

It remains to show that there is exactly one morphism from each rank 1 to each rank 2 object, or equivalently, that the three non-trivial elements in a rank 2 object are non-conjugate in  $PS_2^1$ . If they were conjugate in  $PS_2^1$ , then there would be an element in  $PS_2^1$  of the form  $\omega^{\pm 1}x$  with  $x$  in  $PS_2^1$  which centralizes the rank 1 subgroup generated by one of these elements, respectively its preimage in  $\mathbb{S}_2^1$  would normalize the preimage, and this contradicts what we have seen in the proof of part a) above.  $\square$

**Remark 3.5.** We can choose representatives  $E_2$  and  $E'_2$  for the two conjugacy classes of elementary abelian 2-subgroups of rank 2 such that  $E_2 \cap E'_2$  is cyclic of order 2. In fact, if  $E_2$  is such that  $\tilde{E}_2$  is the subgroup of  $G_{24}$  generated  $i$  and  $j$  then conjugation by  $1 + 2i$  fixes  $i$  and carries  $\tilde{E}_2$  to  $\tilde{E}'_2$ . In  $\mathbb{S}_2^1$  the group  $\tilde{E}_2$  is not conjugate to  $\tilde{E}'_2$ , hence in the quotient  $PS_2^1$  we get that  $E_2$  and  $E'_2$  are non-conjugate and intersect in the subgroup generated by the image of  $i$ .

### 3.3. Quillen's $F$ -isomorphism for $PS_2^1$ and the mod-2 cohomology algebra of $PS_2^1$ .

The inverse limit in Quillen's Theorem 3.1 is always a subalgebra of the product  $\prod_E H^*E$  where  $E$  runs through the maximal elementary abelian subgroups of  $G$ , up to conjugacy. By Theorem 3.4 there are, in the case of  $G = PS_2^1$ , two of them, both of rank 2 with mod-2 cohomology both given by  $\mathbb{F}_2[x, y]$  with  $x$  and  $y$  of cohomological degree 1.

**Proposition 3.6.** *There is an isomorphism of graded  $\mathbb{F}_2$ -algebras*

$$\lim_{\mathcal{A}(PS_2^1)} H^*E \cong \{(p_1, p_2) \in \mathbb{F}_2[x, y] \times \mathbb{F}_2[x, y] \mid p_1 - p_2 \text{ is divisible by } xy(x + y)\}$$

*Proof.* If  $E_1$  and  $E_2$  are two non-conjugate elementary abelian 2-subgroups of rank 2 of  $PS_2^1$  then the non-trivial elements of  $E_1$  and  $E_2$  belong to the three non-conjugate elementary abelian 2-subgroups  $F_1$ ,  $F_2$  and  $F_3$  of rank 1. This gives 6 morphisms in  $\mathcal{A}(PS_2^1)$ , and if we choose the non-trivial elements of  $E_1$  and  $E_2$  as  $e_1^j$ ,  $e_2^j$  and  $e_3^j$  for  $j = 1, 2$  then we have morphisms  $\alpha_{i,j} : F_i \rightarrow E_j$  which send the nontrivial element of  $F_i$  to the element  $e_i^j$  of  $E_j$ .

Then the inverse limit is given by pairs of polynomials  $(p_1, p_2) \in H^*E_1 \times H^*E_2$  such that  $\alpha_{i,1}^* p_1 = \alpha_{i,2}^* p_2$  for  $i = 1, 2, 3$ , or if we identify  $H^*E_1$  with  $H^*E_2$  via the abstract group isomorphism which sends  $e_i^1$  to  $e_i^2$  for  $i = 1, 2, 3$  then  $p_1 - p_2$  must be divisible by the three non-trivial elements in  $\mathbb{F}_2[x, y]$  and the claim follows.  $\square$

The quotient homomorphism  $S_2^1 \rightarrow S_2^1/F_1S_2^1 \cong \mathbb{F}_4$  induces a surjection  $PS_2^1 \rightarrow \mathbb{F}_4$  and the explicit form of the elements  $i$ ,  $j$  and  $k$  given in Lemma 2.1 shows that both subgroups  $E_2$  and  $E'_2$  map isomorphically to this quotient.

**Corollary 3.7.** *As a module over  $H^*(S_2^1/F_1S_2^1) \cong \mathbb{F}_2[x, y]$  the inverse limit is the free submodule of  $\mathbb{F}_2[x, y] \times \mathbb{F}_2[x, y]$  generated by the classes  $(1, 1)$  and  $(xy(x + y), 0)$ .*  $\square$

The following result describes the algebraic centralizer resolution of the trivial  $\mathbb{S}_2^1$ -module  $\mathbb{Z}_2$ . It has been established in Theorem 1.2.1 and 1.2.6 of [2]. The subgroups of  $\mathbb{S}_2^1$  occurring in the statement are those of (1.5) and  $IS_2^1$  is the augmentation ideal of the completed group algebra  $\mathbb{Z}_2[[\mathbb{S}_2^1]]$ . The notation used is analogous to that of Section 1.2. In other words, if  $G$  is a profinite group and  $X$  is a profinite  $G$ -set such that  $X = \lim_i X_i$  with finite  $\mathbb{S}_2^1$ -sets  $X_i$  then we define

$$(3.3) \quad \mathbb{Z}_2[[X]] = \lim_{i,k} \mathbb{Z}/2^k[[X_i]] ,$$

and if  $F$  is a finite subgroup of  $G$  and  $M$  is a  $\mathbb{Z}_2[F]$ -module then  $M \uparrow_F^G$  denotes the  $\mathbb{Z}_2[[G]]$ -module  $\mathbb{Z}_2[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[F]} M$ .

**Theorem 3.8.**

a) There is an exact complex of profinite  $\mathbb{Z}_2[[S_2^1]]$ -modules

$$0 \rightarrow \mathbb{Z}_2 \uparrow_{G_{24}^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{C_6^1} \xrightarrow{\partial_2} \mathbb{Z}_2 \uparrow_{C_6^1} \xrightarrow{\partial_1} \mathbb{Z}_2 \uparrow_{G_{24}^1} \xrightarrow{\varepsilon} \mathbb{Z}_2 \rightarrow 0$$

b) The maps  $\partial_1$ ,  $\partial_2$  and  $\partial_3$  are trivial modulo  $(2, IS_2^1)$ .  $\square$

**Remark 3.9.** a) In fact, the central subgroup  $C_2$  acts trivially in this complex and according to table (1.5) the complex can be considered as a complex of profinite  $PS_2^1$ -modules

$$(3.4) \quad 0 \rightarrow \mathbb{Z}_2 \uparrow_{A_4^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{C_3^1} \xrightarrow{\partial_2} \mathbb{Z}_2 \uparrow_{C_3^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{A_4^1} \xrightarrow{\varepsilon} \mathbb{Z}_2 \rightarrow 0$$

or even as a complex of profinite  $PS_2^1$ -modules

$$(3.5) \quad 0 \rightarrow \mathbb{Z}_2 \uparrow_{E_2^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{\{1\}} \xrightarrow{\partial_2} \mathbb{Z}_2 \uparrow_{\{1\}} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{E_2^1} \xrightarrow{\varepsilon} \mathbb{Z}_2 \rightarrow 0 .$$

b) For every profinite  $PS_2^1$ -module  $M$  there is a duality spectral sequence associated to the complex of (3.5)

$$E_1^{s,t} = \text{Ext}_{\mathbb{Z}_2[[PS_2^1]]}^s(C_t, M) \implies \text{Ext}_{\mathbb{Z}_2[[PS_2^1]]}^{s+t}(\mathbb{Z}_2, M) \cong H^{s+t}(PS_2^1, M)$$

with

$$C_s = \begin{cases} \mathbb{Z}_2 \uparrow_{E_2^1} & s = 0 \\ \mathbb{Z}_2 \uparrow_{\{1\}} & s = 1, 2 \\ \mathbb{Z}_2 \uparrow_{E_2^1} & s = 3 \\ 0 & \text{else .} \end{cases}$$

If  $M = \mathbb{F}_2$  we can identify the  $E_1$ -term via the usual Shapiro-type isomorphisms with

$$(3.6) \quad E_1^{s,*} = \begin{cases} H^*(E_2) \cong \mathbb{F}_2[x, y] & s = 0 \\ H^*(\{1\}) \cong \mathbb{F}_2 & s = 1, 2 \\ H^*(E_2) \cong \mathbb{F}_2[x, y] & s = 3 \\ 0 & \text{else .} \end{cases}$$

**Proposition 3.10.** *The duality spectral sequence for the group  $PS_2^1$  and  $M = \mathbb{F}_2$  collapses at  $E_1$ .*

*Proof.* By part b) of Theorem 3.8 we have  $d_1 = 0$  and by (3.6) any higher differential would have to originate at the vertical edge. However, as we have noted before the composition of the inclusion of  $E_2$  into  $PS_2^1$  followed by the quotient map  $PS_2^1 \rightarrow \mathbb{F}_4$  is an isomorphism. This implies that the vertical edge of the duality spectral sequence survives to  $E_\infty$ , in particular all differentials originating at the vertical edge are trivial.  $\square$

**Theorem 3.11.**

a) The map of Theorem 3.1

$$q_{PS_2^1} : H^*(PS_2^1) \rightarrow \lim_{\mathcal{A}(PS_2^1)^{op}} H^*(E)$$

is surjective with kernel  $\Sigma\mathbb{F}_2 \oplus \Sigma^2\mathbb{F}_2$  where  $\Sigma^k\mathbb{F}_2$  is the graded  $\mathbb{F}_2$ -module  $\mathbb{F}_2$  concentrated in degree  $k$ .

b) The Poincaré series  $\chi := \sum_{n \geq 0} \dim_{\mathbb{F}_2} H^n(PS_2^1)t^n$  is given by

$$\chi = \frac{1 + t^3}{(1 - t)^2 + t + t^2} .$$

c) The Bockstein homomorphism induces an isomorphism between the kernel of  $q_{PS_2^1}$  in cohomological degree 1 and 2.

*Proof.* a) The spectral sequence (3.6) is one of modules over  $H^*(PS_2^1/F_1PS_2^1, \mathbb{F}_2) \cong \mathbb{F}_2[x, y]$ . By Proposition 3.10 we get a filtration

$$0 \subset G_3 \subset G_2 \subset G_1 \subset G_0 = H^*(PS_2^1)$$

by  $H^*(PS_2^1/F_1PS_2^1) \cong \mathbb{F}_2[x, y]$ -modules  $G_i$  with associated graded given by

$$G_0/G_1 = \mathbb{F}_2[x, y], \quad G_1/G_2 \cong \Sigma\mathbb{F}_2, \quad G_2/G_3 = \Sigma^2\mathbb{F}_2, \quad G_3 \cong \Sigma^3\mathbb{F}_2[x, y].$$

Because both inclusions  $E_2 \subset PS_2^1$  and  $E_2' \subset PS_2^1$  split the projection map from  $PS_2^1$  to  $PS_2^1/F_1PS_2^1$  the image of  $q_G$  maps onto the diagonal in  $H^*(E_2) \times H^*(E_2')$ . Furthermore  $G_1$  maps trivially to  $H^*(E_2)$ . By linearity with respect to  $H^*(PS_2^1/F_1PS_2^1)$  the quotient  $G_1/G_3$  maps also trivially to  $H^*(E_2')$  and then the generator of the  $\mathbb{F}_2[x, y]$ -module  $G_3$  must map non-trivially to  $(0, xy(x+y)) \in \lim_{\mathcal{A}(PS_2^1)_{op}} H^*(E)/q_G(G_0/G_2)$  because otherwise Quillen's map could not be an  $F$ -isomorphism. Part a) follows.

b) This is an immediate consequence of part a) and Corollary 3.7.

c) It is enough to show that the class in  $H^1(PS_2^1, \mathbb{F}_2)$  detected in  $G_1/G_2$  lifts to a class in  $H^1(PS_2^1, \mathbb{Z}_2)$  of order 2. The cohomology of the groups  $S_2$  has been first investigated by Ravenel [27]. For a recent account which stresses the group theoretical point of view see [19].

In particular, it follows from Proposition 3.5.3 of [19] that the mod-2 reduction homomorphism  $H_1(PS_2^1, \mathbb{Z}_2) \rightarrow H_1(PS_2^1, \mathbb{F}_2)$  is an isomorphism and both groups are isomorphic to  $(\mathbb{Z}/2)^3$ . As a consequence we find  $H^1(PS_2^1, \mathbb{Z}_2) = 0$ ,  $H^1(PS_2^1, \mathbb{F}_2) \cong (\mathbb{Z}/2)^3$  and the mod-2 Bockstein homomorphism  $H^1(PS_2^1, \mathbb{F}_2) \rightarrow H^2(PS_2^1, \mathbb{F}_2)$  is injective. Part c) follows.  $\square$

## 4. ALGEBRAIC CENTRALIZER RESOLUTIONS

### 4.1. Galois-twisted modules.

We take up the notions introduced in Section 1.2. So we assume that  $n > 1$  is an integer,  $p$  is a prime and  $\text{Gal}$  is the Galois group of the field extension  $\mathbb{F}_p \subset \mathbb{F}_q$  where  $q = p^n$ , and  $\mathbb{W}$  denotes the ring of Witt vectors of  $\mathbb{F}_q$ . Furthermore let  $G$  be a profinite group equipped with a continuous homomorphism  $\phi : G \rightarrow \text{Gal}$  and let  $S$  be the kernel of  $\phi$ . As before we consider the *Galois-twisted completed group ring*  $\mathbb{W}_\phi[[G]]$  of  $G$  and *Galois-twisted  $p$ -profinite  $\mathbb{W}_\phi[[G]]$ -modules*. In order to keep notations simple we will, as before, simply write  $\mathbb{W}[[G]]$  instead of  $\mathbb{W}_\phi[[G]]$ .

Note that for  $n = 1$  we recover the usual group ring  $\mathbb{Z}_p[[G]]$ . In general, the action of  $S$  on a Galois-twisted profinite  $G$ -module is  $\mathbb{W}$ -linear while the action of  $G$  is only  $\mathbb{Z}_p$ -linear. The groups we have in mind are  $\mathbb{G}_n$ ,  $\mathbb{G}_n^1$ ,  $P\mathbb{G}_n$  and closed subgroups of these groups, in particular in the case  $n = p = 2$ .

A crucial input for the sequel is the following result.

**Proposition 4.1.** *Suppose that  $G$  is a finite group and  $P$  is a Galois-twisted  $p$ -profinite  $G$ -module.*

a) *If  $P$  is  $\mathbb{W}[[S]]$ -projective then  $P$  is  $\mathbb{W}[[G]]$ -projective.*

b) *If  $P$  is  $\mathbb{F}_q[[S]]$  projective then  $P$  is  $\mathbb{F}_q[[G]]$ -projective.*



*Proof.* We give the proof of part a). The proof of part b) is completely analogous. Let  $\varphi : P \rightarrow M$  be a homomorphism of Galois-twisted profinite  $G$ -modules and let  $\pi : M \rightarrow N$  be an epimorphism of Galois-twisted profinite  $G$ -modules. Because  $P$  is  $\mathbb{W}[[S]]$ -projective there exist a  $\mathbb{W}[[S]]$ -linear homomorphism  $\tilde{\varphi} : P \rightarrow M$  such that  $\pi\tilde{\varphi} = \varphi$ .

Let  $\sigma$  be the generator of Gal given by the Frobenius homomorphism  $x \mapsto x^p$  of  $\mathbb{F}_q$  and denote its lift to  $\mathbb{W}$  still by  $\sigma$ . The trace map  $tr : \mathbb{F}_q \rightarrow \mathbb{F}_p$  is surjective by Hilbert 90, and therefore the trace map  $tr : \mathbb{W} \rightarrow \mathbb{Z}_p$  is also surjective, in particular there exists an element  $\lambda \in \mathbb{W}$  with  $tr(\lambda) = 1$ . Furthermore, if  $h \in S$ ,  $g \in G$  and  $x \in P$  then

$${}^{gh}(\lambda)gh\tilde{\varphi}(h^{-1}g^{-1}x) = ({}^g\lambda)g\tilde{\varphi}(g^{-1}x),$$

hence  $({}^g\lambda)g\tilde{\varphi}(g^{-1}x)$  is constant on  $S$ -orbits for the translation action of  $S$  on  $G$  on the right and

$$\psi : P \rightarrow M, \quad x \mapsto \sum_{g \in G/S} ({}^g\lambda)g\tilde{\varphi}(g^{-1}x)$$

is a well-defined. Furthermore,  $\psi$  is a  $\mathbb{W}[[G]]$ -linear map. In fact, if  $h$  is in  $G$  then

$$\begin{aligned} h\psi(x) &= \sum_{g \in G/S} {}^h({}^g\lambda)hg\tilde{\varphi}(g^{-1}x) = \sum_{hg \in G/S} ({}^{hg}\lambda)hg\tilde{\varphi}(g^{-1}x) \\ &= \sum_{hg \in G/S} ({}^{hg}\lambda)hg\tilde{\varphi}((hg)^{-1}hx) = \psi(hx). \end{aligned}$$

Furthermore

$$\begin{aligned} \pi\psi(x) &= \sum_{g \in G/S} ({}^g\lambda)g\pi\tilde{\varphi}(g^{-1}x) = \sum_{g \in G/S} ({}^g\lambda)g\varphi(g^{-1}x) \\ &= \sum_{g \in G/S} ({}^g\lambda)\varphi(x) = tr(\lambda)\varphi(x) = \varphi(x) \end{aligned}$$

and this shows that  $P$  is projective.  $\square$

**Corollary 4.2.** *Suppose that  $G$  is finite and  $S$  is of order prime to  $p$ . Then the trivial Galois-twisted profinite  $G$ -module  $\mathbb{W}$  resp. the trivial Galois-twisted profinite  $G$ -module  $\mathbb{F}_q$  is a projective  $\mathbb{W}[[G]]$ -module resp.  $\mathbb{F}_q[[G]]$ -module.  $\square$*

**Lemma 4.3.** *Suppose  $G$  is a finite group and let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be an exact sequences of Galois-twisted  $p$ -profinite  $G$ -modules which is split as sequence of  $\mathbb{W}$ -modules. If  $M_1$  is projective then the sequence is split as a sequence of Galois-twisted profinite  $G$ -modules.*

*Proof.* This proof is actually extracted from the proof of Lemma 16 of [18]. We begin with the following observation. If  $M_1$  is projective as Galois-twisted profinite  $G$ -module then  $M_1$  is a direct summand in the induced module  $\mathbb{W}[G] \otimes_{\mathbb{W}} M_1$ . Furthermore for a finite group the induced module and the coinduced module  $\text{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$  are isomorphic.

The existence of a  $\mathbb{W}$ -linear splitting of the monomorphism  $M_1 \rightarrow M_2$  implies that any  $\mathbb{W}[G]$ -linear map  $\varphi$  from  $M_1$  to the coinduced module  $\text{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$  can be extended to a  $\mathbb{W}[G]$ -linear map  $\tilde{\varphi} : M_2 \rightarrow \text{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$ . Now we take for  $\varphi$  any  $\mathbb{W}[G]$ -split inclusion of  $M_1$  into  $\text{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$ . Then the composition of  $\tilde{\varphi}$  with a  $\mathbb{W}[G]$ -linear splitting of  $\varphi$  provides the desired splitting of the monomorphism  $M_1 \rightarrow M_2$ .  $\square$

The other crucial input in the construction of the centralizer resolution is the following Nakayama type lemma which is analogous to Lemma 4.3 of [14]. We say that a profinite  $p$ -group  $G$  is finitely generated if  $H_1(G, \mathbb{F}_q)$  is a finite dimensional  $\mathbb{F}_q$ -vector space. In the sequel the kernel of the augmentation  $\mathbb{W}[[G]] \rightarrow \mathbb{F}_q$  is denoted by  $I_p G$ , or simply by  $I$  if  $p$  and  $G$  are clear from the context. Then a Galois-twisted  $p$ -profinite  $\mathbb{W}[[G]]$ -module  $M$  is automatically  $I$ -complete, i.e the filtration by the submodules  $I^n M$ ,  $n \geq 0$ , is complete.

**Lemma 4.4.** *Let  $G$  be a finitely generated profinite  $p$ -group and  $f : M \rightarrow N$  a morphism of complete Galois-twisted  $p$ -profinite  $\mathbb{W}[[G]]$ -modules.*

a) *If  $\text{Tor}_0(f) : \text{Tor}_0^{\mathbb{W}[[G]]}(M, \mathbb{F}_q) \rightarrow \text{Tor}_0^{\mathbb{W}[[G]]}(N, \mathbb{F}_q)$  is surjective, then  $f$  is surjective.*

b) If  $\text{Tor}_q(f) : \text{Tor}_q^{\mathbb{W}[[G]]}(M, \mathbb{F}_q) \rightarrow \text{Tor}_q^{\mathbb{W}[[G]]}(N, \mathbb{F}_q)$  is an isomorphism for  $q = 0$  and surjective for  $q = 1$  then  $f$  is an isomorphism.  $\square$

#### 4.2. The algebraic centralizer resolution for $PG_2^1$ and $PG_2$ .

The following theorem establishes Theorem 1.2 of the introduction. The finite subgroups of  $PG_2^1$  occurring in this section are those specified in table (1.5) and section 2.

**Theorem 4.5.** *There are  $\mathcal{F}$ -resolutions of the trivial Galois-twisted profinite  $PG_2^1$ -module resp.  $G_2^1$ -module  $\mathbb{W}$*

$$0 \longrightarrow \mathbb{W} \uparrow_{\mathfrak{S}_3}^{PG_2^1} \xrightarrow{\partial_3} \mathbb{W} \uparrow_{\text{Gal}}^{PG_2^1} \xrightarrow{\partial_2} \mathbb{W} \uparrow_{\mathfrak{S}_3}^{PG_2^1} \oplus \mathbb{W} \uparrow_{C_4}^{PG_2^1} \xrightarrow{\partial_1} \mathbb{W} \uparrow_{\mathfrak{S}_4}^{PG_2^1} \oplus \mathbb{W} \uparrow_{\mathfrak{S}'_4}^{PG_2^1} \xrightarrow{\varepsilon} \mathbb{W}$$

and

$$0 \longrightarrow \mathbb{W} \uparrow_{G_2^1}^{G_2^1} \xrightarrow{\partial_3} \mathbb{W} \uparrow_{C_2 \times \text{Gal}}^{G_2^1} \xrightarrow{\partial_2} \mathbb{W} \uparrow_{G_2^1}^{G_2^1} \oplus \mathbb{W} \uparrow_{C_8}^{G_2^1} \xrightarrow{\partial_1} \mathbb{W} \uparrow_{G_{48}^1}^{G_2^1} \oplus \mathbb{W} \uparrow_{G_{48}^1}^{G_2^1} \xrightarrow{\varepsilon} \mathbb{W} .$$

*Proof.* The second complex is obtained from the first one by simply considering a complex of profinite  $\mathbb{W}[[PG_2^1]]$ -modules as a complex of profinite  $\mathbb{W}[[G_2^1]]$ -modules via the canonical projection  $G_2^1 \rightarrow PG_2^1$ . The property of being an  $\mathcal{F}$ -resolutions is preserved by the analogue of Lemma 14 of [18]. So we concentrate on constructing the first complex. In order to simplify notation we will write this complex in the sequel as

$$0 \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}_2 .$$

The existence of an exact complex follows from splicing the exact sequences of Lemma 4.6, Lemma 4.9 and Lemma 4.10 below. The  $\mathcal{F}$ -projectivity of the resolution will be established in Lemma 4.11 below.  $\square$

Our strategy for the remainder of the proof of Theorem 4.5 is analogous to the strategy used in section 4 of [14] in the construction of the duality resolution for  $p = 3$ . In the following computations we will abbreviate  $\text{Ext}_{\mathbb{W}[[PS_2^1]]}^*(M, \mathbb{F}_4)$  simply by  $\text{Ext}^*(M)$  and  $\text{Tor}_*^{\mathbb{W}[[PS_2^1]]}(\mathbb{F}_4, M)$  simply by  $\text{Tor}_*(M)$ . We observe that we have isomorphisms

$$(4.1) \quad \text{Ext}^q(M) \cong \text{Tor}_q(M)^*$$

for any profinite  $\mathbb{W}[[PS_2^1]]$ -module  $M$  if  $(-)^*$  denotes the  $\mathbb{F}_4$ -linear dual.

We also note that  $\text{Ext}^*(-)$  and  $\text{Tor}_*(-)$  define functors from the category of Galois-twisted profinite  $\mathbb{W}[[PG_2^1]]$ -modules to Galois-twisted  $\mathbb{F}_4[[PG_2^1/PS_2^1]]$ -modules and this will be important in the proof of Lemma 4.8, Lemma 4.9 and Lemma 4.10 below. Note that the quotient group  $PG_2^1/PS_2^1$  is isomorphic to  $PS_2^1/PS_2^1 \times \text{Gal} \cong \mathbb{F}_4^\times \times \text{Gal} \cong \mathfrak{S}_3$ .

As input for our construction we will use Lemma 4.4 and the isomorphisms (4.1) together with the calculation of  $\text{Ext}^*(\mathbb{W}) = H^*(PS_2^1; \mathbb{F}_4)$ . The latter is given by Theorem 3.11 with coefficients extended from  $\mathbb{F}_2$  to  $\mathbb{F}_4$ .

**Lemma 4.6.** *There is a short exact sequence of Galois-twisted profinite  $PG_2^1$ -modules*

$$(4.2) \quad 0 \rightarrow N_1 \rightarrow P_0 := \mathbb{W} \uparrow_{\mathfrak{S}_4}^{PS_2^1} \oplus \mathbb{W} \uparrow_{\mathfrak{S}'_4}^{PS_2^1} \xrightarrow{\varepsilon} \mathbb{W} \rightarrow 0 .$$

where  $\varepsilon$  is given by augmentation. The Poincaré series  $\sum_{n \geq 0} \dim_{\mathbb{F}_4} \text{Coker}(\text{Ext}^n(\varepsilon))t^n$  is equal to  $\frac{1+t+t^2}{1-t}$  while the Poincaré series of  $\sum_{n \geq 0} \dim_{\mathbb{F}_4} \text{Ker}(\text{Ext}^n(\varepsilon))t^n$  is  $t + t^2$  and the Poincaré series  $\chi_1 := \sum_{n \geq 0} \dim_{\mathbb{F}_4} \text{Ext}^n(N_1)t^n$  is given by

$$\chi_1 = \frac{1+t+t^2}{1-t} + 1 + t .$$

*Proof.* It is clear that  $\varepsilon$  is surjective. As modules over  $\mathbb{W}[[PS_2^1]]$  we have

$$P_0 = \mathbb{W}\uparrow_{E_2}^{PS_2^1} \oplus \mathbb{W}\uparrow_{E_2'}^{PS_2^1}$$

where  $E_2$  and  $E_2'$  are the elementary abelian 2-subgroups of rank 2 of table (1.5). By the Shapiro lemma there is an isomorphism

$$\text{Ext}^*(P_0) \cong H^*(E_2, \mathbb{F}_4) \times H^*(E_2'; \mathbb{F}_4)$$

and  $\text{Ext}^*(\varepsilon)$  corresponds via this isomorphism to the restriction homomorphism

$$H^*(PS_2^1, \mathbb{F}_4) \rightarrow H^*(E_2, \mathbb{F}_4) \times H^*(E_2', \mathbb{F}_4).$$

The long exact sequence in  $\text{Ext}^*(-)$  associated to the short exact sequence (4.2) gives a short exact sequence of  $\mathbb{F}_4[PG_2^1/PS_2^1]$ -modules

$$(4.3) \quad 0 \rightarrow \text{Coker}(\text{Ext}^*(\varepsilon)) \rightarrow \text{Ext}^*(N_1) \rightarrow \text{Ker}(\text{Ext}^{*+1}(\varepsilon)) \rightarrow 0$$

and by Theorem 3.11 the Poincaré series of  $\text{Coker}(\text{Ext}^*(\varepsilon))$  is given by

$$\frac{2}{(1-t)^2} - \frac{1+t^3}{(1-t)^2} = \frac{1-t^3}{(1-t)^2} = \frac{1+t+t^2}{1-t}$$

while that of  $\text{Ker}(\text{Ext}^*(\varepsilon))$  is given by  $t+t^2$ . The result follows.  $\square$

**Remark 4.7.** (cf. Remark 3.5) We can and will choose  $\mathfrak{S}_4$  and  $\mathfrak{S}_4'$  such that  $\mathfrak{S}_4 \cap \mathfrak{S}_4' = C_4$ . In fact, if we choose as generator of the subgroup  $C_4$  the image of  $1+i$  (as in Lemma 2.4) and for  $\mathfrak{S}_4$  the group  $PG_{48}$  of Lemma 2.2 then conjugation by  $1+2i$  fixes  $C_4$  and we can take the conjugate copy of  $\mathfrak{S}_4$  as  $\mathfrak{S}_4'$ . Then it is elementary to check that  $\mathfrak{S}_4 \cap \mathfrak{S}_4' = C_4$ .

**Lemma 4.8.** *There is a homomorphism of Galois-twisted profinite  $PG_2^1$ -modules*

$$(4.4) \quad \psi : \mathbb{W}\uparrow_{C_4}^{PG_2^1} \rightarrow N_1$$

with the following properties.

a) *The Poincaré series*

$$\chi_k := \sum_{n \geq 0} \dim_{\mathbb{F}_4} \text{Ker}(\text{Ext}^n(\psi))t^n, \quad \chi_c := \sum_{n \geq 0} \dim_{\mathbb{F}_4} \text{Coker}(\text{Ext}^n(\psi))t^n$$

are given by

$$\chi_k = 1+t, \quad \chi_c = 2+t.$$

b) *As a Galois-twisted  $\mathfrak{S}_3 = PG_2^1/PS_2^1$ -module  $\text{Coker}(\text{Ext}^0(\psi))$  is isomorphic to the cokernel of the  $\mathbb{F}_4[\mathfrak{S}_3]$ -linear inclusion  $\mathbb{F}_4 \cong \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]^{C_3} \rightarrow \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]$ .*

*Proof.* Consider the map of Galois-twisted profinite  $PG_2^1$ -modules

$$(4.5) \quad \varphi : \mathbb{W}\uparrow_{C_4}^{PG_2^1} \rightarrow P_0 = \mathbb{W}\uparrow_{\mathfrak{S}_4}^{PG_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_4'}^{PG_2^1}$$

which sends the generator  $e_1$  to  $(e_0, -e_0')$ . Here  $e_1$  is given as  $e \otimes 1 \in \mathbb{W}[[PG_2^1]] \otimes_{\mathbb{W}[[C_4]]} \mathbb{W}$  and  $e_0$  resp.  $e_0'$  by the corresponding element in  $\mathbb{W}[[PG_2^1]] \otimes_{\mathbb{W}[[\mathfrak{S}_4]]} \mathbb{W}$  resp. in  $\mathbb{W}[[PG_2^1]] \otimes_{\mathbb{W}[[\mathfrak{S}_4']] } \mathbb{W}$ .

Clearly  $\varepsilon\varphi$  is trivial, hence  $\varphi$  factors as composition  $\psi : \mathbb{W}\uparrow_{C_4}^{PG_2^1} \rightarrow N_1$  followed by the inclusion of  $N_1$  into  $P_0$ . In order to analyze  $\text{Ext}^*(\psi)$  we start by analyzing  $\text{Ext}^*(\varphi)$ .

As homomorphism of modules over  $PS_2^1$  the homomorphism  $\varphi$  becomes

$$\mathbb{W}\uparrow_{C_2}^{PS_2^1} \oplus \mathbb{W}\uparrow_{\omega C_2 \omega^{-1}}^{PS_2^1} \oplus \mathbb{W}\uparrow_{\omega^2 C_2 \omega^{-2}}^{PS_2^1} \rightarrow \mathbb{W}\uparrow_{E_2}^{PS_2^1} \oplus \mathbb{W}\uparrow_{E_2'}^{PS_2^1}$$

and the induced map  $\text{Ext}^*(\varphi)$  becomes a homomorphism

$$H^*(E_2, \mathbb{F}_4) \oplus H^*(E'_2, \mathbb{F}_4) \rightarrow \prod_{i=0}^2 H^*(\omega^i C_2 \omega^{-1}, \mathbb{F}_4)$$

whose kernel is the inverse limit of Corollary 3.7 with  $\mathbb{F}_2$ -coefficients replaced by  $\mathbb{F}_4$ -coefficients. Therefore  $\text{Ker}(\text{Ext}^*(\varphi))$  has Poincaré series  $\frac{1+t^3}{(1-t)^2}$  and the exact sequence

$$0 \rightarrow \text{Ker}(\text{Ext}^*(\varphi)) \rightarrow H^*(E_2, \mathbb{F}_4) \oplus H^*(E'_2, \mathbb{F}_4) \rightarrow \prod_{i=0}^2 H^*(\omega^i C_2 \omega^{-1}, \mathbb{F}_4) \rightarrow \text{Coker}(\text{Ext}^*(\varphi)) \rightarrow 0$$

shows that  $\text{Coker}(\text{Ext}^*(\varphi))$  has Poincaré series  $\chi_C$  given by

$$\chi_C := \frac{1+t^3}{(1-t)^2} + \frac{3}{1-t} - \frac{2}{(1-t)^2} = \frac{t^3 + 3(1-t) - 1}{(1-t)^2} = \frac{t^3 - 3t + 2}{(1-t)^2} = t + 2 .$$

Now we turn towards analyzing  $\text{Ext}^*(\psi)$  and we consider the exact sequence (4.3)

$$0 \rightarrow \text{Coker}(\text{Ext}^*(\varepsilon)) \rightarrow \text{Ext}^*(N_1) \rightarrow \text{Ker}(\text{Ext}^{*+1}(\varepsilon)) \rightarrow 0 .$$

Because  $\varepsilon\varphi$  is trivial  $\text{Ext}^*(\varphi)$  factors through  $\text{Coker}(\text{Ext}^*(\varepsilon))$  and the restriction of  $\text{Ext}^*(\psi)$  to the submodule  $\text{Coker}(\text{Ext}^*(\varepsilon))$  is induced by  $\text{Ext}^*(\varphi)$ . If  $\chi_K$  is the Poincaré series of the kernel of this restriction then we have an identity

$$\chi_K + \sum_{n \geq 0} \dim_{\mathbb{F}_4} \text{Ext}^n(\mathbb{W} \uparrow_{C_4}^{PG_2^1}) t^n = \chi_C + \dim_{\mathbb{F}_4} \text{Coker}(\text{Ext}^n(\varepsilon)) t^n$$

We have just seen that  $\chi_C = t + 2$  and therefore Lemma 4.6 implies the following identity of Poincaré series

$$\chi_C + \sum_{n \geq 0} \dim_{\mathbb{F}_4} \text{Coker}(\text{Ext}^n(\varepsilon)) t^n = t + 2 + \frac{1+t+t^2}{1-t} = \frac{3}{1-t} = \sum_{n \geq 0} \dim_{\mathbb{F}_4} (\text{Ext}^n(\mathbb{W} \uparrow_{C_4}^{PG_2^1})) t^n .$$

and this shows that  $\chi_K = 0$ . In other words, the restriction of  $\text{Ext}^*(\psi)$  to  $\text{Coker}(\text{Ext}^*(\varepsilon))$  is injective. Part a) will therefore follow if we can show that there are elements  $\tilde{x} \in \text{Ext}^0(N_1)$  respectively  $\tilde{y} \in \text{Ext}^1(N_1)$  which are both in the kernel of  $\text{Ext}^*(\psi)$  and which project in the exact sequence (4.3) to non-trivial elements in  $\text{Ext}^1(\text{Ker}(\varepsilon)) = \mathbb{F}_4$  respectively in  $\text{Ext}^2(\text{Ker}(\varepsilon)) = \mathbb{F}_4$ .

The short exact sequence (4.3) is a sequence of Galois-twisted  $PG_2^1/PS_2^1 = \mathfrak{S}_3$ -modules and we know from Lemma 4.6 that  $\text{Ker}(\text{Ext}^{*+1}(\varepsilon))$  is trivial unless  $* = 0$  or  $* = 1$  and in this case its value is  $\mathbb{F}_4$ . The Galois-twisting arises from the canonical homomorphism  $\mathbb{G}_2^1 \rightarrow \text{Gal}$  which induces a homomorphism  $PG_2^1/PS_2^1 \cong \mathfrak{S}_3 \rightarrow \text{Gal}$  with kernel  $C_3$  cyclic of order 3. Therefore Corollary 4.2 shows that the short exact sequences

$$0 \rightarrow \text{Coker}(\text{Ext}^0(\varepsilon)) = \mathbb{F}_4 \rightarrow \text{Ext}^0(N_1) \rightarrow \text{Ker}(\text{Ext}^1(\varepsilon)) = \mathbb{F}_4 \rightarrow 0$$

and

$$0 \rightarrow \text{Coker}(\text{Ext}^1(\varepsilon)) = \mathbb{F}_4^2 \rightarrow \text{Ext}^1(N_1) \rightarrow \text{Ker}(\text{Ext}^2(\varepsilon)) = \mathbb{F}_4 \rightarrow 0$$

are split exact as sequences of Galois-twisted  $\mathbb{F}_4[\mathfrak{S}_3]$ -modules. In particular  $\text{Ext}^0(N_1)$  is isomorphic to the trivial Galois-twisted module  $\mathbb{F}_4 \oplus \mathbb{F}_4$ . On the other hand it is clear that as  $\mathbb{F}_4[\mathfrak{S}_3]$ -module  $\text{Ext}^0(\mathbb{W} \uparrow_{C_4}^{PG_2^1})$  is isomorphic to  $\mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]$  and therefore its  $C_3$ -invariants are isomorphic to  $\mathbb{F}_4$ . The image of  $\text{Ext}^0(\varphi)$  is isomorphic to  $\mathbb{F}_4$  and therefore necessarily equal to these invariants. Hence the image of  $\text{Ext}^0(\varphi)$  must agree with the image of  $\text{Ext}^0(\psi)$  because otherwise  $\text{Ext}^0(\mathbb{W} \uparrow_{C_4}^{PG_2^1})$  would contain a  $\mathbb{F}_4[C_3]$ -submodule of dimension 2 with trivial action of  $C_3$ . This shows the existence of  $\tilde{x}$  and proves part (b) because we have just seen that  $\text{Coker}(\text{Ext}^0(\psi))$  is isomorphic to the cokernel of the inclusion

$$\mathbb{F}_4 = \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]^{C_3} \rightarrow \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}] .$$

Finally we use that for each profinite  $\mathbb{W}[[PG_2^1]]$ -module  $M$  the short exact sequence of trivial Galois-twisted  $\mathbb{W}[[PG_2^1]]$ -modules

$$0 \rightarrow \mathbb{W}/(2) \xrightarrow{\times 2} \mathbb{W}/(4) \rightarrow \mathbb{W}/(2) \rightarrow 0$$

induces a connecting homomorphism

$$\delta : \text{Ext}^*(M) = \text{Ext}_{\mathbb{W}[[PS_2^1]]}^*(M, \mathbb{W}/(2)) \rightarrow \text{Ext}_{\mathbb{W}[[PS_2^1]]}^{*+1}(M, \mathbb{W}/(2)) = \text{Ext}^{*+1}(M)$$

which is functorial in  $M$  and commutes with the connecting homomorphisms associated to short exact sequences  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$  in the first variable of  $\text{Ext}$ . If  $M = \mathbb{W}$  is the trivial module then part c) of Theorem 3.11 says that this connecting homomorphism induces an isomorphism between the kernel of  $\text{Ext}^1(\varepsilon)$  and the kernel of  $\text{Ext}^2(\varepsilon)$ . In the exact sequence (4.3) for  $* = 0$  we have just seen that we can choose a lift of a generator  $x \in \text{Ext}^1(\text{Ker}(\varepsilon))$  to an element  $\tilde{x} \in \text{Ext}^0(N_1)$  such that  $\tilde{x}$  is in the kernel of  $\text{Ext}^0(\psi)$ . Then  $\tilde{y} := \delta(\tilde{x})$  is in the kernel of  $\text{Ext}^1(\psi)$  and projects to a non-trivial element in  $\text{Ker}(\text{Ext}^2(\varepsilon))$ .  $\square$

**Lemma 4.9.** *There is a short exact sequence of Galois-twisted profinite  $PG_2^1$ -modules*

$$(4.6) \quad 0 \rightarrow N_2 \rightarrow P_1 := \mathbb{W}\uparrow_{C_4}^{PG_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_3}^{PG_2^1} \xrightarrow{\rho} N_1 \rightarrow 0$$

such that the Poincaré series  $\chi_2 := \sum_{n \geq 0} \dim_{\mathbb{F}_4}(\text{Ext}^n(N_2))t^n$  is given by

$$\chi_2 = 3 + t .$$

and such that there is an isomorphism of Galois-twisted  $\mathbb{F}_4[PG_2^1/PS_2^1] = \mathbb{F}_4[\mathfrak{S}_3]$ -modules

$$\text{Ext}^0(N_2) \cong \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}] .$$

*Proof.* By the isomorphisms of (4.1) we have  $\text{Coker}(\text{Tor}_0(\psi)) \cong \text{Ker}(\text{Ext}^0(\psi))$  and by the previous lemma this is isomorphic to the necessarily trivial Galois-twisted module  $\mathbb{F}_4$ .

By Corollary 4.2  $\mathbb{W}$  is a projective Galois-twisted  $\mathbb{W}[\mathfrak{S}_3]$ -module. Hence, the canonical epimorphism  $\mathbb{W} \rightarrow \mathbb{F}_4$  can be lifted against the canonical projection

$$N_1 \rightarrow \text{Tor}_0(N_1) \rightarrow \text{Coker}(\text{Tor}_0(\psi)) \cong \mathbb{F}_4$$

to a  $\mathbb{W}[[PG_2^1]]$ -linear homomorphism

$$\psi' : \mathbb{W}\uparrow_{\mathfrak{S}_3}^{PG_2^1} \rightarrow N_1 .$$

Then the homomorphism

$$\rho : P_1 = \mathbb{W}\uparrow_{C_4}^{PG_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_3}^{PG_2^1} \rightarrow N_1$$

is defined via its restriction to the two summands given by  $\psi$  and  $\psi'$ . By construction the map  $\rho$  induces an epimorphism on  $\text{Tor}_0(-)$ . By Lemma 4.4 it is therefore surjective and  $N_2$  is defined as its kernel and we have established the short exact sequence (4.6).

The long exact sequence in  $\text{Ext}^*(-)$  associated to the short exact sequence (4.6) gives a short exact sequence

$$(4.7) \quad 0 \rightarrow \text{Coker}(\text{Ext}^*(\rho)) \rightarrow \text{Ext}^*(N_2) \rightarrow \text{Ker}(\text{Ext}^{*+1}(\rho)) \rightarrow 0 .$$

Because  $\mathbb{W}\uparrow_{\mathfrak{S}_3}^{PG_2^1}$  is isomorphic to  $\mathbb{W}[[PS_2^1]]$  as  $PS_2^1$ -module we get  $\text{Ext}^*(\mathbb{W}\uparrow_{\mathfrak{S}_3}^{PG_2^1}) \cong \mathbb{F}_4$  concentrated in degree 0 and hence  $\text{Ext}^*(\rho)$  agrees with  $\text{Ext}^*(\psi)$  for  $* > 0$ . For  $* = 0$  the difference is that

$$\text{Ext}^0(\rho) : \text{Ext}^0(N_1) \cong \mathbb{F}_4 \oplus \mathbb{F}_4 \rightarrow \text{Ext}(P_1) = (\mathbb{F}_2)^4$$

is injective with cokernel isomorphic to the cokernel of  $\text{Ext}^0(\psi)$  while

$$\text{Ext}^0(\psi) : \text{Ext}^0(N_1) \cong \mathbb{F}_4 \oplus \mathbb{F}_4 \rightarrow \text{Ext}(\mathbb{W}\uparrow_{C_4}^{PG_2^1}) = (\mathbb{F}_4)^3$$

has kernel  $\mathbb{F}_4$ . It follows that  $\chi_2 = 3 + t$  as claimed.

For the last statement we use that the short exact sequence (4.7) for  $* = 0$  is one of Galois-twisted  $\mathbb{F}_4[\mathfrak{S}_3]$ -modules and identifies by part b) of Lemma 4.8 with the sequence

$$0 \rightarrow \text{Coker}(\mathbb{F}_4 \rightarrow \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]) \rightarrow \text{Ext}^0(N_2) \rightarrow \text{Ker}(\text{Ext}^1(\rho)) = \mathbb{F}_4 \rightarrow 0 .$$

By Proposition 4.1 the sequence is split just as the sequence

$$0 \rightarrow \mathbb{F}_4 \cong \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]^{C_3} \rightarrow \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}] \rightarrow \text{Coker}(\mathbb{F}_4[\mathfrak{S}_3/\text{Gal}] \rightarrow \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]) \rightarrow 0$$

and this implies  $\text{Ext}^0(N_2) \cong \mathbb{F}_4 \oplus \text{Coker}(\mathbb{F}_4 \rightarrow \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]) \cong \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]$ .  $\square$

**Lemma 4.10.** *There is a short exact sequence of Galois-twisted profinite  $P\mathbb{G}_2^1$ -modules*

$$(4.8) \quad 0 \rightarrow P_3 := \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \rightarrow P_2 := \mathbb{W}\uparrow_{\text{Gal}}^{P\mathbb{S}_2^1} \rightarrow N_2 \rightarrow 0 .$$

*Proof.* By Proposition 4.1 the Galois-twisted  $\mathfrak{S}_3$ -module  $\mathbb{W}[\mathfrak{S}_3/\text{Gal}]$  is projective. Hence the canonical epimorphism  $\mathbb{W}[\mathfrak{S}_3/\text{Gal}] \rightarrow \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]$  can be lifted against the canonical projection

$$N_2 \rightarrow \text{Tor}_0(N_2) \cong \mathbb{F}_4[\mathfrak{S}_3/\text{Gal}]$$

to a  $\mathbb{W}[[P\mathbb{G}_2^1]]$ -linear homomorphism

$$\mathbb{W}\uparrow_{\text{Gal}}^{P\mathbb{G}_2^1} = (\mathbb{W}\uparrow_{\mathfrak{S}_3}^{\mathfrak{S}_3}) \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \rightarrow N_2 .$$

Then the homomorphism  $P_2 \rightarrow N_2$  induces an epimorphism on  $\text{Tor}_0(-)$ . By Lemma 4.4 it is therefore surjective. Let  $N_3$  be its kernel so that (4.8) is a short exact sequence.

By the isomorphisms of (4.1) and by the previous lemma the induced map in  $\text{Ext}^*(-)$  is a monomorphism with cokernel  $\mathbb{F}_4$  concentrated in degree 1. Therefore, if  $N_3$  is the kernel of our map  $P_2 \rightarrow N_2$ , then by the isomorphisms of (4.1) we have

$$\text{Tor}_q(N_3) = \begin{cases} \mathbb{F}_4 & q = 0 \\ 0 & q > 0 \end{cases} .$$

By Corollary 4.2 the Galois-twisted  $\mathfrak{S}_3$ -module  $\mathbb{W}$  is again projective. Hence the canonical epimorphism of Galois twisted  $\mathfrak{S}_3$ -modules  $\mathbb{W} \rightarrow \mathbb{F}_4$  can be lifted against the canonical projection

$$N_3 \rightarrow \text{Tor}_0(N_3) \cong \mathbb{F}_4$$

to a  $\mathbb{W}[[P\mathbb{G}_2^1]]$ -linear homomorphism of Galois-twisted profinite  $P\mathbb{G}_2^1$ -modules

$$\psi' : \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \rightarrow N_3 .$$

By construction the map induces an isomorphism on  $\text{Tor}_q(-)$  for all  $q$  and by Lemma 4.4 it is therefore an isomorphism.  $\square$

**Lemma 4.11.** *The exact complexes of Galois-twisted profinite  $P\mathbb{G}_2^1$ -modules respectively  $\mathbb{G}_2^1$ -modules of Theorem 4.5 are  $\mathcal{F}$ -resolutions of the trivial Galois-twisted module  $\mathbb{W}$ .*

*Proof.* By the obvious analogue of Lemma 14 of [18] it suffices to consider the case of  $P\mathbb{G}_2^1$ . Furthermore it suffices to show that the sequence is split after restriction to any finite subgroup  $F$  and for this it is enough that the short exact sequences of Lemma 4.6, Lemma 4.9 and Lemma 4.10 are split exact after restriction to  $F$ . Furthermore it suffices to consider to restrict attention to maximal finite 2-subgroups which are given by  $\mathfrak{S}_4$  and  $\mathfrak{S}'_4$  (cf. the discussion around (2.5)). The maps

$$\mathbb{W} \rightarrow \mathbb{W}\uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}'_4}^{P\mathbb{G}_2^1}$$

given by sending  $x$  to  $(x\mathfrak{S}_4, 0)$  resp.  $x$  to  $(0, x\mathfrak{S}'_4)$  are clearly  $\mathbb{W}[\mathfrak{S}_4]$  resp.  $\mathbb{W}[\mathfrak{S}'_4]$ -linear and provide splittings of  $\varepsilon$ . Then Lemma 4.3 shows that for any finite subgroup  $F$  of  $P\mathbb{G}_2^1$  the exact sequence of Lemma 4.10 is split as sequence of  $\mathbb{W}[F]$ -modules. In particular, as  $\mathbb{W}[F]$ -module

$N_2$  is a direct summand in  $\mathbb{W}\uparrow_{\text{Gal}}^{PG_2^1}$  and therefore  $N_2$  is projective as  $\mathbb{W}[F]$ -module. So Lemma 4.3 also applies to the exact sequence of Lemma 4.9 and this completes the proof.  $\square$

Finally we turn towards the construction of an  $\mathcal{F}$ -resolution for the trivial Galois-twisted profinite  $G$ -module  $\mathbb{W}$  for  $G = \mathbb{G}_2$  resp.  $G = PG_2$ . We observe that because of  $\mathbb{G}_2/\mathbb{G}_2^1 \cong PG_2/PG_2 \cong \mathbb{Z}_2$  there is a short exact sequence of Galois-twisted profinite  $\mathbb{G}_2$  resp.  $\mathbb{G}_2^1$ -modules

$$(4.9) \quad 0 \rightarrow \mathbb{W}[[\mathbb{Z}_2]] \xrightarrow{g-e} \mathbb{W}[[\mathbb{Z}_2]] \rightarrow \mathbb{W} \rightarrow 0$$

where  $g$  is a topological generator of the group  $\mathbb{Z}_2$ . In fact, for every prime  $p$  here is a well known isomorphism  $\mathbb{Z}_p[[T]] \rightarrow \mathbb{Z}_p[[\mathbb{Z}_p]]$  which sends  $T$  to  $g - e$  where  $g$  is any topological generator of  $\mathbb{Z}_p$  and this extends to an isomorphism  $\mathbb{W}[[T]] \rightarrow \mathbb{W}[[\mathbb{Z}_p]]$ . Via this isomorphism the augmentation is just given by the map which sends  $T$  to 0.

By the analogue of Lemma 15 of [18] induction of the  $\mathcal{F}$ -resolutions of Theorem 4.5 from  $PG_2^1$  to  $PG_2$  gives an  $\mathcal{F}$ -resolution of the  $\mathbb{W}[[PG_2]]$ -module  $\mathbb{W}[[PG_2/PG_2^1]]$  of the form

$$(4.10) \quad 0 \rightarrow Q_3 \xrightarrow{\partial_3} Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\varepsilon} \mathbb{W}\uparrow_{PG_2^1}^{PG_2} \rightarrow 0$$

with

$$Q_3 = \mathbb{W}\uparrow_{\mathbb{G}_3}^{PG_2}, \quad Q_2 = \mathbb{W}\uparrow_{\text{Gal}}^{PG_2}, \quad Q_1 = \mathbb{W}\uparrow_{\mathbb{G}_3}^{PG_2} \oplus \mathbb{W}\uparrow_{C_4}^{PG_2}, \quad Q_0 = \mathbb{W}\uparrow_{\mathbb{G}_4}^{PG_2} \oplus \mathbb{W}\uparrow_{\mathbb{G}_4}^{PG_2}$$

and the monomorphism of the exact sequence (4.10) can be covered by a map of complexes

$$(4.11) \quad \begin{array}{ccccccccccc} 0 & \rightarrow & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\varepsilon} & \mathbb{W}\uparrow_{PG_2^1}^{PG_2} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow g-e & & \\ 0 & \rightarrow & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\varepsilon} & \mathbb{W}\uparrow_{PG_2^1}^{PG_2} & \rightarrow & 0 \end{array}$$

The following result is a more precise form of Theorem 1.1.

**Theorem 4.12.** *The total complex of the double complex  $Q_{*,*}$  of (4.11) is a Galois-twisted  $\mathcal{F}$ -resolution of the trivial  $\mathbb{W}[[PG_2]]$ -module  $\mathbb{W}$ .*

*Proof.* It is clear that  $\text{Tot}(C)_*$  is an exact complex of  $\mathcal{F}$ -projective modules. The fact that the complexes  $\text{Hom}_{\mathbb{W}[[G]]}(P, Q_{i,*})$  are exact for  $i = 1, 0$  and each  $\mathcal{F}$ -projective module  $P$  implies that the complex  $\text{Hom}_{\mathbb{W}[[G]]}(P, \text{Tot}(Q)_*)$  is exact for each  $\mathcal{F}$ -projective module  $P$ .  $\square$

## 5. REALIZING THE CENTRALIZER RESOLUTIONS

### 5.1. Preliminaries on Morava modules.

By the Goerss-Hopkins-Miller theorem the extended Morava stabilizer group  $\mathbb{G}_n(\Gamma)$  acts on the spectrum  $E_n(\Gamma)$  (see [12], [28]); we recall that  $E_n(\Gamma)$  is a Landweber exact spectrum given by a 2-periodic theory with coefficients  $\pi_*(E_n(\Gamma)) = \pi_0(E_n)[u^{\pm 1}]$  (with  $u \in \pi_{-2}(E_2)$ ) whose associated formal group law over  $\pi_0(E_n(\Gamma))$  is a universal deformation of  $\Gamma$  in the sense of Lubin and Tate [LT]. In particular there is a (non-canonical) isomorphism between  $\pi_0(E_n(\Gamma))$  and  $\mathbb{W}[[u_1, \dots, u_{n-1}]]$ , the ring of formal power series over  $\mathbb{W}$  in the variables  $u_1, \dots, u_{n-1}$ . The maximal ideal  $(p, u_1, \dots, u_{n-1})$  of this power series ring will be denoted  $\mathfrak{m}$ . To avoid cluttered notation we will abbreviate in this section  $E_n(\Gamma)$  simply by  $E_n$  and  $\mathbb{G}_n(\Gamma)$  by  $\mathbb{G}_n$ .

For the purposes of this paper, a Morava module is a complete  $(E_n)_*$ -module  $M$  equipped with a continuous  $\mathbb{G}_n$ -action (continuous with respect to the  $\mathfrak{m}$ -adic topology on  $M$  and the

profinite topology on  $\mathbb{G}_n$ ) such that for  $g \in \mathbb{G}_n$ ,  $a \in \pi_*E_n$ ,  $\lambda \in \mathbb{W}$  and  $x \in M$  we have

$$g(ax) = g(a)g(x) \quad \text{and} \quad g(\lambda x) = {}^g\lambda g(x).$$

So a Morava module is a Galois-twisted module but it need not be  $p$ -profinite and therefore not be a module over the (twisted) group algebra  $\mathbb{W}[[\mathbb{G}_n]]$ . The category of Morava modules will be denoted  $\mathcal{E}\mathcal{G}_n$ . A morphism in this category is an  $(E_n)_*$ -linear map  $M \rightarrow M'$  which commutes with the action of  $\mathbb{G}_n$ . We note that such a map will be automatically continuous with respect to the  $\mathfrak{m}$ -adic topologies on  $M_1$  and  $M_2$ .

The Morava module of a spectrum  $X$  is defined as

$$(E_n)_*X = \pi_*L_{K(n)}(E_n \wedge X).$$

This is an  $(E_n)_*$ -module which is complete but not necessarily Hausdorff with respect to the  $\mathfrak{m}$ -adic topology if  $\mathfrak{m}$  denotes the maximal ideal of  $\pi_0(E_n)$ . All Morava modules in this paper will be Hausdorff, in fact they will all be pro-discrete.

For the Honda formal group law the following result is folklore and can be found in [10] or in [31]. We give a proof which is very close to that in [31].

**Proposition 5.1.** *Let  $E_n$  be the Lubin-Tate spectrum associated to a deformation of a formal group law  $\Gamma$  over  $\mathbb{F}_q$  which is already defined over  $\mathbb{F}_p$ . Assume that the Frobenius endomorphism  $\xi_\Gamma$  defined by  $\xi_\Gamma(x) = x^p$  satisfies an equation  $\xi_\Gamma^n = pu$  in the endomorphism ring of  $\Gamma$  (over  $\mathbb{F}_p$ ) where  $u$  is a  $p$ -adic unit. Then there is an isomorphism*

$$(5.1) \quad \phi : \pi_*L_{K(n)}(E_n \wedge E_n) \cong \text{map}_{cts}(\mathbb{G}_n, (E_n)_*)$$

which is adjoint to the map

$$\mathbb{G}_n \times \pi_*L_{K(n)}(E_n \wedge E_n) \rightarrow \pi_*(E_n)$$

given by

$$(x : S^n \rightarrow E_n \wedge E_n, g \in \mathbb{G}_n) \mapsto (S^0 \xrightarrow{x} E_n \wedge E_n \xrightarrow{1 \wedge g} E_n \wedge E_n \xrightarrow{\mu} E_n)$$

where  $\mu$  is multiplication on  $E_n$ .

We prepare the proof of the proposition with two remarks, one on formal group laws and another one on  $q$ -Boolean algebras.

**Remark 5.2.** a) Let  $q = p^n$  and let  $k$  be a field which contains  $\mathbb{F}_q$ . The endomorphism  $\xi_F^n$  commutes with an endomorphism  $\sum_i a_i x^i \in \text{End}_k(\Gamma)$  if and only if  $a_i^q = a_i$  for all  $i$ , i.e.  $a_i \in \mathbb{F}_q$  for all  $i$ . Hence the canonical map

$$\text{End}_{\mathbb{F}_q}(\Gamma) \rightarrow \text{End}_k(\Gamma)$$

is an isomorphism if and only if  $\xi_F^n$  is central in which case it must satisfy an equation  $\xi_F^n = pu$  in the endomorphism ring of  $\Gamma$  (over  $\mathbb{F}_p$ ) for some  $p$ -adic unit.

b) More generally, if  $k$  is a finite field of order  $p^m$  then the endomorphism ring over  $k$  is isomorphic to the centralizer of  $\xi_F^m$  in  $\text{End}_k(\Gamma)$ .

**Remark 5.3.** a) Let  $\Gamma$  be any formal group law over  $\mathbb{F}_q$  and consider as in the proof of Theorem 12 of [31] the functor which sends an  $\mathbb{F}_q$ -algebra  $A$  to the set of pairs  $(\beta, f)$  where  $\alpha : \mathbb{F}_q \rightarrow A$  is the ring homomorphism defining the  $\mathbb{F}_q$ -algebra structure on  $A$ ,  $\beta : \mathbb{F}_q \rightarrow A$  is any other ring homomorphism and  $f$  is an isomorphism  $\beta_*\Gamma \rightarrow \alpha_*\Gamma$  of formal group laws. This functor is corepresented by the  $\mathbb{F}_q$  algebra

$$B(\Gamma) := \mathbb{F}_q \otimes_L L[t_0, t_1, \dots] \otimes_L \mathbb{F}_q$$

where the  $\mathbb{F}_q$ -algebra structure comes from the first tensor factor  $\mathbb{F}_q$ ,  $L$  is the Lazard ring,  $\mathbb{F}_q$  is considered as an  $L$ -algebra via the homomorphism classifying  $\Gamma$  and  $L[t_0, t_1, \dots]$  is an



$L$ -algebra via the usual units  $\eta_L$  and  $\eta_R$  in the Hopf-algebroid  $(L, L[t_0, t_1, \dots])$ . The algebra  $B$  is generated over  $\mathbb{F}_q \otimes \mathbb{F}_q$  by the elements  $t_i$ ,  $i = 0, 1, \dots$ , with respect to complicated relations determined by  $\eta_R$  and  $\Gamma$ .

Now assume that the formal group law  $\Gamma$  is already defined over  $\mathbb{F}_p$ . Then we have  $\beta_*\Gamma = \alpha_*\Gamma$  because there is only one algebra homomorphism  $\mathbb{F}_p \rightarrow A$  and then  $f$  is an automorphism of  $\beta_*\Gamma = \alpha_*\Gamma$ . If furthermore  $\zeta_\Gamma^n = pu$  then for any endomorphism  $\sum_i a_i x^i \in \text{End}_A(\Gamma)$  we have  $a_i^q = a_i$  and the complicated relations must include the relations  $t_i^q = t_i$  for all  $i$ . Therefore  $B(\Gamma)$  is a  $q$ -Boolean algebra, i.e. an  $\mathbb{F}_q$ -algebra which satisfies  $x^q = x$  for any  $x \in B$ .

b) Let  $B$  be a  $q$ -Boolean algebra. A  $q$ -Boolean algebra which is an integral domain only has  $q$  solutions to the equation  $x^q = x$ , hence any prime ideal in such a  $B$  is maximal and is the kernel of a unique  $\mathbb{F}_q$ -algebra morphism  $B \rightarrow \mathbb{F}_q$ . So we can identify the spectrum  $\text{spec}(B)$  with  $\text{Hom}_{\mathbb{F}_q\text{-alg}}(B, \mathbb{F}_q)$ . Furthermore  $B$  is the colimit of its finite  $\mathbb{F}_q$ -subalgebras and this defines a profinite topology on its spectrum  $\text{spec}(B)$ . The structure theorem for  $q$ -Boolean algebra says that the evaluation map from  $B$  to the algebra of continuous functions on its spectrum

$$B \rightarrow \text{map}_{cts}(\text{spec}(B), \mathbb{F}_q), \quad x \mapsto (\varphi \mapsto \varphi(x))$$

is an isomorphism. In fact, if  $x \in B$  is any element then  $x^{q-1}$  is idempotent. Hence, if  $x^{q-1} \neq 1$  then  $B$  is the product of the ideals generated by  $x^{q-1}$  and  $1 - x^{q-1}$ . From this one sees immediately that the evaluation map is an isomorphism if  $B$  is finite. The general case follows by observing that for a profinite set  $S = \lim_i S_i$  with  $S_i$  finite, the set of continuous functions  $\text{map}_{cts}(S, \mathbb{F}_q)$  is equal to  $\text{colim}_i \text{map}(S_i, \mathbb{F}_q)$ .

We are now ready for the proof of Proposition 5.1.

*Proof.* It is enough to prove the isomorphism in degree 0 after reducing modulo the ideal generated by the maximal ideal  $\mathfrak{m}$  in  $\pi_0(E)$ . The  $\mathbb{F}_q$ -algebra  $(E_n)_0 E_n / \mathfrak{m}$  agrees with the algebra  $B(\Gamma)$  considered in part a) of the previous remark. By the assumption on  $\Gamma$  and part b) of the preceding remark  $(E_n)_0 E_n / \mathfrak{m}$  is an  $\mathbb{F}_q$ -Boolean algebra and is therefore isomorphic to the ring of continuous functions on its spectrum. The spectrum of  $(E_n)_0 E_n / \mathfrak{m}$  identifies with the profinite set of pairs  $(\beta, f)$  and this is exactly equal to  $\mathbb{G}_n(\Gamma)$ .  $\square$

In the remainder of this section we assume that  $\Gamma$  satisfies the assumption of Proposition 5.1.

The group  $\mathbb{G}_n \times \mathbb{G}_n$  acts on  $\pi_*(L_{K(n)}(E_n \wedge E_n))$ . The action of the left hand factor is the one used in the definition of the Morava module of  $\pi_*(L_{K(n)}(E_n \wedge E_n))$ . We will also need the action of the right hand factor and we need to know how this action translates to the right hand side of the isomorphism (5.1). We record this in the following lemma whose proof is straightforward.

**Lemma 5.4.** *Let  $g, h_1$  and  $h_2$  be elements of  $\mathbb{G}_n$  and  $x$  be an element of  $\pi_*(L_{K(n)}E_n \wedge E_n)$ . Then*

$$\phi((h_1, h_2)x)(g) = h_1 \phi(x)(h_1^{-1} g h_2) .$$

*In other words, the action on the left hand copy of  $E_n$  corresponds to the diagonal action on the set of continuous maps while the action on the right hand copy of  $E_n$  corresponds to the action on  $\mathbb{G}_n$  on the right.*  $\square$

The results of [10] on homotopy fixed points will now carry over to the case of  $E_n = E_n(\Gamma)$  if  $\Gamma$  satisfies the assumptions of Proposition 5.1. In particular we have the following result.

**Corollary 5.5.** *Let  $K$  be a closed subgroup of  $\mathbb{G}_n$ . Then there is an isomorphism of Morava modules*

$$(E_n)_*(E_n^{hK}) \cong \text{map}_{cts}(\mathbb{G}_n/K, (E_n)_*) \cong \text{Hom}_{\mathbb{W}[\text{Gal}]_{cts}}(\mathbb{W}[[\mathbb{G}_n/K]], (E_n)_*)$$

if  $\mathbb{G}_n$  acts diagonally on the set of continuous maps respectively continuous Galois-twisted homomorphisms.  $\square$

Let  $M$  be a Morava module and let  $x \in M$ . If  $\alpha : M \rightarrow (E_n)_*$  is an  $(E_n)_*$ -linear map let  $\Phi(\alpha) : M \rightarrow \text{map}_{cts}(\mathbb{G}_n, (E_n)_*)$  be given by

$$(\Phi(\alpha)(x))(g) = g\alpha(g^{-1}x) .$$

Conversely, let  $\beta : M \rightarrow \text{map}_{cts}(\mathbb{G}_n, (E_n)_*)$  be a morphism of Morava modules where  $\mathbb{G}_n$  acts on  $\text{map}_{cts}(\mathbb{G}_n, (E_n)_*)$  diagonally. Then let  $\Psi(\beta) : M \rightarrow (E_n)_*$  be given by

$$\Psi(\beta)(x) = (\beta(x))(e) .$$

For  $h \in \mathbb{G}_n$  let  $h * \beta : M \rightarrow \text{map}_{cts}(\mathbb{G}_n, (E_n)_*)$  and  $h * \alpha : M \rightarrow (E_n)_*$  be given by

$$(h * \beta(x))(g) = (\beta(x))(gh) \quad \text{and} \quad (h * \alpha)(x) = h\alpha(h^{-1}x) .$$

The proof of the following lemma is straightforward and left to the reader.

**Lemma 5.6.**

a)  $\Phi(\alpha)$  is a homomorphism of Morava modules.

b)  $\Psi(\beta)$  is  $(E_n)_*$ -linear.

c) The map

$$\Phi : \text{Hom}_{(E_n)_*}(M, (E_n)_*) \rightarrow \text{Hom}_{\mathcal{E}\mathcal{G}_n}(M, \text{map}_{cts}(\mathbb{G}_n, (E_n)_*)), \quad \alpha \mapsto \Phi(\alpha)$$

is an isomorphism with inverse given by

$$\Psi : \text{Hom}_{\mathcal{E}\mathcal{G}_n}(M, \text{map}_{cts}(\mathbb{G}_n, (E_n)_*)) \rightarrow \text{Hom}_{(E_n)_*}(M, (E_n)_*), \quad \beta \mapsto \Psi(\beta) .$$

d) The action of  $\mathbb{G}_n$  on  $\pi_*L_{K(n)}(E_n \wedge E_n)$  on the right hand smash factor translates via the isomorphism of Proposition 5.1 and of the isomorphisms  $\Phi$  and  $\Psi$  of part c) into the diagonal action, i.e. for  $h \in \mathbb{G}_n$  we have  $\Phi(h * \alpha) = h * \Phi(\alpha)$  and  $\Psi(h * \beta) = h * \Psi(\beta)$ .  $\square$

The following two results are taken from [14]. There they were crucial in realizing the duality resolution at  $n = 2$  and  $p = 3$  and here they are crucial for constructing the centralizer resolution for  $n = p = 2$ . In these results we use the following notation: if  $E$  is a spectrum and  $X = \lim_i X_i$  is an inverse limit of a sequence of finite sets then  $E[[X]]$  is defined as  $\text{holim}_i E \wedge (X_i)_+$ . We observe that if  $X$  is such a profinite set with a continuous action of a finite group  $K$  and if  $E$  is a  $K$ -spectrum then  $E[[X]]$  is a  $K$ -spectrum via the diagonal action. If  $X$  and  $Y$  are spectra then we denote the function spectrum by  $F(X, Y)$ .

**Proposition 5.7** ([GHMR1, Prop. 2.6]).

a) Let  $K_1$  be a closed subgroup and  $K_2$  a finite subgroup of  $\mathbb{G}_n$ . Then there is a natural equivalence (where the homotopy fixed points on the left hand side are formed with respect to the diagonal action of  $K_2$ )

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq F(E_n^{hK_1}, E_n^{hK_2}) .$$

b) If  $K_1$  is also an open subgroup then there is a natural decomposition

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq \coprod_{K_2 \setminus \mathbb{G}_n/K_1} E_n^{hK_x}$$

where  $K_x = K_2 \cap xK_1x^{-1}$  is the isotropy subgroup of the coset  $xK_1$  and  $K_2 \backslash \mathbb{G}_n / K_1$  is the finite set of double cosets.

c) If  $K_1$  is a closed subgroup and  $K_1 = \bigcap_i U_i$  for a decreasing sequence of open subgroups  $U_i$  then

$$F(E_n^{hK_1}, E_n^{hK_2}) \simeq \operatorname{holim}_i E_n[[\mathbb{G}_n/U_i]]^{hK_2} \simeq \operatorname{holim}_i \prod_{K_2 \backslash \mathbb{G}_n / U_i} E_n^{hK_{x,i}}$$

where  $K_{x,i} = K_2 \cap xU_i x^{-1}$  is, as before, the isotropy subgroup of the coset  $xU_i$ .  $\square$

The following remark is taken from section 1.3 of [18].

**Remark 5.8.** If  $U_i \subset U_j$  then the map

$$\prod_{K_2 \backslash \mathbb{G}_n / U_i} E_n^{hK_{x,i}} \rightarrow \prod_{K_2 \backslash \mathbb{G}_n / U_j} E_n^{hK_{x,j}}$$

in the inverse system of part (c) of the proposition can be described as follows: if  $x \in \mathbb{G}_n / U_i$  has isotropy group  $K_{x,i}$  and its image  $x' \in \mathbb{G}_n / U_j$  has isotropy group  $K_{x',j}$  then the restriction of the map to the factor determined by  $x$  sends  $E_n^{hK_{x,i}}$  via the transfer to the factor  $E_n^{hK_{x',j}}$  determined by  $x$ . Because  $K_2$  is finite this implies that on homotopy groups the inverse system is Mittag-Leffler.

**Proposition 5.9** ([GHMR1, Prop. 2.7]). *Let  $K_1$  and  $K_2$  be closed subgroups of  $\mathbb{G}_n$  and suppose that  $K_2$  is finite. Then there is an isomorphism*

$$((E_n)_*[[\mathbb{G}_n/K_1]])^{K_2} \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{E}_{\mathcal{G}_n}}((E_n)_*E_n^{hK_1}, (E_n)_*E_n^{hK_2})$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi_* E_n[[\mathbb{G}_n/K_1]]^{hK_2} & \longrightarrow & ((E_n)_*[[\mathbb{G}_n/K_1]])^{K_2} \\ \downarrow \cong & & \downarrow \cong \\ \pi_* F(E_n^{hK_1}, E_n^{hK_2}) & \longrightarrow & \operatorname{Hom}_{\mathcal{E}_{\mathcal{G}_n}}((E_n)_*E_n^{hK_1}, (E_n)_*E_n^{hK_2}) \end{array}$$

where the top horizontal map is the edge homomorphism in the homotopy fixed point spectral sequence, the left-hand vertical map is the isomorphism given by Proposition 5.7 and the bottom horizontal map is the  $E_n$ -Hurewicz homomorphism.  $\square$

We will also need the following result from section 1 of [5].

**Lemma 5.10.** *Let  $K \subset \mathbb{G}_n$  be a closed subgroup and let  $K_0 = K \cap \mathbb{S}_n$ . Suppose the canonical map*

$$K/K_0 \rightarrow \mathbb{G}_n/\mathbb{S}_n \cong \operatorname{Gal}$$

is an isomorphism.

a) *There is a Gal-equivariant equivalence*

$$\operatorname{Gal}_+ \wedge E^{hK} \rightarrow E^{hK_0} .$$

b) *For any profinite Morava module  $M$  we have isomorphisms*

$$H^*(K, M) \cong H_*(K_0, M)^{\operatorname{Gal}}, \quad H_*(K_0, M) \cong \mathbb{W} \otimes_{\mathbb{Z}_p} H_*(K, M) . \quad \square$$

5.2. Realizing the centralizer resolution for  $\mathbb{G}_2^1$  and for  $\mathbb{G}_2$ .

The construction of the topological centralizer resolutions of Theorem 1.4 and Theorem 1.5 comes in two steps. In Proposition 5.11 we first construct a complex of spectra  $X_\bullet$  such that the complex of Morava modules  $(E_2)_*(X_\bullet)$  is isomorphic to the complex  $\text{Hom}_{\mathbb{W}[\text{Gal}]^{-cts}}(P_\bullet, (E_2)_*)$  if  $P_\bullet$  denotes the complexes of Theorem 1.1 resp. of Theorem 1.2. Here  $\text{Hom}_{\mathbb{W}[\text{Gal}]^{-cts}}$  denotes continuous homomorphisms of Galois-twisted continuous Gal-modules. This part is analogous to the first step in the construction of the duality resolution at  $n = 2$  and  $p = 3$  in [14] and the centralizer resolutions in [18]. In the second step we refine the complex of spectra to a resolution, i.e. we construct the necessary factorisations of the maps  $\alpha_i$ . This step follows the strategy used in the proof of Theorem 25 and Theorem 26 of [18]. The crucial result is Proposition 5.12. We give details of the proof of Theorem 1.5. The proof of Theorem 1.4 is completely analogous with details which are less complicated.

**Proposition 5.11.** *There is a complex of spectra*

$$\begin{aligned} X_\bullet : * \rightarrow L_{K(2)}S^0 \rightarrow E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \rightarrow E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \vee E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \\ \rightarrow E_2^{C_2 \times \text{Gal}} \vee E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \rightarrow E_2^{hG_{12}(\Gamma)} \vee E_2^{C_2 \times \text{Gal}} \rightarrow E_2^{hG_{12}(\Gamma)} \rightarrow * \end{aligned}$$

such that the complex of Morava modules  $(E_2)_*(X_\bullet)$  is isomorphic to the complex of Morava modules  $\text{Hom}_{\mathbb{W}[\text{Gal}]^{-cts}}(P_\bullet, (E_2)_*)$  if  $P_\bullet$  denotes the complex given by Theorem 1.1 respectively Theorem 4.12.

*Proof.* By Corollary 5.5 we can choose  $X_i$  as the explicit (wedges of) homotopy fixed point spectra appearing in the statement of the theorem such that we have isomorphisms of Morava modules

$$(E_2)_*(X_i) = \text{Hom}_{\mathbb{W}[\text{Gal}]}(P_i, (E_2)_*)$$

for  $i = -1, 0, 1, 2, 3, 4$ . It is therefore enough to show that the  $E_2$ -Hurewicz homomorphisms

$$(5.2) \quad \pi_0 F(X_i, X_{i+1}) \rightarrow \text{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_*X_i, (E_2)_*X_{i+1})$$

are surjective for  $i = -1, 0, 1, 2, 3$  and the  $E_2$ -Hurewicz homomorphisms

$$(5.3) \quad \pi_0 F(X_i, X_{i+2}) \rightarrow \text{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_*X_i, (E_2)_*X_{i+2})$$

are injective for  $i = -1, 0, 1, 2$ . In fact, we will see that in most cases these homomorphisms are even isomorphisms.

By the explicit nature of the spectra  $X_i$  it is enough to show that the  $E_2$ -Hurewicz homomorphisms

$$(5.4) \quad \pi_0 F(E_2^{hK_1}, E_2^{hK_2}) \rightarrow \text{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_*E_2^{hK_1}, (E_2)_*E_2^{hK_2})$$

are isomorphisms for every combination of  $H_1$  and  $H_2$  with  $H_1$  running through  $\mathbb{G}_2, G_{48}(\Gamma), G'_{48}(\Gamma), C_8, G_{12}(\Gamma), C_2 \times \text{Gal}$  and  $H_2$  running through  $G_{48}(\Gamma), G'_{48}(\Gamma), C_8, G_{12}(\Gamma), C_2 \times \text{Gal}$ , except possibly in the case that  $H_1$  and  $H_2$  are equal to either  $G_{48}(\Gamma)$  or  $G'_{48}(\Gamma)$ . In this case we will see that we still have at least a surjection and this is good enough.

In fact, by Proposition 5.9 it is enough to show that the edge homomorphisms

$$\pi_0(E_2[[\mathbb{G}_2/H_1]])^{hH_2} \rightarrow \pi_0(E_2[[\mathbb{G}_2/H_1]])^{H_2}$$

of the descent spectral sequences is an isomorphism for every combination of  $H_1$  and  $H_2$ . except if  $H_1, H_2 \in \{G_{48}(\Gamma), G'_{48}(\Gamma)\}$  and that in this case it is still surjective.

For this we use Proposition 5.7 and the usual  $\lim\text{-}\lim^1$ -sequence. First we note that the  $\lim^1$ -terms  $\lim^1 \prod_{H_2 \setminus \mathbb{G}_n/U_i} \pi_1(E_2^{hH_{x,i}})$  and  $\lim^1 \prod_{H_2 \setminus \mathbb{G}_n/U_i} \pi_1(E_2)^{H_{x,i}}$  arising from part c) of Proposition 5.7 are trivial. For the second  $\lim^1$ -term this is trivial because  $\pi_1(E_2) = 0$  and

for the first  $\lim^1$ -term this follows from Remark 5.8 because the inverse system satisfies the Mittag-Leffler condition.

Therefore we get the desired isomorphisms in (5.2) respectively the surjection in (5.3) if for every  $H_{x,i}$  the edge homomorphism

$$(5.5) \quad \pi_0(E_2^{hH_{x,i}}) \rightarrow \pi_0(E_2)^{H_{x,i}}$$

of the homotopy fixed point spectral sequence is an isomorphism in degree 0 respectively surjective in degree 0. The groups  $H_{x,i}$  always contain the central  $C_2$ . Furthermore, by Lemma 5.10 it is enough to assume that  $H_2$  is contained in  $\mathbb{S}_2$ . The relevant groups are then  $G_{24}$ ,  $Q_8$ ,  $C_6$ ,  $C_4$  and  $C_2$ , and by Lemma 5.10 the edge homomorphism is a surjection for  $G_{24}$  if and only if this is the case for  $G_{48}(\Gamma)$ .

The relevant calculations can be found in [1] and [11] in the case of  $G_{48}(\Gamma_E)$ , in [25] in the case of  $C_6$ , in [8] in the case of  $C_4$ , and in [5] in the case of  $C_2$ . In these cases the edge homomorphism is always an isomorphism. So it remains to consider the case of  $Q_8$ . The homotopy fixed point spectrum  $E_2^{hG_{24}}$  is 192-periodic with periodicity generator given by  $\Delta^8$  where the modular form  $\Delta$  is the algebraic periodicity generator for the  $G_{24}$ -module  $(E_2)_*$ . With respect to the action of  $Q_8$  there is an invariant  $\tilde{\Delta}$  such that  $\tilde{\Delta}^3 = \Delta$  (cf. Theorem A.4 of [3]). Then  $E_2^{hQ_8}$  will be 64-periodic with periodicity generator  $\tilde{\Delta}^8$  and there is an equivalence

$$E_2^{hQ_8} \cong E_2^{hG_{24}} \vee \Sigma^{64} E_2^{hG_{24}} \vee \Sigma^{128} E_2^{hG_{24}} .$$

So we need to understand the edge homomorphism

$$\pi_k(E_2^{hG_{24}}) \rightarrow \pi_k(E_2)^{G_{24}}$$

not only for  $k = 0$  but also for  $k = 128$  and  $k = 64$ . The calculations in [1] and [11] show that this is still an isomorphism for  $k = 64$  while for  $k = 128$  it is only surjective with kernel isomorphic to  $\mathbb{F}_4$  and given by a class denoted  $\Delta^5 \varepsilon$ . The case of  $Q_8$  can only arise if  $H_1, H_2 \in \{G_{48}(\Gamma), G'_{48}(\Gamma)\}$ ; in case  $H_1 = \mathbb{G}_n$  all  $H_{x,i}$  are equal to  $H_2$ .  $\square$

To complete the proof of Theorem 1.5 it remains to construct the factorizations  $X_{i-1} \xrightarrow{\beta_i} W_i \xrightarrow{\gamma_i} X_i$  of  $\alpha_i$  for  $i = 1, 2, 3, 4$ , such each  $W_{i-1} \xrightarrow{\gamma_{i-1}} X_{i-1} \xrightarrow{\beta_i} W_i$  is a cofibration. We note that these factorisations will realize the splitting of the exact complex of Morava modules  $(E_2)_*(X_\bullet)$  into the usual short exact sequences. In particular, this will show that  $\gamma_4 : (E_2)_* W_4 \rightarrow (E_2)_* X_4$  is an isomorphism, hence  $\gamma_4$  is an equivalence and the resolution is of length 4.

The factorizations are constructed inductively. In the case  $i = 1$  we simply use that the composition  $\alpha_1 \alpha_0$  is null homotopic. Now let  $2 \leq i \leq 4$  and suppose that for  $0 \leq r < i$  we have already constructed factorizations  $X_{r-1} \xrightarrow{\beta_r} W_r \xrightarrow{\gamma_r} X_r$  of  $\alpha_r$  such that  $W_{r-1} \xrightarrow{\gamma_{r-1}} X_{r-1} \xrightarrow{\beta_r} W_r$  is a cofibration. In order to factor  $\alpha_i$  it is enough to show that the composition  $\alpha_i \gamma_{i-1}$  considered as element in  $\pi_0 F(W_{i-1}, X_i)$  is null homotopic.

The already constructed cofibrations can be used to analyze  $\pi_* F(W_{i-1}, X_i)$ . In fact, if  $Z$  is any spectrum then these cofibrations determine a finitely convergent Adams type spectral sequence which has the form

$$(5.6) \quad E_1^{s,t}(W_{i-1}, Z) \implies \pi_{t-s} F(W_{i-1}, Z)$$

with differentials  $d_r^{s,t} : E_r^{s,t}(W_{i-1}, Z) \rightarrow E_r^{s+r, t+r-1}(W_{i-1}, Z)$  and

$$E_1^{s,t}(W_{i-1}, Z) = \begin{cases} \pi_t F(X_{i-2-s}, Z) & 0 \leq s < i \\ 0 & s \geq i . \end{cases}$$

Clearly  $\alpha_i \gamma_{i-1}$  is in the kernel of  $\beta_{i-1}^* : \pi_0(F(W_{i-1}, X_i)) \rightarrow \pi_0(F(X_{i-2}, X_i))$ . This implies that  $\alpha_i \gamma_{i-1}$  must be detected in higher filtration, i.e. in one of the groups  $E_\infty^{s,s}(W_{i-1}, X_i)$  for  $s > 0$ . By the following result these groups are trivial, so the factorization exists and the induction step is complete.

**Proposition 5.12.** *Let  $i \in \{2, 3, 4\}$ . If  $Z = X_i$  then in the spectral sequences (5.6) we have  $E_2^{s,s}(W_{i-1}, X_i) = 0$  for all  $s > 0$ .*

*Proof.* If  $s \geq i$  then we already have  $E_1^{s,s}(W_{i-1}, X_i) = 0$ . The  $d_1$ -differentials are induced by the maps  $\alpha_i$  and hence we need to show that for  $1 \leq s < i - 1$  the sequences

$$(5.7) \quad \pi_s F(X_{i-1-s}, X_i) \xrightarrow{\alpha_{i-s}^*} \pi_s F(X_{i-2-s}, X_i) \xrightarrow{\alpha_{i-s-1}^*} \pi_s F(X_{i-3-s}, X_i)$$

are exact in the middle and that for  $s = i - 1$

$$(5.8) \quad \alpha_0^* : \pi_s F(X_0, X_i) \rightarrow \pi_s F(X_{-1}, X_i)$$

is onto. The following two lemmas imply Proposition 5.12. □

**Lemma 5.13.**  $E_2^{3,t}(W_3, X_4) = 0$  for every  $t$ .

*Proof.* In this case Proposition 5.7 shows that

$$F(X_{-1}, X_4) \simeq E_2^{hG_{12}(\Gamma)} \quad \text{and} \quad F(X_0, X_4) \simeq (E_2[[\mathbb{G}_2/G_{48}(\Gamma)]] \vee E_2[[\mathbb{G}_2/G'_{48}(\Gamma)]])^{hG_{12}(\Gamma)}$$

and the map  $F(X_0, X_4) \rightarrow F(X_{-1}, X_4)$  corresponds to the map

$$(E_2[[\mathbb{G}_2/G_{48}(\Gamma)]] \vee E_2[[\mathbb{G}_2/G'_{48}(\Gamma)]])^{hG_{12}(\Gamma)} \rightarrow E_2[[\mathbb{G}_2/\mathbb{G}_2]]^{hG_{12}(\Gamma)}$$

induced by the canonical maps  $\mathbb{G}_2/G_{48}(\Gamma) \amalg \mathbb{G}_2/G'_{48}(\Gamma) \rightarrow \mathbb{G}_2/\mathbb{G}_2$ . The latter map has a  $G_{12}(\Gamma)$ -equivariant section and this implies that the map

$$\pi_*(E_2[[\mathbb{G}_2/G_{48}(\Gamma)]] \vee E_2[[\mathbb{G}_2/G'_{48}(\Gamma)]])^{hG_{12}(\Gamma)} \rightarrow \pi_* E_2[[\mathbb{G}_2/\mathbb{G}_2]]^{hG_{12}(\Gamma)}$$

is split surjective and hence

$$\alpha_0^* : \pi_t F(X_0, X_i) \rightarrow \pi_t F(X_{-1}, X_i)$$

is surjective for every  $t$ . □

**Lemma 5.14.** *Let  $i \in \{2, 3, 4\}$ ,  $t \in \{0, 1, 2\}$  and  $s > 0$ . Then  $E_2^{s,t}(W_{i-1}, X_i) = 0$ .*

*Proof.* The spectrum  $X_i$  is a wedge of homotopy fixed point spectra  $E_2^{hF_{i,j}}$  for certain explicitly given finite subgroups  $F_{i,j}$ ,  $j = 1, \dots, m_i$

$$X_i = \bigvee_{j=1}^{m_i} E_2^{hF_{i,j}} .$$

Then, for any spectrum  $Y$  we have a homotopy equivalence natural in  $Y$

$$(5.9) \quad F(Y, X_i) \simeq \bigvee_{j=1}^{m_i} F(Y, E_2^{hF_{i,j}}) \simeq \bigvee_{j=1}^{m_i} F(Y, E_2)^{hF_{i,j}} .$$

If  $Y = X_{i-2-s}$  then  $Y$  is again a finite wedge of homotopy fixed point spectra  $E_2^{hG_{s,j}}$  with explicitly given closed subgroups  $G_{s,j}$ ,  $j = 1, \dots, n_s$ . For each of these wedge summands Proposition 5.9 (with  $K_2$  is the trivial group) gives an isomorphism

$$\pi_* F(E_2^{hG_{s,j}}, E_2) \cong \text{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_* E_2^{hG_{s,j}}, (E_2)_* E_2)$$

and because  $(E_2)_* E_2 \cong \text{Hom}_{ct_s}(\mathbb{G}_2, (E_2)_*)$  is coinduced this simplifies by Lemma 5.6 to an isomorphism

$$\pi_* F(E_2^{hG_{s,j}}, E_2) \cong \text{Hom}_{(E_2)_*}((E_2)_* E_2^{hG_{s,j}}, (E_2)_* .)$$

These isomorphisms combine to give an isomorphism

$$(5.10) \quad \pi_* F(X_{i-2-s}, E_2) \cong \text{Hom}_{(E_2)_*}((E_2)_* X_{i-2-s}, (E_2)_*) .$$

Again by Lemma 5.6 this isomorphism is compatible with the action of  $\mathbb{G}_2$  which acts on the left hand side via its action on  $E_2$  and on the right hand side diagonally.

From (5.9) and (5.10) we get a (direct sum of) descent spectral sequence(s) converging to  $\pi_{q-p} F(X_{i-2-s}, X_i)$  with  $E_2$ -term given by

$$(5.11) \quad E_2^{p,q}(s, i) := \bigoplus_{j=1}^{m_i} H^p(F_{i,j}, \text{Hom}_{(E_2)_*}((E_2)_{*+q} X_{i-2-s}, (E_2)_*)) \implies \pi_{p-q} F(X_{i-2-s}, X_i) .$$

From the isomorphism of complexes  $(E_2)_*(X_\bullet) \cong \text{Hom}_{\mathbb{W}[\text{Gal}] - \text{cts}}(C_\bullet, (E_2)_*)$  and the fact that the complex

$$0 \rightarrow N_i \rightarrow C_i \rightarrow \dots \rightarrow C_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

is  $\mathbb{W}[F]$ -split<sup>3</sup> for every finite subgroup of  $\mathbb{G}_2$  we deduce that the complex

$$0 \rightarrow (E_2)_* X_{-1} \rightarrow (E_2)_* X_0 \rightarrow \dots \rightarrow (E_2)_* X_{i-1} \rightarrow (E_2)_* W_i \rightarrow 0$$

is  $(E_2)_*[F]$ -split.

Now let  $i \leq 4$ ,  $s \leq i-1$  and  $t \in \{0, 1, 2\}$ . By the following Lemma there are isomorphisms

$$E_1^{s,t}(W_{i-1}, X_i) = \pi_t F(X_{i-2-s}, X_i) \cong \bigoplus_{j=1}^{m_i} H^t(F_{i,j}, \text{Hom}_{(E_2)_*}((E_2)_{*+2t} X_{i-2-s}, (E_2)_*))$$

and the differential  $d_1^{s,t} : E_1^{s,t}(W_{i-1}, X_i) \rightarrow E_1^{s+1,t}(W_{i-1}, X_i)$  is induced by the map  $\alpha_{i-s-1} : (E_2)_* X_{i-2-s} \rightarrow (E_2)_* X_{i-3-s}$  (with  $X_{-2} = *$ ). Because the complex

$$0 \rightarrow (E_2)_* X_{-1} \rightarrow (E_2)_* X_0 \rightarrow \dots \rightarrow (E_2)_* X_{i-2} \rightarrow (E_2)_* W_{i-1} \rightarrow 0$$

is  $\pi_* E_2[F_{i,j}]$ -split we deduce that for  $t \in \{0, 1, 2\}$

$$E_2^{s,t}(W_{i-1}, X_i) = H_s(E_1^{*,t}, d_1^{*,t}) \cong \begin{cases} \bigoplus_{j=1}^{m_i} H^t(F_{i,j}, \text{Hom}_{(E_2)_*}((E_2)_{*+2t} W_{i-1}, (E_2)_*)) & s = 0 \\ 0 & s > 0 \end{cases}$$

as claimed.  $\square$

**Lemma 5.15.** *Let  $i \in \{2, 3, 4\}$ ,  $s$  and  $t$  be integers,  $0 \leq s < i$  and  $t \in \{0, 1, 2\}$ . In the spectral sequences (5.11) we have*

$$E_2^{t,2t}(s, i) \cong E_\infty^{t,2t}(s, i) \cong \pi_t F(X_{i-2-s}, X_i) .$$

*Proof.* If  $H_1$  is a closed subgroup and  $H_2$  is a finite subgroup of  $\mathbb{G}_2$  then Proposition 5.7 gives an equivalence

$$F(E_2^{hH_1}, E_2^{hH_2}) \simeq E_2[[\mathbb{G}_2/H_1]]^{hH_2} ,$$

in particular an isomorphism

$$\pi_* F(E_2^{hH_1}, E_2) \cong \pi_* E_2[[\mathbb{G}_2/H_1]] .$$

The spectrum  $X_{i-2-s}$  is a wedge of homotopy fixed point spectra with respect to explicitly known closed subgroups  $G_{s,j}$ ,  $j = 1, \dots, n_s$  and  $X_i$  is also a wedge of homotopy fixed point spectra with respect to explicitly known closed subgroups  $F_{i,j}$ ,  $j = 1, \dots, n_i$ . Therefore the  $E_2$ -term of the spectral sequence (5.11) can be rewritten as

$$E_2^{p,q}(s, i) = \bigoplus_{j=1}^{m_i} \bigoplus_{k=1}^{n_s} H^p(F_{i,j}, \pi_q E_2[[\mathbb{G}_2/G_{s,k}]]) \implies \pi_{q-p} F(X_{i-1-s}, X_{i+1})$$

<sup>3</sup>We use this opportunity to point out an annoying typo on line 2 of page 164 of [18]. Instead of “ $\mathbb{Z}_p$ -split” it should have read “ $\mathbb{Z}_p[F]$ -split”.

This spectral sequence is the direct sum of spectral sequences indexed by  $j$  and  $k$ .

If  $H_1$  is an open subgroup of  $\mathbb{G}_2$  and  $H_2$  is a finite subgroup of  $\mathbb{G}_2$  then by part b) of Proposition 2.6 of [14] the function spectrum  $F(E_2^{hH_1}, E_2^{hH_2})$  is identified with  $E[[\mathbb{G}_2/H_1]]^{hH_2}$  and this is a finite product of homotopy fixed point spectra of the form  $E_2^{hF}$  where  $F$  is always a subgroup of  $H_2$ . In our case  $H_2$  is one of the groups  $C_2 \times \text{Gal}$ ,  $C_8$  or  $G_{12}(\Gamma)$  and the homotopy groups  $\pi_t$  for  $t = 0, 1, 2$  of the homotopy fixed point spectra  $E_2^{hF}$  are always given by  $H^{t, 2t}(F, (E_2)_*)$ . In fact, as in the proof of Proposition 5.11 one sees that because of Lemma 5.10 it is enough to consider the cases that  $F$  is either  $C_2$ ,  $C_4$  or  $C_6$  and then the necessary information is provided by [25], [8] and [5].

Therefore for  $t = 0$ ,  $t = 1$  and  $t = 2$  and  $H_1$  open we get isomorphisms

$$(5.12) \quad \pi_t(F(E_2^{hH_1}, E_2^{hH_2})) \cong H^{t, 2t}(H_2; \pi_*(E_2[[\mathbb{G}_2/H_1]])) .$$

A general closed subgroup  $H_1$  can be written as an intersection of a decreasing sequence of open subgroups  $U_i$  and then

$$F(E_2^{hH_1}, E_2^{hH_2}) \simeq \text{holim}_i F(E_2^{hU_i}, E_2^{hH_2})$$

and Remark 5.8 show that for  $t = 0, 1, 2$  the sources of the isomorphisms of (5.12) are given as the obvious inverse limit. For the target one uses that

$$\pi_*(E_2[[\mathbb{G}_2/H_1]]) = \lim_i \pi_*(E_2[[\mathbb{G}_2/U_i]])$$

if  $H_1 = \bigcap_i U_i$  for  $U_i$  a decreasing sequence of open subgroups of  $\mathbb{G}_2$ . Because the inverse limit is an exact functor on the category of profinite abelian groups we find that  $H^*(F, -)$  commutes with inverse limits of profinite coefficient modules and thus the case of open subgroups implies the case of closed subgroups.  $\square$

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