On correctors for linear elliptic homogenization in the presence of local defects: the case of advection-diffusion

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Abstract

We follow-up on our works devoted to homogenization theory for linear second-order elliptic equations with coefficients that are perturbations of periodic coefficients. We have first considered equations in divergence form in [6, 7, 8]. We have next shown, in our recent work [9], using a slightly different strategy of proof than in our earlier works, that we may also address the equation $-a_{ij}\partial_{ij}u = f$. The present work is devoted to advection-diffusion equations: $-a_{ij}\partial_{ij}u + b_j\partial_j u = f$. We prove, under suitable assumptions on the coefficients a_{ij}, b_j, $1 \leq i,j \leq d$ (typically that they are the sum of a periodic function and some perturbation in L^p, for suitable $p < +\infty$), that the equation admits a (unique) invariant measure and that this measure may be used to transform the problem into a problem in divergence form, amenable to the techniques we have previously developed for the latter case.

1 Introduction

We study homogenization theory for the advection-diffusion equation

$$- a_{ij}(x/\varepsilon)\,\partial_{ij}u^\varepsilon + \varepsilon^{-1}\,b_j(x/\varepsilon)\,\partial_j u^\varepsilon = f, \tag{1}$$

when the coefficients a and b in (1) are *perturbations*, formally vanishing at infinity, of periodic coefficients. Equation (1) is supplied with homogeneous Dirichlet boundary conditions and posed on a bounded regular domain $\Omega \subset \mathbb{R}^d$, with a right-hand-side term $f \in L^2(\Omega)$. We assume that the coefficients a and b satisfy

$$a = a^{per} + \tilde{a}, \quad b = b^{per} + \tilde{b} \tag{2}$$

where a^{per}, b^{per} describe a *periodic* unperturbed background, and $\tilde{a}$, $\tilde{b}$ the perturbation, with

$$\begin{cases} a^{per}(x) + \tilde{a}(x) \quad \text{and} \quad a^{per}(x) \quad \text{are both uniformly elliptic, in } x \in \mathbb{R}^d, \\ a^{per} \in \left(L^\infty(\mathbb{R}^d)\right)^{d\times d}, \quad b^{per} \in \left(L^\infty(\mathbb{R}^d)\right)^d, \\ \tilde{a} \in \left(L^\infty(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)\right)^{d\times d}, \quad \tilde{b} \in \left(L^\infty(\mathbb{R}^d) \cap L^s(\mathbb{R}^d)\right)^d, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for some} \quad 1 \le r, s < +\infty, \\ a^{per},\, \tilde{a} \in \left(C^{0,\alpha}_{\text{unif}}(\mathbb{R}^d)\right)^{d\times d}, \quad b^{per},\, \tilde{b} \in \left(C^{0,\alpha}_{\text{unif}}(\mathbb{R}^d)\right)^d \quad \text{for some} \quad \alpha > 0, \end{cases} \tag{3}$$

We also note that, without loss of generality and because of the specific form of the operator $-\,a_{ij}\,\partial_{ij}$, we may always assume that a is symmetric. We aim to show that the solution to (1) may be efficiently approximated using the same ingredients as classical periodic homogenization theory and with the same quality of approximation. In a series of works [6, 7, 8] (see also [5, 19]), we have studied the same issue for the equation in divergence form $-\text{div}\,\left(a(x/\varepsilon)\,\nabla u^\varepsilon\right) = f$. The heart of the matter is the existence of a corrector function w_p, strictly sublinear at infinity (that is, $\dfrac{w_p(x)}{1+|x|} \overset{|x|\to\infty}{\longrightarrow} 0$), solution, for each $p \in \mathbb{R}^d$, to $-\,\text{div}\,(a\,(p + \nabla w_p)) = 0$ in $\mathbb{R}^d$. More precisely, $w_p = w_{p,per} + \tilde{w}_p$ with $w_{p,per}$ the periodic corrector and $\tilde{w}_p$ solution with $\nabla \tilde{w}_p \in L^r$ to $-\,\text{div}\,(a\,\nabla \tilde{w}_p) = \text{div}\,(\tilde{a}\,(p + \nabla w_{p,per}))$ in $\mathbb{R}^d$. Such a situation comes in sharp contrast to the general case of homogenization theory where only a sequence of "approximate" correctors is needed to conclude, but where the rate of convergence of the approximation is then unknown. The existence of $\tilde{w}_p$ above is actually a consequence of the *a priori* estimate

$$\|\nabla u\|_{L^q} \le C_q\, \|f\|_{L^q}, \tag{4}$$

for the exponent $q = r$, and u solution to

$$-\,\text{div}\,(a\,\nabla u) = \text{div}\, f \quad \text{in}\, \mathrm{R}^{\mathrm{d}}. \tag{5}$$

The case of the equation (1) with a vanishing advection field $b \equiv 0$ has been studied in [9]. The corrector associated to this equation identically vanishes, but the issue remains to assess the rate of convergence of the homogenized approximation. This can be achieved proving the estimate

$$\left\|D^2 u\right\|_{(L^q(\mathbb{R}^d))^{d\times d}} \le C_q \left\|f\right\|_{L^q(\mathbb{R}^d)}. \tag{6}$$

for solutions to $-a_{ij}\,\partial_{ij}u = f$. From that estimate follows the existence of an invariant measure solution to $-\partial_{ij}(a_{ij}\,m) = 0$. Using it to transform equation (1) into an equation in divergence form, we may apply the previous results and conclude.

In all the article, we assume that the dimension satisfies $d \ge 3$.

Our purpose here is to study the general case in (1). Of course, besides $b \equiv 0$, the other particular case is when $b_j = \partial_i a_{ij}$ in which case equation (1) is actually in divergence form $-\partial_i\left(a_{ij}(x/\varepsilon)\,\partial_j u^\varepsilon\right) = f$. Otherwise than that, the equation requires a specific treatment. As in [9], our strategy of proof is based upon establishing an a priori estimate of the type (6). Because of the presence of the advection field b, a loss in the Lebesgue exponent q will be observed (see our precise statement in Proposition 2.1 below). Intuitively, and again as in our previous work, the estimate holds true because the perturbations $\tilde{a}$, and now respectively $\tilde{b}$, within the coefficients a and b respectively, both formally vanish at infinity, while the estimate holds true when $a = a^{per}$, $b = b^{per}$ (using the results of Avellaneda and Lin [1, 2, 3]). To the best of our knowledge, it has never been remarked with such a degree of generality that, using an adequate invariant measure, homogenization for the equation (1) can be studied and rates can be made precise, simply by transforming the equation into an equation in divergence form.

Our article is organized as follows. We prove in Section 2, Proposition 2.1, our central estimate. We also explain the loss of integrability we necessarily observe in comparison to the case of an equation in divergence form or to the case when the advection field b vanishes. The estimate is then used in Section 3 to study the adjoint equation to (1), and prove it admits an invariant measure solution. Various remarks on possible, very specific cases of coefficients a and b are considered. The invariant measure is in turn employed in Section 4 to transform equation (1) in an equation in divergence form. This allows to apply the results of [8, 9] about the properties of the corrector and the results of [5, 19] on the approximation of the solution of (1) by homogenization theory.

2 The central estimate

We begin by stating and proving our central result (Proposition 2.1) for solutions to the advection-diffusion equation on the whole space. We will next use the result to prove the existence of an adequate corrector and conclude the section by some remarks on the optimality of our results.

Proposition 2.1 *Assume* (2)-(3) *for some* $1 \leq r < d$ *and* $1 \leq s < d$. *Assume also that* $\langle m_{\rm per} b^{\rm per}\rangle = 0$, *where* $m_{\rm per}$ *is the invariant measure associated with the periodic operator* $a_{ij}^{\rm per}\partial_{ij} + b_j^{\rm per}\partial_j$ *(see* (32) *below for its definition). Fix* $1 < q < d$ *and set* $\dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{d}$. *Then, for all* $f \in \left(L^{q^*} \cap L^q\right)(\mathbb{R}^d)$, *there exists* $u \in L^1_{loc}(\mathbb{R}^d)$ *such that* $D^2 u \in L^q(\mathbb{R}^d)$, *solution to*

$$-a_{ij}\partial_{ij}u + b_j\,\partial_j u = f \quad \text{in} \quad \mathbb{R}^{\rm d}. \tag{7}$$

Such a solution is unique up to the addition of an (at most) affine function. In addition, there exists a constant C_q, *independent on* f *and* u, *and only depending on* q, d *and the coefficients* a *and* b, *such that* u *satisfies*

$$\left\|D^2 u\right\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}} + \|\nabla u\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^d} \leq C_q \, \|f\|_{\left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)}\,. \tag{8}$$

Remark 1 *In the general case, the above inequality is sharp. However, in the particular case* $b = 0$, *it is not optimal, and this observation is not related to the presence of defects. It is already true in the purely periodic case. Indeed, [9, Proposition 3.1] gives, in the case* $b = 0$, *the estimate, for any* $f \in L^q(\mathbb{R}^d)$,

$$\left\|D^2 u\right\|_{(L^q(\mathbb{R}^d))^{d\times d}} \leq C_q \, \|f\|_{L^q(\mathbb{R}^d)}\,,$$

where u *is the solution of* (7). *On the other hand, [3, Theorem B] exactly states that the result is true (in the periodic case) if and only if the field* b *vanishes. This will be made precise in Remarks 4 and 5 below. Put differently, this loss of decay at infinity is necessary as soon as a non-trivial transport field* b *is considered.*

Proof of Proposition 2.1

As in [9] for the proof of the analogous estimates for the equations in divergence form or the equation (7) with $b \equiv 0$, we argue by continuation. We henceforth fix some $1 \leq q < d$. We define $a_t = a^{per} + t\,\tilde{a}$, $b_t = b^{per} + t\,\tilde{b}$ and intend to prove the statements of Proposition 2.1 for $t = 1$. For this purpose, we introduce the property $\mathcal{P}$ defined by: *we say that the coefficients* a *and* b, *satisfying the assumptions* (2)-(3) *(for some* $1 \leq r < +\infty$*) satisfy* $\mathcal{P}$ *if the statements of Proposition 2.1 hold true for equation* (5) *with coefficient* a *and* b. We next define the interval

$$\mathcal{I} = \{t \in [0,1] \,/\, \forall s \in [0,t], \text{Property}\,\mathcal{P}\,\text{is true for}\,a_s\,\text{and}\,b_s\}\,. \tag{9}$$

We intend to successively prove that $\mathcal{I}$ is not empty, open and closed (both notions being understood relatively to the closed interval $[0,1]$), which will show that $\mathcal{I} = [0,1]$, and thus the result claimed.

Step 1: $0 \in \mathcal{I}$. To start with, we show that $0 \in \mathcal{I}$. In the particular case when $b^{per} \equiv 0$ (a case considered in [9, Proposition 3.1]), the fact that $0 \in \mathcal{I}$ is shown to be a consequence of the results of [3, Theorem B]. Indeed, the adjoint equation (44) associated to (7), which reads as $-\partial_{ij}(a_{ij}^{per}\, m_{per}) = 0$, admits (see e.g. [4]), a unique nonnegative periodic solution m_{per} that is normalized, regular and bounded away from zero. Multiplying (7) by m_{per}, we may write this equation in the divergence form

$$-\operatorname{div}\left(\mathcal{A}^{per}\nabla u\right) = m_{per} f, \tag{10}$$

with

$$\mathcal{A}^{per} = m_{per}\, a^{per} - \mathcal{B}^{per}, \tag{11}$$

$\mathcal{B}^{per}$ the skew-symmetric matrix defined by $\operatorname{div}\left(\mathcal{B}^{per}\right) = \operatorname{div}(m_{per}\, a^{per})$, and where

$$\operatorname{div}\mathcal{A}^{per} = \operatorname{div}(m_{per}\, a^{per}) - \operatorname{div}\mathcal{B}^{per} = 0. \tag{12}$$

A proof of the existence (and uniqueness) of $\mathcal{B}^{per}$ may be found in [18, Chapter 1]. We thus deduce from [3, Theorem B] that

$$\left\|D^2\, u\right\|_{(L^q(\mathbb{R}^d))^{d\times d}} \leq C\, \|f\|_{L^q(\mathbb{R}^d)}\,. \tag{13}$$

This inequality is actually valid for any $q > 1$, hence it holds also for q^*:

$$\left\|D^2\, u\right\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}} \leq C\, \|f\|_{L^{q^*}(\mathbb{R}^d)}\,.$$

Using (13) and Gagliardo-Nirenberg-Sobolev inequality (see for instance [13, Section 5.6.1, Theorem 1]), we infer $\|\nabla u\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^d} \leq C\|f\|_{L^q(\mathbb{R}^d)}$. The local integrability $u \in L^1_{loc}(\mathbb{R}^d)$ is obtained by elliptic regularity using $f \in L^1_{loc}(\mathbb{R}^d)$ and the Hölder regularity of the coefficient a^{per} stated in (3). This property immediately carries over to all the other cases we henceforth consider as soon we know there is a solution.

We next insert a non vanishing advection field b^{per}. Unless $b^{per}_j = \partial_i\, a^{per}_{ij}$ (and the equation is then in divergence form), we have to work more. We still have, as above again because of the classical results exposed in [4], the existence of an invariant measure, this time solution to $-\partial_j(\partial_i(m_{per}\, a^{per}_{ij}) + m_{per} b^{per}_j) = 0$, with all the suitable properties. This allows again to write the original equation in the divergence form (10), but this time, (12) is not satisfied and we cannot apply [3, Theorem B]. However, since (10) holds, and since the matrix-valued coefficient is periodic and regular (because of (3)), we know that the Green function $G^{per}(x, y)$ associated to the operator $-\operatorname{div}\left(\mathcal{A}^{per}\nabla\, .\right)$ satisfies, for all $x, y \in \mathbb{R}^d$, (see [17, Theorem 1.1] and [10, Proposition 2])

$$|\nabla G^{per}(x, y)| \leq \frac{C}{|x-y|^{d-1}}\ . \tag{14}$$

Hence,

$$\nabla u(x) = \int_{\mathbb{R}^d} \nabla_x G^{per}(x, y)\ m_{per}(y) f(y) dy$$

satisfies

$$|\nabla u(x)| \leq \|m_{per}\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{C}{|x-y|^{d-1}} |f(y)| dy.$$

Now, the O'Neil-Young inequality [22, 23] states that $\forall f \in L^{p_1,q_1}(\mathbb{R}^d)$, $\forall g \in L^{p_2,q_2}(\mathbb{R}^d)$,

$$\|f * g\|_{L^{\sigma,\theta}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_1,q_1}(\mathbb{R}^d)}\,\|g\|_{L^{p_2,q_2}(\mathbb{R}^d)}, \tag{15}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{\sigma}$ and $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{\theta}$, $1 \leq p_i \leq \infty$, $1 \leq q_i \leq \infty$ (except for the case $(p_i = 1, q_i = \infty)$)and $L^{p,q}$ denotes the Lorentz space of exponent (p,q) (see [16, 21].) The constant C in (15) does not depend on f and g. It is easily proved that $|x|^{-(d-1)} \in L^{d/(d-1),\infty}(\mathbb{R}^d)$, hence, since $f \in L^q(\mathbb{R}^d)$,

$$\|\nabla u\|_{\left(L^{q^*,\theta}(\mathbb{R}^d)\right)^d} \leq C\ \|m_{per}\|_{L^\infty(\mathbb{R}^d)}\ \|f\|_{L^q(\mathbb{R}^d)}\ \left\|\frac{1}{|x|^{d-1}}\right\|_{L^{d/(d-1),\infty}(\mathbb{R}^d)}, \tag{16}$$

provided $\frac{1}{\theta} \leq \frac{1}{q}$. Since $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}$, $\theta = q^*$ is allowed in (15). Therefore,

$$\|\nabla u\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^d} \leq C\ \|m_{per}\|_{L^\infty(\mathbb{R}^d)}\ \|f\|_{L^q(\mathbb{R}^d)}\,. \tag{17}$$

We next rewrite $-a_{ij}^{per}\partial_{ij}u + b_j^{per}\,\partial_j u = f$ as $-a_{ij}^{per}\partial_{ij}u = f - b_j^{per}\,\partial_j u$. In the right-hand side of the latter equation, we note that

$$\left\|f - b_j^{per}\,\partial_j u\right\|_{L^{q^*}(\mathbb{R}^d)} \leq \|f\|_{L^{q^*}(\mathbb{R}^d)} + \|b^{per}\|_{L^\infty(\mathbb{R}^d)}\ \|\nabla u\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^d} \tag{18}$$

We may therefore apply [3, Theorem B]: inserting (17) into (18), we obtain (8) in the specific case of periodic coefficients.

Step 2: $\mathcal{I}$ is open. The fact that $\mathcal{I}$ is open (relatively to the interval $[0,1]$) is a straightforward consequence of the Banach fixed point Theorem. We solve, for $f \in L^q(\mathbb{R}^d)$ fixed and $\varepsilon > 0$ presumably small,

$$-((a_t)_{ij} + \varepsilon\,\tilde{a}_{ij})\partial_{ij}u + ((b_t)_j + \varepsilon\,\tilde{b}_j)\,\partial_j u = f \quad \text{in} \quad \mathrm{R}^{\mathrm{d}},$$

using the iterations $u^0 = 0$ and, for all $n \in \mathbb{N}$,

$$-(a_t)_{ij}\partial_{ij}u^{n+1} + (b_t)_j\,\partial_j u^{n+1} = f\ +\ \varepsilon\,\left(\tilde{a}_{ij}\partial_{ij}u^n - \tilde{b}_j\,\partial_j u^n\right)$$

The point is to prove that the right-hand side belongs to $\left(L^{q^*} \cap L^q\right)(\mathbb{R}^d)$. By assumption, $f \in \left(L^{q^*} \cap L^q\right)(\mathbb{R}^d)$. We also have, by the Hölder inequality, (i) because $r \leq d$, $\tilde{a} \in \left(\left(L^d \cap L^\infty\right)(\mathbb{R}^d)\right)^{d\times d}$ and, by inductive hypothesis, $D^2u^n \in \left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}$, thus $\tilde{a} : D^2u^n \in \left(L^q \cap L^{q^*}\right)(\mathbb{R}^d)$, (ii) because $s \leq d$, $\tilde{b} \in \left(L^d \cap L^\infty(\mathbb{R}^d)\right)^d$ and, by inductive hypothesis and the Sobolev embedding Theorem, $\nabla u^n \in \left(L^{q^*}(\mathbb{R}^d)\right)^d$ (we recall that $\dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{d}$,) thus $\tilde{b}\,.\,\nabla u^n \in \left(L^q \cap L^{q^*}\right)(\mathbb{R}^d)$. By induction, the iterate u^{n+1} is thus well defined (up to an irrelevant, at most affine, function) with $D^2u^{n+1} \in \left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}$ and $\nabla u^{n+1} \in \left(L^{q^*}(\mathbb{R}^d)\right)^d$, precisely applying Property $\mathcal{P}$ for the coefficients a_t, b_t. Also because of that property, we have, for ε sufficiently small, a geometric convergence of the series $\sum_n (u^{n+1} - u^n)$. Existence of the solution u follows. The uniqueness of a solution (again up to the addition of an irrelevant at most affine function) is proven similarly.

Step 3: $\mathcal{I}$ is closed. We now show, and this is the key point of the proof, that $\mathcal{I}$ is closed. We assume that $t_n \in \mathcal{I}$, $t_n \le t$, $t_n \longrightarrow t$ as $n \longrightarrow +\infty$. For all $n \in \mathbb{N}$, we know that, for any $f \in \left(L^{q^*} \cap L^q\right)(\mathbb{R}^d)$, we have a solution (unique to the addition of an irrelevant function) u^n with $D^2 u^n \in \left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}$ and $\nabla u \in \left(L^{q^*}(\mathbb{R}^d)\right)^d$ of the equation

$$-(a_{t_n})_{ij}\partial_{ij}u^n + (b_{t_n})_j\,\partial_j u^n = f \quad \text{in} \quad \mathrm{R}^{\mathrm{d}},$$

and that this solution satisfies

$$\left\|D^2 u^n\right\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}} + \|\nabla u^n\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^d} \le C_n\,\|f\|_{\left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)},$$

for a constant C_n depending on n but not on f nor on u^n. We want to show the same properties for t.

We first conclude temporarily admitting that the constants C_n are bounded uniformly in n. Next, we will prove this is indeed the case. For $f \in \left(L^q(\mathbb{R}^d)\right)^d$ fixed, we consider the sequence of solutions u^n to

$$-(a_{t_n})_{ij}\partial_{ij}u^n + (b_{t_n})_j\,\partial_j u^n = f \quad \text{in} \quad \mathbb{R}^d,$$

which we may write as

$$-(a_t)_{ij}\partial_{ij}u^n + (b_t)_j\,\partial_j u^n = f + (t - t_n)\left(-\tilde{a}_{ij}\partial_{ij}u^n + \tilde{b}_j\,\partial_j u^n\right) \quad \text{in} \quad \mathbb{R}^d,$$

Since $t_n \in \mathcal{I}$ for all $n \in \mathbb{N}$ and the constants C_n are uniformly bounded, we know that the sequences $D^2 u^n$ and ∇u^n are bounded in $\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}$ and in $\left(L^{q^*}(\mathbb{R}^d)\right)^d$, respectively. We may pass to the weak limit in the above equation and find a solution u to $-(a_t)_{ij}\partial_{ij}u + (b_t)_j\,\partial_j u = f$. The solution also satisfies the estimate (because the sequence C_n is bounded and because the norm is weakly lower semi continuous).

In order to prove that the constants C_n are indeed bounded uniformly in n, we argue by contradiction. We assume we have $f^n \in \left(L^q(\mathbb{R}^d)\right)^d$ and u^n with $D^2 u^n \in \left(L^q(\mathbb{R}^d)\right)^{d\times d}$, such that

$$-(a_{t_n})_{ij}\partial_{ij}u^n + (b_{t_n})_j\,\partial_j u^n = f^n \quad \text{in} \quad \mathbb{R}^d, \tag{19}$$

$$\|f^n\|_{\left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)} \overset{n\longrightarrow+\infty}{\longrightarrow} 0, \tag{20}$$

$$\|\nabla u^n\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^d} + \left\|D^2 u^n\right\|_{\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}} = 1, \quad \text{for all}\, n \in \mathbb{N}. \tag{21}$$

To start with, we rewrite (19) as

$$-(a_t)_{ij}\partial_{ij}u^n + (b_t)_j\,\partial_j u^n = f^n + (t_n - t)\,\tilde{a}_{ij}\,\partial_{ij}u^n + (t_n - t)\,\tilde{b}_j\,\partial_j u^n,$$

where, as $n \longrightarrow 0$, the rightmost two terms vanish in $\left(L^{q^*} \cap L^q\right)(\mathbb{R}^d)$ using the bound (21) and the same argument as above for the openness of $\mathcal{I}$. Therefore, without loss of generality, we may change the definition of f^n and replace (19) by

$$-(a_t)_{ij}\partial_{ij}u^n + (b_t)_j\,\partial_j u^n = f^n \quad \text{in}\,\mathbb{R}^d, \tag{22}$$

In the spirit of the method of concentration-compactness [20], we now claim that the sequence u^n satisfies

$$\exists\,\eta > 0, \quad \exists\,0 < R < +\infty, \quad \forall\,n \in \mathbb{N},$$
$$\left\|D^2 u^n\right\|_{\left(L^{q^*}(B_R)\right)^{d\times d}} + \|\nabla u^n\|_{\left(L^{q^*}(B_R)\right)^d} \geq \eta > 0, \tag{23}$$

where B_R of course denotes the ball of radius R centered at the origin. We again argue by contradiction and assume that, contrary to (23),

$$\forall\,0 < R < +\infty, \quad \left\|D^2 u^n\right\|_{\left(L^{q^*}(B_R)\right)^{d\times d}} + \|\nabla u^n\|_{\left(L^{q^*}(B_R)\right)^d} \overset{n\longrightarrow+\infty}{\longrightarrow} 0. \tag{24}$$

Since both $\tilde{a}$ and $\tilde{b}$ satisfy the properties in (3), they vanish at infinity and thus, for any $\delta > 0$, we may find some sufficiently large radius R such that

$$\|\tilde{a}\|_{\left((L^d\cap L^\infty)(B_R^c)\right)^{d\times d}} \leq \delta, \quad \left\|\tilde{b}\right\|_{\left((L^d\cap L^\infty)(B_R^c)\right)^d} \leq \delta, \tag{25}$$

where B_R^c denotes the complement set of the ball B_R. We then estimate

$$\begin{aligned}
\left\|\tilde{a}\,D^2u^n\right\|^q_{L^q(\mathbb{R}^d)} &= \left\|\tilde{a}\,D^2u^n\right\|^q_{L^q(B_R)} + \left\|\tilde{a}\,D^2u^n\right\|^q_{L^q(B_R^c)} \\
&\leq \|\tilde{a}\|^q_{(L^d(\mathbb{R}^d))^{d\times d}} \left\|D^2u^n\right\|^{q^*}_{\left(L^{q^*}(B_R)\right)^{d\times d}} \\
&\quad + \|\tilde{a}\|^q_{\left(L^d(B_R^c)\right)^{d\times d}} \left\|D^2u^n\right\|^{q^*}_{\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}} \\
&\leq \|\tilde{a}\|^q_{(L^d(\mathbb{R}^d))^{d\times d}} \left\|D^2u^n\right\|^q_{\left(L^{q^*}(B_R)\right)^{d\times d}} + \delta,
\end{aligned} \tag{26}$$

using (21) and (25) for the latter majoration. Given that (24) implies that the first term in the right hand side of (26) vanishes, and since δ is arbitrary, this shows that $\tilde{a}\,D^2u^n \to 0$ in $L^q(\mathbb{R}^d)$. By the exact same argument, this time using the L^∞ estimate of $\tilde{a}$ in (25), we likewise obtain that $\tilde{a}\,D^2u^n$ vanishes in $L^{q^*}(\mathbb{R}^d)$. Therefore

$$\left\|\tilde{a}\,D^2u^n\right\|_{\left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)} \overset{n\longrightarrow+\infty}{\longrightarrow} 0. \tag{27}$$

We address the first order term similarly, getting

$$\left\|\tilde{b}\,\nabla u^n\right\|_{\left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)} \overset{n\longrightarrow+\infty}{\longrightarrow} 0. \tag{28}$$

We next notice that (22) also reads as

$$-a_{ij}^{per}\,\partial_{ij}u^n + b_j^{per}\,\partial_j u^n = f^n + t\,\tilde{a}_{ij}\,\partial_{ij}u^n - t\,\tilde{b}_j\,\partial_j u^n,$$

and use (20), (27) and (28) to estimate its right-hand side. In view of the estimate (8) which, as mentioned above, holds for the operator for periodic coefficients, this implies that D^2u^n and ∇u^n (strongly) converge to zero in $\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}$ and $\left(L^{q^*}(\mathbb{R}^d)\right)^{d}$, respectively. This evidently contradicts (21) . We therefore have established (23).

We are now in position to finally reach a contradiction. Because of the bound (21), we may claim that, up to an extraction, D^2u^n weakly converges in $\left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}$, to some D^2u. This convergence is actually strong in $\left(L^{q^*}_{loc}(\mathbb{R}^d)\right)^{d\times d}$. This is proven combining Sobolev compact embeddings and estimates for general elliptic operators (see e.g. [14, Theorem 7.3]). Passing to the weak limit in (22), we obtain $-(a_t)_{ij}\partial_{ij}u + (b_t)_j\partial_j u = 0$ for u that does not identically vanish. This is a contradiction with the uniqueness we prove below.

There remains to prove uniqueness. We thus consider a solution u to (7) with $f = 0$, $D^2u \in L^{q^*}(\mathbb{R}^d)^{d\times d}$ and $\nabla u \in L^{q^*}(\mathbb{R}^d)^d$.

We first consider the case $q < d/2$, i.e $q^* < d$. In such a case, the Gagliardo-Nirenberg-Sobolev inequality implies that, up to the addition of a constant, $u \in L^{q^{**}}(\mathbb{R}^d)$. Moreover, using elliptic regularity [15, Theorem 9.11], one easily proves that

$$\sup_{x\in\mathbb{R}^d} \|u\|_{W^{2,q^{**}}(B_1(x))} < +\infty, \tag{29}$$

where we recall that $\frac{1}{q^{**}} = \frac{1}{q} - \frac{2}{d}$. If $q^{**} > d$, we apply Morrey's theorem, proving that u is Hölder continuous. If not, we repeat the above argument, obtaining (29) with q^{***}, and so on, that is, for any integer n:

$$\sup_{x\in\mathbb{R}^d} \|u\|_{W^{2,q_n}(B_1(x))} < +\infty, \qquad \frac{1}{q_n} = \frac{1}{q} - \frac{n}{d}, \quad \text{as long as} \quad n < \frac{d}{q}.$$

For m such that $m < \frac{d}{q} < m+1$ (if $\frac{d}{q} \in \mathbb{N}$, slightly decrease q such that it is no longer the case; this is possible because the estimate is local), we have $q_m > d$, hence, by Morrey's theorem, $u \in C^{0,\alpha}_{\text{unif}}(\mathbb{R}^d)$, for some $\alpha > 0$. This and $u \in L^{q^{**}}(\mathbb{R}^d)$ imply that, for any $\delta > 0$, there exists $R > 0$ such that

$$\sup_{|x|>R} |u(x)| \le \delta.$$

Applying the maximum principle on the ball B_R, we thus have $|u| \le \delta$ in $\mathbb{R}^d$. Since this is valid for any $\delta > 0$, we conclude that $u = 0$.

In order to address the case $d/2 \le q < d$, we write (7) as

$$-a_{ij}^{per}\partial_{ij}u + b_j^{per}\partial_j u = t\left(\tilde{a}_{ij}\partial_{ij}u - \tilde{b}_j\partial_j u\right) \tag{30}$$

Here again, the fact that $\tilde{a} \in \left(L^r \cap L^\infty(\mathbb{R}^d)\right)^{d\times d}$ and $D^2 u \in L^{q^*}(\mathbb{R}^d)^{d\times d}$ implies that $\tilde{a}_{ij}\partial_{ij} u \in L^{r_1} \cap L^{q^*}(\mathbb{R}^d)$, with $\frac{1}{r_1} = \frac{1}{r} + \frac{1}{q^*} = \frac{1}{r} + \frac{1}{q} - \frac{1}{d}$. Similarly, $\tilde{b}_j \partial_j u \in L^{s_1} \cap L^{q^*}(\mathbb{R}^d)$, with $\frac{1}{s_1} = \frac{1}{s} + \frac{1}{q} - \frac{1}{d}$. Since $r, s < d$, we have $r_1, s_1 < q$. Applying step 1 of the present proof, we conclude that

$$D^2 u \in \left(L^{\max(r_1,s_1)^*} \cap L^{q^*}(\mathbb{R}^d)\right)^{d\times d}, \quad \nabla u \in \left(L^{\max(r_1,s_1)^*} \cap L^{q^*}(\mathbb{R}^d)\right)^{d}. \quad (31)$$

Repeating this argument, (31) is also valid for s_n and r_n defined by

$$\frac{1}{s_n} = \frac{1}{q} + \frac{n}{s} - \frac{n}{d}, \quad \frac{1}{r_n} = \frac{1}{q} + \frac{n}{r} - \frac{n}{d}.$$

Hence, for n sufficiently large, we have $\max(r_n, s_n) < d/2$, and we may apply the argument of the case $q < d/2$.

We reach a final contradiction. This shows that $\mathcal{I}$ is closed. As it is also open and non empty, it is equal to $[0,1]$ and this concludes the proof of Proposition 2.1. $\diamondsuit$

Remark 2 *It is clear from the above proof that Proposition 2.1 is also valid in the case $r = d$ and/or $s = d$ if we assume in addition that $q < \frac{d}{2}$. The only stage where $s, r < d$ is used is to prove uniqueness in the case $q \geq \frac{d}{2}$.*

We now use Proposition 2.1 to prove the existence of a corrector for our problem. We first recall the following facts for the periodic case (see e.g. [4]). There exists a unique positive measure, bounded away from zero, with normalized periodic average $\langle m_{per}\rangle = 1$, that solves

$$-\partial_i \left(a_{ij}^{per}\partial_j m_{per} + b_j^{per}\, m_{per}\right) = 0 \quad \text{in}\, \mathbb{R}^{\mathrm{d}}. \quad (32)$$

If the condition

$$\langle m_{per} b^{per}\rangle = 0 \quad (33)$$

holds true, then, for all $p \in \mathbb{R}^d$, there exists a periodic corrector function, with normalized average $\langle w_{p,per}\rangle = 0$, solution to

$$-a_{ij}^{per}\partial_{ij} w_{p,per} + b_j^{per}\,\partial_j w_{p,per} = -b^{per}.p \quad \text{in}\, \mathbb{R}^{\mathrm{d}}. \quad (34)$$

By elliptic regularity (see for instance [15, Theorem 9.11]), this function satisfies $w_{p,per} \in L^\infty(\mathbb{R}^d)$, $\nabla\, w_{p,per} \in \left(L^\infty(\mathbb{R}^d)\right)^d$ and $D^2 w_{p,per} \in \left(L^\infty(\mathbb{R}^d)\right)^{d\times d}$.

Corollary 2.2 *As in Proposition 2.1, we assume* (2)-(3) *for some $1 \leq r < d$ and $1 \leq s < d$. We additionally assume the condition* (33). *Then, for all $p \in \mathbb{R}^d$, there exists a corrector function, solution to*

$$-a_{ij}\partial_{ij} w_p + b_j\,\partial_j w_p = -b.p \quad \text{in}\, \mathbb{R}^{\mathrm{d}}. \quad (35)$$

Such a solution is unique up to the addition of an (at most) affine function. It reads as

$$w_p = w_{p,per} + \tilde{w}_p \tag{36}$$

where $w_{p,per}$ *is the periodic corrector solution to* (34) *with normalized average* $\langle w_{p,per} \rangle = 0$*, and where (again up to the addition of an at most affine function)* $\tilde{w}_p \in L^1_{loc}(\mathbb{R}^d)$*,* $\nabla \tilde{w}_p \in \left(L^{q^*}(\mathbb{R}^d)\right)^d$*,* $D^2\, \tilde{w}_p \in \left(L^{q^*}(\mathbb{R}^d)\right)^{d\times d}$ *for* $\frac{1}{q^*} = \frac{1}{\max(r,s)} - \frac{1}{d}$*. In particular, the corrector* w_p *is thus strictly sub-linear at infinity.*

Proof of Corollary 2.2 Using (34), we notice that (35) also reads as

$$-a_{ij}\partial_{ij}\tilde{w}_p + b_j\,\partial_j\tilde{w}_p = -\tilde{b}.p + \tilde{a}_{ij}\partial_{ij}w_{p,per} - \tilde{b}_j\,\partial_j w_{p,per} \quad \text{in}\,\mathbb{R}^{\mathrm{d}}. \tag{37}$$

Given the properties of boundedness of $w_{p,per}$ and its first and second derivatives and our assumptions (2)-(3), the right-hand side of (37) belongs to $\left(L^{\max(r,s)} \cap L^\infty\right)(\mathbb{R}^d)$. Since we have assumed $\max(r,s) < d$, we may apply Proposition 2.1 for the exponent $q = \max(r,s)$ and we obtain the results stated in Corollary 2.2. $\diamondsuit$

We conclude this section with a series of remarks on our assumptions and results of Proposition 2.1 and Corollary 2.2.

Remark 3 *One should not be surprised by the fact that, in the left-hand side of* (8)*, the first derivative* ∇u *and the second derivative* $D^2 u$ *share the* same *integration exponent. One could think that, because of Gagliardo-Nirenberg-Sobolev inequality, the exponent of the first derivative and that of the second derivative are related by* $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}$*. Because of the* structure *of the differential operator, it is indeed possible to have the same exponent. To illustrate this idea, we consider the simple example where* $a = \mathbf{Id}$*, and* $b_i(x) = b_0(|x|)\frac{x_i}{|x|}$*. We assume* $r \mapsto b_0(r)$ *to be smooth and vanish at* 0 *in order to have* $b \in C^{0,\alpha}_{\mathrm{unif}}(\mathbb{R}^d)$*, and* $b_0(r) = 1$ *for* $r \geq 1$*. We also assume that an estimate of the form* $\|D^2 u\|_{L^\beta} \leq C\|f\|_{L^q}$ *holds for the solution of* (7)*, for some exponent* β*. If the right-hand side* f *is radially symmetric, so is the solution* u*, and the equation reads*

$$-\frac{d^2u}{dr^2} - \frac{d-1}{r}\frac{du}{dr} + b_0(r)\frac{du}{dr} = f(r). \tag{38}$$

Since $b_0(r) + \frac{d-1}{r} \to 1$ *as* $r \to +\infty$*, the estimate* $\|D^2 u\|_{L^\beta} \leq C\|f\|_{L^q}$*, together with equation* (38)*, imply* $\|\nabla u\|_{L^\beta} \leq C\|f\|_{L^q}$*.*

Remark 4 *One should not be surprised either by the fact that the exponent of the rignt-hand side of* (8) *is equal to* q^**, which is larger than* q*. Indeed, this is already a necessary condition in the periodic case: if we assume that, for equation* (7)*, an estimate of the type*

$$\left\|D^2\, u\right\|_{(L^\beta(\mathbb{R}^d))^{d\times d}} + \|\nabla u\|_{(L^\beta(\mathbb{R}^d))^d} \leq C_q\, \|f\|_{(L^\beta \cap L^q)(\mathbb{R}^d)}\,,$$

holds for some $\beta \geq 1$*, then one necessarily has*

$$d > q, \quad \beta \geq q^*, \tag{39}$$

unless $b = 0$. Indeed, this estimate is equivalent to the following:

$$\left\|D^2 u\right\|_{(L^\beta(\Omega/\varepsilon))^{d\times d}} + \|\nabla u\|_{(L^\beta(\Omega/\varepsilon))^d} \le C_q \, \|f\|_{(L^\beta\cap L^q)(\Omega/\varepsilon)}\,, \tag{40}$$

for some constant C independent of $\varepsilon \in (0,1)$ and of Ω, where u is a solution to $-a_{ij}\partial_{ij}u + b_i\partial_i u = f$ in Ω/ε with, say, homogeneous Dirichlet boundary condition. Next, let us consider the solution u_ε of the problem

$$a_{ij}\left(\frac{x}{\varepsilon}\right)\partial_{ij}u_\varepsilon + b_i\left(\frac{x}{\varepsilon}\right)\partial_i u_\varepsilon = f$$

in Ω, with homogeneous Dirichlet boundary conditions. In particular, this estimate implies

$$\left\|D^2 u\right\|_{(L^\beta(\Omega/\varepsilon))^{d\times d}} \le C_q \, \|f\|_{(L^\beta\cap L^q)(\Omega/\varepsilon)}\,. \tag{41}$$

Rescaling this equation, we find that $v_\varepsilon(x) := u_\varepsilon(\varepsilon x)$ is solution to

$$a_{ij}\partial_{ij}v_\varepsilon + \varepsilon b_i\partial_i v_\varepsilon = \varepsilon^2 f(\varepsilon x),$$

in Ω/ε. Hence, applying (41), we have

$$\begin{aligned}\left\|D^2 v_\varepsilon\right\|_{(L^\beta(\Omega/\varepsilon))^{d\times d}} &\le C_q \, \varepsilon^2 \, \|f(\varepsilon\cdot)\|_{(L^\beta\cap L^q)(\Omega/\varepsilon)} \\ &= C_q\left(\varepsilon^{2-\frac{d}{q}}\|f\|_{L^q(\Omega)} + \varepsilon^{2-\frac{d}{\beta}}\|f\|_{L^\beta(\Omega)}\right),\end{aligned}$$

Going back to u_ε, this reads

$$\varepsilon^{2-\frac{d}{\beta}}\left\|D^2 u_\varepsilon\right\|_{L^\beta(\Omega)} \le C_q\left(\varepsilon^{2-\frac{d}{q}}\|f\|_{L^q(\Omega)} + \varepsilon^{2-\frac{d}{\beta}}\|f\|_{L^\beta(\Omega)}\right), \tag{42}$$

Finally, assuming that a and b are periodic, we apply standard homogenization technique to u_ε, getting

$$u_\varepsilon(x) = u^*(x) + \varepsilon\partial_j u^*(x) w_j\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 g\left(x, \frac{x}{\varepsilon}\right).$$

Here, w_j denotes the corrector associated to the above equation, and u^* is the solution of the homogenized problem, that is, the limit of u_ε as $\varepsilon \to 0$. If all the data are smooth, we may assume that g is smooth, hence,

$$\partial_i u_\varepsilon(x) = \partial_i u^*(x) + \partial_j u^*(x)\partial_i w_j\left(\frac{x}{\varepsilon}\right) + O(\varepsilon),$$

$$\partial_{ik} u_\varepsilon(x) = \partial_{ik} u^*(x) + \frac{1}{\varepsilon}\partial_j u^*(x)\partial_{ik} w_j\left(\frac{x}{\varepsilon}\right) + O(1).$$

These estimates imply that

$$\|D^2 u_\varepsilon\|_{L^\beta(\Omega)} \quad \textit{scales as} \quad \frac{1}{\varepsilon}, \tag{43}$$

unless $D^2 w_j = 0$, for all j (recall that u^* is independent of w_j). In the periodic case we are studying here, this implies $\nabla w_j = 0$, that is, $b = 0$. Inserting (43) *into* (42)*, we find that*

$$\varepsilon^{1-\frac{d}{\beta}} \le C\left(\varepsilon^{2-\frac{d}{q}} + \varepsilon^{2-\frac{d}{\beta}}\right),$$

where the constant C depends on a, b, f, but not on ε. Letting $\varepsilon \to 0$, we thus find $1 - \frac{d}{\beta} \ge \min\left(2 - \frac{d}{q}, 2 - \frac{d}{\beta}\right)$, that is, (39).

Remark 5 *The case $b = 0$ discussed in Remark 4 exactly corresponds to the condition given in [3, Theorem B], which states that for the equation*

$$-\operatorname{div}(\mathcal{A}^{per}\nabla u) = f,$$

an estimate of the form $\|D^2 u\|_{(L^q(\mathbb{R}^d))^{d\times d}} \le C\|f\|_{L^q(\mathbb{R}^d)}$ can hold if and only if $\operatorname{div}(\mathcal{A}^{per}) = 0$*. Actually, in the calculations of [9, Section 3.1], which are recalled in Step 1 of the proof of Proposition 2.1 above, we recover this fact. The condition on $\mathcal{A}^{per}$ is equivalent to $\nabla w_j = 0$ for all j, since* $-\operatorname{div}\left(\mathcal{A}^{per}(\nabla w_j + e_j)\right) = 0$*, while* $\operatorname{div}(\mathcal{A}^{per}) = \operatorname{div}(m_{per}a^{per} - \mathcal{B}^{per}) = m_{per}b^{per}$ *shows that* $\operatorname{div}\left(\mathcal{A}^{per}\right) = 0$ *if and only if $b^{per} = 0$.*

Remark 6 *The norm of the right-hand side in* (8) *is by definition*

$$\|f\|_{\left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)} = \|f\|_{L^q(\mathbb{R}^d)} + \|f\|_{L^{q^*}(\mathbb{R}^d)}.$$

The presence of the second term is in fact necessary for the estimate (8) *to hold true. Indeed, if $a_{ij}\partial_{ij}u = f$ and $D^2u \in L^{q^*}$ then $f \in L^{q^*}$. Moreover, we are going to use this estimate for functions belonging to spaces of the form $L^q \cap L^\infty$. Having to consider functions that, in addition to being in L^q, belong to L^{q^*}, is not a constraint in our setting.*

Remark 7 (On our assumptions r and s sufficiently small) *Proposition 2.1 and Corollary 2.2 hold true under the assumption, in particular, that the perturbations $\tilde{a}$ and $\tilde{b}$ decay sufficiently fast to zero at infinity, namely that they belong to L^r and L^s with r and s smaller than d. Such a condition turns out to be, qualitatively, necessary. And it is necessary not only to obtain a corrector with perturbation $\nabla\tilde{w}$ in some L^p space, but to obtain a corrector that satisfies the sharp condition to be imposed to a corrector, which is only a* consequence *of the condition $\nabla\tilde{w} \in L^p$: to be (strictly) sub-linear at infinity. In order to show this is the case, we consider the simplistic one-dimensional situation where $a^{per} = 1$, $\tilde{a} = 0$, $b = b^{per} + \tilde{b}$. The corrector equation then reads $-w'' + b(1 + w') = 0$. The sufficient and necessary condition for a periodic corrector w_{per} to exist is $\langle b^{per}\rangle = 0$ (note that this condition is indeed equivalent to $\langle m_{per}b^{per}\rangle = 0$ in this specific situation). That corrector is defined by $w'_{per} = -1 + \left\langle e^{B^{per}}\right\rangle^{-1} e^{B^{per}}$,* *where $B^{per} = \int_0^x b^{per}(t)\,dt$. Then, any solution to the corrector equation reads (up to the addition of an irrelevant constant) as $w = w_{per} + \tilde{w}$, where*

$$(\tilde{w})'(x) = \left\langle e^{B^{per}} \right\rangle^{-1} e^{B^{per}(x)}(-1 + e^{\tilde{B}(x)})$$

and $(\tilde{B})' = -\tilde{b}$. If we then impose on $\tilde{w}$ to be strictly sub-linear at $x = \pm\infty$, then we must *have $\tilde{B}(\pm\infty) = 0$. In other words, both integrals $\int_{-\infty}^0 \tilde{b}(t)\,dt$ and $\int_0^{+\infty} \tilde{b}(t)\,dt$ must be well defined and $\int_{-\infty}^{+\infty} \tilde{b}(t)\,dt = 0$. It therefore in particular implies that $\tilde{b}$ has necessarily some integrability at infinity. For completeness, we check that the above conditions are indeed sufficient: the derivative $(\tilde{w})'$ then behaves as $\displaystyle\int_x^{+\infty} \tilde{b}(t)\,dt$ as $|x| \longrightarrow +\infty$, and $\tilde{w}$ is strictly sub-linear at infinity since $\frac{\tilde{w}(x)-\tilde{w}(0)}{x} \approx \int_x^{+\infty} \tilde{b}(t)\,dt + \int_0^x \frac{t}{x}\tilde{b}(t)dt$. On the other hand, it is easy to build an example of $\tilde{b} \in L^p$ for some $p > 1$, for which $\tilde{b} \notin L^1$, and $\tilde{w}'$ grows exponentially at infinity. Think for instance of $\tilde{b}(x) \approx |x|^{-1/2}$ at infinity, for which $\tilde{w}'(x) \approx e^{2\sqrt{|x|}}$ at infinity.*

3 Existence of the invariant measure

We now consider the issue of existence (and uniqueness in a suitable class) of an invariant measure associated to equation (1), that is a positive function m, actually bounded away from zero, $\inf m > 0$, unique when appropriately normalized, solution to the equation

$$-\partial_i(\partial_j(a_{ij}\,m) + b_i\,m) = 0, \tag{44}$$

on $\mathbb{R}^d$. We know from our previous study [9, Section 3] that this issue is a straightforward consequence of the general estimate of the type (8). The argument essentially goes by duality.

First of all, we know from the general theory (see e.g. [4]), that there exists a unique, periodic measure m_{per}, with normalized periodic average $\langle m_{per}\rangle = 1$, solution to

$$-\partial_i(\partial_j(a_{ij}^{per}\,m_{per}) + b_i^{per}\,m_{per}) = 0. \tag{45}$$

Then we look for m solution to (44) as $m = m_{per} + \tilde{m}$ and rewrite (44) as

$$-\partial_i(\partial_j(a_{ij}\,\tilde{m}) + b_i\,\tilde{m}) = \partial_i(\partial_j(\tilde{a}_{ij}\,m_{per}) + \tilde{b}_i\,m_{per}), \tag{46}$$

The key point for establishing the well-posedness of (46) is to show an *a priori* estimate on the solution to that equation. The conclusion follows by standard arguments made explicit in [9, Section 3].

Let us fix, as in Proposition 2.1, $1 \le r < d$, $1 \le s < d$, $1 < q < d$. Recall our notation $\dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{d}$. We denote by q' the conjugate exponent of q and by

$(q^*)'$ that of q^*. We have $\dfrac{1}{q}+\dfrac{1}{q'}=1$, $\dfrac{1}{q^*}+\dfrac{1}{(q^*)'}=1$. We consider the integral $\displaystyle\int \tilde{m}\, f$ for some arbitrary function $f \in \left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)$.

Introducing the solution u to $-a_{ij}\partial_{ij}u + b_j\,\partial_j u = f$ provided by Proposition 2.1 and using (46), we have

$$\begin{aligned}
\int \tilde{m}\, f &= \int \tilde{m}\,\left(-a_{ij}\partial_{ij}u + b_j\,\partial_j u\right)\\
&= \int \left(-\partial_i(\partial_j(a_{ij}\,\tilde{m}) + b_i\,\tilde{m})\right)\, u\\
&= \int \Big(\partial_i(\partial_j(\tilde{a}_{ij}\, m_{per}) + \tilde{b}_i\, m_{per})\Big)\, u\\
&= \int m_{per}\,\Big(\tilde{a}_{ij}\partial_{ij}u - \tilde{b}_j\,\partial_j u\Big).
\end{aligned}$$

The Hölder inequality and the estimate (8) successively yield

$$\begin{aligned}
\left|\int \tilde{m}\, f\right| &\leq \|m_{per}\|_{L^\infty(\mathbb{R}^d)}\Big(\|\tilde{a}\|_{L^{(q^*)'}(\mathbb{R}^d)^{d\times d}}\,\left\|D^2u\right\|_{L^{q^*}(\mathbb{R}^d)^{d\times d}}\\
&\quad + \left\|\tilde{b}\right\|_{L^{(q^*)'}(\mathbb{R}^d)^d}\,\|\nabla u\|_{L^{q^*}(\mathbb{R}^d)^d}\Big)\\
&\leq C\,\|m_{per}\|_{L^\infty(\mathbb{R}^d)}\left(\|\tilde{a}\|_{L^{(q^*)'}(\mathbb{R}^d)^{d\times d}} + \left\|\tilde{b}\right\|_{L^{(q^*)'}(\mathbb{R}^d)^d}\right)\\
&\quad \times\left(\left\|D^2u\right\|_{L^{q^*}(\mathbb{R}^d)^{d\times d}} + \|\nabla u\|_{L^{q^*}(\mathbb{R}^d)^d}\right)\\
&\leq C\,\|m_{per}\|_{L^\infty(\mathbb{R}^d)}\left(\|\tilde{a}\|_{L^{(q^*)'}(\mathbb{R}^d)^{d\times d}} + \left\|\tilde{b}\right\|_{L^{(q^{**})'}(\mathbb{R}^d)^d}\right)\\
&\quad \times\|f\|_{\left(L^{q^*}\cap L^q\right)(\mathbb{R}^d)},
\end{aligned} \tag{47}$$

for some irrelevant constants C. By definition,

$$\|\tilde{m}\|_{\left(L^{q'}+L^{(q^*)'}\right)(\mathbb{R}^d)} = \sup_{f\neq 0\,\in\left(L^q\cap L^{q^*}\right)(\mathbb{R}^d)} \frac{\left|\int \tilde{m}\, f\right|}{\|f\|_{\left(L^q\cap L^{q^*}\right)(\mathbb{R}^d)}} \tag{48}$$

We therefore infer from (47) and (48) that $\tilde{m} \in \left(L^{q'}+L^{(q^*)'}\right)(\mathbb{R}^d)$ with $1 \leq q' < +\infty$, $1 \leq (q^*)' < +\infty$, provided q is such that our assumptions (3) on the integrability of $\tilde{a}$ and $\tilde{b}$ imply that $\tilde{a} \in \left(L^{(q^*)'}(\mathbb{R}^d)\right)^{d\times d}$ and $\tilde{b} \in \left(L^{(q^*)'}(\mathbb{R}^d)\right)^d$. This is the case when $r \leq (q^*)' < +\infty$ and $s \leq (q^*)' < +\infty$. The four conditions

$$\begin{cases} 1 \leq q' < +\infty,\\ 1 \leq (q^*)' < +\infty,\\ r \leq (q^*)' < +\infty,\\ s \leq (q^*)' < +\infty,\end{cases}$$

reduce to $\frac{1}{q} \geq 1 - \min\left(\frac{1}{r} - \frac{1}{d}, \frac{1}{s} - \frac{1}{d}\right)$. And we therefore obtain the best possible information on the integrability at infinity of $\tilde{m}$ when minimizing q', that is maximizing q, that is taking an *equality* in that equation, namely:

$$\frac{1}{q} = 1 - \min\left(\frac{1}{r} - \frac{1}{d}, \frac{1}{s} - \frac{1}{d}\right). \tag{49}$$

On the other hand, we recall that by classical elliptic regularity results (see [15, Theorem 9.11]), $m_{per} \in C^{0,\alpha}(\mathbb{R}^d)$ and $\tilde{m} \in C^{0,\alpha}(\mathbb{R}^d)$. Moreover, standard results of periodic homogenization [4, Chapter 3, Section 3.3, Theorem 3.4] imply that m_{per} is bounded away from 0. Since $\tilde{m} \in L^q(\mathbb{R}^d)$ and is Hölder continuous, we have

$$\|\tilde{m}\|_{L^\infty(B_R^c)} \overset{R\to+\infty}{\longrightarrow} 0.$$

Hence, for R sufficiently large,

$$\forall x \in B_R^c, \quad m(x) \geq \frac{1}{2} \inf m_{per} > 0. \tag{50}$$

Applying the maximum principle on B_R, we deduce that $m \geq 0$ is valid in the whole space $\mathbb{R}^d$. Next, we apply Harnack inequality [11, 12], which implies that m is bounded away from 0. For the value of q set in (49), we therefore obtain

$$\tilde{m} \in \left(L^{q'} \cap L^\infty\right)(\mathbb{R}^d) \quad \text{for} \quad \frac{1}{q'} = \min\left(\frac{1}{r} - \frac{1}{d}, \frac{1}{s} - \frac{1}{d}\right). \tag{51}$$

We collect our results in the following.

Corollary 3.1 *We assume* (2)-(3) *for some* $1 \leq r < d$, $1 \leq s < d$, *and the condition* (33). *Then there exists an invariant mesure* m, *solution to* (44), *that is* $-\partial_i(\partial_j(a_{ij}\, m) + b_i\, m) = 0$. *It reads as* $m = m_{per} + \tilde{m}$, *where* m_{per} *is the unique, normalized periodic invariant measure defined in* (45), *and* $\tilde{m}$ *belongs to* $\left(L^{q'} \cap L^\infty\right)(\mathbb{R}^d)$ *where* q' *is made precise in* (51). *Such a measure is unique, positive, bounded away from zero and Hölder continuous.*

Remark 8 (On coefficients with specific structure) *Our assumptions above are quite general. They apply without specific structure of the coefficients* a *and* b. *If* some *structure is assumed on these coefficients, then we suspect that the existence of an invariant measure may be proven using a different, more constructive approach. A simplistic example is* $a_{ij}^{per} = \delta_{ij}$, $\tilde{a}_{ij} \equiv 0$ *and* b *(thus in particular* b^{per}*) is divergence-free. Then we immediately observe that the periodic invariant measure is constant, and we normalize it to* $m_{per} \equiv 1$, *while* $\tilde{m} \equiv 0$ *(since we look for it in some* $L^q(\mathbb{R}^d)$*). Similar examples may be constructed using different adequate coefficients* a *and by "dividing"* b *by* a. *This suffices to show that the presence of structure in the coefficients significantly changes* *the landscape. We wish to concentrate here on an example which, although also simple, is more instructive. We again fix* $a_{ij}^{per} = \delta_{ij}$, $\tilde{a}_{ij} \equiv 0$, *and this time set* $b_i^{per} \equiv 0$, *and* $\tilde{b} = \nabla \tilde{\psi}$ *for some function* $\tilde{\psi} \in L^q(\mathbb{R}^d)$ *for some* $1 \leq q < +\infty$, ψ *sufficiently regular (typically Hölder continuous,* $C^{1,\alpha}$ *so that the regularity assumed in* (3) *is satisfied). The perturbation* $\tilde{m}$ *of the periodic measure* $m^{per} = 1$ *solves* $\partial_j \left(\partial_j \tilde{m} + (1 + \tilde{m})\, \partial_j \tilde{\psi} \right) = 0$. *It is readily seen that* $\tilde{m} = \exp(-\tilde{\psi}) - 1$, *so that the full invariant measure is* $m = 1 + (\exp(-\tilde{\psi}) - 1) = \exp(-\tilde{\psi})$. *Since* $\tilde{\psi}$ *vanishes at infinity (by regularity and integrability),* $\tilde{m}$ *behaves like* $\tilde{\psi}$ *at infinity and also belongs to* $L^q(\mathbb{R}^d)$. *The point of this remark is that the exponent* $1 \leq q < +\infty$ *may be arbitrarily large, in sharp contrast with both our "general" assumption* $\tilde{b} \in \left(L^s(\mathbb{R}^d)\right)^d$ *for* s *sufficiently small and our conclusion on* $\tilde{m} \in L^\beta(\mathbb{R}^d)$ *again with* β *small. Notice also that this observation does not contradict our considerations of Remark 7. Indeed, with this specific structure,* $b^{per} \equiv 0$, $\tilde{b} = (\tilde{\psi})'$ *in our one-dimensional example there, and thus* $(\tilde{w})' = \exp(-\tilde{\psi}) - 1$ *does belong to* $L^q(\mathbb{R}^d)$.

4 Application to homogenization

It is classical in the periodic case that the invariant measure allows one to recast (by multiplication) the original problem as a problem for an equation in divergence form. We have recalled the standard argument in [9] and above in Step 1 of the proof of Proposition 2.1. In the present section, we extend it to the perturbed case with a drift and for simplicity we proceed in dimension $d \geq 3$.

More precisely, we may rewrite (1), and the associated corrector equation (35), respectively as

$$-\operatorname{div}(\mathcal{A}_\varepsilon \nabla u^\varepsilon) = m_\varepsilon f, \tag{52}$$

and

$$-\operatorname{div}(\mathcal{A}(p + \nabla w_p)) = 0, \tag{53}$$

with $m_\varepsilon(x) = m(x/\varepsilon)$, with the elliptic matrix valued coefficient $\mathcal{A}_\varepsilon(x) = \mathcal{A}(x/\varepsilon)$ defined by

$$\mathcal{A} = m\, a - \mathcal{B} \tag{54}$$

and the skew-symmetric matrix-valued coefficient $\mathcal{B}$ is defined by

$$\operatorname{div}(\mathcal{B}) = mb + \operatorname{div}(ma).$$

Such a matrix may be proved to exist using the fact that $\operatorname{div}(mb + \operatorname{div}(ma)) = 0$, by definition of the measure m. In the specific case of dimension $d = 3$, we have

$$\mathcal{B} = \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}.$$

where the vector field $B = (B_1, B_2, B_3)$ is defined by $\operatorname{curl} B = m\, b + \operatorname{div}(m\, a)$. In our specific case, where $m = m_{per} + \tilde{m}$, $\mathcal{B}$ is defined as the sum $\mathcal{B} = \mathcal{B}^{per} + \tilde{\mathcal{B}}$, where the periodic part $\mathcal{B}^{per}$ is obtained solving the periodic equation div $\mathcal{B}^{per} = m_{per}\, b^{per} + \operatorname{div}(m_{per}\, a^{per})$ (the right-hand side being divergence-free because of (45), we recall) and where

$$\operatorname{div}\, \tilde{\mathcal{B}} = \tilde{m}\, b^{per} + (m_{per} + \tilde{m})\ \tilde{b} + \operatorname{div}(\tilde{m}\, a^{per} + (m_{per} + \tilde{m})\ \tilde{a}). \tag{55}$$

The latter equation also has a divergence-free right-hand side by subtraction of (46) to (45). The matrix $\tilde{\mathcal{B}}$, which is unique up to the addition of a constant, is found upon solving

$$\begin{aligned} -\Delta \tilde{\mathcal{B}}_{ij} = {} & \partial_{jk}\left(\tilde{m} a_{ik}^{per} + (m_{per} + \tilde{m})\, \tilde{a}_{ik}\right) - \partial_{ik}\left(\tilde{m} a_{jk}^{per} + (m_{per} + \tilde{m})\, \tilde{a}_{jk}\right) \\ & + \partial_j\left(\tilde{m} b_i^{per} + (m_{per} + \tilde{m})\, \tilde{b}_i\right) - \partial_i\left(\tilde{m} b_j^{per} + (m_{per} + \tilde{m})\, \tilde{b}_j\right). \end{aligned} \tag{56}$$

Existence and uniqueness of the solution of this equation is proved using Calderón-Zygmund theory. The detailed argument may be found in [9] for the case $b = 0$, with the result that $\tilde{\mathcal{B}}^{b=0} \in L^{q'}(\mathbb{R}^d)$, with $\frac{1}{q'} = \min\left(\frac{1}{r} - \frac{1}{d}, \frac{1}{s} - \frac{1}{d}\right)$. In order to deal with b, since the equation is linear, we only need to solve (55) in the case $a = 0$. For this purpose, we simply use the following representation theorem:

$$\begin{aligned} \tilde{\mathcal{B}}_{ij}^{a=0} = {} & (d-2)\frac{x_j}{|x|^d} * \left(\tilde{m} b_i^{per} + (m_{per} + \tilde{m})\, \tilde{b}_i\right) \\ & - (d-2)\frac{x_i}{|x|^d} * \left(\tilde{m} b_j^{per} + (m_{per} + \tilde{m})\, \tilde{b}_j\right). \end{aligned}$$

Since $\frac{x_j}{|x|^d} \in L^{d/(d-1),\infty}(\mathbb{R}^d)$ and $\tilde{m} b_i^{per} + (m_{per} + \tilde{m})\, \tilde{b}_i \in L^{q'}(\mathbb{R}^d)$, with $\frac{1}{q'} = \min\left(\frac{1}{r} - \frac{1}{d}, \frac{1}{s} - \frac{1}{d}\right)$, the Young-O'Neil inequality for Lorentz spaces (15) implies that $\tilde{\mathcal{B}}^{a=0} \in L^{\alpha}(\mathbb{R}^d)$, with $\frac{1}{\alpha} = \frac{1}{(q')^*} = \min\left(\frac{1}{r} - \frac{2}{d}, \frac{1}{s} - \frac{2}{d}\right)$. Finally, $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}^{a=0} + \tilde{\mathcal{B}}^{b=0}$ satisfies

$$\tilde{\mathcal{B}} \in \left(L^{\alpha}(\mathbb{R}^d)\right)^d \quad \text{for} \quad \frac{1}{\alpha} = \min\left(\frac{1}{r} - \frac{2}{d}, \frac{1}{s} - \frac{2}{d}\right). \tag{57}$$

We end up with the corrector problem (53), where

$$\mathcal{A} = \underbrace{m_{per} a^{per} - \mathcal{B}^{per}}_{:=\mathcal{A}^{per}} + \underbrace{\tilde{m} a^{per} + (m_{per} + \tilde{m})\, \tilde{a} - \tilde{\mathcal{B}}.}_{:=\tilde{\mathcal{A}}}$$

The above considerations imply that $\tilde{\mathcal{A}} \in (L^{\alpha} \cap L^{\infty})(\mathbb{R}^d)$, with α defined by (57). Hence, applying Proposition 2.1 of [9], or Theorem 4.1 of [8], we therefore find that the solution w_p of (53) exists, is unique up to the addition of a constant, and reads $\nabla w_p = \nabla w_{p,per} + \nabla \tilde{w}_p$, where $w_{p,per}$ is the periodic corrector associated with $\mathcal{A}^{per}$, and

$$\nabla \tilde{w}_p \in L^{\alpha}(\mathbb{R}^d)^d. \tag{58}$$

Compared to Corollary 2.2, we have seemingly lost some decay at infinity, since there, we have $\nabla \tilde{w}_p \in L^{q^*}$, and $\frac{1}{\alpha} = \frac{1}{q^*} - \frac{1}{d}$, hence $\alpha > q^*$.

However, it is possible to recover the fact that $\nabla \tilde{w}_p \in L^{q^*}(\mathbb{R}^d)$ as follows: inserting $w_p = w_{p,per} + \tilde{w}_p$ into (53), and using the fact that $-\operatorname{div}\left(\mathcal{A}^{per}(\nabla w_{p,per} + p)\right) = 0$, we write the equation satisfied by $\tilde{w}_p$:

$$-\operatorname{div}\left((\mathcal{A}^{per} + \tilde{\mathcal{A}})\nabla \tilde{w}_p\right) = \operatorname{div}\left(\tilde{\mathcal{A}}(\nabla w_{p,per} + p)\right).$$

That is,

$$\begin{aligned}-\operatorname{div}\left((\mathcal{A}^{per} + \tilde{\mathcal{A}})\nabla \tilde{w}_p\right) &= \operatorname{div}\left[(\tilde{m}a^{per} + (m_{per} + \tilde{m})\,\tilde{a})(\nabla w_{p,per} + p)\right] \\ &\quad - \operatorname{div}\left[\tilde{\mathcal{B}}\,(\nabla w_{p,per} + p)\right] \end{aligned} \tag{59}$$

Actually, the right-hand side of (59) is exactly the right-hand side of (37) multiplied by m. Hence, (59) also reads

$$-\operatorname{div}\left((\mathcal{A}^{per} + \tilde{\mathcal{A}})\nabla \tilde{w}_p\right) = m\left(-\tilde{b}.p + \tilde{a}_{ij}\partial_{ij} w_{p,per} - \tilde{b} \cdot \nabla w_{p,per}\right). \tag{60}$$

Next, we solve the following equation:

$$-\Delta g = m\left(-\tilde{b}.p + \tilde{a}_{ij}\partial_{ij} w_{p,per} - \tilde{b} \cdot \nabla w_{p,per}\right) \in L^{\max(r,s)}(\mathbb{R}^d),$$

by defining

$$g = \frac{1}{|x|^{d-2}} * m\left(-\tilde{b}.p + \tilde{a}_{ij}\partial_{ij} w_{p,per} - \tilde{b} \cdot \nabla w_{p,per}\right).$$

Since $\nabla \frac{1}{|x|^{d-2}} \in L^{d/(d-1),\infty}(\mathbb{R}^d)$, the Young-O'Neil inequality (15) implies that

$$\nabla g \in L^{q'}(\mathbb{R}^d), \quad \frac{1}{q'} = \min\left(\frac{1}{r} - \frac{1}{d}, \frac{1}{s} - \frac{1}{d}\right).$$

Hence, (60) also reads

$$-\operatorname{div}\left((\mathcal{A}^{per} + \tilde{\mathcal{A}})\nabla \tilde{w}_p\right) = \operatorname{div}\left(-\nabla g\right), \quad \nabla g \in L^{q'}(\mathbb{R}^d).$$

Applying Proposition 2.1 of [9], we thus have $\nabla \tilde{w}_p \in L^{q'}(\mathbb{R}^d)^d$. Thus, we recover the result of Corollary 2.2.

Moreover, the fact that $\nabla \tilde{w} \in L^{q'}(\mathbb{R}^d)$ allows to apply the theory of [5, 19], in order to find approximation results for the homogenization of equation (1). We therefore find convergence theorems in $W^{1,p}$.

Let us mention that, as pointed out in [9], if G is the Green function associated to (7), and if $\mathcal{G}$ is the Green function associated to $-\operatorname{div}(\mathcal{A}\nabla \cdot)$, we have

$$\mathcal{G}(x,y) = m(y)G(x,y).$$

Therefore, all the estimates that are valid for the Green function G yield adequate estimates on $\mathcal{G}$, given the assumptions on a, b and the regularity that they imply on m.

Remark 9 (Again on the case of coefficients with some specific structure) *We return here to the specific case we have examined in Remark 8, that is $a_{ij}^{per} = \delta_{ij}$, $\tilde{a}_{ij} \equiv 0$, $b_i^{per} \equiv 0$, and $\tilde{b} = \nabla\tilde{\psi}$ for some function $\tilde{\psi} \in L^q(\mathbb{R}^d)$. We now look at the corrector functions. In this case, the corrector equation reads, for $p \in \mathbb{R}^d$, as $-\Delta w_p + \nabla\tilde{\psi} \,.\, \nabla w_p = -p \,.\, \nabla w_p$. On the one hand, we evidently have $w_{p,per} = 0$. On the other hand, multiplying the equation by the invariant measure $m = \exp(-\tilde{\psi})$ yields $-\operatorname{div}\Big(\exp(-\tilde{\psi})\,(p + \nabla\tilde{w}_p)\Big) = 0$. Using our results on the equations in divergence form, we conclude to the existence of a corrector $\tilde{w}_p$ with $\nabla\tilde{w}_p \in L^q(\mathbb{R}^d)$. Once again, we notice that $1 \leq q < +\infty$ is arbitrary.*

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