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Abstract. We consider the kinetic Fokker-Planck equation with weak confinement force. We proved some (polynomial and sub-exponential) rate of convergence to the equilibrium (depending on the space to which the initial datum belongs). Our results generalized the result in [4, 5, 21, 11, 10, 9, 1, 14] to weak confinement case.

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Keywords: weak hypocoercivity; weak hypodissipativity; Fokker-Planck equation; semigroup; weak Poincaré inequality; rate of convergence.

Contents

1. Introduction

In this paper, we consider the weak hypocoercivity issue for the kinetic Fokker-Planck (KFP for short) equation

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(vf), \tag{1.1}$$

for a density function $f = f(t, x, v)$, with $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$. The evolution equation is complemented with an initial datum

$$f(0, \cdot) = f_0 \quad on\ \mathbb{R}^{2d}.$$

We make the fundamental assumption on the confinement potential V

$$V(x) = \langle x \rangle^\gamma, \quad \gamma \in (0, 1),$$

where $\langle x \rangle^2 := 1 + |x|^2$.

Let us make some elementary but fundamental observations. First, the equation is mass conservative, that is

$$\mathcal{M}(f_0) = \mathcal{M}(f(t, \cdot)),$$

where we define the mass of f by

$$\mathcal{M}(f) = \int_{R^d \times R^d} f dx dv.$$

Next, we observe that

$$G = Z^{-1} e^{-W}, \quad W = \frac{v^2}{2} + V(x), \quad Z \in \mathbb{R}_+ \tag{1.2}$$

is a positive normalized steady state of the KFP model, precisely

$$LG = 0, \quad G > 0, \quad \mathcal{M}(G) = 1,$$

by choosing the normalizing constant $Z > 0$ appropriately. Finally we observe that, contrary to the case $\gamma \geq 1$, a Poincaré inequality of the type

$$\exists c > 0, \quad \int_{\mathbb{R}^d} |f(x)|^2 \exp(-V(x)) dx \leq c \int_{\mathbb{R}^d} |\nabla f(x)|^2 \exp(-V(x)) dx,$$

for any smooth function $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} f(x) \exp(-V(x)) dx = 0,$$

does not hold. Only a weaker version of this inequality remains true (see [20], or below Section 2). In particular, there is no spectral gap for the associated operator $\mathcal{L}$, nor is there an exponential trend to the equilibrium for the associated semigroup.

For a given weight function m, we will denote $L^p(m) = \{f | fm \in L^p\}$ the associated Lebesgue space and $\|f\|_{L^p(m)} = \|fm\|_{L^p}$ the associated norm.

The notation $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$.

With these notations, we can introduce the main result of this paper.

Theorem 1.1. *(1) For any initial datum $f_0 \in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p \in [1, \infty)$, $\epsilon > 0$ small, the associated solution $f(t, \cdot)$ of the kinetic Fokker-Planck equation (1.1) satisfies*

$$\|f(t, \cdot) - \mathcal{M}(f_0)G\|_{L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-Ct^b} \|f_0 - \mathcal{M}(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})},$$

for any $b \in (0, \frac{\gamma}{2-\gamma})$ *and some constant* $C > 0$.
(2) For any initial datum $f_0 \in L^1(m)$, $m = H^k$, $H = x^2 + v^2$, $1 \le k$, *the associated solution* $f(t,\cdot)$ *of the kinetic Fokker-Planck equation (1.1) satisfies*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1} \lesssim (1+t)^{-a}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)},$$

for any $0 < a < \frac{k}{1-\frac{\gamma}{2}}$. *The constants in the estimates do not depend on* f_0, *but rely on* $\gamma, d, \epsilon, \theta, p, k$.

Remark 1.2. Theorem 1.1 is also true when $V(x)$ behaves like $\langle x\rangle^\gamma$, that is for any $V(x)$ satisfying

$$C_1\langle x\rangle^\gamma \le V(x) \le C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \le x\cdot\nabla_x V(x) \le C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

and

$$|D_x^2 V(x)| \le C_5\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

for some constant $C_i > 0$, $R > 0$.

Remark 1.3. There are many classical results on the case $\gamma \ge 1$. In this case there is an exponentially decay, and we refer the interested readers to [21, 4, 5, 11, 9, 10, 1].

Remark 1.4. There are already some convergence results for the weak confinement case proved by probability method on some particular L^1 or L^2 spaces in [1] and [6], this paper extend the result to L^p spaces and more larger spaces.

Let us briefly explain the main ideas behind our method of proof.

We first introduce four spaces $E_1 = L^2(G^{-1/2})$, $E_2 = L^2(G^{-1/2}e^{\epsilon_1 V(x)})$, $E_3 = L^2(G^{-(1+\epsilon_2)/2})$ and $E_0 = L^2(G^{-1/2}\langle x\rangle^{\gamma-1})$, with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ small such that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset L^2$. Thus E_1 is an interpolation space between E_0 and E_2. We first use a hypocorecivity argument as in [4, 5] to prove that, for any $f_0 \in E_3$, the solution to the KFP equation (1.1) satisfies

$$\frac{d}{dt}\|f(t)\|_{E_1} \le -\lambda\|f(t)\|_{E_0},$$

for some constant $\lambda > 0$. We use this and the Duhamel formula to prove

$$\|f(t)\|_{E_2} \lesssim \|f_0\|_{E_3}.$$

Combining the two inequalities and using a interpolation argument as in [12], we get

$$\|f(t)\|_{E_1} \lesssim e^{-at^b}\|f_0\|_{E_3}, \tag{1.3}$$

for some $a > 0, b \in (0,1)$.

We then generalize the decay estimate to a wider class of Banach spaces by adapting the extension theory introduced in [18] and developed in [14, 8]. For any operator $\mathcal{L}$, denote $S_{\mathcal{L}}(t)$ the associated semigroup. We introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is an appropriately defined bounded operator so that $\mathcal{B}$ becomes a dissipative operator. By proving some regularization estimate in $S_{\mathcal{B}}$ in L^p

$$\|S_{\mathcal{B}}(t)\|_{L^p(m_1)\to L^2(m_2)} \lesssim t^{-\alpha}, \quad \forall t \in [0, \eta],$$

for some weight function m_1, m_2 and some $\alpha, \eta > 0$, and using the iterated Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \sum_{l=1}^{n-1} (S_{\mathcal{B}}) * (\mathcal{A}S_{\mathcal{B}})^{(*l)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}(t))^{*n}, \tag{1.4}$$

we deduce the L^p convergence on $S_{\mathcal{L}}$, where the convolution of two semigroups $S_{\mathcal{A}}(t)$ $S_{\mathcal{B}}(t)$is defined by

$$(S_{\mathcal{A}} * S_{\mathcal{B}})(t) = \int_0^t S_{\mathcal{A}}(s) S_{\mathcal{B}}(t-s) ds.$$

Let us end the introduction by describing the plan of the paper. In Section 2, we will develop a hypocoercivity argument to prove a weighted L^2 estimate for the KFP model. In section 3, we introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and using the L^2 estimate, we prove a L^2 convergence. In Section 4 we present the proof of a regularization estimate on $S_{\mathcal{B}}$ from L^2 to L^p. In Section 5 we prove some L^1 estimate on the semigroup $S_{\mathcal{B}}$. Finally in Section 6 we use the above regularization estimate to conclude the L^p convergence for KFP equation.

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2. L^2 framework: Dirichlet form and rate of convergence estimate

For later discussion, we introduce some notations for the whole paper. We split the KFP operator as

$$\mathcal{L} = \mathcal{T} + \mathcal{S},$$

where $\mathcal{T}$ stands for the transport part

$$\mathcal{T} f = -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f,$$

and $\mathcal{S}$ stands for the collision part

$$\mathcal{S} f = \Delta_v f + div_v(vf).$$

We will denote the cut-off function χ such that $\chi(x,v) \in [0,1]$, $\chi(x,v) \in C^\infty$, $\chi(x,v) = 1$ when $x^2+v^2 \leq 1$, $\chi(x,v) = 0$ when $x^2+v^2 \geq 2$, and then denote $\chi_R = \chi(x/R, v/R)$.

We may also define another splitting of the KFP operator $\mathcal{L}$ by

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} = K\chi_R(x,v). \tag{2.1}$$

with $K, R > 0$ to be chosen later.

We use $\int f$ in place of $\int_{\mathbb{R}^d \times R^d} f dxdv$ for short, similarly $\int f dx$ means $\int_{\mathbb{R}^d} f dx$, $\int f dv$ means $\int_{\mathbb{R}^d} f dv$. $B_{|x|\leq\rho}$ is used to denote the ball such that $\{x \in \mathbb{R}^d || x| \leq \rho\}$, similarly B_ρ means the ball such that $\{x, v \in \mathbb{R}^d || x|^2 + v^2 \leq \rho\}$.

For $V(x) = \langle x\rangle^\gamma, 0 < \gamma < 1$, we also denote $\langle \nabla V\rangle$ for $\langle x\rangle^{\gamma-1}$, and $\langle \nabla V\rangle^{-1}$ for $\langle x\rangle^{1-\gamma}$.

With these notations we introduce the Dirichlet form adapted to our problem. We define the 0 order and first order moments

$$\rho_f = \rho[f] = \int f dv, \quad j_f = j[f] = \int v f dv,$$

then we define a projection operator π by

$$\pi f = M\rho_f, \quad M = Ce^{-v^2/2}, \quad \int M dv = 1,$$

and the complement of π by

$$\pi^\perp = 1 - \pi, \quad f^\perp = \pi^\perp f.$$

We define an elliptic operator Δ_V and its dual Δ_V^* by

$$\Delta_V u := div_x(\nabla_x u + \nabla_x V u), \quad \Delta_V^* u = \Delta_x u - \nabla_x V \cdot \nabla_x u,$$

let $u = (\Delta_V^*)^{-1}\xi$ be the solution to the above elliptic equation

$$\Delta_V^* u = \xi \ on \ \mathbb{R}^d,$$

note that u can differ by a constant, we also requires that

$$\int u e^{-V} \langle \nabla V\rangle^{-2} dx = 0,$$

using these notations, define a scalar product by

$$\begin{aligned} ((f,g)) &:= (f,g)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j_f, (\rho_g e^V \langle \nabla V\rangle^2))_{L^2} \\ &\quad + \epsilon((\rho_f e^V \langle \nabla V\rangle^2), \Delta_V^{-1}\nabla_x j_g)_{L^2} \\ &= (f,g)_{\mathcal{H}} + \epsilon(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V\rangle^2))_{L^2} \\ &\quad + \epsilon((\nabla_x(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V\rangle^2), j_g)_{L^2}, \end{aligned}$$

for some $\epsilon > 0$ to be specified later.

We then define the Dirichlet form

$$\begin{aligned} D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], (\rho_f e^V \langle \nabla V\rangle^2))_{L^2} \\ &\quad + \epsilon((\rho[-\mathcal{L}f] e^V \langle \nabla V\rangle^2), \Delta_V^{-1}\nabla_x j_f)_{L^2}. \end{aligned}$$

Finally we define $\mathcal{H} = L^2(G^{-1/2})$, $\mathcal{H}_1 = L^2(G^{-1/2}\langle\nabla V\rangle)$ and

$$\mathcal{H}_0 = \{h \in \mathcal{H}, \int f dxdv = 0\}$$

where we recall that G has been introduced in (1.2). With these notations we can come to our first theorem.

Theorem 2.1. *There exists $\epsilon > 0$ small enough, such that on $\mathcal{H}_0$ the norm $((f,f))^{\frac{1}{2}}$ defined above is equivalent to the norm of $\mathcal{H}$, moreover there exist $\lambda > 0$, such that*

$$D[f] \geq \lambda \|f\|^2_{\mathcal{H}_1}, \quad \forall f \in \mathcal{H}_0.$$

As a consequence, for any $f_0 \in \mathcal{H}_0$, we have

$$\frac{d}{dt}((f,f)) \leq -C \int f^2 G^{-1} \langle x\rangle^{2(\gamma-1)}, \tag{2.2}$$

for some constant $C > 0$. In particular for any $f_0 \in \mathcal{H}_0$, we have

$$\|f(t,\cdot)\|_{L^2(G^{-\frac{1}{2}})} \leq C \|f_0\|_{L^2(G^{-\frac{1}{2}})}, \tag{2.3}$$

for some constant $C > 0$.

Remark 2.2. In $\mathcal{H}_0$ we have

$$\int \rho_f e^V \langle\nabla V\rangle^2 e^{-V} \langle\nabla V\rangle^{-2} dx = \int \rho_f dx = \int f dxdv = 0,$$

so the term $(\Delta_V^*)^{-1}(\rho_g e^V \langle\nabla V\rangle^2)$ is well defined in $\mathcal{H}_0$.

Remark 2.3. (1) By little modifying the method in Villani's paper [21], a H^1 version of our theorem can be established.
(2) Our statement is a generalization of [4, 5].

Before proving the theorem, we need some lemmas.

We say that W satisfies a local Poincaré inequality on a bounded open set Ω if there exist some constant $\kappa_\Omega > 0$ such that:

$$\int_\Omega h^2 W \leq k_\Omega \int_\Omega |\nabla h|^2 W + \frac{1}{W(\Omega)} (\int_\Omega hW)^2,$$

for any nice function $h : \mathbb{R}^d \to \mathbb{R}$ and where we denote $W(\Omega) := \langle W 1_\Omega\rangle$.

Lemma 2.4. *Under the assumption $W, W^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$, the function W satisfies the local Poincaré inequality for any ball $\Omega \in \mathbb{R}^d$.*

For the proof of Lemma 2.4 we refer to [19] Lemma 2.3.

Lemma 2.5. *(weak Poincaré inequality) There exist a constant $\lambda > 0$ such that*

$$\|u\|_{L^2(\langle\nabla V\rangle e^{-V/2})} \leq \lambda \|\nabla u\|_{L^2(e^{-V/2})}$$

for any $u \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} u e^{-V} \langle\nabla V\rangle^{-2} dx = 0$$

Proof of Lemma 2.5. We prove for any $h \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} h e^{-V} \langle \nabla V \rangle^{-2} = 0, \tag{2.4}$$

we have

$$\int_{\mathbb{R}^d} |\nabla h|^2 e^{-V} \geq \lambda \int_{\mathbb{R}^d} h^2 e^{-V} \langle x \rangle^{2(\gamma-1)},$$

for some $\lambda > 0$. Taking $g = he^{-1/2V}$, we have $\nabla g = \nabla h e^{-\frac{1}{2}V} - \frac{1}{2}\nabla V h e^{-\frac{1}{2}V}$, so that

$$\begin{aligned} 0 \leq \int |\nabla g|^2 &= \int |\nabla h|^2 e^{-V} + \int h^2 \frac{1}{4} |\nabla V|^2 e^{-V} - \int \frac{1}{2} \nabla (h^2) \cdot \nabla V e^{-V} \\ &= \int |\nabla h|^2 e^{-V} + \int h^2 (\frac{1}{2} \Delta V - \frac{1}{4} |\nabla V|^2) e^{-V}. \end{aligned}$$

We deduce for some $K, R_0 > 0$

$$\int |\nabla h|^2 e^{-V} \geq \int \frac{1}{8} h^2 \langle \nabla V \rangle^2 e^{-V} - K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2}.$$

Defining

$$\epsilon_R := \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6}, \quad Z_R := \int_{B_R} e^{-V} \langle \nabla V \rangle^{-2},$$

and using (2.4), we get

$$\begin{aligned} (\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2})^2 &= (\int_{B_R^c} h e^{-V} \langle \nabla V \rangle^{-2})^2 \\ &\leq \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2 \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6} \\ &\leq \epsilon_R \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Using the local Poincaré inequality in Lemma 2.4, we deduce

$$\begin{aligned} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^{-2} &\leq C_R \int_{B_R} |\nabla h|^2 e^{-V} \langle \nabla V \rangle^{-2} + \frac{1}{Z_R} (\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2})^2 \\ &\leq C_R' \int_{B_R} |\nabla h|^2 e^{-V} + \frac{\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Putting all the inequalities together and taking $R > R_0$, we finally get

$$\begin{aligned} \int h^2 e^{-V} \langle \nabla V \rangle^2 &\leq 8 \int |\nabla h|^2 e^{-V} + 8K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2} \\ &\leq 8(1 + KC_R') \int |\nabla h|^2 e^{-V} + \frac{8K\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2, \end{aligned}$$

and we conclude by taking R large such that: $\frac{8K\epsilon_R}{Z_R} \leq \frac{1}{2}$. □

Lemma 2.6. *(Elliptic Estimate) For any $\xi_1 \in L^2(\langle\nabla V\rangle^{-1}e^{-V/2})$ and $\xi_2 \in L^2(e^{-V/2})$, the solution $u \in L^2(e^{-V/2})$ to the elliptic equation*

$$-\Delta_V^* u = \xi_1 + \nabla\xi_2, \quad \int ue^{-V}\langle\nabla V\rangle^{-2}dx = 0, \tag{2.5}$$

satisfies

$$\|u\|_{L^2(\langle\nabla V\rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})} \lesssim \|\xi_1\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})} + \|\xi_2\|_{L^2(e^{-V/2})}. \tag{2.6}$$

Similarly for any $\xi \in L^2(e^{-V/2})$, the solution $u \in L(e^{-V/2})$ to the elliptic problem

$$-\Delta_V^* u = \xi, \quad \int ue^{-V}\langle\nabla V\rangle^{-2} = 0,$$

satisfies

$$\|u\|_{L^2(\langle\nabla V\rangle^2 e^{-V/2})} + \|\nabla u\|_{L^2(\langle\nabla V\rangle e^{-V/2})} + \|D^2 u\|_{L^2(e^{-V/2})} \lesssim \|\xi\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})}. \tag{2.7}$$

Proof of Lemma 2.6. Multiply (2.5) by ue^{-V} and observes that

$$e^V \mathrm{div}_x[e^{-V}\nabla_x u] = \Delta_x u - \nabla_x V\cdot\nabla_x u = \Delta_V^* u, \tag{2.8}$$

we have after integration

$$-\int e^V \mathrm{div}_x[e^{-V}\nabla_x u]ue^{-V} = \int(\xi_1 + \nabla\cdot\xi_2)ue^{-V}.$$

Performing one integration by parts, we deduce

$$\int e^{-V}|\nabla_x u|^2 = \int(\xi_1 u + \xi_2\cdot\nabla u - \xi_2\cdot\nabla V u)e^{-V},$$

using Lemma 2.5 we obtain (2.6). In inequality (2.7), the first two terms are easily bounded by (2.6) and $\langle\nabla V\rangle \le 1$, we then only need to prove the bound for the third term. By integration by parts, we have

$$\begin{aligned}
\int |D^2 u|^2 e^{-V} &= \sum_{i,j=1}^d \int (\partial_{ij}^2 u)^2 e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_i u(\partial_{ij}^2 u\partial_j V - \partial_{ijj}^3 u)e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_{jj} u\partial_i(\partial_i u e^{-V}) - \frac{1}{2}\int(\partial_i u)^2\partial_j(\partial_j V e^{-V}) \\
&= \int(\Delta u)(-\Delta_V^* u)e^{-V} + \int|\nabla u|^2(|\nabla V|^2 - \Delta V)e^{-V} \\
&\lesssim \|D^2 u\|_{L^2(e^{-V/2})}\|\xi\|_{L^2(e^{-V/2})} + \|\langle\nabla V\rangle\nabla u\|_{L^2(e^{-V/2})},
\end{aligned}$$

where in the third equality we have used

$$\begin{aligned}\int \partial^2_{ij}u\partial_i u\partial_j V e^{-V} &= -\int \partial_i u\partial_j(\partial_i u\partial_j V e^{-V})\\ &= -\int \partial^2_{ij}u\partial_i u\partial_j V e^{-V} - \int (\partial_i u)^2\partial_j(\partial_j V e^{-V}),\end{aligned}$$

which implies

$$\int \partial^2_{ij}u\partial_i u\partial_j V e^{-V} = -\frac{1}{2}\int (\partial_i u)^2\partial_j(\partial_j V e^{-V}),$$

and in the fourth equality we have used (2.8). That concludes the proof. □

Now we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. First we prove the equivalence of the norms associated to $((\ , \ ))$ and $(\ , \ )_{\mathcal{H}}$. By Cauchy-Schwarz inequality and Lemma 2.4, we have

$$(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V\langle\nabla V\rangle^2))_{L^2} \le \|j_f\|_{L^2(e^{V/2})}\|\rho_g e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})},$$

and obviously

$$\|\rho_g e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})} = \|\rho_g\|_{L^2(\langle\nabla V\rangle e^{V/2})} \le \|\rho_g\|_{L^2(e^{V/2})} \lesssim \|g\|_{\mathcal{H}}.$$

Using the elementary observations

$$|j_f| \lesssim \|f\|_{L^2(e^{v^2/4})} \quad |\rho_f| \lesssim \|f\|_{L^2(e^{v^2/4})},$$

we deduce

$$(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V\langle\nabla V\rangle^2))_{L^2} \lesssim \|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}},$$

The third term in the definition of $((\ , \ ))$ can be estimated in the same way and that ends the proof of equivalence of norms. □

Now we prove the main estimate of the theorem. We split the Dirichlet term $D[f]$ into 3 parts

$$D[f] = T_1 + \epsilon T_2 + \epsilon T_3,$$

with

$$\begin{aligned}T_1 &:= (\mathcal{L}f, f)_{\mathcal{H}}\\ T_2 &:= (\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle\nabla V\rangle)}\\ T_3 &:= ((\Delta_V)^{-1}\nabla_x j_f, \rho[-\mathcal{L}f])_{L^2(e^{V/2}\langle\nabla V\rangle)}\ ,\end{aligned}$$

and compute them separately.

For the T_1 term, using the classical Poincaré inequality, we have

$$\begin{aligned}T_1 &:= (-\mathcal{T}f + \mathcal{S}f, f)_{\mathcal{H}} = (-\mathcal{S}f, f)_{\mathcal{H}}\\ &= -\int [\Delta_v f + div_v(vf)]fM^{-1}e^V = \int |\nabla_v(f/M)|^2 M e^V\\ &\ge k_p\int |f/M - \rho_f|^2 M e^V = k_p\|f - \rho_f M\|^2_{\mathcal{H}} = k_p\|f^\perp\|^2_{\mathcal{H}},\end{aligned}$$

for some $k_p > 0$. We split the T_2 term as

$$\begin{aligned} T_2 &:= (\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle\nabla V\rangle)} \\ &= (\Delta_V^{-1}\nabla_x j[-\mathcal{T}\pi f], \rho_f)_{L^2(e^{V/2}\langle\nabla V\rangle)} \\ &\quad +(\Delta_V^{-1}\nabla_x j[-\mathcal{T}f^\perp], \rho_f)_{L^2(e^{V/2}\langle\nabla V\rangle)} \\ &\quad +(\Delta_V^{-1}\nabla_x j[-\mathcal{S}f], \rho_f)_{L^2(e^{V/2}\langle\nabla V\rangle)} \\ &:= T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned}$$

First observe

$$\mathcal{T}\pi f = -v\cdot\nabla_x\rho_f M - \nabla_x V\cdot v\rho_f M = -e^{-V}Mv\cdot\nabla_x(\rho_f/e^{-V}),$$

so that we have

$$j[-\mathcal{T}\pi f] = \langle vv_k M\rangle e^{-V}\partial_{x_k}(\rho_f/e^{-V}) = e^{-V}\nabla_x(\rho_f/e^{-V}).$$

Next by (2.8), we have

$$\begin{aligned} T_{2,1} &= (j[-\mathcal{T}\pi f], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &= (\rho_f, [e^V div_x(e^{-V}\nabla)][(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)])_{L^2} \\ &= \|\rho_f e^{V/2}\langle\nabla V\rangle\|_{L^2}^2 = \|\pi f\|_{\mathcal{H}_1}^2. \end{aligned}$$

Using the notation $\eta_1 = \langle v\otimes vf^\perp\rangle$ and $\eta_{2,\alpha\beta} = \langle v_\alpha\partial_{v_\beta}f^\perp\rangle$, and observing that

$$|\eta_1| \lesssim \|f^\perp\|_{L^2(e^{v^2/4})}, |\eta_2| \lesssim \|f^\perp\|_{L^2(e^{v^2/4})},$$

we compute

$$\begin{aligned} T_{2,2} &= (j[-\mathcal{T}f^\perp], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &= (D\eta_1 + \eta_2\nabla V, \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &= (\eta_1, D^2(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} + (\eta_2, \nabla V\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &= \|\eta_1\|_{L^2(e^{V/2})}\|D^2(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})} \\ &\quad +\|\eta_2\|_{L^2(e^{V/2})}\|\nabla V\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})}. \end{aligned}$$

By Lemma 2.6, we estimate

$$\begin{aligned} T_{2,2} &\lesssim \|\eta_1\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} \\ &\quad +\|\eta_2\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

Using

$$\begin{aligned} j[-\mathcal{S}f] = j[-\mathcal{S}f^\perp] &= -\int v[\Delta_v f^\perp + div_v(vf^\perp)]dv \\ &= d\int f^\perp v dv \lesssim \|f^\perp\|_{L^2(e^{v^2/4})}, \end{aligned}$$

and Lemma 2.6, we have

$$\begin{aligned} T_{2,3} &= (j[-Sf], \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V\rangle^2))_{L^2} \\ &\leq \|j[-Sf]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V\rangle^2)\|_{L^2(e^{-V/2})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|\rho_f e^V \langle \nabla V\rangle^2\|_{L^2(\langle \nabla V\rangle^{-1} e^{-V/2})} \\ &= \|f^\perp\|_{\mathcal{H}} \|\rho_f\|_{L^2(\langle \nabla V\rangle e^{V/2})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

Finally we come to the T_3 term. Using

$$\rho[-Sf] = \int \nabla_v \cdot (\nabla_v f + vf) dv = 0,$$

and

$$\begin{aligned} \rho[-Tf] &= \rho[v\nabla_x f - \nabla_x V(x) \nabla_v f] \\ &= \int v\nabla_x f - \nabla_x V(x) \nabla_v f dv \\ &= \nabla_x j[f], \end{aligned}$$

because $\nabla(\langle \nabla V\rangle^2) \lesssim \langle \nabla V\rangle^2$ and $\langle \nabla V\rangle^2 \lesssim \langle \nabla V\rangle$, we get

$$\begin{aligned} T_3 &= ((\Delta_V)^{-1} \nabla_x j_f, \rho[-\mathcal{L} f])_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &= ((\Delta_V)^{-1} \nabla_x j[f^\perp], \rho[-\mathcal{T} f])_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &= (j[-f^\perp], \nabla(\Delta_V^*)^{-1}(\nabla_x j[f] e^V \langle \nabla V\rangle^2)_{L^2} \\ &= \|j[f^\perp]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}[\nabla_x (j_f e^V \langle \nabla V\rangle^2) \\ &\quad -\nabla V j_f e^V \langle \nabla V\rangle^2 - \nabla(\langle \nabla V\rangle^2) j_f e^V]\|_{L^2(e^{-V/2})}, \end{aligned}$$

using again Lemma 2.6, we have

$$\begin{aligned} T_3 &\lesssim \|j[f^\perp]\|_{L^2(e^{V/2})} (\|j_f e^V \langle \nabla V\rangle^2\|_{L^2(e^{-V/2}\langle \nabla V\rangle^{-1})} \\ &\quad + \|j_f e^V \nabla(\langle \nabla V\rangle^2)\|_{L^2(\langle \nabla V\rangle^{-1} e^{-V/2})}) \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1}. \end{aligned}$$

Putting all the terms together and choosing $\epsilon > 0$ small enough, we can deduce

$$\begin{aligned} D[f] &\geq k_p \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1} - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1} \\ &\geq k_p \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - (2\epsilon + 4\epsilon^{1/2}) K \|f^\perp\|_{\mathcal{H}}^2 - \epsilon^{3/2} 4K \|\pi f\|_{\mathcal{H}_1}^2 \\ &\geq \frac{k_p}{2} (\|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2) \geq \frac{\epsilon}{M} \|f\|_{\mathcal{H}_1}, \end{aligned}$$

for some $M > 0$. □

3. L^2 sub-exponential decay for the kinetic Fokker-Planck equation based on a splitting trick

In this section we establish a first decay estimate on $S_{\mathcal{L}}$ which is a particular case in the result of Theorem 1.1.

Theorem 3.1. *Using the notation and results in Theorem 2.1, we have*

$$\|S_{\mathcal{L}}(t)f_0\|_{L^2(G^{-\frac{1}{2}})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}}\|f_0\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})},$$

for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)}) \cap \mathcal{H}_0$, $\epsilon > 0$ small enough.

Remark 3.2. It's worth emphasizing that we deduce immediately part (1) of Theorem 1.1 in the case $p = 2$ by considering the initial datum $f_0 - \mathcal{M}(f_0)$ for any $f_0 \in L^2(G^{-\frac{1}{2}+\epsilon})$.

Recall the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ introduced in (2.1), we first prove some decay estimate on the semigroup $S_{\mathcal{B}}$.

Lemma 3.3. *Let us fix $p \in [1, \infty)$.*

(1) For any given smooth weight function m, we have

$$\int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m = \frac{1}{p}\int f^p G^{-(p-1)}\tilde{m}, \tag{3.1}$$

with

$$\tilde{m} = \Delta_v m - \nabla_v m \cdot v - \nabla V(x) \cdot \nabla_v m + v \cdot \nabla_x m.$$

(2) Taking $m = e^{\epsilon H^\delta}$, $\epsilon > 0$ if $0 < \delta < \frac{\gamma}{2}$, ϵ small enough if $\delta = \frac{\gamma}{2}$, $H = 3v^2 + 2x \cdot v + x^2 + 1$, we have

$$\int f^{p-1}(\mathcal{B}f)G^{-(p-1)}e^{\epsilon H^\delta} \leq -C\int f^p G^{-(p-1)}e^{\epsilon H^\delta}H^{\frac{\delta}{2}+\gamma-1}, \tag{3.2}$$

for some K and R large.

(3) With the same notation as above, there holds

$$\|S_{\mathcal{B}}(t)\|_{L^p(e^{2\epsilon H^\delta}G^{-\frac{p-1}{p}})\to L^p(e^{\epsilon H^\delta}G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}}, \tag{3.3}$$

for some $a > 0$. In particular, this implies

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})\to L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Proof of Lemma 3.3. Step 1. Recall (1.2), we write

$$\int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m = \int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m + \int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m.$$

We first compute the contribution of the term with operator $\mathcal{T}$

$$\begin{aligned}\int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m &= \frac{1}{p}\int \mathcal{T}(f^p)G^{-(p-1)}m \\ &= -\frac{1}{p}\int f^p\mathcal{T}(G^{-(p-1)}m) \\ &= \frac{1}{p}\int f^pG^{-(p-1)}(v\cdot\nabla_x m - \nabla V(x)\cdot\nabla_v m).\end{aligned}$$

For the the term with operator $\mathcal{S}$, we use one integration by parts, and we get

$$\begin{aligned}&\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ =\ &\int f^{p-1}(\Delta_v f + \mathrm{div}_v(vf))G^{-(p-1)}m \\ =\ &-\int \nabla_v((fG^{-1})^{p-1}m)\cdot\nabla_v(fG^{-1})G \\ =\ &-\int (p-1)|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm - \frac{1}{p}\nabla_v((fG^{-1})^p)\cdot(\nabla_v m)G.\end{aligned}$$

Performing another integration by parts on the latter term, we have

$$\begin{aligned}&\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ =\ &\int -(p-1)|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm + \frac{1}{p}\nabla_v\cdot(G\nabla_v m)(fG^{-1})^p \\ =\ &\int -(p-1)|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm + \frac{1}{p}(\Delta_v m - v\cdot\nabla_v m)f^pG^{-(p-1)}.\end{aligned}$$

Identity (3.1) follows by putting together the two identities.
Step 2. We particular use $m = e^{\epsilon H^\delta}$ and we easily compute

$$\frac{\nabla_v m}{m} = \delta\epsilon\frac{\nabla_v H}{H^{1-\delta}}, \quad \frac{\nabla_x m}{m} = \delta\epsilon\frac{\nabla_x H}{H^{1-\delta}},$$

and

$$\frac{\Delta_v m}{m} \le \delta\epsilon\frac{\Delta_v H}{H^{1-\delta}} + (\delta\epsilon)^2\frac{|\nabla_v H|^2}{H^{2(1-\delta)}}.$$

We deduce that $\phi = \frac{\tilde{m}}{m}$ satisfies

$$\frac{\phi H^{1-\delta}}{\epsilon\delta} \le \Delta_v H + \epsilon\delta\frac{|\nabla_v H|^2}{H^{1-\delta}} - v\cdot\nabla_v H + v\cdot\nabla_x H - \nabla_x V(x)\cdot\nabla_v H.$$

From the very definition of H, we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6.$$

Choosing $\epsilon > 0$ arbitrary if $0 < 2\delta < \gamma$, ϵ small enough if $2\delta = \gamma$,we deduce

$$\begin{aligned}
&\Delta_v H + 2\epsilon\delta\frac{|\nabla_v H|^2}{H^{1-\delta}} + v\cdot\nabla_x H - v\cdot\nabla_v H - \nabla_x V(x)\cdot\nabla_v H \\
= \;& 6 + \epsilon\delta\frac{(6v+2x)^2}{H^{1-\delta}} + 2v^2 + 2x\cdot v - 6v^2 - 2x\cdot v - 6v\cdot\nabla_x V(x) - 2x\cdot\nabla_x V(x) \\
\leq \;& (2v^2 + C_1 v + C_2 v^{2\delta} - 6v^2) + (C_3\epsilon\delta x^{2\delta} - 2x\cdot\nabla_x V(x)) + C \\
\leq \;& -C_4 v^2 - C_5 x\cdot\nabla_x V(x) + C_6 \\
\leq \;& -C_7 H^{\frac{\gamma}{2}} + K\chi_R,
\end{aligned}$$

for some constants $C_i, K, R > 0$. As a consequence, we have proved

$$\phi - K\chi_R \leq \frac{-C}{H^{1-\delta-\frac{\gamma}{2}}} \leq 0,$$

which is nothing but (3.2).
Step 3. In the following, denote $f_t = S_{\mathcal{B}}(t)f_0$ the solution to the evolution equation $\partial_t f = \mathcal{B}f, f(0) = f_0$. On the one hand, by (3.2) we have

$$\frac{d}{dt}\int f_t^p G^{-(p-1)} e^{2\epsilon H^\delta} = \int f_t^{p-1}(\mathcal{B}f_t) G^{-(p-1)} e^{2\epsilon H^\delta} \leq 0,$$

which implies

$$\int f_t^p G^{-(p-1)} e^{2\epsilon H^\delta} \leq \int f_0^p G^{-(p-1)} e^{2\epsilon H^\delta} := Y_1, \quad \forall t \geq 0$$

On the other hand, defining

$$Y := \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta},$$

using again (3.2), we have

$$\begin{aligned}
\frac{d}{dt}Y &= p\int f_t^{p-1}\mathcal{B}f_t G^{-(p-1)} e^{\epsilon H^\delta} \\
&\leq -a\int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} H^{\delta+\frac{\gamma}{2}-1} \\
&\leq -a\int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x\rangle^{2\delta+\gamma-2} \\
&\leq -a\int_{B_{|x|\leq\rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x\rangle^{2\delta+\gamma-2},
\end{aligned}$$

for any $\rho > 0$ and for some $a > 0$. As $2\delta + \gamma < 2$, $0 \leq |x| \leq \rho$ implies $\langle x\rangle^{2\delta+\gamma-2} \geq \langle\rho\rangle^{2\delta+\gamma-2}$, we deduce

$$\begin{aligned}
\frac{d}{dt}Y &\leq -a\langle\rho\rangle^{2\delta+\gamma-2}\int_{B_{|x|\leq\rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \\
&\leq -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2}\int_{B_{|x|\geq\rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta},
\end{aligned}$$

Using that $e^{\epsilon\langle x\rangle^{2\delta}} \geq e^{\epsilon\langle\rho\rangle^{2\delta}}$ on $|x| \geq \rho$, we get

$$\begin{aligned}\frac{d}{dt}Y &\leq -a\langle\rho\rangle^{2\delta+\gamma-2}Y + a\langle\rho\rangle^{2\delta+\gamma-2}e^{-\epsilon\langle\rho\rangle^{2\delta}}\int_{B_{|x|\geq\rho}} f_t^p G^{-(p-1)}e^{\epsilon H^\delta}e^{\epsilon\langle x\rangle^{2\delta}}\\ &\leq -a\langle\rho\rangle^{2\delta+\gamma-2}Y + a\langle\rho\rangle^{2\delta+\gamma-2}e^{-\epsilon\langle\rho\rangle^{2\delta}}\int f_t^p G^{-(p-1)}e^{\epsilon H^\delta}e^{\epsilon\langle x\rangle^{2\delta}}\\ &\leq -a\langle\rho\rangle^{2\delta+\gamma-2}Y + a\langle\rho\rangle^{2\delta+\gamma-2}e^{-\epsilon\langle\rho\rangle^{2\delta}}CY_1.\end{aligned}$$

Thanks to Grönwall's Lemma, we obtain

$$\begin{aligned}Y(t) &\leq e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t}Y(0) + Ce^{-\epsilon\langle\rho\rangle^{2\delta}}Y_1\\ &\lesssim (e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t} + e^{-\epsilon\langle\rho\rangle^{2\delta}})Y_1,\end{aligned}$$

Choosing finally ρ such that $a\langle\rho\rangle^{2\delta+\gamma-2}t = \epsilon\langle\rho\rangle^{2\delta}$, that is $\langle\rho\rangle^{2-\gamma} = Ct$, we deduce

$$Y(t) \leq C_1 e^{-C_2 t^{\frac{2\delta}{2-\gamma}}}Y_2,$$

for some $C_i > 0$, and we deduce the proof of (3.3). □

Now we come to prove Theorem 3.1.

Proof of Theorem 3.1. We recall that from (2.3), we have

$$\|S_{\mathcal{L}}(t)\|_{L^2(G^{-1/2})\to L^2(G^{-1/2})} \lesssim 1, \quad \forall t \geq 0$$

From the very definition of $\mathcal{A}$ we have

$$\|\mathcal{A}\|_{L^2(G^{-1/2})\to L^2(e^{2\epsilon H^\delta}G^{-1/2})} \lesssim 1.$$

From Lemma 3.3 case $p = 2$, we have

$$\|S_{\mathcal{B}}(t)\|_{L^2(e^{2\epsilon H^\delta}G^{-1/2})\to L^2(e^{\epsilon H^\delta}G^{-1/2})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}}, \quad \forall t \geq 0.$$

Gathering the three estimates and using Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}}$$

we deduce

$$\|S_{\mathcal{L}}(t)\|_{L^2(e^{2\epsilon H^\delta}G^{-1/2})\to L^2(e^{\epsilon H^\delta}G^{-1/2})} \lesssim 1, \quad \forall t \geq 0.$$

In the following, we denote $f_t = S_{\mathcal{L}}(t)f_0$ the solution to the evolution equation $\partial_t f = \mathcal{L}f, f(0,\cdot) = f_0$. Taking $2\delta = \gamma$, ϵ small enough, we have in particular

$$\int f_t^2 G^{-1}e^{\epsilon H^{\frac{\gamma}{2}}} \leq C\int f_0^2 G^{-1}e^{2\epsilon H^{\frac{\gamma}{2}}} =: Y_3.$$

We define

$$Y_2(t) := ((f,f)),$$

with $((,))$ is defined in Theorem 2.1. Thanks to the result in (2.2), we have

$$\begin{aligned}\frac{d}{dt}Y_2 &\leq -a\int f_t^2 G^{-1}\langle x\rangle^{2(\gamma-1)}\\ &\leq -a\int_{B_{|x|\leq\rho}} f_t^2 G^{-1}\langle x\rangle^{2(\gamma-1)},\end{aligned}$$

for any $\rho \geq 0$, using the same argument as Lemma 3.3, we deduce

$$\begin{aligned} Y_2(t) &\leq Ce^{-a\langle\rho\rangle^{2(\gamma-1)}t}Y_2(0) + Ce^{-\epsilon_2\langle\rho\rangle^{\gamma}}Y_3 \\ &\lesssim (e^{-a\langle\rho\rangle^{2(\gamma-1)}t} + e^{-\epsilon_2\langle\rho\rangle^{\gamma}})Y_3. \end{aligned}$$

Choosing ρ such that $a\langle\rho\rangle^{2(\gamma-1)}t = \epsilon_2\langle\rho\rangle^{\gamma}$, that is $\langle\rho\rangle^{2-\gamma} = Ct$, we conclude

$$Y_2(t) \leq C_1 e^{-C_2 t^{\gamma/(2-\gamma)}}Y_3,$$

for some constants $C_i > 0$. As $H^{\frac{\gamma}{2}} \lesssim C(\frac{v^2}{2} + V(x))$, we have

$$e^{\epsilon H^{\frac{\gamma}{2}}} \leq G^{-C\epsilon},$$

Taking ϵ small, the proof of Theorem 3.1 is done. □

4. Regularization property of $S_{\mathcal{B}}$

In this section we will denote $\mathcal{L}^* = \mathcal{L}^*_{G^{-1/2}} = \mathcal{S} - \mathcal{T}$ be the dual operator of $\mathcal{L}$ on $L^2(G^{-1/2})$. In other words, L^* is defined by the identity

$$\int (\mathcal{L}f)gG^{-1} = \int (\mathcal{L}^*g)fG^{-1}.$$

for any smooth function f, g. We also denote $\mathcal{B}^* = \mathcal{L}^* - K\chi_R$. The aim of this section is to establish the following regularization property. The proof closely follows the proof of similar results in [10, 14, 21]

Theorem 4.1. *For any $0 \leq \delta < 1$, there exist $\eta > 0$ such that*

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}}\|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

Similarly, for any $0 \leq \delta < 1$, there exist $\eta > 0$ such that

$$\|\mathcal{S}_{\mathcal{B}^*}(t)f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}}\|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

As a consequence, for any $0 \leq \delta < 1$, there exist $\eta > 0$ such that

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^\infty(G^{-1/2})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}}\|f\|_{L^2(G^{-1/2})}, \quad \forall t \in [0, \eta].$$

We start with some elementary lemmas.

Lemma 4.2. *For any $0 \leq \delta < 1$, we have*

$$\begin{aligned} \int (f(\mathcal{L}g) + g(\mathcal{L}f))G^{-(1+\delta)} &= -2\int \nabla_v(fG^{-1}) \cdot \nabla_v(gG^{-1})G^{1-\delta} \\ &+ \int (\delta d - \delta(1-\delta)v^2)fgG^{-(1+\delta)} \end{aligned} \tag{4.1}$$

in particular, this implies

$$\begin{aligned}\int f(\mathcal{L}f)G^{-(1+\delta)} &= -\int |\nabla_v(fG^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2}\int f^2 G^{-(1+\delta)} \\ &- \frac{\delta(1-\delta)}{2}\int v^2 f^2 G^{-(1+\delta)}, \end{aligned} \tag{4.2}$$

similarly, for any $0 \le \delta < 1$, *we have*

$$\begin{aligned}\int f(\mathcal{L}f)G^{-(1+\delta)} &= -\int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2}\int v^2 f^2 G^{-(1+\delta)} \\ &+ \frac{(2+\delta)d}{2}\int f^2 G^{-(1+\delta)}. \end{aligned} \tag{4.3}$$

All the equalities remain true when $\mathcal{L}$ *is replaced by* $\mathcal{L}^*$.

Proof of Lemma 4.2. Recall $\mathcal{T}(G^{-(1+\delta)}) = 0$, we have

$$\int f(\mathcal{T}g)G^{-(1+\delta)} = \int \mathcal{T}(fG^{-(1+\delta)})g = -\int (\mathcal{T}f)gG^{-(1+\delta)},$$

which implies

$$\int f(\mathcal{T}g)G^{-(1+\delta)} + \int (\mathcal{T}f)gG^{-(1+\delta)} = 0.$$

for the term with operator $\mathcal{S}$ we have

$$\begin{aligned}\int f(\mathcal{S}g)G^{-(1+\delta)} &= -\int \nabla_v(fG^{-(1+\delta)})\cdot(\nabla_v g + vg) \\ &= -\int (\nabla_v f + (1+\delta)vf)\cdot(\nabla_v g + vg)G^{-(1+\delta)} \\ &= -\int \nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta} \\ &\quad -\int (\delta v^2 fg + \delta f v\cdot\nabla_v g)G^{-(1+\delta)},\end{aligned}$$

using integration by parts

$$\begin{aligned}\int \delta f v\cdot\nabla_v g G^{-(1+\delta)} &= -\int \delta g \nabla_v\cdot(vfG^{-(1+\delta)}) \\ &= -\int \delta g v\cdot\nabla_v f G^{-(1+\delta)} \\ &\quad -\int (\delta d + \delta(1+\delta)v^2) fg G^{-(1+\delta)},\end{aligned}$$

so we deduce

$$\begin{aligned}&\int (f(\mathcal{S}g) + g(\mathcal{S}f))G^{-(1+\delta)} \\ =\ & -2\int \nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta} + \int (\delta d - \delta(1-\delta)v^2) fg G^{-(1+\delta)},\end{aligned}$$

so (4.1) and (4.2) are thus proved by combining the two terms above. Finally, we compute

$$\begin{aligned}
&\int f\mathcal{S}fG^{-(1+\delta)}\\
=\ &-\int(\nabla_v f+(1+\delta)vf)\cdot(\nabla_v f+vf)G^{-(1+\delta)}\\
=\ &-\int|\nabla_v f|^2G^{-(1+\delta)}-\int(1+\delta)v^2f^2G^{-(1+\delta)}-\int(2+\delta)fv\cdot\nabla_v fG^{-(1+\delta)}\\
=\ &-\int|\nabla_v f|^2G^{-(1+\delta)}-\int(1+\delta)v^2f^2G^{-(1+\delta)}+\frac{2+\delta}{2}\int\nabla_v\cdot(vG^{-(1+\delta)})f^2\\
=\ &-\int|\nabla_v f|^2G^{-(1+\delta)}+\frac{\delta(1+\delta)}{2}\int v^2f^2G^{-(1+\delta)}+\frac{(2+\delta)d}{2}\int f^2G^{-(1+\delta)},
\end{aligned}$$

so (4.3) follows by putting together the above equality with

$$\int f\mathcal{T}fG^{-(1+\delta)}=0.$$

Since the term associated with $\mathcal{T}$ is 0, by $\mathcal{L}=\mathcal{S}+\mathcal{T},\mathcal{L}^*=\mathcal{S}-\mathcal{T}$, we know the same equalities will remain true when $\mathcal{L}$ is replaced by $\mathcal{L}^*$. □

Lemma 4.3. *When $f_t=S_{\mathcal{B}}(t)f_0$, define an energy functional*

$$\begin{aligned}
\mathcal{F}(t,f_t)\ &:=\ A\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})}+at^2\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta))})}\\
&+\ 2ct^4(\nabla_v f_t,\nabla_x f_t)^2_{L^2(G^{-1/2(1+\delta)})}+bt^6\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta))})},
\end{aligned}\tag{4.4}$$

when $f_t=S_{\mathcal{B}^*}(t)f_0$, define another energy functional

$$\begin{aligned}
\mathcal{F}^*(t,f_t)\ &:=\ A\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})}+at^2\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta))})}\\
&-\ 2ct^4(\nabla_v f_t,\nabla_x f_t)^2_{L^2(G^{-1/2(1+\delta)})}+bt^6\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta))})},
\end{aligned}\tag{4.5}$$

with $a,b,c>0, c\le\sqrt{ab}$ and A large enough. Then for both cases we have

$$\frac{d}{dt}F(t,f_t)\le -L(\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})}+t^4\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})})+\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})},$$

for all $t\in[0,\eta]$, for some $L>0,C>0$ and $F=\mathcal{F}$ or $\mathcal{F}^*$.

Proof of Lemma 4.3. We only prove the case $F=\mathcal{F}$, the proof for $F=\mathcal{F}^*$ is the same. We split the computation into several parts and then put them

together. First using (4.2) and (4.3) we have

$$\begin{aligned}
&\frac{d}{dt}\int f^2 G^{-(1+\delta)} \\
=\ & \int f(\mathcal{L}-K\chi_R) f G^{-(1+\delta)} \\
=\ & \frac{1-\delta}{2}\int f\mathcal{L}f G^{-(1+\delta)} + \frac{1+\delta}{2}\int f\mathcal{L}f G^{-(1+\delta)} - \int K\chi_R f^2 G^{-(1+\delta)} \\
\le\ & -\frac{1-\delta}{2}\int |\nabla_v f|^2 G^{-(1+\delta)} - \frac{1+\delta}{2}\int |\nabla_v (fG^{-1})|^2 G^{1-\delta} + C\int f^2 G^{-(1+\delta)} \\
\le\ & -\frac{1-\delta}{2}\int |\nabla_v f|^2 G^{-(1+\delta)} + C\int f^2 G^{-(1+\delta)}.
\end{aligned}$$

By

$$\partial_{x_i}\mathcal{L}f = \mathcal{L}\partial_{x_i} f + \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f, \tag{4.6}$$

and (4.2) we have

$$\begin{aligned}
&\frac{d}{dt}\int (\partial_{x_i} f)^2 G^{-(1+\delta)} \\
=\ & \int \partial_{x_i} f \partial_{x_i}(\mathcal{L}-K\chi_R) f G^{-(1+\delta)} \\
=\ & -\int |\nabla_v(\partial_{x_i} f G^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2}\int (\partial_{x_i} f)^2 G^{-(1+\delta)} \\
& -\frac{\delta(1-\delta)}{2}\int v^2 (\partial_{x_i} f)^2 G^{-(1+\delta)} + \int \partial_{x_i} f \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f G^{-(1+\delta)} \\
& -\int K\chi_R |\partial_{x_i} f|^2 G^{-(1+\delta)} - \int K \partial_{x_i} f \partial_{x_i}\chi_R f G^{-(1+\delta)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing up by i, we get

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla_x f|^2 G^{-(1+\delta)} \\
\le\ & -\sum_{i=1}^{d}\int |\nabla_v(\partial_{x_i} f G^{-1})|^2 G^{1-\delta} - \frac{\delta(1-\delta)}{2}\int v^2 (\nabla_x f)^2 G^{-(1+\delta)} \\
& + C\int |\nabla_v f|^2 G^{-(1+\delta)} + C\int |\nabla_x f|^2 G^{-(1+\delta)} + C\int |f|^2 G^{-(1+\delta)},
\end{aligned}$$

for some $C > 0$. Similarly using

$$\partial_{v_i}\mathcal{L}f = \mathcal{L}\partial_{v_i} f - \partial_{x_i} f + \partial_{v_i} f, \tag{4.7}$$

and (4.2), we have

$$\begin{aligned}
&\frac{d}{dt}\int(\partial_{v_i}f)^2G^{-(1+\delta)}\\
=\ &\int\partial_{v_i}f\partial_{v_i}(\mathcal{L}-K\chi_R)fG^{-(1+\delta)}\\
=\ &-\int|\nabla_v(\partial_{v_i}fG^{-1})|^2G^{1-\delta}+\frac{\delta d}{2}\int(\partial_{v_i}f)^2G^{-(1+\delta)}\\
&-\frac{\delta(1-\delta)}{2}\int v^2(\partial_{v_i}f)^2G^{-(1+\delta)}-\int\partial_{x_i}f\partial_{v_i}fG^{-(1+\delta)}\\
&+\int|\partial_{v_i}f|^2G^{-(1+\delta)}-\int K\chi_R|\partial_{v_i}f|^2G^{-(1+\delta)}-\int K\partial_{v_i}f\partial_{v_i}\chi_RfG^{-(1+\delta)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing up by i we get

$$\begin{aligned}
&\frac{d}{dt}\int|\nabla_vf|^2G^{-(1+\delta)}\\
\leq\ &-\sum_{i=1}^{d}\int|\nabla_v(\partial_{v_i}fG^{-1})|^2G^{1-\delta}+C\int|\nabla_xf||\nabla_vf|G^{-(1+\delta)}\\
&+C\int|\nabla_vf|^2G^{-(1+\delta)}+C\int|f|^2G^{-(1+\delta)}-\frac{\delta(1-\delta)}{2}\int v^2(\nabla_vf)^2G^{-(1+\delta)}.
\end{aligned}$$

For the crossing term, we split it also into two parts

$$\begin{aligned}
&\frac{d}{dt}\int2\partial_{v_i}f\partial_{x_i}fG^{-(1+\delta)}\\
=\ &(\int\partial_{v_i}f\partial_{x_i}\mathcal{L}fG^{-(1+\delta)}+\int\partial_{v_i}\mathcal{L}f\partial_{x_i}fG^{-(1+\delta)})\\
&-(\int\partial_{v_i}f\partial_{x_i}(K\chi_Rf)G^{-(1+\delta)}+\int\partial_{x_i}(K\chi_Rf)\partial_{v_i}fG^{-(1+\delta)})\\
:=\ &W_1+W_2.
\end{aligned}$$

Using (4.6) and (4.7) we have

$$\begin{aligned}
W_1\ =\ &\int\partial_{v_i}f\mathcal{L}(\partial_{x_i}f)G^{-(1+\delta)}+\int\mathcal{L}(\partial_{v_i}f)\partial_{x_i}fG^{-(1+\delta)}\\
&+\int\partial_{v_i}f\sum_{j=1}^{d}\partial_{x_ix_j}V(x)\partial_{v_j}fG^{-(1+\delta)}-\int|\partial_{x_i}f|^2G^{-(1+\delta)}\\
&+\int\partial_{x_i}f\partial_{v_i}fG^{-(1+\delta)}.
\end{aligned}$$

By (4.1), we deduce

$$\begin{aligned} W_1 &= -\int 2\nabla_v(\partial_{v_i} f G^{-1})\cdot\nabla_v(\partial_{x_i} f G^{-1})G^{1-\delta} + \delta d\int \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} \\ &\quad -\delta(1-\delta)\int v^2 \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} + \int \partial_{v_i} f \sum_{j=1}^{d} \partial_{x_i x_j} V(x) \partial_{v_j} f G^{-(1+\delta)} \\ &\quad -\int |\partial_{x_i} f|^2 G^{-(1+\delta)} + \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)}. \end{aligned}$$

For the W_2 term we have

$$\begin{aligned} W_2 &= -\int \partial_{v_i} f \partial_{x_i}(K\chi_R f) G^{-(1+\delta)} - \int \partial_{x_i}(K\chi_R f)\partial_{v_i} f G^{-(1+\delta)} \\ &= -\int 2K\chi_R \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)} + \int Kf(\partial_{v_i}\chi_R \partial_{x_i} f + \partial_{v_i} f \partial_{x_i}\chi_R) G^{-(1+\delta)} \\ &\le C\int |\partial_{x_i} f||\partial_{v_i} f| G^{-(1+\delta)} + C\int |\partial_{v_i} f||f| G^{-(1+\delta)} + C\int |f||\partial_{x_i} f| G^{-(1+\delta)}, \end{aligned}$$

Combining the two parts, using Cauchy-Schwarz inequality, and summing up by i we get

$$\begin{aligned} &\frac{d}{dt}\int 2\nabla_x f\cdot\nabla_v f G^{-(1+\delta)} \\ \le\ & -\sum_{i=1}^{d}\int 2\nabla_v(\partial_{v_i} f G^{-1})\cdot\nabla_v(\partial_{x_i} f G^{-1})G^{1-\delta} - \frac{1}{2}\int |\nabla_x f|^2 G^{-(1+\delta)} \\ & +C\int |\nabla_v f|^2 G^{-(1+\delta)} + C\int |f|^2 G^{-(1+\delta)} - \delta(1-\delta)\int v^2 \nabla_v f\cdot\nabla_x f G^{-(1+\delta)}. \end{aligned}$$

For the very definition of $\mathcal{F}$ in (4.4), we easily compute

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}(t, f_t) \\ =\ & A\frac{d}{dt}\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + at^2\frac{d}{dt}\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \\ & +2ct^4\frac{d}{dt}\langle\nabla_v f_t, \nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})} + bt^6\frac{d}{dt}\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \\ & +2at\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + 8ct^3\langle\nabla_v f_t, \nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})} \\ & +6bt^5\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})}. \end{aligned}$$

Gathering all the inequalities above together, we have

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}(t,f_t)\\
\leq\ & (2at-\frac{A(1-\delta)}{2}+Cat^2+2Ct^4c+Cbt^6)\int|\nabla_v f_t|^2G^{-(1+\delta)}\\
&+(6bt^5-\frac{c}{2}t^4+Cbt^6)\int|\nabla_x f_t|^2G^{-(1+\delta)}+(8ct^3+Cat^2)\int|\nabla_v f_t||\nabla_x f_t|G^{-(1+\delta)}\\
&-(at^2\sum_{i=1}^d\int|\nabla_v(\partial_{v_i}f_tG^{-1})|^2G^{1-\delta}+bt^6\sum_{i=1}^d\int|\nabla_v(\partial_{x_i}f_tG^{-1})|^2G^{1-\delta}\\
&+2ct^4\sum_{i=1}^d\int\nabla_v(\partial_{v_i}f_tG^{-1})\cdot\nabla_v(\partial_{x_i}f_tG^{-1})G^{1-\delta})-\frac{\delta(1-\delta)}{2}(at^2\int v^2(\nabla_v f)^2G^{-(1+\delta)}\\
&+bt^6\int v^2(\nabla_x f)^2G^{-(1+\delta)}+2ct^4\int v^2\nabla_v f\cdot\nabla_x fG^{-(1+\delta)})+C\int f_t^2G^{-(1+\delta)},
\end{aligned}$$

for some $C>0$. We observe that

$$\begin{aligned}
&|2ct^4\int v^2\nabla_v f\cdot\nabla_x fG^{-(1+\delta)}|\\
\leq\ & at^2\int v^2(\nabla_v f)^2G^{-(1+\delta)}+bt^6\int v^2(\nabla_x f)^2G^{-(1+\delta)},
\end{aligned}$$

and

$$\begin{aligned}
&|2ct^4\sum_{i=1}^d\int 2\nabla_v(\partial_{v_i}f_tG^{-1})\cdot\nabla_v(\partial_{x_i}f_tG^{-1})G^{1-\delta}|\\
\leq\ & at^2\sum_{i=1}^d\int|\nabla_v(\partial_{v_i}f_tG^{-1})|^2G^{1-\delta}+bt^6\sum_{i=1}^d\int|\nabla_v(\partial_{x_i}f_tG^{-1})|^2G^{1-\delta}.
\end{aligned}$$

by our choice on a,b,c. So by taking A large and $0<\eta$ small ($t\in[0,\eta]$), as a consequence

$$\frac{d}{dt}\mathcal{F}(t,f_t)\leq -L(\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})}+t^4\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})})+C\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})},$$

for some $L,C>0$, and that ends the proof. □

Remark 4.4. For the case $F=\mathcal{F}^*$, the only difference in the proof is to change (4.6) and (4.7) into

$$\partial_{x_i}\mathcal{L}^*f=\mathcal{L}^*\partial_{x_i}f-\partial_{x_i}(\nabla_x V(x)\cdot\nabla_v f)=\mathcal{L}^*\partial_{x_i}f-\sum_{j=1}^d\partial^2_{x_ix_j}V\partial_{v_j}f,$$

and

$$\partial_{v_i}\mathcal{L}^*f=\mathcal{L}^*\partial_{v_i}f+\partial_{x_i}f+\partial_{v_i}f.$$

The following proof of this section is true for both cases.

Lemma 4.5. *For any $0 \leq \delta < 1$, we have*

$$\int |\nabla_{x,v}(fG^{-1/2(1+\delta)})|^2 \leq \int |\nabla_{x,v} f|^2 G^{-(1+\delta)} + C\int f^2 G^{-(1+\delta)},$$

Prove of Lemma 4.5. For any weight function m we have

$$\begin{aligned}\int |\nabla_x(fm)|^2 &= \int |\nabla_x f m + \nabla_x m f|^2 \\ &= \int |\nabla_x f|^2 m^2 + \int |\nabla_x m|^2 f^2 + \int 2f\nabla_x f m \nabla_x m \\ &= \int |\nabla_x f|^2 m^2 + \int (|\nabla_x m|^2 - \frac{1}{2}\Delta_x(m^2))f^2,\end{aligned}$$

taking $m = G^{-1/2(1+\delta)}$ we have

$$\begin{aligned}&\int |\nabla_x(fG^{-1/2(1+\delta)})|^2 \\ =\ &\int |\nabla_x f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4}|\nabla_x V(x)|^2 + \frac{1+\delta}{2}\Delta_x V(x))f^2 G^{-(1+\delta)} \\ \leq\ &\int |\nabla_x f|^2 G^{-(1+\delta)} + C\int f^2 G^{-(1+\delta)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int |\nabla_v(fG^{-1/2(1+\delta)})|^2 \\ =\ &\int |\nabla_v f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4}v^2 + \frac{1+\delta}{2}d)f^2 G^{-(1+\delta)} \\ \leq\ &\int |\nabla_v f|^2 G^{-(1+\delta)}.\end{aligned}$$

Putting together the two inequalities we obtain the result. □

Lemma 4.6. *Nash's inequality: for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$,there exist a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}} \leq C_d \|f\|_{L^1}^{2/d} \|\nabla_v f\|_{L^2},$$

For the proof of Nash's inequality, we refer to [13], Section 8.13 for instance. □

Lemma 4.7. *For any $0 \leq \delta < 1$ we have*

$$\frac{d}{dt}\int |f| G^{-1/2(1+\delta)} \leq d\int |f| G^{-1/2(1+\delta)}, \tag{4.8}$$

which implies

$$\int |f_t| G^{-1/2(1+\delta)} \leq Ce^{dt}\int |f_0| G^{-1/2(1+\delta)}.$$

In particular we have

$$\int |f_t| G^{-1/2(1+\delta)} \leq C \int |f_0| G^{-1/2(1+\delta)}, \quad \forall t \in [0, \eta], \tag{4.9}$$

for some constant $C > 0$.

Proof of Lemma 4.7. By Lemma 5.2 in the next section, letting $p = 1$, we have

$$\begin{aligned} & \frac{d}{dt} \int |f| G^{-1/2(1+\delta)} \\ = & \int |f| (\Delta_v G^{-1/2(1+\delta)} - v \cdot \nabla_v G^{-1/2(1+\delta)} \\ & + v \cdot \nabla_x G^{-1/2(1+\delta)} - \nabla V(x) \cdot \nabla_v G^{-1/2(1+\delta)} - K \chi_R G^{-1/2(1+\delta)}) \\ \leq & \int |f| (\frac{1+\delta}{2} d - \frac{(1+\delta)(1-\delta)}{4} v^2) G^{-1/2(1+\delta)} \leq \int |f| d G^{-1/2(1+\delta)}. \end{aligned}$$

so (4.8) is proved. As $\mathcal{T} G^{-1/2(1+\delta)} = 0$, the result is still true when $F = \mathcal{F}^*$. □

Now we come to the proof of Theorem 4.1.

Proof of Theorem 4.1. We define

$$\mathcal{G}(t, f_t) = B \|f_t\|^2_{L^1(G^{-1/2(1+\delta)})} + t^Z \mathcal{F}(t, f_t),$$

with $B, Z > 0$ to be fixed and $\mathcal{F}$ defined in Lemma 4.3. We choose $t \in [0, \eta]$, η small such that $(a + b + c) Z \eta^{Z+1} \leq \frac{1}{2} L \eta^Z$ (a, b, c, L are also defined Lemma 4.3), by (4.8) and Lemma 4.3 we have

$$\begin{aligned} \frac{d}{dt} \mathcal{G}(t, f_t) \leq & \; dB \|f_t\|^2_{L^1(G^{-1/2(1+\delta)})} + Z t^{Z-1} \mathcal{F}(t, f_t) \\ & - L t^Z (\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + t^4 \|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})}) \\ & + C t^Z \|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \\ \leq & \; dB \|f_t\|^2_{L^1(G^{-1/2(1+\delta)})} + C t^{Z-1} \|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \\ & - \frac{L}{2} t^Z (\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + t^4 \|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})}). \end{aligned}$$

Nash's inequality and Lemma 4.3 implies

$$\begin{aligned} \int f_t^2 G^{-(1+\delta)} \leq & \; (\int |f_t| G^{-1/2(1+\delta)})^{\frac{4}{d+2}} (\int |\nabla_{x,v} (f_t G^{-1/2(1+\delta)})|^2)^{\frac{d}{d+2}} \\ \leq & \; (\int |f_t| G^{-1/2(1+\delta)})^{\frac{4}{d+2}} (\int |\nabla_{x,v} f_t|^2 G^{-(1+\delta)} + C \int f_t^2 G^{-(1+\delta)})^{\frac{d}{d+2}}. \end{aligned}$$

Using Young's inequality, we have

$$\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \leq C_\epsilon t^{-5d} \|f\|^2_{L^1(G^{-1/2(1+\delta)})} + \epsilon t^5 (\|\nabla_{x,v} f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + C \|f_t\|^2_{L^2(G^{-1/2(1+\delta)})}).$$

Taking ϵ small such that $C \epsilon \eta^5 \leq \frac{1}{2}$, we deduce

$$\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \leq 2 C_\epsilon t^{-5d} \|f\|^2_{L^1(G^{-1/2(1+\delta)})} + 2 \epsilon t^5 \|\nabla_{x,v} f_t\|^2_{L^2(G^{-1/2(1+\delta)})}.$$

Taking ϵ small we have

$$\frac{d}{dt}\mathcal{G}(t,f_t) \leq dB\|f_t\|^2_{L^1(G^{-1/2(1+\delta)})} + C_1 t^{Z-1-5d}\|f_t\|^2_{L^1(G^{-1/2(1+\delta)})},$$

for some $C_1 > 0$. Choosing $Z = 1 + 5d$, and using (4.9), we deduce

$$\forall t \in [0,\eta], \quad \mathcal{G}(t,f_t) \leq \mathcal{G}(0,f_0) + C_2\|f_0\|^2_{L^1(G^{-1/2(1+\delta)})} \leq C_3\|f_0\|^2_{L^1(G^{-1/2(1+\delta)})},$$

which ends the proof. □

5. $S_{\mathcal{B}}$ decay in larger spaces

The aim of this section is to prove the following decay estimate for the semigroup $S_{\mathcal{B}}$ which will be useful in the last section where we will prove Theorem 1.1 in full generally.

Theorem 5.1. *Let $H = 1 + x^2 + 2v\cdot x + 3v^2$, for any $\theta \in (0,1)$ and for any $l > 0$, we have*

$$\|\mathcal{S}_{\mathcal{B}}(t)\|_{L^1(H^l)\to L^1(H^{l\theta})} \lesssim (1+t)^{-a},$$

where

$$a = \frac{l(1-\theta)}{1-\frac{\gamma}{2}}.$$

We start with an elementary identity.

Lemma 5.2. *For the kinetic Fokker Planck operator $\mathcal{L}$, let m be a weight function, for any $p \in [1,\infty]$ we have*

$$\int (\mathcal{L}f) f^{p-1} m^p = -(p-1)\int |\nabla_v (mf)|^2 (mf)^{p-2} + \int f^p m^p \phi,$$

with

$$\phi = \frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + (\frac{2}{p}-1)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

In particular when $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

Proof of Lemma 5.2. We split the integral as

$$\int (\mathcal{L}f) f^{p-1} m^p = \int f^{p-1}\mathcal{S}f m^p + \int f^{p-1}\mathcal{T}f m^p.$$

First compute the contribution of the term with operator $\mathcal{T}$

$$\int f^{p-1}\mathcal{T}f m^p = \frac{1}{p}\int \mathcal{T}(f^p) m^p = -\int f^p m^p \frac{\mathcal{T}m}{m}.$$

Concerning the term with operator $\mathcal{S}$, we split it also into two parts

$$\int (\mathcal{S}f) f^{p-1} m^p = \int f^{p-1} m^p (\Delta_v f + div_v(vf)) := C_1 + C_2.$$

We first compute the C_2 term, we have

$$\begin{aligned} C_2 &= \int f^{p-1}m^p(df + v\cdot\nabla_v f) \\ &= \int df^p m^p - \frac{1}{p}\int f^p \mathrm{div}_v(vm^p) \\ &= \int f^p[(1-\frac{1}{p})d - v\cdot\frac{\nabla_v m}{m}]m^p. \end{aligned}$$

We turn to the C_1 term, we have

$$\begin{aligned} C_1 &= \int f^{p-1}m^p\Delta_v f = -\int \nabla_v(f^{p-1}m^p)\cdot\nabla_v f \\ &= \int -(p-1)|\nabla_v f|^2 f^{p-2}m^p - \frac{1}{p}\int \nabla_v f^p\cdot\nabla_v m^p. \end{aligned}$$

Using $\nabla_v(mf) = m\nabla_v f + f\nabla_v m$, we deduce

$$\begin{aligned} C_1 &= -(p-1)\int |\nabla_v(mf)|^2 f^p m^{p-2} + (p-1)\int |\nabla_v m|^2 f^p m^{p-2} \\ &\quad + \frac{2(p-1)}{p^2}\int \nabla_v(f^p)\cdot\nabla_v(m^p) - \frac{1}{p}\int \nabla_v(f^p)\cdot\nabla_v(m^p) \\ &= -(p-1)\int |\nabla_v(mf)|^2 f^{p-2}m^p + (p-1)\int |\nabla_v m|^2 f^p m^{p-2} \\ &\quad + \frac{p-2}{p^2}\int f^p\Delta_v m^p. \end{aligned}$$

Using that $\Delta_v m^p = p\Delta_v m\ m^{p-1} + p(p-1)|\nabla_v m|^2 m^{p-2}$, we obtain

$$C_1 = -(p-1)\int |\nabla_v(mf)|^2 f^{p-2}m^{p-2} + \int f^p m^p[(\frac{2}{p}-1)\frac{\Delta_v m}{m} + 2(1-\frac{1}{p})\frac{|\nabla_v m|^2}{m^2}].$$

We conclude by combining the above equalities. □

Proof of Theorem 5.1. From Lemma 5.2, we have

$$\begin{aligned} (5.1)\ \int(\mathcal{B}f)f^{p-1}m^p &= \int(\mathcal{L} - M\chi_R)f^{p-1}m^p \\ &= -(p-1)\int |\nabla_v(mf)|^2(mf)^{p-2} + \int f^p m^p\phi, \end{aligned}$$

with

$$\phi = [\frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + (\frac{2}{p}-1)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R].$$

When $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R.$$

Let $m = H^k$. We have

$$\frac{\nabla_v m}{m} = k\frac{\nabla_v H}{H}, \quad \frac{\nabla_x m}{m} = k\frac{\nabla_x H}{H},$$

and

$$\frac{\Delta_v m}{m} = \frac{k\Delta_v H}{H} + \frac{k(k-1)|\nabla_v H|^2}{H^2}.$$

Summing up, we have for ϕ

$$\frac{\phi H}{k} = \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} - v\cdot\nabla_v H + v\cdot\nabla_x H - \nabla_x V(x)\cdot\nabla_v H - M\chi_R,$$

From the very definition of H, we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6.$$

We then compute

$$\begin{aligned}
& \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} + v\cdot\nabla_x H - v\cdot\nabla_v H - \nabla_x V(x)\cdot\nabla_v H \\
= \;& 6 + (k-1)\frac{(6v+2x)^2}{H} + 2v^2 + 2x\cdot v - 6v^2 \\
& -2x\cdot v - 6v\cdot\nabla_x V(x) - 2x\cdot\nabla_x V(x) \\
\leq \;& (2v^2 + Cv - 6v^2) - 2x\cdot\nabla_x V(x) + C \\
\leq \;& -C_1 v^2 - C_2 x\cdot\nabla_x V(x) + C_3 \\
\leq \;& -C_4 H^{\frac{\gamma}{2}} + K_1\chi_{R_1},
\end{aligned}$$

for some $C_i > 0$. Taking K and R large enough, we have $\phi \leq -CH^{\frac{\gamma}{2}-1}$, using this inequality in equation (5.1), we deduce

$$\begin{aligned}
\frac{d}{dt}Y_4(t) := \frac{d}{dt}\int |f_{\mathcal{B}}(t)|H^k \;&=\; \int sign(f_{\mathcal{B}}(t))\mathcal{B}f_{\mathcal{B}}(t)H^k \\
&\leq\; -C\int |f_B(t)|H^{k-1+\frac{\gamma}{2}},
\end{aligned} \tag{5.2}$$

for any $k > 1$. In particular for any $l \geq 1$, we can find K and R large enough such that

$$\frac{d}{dt}\int |f_{\mathcal{B}}(t)|H^l \leq 0,$$

which readily implies

$$\int |f_{\mathcal{B}}(t)|H^l \leq \int |f_0|H^l := Y_5.$$

Denoting

$$\alpha = \frac{l-k}{l-k+1-\frac{\gamma}{2}} \in (0,1),$$

the Hölder's inequality

$$\int |f_B(t)|H^k \leq (\int |f_B(t)|H^{k-1+\frac{\gamma}{2}})^{\alpha}(\int |f_B(t)|H^l)^{1-\alpha},$$

implies

$$(\int |f_B(t)|H^k)^{\frac{1}{\alpha}}(\int |f_B(t)|H^l)^{\frac{\alpha-1}{\alpha}} \leq \int |f_B(t)|H^{k-1+\frac{\gamma}{2}},$$

From this inequality and (5.2), we get

$$\frac{d}{dt}Y_4(t) \leq -C(Y_4(t))^{\frac{1}{\alpha}}Y_5^{\frac{\alpha-1}{\alpha}}.$$

Using $Y_4(0) \leq Y_5$, after an integration, we deduce

$$Y_4(t) \leq C_\alpha \frac{1}{(1+t)^{\frac{\alpha}{1-\alpha}}}Y_5,$$

which is nothing but the polynomial decay on $S_{\mathcal{B}}$

$$\|\mathcal{S}_B(t)\|_{L^p(H^l)\to L^p(H^k)} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l-k}{1-\frac{\gamma}{2}}, \quad \forall 0 < k < l, \quad 1 \leq l.$$

We conclude Theorem 5.1 by writing $k = l\theta$, $0 < \theta < 1$.

6. L^p convergence for the KFP model

Before going to the proof of our main theorem, we need two last deduced results.

Lemma 6.1. *For any $\epsilon > 0$ small enough, we have*

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

and

$$\|\mathcal{A}S_B(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

for some $a > 0$. Similarly for any $0 < b < \frac{\gamma}{2-\gamma}$ and for any $\epsilon > 0$ small enough, we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha}e^{-at^b}, \quad \forall t \geq 0,$$

for $\alpha = \frac{5d+1}{2}$ and some $a > 0$.

Proof of Lemma 6.1. The first two inequalities are obtained obviously by Lemma 3.3 and the property of $\mathcal{A} = M\chi_R$. For the third inequality we split it into two parts, $t \in [0, \eta]$ and $t > \eta$, where η is defined in Theorem 4.1. When $t \in [0, \eta]$, we have $e^{-at^{\frac{\gamma}{2-\gamma}}} \geq e^{-a\eta^{\frac{\gamma}{2-\gamma}}}$, by Theorem 4.1, we have

$$\|\mathcal{A}S_B(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} \lesssim t^{-\alpha}e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \in [0, \eta],$$

for some $a > 0$. When $t \geq \eta$, by Theorem 4.1, we have

$$\|S_{\mathcal{B}}(\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim \eta^\alpha \lesssim 1,$$

and by Lemma 3.3

$$\|\mathcal{S}_{\mathcal{B}}(t-\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})} \lesssim e^{-a(t-\eta)^{\frac{\gamma}{2-\gamma}}} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

gathering the two inequalities, we have

$$\|\mathcal{A}\mathcal{S}_B(t)\|_{L^1(G^{-1/2(1+2\epsilon)})\to L^2(G^{-1/2(1+2\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}} \lesssim t^{-\alpha}e^{-at^b}, \quad \forall t > \eta,$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, the proof is ended by combining the two cases above. □

Lemma 6.2. *Similarly as Lemma 6.1. For any $p \in (2, \infty)$, we have*

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-1/2})\to L^2(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0.$$

and

$$\|\mathcal{S}_B(t)\mathcal{A}\|_{L^p(G^{-1/2})\to L^p(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0.$$

for some $a > 0$. And for any $0 < b < \frac{\gamma}{2-\gamma}$ we have

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-1/2})\to L^p(G^{-1/2})} \lesssim t^{-\beta}e^{-at^b}, \quad \forall t \geq 0.$$

for some $\beta > 0$ and some $a > 0$.

The proof of Lemma 6.2 is similar to the proof of Lemma 6.1 and thus skipped.

Lemma 6.3. *let X, Y be two Banach spaces, $S(t)$ a semigroup such that for all $t \geq 0$and some $0 < a, 0 < b < 1$ we have*

$$\|S(t)\|_{X\to X} \leq C_X e^{-at^b}, \quad \|S(t)\|_{Y\to Y} \leq C_Y e^{-at^b},$$

and for some $0 < \alpha$, we have

$$\|S(t)\|_{X\to Y} \leq C_{X,Y} t^{-\alpha} e^{-at^b}.$$

Then we can have that for all integer $n > 0$

$$\|S^{(*n)}(t)\|_{X\to X} \leq C_{X,n} t^{n-1} e^{-at^b},$$

similarly

$$\|S^{(*n)}(t)\|_{Y\to Y} \leq C_{Y,n} t^{n-1} e^{-at^b},$$

and

$$\|S^{(*n)}(t)\|_{X\to Y} \leq C_{X,Y,n} t^{n-\alpha-1} e^{-at^b}.$$

In particular for $\alpha + 1 < n$, and for any $b^* < b$

$$\|S^{(*n)}(t)\|_{X\to Y} \leq C_{X,Y,n} e^{-at^{b^*}}.$$

Proof of Lemma 6.3. The proof is the same as Lemma 2.5 in [15], plus the fact $t^b \leq s^b + (t-s)^b$ for any $0 \leq s \leq t, 0 < b < 1$. □

Then we come to the final proof.

Proof of Theorem 1.1. We only prove the case when $m = G^{\frac{p-1}{p}(1+\epsilon)}, \quad p \in [1, 2]$, for the proof of the other cases, one need only replace the use of Lemma 6.1 in the following proof by Lemma 6.2 and Theorem 4.1. We will prove $p = 1$ first, this time we need to prove

$$\|S_{\mathcal{L}}(I - \Pi)(t)\|_{L^1(G^{-\epsilon})\to L^1} \lesssim e^{-at^b},$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, where I is the identity operator and Π is a projection operator defined by

$$\Pi(f) = \mathcal{M}(f)G.$$

First, Iterating the Duhamel's formula we split it into 3 terms

$$\begin{aligned} S_{\mathcal{L}}(I-\Pi) &= (I-\Pi)\{S_{\mathcal{B}} + \sum_{l=1}^{n-1}(S_{\mathcal{B}}\mathcal{A})^{(*l)} * (S_{\mathcal{B}})\} \\ &\quad + \{(I-\Pi)S_{\mathcal{L}}\} * (\mathcal{A}S_{\mathcal{B}}(t))^{*n}, \end{aligned}$$

and we will estimate them separately. By Lemma 3.3, we have

$$\|S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon})\to L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

the first term is thus estimated. For the second term, still using Lemma 3.3, we get

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^1(G^{-\epsilon})\to L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

by Lemma 6.3, we have

$$\|(S_{\mathcal{B}}(t)\mathcal{A})^{*l}\|_{L^1(G^{-\epsilon})\to L^1} \lesssim t^{l-1}e^{-at^{\frac{\gamma}{2-\gamma}}},$$

thus the second term is estimated. For the last term by Lemma 3.3

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon})\to L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

By Lemma 6.1 and 6.3, for any $0 < b < \frac{\gamma}{2-\gamma}$, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n-1)}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{n-\alpha-2}e^{-at^b},$$

finally by Theorem 3.1, we have

$$\|S_{\mathcal{L}}(t)(I-\Pi)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Taking $n > \alpha + 2$ the third term is estimated thus the proof of case $p = 1$ is concluded by gathering the inequalities above. As the case $p = 2$ ia already proved in Theorem 3.1, the case $p \in (1, 2)$ follows by interpolation. □

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