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THE KINETIC FOKKER-PLANCK EQUATION WITH WEAK CONFINEMENT FORCE

CHUQI CAO

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ABSTRACT. We consider the kinetic Fokker-Planck equation with weak confinement force. We proved some (polynomial and sub-exponential) rate of convergence to the equilibrium (depending on the space to which the initial datum belongs). Our results generalized the result in [4, 5, 21, 11, 10, 9, 1, 14] to weak confinement case.

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Keywords: weak hypocoercivity; weak hypodissipativity; Fokker-Planck equation; semigroup; weak Poincaré inequality; rate of convergence.

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1. INTRODUCTION

In this paper, we consider the weak hypocoercivity issue for the kinetic Fokker-Planck (KFP for short) equation

$$(1.1) \quad \partial_t f = \mathcal{L}f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \operatorname{div}_v(vf),$$

for a density function $f = f(t, x, v)$, with $t \geq 0$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$. The evolution equation is complemented with an initial datum

$$f(0, \cdot) = f_0 \text{ on } \mathbb{R}^{2d}.$$

We make the fundamental assumption on the confinement potential V

$$V(x) = \langle x \rangle^\gamma, \quad \gamma \in (0, 1),$$

where $\langle x \rangle^2 := 1 + |x|^2$.

Let us make some elementary but fundamental observations. First, the equation is mass conservative, that is

$$\mathcal{M}(f_0) = \mathcal{M}(f(t, \cdot)),$$

where we define the mass of f by

$$\mathcal{M}(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv.$$

Next, we observe that

$$(1.2) \quad G = Z^{-1} e^{-W}, \quad W = \frac{v^2}{2} + V(x), \quad Z \in \mathbb{R}_+$$

is a positive normalized steady state of the KFP model, precisely

$$LG = 0, \quad G > 0, \quad \mathcal{M}(G) = 1,$$

by choosing the normalizing constant $Z > 0$ appropriately. Finally we observe that, contrary to the case $\gamma \geq 1$, a Poincaré inequality of the type

$$\exists c > 0, \quad \int_{\mathbb{R}^d} |f(x)|^2 \exp(-V(x)) dx \leq c \int_{\mathbb{R}^d} |\nabla f(x)|^2 \exp(-V(x)) dx,$$

for any smooth function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} f(x) \exp(-V(x)) dx = 0,$$

does not hold. Only a weaker version of this inequality remains true (see [20], or below Section 2). In particular, there is no spectral gap for the associated operator \mathcal{L} , nor is there an exponential trend to the equilibrium for the associated semigroup.

For a given weight function m , we will denote $L^p(m) = \{f | fm \in L^p\}$ the associated Lebesgue space and $\|f\|_{L^p(m)} = \|fm\|_{L^p}$ the associated norm.

The notation $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$.

With these notations, we can introduce the main result of this paper.

Theorem 1.1. (1) For any initial datum $f_0 \in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p \in [1, \infty)$, $\epsilon > 0$ small, the associated solution $f(t, \cdot)$ of the kinetic Fokker-Planck equation (1.1) satisfies

$$\|f(t, \cdot) - \mathcal{M}(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})} \lesssim e^{-Ct^b} \|f_0 - \mathcal{M}(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})},$$

for any $b \in (0, \frac{\gamma}{2-\gamma})$ and some constant $C > 0$.

(2) For any initial datum $f_0 \in L^1(m)$, $m = H^k$, $H = x^2 + v^2$, $1 \leq k$, the associated solution $f(t, \cdot)$ of the kinetic Fokker-Planck equation (1.1) satisfies

$$\|f(t, \cdot) - \mathcal{M}(f_0)G\|_{L^1} \lesssim (1+t)^{-a} \|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)},$$

for any $0 < a < \frac{k}{1-\frac{\gamma}{2}}$. The constants in the estimates do not depend on f_0 , but rely on $\gamma, d, \epsilon, \theta, p, k$.

Remark 1.2. Theorem 1.1 is also true when $V(x)$ behaves like $\langle x \rangle^\gamma$, that is for any $V(x)$ satisfying

$$C_1 \langle x \rangle^\gamma \leq V(x) \leq C_2 \langle x \rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3 |x| \langle x \rangle^{\gamma-1} \leq x \cdot \nabla_x V(x) \leq C_4 |x| \langle x \rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

and

$$|D_x^2 V(x)| \leq C_5 \langle x \rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

for some constant $C_i > 0$, $R > 0$.

Remark 1.3. There are many classical results on the case $\gamma \geq 1$. In this case there is an exponentially decay, and we refer the interested readers to [21, 4, 5, 11, 9, 10, 1].

Remark 1.4. There are already some convergence results for the weak confinement case proved by probability method on some particular L^1 or L^2 spaces in [1] and [6], this paper extend the result to L^p spaces and more larger spaces.

Let us briefly explain the main ideas behind our method of proof.

We first introduce four spaces $E_1 = L^2(G^{-1/2})$, $E_2 = L^2(G^{-1/2}e^{\epsilon_1 V(x)})$, $E_3 = L^2(G^{-(1+\epsilon_2)/2})$ and $E_0 = L^2(G^{-1/2}\langle x \rangle^{\gamma-1})$, with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ small such that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset L^2$. Thus E_1 is an interpolation space between E_0 and E_2 . We first use a hypocoercivity argument as in [4, 5] to prove that, for any $f_0 \in E_3$, the solution to the KFP equation (1.1) satisfies

$$\frac{d}{dt} \|f(t)\|_{E_1} \leq -\lambda \|f(t)\|_{E_0},$$

for some constant $\lambda > 0$. We use this and the Duhamel formula to prove

$$\|f(t)\|_{E_2} \lesssim \|f_0\|_{E_3}.$$

Combining the two inequalities and using a interpolation argument as in [12], we get

$$(1.3) \quad \|f(t)\|_{E_1} \lesssim e^{-at^b} \|f_0\|_{E_3},$$

for some $a > 0, b \in (0, 1)$.

We then generalize the decay estimate to a wider class of Banach spaces by adapting the extension theory introduced in [18] and developed in [14, 8]. For any operator \mathcal{L} , denote $S_{\mathcal{L}}(t)$ the associated semigroup. We introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where \mathcal{A} is an appropriately defined bounded operator so that \mathcal{B} becomes a dissipative operator. By proving some regularization estimate in $S_{\mathcal{B}}$ in L^p

$$\|S_{\mathcal{B}}(t)\|_{L^p(m_1) \rightarrow L^2(m_2)} \lesssim t^{-\alpha}, \quad \forall t \in [0, \eta],$$

for some weight function m_1, m_2 and some $\alpha, \eta > 0$, and using the iterated Duhamel's formula

$$(1.4) \quad S_{\mathcal{L}} = S_{\mathcal{B}} + \sum_{l=1}^{n-1} (S_{\mathcal{B}})^{*l} * (\mathcal{A}S_{\mathcal{B}})^{*l} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{*n},$$

we deduce the L^p convergence on $S_{\mathcal{L}}$, where the convolution of two semigroups $S_{\mathcal{A}}(t) S_{\mathcal{B}}(t)$ is defined by

$$(S_{\mathcal{A}} * S_{\mathcal{B}})(t) = \int_0^t S_{\mathcal{A}}(s) S_{\mathcal{B}}(t-s) ds.$$

Let us end the introduction by describing the plan of the paper. In Section 2, we will develop a hypocoercivity argument to prove a weighted L^2 estimate for the KFP model. In section 3, we introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and using the L^2 estimate, we prove a L^2 convergence. In Section 4 we present the proof of a regularization estimate on $S_{\mathcal{B}}$ from L^2 to L^p . In Section 5 we prove some L^1 estimate on the semigroup $S_{\mathcal{B}}$. Finally in Section 6 we use the above regularization estimate to conclude the L^p convergence for KFP equation.

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2. L^2 FRAMEWORK: DIRICHLET FORM AND RATE OF CONVERGENCE ESTIMATE

For later discussion, we introduce some notations for the whole paper. We split the KFP operator as

$$\mathcal{L} = \mathcal{T} + \mathcal{S},$$

where \mathcal{T} stands for the transport part

$$\mathcal{T}f = -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f,$$

and \mathcal{S} stands for the collision part

$$\mathcal{S}f = \Delta_v f + \text{div}_v(vf).$$

We will denote the cut-off function χ such that $\chi(x, v) \in [0, 1]$, $\chi(x, v) \in C^\infty$, $\chi(x, v) = 1$ when $x^2 + v^2 \leq 1$, $\chi(x, v) = 0$ when $x^2 + v^2 \geq 2$, and then denote $\chi_R = \chi(x/R, v/R)$.

We may also define another splitting of the KFP operator \mathcal{L} by

$$(2.1) \quad \mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} = K\chi_R(x, v).$$

with $K, R > 0$ to be chosen later.

We use $\int f$ in place of $\int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv$ for short, similarly $\int f dx$ means $\int_{\mathbb{R}^d} f dx$, $\int f dv$ means $\int_{\mathbb{R}^d} f dv$. $B_{|x| \leq \rho}$ is used to denote the ball such that $\{x \in \mathbb{R}^d \mid |x| \leq \rho\}$, similarly B_ρ means the ball such that $\{x, v \in \mathbb{R}^d \mid |x|^2 + v^2 \leq \rho\}$.

For $V(x) = \langle x \rangle^\gamma$, $0 < \gamma < 1$, we also denote $\langle \nabla V \rangle$ for $\langle x \rangle^{\gamma-1}$, and $\langle \nabla V \rangle^{-1}$ for $\langle x \rangle^{1-\gamma}$.

With these notations we introduce the Dirichlet form adapted to our problem. We define the 0 order and first order moments

$$\rho_f = \rho[f] = \int f dv, \quad j_f = j[f] = \int v f dv,$$

then we define a projection operator π by

$$\pi f = M \rho_f, \quad M = C e^{-v^2/2}, \quad \int M dv = 1,$$

and the complement of π by

$$\pi^\perp = 1 - \pi, \quad f^\perp = \pi^\perp f.$$

We define an elliptic operator Δ_V and its dual Δ_V^* by

$$\Delta_V u := \operatorname{div}_x(\nabla_x u + \nabla_x V u), \quad \Delta_V^* u = \Delta_x u - \nabla_x V \cdot \nabla_x u,$$

let $u = (\Delta_V^*)^{-1} \xi$ be the solution to the above elliptic equation

$$\Delta_V^* u = \xi \text{ on } \mathbb{R}^d,$$

note that u can differ by a constant, we also requires that

$$\int u e^{-V} \langle \nabla V \rangle^{-2} dx = 0,$$

using these notations, define a scalar product by

$$\begin{aligned} ((f, g)) &:= (f, g)_{\mathcal{H}} + \epsilon (\Delta_V^{-1} \nabla_x j_f, (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon ((\rho_f e^V \langle \nabla V \rangle^2), \Delta_V^{-1} \nabla_x j_g)_{L^2} \\ &= (f, g)_{\mathcal{H}} + \epsilon (j_f, \nabla_x (\Delta_V^*)^{-1} (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon ((\nabla_x (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2), j_g)_{L^2}, \end{aligned}$$

for some $\epsilon > 0$ to be specified later.

We then define the Dirichlet form

$$\begin{aligned} D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f)_{\mathcal{H}} + \epsilon (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon ((\rho[-\mathcal{L}f] e^V \langle \nabla V \rangle^2), \Delta_V^{-1} \nabla_x j_f)_{L^2}. \end{aligned}$$

Finally we define $\mathcal{H} = L^2(G^{-1/2})$, $\mathcal{H}_1 = L^2(G^{-1/2}\langle\nabla V\rangle)$ and

$$\mathcal{H}_0 = \{h \in \mathcal{H}, \int f dx dv = 0\}$$

where we recall that G has been introduced in (1.2). With these notations we can come to our first theorem.

Theorem 2.1. *There exists $\epsilon > 0$ small enough, such that on \mathcal{H}_0 the norm $((f, f))^{\frac{1}{2}}$ defined above is equivalent to the norm of \mathcal{H} , moreover there exist $\lambda > 0$, such that*

$$D[f] \geq \lambda \|f\|_{\mathcal{H}_1}^2, \quad \forall f \in \mathcal{H}_0.$$

As a consequence, for any $f_0 \in \mathcal{H}_0$, we have

$$(2.2) \quad \frac{d}{dt}((f, f)) \leq -C \int f^2 G^{-1} \langle x \rangle^{2(\gamma-1)},$$

for some constant $C > 0$. In particular for any $f_0 \in \mathcal{H}_0$, we have

$$(2.3) \quad \|f(t, \cdot)\|_{L^2(G^{-\frac{1}{2}})} \leq C \|f_0\|_{L^2(G^{-\frac{1}{2}})},$$

for some constant $C > 0$.

Remark 2.2. In \mathcal{H}_0 we have

$$\int \rho_f e^V \langle \nabla V \rangle^2 e^{-V} \langle \nabla V \rangle^{-2} dx = \int \rho_f dx = \int f dx dv = 0,$$

so the term $(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)$ is well defined in \mathcal{H}_0 .

Remark 2.3. (1) By little modifying the method in Villani's paper [21], a H^1 version of our theorem can be established.

(2) Our statement is a generalization of [4, 5].

Before proving the theorem, we need some lemmas.

We say that W satisfies a local Poincaré inequality on a bounded open set Ω if there exist some constant $\kappa_\Omega > 0$ such that:

$$\int_\Omega h^2 W \leq \kappa_\Omega \int_\Omega |\nabla h|^2 W + \frac{1}{W(\Omega)} \left(\int_\Omega h W \right)^2,$$

for any nice function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and where we denote $W(\Omega) := \langle W 1_\Omega \rangle$.

Lemma 2.4. *Under the assumption $W, W^{-1} \in L_{loc}^\infty(\mathbb{R}^d)$, the function W satisfies the local Poincaré inequality for any ball $\Omega \in \mathbb{R}^d$.*

For the proof of Lemma 2.4 we refer to [19] Lemma 2.3.

Lemma 2.5. *(weak Poincaré inequality) There exist a constant $\lambda > 0$ such that*

$$\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} \leq \lambda \|\nabla u\|_{L^2(e^{-V/2})}$$

for any $u \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} u e^{-V} \langle \nabla V \rangle^{-2} dx = 0$$

Proof of Lemma 2.5. We prove for any $h \in \mathcal{D}(\mathbb{R}^d)$ such that

$$(2.4) \quad \int_{\mathbb{R}^d} h e^{-V} \langle \nabla V \rangle^{-2} = 0,$$

we have

$$\int_{\mathbb{R}^d} |\nabla h|^2 e^{-V} \geq \lambda \int_{\mathbb{R}^d} h^2 e^{-V} \langle x \rangle^{2(\gamma-1)},$$

for some $\lambda > 0$. Taking $g = h e^{-1/2V}$, we have $\nabla g = \nabla h e^{-1/2V} - \frac{1}{2} \nabla V h e^{-1/2V}$, so that

$$\begin{aligned} 0 \leq \int |\nabla g|^2 &= \int |\nabla h|^2 e^{-V} + \int h^2 \frac{1}{4} |\nabla V|^2 e^{-V} - \int \frac{1}{2} \nabla(h^2) \cdot \nabla V e^{-V} \\ &= \int |\nabla h|^2 e^{-V} + \int h^2 \left(\frac{1}{2} \Delta V - \frac{1}{4} |\nabla V|^2 \right) e^{-V}. \end{aligned}$$

We deduce for some $K, R_0 > 0$

$$\int |\nabla h|^2 e^{-V} \geq \int \frac{1}{8} h^2 \langle \nabla V \rangle^2 e^{-V} - K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2}.$$

Defining

$$\epsilon_R := \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6}, \quad Z_R := \int_{B_R} e^{-V} \langle \nabla V \rangle^{-2},$$

and using (2.4), we get

$$\begin{aligned} \left(\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2} \right)^2 &= \left(\int_{B_R^c} h e^{-V} \langle \nabla V \rangle^{-2} \right)^2 \\ &\leq \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2 \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6} \\ &\leq \epsilon_R \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Using the local Poincaré inequality in Lemma 2.4, we deduce

$$\begin{aligned} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^{-2} &\leq C_R \int_{B_R} |\nabla h|^2 e^{-V} \langle \nabla V \rangle^{-2} + \frac{1}{Z_R} \left(\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2} \right)^2 \\ &\leq C'_R \int_{B_R} |\nabla h|^2 e^{-V} + \frac{\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Putting all the inequalities together and taking $R > R_0$, we finally get

$$\begin{aligned} \int h^2 e^{-V} \langle \nabla V \rangle^2 &\leq 8 \int |\nabla h|^2 e^{-V} + 8K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2} \\ &\leq 8(1 + KC'_R) \int |\nabla h|^2 e^{-V} + \frac{8K\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2, \end{aligned}$$

and we conclude by taking R large such that: $\frac{8K\epsilon_R}{Z_R} \leq \frac{1}{2}$. \square

Lemma 2.6. (*Elliptic Estimate*) For any $\xi_1 \in L^2(\langle \nabla V \rangle^{-1} e^{-V/2})$ and $\xi_2 \in L^2(e^{-V/2})$, the solution $u \in L^2(e^{-V/2})$ to the elliptic equation

$$(2.5) \quad -\Delta_V^* u = \xi_1 + \nabla \xi_2, \quad \int u e^{-V} \langle \nabla V \rangle^{-2} dx = 0,$$

satisfies

$$(2.6) \quad \|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})} \lesssim \|\xi_1\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})} + \|\xi_2\|_{L^2(e^{-V/2})}.$$

Similarly for any $\xi \in L^2(e^{-V/2})$, the solution $u \in L^2(e^{-V/2})$ to the elliptic problem

$$-\Delta_V^* u = \xi, \quad \int u e^{-V} \langle \nabla V \rangle^{-2} = 0,$$

satisfies

$$(2.7) \quad \|u\|_{L^2(\langle \nabla V \rangle^2 e^{-V/2})} + \|\nabla u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|D^2 u\|_{L^2(e^{-V/2})} \lesssim \|\xi\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})}.$$

Proof of Lemma 2.6. Multiply (2.5) by $u e^{-V}$ and observes that

$$(2.8) \quad e^V \operatorname{div}_x [e^{-V} \nabla_x u] = \Delta_x u - \nabla_x V \cdot \nabla_x u = \Delta_V^* u,$$

we have after integration

$$-\int e^V \operatorname{div}_x [e^{-V} \nabla_x u] u e^{-V} = \int (\xi_1 + \nabla \cdot \xi_2) u e^{-V}.$$

Performing one integration by parts, we deduce

$$\int e^{-V} |\nabla_x u|^2 = \int (\xi_1 u + \xi_2 \cdot \nabla u - \xi_2 \cdot \nabla V u) e^{-V},$$

using Lemma 2.5 we obtain (2.6). In inequality (2.7), the first two terms are easily bounded by (2.6) and $\langle \nabla V \rangle \leq 1$, we then only need to prove the bound for the third term. By integration by parts, we have

$$\begin{aligned} \int |D^2 u|^2 e^{-V} &= \sum_{i,j=1}^d \int (\partial_{ij}^2 u)^2 e^{-V} \\ &= \sum_{i,j=1}^d \int \partial_i u (\partial_{ij}^2 u \partial_j V - \partial_{ijj}^3 u) e^{-V} \\ &= \sum_{i,j=1}^d \int \partial_{jj} u \partial_i (\partial_i u e^{-V}) - \frac{1}{2} \int (\partial_i u)^2 \partial_j (\partial_j V e^{-V}) \\ &= \int (\Delta u) (-\Delta_V^* u) e^{-V} + \int |\nabla u|^2 (|\nabla V|^2 - \Delta V) e^{-V} \\ &\lesssim \|D^2 u\|_{L^2(e^{-V/2})} \|\xi\|_{L^2(e^{-V/2})} + \|\langle \nabla V \rangle \nabla u\|_{L^2(e^{-V/2})}, \end{aligned}$$

where in the third equality we have used

$$\begin{aligned} \int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} &= - \int \partial_i u \partial_j (\partial_i u \partial_j V e^{-V}) \\ &= - \int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} - \int (\partial_i u)^2 \partial_j (\partial_j V e^{-V}), \end{aligned}$$

which implies

$$\int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} = -\frac{1}{2} \int (\partial_i u)^2 \partial_j (\partial_j V e^{-V}),$$

and in the fourth equality we have used (2.8). That concludes the proof. \square

Now we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. First we prove the equivalence of the norms associated to $((\cdot, \cdot))$ and $(\cdot, \cdot)_{\mathcal{H}}$. By Cauchy-Schwarz inequality and Lemma 2.4, we have

$$(jf, \nabla_x (\Delta_V^*)^{-1} (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \leq \|jf\|_{L^2(e^{V/2})} \|\rho_g e^V \langle \nabla V \rangle^2\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})},$$

and obviously

$$\|\rho_g e^V \langle \nabla V \rangle^2\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})} = \|\rho_g\|_{L^2(\langle \nabla V \rangle e^{V/2})} \leq \|\rho_g\|_{L^2(e^{V/2})} \lesssim \|g\|_{\mathcal{H}}.$$

Using the elementary observations

$$|jf| \lesssim \|f\|_{L^2(e^{v^2/4})} \quad |\rho f| \lesssim \|f\|_{L^2(e^{v^2/4})},$$

we deduce

$$(jf, \nabla_x (\Delta_V^*)^{-1} (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \lesssim \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}},$$

The third term in the definition of $((\cdot, \cdot))$ can be estimated in the same way and that ends the proof of equivalence of norms. \square

Now we prove the main estimate of the theorem. We split the Dirichlet term $D[f]$ into 3 parts

$$D[f] = T_1 + \epsilon T_2 + \epsilon T_3,$$

with

$$\begin{aligned} T_1 &:= (\mathcal{L}f, f)_{\mathcal{H}} \\ T_2 &:= (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], \rho f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ T_3 &:= ((\Delta_V)^{-1} \nabla_x jf, \rho[-\mathcal{L}f])_{L^2(e^{V/2} \langle \nabla V \rangle)}, \end{aligned}$$

and compute them separately.

For the T_1 term, using the classical Poincaré inequality, we have

$$\begin{aligned} T_1 &:= (-\mathcal{T}f + \mathcal{S}f, f)_{\mathcal{H}} = (-\mathcal{S}f, f)_{\mathcal{H}} \\ &= - \int [\Delta_v f + \operatorname{div}_v(vf)] f M^{-1} e^V = \int |\nabla_v(f/M)|^2 M e^V \\ &\geq k_p \int |f/M - \rho f|^2 M e^V = k_p \|f - \rho f M\|_{\mathcal{H}}^2 = k_p \|f^\perp\|_{\mathcal{H}}^2, \end{aligned}$$

for some $k_p > 0$. We split the T_2 term as

$$\begin{aligned}
T_2 &:= (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\
&= (\Delta_V^{-1} \nabla_x j[-\mathcal{T}\pi f], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\
&\quad + (\Delta_V^{-1} \nabla_x j[-\mathcal{T}f^\perp], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\
&\quad + (\Delta_V^{-1} \nabla_x j[-\mathcal{S}f], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\
&:= T_{2,1} + T_{2,2} + T_{2,3}.
\end{aligned}$$

First observe

$$\mathcal{T}\pi f = -v \cdot \nabla_x \rho_f M - \nabla_x V \cdot v \rho_f M = -e^{-V} M v \cdot \nabla_x (\rho_f / e^{-V}),$$

so that we have

$$j[-\mathcal{T}\pi f] = \langle v v_k M \rangle e^{-V} \partial_{x_k} (\rho_f / e^{-V}) = e^{-V} \nabla_x (\rho_f / e^{-V}).$$

Next by (2.8), we have

$$\begin{aligned}
T_{2,1} &= (j[-\mathcal{T}\pi f], \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\
&= (\rho_f, [e^V \operatorname{div}_x(e^{-V} \nabla)] [(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)])_{L^2} \\
&= \|\rho_f e^{V/2} \langle \nabla V \rangle\|_{L^2}^2 = \|\pi f\|_{\mathcal{H}_1}^2.
\end{aligned}$$

Using the notation $\eta_1 = \langle v \otimes v f^\perp \rangle$ and $\eta_{2,\alpha\beta} = \langle v_\alpha \partial_{v_\beta} f^\perp \rangle$, and observing that

$$|\eta_1| \lesssim \|f^\perp\|_{L^2(e^{v^2/4})}, |\eta_2| \lesssim \|f^\perp\|_{L^2(e^{v^2/4})},$$

we compute

$$\begin{aligned}
T_{2,2} &= (j[-\mathcal{T}f^\perp], \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\
&= (D\eta_1 + \eta_2 \nabla V, \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\
&= (\eta_1, D^2(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} + (\eta_2, \nabla V \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\
&= \|\eta_1\|_{L^2(e^{V/2})} \|D^2(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})} \\
&\quad + \|\eta_2\|_{L^2(e^{V/2})} \|\nabla V \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})}.
\end{aligned}$$

By Lemma 2.6, we estimate

$$\begin{aligned}
T_{2,2} &\lesssim \|\eta_1\|_{L^2(e^{V/2})} \|\rho_f e^V \langle \nabla V \rangle^2\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})} \\
&\quad + \|\eta_2\|_{L^2(e^{V/2})} \|\rho_f e^V \langle \nabla V \rangle^2\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})} \\
&\lesssim \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1}.
\end{aligned}$$

Using

$$\begin{aligned}
j[-\mathcal{S}f] = j[-\mathcal{S}f^\perp] &= - \int v [\Delta_v f^\perp + \operatorname{div}_v(v f^\perp)] dv \\
&= d \int f^\perp v dv \lesssim \|f^\perp\|_{L^2(e^{v^2/4})},
\end{aligned}$$

and Lemma 2.6, we have

$$\begin{aligned}
T_{2,3} &= (j[-Sf], \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\
&\leq \|j[-Sf]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})} \\
&\lesssim \|f^\perp\|_{\mathcal{H}} \|\rho_f e^V \langle \nabla V \rangle^2\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})} \\
&= \|f^\perp\|_{\mathcal{H}} \|\rho_f\|_{L^2(\langle \nabla V \rangle e^{V/2})} \\
&\lesssim \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1}.
\end{aligned}$$

Finally we come to the T_3 term. Using

$$\rho[-Sf] = \int \nabla_v \cdot (\nabla_v f + v f) dv = 0,$$

and

$$\begin{aligned}
\rho[-Tf] &= \rho[v \nabla_x f - \nabla_x V(x) \nabla_v f] \\
&= \int v \nabla_x f - \nabla_x V(x) \nabla_v f dv \\
&= \nabla_x j[f],
\end{aligned}$$

because $\nabla(\langle \nabla V \rangle^2) \lesssim \langle \nabla V \rangle^2$ and $\langle \nabla V \rangle^2 \lesssim \langle \nabla V \rangle$, we get

$$\begin{aligned}
T_3 &= ((\Delta_V)^{-1} \nabla_x j f, \rho[-\mathcal{L}f])_{L^2(e^{V/2} \langle \nabla V \rangle)} \\
&= ((\Delta_V)^{-1} \nabla_x j[f^\perp], \rho[-\mathcal{T}f])_{L^2(e^{V/2} \langle \nabla V \rangle)} \\
&= (j[-f^\perp], \nabla(\Delta_V^*)^{-1}(\nabla_x j[f] e^V \langle \nabla V \rangle^2))_{L^2} \\
&= \|j[f^\perp]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}[\nabla_x(j_f e^V \langle \nabla V \rangle^2) \\
&\quad - \nabla V j_f e^V \langle \nabla V \rangle^2 - \nabla(\langle \nabla V \rangle^2) j_f e^V]\|_{L^2(e^{-V/2})},
\end{aligned}$$

using again Lemma 2.6, we have

$$\begin{aligned}
T_3 &\lesssim \|j[f^\perp]\|_{L^2(e^{V/2})} (\|j_f e^V \langle \nabla V \rangle^2\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})} \\
&\quad + \|j_f e^V \nabla(\langle \nabla V \rangle^2)\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})}) \\
&\lesssim \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1}.
\end{aligned}$$

Putting all the terms together and choosing $\epsilon > 0$ small enough, we can deduce

$$\begin{aligned}
D[f] &\geq k_p \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1} - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1} \\
&\geq k_p \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - (2\epsilon + 4\epsilon^{1/2})K \|f^\perp\|_{\mathcal{H}}^2 - \epsilon^{3/2} 4K \|\pi f\|_{\mathcal{H}_1}^2 \\
&\geq \frac{k_p}{2} (\|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2) \geq \frac{\epsilon}{M} \|f\|_{\mathcal{H}_1},
\end{aligned}$$

for some $M > 0$. □

3. L^2 SUB-EXPONENTIAL DECAY FOR THE KINETIC FOKKER-PLANCK EQUATION BASED ON A SPLITTING TRICK

In this section we establish a first decay estimate on $S_{\mathcal{L}}$ which is a particular case in the result of Theorem 1.1.

Theorem 3.1. *Using the notation and results in Theorem 2.1, we have*

$$\|S_{\mathcal{L}}(t)f_0\|_{L^2(G^{-\frac{1}{2}})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}} \|f_0\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})},$$

for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)}) \cap \mathcal{H}_0$, $\epsilon > 0$ small enough.

Remark 3.2. It's worth emphasizing that we deduce immediately part (1) of Theorem 1.1 in the case $p = 2$ by considering the initial datum $f_0 - \mathcal{M}(f_0)$ for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)})$.

Recall the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ introduced in (2.1), we first prove some decay estimate on the semigroup $S_{\mathcal{B}}$.

Lemma 3.3. *Let us fix $p \in [1, \infty)$.*

(1) *For any given smooth weight function m , we have*

$$(3.1) \quad \int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m = \frac{1}{p} \int f^p G^{-(p-1)}\tilde{m},$$

with

$$\tilde{m} = \Delta_v m - \nabla_v m \cdot v - \nabla V(x) \cdot \nabla_v m + v \cdot \nabla_x m.$$

(2) *Taking $m = e^{\epsilon H^\delta}$, $\epsilon > 0$ if $0 < \delta < \frac{\gamma}{2}$, ϵ small enough if $\delta = \frac{\gamma}{2}$, $H = 3v^2 + 2x \cdot v + x^2 + 1$, we have*

$$(3.2) \quad \int f^{p-1}(\mathcal{B}f)G^{-(p-1)}e^{\epsilon H^\delta} \leq -C \int f^p G^{-(p-1)}e^{\epsilon H^\delta} H^{\frac{\delta}{2}+\gamma-1},$$

for some K and R large.

(3) *With the same notation as above, there holds*

$$(3.3) \quad \|S_{\mathcal{B}}(t)\|_{L^p(e^{2\epsilon H^\delta} G^{-\frac{p-1}{p}}) \rightarrow L^p(e^{\epsilon H^\delta} G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

for some $a > 0$. In particular, this implies

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)}) \rightarrow L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Proof of Lemma 3.3. Step 1. Recall (1.2), we write

$$\int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m = \int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m + \int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m.$$

We first compute the contribution of the term with operator \mathcal{T}

$$\begin{aligned} \int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m &= \frac{1}{p} \int \mathcal{T}(f^p)G^{-(p-1)}m \\ &= -\frac{1}{p} \int f^p \mathcal{T}(G^{-(p-1)}m) \\ &= \frac{1}{p} \int f^p G^{-(p-1)}(v \cdot \nabla_x m - \nabla V(x) \cdot \nabla_v m). \end{aligned}$$

For the the term with operator \mathcal{S} , we use one integration by parts, and we get

$$\begin{aligned} &\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ &= \int f^{p-1}(\Delta_v f + \operatorname{div}_v(vf))G^{-(p-1)}m \\ &= -\int \nabla_v((fG^{-1})^{p-1}m) \cdot \nabla_v(fG^{-1})G \\ &= -\int (p-1)|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm - \frac{1}{p}\nabla_v((fG^{-1})^p) \cdot (\nabla_v m)G. \end{aligned}$$

Performing another integration by parts on the latter term, we have

$$\begin{aligned} &\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ &= \int -(p-1)|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm + \frac{1}{p}\nabla_v \cdot (G\nabla_v m)(fG^{-1})^p \\ &= \int -(p-1)|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm + \frac{1}{p}(\Delta_v m - v \cdot \nabla_v m)f^p G^{-(p-1)}. \end{aligned}$$

Identity (3.1) follows by putting together the two identities.

Step 2. We particular use $m = e^{\epsilon H^\delta}$ and we easily compute

$$\frac{\nabla_v m}{m} = \delta\epsilon \frac{\nabla_v H}{H^{1-\delta}}, \quad \frac{\nabla_x m}{m} = \delta\epsilon \frac{\nabla_x H}{H^{1-\delta}},$$

and

$$\frac{\Delta_v m}{m} \leq \delta\epsilon \frac{\Delta_v H}{H^{1-\delta}} + (\delta\epsilon)^2 \frac{|\nabla_v H|^2}{H^{2(1-\delta)}}.$$

We deduce that $\phi = \frac{\tilde{m}}{m}$ satisfies

$$\frac{\phi H^{1-\delta}}{\epsilon\delta} \leq \Delta_v H + \epsilon\delta \frac{|\nabla_v H|^2}{H^{1-\delta}} - v \cdot \nabla_v H + v \cdot \nabla_x H - \nabla_x V(x) \cdot \nabla_v H.$$

From the very definition of H , we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6.$$

Choosing $\epsilon > 0$ arbitrary if $0 < 2\delta < \gamma$, ϵ small enough if $2\delta = \gamma$, we deduce

$$\begin{aligned}
& \Delta_v H + 2\epsilon\delta \frac{|\nabla_v H|^2}{H^{1-\delta}} + v \cdot \nabla_x H - v \cdot \nabla_v H - \nabla_x V(x) \cdot \nabla_v H \\
&= 6 + \epsilon\delta \frac{(6v + 2x)^2}{H^{1-\delta}} + 2v^2 + 2x \cdot v - 6v^2 - 2x \cdot v - 6v \cdot \nabla_x V(x) - 2x \cdot \nabla_x V(x) \\
&\leq (2v^2 + C_1 v + C_2 v^{2\delta} - 6v^2) + (C_3 \epsilon \delta x^{2\delta} - 2x \cdot \nabla_x V(x)) + C \\
&\leq -C_4 v^2 - C_5 x \cdot \nabla_x V(x) + C_6 \\
&\leq -C_7 H^{\frac{\gamma}{2}} + K \chi_R,
\end{aligned}$$

for some constants $C_i, K, R > 0$. As a consequence, we have proved

$$\phi - K \chi_R \leq \frac{-C}{H^{1-\delta-\frac{\gamma}{2}}} \leq 0,$$

which is nothing but (3.2).

Step 3. In the following, denote $f_t = S_{\mathcal{B}}(t)f_0$ the solution to the evolution equation $\partial_t f = \mathcal{B}f$, $f(0) = f_0$. On the one hand, by (3.2) we have

$$\frac{d}{dt} \int f_t^p G^{-(p-1)} e^{2\epsilon H^\delta} = \int f_t^{p-1} (\mathcal{B}f_t) G^{-(p-1)} e^{2\epsilon H^\delta} \leq 0,$$

which implies

$$\int f_t^p G^{-(p-1)} e^{2\epsilon H^\delta} \leq \int f_0^p G^{-(p-1)} e^{2\epsilon H^\delta} := Y_1, \quad \forall t \geq 0$$

On the other hand, defining

$$Y := \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta},$$

using again (3.2), we have

$$\begin{aligned}
\frac{d}{dt} Y &= p \int f_t^{p-1} \mathcal{B}f_t G^{-(p-1)} e^{\epsilon H^\delta} \\
&\leq -a \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} H^{\delta+\frac{\gamma}{2}-1} \\
&\leq -a \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x \rangle^{2\delta+\gamma-2} \\
&\leq -a \int_{B_{|x| \leq \rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x \rangle^{2\delta+\gamma-2},
\end{aligned}$$

for any $\rho > 0$ and for some $a > 0$. As $2\delta + \gamma < 2$, $0 \leq |x| \leq \rho$ implies $\langle x \rangle^{2\delta+\gamma-2} \geq \langle \rho \rangle^{2\delta+\gamma-2}$, we deduce

$$\begin{aligned}
\frac{d}{dt} Y &\leq -a \langle \rho \rangle^{2\delta+\gamma-2} \int_{B_{|x| \leq \rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \\
&\leq -a \langle \rho \rangle^{2\delta+\gamma-2} Y + a \langle \rho \rangle^{2\delta+\gamma-2} \int_{B_{|x| \geq \rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta},
\end{aligned}$$

Using that $e^{\epsilon\langle x \rangle^{2\delta}} \geq e^{\epsilon\langle \rho \rangle^{2\delta}}$ on $|x| \geq \rho$, we get

$$\begin{aligned} \frac{d}{dt}Y &\leq -a\langle \rho \rangle^{2\delta+\gamma-2}Y + a\langle \rho \rangle^{2\delta+\gamma-2}e^{-\epsilon\langle \rho \rangle^{2\delta}} \int_{B_{|x|\geq\rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon\langle x \rangle^{2\delta}} \\ &\leq -a\langle \rho \rangle^{2\delta+\gamma-2}Y + a\langle \rho \rangle^{2\delta+\gamma-2}e^{-\epsilon\langle \rho \rangle^{2\delta}} \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon\langle x \rangle^{2\delta}} \\ &\leq -a\langle \rho \rangle^{2\delta+\gamma-2}Y + a\langle \rho \rangle^{2\delta+\gamma-2}e^{-\epsilon\langle \rho \rangle^{2\delta}} CY_1. \end{aligned}$$

Thanks to Grönwall's Lemma, we obtain

$$\begin{aligned} Y(t) &\leq e^{-a\langle \rho \rangle^{2\delta+\gamma-2}t}Y(0) + Ce^{-\epsilon\langle \rho \rangle^{2\delta}}Y_1 \\ &\lesssim (e^{-a\langle \rho \rangle^{2\delta+\gamma-2}t} + e^{-\epsilon\langle \rho \rangle^{2\delta}})Y_1, \end{aligned}$$

Choosing finally ρ such that $a\langle \rho \rangle^{2\delta+\gamma-2}t = \epsilon\langle \rho \rangle^{2\delta}$, that is $\langle \rho \rangle^{2-\gamma} = Ct$, we deduce

$$Y(t) \leq C_1 e^{-C_2 t^{\frac{2\delta}{2-\gamma}}} Y_2,$$

for some $C_i > 0$, and we deduce the proof of (3.3). \square

Now we come to prove Theorem 3.1.

Proof of Theorem 3.1. We recall that from (2.3), we have

$$\|S_{\mathcal{L}}(t)\|_{L^2(G^{-1/2}) \rightarrow L^2(G^{-1/2})} \lesssim 1, \quad \forall t \geq 0$$

From the very definition of \mathcal{A} we have

$$\|\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^2(e^{2\epsilon H^\delta} G^{-1/2})} \lesssim 1.$$

From Lemma 3.3 case $p = 2$, we have

$$\|S_{\mathcal{B}}(t)\|_{L^2(e^{2\epsilon H^\delta} G^{-1/2}) \rightarrow L^2(e^{\epsilon H^\delta} G^{-1/2})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}}, \quad \forall t \geq 0.$$

Gathering the three estimates and using Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}}$$

we deduce

$$\|S_{\mathcal{L}}(t)\|_{L^2(e^{2\epsilon H^\delta} G^{-1/2}) \rightarrow L^2(e^{\epsilon H^\delta} G^{-1/2})} \lesssim 1, \quad \forall t \geq 0.$$

In the following, we denote $f_t = S_{\mathcal{L}}(t)f_0$ the solution to the evolution equation $\partial_t f = \mathcal{L}f$, $f(0, \cdot) = f_0$. Taking $2\delta = \gamma$, ϵ small enough, we have in particular

$$\int f_t^2 G^{-1} e^{\epsilon H^{\frac{\gamma}{2}}} \leq C \int f_0^2 G^{-1} e^{2\epsilon H^{\frac{\gamma}{2}}} =: Y_3.$$

We define

$$Y_2(t) := ((f, f)),$$

with $((,))$ is defined in Theorem 2.1. Thanks to the result in (2.2), we have

$$\begin{aligned} \frac{d}{dt}Y_2 &\leq -a \int f_t^2 G^{-1} \langle x \rangle^{2(\gamma-1)} \\ &\leq -a \int_{B_{|x|\leq\rho}} f_t^2 G^{-1} \langle x \rangle^{2(\gamma-1)}, \end{aligned}$$

for any $\rho \geq 0$, using the same argument as Lemma 3.3, we deduce

$$\begin{aligned} Y_2(t) &\leq C e^{-a\langle \rho \rangle^{2(\gamma-1)}t} Y_2(0) + C e^{-\epsilon_2 \langle \rho \rangle^\gamma} Y_3 \\ &\lesssim (e^{-a\langle \rho \rangle^{2(\gamma-1)}t} + e^{-\epsilon_2 \langle \rho \rangle^\gamma}) Y_3. \end{aligned}$$

Choosing ρ such that $a\langle \rho \rangle^{2(\gamma-1)}t = \epsilon_2 \langle \rho \rangle^\gamma$, that is $\langle \rho \rangle^{2-\gamma} = Ct$, we conclude

$$Y_2(t) \leq C_1 e^{-C_2 t^{\gamma/(2-\gamma)}} Y_3,$$

for some constants $C_i > 0$. As $H^{\frac{\gamma}{2}} \lesssim C(\frac{v^2}{2} + V(x))$, we have

$$e^{\epsilon H^{\frac{\gamma}{2}}} \leq G^{-C\epsilon},$$

Taking ϵ small, the proof of Theorem 3.1 is done. \square

4. REGULARIZATION PROPERTY OF $S_{\mathcal{B}}$

In this section we will denote $\mathcal{L}^* = \mathcal{L}_{G^{-1/2}}^* = \mathcal{S} - \mathcal{T}$ be the dual operator of \mathcal{L} on $L^2(G^{-1/2})$. In other words, L^* is defined by the identity

$$\int (\mathcal{L}f)gG^{-1} = \int (\mathcal{L}^*g)fG^{-1}.$$

for any smooth function f, g . We also denote $\mathcal{B}^* = \mathcal{L}^* - K\chi_R$. The aim of this section is to establish the following regularization property. The proof closely follows the proof of similar results in [10, 14, 21]

Theorem 4.1. *For any $0 \leq \delta < 1$, there exist $\eta > 0$ such that*

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

Similarly, for any $0 \leq \delta < 1$, there exist $\eta > 0$ such that

$$\|\mathcal{S}_{\mathcal{B}^*}(t)f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

As a consequence, for any $0 \leq \delta < 1$, there exist $\eta > 0$ such that

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^\infty(G^{-1/2})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^2(G^{-1/2})}, \quad \forall t \in [0, \eta].$$

We start with some elementary lemmas.

Lemma 4.2. *For any $0 \leq \delta < 1$, we have*

$$\begin{aligned} \int (f(\mathcal{L}g) + g(\mathcal{L}f))G^{-(1+\delta)} &= -2 \int \nabla_v(fG^{-1}) \cdot \nabla_v(gG^{-1})G^{1-\delta} \\ (4.1) \qquad \qquad \qquad &+ \int (\delta d - \delta(1-\delta)v^2)f g G^{-(1+\delta)} \end{aligned}$$

in particular, this implies

$$(4.2) \quad \begin{aligned} \int f(\mathcal{L}f)G^{-(1+\delta)} &= - \int |\nabla_v(fG^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2} \int f^2 G^{-(1+\delta)} \\ &- \frac{\delta(1-\delta)}{2} \int v^2 f^2 G^{-(1+\delta)}, \end{aligned}$$

similarly, for any $0 \leq \delta < 1$, we have

$$(4.3) \quad \begin{aligned} \int f(\mathcal{L}f)G^{-(1+\delta)} &= - \int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2} \int v^2 f^2 G^{-(1+\delta)} \\ &+ \frac{(2+\delta)d}{2} \int f^2 G^{-(1+\delta)}. \end{aligned}$$

All the equalities remain true when \mathcal{L} is replaced by \mathcal{L}^* .

Proof of Lemma 4.2. Recall $\mathcal{T}(G^{-(1+\delta)}) = 0$, we have

$$\int f(\mathcal{T}g)G^{-(1+\delta)} = \int \mathcal{T}(fG^{-(1+\delta)})g = - \int (\mathcal{T}f)gG^{-(1+\delta)},$$

which implies

$$\int f(\mathcal{T}g)G^{-(1+\delta)} + \int (\mathcal{T}f)gG^{-(1+\delta)} = 0.$$

for the term with operator \mathcal{S} we have

$$\begin{aligned} \int f(\mathcal{S}g)G^{-(1+\delta)} &= - \int \nabla_v(fG^{-(1+\delta)}) \cdot (\nabla_v g + vg) \\ &= - \int (\nabla_v f + (1+\delta)vf) \cdot (\nabla_v g + vg)G^{-(1+\delta)} \\ &= - \int \nabla_v(fG^{-1}) \cdot \nabla_v(gG^{-1})G^{1-\delta} \\ &\quad - \int (\delta v^2 fg + \delta fv \cdot \nabla_v g)G^{-(1+\delta)}, \end{aligned}$$

using integration by parts

$$\begin{aligned} \int \delta fv \cdot \nabla_v g G^{-(1+\delta)} &= - \int \delta g \nabla_v \cdot (vfG^{-(1+\delta)}) \\ &= - \int \delta gv \cdot \nabla_v f G^{-(1+\delta)} \\ &\quad - \int (\delta d + \delta(1+\delta)v^2) fg G^{-(1+\delta)}, \end{aligned}$$

so we deduce

$$\begin{aligned} &\int (f(\mathcal{S}g) + g(\mathcal{S}f))G^{-(1+\delta)} \\ &= -2 \int \nabla_v(fG^{-1}) \cdot \nabla_v(gG^{-1})G^{1-\delta} + \int (\delta d - \delta(1-\delta)v^2) fg G^{-(1+\delta)}, \end{aligned}$$

so (4.1) and (4.2) are thus proved by combining the two terms above. Finally, we compute

$$\begin{aligned}
& \int f \mathcal{S} f G^{-(1+\delta)} \\
&= - \int (\nabla_v f + (1+\delta)vf) \cdot (\nabla_v f + vf) G^{-(1+\delta)} \\
&= - \int |\nabla_v f|^2 G^{-(1+\delta)} - \int (1+\delta)v^2 f^2 G^{-(1+\delta)} - \int (2+\delta)fv \cdot \nabla_v f G^{-(1+\delta)} \\
&= - \int |\nabla_v f|^2 G^{-(1+\delta)} - \int (1+\delta)v^2 f^2 G^{-(1+\delta)} + \frac{2+\delta}{2} \int \nabla_v \cdot (vG^{-(1+\delta)}) f^2 \\
&= - \int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2} \int v^2 f^2 G^{-(1+\delta)} + \frac{(2+\delta)d}{2} \int f^2 G^{-(1+\delta)},
\end{aligned}$$

so (4.3) follows by putting together the above equality with

$$\int f \mathcal{T} f G^{-(1+\delta)} = 0.$$

Since the term associated with \mathcal{T} is 0, by $\mathcal{L} = \mathcal{S} + \mathcal{T}$, $\mathcal{L}^* = \mathcal{S} - \mathcal{T}$, we know the same equalities will remain true when \mathcal{L} is replaced by \mathcal{L}^* . \square

Lemma 4.3. *When $f_t = S_{\mathcal{B}}(t)f_0$, define an energy functional*

$$\begin{aligned}
\mathcal{F}(t, f_t) &:= A \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2 \|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\
(4.4) \quad &+ 2ct^4 (\nabla_v f_t, \nabla_x f_t)_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2,
\end{aligned}$$

when $f_t = S_{\mathcal{B}^*}(t)f_0$, define another energy functional

$$\begin{aligned}
\mathcal{F}^*(t, f_t) &:= A \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2 \|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\
(4.5) \quad &- 2ct^4 (\nabla_v f_t, \nabla_x f_t)_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2,
\end{aligned}$$

with $a, b, c > 0$, $c \leq \sqrt{ab}$ and A large enough. Then for both cases we have

$$\frac{d}{dt} F(t, f_t) \leq -L (\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + t^4 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2) + \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2,$$

for all $t \in [0, \eta]$, for some $L > 0, C > 0$ and $F = \mathcal{F}$ or \mathcal{F}^* .

Proof of Lemma 4.3. We only prove the case $F = \mathcal{F}$, the proof for $F = \mathcal{F}^*$ is the same. We split the computation into several parts and then put them

together. First using (4.2) and (4.3) we have

$$\begin{aligned}
& \frac{d}{dt} \int f^2 G^{-(1+\delta)} \\
&= \int f(\mathcal{L} - K\chi_R) f G^{-(1+\delta)} \\
&= \frac{1-\delta}{2} \int f \mathcal{L} f G^{-(1+\delta)} + \frac{1+\delta}{2} \int f \mathcal{L} f G^{-(1+\delta)} - \int K\chi_R f^2 G^{-(1+\delta)} \\
&\leq -\frac{1-\delta}{2} \int |\nabla_v f|^2 G^{-(1+\delta)} - \frac{1+\delta}{2} \int |\nabla_v (fG^{-1})|^2 G^{1-\delta} + C \int f^2 G^{-(1+\delta)} \\
&\leq -\frac{1-\delta}{2} \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int f^2 G^{-(1+\delta)}.
\end{aligned}$$

By

$$(4.6) \quad \partial_{x_i} \mathcal{L} f = \mathcal{L} \partial_{x_i} f + \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f,$$

and (4.2) we have

$$\begin{aligned}
& \frac{d}{dt} \int (\partial_{x_i} f)^2 G^{-(1+\delta)} \\
&= \int \partial_{x_i} f \partial_{x_i} (\mathcal{L} - K\chi_R) f G^{-(1+\delta)} \\
&= -\int |\nabla_v (\partial_{x_i} f G^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2} \int (\partial_{x_i} f)^2 G^{-(1+\delta)} \\
&\quad - \frac{\delta(1-\delta)}{2} \int v^2 (\partial_{x_i} f)^2 G^{-(1+\delta)} + \int \partial_{x_i} f \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f G^{-(1+\delta)} \\
&\quad - \int K\chi_R |\partial_{x_i} f|^2 G^{-(1+\delta)} - \int K \partial_{x_i} f \partial_{x_i} \chi_R f G^{-(1+\delta)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing up by i , we get

$$\begin{aligned}
& \frac{d}{dt} \int |\nabla_x f|^2 G^{-(1+\delta)} \\
&\leq -\sum_{i=1}^d \int |\nabla_v (\partial_{x_i} f G^{-1})|^2 G^{1-\delta} - \frac{\delta(1-\delta)}{2} \int v^2 (\nabla_x f)^2 G^{-(1+\delta)} \\
&\quad + C \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int |\nabla_x f|^2 G^{-(1+\delta)} + C \int |f|^2 G^{-(1+\delta)},
\end{aligned}$$

for some $C > 0$. Similarly using

$$(4.7) \quad \partial_{v_i} \mathcal{L} f = \mathcal{L} \partial_{v_i} f - \partial_{x_i} f + \partial_{v_i} f,$$

and (4.2), we have

$$\begin{aligned}
& \frac{d}{dt} \int (\partial_{v_i} f)^2 G^{-(1+\delta)} \\
&= \int \partial_{v_i} f \partial_{v_i} (\mathcal{L} - K\chi_R) f G^{-(1+\delta)} \\
&= - \int |\nabla_v (\partial_{v_i} f G^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2} \int (\partial_{v_i} f)^2 G^{-(1+\delta)} \\
&\quad - \frac{\delta(1-\delta)}{2} \int v^2 (\partial_{v_i} f)^2 G^{-(1+\delta)} - \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)} \\
&\quad + \int |\partial_{v_i} f|^2 G^{-(1+\delta)} - \int K\chi_R |\partial_{v_i} f|^2 G^{-(1+\delta)} - \int K \partial_{v_i} f \partial_{v_i} \chi_R f G^{-(1+\delta)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing up by i we get

$$\begin{aligned}
& \frac{d}{dt} \int |\nabla_v f|^2 G^{-(1+\delta)} \\
&\leq - \sum_{i=1}^d \int |\nabla_v (\partial_{v_i} f G^{-1})|^2 G^{1-\delta} + C \int |\nabla_x f| |\nabla_v f| G^{-(1+\delta)} \\
&\quad + C \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int |f|^2 G^{-(1+\delta)} - \frac{\delta(1-\delta)}{2} \int v^2 (\nabla_v f)^2 G^{-(1+\delta)}.
\end{aligned}$$

For the crossing term, we split it also into two parts

$$\begin{aligned}
& \frac{d}{dt} \int 2\partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} \\
&= \left(\int \partial_{v_i} f \partial_{x_i} \mathcal{L} f G^{-(1+\delta)} + \int \partial_{v_i} \mathcal{L} f \partial_{x_i} f G^{-(1+\delta)} \right) \\
&\quad - \left(\int \partial_{v_i} f \partial_{x_i} (K\chi_R f) G^{-(1+\delta)} + \int \partial_{x_i} (K\chi_R f) \partial_{v_i} f G^{-(1+\delta)} \right) \\
&:= W_1 + W_2.
\end{aligned}$$

Using (4.6) and (4.7) we have

$$\begin{aligned}
W_1 &= \int \partial_{v_i} f \mathcal{L} (\partial_{x_i} f) G^{-(1+\delta)} + \int \mathcal{L} (\partial_{v_i} f) \partial_{x_i} f G^{-(1+\delta)} \\
&\quad + \int \partial_{v_i} f \sum_{j=1}^d \partial_{x_i x_j} V(x) \partial_{v_j} f G^{-(1+\delta)} - \int |\partial_{x_i} f|^2 G^{-(1+\delta)} \\
&\quad + \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)}.
\end{aligned}$$

By (4.1), we deduce

$$\begin{aligned}
W_1 &= - \int 2\nabla_v(\partial_{v_i} f G^{-1}) \cdot \nabla_v(\partial_{x_i} f G^{-1}) G^{1-\delta} + \delta d \int \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} \\
&\quad - \delta(1-\delta) \int v^2 \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} + \int \partial_{v_i} f \sum_{j=1}^d \partial_{x_i x_j} V(x) \partial_{v_j} f G^{-(1+\delta)} \\
&\quad - \int |\partial_{x_i} f|^2 G^{-(1+\delta)} + \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)}.
\end{aligned}$$

For the W_2 term we have

$$\begin{aligned}
W_2 &= - \int \partial_{v_i} f \partial_{x_i} (K \chi_R f) G^{-(1+\delta)} - \int \partial_{x_i} (K \chi_R f) \partial_{v_i} f G^{-(1+\delta)} \\
&= - \int 2K \chi_R \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)} + \int K f (\partial_{v_i} \chi_R \partial_{x_i} f + \partial_{v_i} f \partial_{x_i} \chi_R) G^{-(1+\delta)} \\
&\leq C \int |\partial_{x_i} f| |\partial_{v_i} f| G^{-(1+\delta)} + C \int |\partial_{v_i} f| |f| G^{-(1+\delta)} + C \int |f| |\partial_{x_i} f| G^{-(1+\delta)},
\end{aligned}$$

Combining the two parts, using Cauchy-Schwarz inequality, and summing up by i we get

$$\begin{aligned}
&\frac{d}{dt} \int 2\nabla_x f \cdot \nabla_v f G^{-(1+\delta)} \\
&\leq - \sum_{i=1}^d \int 2\nabla_v(\partial_{v_i} f G^{-1}) \cdot \nabla_v(\partial_{x_i} f G^{-1}) G^{1-\delta} - \frac{1}{2} \int |\nabla_x f|^2 G^{-(1+\delta)} \\
&\quad + C \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int |f|^2 G^{-(1+\delta)} - \delta(1-\delta) \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}.
\end{aligned}$$

For the very definition of \mathcal{F} in (4.4), we easily compute

$$\begin{aligned}
&\frac{d}{dt} \mathcal{F}(t, f_t) \\
&= A \frac{d}{dt} \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2 \frac{d}{dt} \|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\
&\quad + 2ct^4 \frac{d}{dt} \langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6 \frac{d}{dt} \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\
&\quad + 2at \|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + 8ct^3 \langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 \\
&\quad + 6bt^5 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2.
\end{aligned}$$

Gathering all the inequalities above together, we have

$$\begin{aligned}
& \frac{d}{dt} \mathcal{F}(t, f_t) \\
\leq & \left(2at - \frac{A(1-\delta)}{2} + Cat^2 + 2Ct^4c + Cbt^6 \right) \int |\nabla_v f_t|^2 G^{-(1+\delta)} \\
& + (6bt^5 - \frac{c}{2}t^4 + Cbt^6) \int |\nabla_x f_t|^2 G^{-(1+\delta)} + (8ct^3 + Cat^2) \int |\nabla_v f_t| |\nabla_x f_t| G^{-(1+\delta)} \\
& - (at^2 \sum_{i=1}^d \int |\nabla_v(\partial_{v_i} f_t G^{-1})|^2 G^{1-\delta} + bt^6 \sum_{i=1}^d \int |\nabla_v(\partial_{x_i} f_t G^{-1})|^2 G^{1-\delta} \\
& + 2ct^4 \sum_{i=1}^d \int \nabla_v(\partial_{v_i} f_t G^{-1}) \cdot \nabla_v(\partial_{x_i} f_t G^{-1}) G^{1-\delta}) - \frac{\delta(1-\delta)}{2} (at^2 \int v^2 (\nabla_v f)^2 G^{-(1+\delta)} \\
& + bt^6 \int v^2 (\nabla_x f)^2 G^{-(1+\delta)} + 2ct^4 \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}) + C \int f_t^2 G^{-(1+\delta)},
\end{aligned}$$

for some $C > 0$. We observe that

$$\begin{aligned}
& |2ct^4 \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}| \\
\leq & at^2 \int v^2 (\nabla_v f)^2 G^{-(1+\delta)} + bt^6 \int v^2 (\nabla_x f)^2 G^{-(1+\delta)},
\end{aligned}$$

and

$$\begin{aligned}
& |2ct^4 \sum_{i=1}^d \int 2 \nabla_v(\partial_{v_i} f_t G^{-1}) \cdot \nabla_v(\partial_{x_i} f_t G^{-1}) G^{1-\delta}| \\
\leq & at^2 \sum_{i=1}^d \int |\nabla_v(\partial_{v_i} f_t G^{-1})|^2 G^{1-\delta} + bt^6 \sum_{i=1}^d \int |\nabla_v(\partial_{x_i} f_t G^{-1})|^2 G^{1-\delta}.
\end{aligned}$$

by our choice on a, b, c . So by taking A large and $0 < \eta$ small ($t \in [0, \eta]$), as a consequence

$$\frac{d}{dt} \mathcal{F}(t, f_t) \leq -L(\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + t^4 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2) + C\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2,$$

for some $L, C > 0$, and that ends the proof. \square

Remark 4.4. For the case $F = \mathcal{F}^*$, the only difference in the proof is to change (4.6) and (4.7) into

$$\partial_{x_i} \mathcal{L}^* f = \mathcal{L}^* \partial_{x_i} f - \partial_{x_i} (\nabla_x V(x) \cdot \nabla_v f) = \mathcal{L}^* \partial_{x_i} f - \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f,$$

and

$$\partial_{v_i} \mathcal{L}^* f = \mathcal{L}^* \partial_{v_i} f + \partial_{x_i} f + \partial_{v_i} f.$$

The following proof of this section is true for both cases.

Lemma 4.5. *For any $0 \leq \delta < 1$, we have*

$$\int |\nabla_{x,v}(fG^{-1/2(1+\delta)})|^2 \leq \int |\nabla_{x,v}f|^2 G^{-(1+\delta)} + C \int f^2 G^{-(1+\delta)},$$

Prove of Lemma 4.5. For any weight function m we have

$$\begin{aligned} \int |\nabla_x(fm)|^2 &= \int |\nabla_x fm + \nabla_x mf|^2 \\ &= \int |\nabla_x f|^2 m^2 + \int |\nabla_x m|^2 f^2 + \int 2f \nabla_x f m \nabla_x m \\ &= \int |\nabla_x f|^2 m^2 + \int (|\nabla_x m|^2 - \frac{1}{2} \Delta_x(m^2)) f^2, \end{aligned}$$

taking $m = G^{-1/2(1+\delta)}$ we have

$$\begin{aligned} &\int |\nabla_x(fG^{-1/2(1+\delta)})|^2 \\ &= \int |\nabla_x f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4} |\nabla_x V(x)|^2 + \frac{1+\delta}{2} \Delta_x V(x)) f^2 G^{-(1+\delta)} \\ &\leq \int |\nabla_x f|^2 G^{-(1+\delta)} + C \int f^2 G^{-(1+\delta)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\int |\nabla_v(fG^{-1/2(1+\delta)})|^2 \\ &= \int |\nabla_v f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4} v^2 + \frac{1+\delta}{2} d) f^2 G^{-(1+\delta)} \\ &\leq \int |\nabla_v f|^2 G^{-(1+\delta)}. \end{aligned}$$

Putting together the two inequalities we obtain the result. \square

Lemma 4.6. *Nash's inequality: for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, there exist a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}} \leq C_d \|f\|_{L^1}^{2/d} \|\nabla_v f\|_{L^2},$$

For the proof of Nash's inequality, we refer to [13], Section 8.13 for instance. \square

Lemma 4.7. *For any $0 \leq \delta < 1$ we have*

$$(4.8) \quad \frac{d}{dt} \int |f| G^{-1/2(1+\delta)} \leq d \int |f| G^{-1/2(1+\delta)},$$

which implies

$$\int |f_t| G^{-1/2(1+\delta)} \leq C e^{dt} \int |f_0| G^{-1/2(1+\delta)}.$$

In particular we have

$$(4.9) \quad \int |f_t|G^{-1/2(1+\delta)} \leq C \int |f_0|G^{-1/2(1+\delta)}, \quad \forall t \in [0, \eta],$$

for some constant $C > 0$.

Proof of Lemma 4.7. By Lemma 5.2 in the next section, letting $p = 1$, we have

$$\begin{aligned} & \frac{d}{dt} \int |f|G^{-1/2(1+\delta)} \\ &= \int |f|(\Delta_v G^{-1/2(1+\delta)} - v \cdot \nabla_v G^{-1/2(1+\delta)} \\ & \quad + v \cdot \nabla_x G^{-1/2(1+\delta)} - \nabla V(x) \cdot \nabla_v G^{-1/2(1+\delta)} - K\chi_R G^{-1/2(1+\delta)}) \\ & \leq \int |f| \left(\frac{1+\delta}{2}d - \frac{(1+\delta)(1-\delta)}{4}v^2 \right) G^{-1/2(1+\delta)} \leq \int |f|dG^{-1/2(1+\delta)}. \end{aligned}$$

so (4.8) is proved. As $\mathcal{T}G^{-1/2(1+\delta)} = 0$, the result is still true when $F = \mathcal{F}^*$. \square

Now we come to the proof of Theorem 4.1.

Proof of Theorem 4.1. We define

$$\mathcal{G}(t, f_t) = B\|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + t^Z \mathcal{F}(t, f_t),$$

with $B, Z > 0$ to be fixed and \mathcal{F} defined in Lemma 4.3. We choose $t \in [0, \eta]$, η small such that $(a+b+c)Z\eta^{Z+1} \leq \frac{1}{2}L\eta^Z$ (a, b, c, L are also defined Lemma 4.3), by (4.8) and Lemma 4.3 we have

$$\begin{aligned} \frac{d}{dt} \mathcal{G}(t, f_t) & \leq dB\|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + Zt^{Z-1}\mathcal{F}(t, f_t) \\ & \quad - Lt^Z (\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + t^4 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2) \\ & \quad + Ct^Z \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\ & \leq dB\|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + Ct^{Z-1}\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\ & \quad - \frac{L}{2}t^Z (\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + t^4 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2). \end{aligned}$$

Nash's inequality and Lemma 4.3 implies

$$\begin{aligned} \int f_t^2 G^{-(1+\delta)} & \leq \left(\int |f_t|G^{-1/2(1+\delta)} \right)^{\frac{4}{d+2}} \left(\int |\nabla_{x,v}(f_t G^{-1/2(1+\delta)})|^2 \right)^{\frac{d}{d+2}} \\ & \leq \left(\int |f_t|G^{-1/2(1+\delta)} \right)^{\frac{4}{d+2}} \left(\int |\nabla_{x,v} f_t|^2 G^{-(1+\delta)} + C \int f_t^2 G^{-(1+\delta)} \right)^{\frac{d}{d+2}}. \end{aligned}$$

Using Young's inequality, we have

$$\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \leq C\epsilon t^{-5d} \|f\|_{L^1(G^{-1/2(1+\delta)})}^2 + \epsilon t^5 (\|\nabla_{x,v} f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + C\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2).$$

Taking ϵ small such that $C\epsilon\eta^5 \leq \frac{1}{2}$, we deduce

$$\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \leq 2C\epsilon t^{-5d} \|f\|_{L^1(G^{-1/2(1+\delta)})}^2 + 2\epsilon t^5 \|\nabla_{x,v} f_t\|_{L^2(G^{-1/2(1+\delta)})}^2.$$

Taking ϵ small we have

$$\frac{d}{dt} \mathcal{G}(t, f_t) \leq dB \|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + C_1 t^{Z-1-5d} \|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2,$$

for some $C_1 > 0$. Choosing $Z = 1 + 5d$, and using (4.9), we deduce

$$\forall t \in [0, \eta], \quad \mathcal{G}(t, f_t) \leq \mathcal{G}(0, f_0) + C_2 \|f_0\|_{L^1(G^{-1/2(1+\delta)})}^2 \leq C_3 \|f_0\|_{L^1(G^{-1/2(1+\delta)})}^2,$$

which ends the proof. \square

5. $\mathcal{S}_{\mathcal{B}}$ DECAY IN LARGER SPACES

The aim of this section is to prove the following decay estimate for the semigroup $\mathcal{S}_{\mathcal{B}}$ which will be useful in the last section where we will prove Theorem 1.1 in full generality.

Theorem 5.1. *Let $H = 1 + x^2 + 2v \cdot x + 3v^2$, for any $\theta \in (0, 1)$ and for any $l > 0$, we have*

$$\|\mathcal{S}_{\mathcal{B}}(t)\|_{L^1(H^l) \rightarrow L^1(H^{l\theta})} \lesssim (1+t)^{-a},$$

where

$$a = \frac{l(1-\theta)}{1 - \frac{\gamma}{2}}.$$

We start with an elementary identity.

Lemma 5.2. *For the kinetic Fokker Planck operator \mathcal{L} , let m be a weight function, for any $p \in [1, \infty]$ we have*

$$\int (\mathcal{L}f) f^{p-1} m^p = -(p-1) \int |\nabla_v(mf)|^2 (mf)^{p-2} + \int f^p m^p \phi,$$

with

$$\phi = \frac{2}{p'} \frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p} - 1\right) \frac{\Delta_v m}{m} + \frac{d}{p'} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

In particular when $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

Proof of Lemma 5.2. We split the integral as

$$\int (\mathcal{L}f) f^{p-1} m^p = \int f^{p-1} \mathcal{S}f m^p + \int f^{p-1} \mathcal{T}f m^p.$$

First compute the contribution of the term with operator \mathcal{T}

$$\int f^{p-1} \mathcal{T}f m^p = \frac{1}{p} \int \mathcal{T}(f^p) m^p = - \int f^p m^p \frac{\mathcal{T}m}{m}.$$

Concerning the term with operator \mathcal{S} , we split it also into two parts

$$\int (\mathcal{S}f) f^{p-1} m^p = \int f^{p-1} m^p (\Delta_v f + \text{div}_v(vf)) := C_1 + C_2.$$

We first compute the C_2 term, we have

$$\begin{aligned} C_2 &= \int f^{p-1} m^p (df + v \cdot \nabla_v f) \\ &= \int df^p m^p - \frac{1}{p} \int f^p \operatorname{div}_v (v m^p) \\ &= \int f^p \left[\left(1 - \frac{1}{p}\right) d - v \cdot \frac{\nabla_v m}{m} \right] m^p. \end{aligned}$$

We turn to the C_1 term, we have

$$\begin{aligned} C_1 &= \int f^{p-1} m^p \Delta_v f = - \int \nabla_v (f^{p-1} m^p) \cdot \nabla_v f \\ &= \int -(p-1) |\nabla_v f|^2 f^{p-2} m^p - \frac{1}{p} \int \nabla_v f^p \cdot \nabla_v m^p. \end{aligned}$$

Using $\nabla_v(mf) = m \nabla_v f + f \nabla_v m$, we deduce

$$\begin{aligned} C_1 &= -(p-1) \int |\nabla_v(mf)|^2 f^p m^{p-2} + (p-1) \int |\nabla_v m|^2 f^p m^{p-2} \\ &\quad + \frac{2(p-1)}{p^2} \int \nabla_v(f^p) \cdot \nabla_v(m^p) - \frac{1}{p} \int \nabla_v(f^p) \cdot \nabla_v(m^p) \\ &= -(p-1) \int |\nabla_v(mf)|^2 f^{p-2} m^p + (p-1) \int |\nabla_v m|^2 f^p m^{p-2} \\ &\quad + \frac{p-2}{p^2} \int f^p \Delta_v m^p. \end{aligned}$$

Using that $\Delta_v m^p = p \Delta_v m m^{p-1} + p(p-1) |\nabla_v m|^2 m^{p-2}$, we obtain

$$C_1 = -(p-1) \int |\nabla_v(mf)|^2 f^{p-2} m^{p-2} + \int f^p m^p \left[\left(\frac{2}{p} - 1\right) \frac{\Delta_v m}{m} + 2\left(1 - \frac{1}{p}\right) \frac{|\nabla_v m|^2}{m^2} \right].$$

We conclude by combining the above equalities. \square

Proof of Theorem 5.1. From Lemma 5.2, we have

$$\begin{aligned} (5.1) \quad \int (\mathcal{B}f) f^{p-1} m^p &= \int (\mathcal{L} - M\chi_R) f^{p-1} m^p \\ &= -(p-1) \int |\nabla_v(mf)|^2 (mf)^{p-2} + \int f^p m^p \phi, \end{aligned}$$

with

$$\phi = \left[\frac{2}{p'} \frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p} - 1\right) \frac{\Delta_v m}{m} + \frac{d}{p'} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R \right].$$

When $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R.$$

Let $m = H^k$. We have

$$\frac{\nabla_v m}{m} = k \frac{\nabla_v H}{H}, \quad \frac{\nabla_x m}{m} = k \frac{\nabla_x H}{H},$$

and

$$\frac{\Delta_v m}{m} = \frac{k\Delta_v H}{H} + \frac{k(k-1)|\nabla_v H|^2}{H^2}.$$

Summing up, we have for ϕ

$$\frac{\phi H}{k} = \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} - v \cdot \nabla_v H + v \cdot \nabla_x H - \nabla_x V(x) \cdot \nabla_v H - M\chi_R,$$

From the very definition of H , we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6.$$

We then compute

$$\begin{aligned} & \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} + v \cdot \nabla_x H - v \cdot \nabla_v H - \nabla_x V(x) \cdot \nabla_v H \\ &= 6 + (k-1)\frac{(6v+2x)^2}{H} + 2v^2 + 2x \cdot v - 6v^2 \\ & \quad - 2x \cdot v - 6v \cdot \nabla_x V(x) - 2x \cdot \nabla_x V(x) \\ & \leq (2v^2 + Cv - 6v^2) - 2x \cdot \nabla_x V(x) + C \\ & \leq -C_1 v^2 - C_2 x \cdot \nabla_x V(x) + C_3 \\ & \leq -C_4 H^{\frac{7}{2}} + K_1 \chi_{R_1}, \end{aligned}$$

for some $C_i > 0$. Taking K and R large enough, we have $\phi \leq -CH^{\frac{7}{2}-1}$, using this inequality in equation (5.1), we deduce

$$(5.2) \quad \begin{aligned} \frac{d}{dt} Y_4(t) &:= \frac{d}{dt} \int |f_{\mathcal{B}}(t)| H^k = \int \text{sign}(f_{\mathcal{B}}(t)) \mathcal{B} f_{\mathcal{B}}(t) H^k \\ &\leq -C \int |f_{\mathcal{B}}(t)| H^{k-1+\frac{7}{2}}, \end{aligned}$$

for any $k > 1$. In particular for any $l \geq 1$, we can find K and R large enough such that

$$\frac{d}{dt} \int |f_{\mathcal{B}}(t)| H^l \leq 0,$$

which readily implies

$$\int |f_{\mathcal{B}}(t)| H^l \leq \int |f_0| H^l := Y_5.$$

Denoting

$$\alpha = \frac{l-k}{l-k+1-\frac{7}{2}} \in (0, 1),$$

the Hölder's inequality

$$\int |f_{\mathcal{B}}(t)| H^k \leq \left(\int |f_{\mathcal{B}}(t)| H^{k-1+\frac{7}{2}} \right)^\alpha \left(\int |f_{\mathcal{B}}(t)| H^l \right)^{1-\alpha},$$

implies

$$\left(\int |f_{\mathcal{B}}(t)| H^k \right)^\frac{1}{\alpha} \left(\int |f_{\mathcal{B}}(t)| H^l \right)^\frac{\alpha-1}{\alpha} \leq \int |f_{\mathcal{B}}(t)| H^{k-1+\frac{7}{2}},$$

From this inequality and (5.2), we get

$$\frac{d}{dt}Y_4(t) \leq -C(Y_4(t))^{\frac{1}{\alpha}}Y_5^{\frac{\alpha-1}{\alpha}}.$$

Using $Y_4(0) \leq Y_5$, after an integration, we deduce

$$Y_4(t) \leq C_\alpha \frac{1}{(1+t)^{\frac{\alpha}{1-\alpha}}} Y_5,$$

which is nothing but the polynomial decay on $S_{\mathcal{B}}$

$$\|\mathcal{S}_{\mathcal{B}}(t)\|_{L^p(H^l) \rightarrow L^p(H^k)} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l-k}{1-\frac{\gamma}{2}}, \quad \forall 0 < k < l, \quad 1 \leq l.$$

We conclude Theorem 5.1 by writing $k = l\theta$, $0 < \theta < 1$.

6. L^p CONVERGENCE FOR THE KFP MODEL

Before going to the proof of our main theorem, we need two last deduced results.

Lemma 6.1. *For any $\epsilon > 0$ small enough, we have*

$$\|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

and

$$\|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

for some $a > 0$. Similarly for any $0 < b < \frac{\gamma}{2-\gamma}$ and for any $\epsilon > 0$ small enough, we have

$$\|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} e^{-at^b}, \quad \forall t \geq 0,$$

for $\alpha = \frac{5d+1}{2}$ and some $a > 0$.

Proof of Lemma 6.1. The first two inequalities are obtained obviously by Lemma 3.3 and the property of $\mathcal{A} = M\chi_R$. For the third inequality we split it into two parts, $t \in [0, \eta]$ and $t > \eta$, where η is defined in Theorem 4.1.

When $t \in [0, \eta]$, we have $e^{-at^{\frac{\gamma}{2-\gamma}}} \geq e^{-a\eta^{\frac{\gamma}{2-\gamma}}}$, by Theorem 4.1, we have

$$\|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} \lesssim t^{-\alpha} e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \in [0, \eta],$$

for some $a > 0$. When $t \geq \eta$, by Theorem 4.1, we have

$$\|\mathcal{S}_{\mathcal{B}}(\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim \eta^\alpha \lesssim 1,$$

and by Lemma 3.3

$$\|\mathcal{S}_{\mathcal{B}}(t-\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-\frac{1}{2}})} \lesssim e^{-a(t-\eta)^{\frac{\gamma}{2-\gamma}}} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

gathering the two inequalities, we have

$$\|\mathcal{A}S_B(t)\|_{L^1(G^{-1/2(1+2\epsilon)}) \rightarrow L^2(G^{-1/2(1+2\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}} \lesssim t^{-\alpha} e^{-at^b}, \quad \forall t > \eta,$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, the proof is ended by combining the two cases above. \square

Lemma 6.2. *Similarly as Lemma 6.1. For any $p \in (2, \infty)$, we have*

$$\|S_B(t)\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^2(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0.$$

and

$$\|S_B(t)\mathcal{A}\|_{L^p(G^{-1/2}) \rightarrow L^p(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0.$$

for some $a > 0$. And for any $0 < b < \frac{\gamma}{2-\gamma}$ we have

$$\|S_B(t)\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^p(G^{-1/2})} \lesssim t^{-\beta} e^{-at^b}, \quad \forall t \geq 0.$$

for some $\beta > 0$ and some $a > 0$.

The proof of Lemma 6.2 is similar to the proof of Lemma 6.1 and thus skipped.

Lemma 6.3. *let X, Y be two Banach spaces, $S(t)$ a semigroup such that for all $t \geq 0$ and some $0 < a, 0 < b < 1$ we have*

$$\|S(t)\|_{X \rightarrow X} \leq C_X e^{-at^b}, \quad \|S(t)\|_{Y \rightarrow Y} \leq C_Y e^{-at^b},$$

and for some $0 < \alpha$, we have

$$\|S(t)\|_{X \rightarrow Y} \leq C_{X,Y} t^{-\alpha} e^{-at^b}.$$

Then we can have that for all integer $n > 0$

$$\|S^{(*n)}(t)\|_{X \rightarrow X} \leq C_{X,n} t^{n-1} e^{-at^b},$$

similarly

$$\|S^{(*n)}(t)\|_{Y \rightarrow Y} \leq C_{Y,n} t^{n-1} e^{-at^b},$$

and

$$\|S^{(*n)}(t)\|_{X \rightarrow Y} \leq C_{X,Y,n} t^{n-\alpha-1} e^{-at^b}.$$

In particular for $\alpha + 1 < n$, and for any $b^* < b$

$$\|S^{(*n)}(t)\|_{X \rightarrow Y} \leq C_{X,Y,n} e^{-at^{b^*}}.$$

Proof of Lemma 6.3. The proof is the same as Lemma 2.5 in [15], plus the fact $t^b \leq s^b + (t-s)^b$ for any $0 \leq s \leq t, 0 < b < 1$. \square

Then we come to the final proof.

Proof of Theorem 1.1. We only prove the case when $m = G^{\frac{p-1}{p}(1+\epsilon)}$, $p \in [1, 2]$, for the proof of the other cases, one need only replace the use of Lemma 6.1 in the following proof by Lemma 6.2 and Theorem 4.1. We will prove $p = 1$ first, this time we need to prove

$$\|S_{\mathcal{L}}(I - \Pi)(t)\|_{L^1(G^{-\epsilon}) \rightarrow L^1} \lesssim e^{-at^b},$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, where I is the identity operator and Π is a projection operator defined by

$$\Pi(f) = \mathcal{M}(f)G.$$

First, Iterating the Duhamel's formula we split it into 3 terms

$$\begin{aligned} S_{\mathcal{L}}(I - \Pi) &= (I - \Pi)\{S_{\mathcal{B}} + \sum_{l=1}^{n-1} (S_{\mathcal{B}}\mathcal{A})^{*l} * (S_{\mathcal{B}})\} \\ &\quad + \{(I - \Pi)S_{\mathcal{L}}\} * (\mathcal{A}S_{\mathcal{B}}(t))^{*n}, \end{aligned}$$

and we will estimate them separately. By Lemma 3.3, we have

$$\|S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon}) \rightarrow L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

the first term is thus estimated. For the second term, still using Lemma 3.3, we get

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^1(G^{-\epsilon}) \rightarrow L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

by Lemma 6.3, we have

$$\|(S_{\mathcal{B}}(t)\mathcal{A})^{*l}\|_{L^1(G^{-\epsilon}) \rightarrow L^1} \lesssim t^{l-1}e^{-at^{\frac{\gamma}{2-\gamma}}},$$

thus the second term is estimated. For the last term by Lemma 3.3

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon}) \rightarrow L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

By Lemma 6.1 and 6.3, for any $0 < b < \frac{\gamma}{2-\gamma}$, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{*(n-1)}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{n-\alpha-2}e^{-at^b},$$

finally by Theorem 3.1, we have

$$\|S_{\mathcal{L}}(t)(I - \Pi)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Taking $n > \alpha + 2$ the third term is estimated thus the proof of case $p = 1$ is concluded by gathering the inequalities above. As the case $p = 2$ is already proved in Theorem 3.1, the case $p \in (1, 2)$ follows by interpolation. \square

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