



HAL
open science

THE KINETIC FOKKER-PLANCK EQUATION WITH WEAK CONFINEMENT FORCE

Chuqi Cao

► **To cite this version:**

Chuqi Cao. THE KINETIC FOKKER-PLANCK EQUATION WITH WEAK CONFINEMENT FORCE. 2018. hal-01697058v1

HAL Id: hal-01697058

<https://hal.science/hal-01697058v1>

Preprint submitted on 30 Jan 2018 (v1), last revised 7 Jun 2018 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE KINETIC FOKKER-PLANCK EQUATION WITH WEAK CONFINEMENT FORCE

CHUQI CAO

Version of January 30, 2018

ABSTRACT. We consider the kinetic Fokker-Planck equation with weak confinement force which do not have an exponential convergence to the equilibrium. We proved some (polynomial and sub-exponential) rate of convergence to the equilibrium (depending on the space to which the initial datum belongs). Our results generalized the result in [5] [1] [2] to weak confinement case.

CONTENTS

1.	Introduction	1
2.	L^2 framework: Dirichlet form and rate of convergence estimate	4
3.	L^2 sub-exponential decay for the kinetic Fokker-Planck equation based on a splitting	12
4.	L^1 convergence on $S_{\mathcal{B}}$	17
5.	Regularization Lemma	20
6.	L^p convergence for the KFP model	30
	References	34

1. INTRODUCTION

In this paper, we consider the weak hypocoercivity issue for the kinetic Fokker-Planck(KFP for short) equation

$$(1.1) \quad \partial_t f = \mathcal{L}f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \operatorname{div}_v(vf),$$

for a density function $f = f(t, x, v)$, with $t \geq 0$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$. The evolution equations are complemented with an initial data

$$f(0, \cdot) = f_0 \quad \text{on } \mathbb{R}^{2d}.$$

We make the fundamental assumption on the confinement potential V

$$V(x) = \langle x \rangle^\gamma, \quad \gamma \in (0, 1],$$

where $\langle x \rangle^2 := 1 + |x|^2$.

We make some fundamental observations. One is mass conservative, $M(f_0) = M(f(t, \cdot))$, where we define

$$M(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv.$$

We also observe the existence of steady state G for the KFP models $\mathcal{L}G = 0$, given by $G = Ce^{-W}$, $W = \frac{v^2}{2} + V(x)$, C is the normalizing constant such that

$$M(G) = 1.$$

Also observe that contrary to the case $\gamma \geq 1$, a Poincaré inequality of the type

$$\exists c > 0, \quad \int_{\mathbb{R}^d} |f(x)|^2 \exp(-V(x)) dx \leq c \int_{\mathbb{R}^d} |\nabla f(x)|^2 \exp(-V(x)) dx,$$

for f such that

$$\int_{\mathbb{R}^d} f(x) \exp(-V(x)) dx = 0,$$

does not hold, but only a weaker version of this inequality remains true (see [10], or below section 2). In particular, there is no spectral gap for the associated operator \mathcal{L} , nor is there an exponential trend to the equilibrium for the associated semigroup.

In order to state our main result, we introduce some notations.

We will denote $H = 1 + J + K + Q$, where $J = x^2$, $K = 2v \cdot x$, $Q = 3v^2$. Observe that $\frac{1}{2}(x^2 + v^2 + 1) \leq H \leq 4(x^2 + v^2 + 1)$, which means H is equivalent to $x^2 + v^2 + 1$

For a given weight function m , we will denote $L^p(m) = \{f | fm \in L^p\}$ and $\|f\|_{L^p(m)} = \|fm\|_{L^p}$.

With these notations we can introduce the main result of this paper:

Theorem 1.1. (1) For any initial data $f_0 \in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p \in [1, \infty)$, $\epsilon > 0$ small, the associated solution $f(t, \cdot)$ of the kinetic Fokker-Planck equation (1.1) satisfies

$$\|f(t, \cdot) - M(f_0)G\|_{L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-Ct^b} \|f_0 - M(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})}.$$

$$\forall 0 < b < \frac{\gamma}{2 - \gamma}.$$

(2) Let $m = H^k$, $0 < k$, For any initial data $f_0 \in L^1(m)$, the associated solution $f(t, \cdot)$ of the kinetic Fokker-Planck equation (1.1) satisfies

$$\|f(t, \cdot) - M(f_0)G\|_{L^1(m^\theta)} \lesssim (1+t)^{-a} \|f_0 - M(f_0)G\|_{L^1(m)},$$

$$\forall 0 < \theta < 1, \quad \forall 0 < a < \frac{k(1-\theta)}{1 - \frac{\gamma}{2}}.$$

Remark 1.2. The constant in the theorem does not depend on f_0 , only depends on $\gamma, d, \epsilon, \theta$

Remark 1.3. The Theorem 1.1 also true when V behaves like $\langle x \rangle^\gamma$, that is for any V satisfying

$$C_1 \langle x \rangle^\gamma \leq V(x) \leq C_2 \langle x \rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3 |x| \langle x \rangle^{\gamma-1} \leq x \cdot \nabla_x V(x) \leq C_4 |x| \langle x \rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

and

$$|D_x^2 V(x)| \leq C_5 \langle x \rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

for some constant $C_i > 0$, $R > 0$.

Remark 1.4. There are many classical results on the case $1 \leq \gamma$. In this case there is an exponentially decay proved by Villani [5] and Dolbeault, Mouhot, Schmeiser [1] [2]

For any initial data $f_0 \in L^2(G^{-1/2})$, $f(t, \cdot)$ is the solution of (1.1) correspond to this f_0 , we have:

$$\|f(t, \cdot) - M(f_0)G\|_{L^2(G^{-1/2})} \lesssim e^{-Ct} \|f_0 - M(f_0)G\|_{L^2(G^{-1/2})}.$$

Let us briefly explain the main ideas behind our method of proof.

We first introduce four spaces $E_1 = L^2(G^{-1/2})$, $E_2 = L^2(G^{-1/2}e^{\epsilon_1 V(x)})$, $E_3 = L^2(G^{-(1+\epsilon_2)/2})$ and $E_0 = L^2(G^{-1/2}\langle x \rangle^{\gamma-1})$ (remember $\gamma \in (0, 1)$), take $\epsilon_1 > 0$ and $\epsilon_2 > 0$ small such that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset L^2$, and E_1 is an exponentially interpolation between E_0 and E_2 . We first use an argument as in [1][2] to prove, for any $f_0 \in E_3$, the solution to the KFP equation 1.1 satisfies

$$\frac{d}{dt} \|f(t)\|_{E_1} \leq -\lambda \|f(t)\|_{E_0},$$

for some constant $\lambda > 0$. We use this and Duhamel formula to prove

$$\|f(t)\|_{E_2} \lesssim \|f_0\|_{E_3}.$$

Combine the two inequalities and using a exponentially interpolation argument as in [3], we can have

$$(1.2) \quad \|f(t)\|_{E_1} \lesssim e^{-at^b} \|f_0\|_{E_3},$$

for some $a > 0$, $b \in (0, 1)$.

We then generalize the decay estimate to a wider class of Banach spaces by adapting the extension theory introduced in [11] and developed in [4] [12]. We first introduce a notation.

If \mathcal{T}_i with $i = 1, 2$ are two given operator valued measurable functions defined on $(0, \infty)$, we denote by

$$(\mathcal{T}_1 * \mathcal{T}_2)(t) = \int_0^t \mathcal{T}_1(s) \mathcal{T}_2(t-s) ds,$$

their convolution on \mathbb{R}_+ . We then denote $\mathcal{T}^{(*0)} := 1$, $\mathcal{T}^{(*1)} := \mathcal{T}$ and for any $k \geq 2$, $\mathcal{T}^{(*k)} := \mathcal{T}^{*(k-1)} * \mathcal{T}$.

Then we consider two Banach spaces $E_4 = L^1(m)$ and $E_5 = L^1(m^\theta)$, $\theta \in (0, 1)$. We introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where \mathcal{A} is an appropriately defined bounded operator so that \mathcal{B} becomes a dissipative operator. Then we can show that, for any integer $k \geq 0$, we have

$$(1.3) \quad \|((S_{\mathcal{B}}\mathcal{A})^{*k} * S_{\mathcal{B}})(t)\|_{E_4 \rightarrow E_5} \in L^1(\mathbb{R}^+),$$

and for some n large, we have

$$(1.4) \quad \|(\mathcal{A}S_{\mathcal{B}})^{*n}(t)\|_{E_4 \rightarrow E_3} \in L^1(\mathbb{R}_+),$$

and our weight m is small such that

$$(1.5) \quad \|I\|_{E_1 \rightarrow E_5} \lesssim 1.$$

Using the semigroup version of Duhamel formula

$$\begin{aligned} S_{\mathcal{L}} &= S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{L}}) = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}} \\ &= S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}) = S_{\mathcal{B}} + (S_{\mathcal{L}}\mathcal{A}) * S_{\mathcal{B}} \end{aligned}$$

Iterating this formula we get

$$(1.6) \quad S_{\mathcal{L}} = S_{\mathcal{B}} + \sum_{l=1}^{n-1} (S_{\mathcal{B}}) * (\mathcal{A}S_{\mathcal{B}})^{*(l)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}(t))^{*n},$$

using the above formula (1.6) and the estimates (1.3) (1.4) (1.5) we can conclude $\|S_{\mathcal{L}}\|_{E_4 \rightarrow E_5} \in L^1(\mathbb{R}_+)$, which is a rough version of the estimates presented in Theorem 1.1.

The organization of the following paper is as follows:

Section 2 we will prove a L^2 estimate for the KFP model.

In section 3 we will use the result in Section 2 to prove the L^2 convergence for the KFP equation, which is the (1) $p = 2$ of the Theorem 1.1

Section 4 we introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and prove some L^1 estimate on semigroup $S_{\mathcal{B}}$.

Section 5 is the proof of a regularization estimate on $S_{\mathcal{B}}$ to go from L^2 to L^1 and L^∞

Section 6 use the regularization lemma to prove L^p convergence for KFP equation, which is Theorem 1.1.

2. L^2 FRAMEWORK: DIRICHLET FORM AND RATE OF CONVERGENCE ESTIMATE

For later discussion we introduce some notations for the whole paper. The first is the splitting of the KFP operator $\mathcal{L} = \mathcal{T} + \mathcal{S}$, where \mathcal{T} stand for the transport part:

$$\mathcal{T}f = -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f,$$

and \mathcal{S} stand for the collision operator

$$\mathcal{S}f = \Delta_v f + \text{div}_v(vf),$$

We will denote the cut-off function χ such that $\chi(x, v) \in [0, 1]$, $\chi(x, v) \in C^\infty$, $\chi(x, v) = 1$ when $x^2 + v^2 \leq 1$, $\chi(x, v) = 0$ when $x^2 + v^2 \geq 2$, and denote $\chi_R = \chi(x/R, v/R)$.

We introduce the splitting of the KFP operator \mathcal{L} and denote it by

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} = M\chi_R(x, v).$$

We will denote $S_{\mathcal{L}}(t)$ the semigroup associated with operator \mathcal{L} , similarly $S_{\mathcal{B}}(t)$ the semigroup associated with operator \mathcal{B} . We use $\int f$ in place of $\int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv$ for short, similarly $\int f dx$ means $\int_{\mathbb{R}^d} f dx$, $\int f dv$ means $\int_{\mathbb{R}^d} f dv$. $B_{|x| \leq \rho}$ is used to denote the ball such that $\{x \in \mathbb{R}^d \mid |x| \leq \rho\}$, similarly B_ρ means the ball such that $\{x, v \in \mathbb{R}^d \mid |x|^2 + v^2 \leq \rho\}$.

For $V(x) = \langle x \rangle^\gamma$, $0 < \gamma \leq 1$, when we write the term $\langle \nabla V \rangle^{-1}$ and $\langle \nabla V \rangle$, $\langle \nabla V \rangle^{-1}$ means $\langle x \rangle^{1-\gamma}$, $\langle \nabla V \rangle$ means $\langle x \rangle^{\gamma-1}$.

And we denote the projection operator $\Pi f = M(f)G$.

With these notations we introduce the Dirichlet form adapted to our problem. We define

$$\rho_f = \rho[f] = \int f dv, \quad j_f = j[f] = \int v f dv.$$

$$M = Ce^{-v^2/2}, \quad \int M dv = 1, \quad \pi f = M\rho_f, \quad \pi^\perp = 1 - \pi, \quad f^\perp = \pi^\perp f.$$

Recall for the kinetic Fokker-Planck model

$$\mathcal{S}f = \Delta_v f + \operatorname{div}_v(vf), \quad \mathcal{T}f = -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f,$$

define an elliptic operator and its dual

$$\Delta_V u := \operatorname{div}_x(\nabla_x u + \nabla_x V u), \quad \Delta_V^* u = \Delta_x u - \nabla_x V \cdot \nabla_x u,$$

we define $u = (\Delta_V^*)^{-1} \xi$ the solution to the above elliptic equation (note that u can differ by a constant)

$$\Delta_V^* u = \xi \text{ on } \mathbb{R}^d, \text{ with } 0 = \langle u e^{-V} \langle \nabla V \rangle^{-2} \rangle := \int u e^{-V} \langle \nabla V \rangle^{-2} dx,$$

we use this notation to define a scalar product by

$$\begin{aligned} ((f, g)) &:= (f, g)_{\mathcal{H}} + \epsilon(\Delta_V^{-1} \nabla_x j_f, (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon((\rho_f e^V \langle \nabla V \rangle^2), \Delta_V^{-1} \nabla_x j_g)_{L^2} \\ &= (f, g)_{\mathcal{H}} + \epsilon(j_f, \nabla_x (\Delta_V^*)^{-1} (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon((\nabla_x (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2), j_g)_{L^2}, \end{aligned}$$

and we define

$$\begin{aligned} D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f)_{\mathcal{H}} + \epsilon(\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon((\rho[-\mathcal{L}f] e^V \langle \nabla V \rangle^2), \Delta_V^{-1} \nabla_x j_f)_{L^2}. \end{aligned}$$

With these notations we can come to our first theorem.

Theorem 2.1. *For the kinetic Fokker-Planck model with $V(x) = \langle x \rangle^\gamma$, $0 < \gamma < 1$, $\mathcal{H} = L^2(G^{-1/2})$, $\mathcal{H}_1 = L^2(G^{-1/2}\langle \nabla V \rangle)$, we define $\mathcal{H}_0 = \{h \in \mathcal{H}, \int f dx dv = 0\}$. Then there exists $\epsilon > 0$ small such that on \mathcal{H}_0 the norm $((f, f))$ defined above is equivalent to the norm of \mathcal{H} , moreover there exist $\lambda > 0$, such that*

$$D[f] \geq \lambda \|f\|_{\mathcal{H}_1}^2, \quad \forall f \in \mathcal{H}_0$$

Remark 2.2. Note that

$$\langle \rho_f e^V \langle \nabla V \rangle^2 e^{-V} \langle \nabla V \rangle^{-2} \rangle dx = \int \rho_f dx = 0,$$

this is one of the core of the construction.

Remark 2.3. in Villani's paper [5], a H^1 version of our theorem is established for the kinetic Fokker-Planck operator for both the case $r \geq 1$ and $0 < r \leq 1$.

Remark 2.4. Our first statement is a generalization of [1] [2], and the method of our second estimate is based on [3].

Before proving the theorem, we need some lemmas.

First we need local Poincaré inequality:

Lemma 2.5. *We say that W satisfies a local Poincaré inequality on a bounded open set Ω if there exist some constant $\kappa_\Omega > 0$ such that:*

$$\int_\Omega h^2 W \leq \kappa_\Omega \int_\Omega |\nabla h|^2 W + \frac{1}{W(\Omega)} \left(\int_\Omega h W \right)^2,$$

for any nice function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and where we denote $W(\Omega) := \langle W 1_\Omega \rangle$.

Under the assumption $W, W^{-1} \in L_{loc}^\infty(\mathbb{R}^d)$, we can prove that the function W satisfies the local Poincaré inequality.

Prove of Lemma 2.5: see [9]

And the weak Poincaré inequality.

Lemma 2.6. *For any $u \in \mathcal{D}(\mathbb{R}^d)$ such that $\langle u e^{-V} \langle \nabla V \rangle^{-2} \rangle = 0$, we have*

$$\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} \lesssim \|\nabla u\|_{L^2(e^{-V/2})}$$

Proof of Lemma 2.6 : We need to prove for any $h \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} h e^{-V} \langle \nabla V \rangle^{-2} = 0,$$

we have

$$\int_{\mathbb{R}^d} |\nabla h|^2 e^{-V} \geq C \int_{\mathbb{R}^d} h^2 e^{-V} \langle x \rangle^{2(\gamma-1)}.$$

Taking $g = h e^{-1/2V}$, we can have $\nabla g = \nabla h e^{-1/2V} - \frac{1}{2} \nabla V h e^{-1/2V}$, so

$$\begin{aligned} \int |\nabla g|^2 &= \int |\nabla h|^2 e^{-V} + \int h^2 \frac{1}{4} |\nabla V|^2 e^{-V} - \int \frac{1}{2} \nabla(h^2) \cdot \nabla V e^{-V} \\ &= \int |\nabla h|^2 e^{-V} + \int h^2 \left(\frac{1}{2} \Delta V - \frac{1}{4} |\nabla V|^2 \right) e^{-V}, \end{aligned}$$

from this we can deduce for some K and $R_0 > 0$

$$\int |\nabla h|^2 e^{-V} \geq \int \frac{1}{8} h^2 \langle \nabla V \rangle^2 e^{-V} - K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2}.$$

Define

$$\epsilon_R := \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6},$$

and

$$Z_R := \int_{B_R} e^{-V} \langle \nabla V \rangle^{-2},$$

then by

$$\int_{\mathbb{R}^d} h e^{-V} \langle \nabla V \rangle^{-2} = 0,$$

we can get

$$\begin{aligned} \left(\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2} \right)^2 &= \left(\int_{B_R^c} h e^{-V} \langle \nabla V \rangle^{-2} \right)^2 \\ &\leq \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2 \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6} \\ &\leq \epsilon_R \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Using Lemma 2.5 we have

$$\begin{aligned} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^{-2} &\leq C_R \int_{B_R} |\nabla h|^2 e^{-V} \langle \nabla V \rangle^{-2} + \frac{1}{Z_R} \left(\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2} \right)^2 \\ &\leq C'_R \int_{B_R} |\nabla h|^2 e^{-V} + \frac{\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Putting all the inequalities together we have

$$\begin{aligned} \int h^2 e^{-V} \langle \nabla V \rangle^2 &\leq 8 \int |\nabla h|^2 e^{-V} + 8K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2} \\ &\leq 8(1 + KC'_R) \int_{B_R} |\nabla h|^2 e^{-V} + \frac{8K\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2, \end{aligned}$$

by taking R large such that: $\frac{8K\epsilon_R}{Z_R} \leq \frac{1}{2}$, we are done. \square

Lemma 2.7. (*Elliptic Estimate*) For any $\xi_1 \in L^2(\langle \nabla V \rangle^{-1} e^{-V/2})$ and $\xi_2 \in L^2(e^{-V/2})$, the solution $u \in L^2(e^{-V/2})$ to the elliptic equation

$$-\Delta_V^* u = \xi_1 + \nabla \xi_2, \quad \langle u e^{-V} \langle \nabla V \rangle^{-2} \rangle = 0,$$

satisfies

$$\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})} \lesssim A := \|\xi_1\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})} + \|\xi_2\|_{L^2(e^{-V/2})}.$$

Similarly for any $\xi \in L^2(e^{-V/2})$, the solution $u \in L(e^{-V/2})$ to the elliptic problem

$$-\Delta_V^* u = \xi, \quad \langle u e^{-V} \langle \nabla V \rangle^{-2} \rangle = 0,$$

satisfies

$$(2.1) \quad \|u\|_{L^2(\langle \nabla V \rangle^2 e^{-V/2})} + \|\nabla u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|D^2 u\|_{L^2(e^{-V/2})} \lesssim \|\xi\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})}.$$

Remark 2.8. I believe in (2.1) we can improve the last term from $\|\xi\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})}$ to $\|\xi\|_{L^2(e^{-V/2})}$, but this result is enough to continue our proof.

Proof of Lemma 2.7: First observe

$$e^V \operatorname{div}_x [e^{-V} \nabla_x u] = \Delta_x u - \nabla_x V \cdot \nabla_x u = \Delta_V^* u,$$

integrate the equation by $u e^{-V}$ and use this observation we get

$$-\int e^V \operatorname{div}_x [e^{-V} \nabla_x u] u e^{-V} = \int (\xi_1 + \nabla \cdot \xi_2) u e^{-V}.$$

Performing integration by parts we have

$$\int e^{-V} |\nabla_x u|^2 = \int (\xi_1 u + \xi_2 \cdot \nabla u - \xi_2 \cdot \nabla V u) e^{-V},$$

by Lemma 2.6

$$\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} \lesssim \|\nabla u\|_{L^2(e^{-V/2})},$$

we deduce

$$\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})}^2 + \|\nabla u\|_{L^2(e^{-V/2})}^2 \lesssim A \times (\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})}),$$

so the proof for the first inequality is done.

For the second inequality, since $\langle \nabla V \rangle \leq 1$ so the only thing need to be proved is the $\|D^2 u\|_{L^2(e^{-V/2})}$ term, by integration by parts

$$\begin{aligned} \int |D^2 u|^2 e^{-V} &= \sum_{i,j=1}^d \int (\partial_{ij}^2 u)^2 e^{-V} \\ &= \sum_{i,j=1}^d \int \partial_i u (\partial_{ij}^2 u \partial_j V - \partial_{ijj}^3 u) e^{-V} \\ &= \sum_{i,j=1}^d \int \partial_{jj} u \partial_i (\partial_i u e^{-V}) - \frac{1}{2} \int (\partial_i u)^2 \partial_j (\partial_j V e^{-V}) \\ &= \int (\Delta u) (-\Delta_V^* u) e^{-V} + \int |\nabla u|^2 (|\nabla V|^2 - \Delta V) e^{-V} \\ &\lesssim \|D^2 u\|_{L^2(e^{-V/2})} \|\xi\|_{L^2(e^{-V/2})} + \|\langle \nabla V \rangle \nabla u\|_{L^2(e^{-V/2})}, \end{aligned}$$

where in the third equality we have used

$$\begin{aligned} \int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} &= - \int \partial_i u \partial_j (\partial_i u \partial_j V e^{-V}) \\ &= - \int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} - \int (\partial_i u)^2 \partial_j (\partial_j V e^{-V}), \end{aligned}$$

which implies

$$\int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} = -\frac{1}{2} \int (\partial_i u)^2 \partial_j (\partial_j V e^{-V}),$$

and in the fourth equality we have used

$$e^V \sum_{i=1}^d \partial_i (e^{-V} \partial_i u) = e^V \operatorname{div}_x [e^{-V} \nabla_x u] = u = \Delta_V^* u,$$

so our proof is done. \square

Now we can turn to the proof of Theorem 2.1.

Proof of Theorem 2.1 :First, we prove the equivalence of the norms $((\cdot, \cdot))$ and $(\cdot, \cdot)_{\mathcal{H}}$. by Cauchy-Schwarz inequality and Lemma 2.7, we have

$$(j_f, \nabla_x (\Delta_V^*)^{-1} (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \leq \|j_f\|_{L^2(e^{V/2})} \|\rho_g e^V \langle \nabla V \rangle^2\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})},$$

and obviously

$$\|\rho_g e^V \langle \nabla V \rangle^2\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})} = \|\rho_g\|_{L^2(\langle \nabla V \rangle e^{V/2})} \leq \|\rho_g\|_{L^2(e^{V/2})} \lesssim \|g\|_{\mathcal{H}}.$$

Using elementary observation

$$|j_f| \lesssim \|f\|_{L^2(e^{v^2/4})} \quad |\rho_f| \lesssim \|f\|_{L^2(e^{v^2/4})},$$

we deduce

$$(j_f, \nabla_x (\Delta_V^*)^{-1} (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \lesssim \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}.$$

The third term in the definition of $((\cdot, \cdot))$ can be estimated in the same way and that ends the proof of equivalence of norms. \square

Now we turn to prove the main estimate of the theorem. We split the $D[f]$ into 3 parts

$$D[f] = T_1 + \epsilon T_2 + \epsilon T_3$$

with

$$\begin{aligned} T_1 &= (\mathcal{L}f, f)_{\mathcal{H}} \\ T_2 &= (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ T_3 &= ((\Delta_V)^{-1} \nabla_x j_f, \rho[-\mathcal{L}f])_{L^2(e^{V/2} \langle \nabla V \rangle)}, \end{aligned}$$

and compute them separately.

For the T_1 term, we have

$$\begin{aligned} T_1 &:= (-\mathcal{T}f + \mathcal{S}f, f)_{\mathcal{H}} = (-\mathcal{S}f, f)_{\mathcal{H}} \\ &= - \int [\Delta_v f + \operatorname{div}_v(vf)] f M^{-1} e^V = \int |\nabla_v(f/M)|^2 M e^V \\ &\geq k_p \int |f/M - \rho_f|^2 M e^V = k_p \|f - \rho_f M\|_{\mathcal{H}}^2 = k_p \|f^\perp\|_{\mathcal{H}}^2, \end{aligned}$$

note in the last line we use the classical Poincare inequality.

For the T_2 term, we split it as

$$\begin{aligned} T_2 &:= (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ &= (\Delta_V^{-1} \nabla_x j[-\mathcal{T}\pi f], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ &\quad + (\Delta_V^{-1} \nabla_x j[-\mathcal{T}f^\perp], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ &\quad + (\Delta_V^{-1} \nabla_x j[-\mathcal{S}f], \rho_f)_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ &:= T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned}$$

We first observe

$$\mathcal{T}\pi f = -v \cdot \nabla_x \rho_f M - \nabla_x V \cdot v \rho_f M = -e^{-V} M v \cdot \nabla_x (\rho_f / e^{-V}),$$

so we can have

$$j[-\mathcal{T}\pi f] = \langle v v_k M \rangle e^{-V} \partial_{x_k} (\rho_f / e^{-V}) = e^{-V} \nabla_x (\rho_f / e^{-V}),$$

next observe that

$$e^V \operatorname{div}_x [e^{-V} \nabla_x u] = \Delta_V^* u,$$

then we have

$$\begin{aligned} T_{2,1} &= (j[-\mathcal{T}\pi f], \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= (\rho_f, [e^V \operatorname{div}_x (e^{-V} \nabla)] [(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)])_{L^2} \\ &= \|\rho_f e^{V/2} \langle \nabla V \rangle\|_{L^2}^2 = \|\pi f\|_{\mathcal{H}_1}^2. \end{aligned}$$

Using the notation $\eta_1 = \langle v \otimes v f^\perp \rangle$ and $\eta_{2,\alpha\beta} = \langle v_\alpha \partial_{v_\beta} f^\perp \rangle$, observe that

$$|\eta_1| \lesssim \|f^\perp\|_{L^2(e^{v^2/4})}, |\eta_2| \lesssim \|f^\perp\|_{L^2(e^{v^2/4})},$$

we compute

$$\begin{aligned} T_{2,2} &= (j[-\mathcal{T}f^\perp], \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= (D\eta_1 + \eta_2 \nabla V, \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= (\eta_1, D^2(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} + (\eta_2, \nabla V \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= \|\eta_1\|_{L^2(e^{V/2})} \|D^2(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})} \\ &\quad + \|\eta_2\|_{L^2(e^{V/2})} \|\nabla V \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})}, \end{aligned}$$

and by Lemma 2.7

$$\begin{aligned} T_{2,2} &\lesssim \|\eta_1\|_{L^2(e^{V/2})} \|\rho_f e^V \langle \nabla V \rangle^2\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})} + \|\eta_2\|_{L^2(e^{V/2})} \|\rho_f e^V \langle \nabla V \rangle^2\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

For $T_{2,3}$ using

$$\begin{aligned} j[-Sf] = j[-Sf^\perp] &= - \int v[\Delta_v f^\perp + \operatorname{div}_v(v f^\perp)] dv \\ &= d \int f^\perp v dv \lesssim \|f^\perp\|_{L^2(e^{v^2/4})}, \end{aligned}$$

we have by Lemma 2.7

$$\begin{aligned} T_{2,3} &= (j[-Sf], \nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &\leq \|j[-Sf]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|\rho_f e^V \langle \nabla V \rangle^2\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})} \\ &= \|f^\perp\|_{\mathcal{H}} \|\rho_f\|_{L^2(\langle \nabla V \rangle e^{V/2})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

Finally we come to the T_3 term using

$$\rho[-Sf] = \int \nabla_v \cdot (\nabla_v f + v f) dv = 0,$$

and

$$\begin{aligned} \rho[-Tf] &= \rho[v \nabla_x f - \nabla_x V(x) \nabla_v f] \\ &= \int v \nabla_x f - \nabla_x V(x) \nabla_v f dv \\ &= \nabla_x \int v f dv \\ &= \nabla_x j[f], \end{aligned}$$

we have by $\nabla(\langle \nabla V \rangle^2) \lesssim \langle \nabla V \rangle^2$ and $\langle \nabla V \rangle^2 \lesssim \langle \nabla V \rangle$ we get

$$\begin{aligned} T_3 &= ((\Delta_V)^{-1} \nabla_x j f, \rho[-\mathcal{L}f])_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ &= ((\Delta_V)^{-1} \nabla_x j[f^\perp], \rho[-\mathcal{T}f])_{L^2(e^{V/2} \langle \nabla V \rangle)} \\ &= (j[-f^\perp], \nabla(\Delta_V^*)^{-1}(\rho[-\mathcal{T}f] e^V \langle \nabla V \rangle^2))_{L^2} \\ &= (j[-f^\perp], \nabla(\Delta_V^*)^{-1}(\nabla_x j[f] e^V \langle \nabla V \rangle^2))_{L^2} \\ &= \|j[f^\perp]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}[\nabla_x(j f e^V \langle \nabla V \rangle^2) - \nabla V j f e^V \langle \nabla V \rangle^2 - \nabla(\langle \nabla V \rangle^2) j f e^V]\|_{L^2(e^{-V/2})}, \end{aligned}$$

again by Lemma 2.7 we have

$$\begin{aligned} T_3 &\lesssim \|j[f^\perp]\|_{L^2(e^{V/2})} (\|j f e^V \langle \nabla V \rangle^2\|_{L^2(e^{-V/2} \langle \nabla V \rangle^{-1})} + \|j f e^V \nabla(\langle \nabla V \rangle^2)\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})}) \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1}. \end{aligned}$$

Gathering all the term together we have for $\epsilon > 0$ small enough

$$\begin{aligned} D[f] &\geq \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1} - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1} \\ &\geq \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - (2\epsilon + 4\epsilon^{1/2})K \|f^\perp\|_{\mathcal{H}}^2 - \epsilon^{3/2} 4K \|\pi f\|_{\mathcal{H}_1}^2 \\ &\geq \frac{1}{2} (\|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2) \geq \frac{\epsilon}{M} \|f\|_{\mathcal{H}_1}, \end{aligned}$$

so the proof is ended. \square

3. L^2 SUB-EXPONENTIAL DECAY FOR THE KINETIC FOKKER-PLANCK EQUATION BASED ON A SPLITTING

Theorem 3.1. *Using the notation and results in Theorem 2.1, we have*

$$\|S_{\mathcal{L}}(t)f_0\|_{L^2(G^{-\frac{1}{2}})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}} \|f_0\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})},$$

for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)}) \cap \mathcal{H}_0$, $\epsilon > 0$ small or equivalently

$$\|S_{\mathcal{L}}(t)(I - \Pi)f_0\|_{L^2(G^{-\frac{1}{2}})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}} \|(I - \Pi)f_0\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})},$$

for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)})$, $\epsilon > 0$ small.

During the proof of Theorem 3.1 we will make use of the following Gronwall's Lemma.

Lemma 3.2. *(Gronwall's Lemma) Suppose $0 < b$, $0 < c$, and $0 \leq u(t) \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ satisfying:*

$$u'(t) \leq -cu(t) + b,$$

in the sense of distribution $\mathcal{D}'((0, \infty))$ we have:

$$u(t) \leq u(0)e^{-ct} + \frac{b}{c}.$$

Since the result is well-known, the proof is omitted. \square

Recall the splitting of \mathcal{L}

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} = M\chi_R(x, v),$$

we first prove some convergence on the semigroup associated with \mathcal{B} denoted by $S_{\mathcal{B}}(t)$.

Lemma 3.3. *For any $p \in [1, \infty)$ we have*

(1) *For a given weight function m , we have*

$$\int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m \leq \frac{1}{p} \int f^p G^{-(p-1)}\tilde{m}$$

with

$$\tilde{m} = \Delta_v m - \nabla_v m \cdot v - \nabla V(x) \cdot \nabla_v m + v \cdot \nabla_x m,$$

(2) Taking $m = e^{\epsilon H^\delta}$, $\epsilon > 0$ if $0 < \delta < \frac{\gamma}{2}$, ϵ small enough if $\delta = \frac{\gamma}{2}$, $H = 3v^2 + 2x \cdot v + x^2 + 1$, we have

$$\int f^{p-1}(\mathcal{B}f)G^{-(p-1)}e^{\epsilon H^\delta} \leq -C \int f^p G^{-(p-1)}e^{\epsilon H^\delta} H^{\frac{\delta}{2} + \gamma - 1},$$

for some M and R large.

(3) Using the results in (2), we can prove

$$\|S_{\mathcal{B}}(t)\|_{L^p(e^{2\epsilon H^\delta} G^{-\frac{p-1}{p}}) \rightarrow L^p(e^{\epsilon H^\delta} G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

for some $a > 0$. In particular, this implies

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p} + \epsilon)}) \rightarrow L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Prove of Lemma 3.3: Step 1: Recall $G^{-1} = Ce^{v^2/2 + V(x)}$. We write

$$\int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m = \int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m + \int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m.$$

We first compute the contribution of the term with operator \mathcal{T}

$$\begin{aligned} \int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m &= \frac{1}{p} \int \mathcal{T}(f^p)G^{-(p-1)}m \\ &= -\frac{1}{p} \int f^p \mathcal{T}(G^{-(p-1)}m) \\ &= \frac{1}{p} \int f^p G^{-(p-1)}(v \cdot \nabla_x m - \nabla V(x) \cdot \nabla_v m), \end{aligned}$$

for the term with operator \mathcal{S} , using integration by parts and we get

$$\begin{aligned} &\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ &= \int f^{p-1}(\Delta_v f + \operatorname{div}_v(vf))G^{-(p-1)}m \\ &= -\int \nabla_v((fG^{-1})^{p-1}m) \cdot (\nabla_v f + vf) \\ &= -\int ((fG^{-1})^{p-1} \nabla_v m + (p-1)(fG^{-1})^{p-2} \nabla_v(fG^{-1})m) \cdot \nabla_v(fG^{-1})G \\ &= -\int (p-1)|\nabla_v(fG^{-1})|^2 (fG^{-1})^{p-2} Gm - \frac{1}{p} \nabla_v((fG^{-1})^p) \cdot (\nabla_v m)G, \end{aligned}$$

performing another integration by parts on the latter term we have

$$\begin{aligned} &\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ &= \int -C|\nabla_v(fG^{-1})|^2 (fG^{-1})^{p-2} Gm + \frac{1}{p} \nabla_v \cdot (G \nabla_v m)(fG^{-1})^p \\ &= \int -C|\nabla_v(fG^{-1})|^2 (fG^{-1})^{p-2} Gm + \frac{1}{p} (\Delta_v m - v \cdot \nabla_v m) f^p G^{-(p-1)}, \end{aligned}$$

putting together the two identities, we obtain the result. \square

Step 2: We particular use $m = e^{\epsilon H^\delta}$ now, we easily compute:

$$\frac{\nabla_v m}{m} = \delta \epsilon \frac{\nabla_v H}{H^{1-\delta}}, \quad \frac{\nabla_x m}{m} = \delta \epsilon \frac{\nabla_x H}{H^{1-\delta}},$$

$$\frac{\Delta_v m}{m} \leq \delta \epsilon \frac{\Delta_v H}{H^{1-\delta}} + (\delta \epsilon)^2 \frac{|\nabla_v H|^2}{H^{2(1-\delta)}}.$$

We deduce that $\phi = \frac{\tilde{m}}{m}$ satisfies

$$\frac{\phi H^{1-\delta}}{\epsilon \delta} \leq \Delta_v H + \epsilon \delta \frac{|\nabla_v H|^2}{H^{1-\delta}} - v \cdot \nabla_v H + v \cdot \nabla_x H - \nabla_x V(x) \cdot \nabla_v H,$$

recall $H = 3v^2 + 2v \cdot x + x^2 + 1$, we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6,$$

so let $\epsilon > 0$ arbitrary if $0 < 2\delta < \gamma$, ϵ small if $2\delta = \gamma$, we have

$$\begin{aligned} & \Delta_v H + 2\epsilon \delta \frac{|\nabla_v H|^2}{H^{1-\delta}} + v \cdot \nabla_x H - v \cdot \nabla_v H - \nabla_x V(x) \cdot \nabla_v H \\ &= 6 + \epsilon \delta \frac{(6v + 2x)^2}{H^{1-\delta}} + 2v^2 + 2x \cdot v - 6v^2 - 2x \cdot v - 6v \cdot \nabla_x V(x) - 2x \cdot \nabla_x V(x) \\ &< (2v^2 + C_1 v + C_2 v^{2\delta} - 6v^2) + (C_3 \epsilon \delta x^{2\delta} - 2x \cdot \nabla_x V(x)) + C \\ &< -C_4 v^2 - C_5 x \cdot \nabla_x V(x) + C_6 \\ &< -C_7 H^{\frac{\gamma}{2}} + M \chi_R, \end{aligned}$$

for some $C_i > 0$. Then we get $\phi \leq \frac{-C}{H^{1-\delta-\frac{\gamma}{2}}} + M \chi_R$, the proof of (2) is ended. \square

Step 3: During the proof of this part, we will denote $f_t = S_{\mathcal{B}}(t)f_0$ the solution to $\partial_t f = \mathcal{B}f$, $f(0) = f_0$. First we have by (2)

$$\int f^{p-1}(\mathcal{B}f) G^{-(p-1)} e^{2\epsilon H^\delta} \leq 0,$$

which implies

$$\int f_t^p G^{-(p-1)} e^{2\epsilon H^\delta} \leq \int f_0^p G^{-(p-1)} e^{2\epsilon H^\delta} := Y_1,$$

define

$$Y := \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta},$$

and making use of the results in (2), for any $\rho > 0$ we have

$$\begin{aligned} \frac{d}{dt}Y &= 2 \int f_t^{p-1} \mathcal{B} f_t G^{-(p-1)} e^{\epsilon H^\delta} \\ &\leq -a \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} H^{\delta + \frac{\gamma}{2} - 1} \\ &\leq -a \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x \rangle^{2\delta + \gamma - 2} \\ &\leq -a \int_{B_{|x| \leq \rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x \rangle^{2\delta + \gamma - 2}. \end{aligned}$$

As $2\delta + \gamma < 2$, so $0 \leq |x| \leq \rho$ can imply $\langle x \rangle^{2\delta + \gamma - 2} \geq \langle \rho \rangle^{2\delta + \gamma - 2}$, we get

$$\begin{aligned} \frac{d}{dt}Y &\leq -a \langle \rho \rangle^{2\delta + \gamma - 2} \int_{B_{|x| \leq \rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} \\ &\leq -a \langle \rho \rangle^{2\delta + \gamma - 2} Y + a \langle \rho \rangle^{2\delta + \gamma - 2} \int_{B_{|x| \geq \rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta}, \end{aligned}$$

and $|x| \geq \rho$ can imply $e^{\epsilon \langle x \rangle^{2\delta}} \geq e^{\epsilon \langle \rho \rangle^{2\delta}}$, we have

$$\begin{aligned} \frac{d}{dt}Y &\leq -a \langle \rho \rangle^{2\delta + \gamma - 2} Y + a \langle \rho \rangle^{2\delta + \gamma - 2} e^{-\epsilon \langle \rho \rangle^{2\delta}} \int_{B_{|x| \geq \rho}} f_t^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon \langle x \rangle^{2\delta}} \\ &\leq -a \langle \rho \rangle^{2\delta + \gamma - 2} Y + a \langle \rho \rangle^{2\delta + \gamma - 2} e^{-\epsilon \langle \rho \rangle^{2\delta}} \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon \langle x \rangle^{2\delta}} \\ &\leq -a \langle \rho \rangle^{2\delta + \gamma - 2} Y + a \langle \rho \rangle^{2\delta + \gamma - 2} e^{-\epsilon \langle \rho \rangle^{2\delta}} C Y_1. \end{aligned}$$

We can deduce from Gronwall's Lemma

$$\begin{aligned} Y(t) &\leq e^{-a \langle \rho \rangle^{2\delta + \gamma - 2} t} Y(0) + C e^{-\epsilon \langle \rho \rangle^{2\delta}} Y_1 \\ &\lesssim (e^{-a \langle \rho \rangle^{2\delta + \gamma - 2} t} + e^{-\epsilon \langle \rho \rangle^{2\delta}}) Y_1, \end{aligned}$$

choosing ρ such that $a \langle \rho \rangle^{2\delta + \gamma - 2} t = \epsilon \langle \rho \rangle^{2\delta}$, that is $\langle \rho \rangle^{2-\gamma} = Ct$, then

$$Y(t) \leq C_1 e^{-C_2 t^{\frac{2\delta}{2-\gamma}}} Y_2,$$

for some $C_i > 0$, then the proof of Lemma 3.3 is complete. \square

Now we come to prove Theorem 3.1.

Proof of Theorem 3.1: By Theorem 2.1, we have

$$\|S_{\mathcal{L}}(t)\|_{L^2(G^{-1/2}) \rightarrow L^2(G^{-1/2})} \lesssim 1,$$

recall $\mathcal{A} = M\chi_R$, so we have

$$\|\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^2(e^{2\epsilon H^\delta} G^{-1/2})} \lesssim 1,$$

and by Lemma 3.3 when $p = 2$

$$\|S_{\mathcal{B}}(t)\|_{L^2(e^{2\epsilon H^\delta} G^{-1/2}) \rightarrow L^2(e^{\epsilon H^\delta} G^{-1/2})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

gathering the three estimates and using Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}},$$

we deduce

$$\|S_{\mathcal{L}}(t)\|_{L^2(e^{2\epsilon H^\delta} G^{-1/2}) \rightarrow L^2(e^{\epsilon H^\delta} G^{-1/2})} \lesssim 1,$$

taking $2\delta = \gamma$, ϵ small enough we have

$$\int f_t^2 G^{-1} e^{\epsilon H^{\frac{\gamma}{2}}} \leq C \int f_0^2 G^{-1} e^{2\epsilon H^{\frac{\gamma}{2}}} =: Y_3,$$

and we define

$$Y_2(t) := \int f_t^2 G^{-1}.$$

Using the result from Theorem 2.1, we have

$$\begin{aligned} \frac{d}{dt} Y_2 &= 2 \int f_t \mathcal{L} f_t G^{-1} \\ &\leq -a \int f_t^2 G^{-1} \langle x \rangle^{2(\gamma-1)} \\ &\leq -a \int_{B_{|x| \leq \rho}} f_t^2 G^{-1} \langle x \rangle^{2(\gamma-1)}, \end{aligned}$$

as $2\gamma < 2$, so $0 \leq |x| \leq \rho$ can imply $\langle x \rangle^{2(\gamma-1)} \geq \langle \rho \rangle^{2(\gamma-1)}$, we have

$$\begin{aligned} \frac{d}{dt} Y_2 &\leq -a \langle \rho \rangle^{2(\gamma-1)} \int_{B_{|x| \leq \rho}} f_t^2 G^{-1} \\ &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} \int_{B_{|x| \geq \rho}} f_t^2 G^{-1}, \end{aligned}$$

for any $\epsilon > 0$ small, we can find $\epsilon_2 > 0$ such that $:\epsilon_2 V(x) \leq \epsilon H^{\frac{\gamma}{2}}$, and $|x| \geq \rho$ can imply $e^{\epsilon_2 \langle x \rangle^\gamma} \geq e^{\epsilon_2 \langle \rho \rangle^\gamma}$, so

$$\begin{aligned} \frac{d}{dt} Y_2 &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} e^{-\epsilon_2 \langle \rho \rangle^\gamma} \int_{B_{|x| \geq \rho}} f_t^2 G^{-1} e^{\epsilon_2 V(x)} \\ &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} e^{-\epsilon_2 \langle \rho \rangle^\gamma} \int f_t^2 G^{-1} e^{\epsilon H^{\frac{\gamma}{2}}} \\ &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} e^{-\epsilon_2 \langle \rho \rangle^\gamma} C Y_3, \end{aligned}$$

we can deduce thanks to Gronwall's Lemma

$$\begin{aligned} Y_2(t) &\leq e^{-a \langle \rho \rangle^{2(\gamma-1)} t} Y_2(0) + C e^{-\epsilon_2 \langle \rho \rangle^\gamma} Y_3 \\ &\lesssim (e^{-a \langle \rho \rangle^{2(\gamma-1)} t} + e^{-\epsilon_2 \langle \rho \rangle^\gamma}) Y_3, \end{aligned}$$

choosing ρ such that $a \langle \rho \rangle^{2(\gamma-1)} t = \epsilon_2 \langle \rho \rangle^\gamma$, that is $\langle \rho \rangle^{2-\gamma} = Ct$, then

$$Y_2(t) \leq C_1 e^{-C_2 t^{\gamma/(2-\gamma)}} Y_3.$$

As $H^{\frac{\gamma}{2}} \lesssim C(\frac{v^2}{2} + V(x))$, we have:

$$e^{\epsilon H^{\frac{\gamma}{2}}} \leq G^{-C\epsilon},$$

so taking ϵ small the proof of Theorem 3.1 is done. \square

4. L^1 CONVERGENCE ON $S_{\mathcal{B}}$

This section we prove the decay rate for $S_{\mathcal{B}}$.

Theorem 4.1. *Let $H = 1 + x^2 + 2\langle v, x \rangle + 3v^2$, for any $\theta \in (0, 1)$ and for $\forall l > 0$, we have*

$$\|S_{\mathcal{B}}(t)\|_{L^1(H^l) \rightarrow L^1(H^{l\theta})} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l(1-\theta)}{1-\frac{\gamma}{2}},$$

To continue our proof we first need a lemma.

Lemma 4.2. *For the kinetic Fokker Planck operator \mathcal{L} , let m be a weight function, for any $p \in [1, \infty]$ we have*

$$\int (\mathcal{L}f) f^{p-1} m^p = -(p-1) \int |\nabla_v(mf)|^2 (mf)^{p-2} + \int f^p m^p \phi,$$

with

$$\phi = \frac{2}{p'} \frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p} - 1\right) \frac{\Delta_v m}{m} + \frac{d}{p'} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m},$$

in particular when $p = 1$

$$\phi = \frac{\Delta_v m}{m} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

Proof of Lemma 4.2: First we have

$$\int (\mathcal{L}f) f^{p-1} m^p = \int f^{p-1} \mathcal{S}f m^p + \int f^{p-1} \mathcal{T}f m^p,$$

we first compute the contribution of the term with operator \mathcal{T} , we have

$$\int f^{p-1} \mathcal{T}f m^p = \frac{1}{p} \int \mathcal{T}(f^p) m^p = - \int f^p m^{p-1} \mathcal{T}m = - \int f^p m^p \frac{\mathcal{T}m}{m},$$

then we come to the term with operator \mathcal{S}

$$\int (\mathcal{S}f) f^{p-1} m^p = \int f^{p-1} m^p (\Delta_v f + \text{div}_v(vf)) := C_1 + C_2,$$

we first compute the C_2 term

$$\begin{aligned} C_2 &= \int f^{p-1} m^p (\text{div}_v v f + v \cdot \nabla_v f) \\ &= \int f^p (\text{div}_v v) m^p - \frac{1}{p} \int f^p \text{div}_v (v m^p) \\ &= \int f^p \left[\left(1 - \frac{1}{p}\right) d - v \cdot \frac{\nabla_v m}{m} \right] m^p, \end{aligned}$$

and we turn to the C_1 term

$$\begin{aligned} C_1 &= \int f^{p-1} m^p \Delta_v f = - \int \nabla_v (f^{p-1} m^p) \cdot \nabla_v f \\ &= \int -(p-1) |\nabla_v f|^2 f^{p-2} m^p - \frac{1}{p} \int \nabla_v f^p \cdot \nabla_v m^p, \end{aligned}$$

using $\nabla_v(mf) = m\nabla_v f + f\nabla_v m$, we have

$$\begin{aligned} C_1 &= -(p-1) \int |\nabla_v(mf)|^2 f^p m^{p-2} + (p-1) \int |\nabla_v m|^2 f^p m^{p-2} \\ &\quad + \frac{2(p-1)}{p^2} \int \nabla_v(f^p) \cdot \nabla_v(m^p) - \frac{1}{p} \int \nabla_v(f^p) \cdot \nabla_v(m^p) \\ &= -(p-1) \int |\nabla_v(mf)|^2 f^{p-2} m^p + (p-1) \int |\nabla_v m|^2 f^p m^{p-2} \\ &\quad + \frac{p-2}{p^2} \int f^p \Delta_v m^p, \end{aligned}$$

using $\Delta_v m^p = p\Delta_v m m^{p-1} + p(p-1)|\nabla_v m|^2 m^{p-2}$, we can conclude

$$C_1 = -(p-1) \int |\nabla_v(mf)|^2 f^{p-2} m^{p-2} + \int f^p m^p \left[\left(\frac{2}{p} - 1\right) \frac{\Delta_v m}{m} + 2\left(1 - \frac{1}{p}\right) \frac{|\nabla_v m|^2}{m^2} \right].$$

Combining the equalities above we are done. \square

Proof of Theorem 4.1 : From Lemma 4.2 we have

$$\begin{aligned} (4.1) \quad \int (\mathcal{B}f) f^{p-1} m^p &= \int (\mathcal{L} - M\chi_R) f^{p-1} m^p \\ &= -(p-1) \int |\nabla_v(mf)|^2 (mf)^{p-2} + \int f^p m^p \phi, \end{aligned}$$

with

$$\phi = \left[\frac{2}{p'} \frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p} - 1\right) \frac{\Delta_v m}{m} + \frac{d}{p'} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R \right],$$

for $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v \cdot \frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R.$$

(1) Let $m = H^k$, we have

$$\frac{\nabla_v m}{m} = k \frac{\nabla_v H}{H}, \quad \frac{\nabla_x m}{m} = k \frac{\nabla_x H}{H},$$

and

$$\frac{\Delta_v m}{m} = \frac{k\Delta_v H}{H} + \frac{k(k-1)|\nabla_v H|^2}{H^2},$$

in sum we have for ϕ

$$\frac{\phi H}{k} = \Delta_v H + (k-1) \frac{|\nabla_v H|^2}{H} - v \cdot \nabla_v H + v \cdot \nabla_x H - \nabla_x V(x) \cdot \nabla_v H - M\chi_R,$$

Recall $H = 3v^2 + 2\langle v, x \rangle + x^2$, we have

$$\nabla_v H = 6v + 2x, \nabla_x H = 2v + 2x, \Delta_v H = 6,$$

we compute

$$\begin{aligned} & \Delta_v H + (k-1) \frac{|\nabla_v H|^2}{H} + v \cdot \nabla_x H - v \cdot \nabla_v H - \nabla_x V(x) \cdot \nabla_v H \\ &= 6 + (k-1) \frac{(6v+2x)^2}{H} + 2v^2 + 2x \cdot v - 6v^2 \\ & \quad - 2x \cdot v - 6v \cdot \nabla_x V(x) - 2x \cdot \nabla_x V(x) \\ & < (2v^2 + Cv - 6v^2) - 2x \cdot \nabla_x V(x) + C \\ & < -C_1 v^2 - C_2 x \cdot \nabla_x V(x) + C_3 \\ & < -C_4 H^{\frac{\gamma}{2}} + M_1 \chi_{R_1}, \end{aligned}$$

so taking M and R very large, we have $\phi \leq -CH^{\frac{\gamma}{2}-1}$, taking this result into equation (4.1), we have:

$$(4.2) \quad \frac{d}{dt} Y_4(t) := \frac{d}{dt} \int |f_{\mathcal{B}}(t)| H^k = \int \text{sign}(f_{\mathcal{B}}(t)) \mathcal{B} f_{\mathcal{B}}(t) H^k \leq -C \int |f_{\mathcal{B}}(t)| H^{k-1+\frac{\gamma}{2}}, k > 1,$$

using this result, first we have for any l large fixed, we can take M and R large such that

$$\frac{d}{dt} \int |f_{\mathcal{B}}(t)| H^l \leq 0,$$

which implies

$$\int |f_{\mathcal{B}}(t)| H^l \leq \int |f_0| H^l := Y_5,$$

and using Holder's inequality:

$$\int |f_{\mathcal{B}}(t)| H^k \leq \left(\int |f_{\mathcal{B}}(t)| H^{k-1+\frac{\gamma}{2}} \right)^\alpha \left(\int |f_{\mathcal{B}}(t)| H^l \right)^{1-\alpha},$$

with

$$\alpha = \frac{l-k}{l-k+1-\frac{\gamma}{2}} \in [0, 1],$$

which means

$$\left(\int |f_{\mathcal{B}}(t)| H^k \right)^\frac{1}{\alpha} \left(\int |f_{\mathcal{B}}(t)| H^l \right)^\frac{\alpha-1}{\alpha} \leq \int |f_{\mathcal{B}}(t)| H^{k-1+\frac{\gamma}{2}},$$

with

$$\alpha = \frac{l-k}{l-k+1-\frac{\gamma}{2}} \in [0, 1],$$

taking the Holder's inequality into 4.2 we have

$$\frac{d}{dt} Y_4(t) \leq -C (Y_4(t))^\frac{1}{\alpha} Y_5^\frac{\alpha-1}{\alpha},$$

using $Y_4(0) \leq Y_5$, we deduce by a simple argument

$$Y_4(t) \leq C_\alpha \frac{1}{(1+t)^{\frac{\alpha}{1-\alpha}}} Y_5,$$

which means we get a polynomial decay

$$\|\mathcal{S}_B(t)\|_{L^p(H^l) \rightarrow L^p(H^k)} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l-k}{1-\frac{\gamma}{2}}, \quad \forall 0 < k < l,$$

we can write it in this way

$$\|\mathcal{S}_B(t)\|_{L^p(H^l) \rightarrow L^p(H^{l\theta})} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l(1-\theta)}{1-\frac{\gamma}{2}}, \quad \forall 0 < \theta < 1, 0 < l,$$

so the proof is ended □

5. REGULARIZATION LEMMA

This section we prove a regularization lemma.

In this section we will denote $\mathcal{L}^* = \mathcal{L}_{G^{-1/2}}^* = \mathcal{S} - \mathcal{T}$ be the dual operator of \mathcal{L} on $L^2(G^{-1/2})$ such that

$$\int (\mathcal{L}f)gG^{-1} = \int (\mathcal{L}^*g)fG^{-1}.$$

for any smooth function f, g , and denote $\mathcal{B}^* = \mathcal{L}^* - M\chi_R$

Lemma 5.1. *For any $0 \leq \delta < 1$, there exist $\eta > 0$ such that*

$$\|\mathcal{S}_B(t)f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

and for \mathcal{B}^* we have similar result

$$\|\mathcal{S}_{B^*}(t)f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

in particular by duality this implies

$$\|\mathcal{S}_B(t)f\|_{L^\infty(G^{-1/2})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^2(G^{-1/2})}, \quad \forall t \in [0, \eta].$$

We start with some elementary lemmas.

Lemma 5.2. *For any $0 \leq \delta < 1$, we have*

$$\begin{aligned} \int f(\mathcal{L}g)G^{-(1+\delta)} + \int g(\mathcal{L}f)G^{-(1+\delta)} &= -2 \int \nabla_v(fG^{-1}) \cdot \nabla_v(gG^{-1})G^{1-\delta} \\ &\quad + \delta d \int fgG^{-(1+\delta)} - \delta(1-\delta) \int v^2 fgG^{-(1+\delta)}, \end{aligned}$$

in particular

$$\begin{aligned} \int f(\mathcal{L}f)G^{-(1+\delta)} &= - \int |\nabla_v(fG^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2} \int f^2 G^{-(1+\delta)} \\ &\quad - \frac{\delta(1-\delta)}{2} \int v^2 f^2 G^{-(1+\delta)}, \end{aligned}$$

we also have another version of $\int f(\mathcal{L}f)G^{-(1+\delta)}$

$$\begin{aligned} \int f(\mathcal{L}f)G^{-(1+\delta)} &= - \int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2} \int v^2 f^2 G^{-(1+\delta)} \\ &\quad + \frac{(2+\delta)d}{2} \int f^2 G^{-(1+\delta)}, \end{aligned}$$

and all the results above remain true when \mathcal{L} is replaced by \mathcal{L}^* .

Proof of Lemma 5.2: First remember $\mathcal{T}(G^{-(1+\delta)}) = 0$, so we have

$$\int f(\mathcal{T}g)G^{-(1+\delta)} = \int \mathcal{T}(fG^{-(1+\delta)})g = - \int (\mathcal{T}f)gG^{-(1+\delta)},$$

which implies

$$\int f(\mathcal{T}g)G^{-(1+\delta)} + \int (\mathcal{T}f)gG^{-(1+\delta)} = 0,$$

As $\mathcal{L} = \mathcal{S} + \mathcal{T}$, $\mathcal{L}^* = \mathcal{S} - \mathcal{T}$, as the \mathcal{T} term is 0, so the computation result of \mathcal{L} and \mathcal{L}^* is the same, for the term with operator \mathcal{S} we have

$$\begin{aligned} \int f(\mathcal{S}g)G^{-(1+\delta)} &= - \int \nabla_v(fG^{-(1+\delta)}) \cdot (\nabla_v g + vg) \\ &= - \int (\nabla_v f + (1+\delta)vf) \cdot (\nabla_v g + vg)G^{-(1+\delta)} \\ &= - \int \nabla_v(fG^{-1}) \cdot \nabla_v(gG^{-1})G^{1-\delta} - \int \delta v^2 fgG^{-(1+\delta)} - \int \delta f v \cdot \nabla_v g G^{-(1+\delta)}, \end{aligned}$$

using integration by parts, we have

$$\begin{aligned} \int \delta v f \nabla_v g G^{-(1+\delta)} &= - \int \delta g \nabla_v \cdot (vfG^{-(1+\delta)}) \\ &= - \int \delta v \cdot \nabla_v f g G^{-(1+\delta)} - \int \delta d f g G^{-(1+\delta)} - \int \delta(1+\delta)v^2 f g G^{-(1+\delta)}, \end{aligned}$$

so

$$\begin{aligned} &\int (f(\mathcal{S}g) + g(\mathcal{S}f))G^{-(1+\delta)} \\ &= -2 \int \nabla_v(fG^{-1}) \cdot \nabla_v(gG^{-1})G^{1-\delta} + \delta d \int f g G^{-(1+\delta)} - \delta(1-\delta) \int v^2 f g G^{-(1+\delta)}, \end{aligned}$$

Gathering the two terms, the proof of the first two equalities are done. \square
 For the last equality, we have:

$$\begin{aligned}
 \int f \mathcal{S} f G^{-(1+\delta)} &= - \int (\nabla_v f + (1+\delta)vf) \cdot (\nabla_v f + vf) G^{-(1+\delta)} \\
 &= - \int |\nabla_v f|^2 G^{-(1+\delta)} - \int (1+\delta)v^2 f^2 G^{-(1+\delta)} - \int (2+\delta)fv \cdot \nabla_v f G^{-(1+\delta)} \\
 &= - \int |\nabla_v f|^2 G^{-(1+\delta)} - \int (1+\delta)v^2 f^2 G^{-(1+\delta)} + \frac{2+\delta}{2} \int \nabla_v \cdot (vG^{-(1+\delta)}) f^2 \\
 &= - \int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2} \int v^2 f^2 G^{-(1+\delta)} + \frac{(2+\delta)d}{2} \int f^2 G^{-(1+\delta)}.
 \end{aligned}$$

Putting together the above equality with $\int f \mathcal{T} f G^{-(1+\delta)} = 0$ the proof is done. \square

Note in the following of this section we will split into two cases, and denote f_t in two ways, $f_t = S_{\mathcal{B}}(t)f_0$ or $f_t = S_{\mathcal{B}^*}(t)f_0$.

Lemma 5.3. *When $f_t = S_{\mathcal{B}}(t)f_0$, we define an energy functional*

$$\begin{aligned}
 \mathcal{F}(t, f_t) &:= A \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2 \|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\
 &\quad + 2ct^4 \langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2,
 \end{aligned}$$

when $f_t = S_{\mathcal{B}^*}(t)f_0$, we define an energy functional

$$\begin{aligned}
 \mathcal{F}(t, f_t) &:= A \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2 \|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\
 &\quad - 2ct^4 \langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2,
 \end{aligned}$$

with $a, b, c > 0, c \leq \sqrt{ab}$ and A large enough. Then for both cases we have

$$\frac{d}{dt} F(t, f_t) \leq -K (\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})} + t^4 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}) + C \int f_t^2 G^{-(1+\delta)},$$

for all $t \in [0, \eta]$ and for some $K > 0, C > 0$.

Proof of Lemma 5.3

We split the computation into several parts and then put them together.

Step 1: (1) For the case $f_t = S_{\mathcal{B}}(t)f_0$, first using both two equalities in Lemma 5.2 we have

$$\begin{aligned}
 \frac{d}{dt} \int f^2 G^{-(1+\delta)} &= \int f(\mathcal{L} - M\chi_R) f G^{-(1+\delta)} \\
 &= \frac{1-\delta}{2} \int f \mathcal{L} f G^{-(1+\delta)} + \frac{1+\delta}{2} \int f \mathcal{L} f G^{-(1+\delta)} - \int M\chi_R f^2 G^{-(1+\delta)} \\
 &\leq -\frac{1-\delta}{2} \int |\nabla_v f|^2 G^{-(1+\delta)} - \frac{1+\delta}{2} \int |\nabla_v (fG^{-1})|^2 G^{1-\delta} + C \int f^2 G^{-(1+\delta)} \\
 &\leq -\frac{1-\delta}{2} \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int f^2 G^{-(1+\delta)}.
 \end{aligned}$$

(2) As Lemma 5.2 is also true for \mathcal{L}^* , so the above inequality is also true for \mathcal{L}^* .

Step 2 : (1) For the case $f_t = S_{\mathcal{B}}(t)f_0$, we easily compute

$$\partial_{x_i}\mathcal{L}f = \mathcal{L}\partial_{x_i}f + \partial_{x_i}(\nabla_x V(x) \cdot \nabla_v f) = \mathcal{L}\partial_{x_i}f + \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f,$$

using this and Lemma 5.2 we have

$$\begin{aligned} \frac{d}{dt} \int (\partial_{x_i} f)^2 G^{-(1+\delta)} &= \int \partial_{x_i} f \partial_{x_i} (\mathcal{L} - M\chi_R) f G^{-(1+\delta)} \\ &= - \int |\nabla_v (\partial_{x_i} f G^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2} \int (\partial_{x_i} f)^2 G^{-(1+\delta)} \\ &\quad - \frac{\delta(1-\delta)}{2} \int v^2 (\partial_{x_i} f)^2 G^{-(1+\delta)} + \int \partial_{x_i} f \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f G^{-(1+\delta)} \\ &\quad - \int M\chi_R |\partial_{x_i} f|^2 G^{-(1+\delta)} + \int M \partial_{x_i} f \partial_{x_i} \chi_R f G^{-(1+\delta)}, \end{aligned}$$

using Cauchy-Schwarz inequality and summing up by i, we get

$$\begin{aligned} &\frac{d}{dt} \int |\nabla_x f|^2 G^{-(1+\delta)} \\ &\leq - \sum_{i=1}^d \int |\nabla_v (\partial_{x_i} f G^{-1})|^2 G^{1-\delta} - \frac{\delta(1-\delta)}{2} \int v^2 (\nabla_x f)^2 G^{-(1+\delta)} \\ &\quad + C \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int |\nabla_x f|^2 G^{-(1+\delta)} + C \int |f|^2 G^{-(1+\delta)}, \end{aligned}$$

for some $C > 0$.

(2) For the case $f_t = S_{\mathcal{B}^*}(t)f_0$, we have

$$\partial_{x_i}\mathcal{L}^*f = \mathcal{L}^*\partial_{x_i}f - \partial_{x_i}(\nabla_x V(x) \cdot \nabla_v f) = \mathcal{L}^*\partial_{x_i}f - \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f,$$

using the same argument we still have the same result as above.

Step 3: (1) For the case $f_t = S_{\mathcal{B}}(t)f_0$, we similarly compute

$$\partial_{v_i}\mathcal{L}f = \mathcal{L}\partial_{v_i}f - \partial_{x_i}f + \partial_{v_i}f,$$

using this inequality and Lemma 5.2, we have

$$\begin{aligned}
\frac{d}{dt} \int (\partial_{v_i} f)^2 G^{-(1+\delta)} &= \int \partial_{v_i} f \partial_{v_i} (\mathcal{L} - M\chi_R) f G^{-(1+\delta)} \\
&= - \int |\nabla_v (\partial_{v_i} f G^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2} \int (\partial_{v_i} f)^2 G^{-(1+\delta)} \\
&\quad - \frac{\delta(1-\delta)}{2} \int v^2 (\partial_{v_i} f)^2 G^{-(1+\delta)} - \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)} \\
&\quad + \int |\partial_{v_i} f|^2 G^{-(1+\delta)} - \int M\chi_R |\partial_{v_i} f|^2 G^{-(1+\delta)} + M \partial_{v_i} f \partial_{v_i} \chi_R f G^{-(1+\delta)},
\end{aligned}$$

using Cauchy-Schwarz inequality and summing up by i we get

$$\begin{aligned}
&\frac{d}{dt} \int |\nabla_v f|^2 G^{-(1+\delta)} \\
&\leq - \sum_{i=1}^d \int |\nabla_v (\partial_{v_i} f G^{-1})|^2 G^{1-\delta} + C \int |\nabla_x f| |\nabla_v f| G^{-(1+\delta)} \\
&\quad + C \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int |f|^2 G^{-(1+\delta)} - \frac{\delta(1-\delta)}{2} \int v^2 (\nabla_v f)^2 G^{-(1+\delta)}.
\end{aligned}$$

For the case $f_t = S_{\mathcal{B}^*}(t)f_0$, we have

$$\partial_{v_i} \mathcal{L}^* f = \mathcal{L}^* \partial_{v_i} f + \partial_{x_i} f + \partial_{v_i} f,$$

using the same argument we still have the same result as above.

Step 4:(1) For the case $f_t = S_{\mathcal{B}}(t)f_0$, for the crossing term we have

$$\begin{aligned}
\frac{d}{dt} \int 2\partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} &= \left(\int \partial_{v_i} f \partial_{x_i} \mathcal{L} f G^{-(1+\delta)} + \int \partial_{v_i} \mathcal{L} f \partial_{x_i} f G^{-(1+\delta)} \right) \\
&\quad - \left(\int \partial_{v_i} f \partial_{x_i} (M\chi_R f) G^{-(1+\delta)} + \int \partial_{x_i} (M\chi_R f) \partial_{v_i} f G^{-(1+\delta)} \right) \\
&:= W_1 + W_2,
\end{aligned}$$

we split it into two parts, for the first part still using

$$\partial_{x_i} \mathcal{L} f = \mathcal{L} \partial_{x_i} f + \partial_{x_i} (\nabla_x V(x) \cdot \nabla_v f) = \mathcal{L} \partial_{x_i} f + \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f,$$

and

$$\partial_{v_i} \mathcal{L} f = \mathcal{L} \partial_{v_i} f - \partial_{x_i} f + \partial_{v_i} f,$$

we have

$$\begin{aligned}
W_1 &= \int \partial_{v_i} f \mathcal{L} (\partial_{x_i} f) G^{-(1+\delta)} + \int \mathcal{L} (\partial_{v_i} f) \partial_{x_i} f G^{-(1+\delta)} \\
&\quad + \int \partial_{v_i} f \sum_{j=1}^d \partial_{x_i x_j} V(x) \partial_{v_j} f G^{-(1+\delta)} - \int |\partial_{x_i} f|^2 G^{-(1+\delta)} + \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)},
\end{aligned}$$

using Lemma 5.2, we have

$$\begin{aligned}
 W_1 &= - \int 2\nabla_v(\partial_{v_i} f G^{-1}) \cdot \nabla_v(\partial_{x_i} f G^{-1}) G^{1-\delta} + \delta d \int \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} \\
 &\quad - \delta(1-\delta) \int v^2 \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} + \int \partial_{v_i} f \sum_{j=1}^d \partial_{x_i x_j} V(x) \partial_{v_j} f G^{-(1+\delta)} \\
 &\quad - \int |\partial_{x_i} f|^2 G^{-(1+\delta)} + \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)}.
 \end{aligned}$$

For the W_2 term we have

$$\begin{aligned}
 W_2 &= - \int \partial_{v_i} f \partial_{x_i} (M \chi_R f) G^{-(1+\delta)} - \int \partial_{x_i} (M \chi_R f) \partial_{v_i} f G^{-(1+\delta)} \\
 &= - \int 2M \chi_R \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)} + \int \partial_{v_i} f M \partial_{x_i} \chi_R f G^{-(1+\delta)} + \int M \partial_{v_i} \chi_R f \partial_{x_i} f G^{-(1+\delta)} \\
 &\leq C \int |\partial_{x_i} f| |\partial_{v_i} f| G^{-(1+\delta)} + C \int |\partial_{v_i} f| |f| G^{-(1+\delta)} + C \int |f| |\partial_{x_i} f| G^{-(1+\delta)},
 \end{aligned}$$

combining the two parts, using Cauchy-Schwarz inequality and summing up by i we get

$$\begin{aligned}
 &\frac{d}{dt} \int 2\nabla_x f \cdot \nabla_v f G^{-(1+\delta)} \\
 &\leq - \sum_{i=1}^d \int 2\nabla_v(\partial_{v_i} f G^{-1}) \cdot \nabla_v(\partial_{x_i} f G^{-1}) G^{1-\delta} - \frac{1}{2} \int |\nabla_x f|^2 G^{-(1+\delta)} \\
 &\quad + C \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int |f|^2 G^{-(1+\delta)} - \delta(1-\delta) \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}.
 \end{aligned}$$

(2) For the case $f_t = S_{\mathcal{B}^*}(t) f_0$ using

$$\partial_{x_i} \mathcal{L}^* f = \mathcal{L}^* \partial_{x_i} f - \partial_{x_i} (\nabla_x V(x) \cdot \nabla_v f) = \mathcal{L}^* \partial_{x_i} f - \sum_{j=1}^d \partial_{x_i x_j}^2 V \partial_{v_j} f,$$

and

$$\partial_{v_i} \mathcal{L}^* f = \mathcal{L}^* \partial_{v_i} f + \partial_{x_i} f + \partial_{v_i} f,$$

we will get a slightly different result

$$\begin{aligned}
 &-\frac{d}{dt} \int 2\nabla_x f \cdot \nabla_v f G^{-(1+\delta)} \\
 &\leq \sum_{i=1}^d \int 2\nabla_v(\partial_{v_i} f G^{-1}) \cdot \nabla_v(\partial_{x_i} f G^{-1}) G^{1-\delta} - \frac{1}{2} \int |\nabla_x f|^2 G^{-(1+\delta)} \\
 &\quad + C \int |\nabla_v f|^2 G^{-(1+\delta)} + C \int |f|^2 G^{-(1+\delta)} + \delta(1-\delta) \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}.
 \end{aligned}$$

Step 5: (1) For the case $f_t = S_{\mathcal{B}}(t)f_0$, gathering all the terms together, recall

$$\begin{aligned} \mathcal{F}(t, f_t) &:= A\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\ &\quad + 2ct^4\langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6\|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2, \end{aligned}$$

with $a, b, c > 0, c \leq \sqrt{ab}$ and A large enough. We easily compute

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}(t, f_t) \\ &= A\frac{d}{dt}\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2\frac{d}{dt}\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\ &\quad + 2ct^4\frac{d}{dt}\langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6\frac{d}{dt}\|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\ &\quad + 2at\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + 8ct^3\langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 + 6bt^5\|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2. \end{aligned}$$

Using the results in Step 1-4 and gathering all the terms together we have

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}(t, f_t) \\ &\leq (2at - \frac{A(1-\delta)}{2} + Cat^2 + 2Ct^4c + Cbt^6) \int |\nabla_v f_t|^2 G^{-(1+\delta)} \\ &\quad + (6bt^5 - ct^4 + Cbt^6) \int |\nabla_x f_t|^2 G^{-(1+\delta)} + C \int f_t^2 G^{-(1+\delta)} \\ &\quad + (8ct^3 + Cat^2) \int |\nabla_v f_t| |\nabla_x f_t| G^{-(1+\delta)} \\ &\quad - (at^2 \sum_{i=1}^d \int |\nabla_v(\partial_{v_i} f_t G^{-1})|^2 G^{1-\delta} + bt^6 \sum_{i=1}^d \int |\nabla_v(\partial_{x_i} f_t G^{-1})|^2 G^{1-\delta} \\ &\quad + 2ct^4 \sum_{i=1}^d \int 2\nabla_v(\partial_{v_i} f_t G^{-1}) \cdot \nabla_v(\partial_{x_i} f_t G^{-1}) G^{1-\delta} - \frac{\delta(1-\delta)}{2} (at^2 \int v^2 (\nabla_v f)^2 G^{-(1+\delta)} \\ &\quad + bt^6 \int v^2 (\nabla_x f)^2 G^{-(1+\delta)} + 2ct^4 \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}), \end{aligned}$$

by our taking on a, b, c we can have

$$|2ct^4 \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}| \leq at^2 \int v^2 (\nabla_v f)^2 G^{-(1+\delta)} + bt^6 \int v^2 (\nabla_x f)^2 G^{-(1+\delta)},$$

and

$$\begin{aligned} &|2ct^4 \sum_{i=1}^d \int 2\nabla_v(\partial_{v_i} f_t G^{-1}) \cdot \nabla_v(\partial_{x_i} f_t G^{-1}) G^{1-\delta}| \\ &\leq at^2 \sum_{i=1}^d \int |\nabla_v(\partial_{v_i} f_t G^{-1})|^2 G^{1-\delta} + bt^6 \sum_{i=1}^d \int |\nabla_v(\partial_{x_i} f_t G^{-1})|^2 G^{1-\delta}, \end{aligned}$$

by taking A large and $0 < \eta$ small ($t \in [0, \eta]$), we have for some $K > 0$:

$$\frac{d}{dt}\mathcal{F}(t, f_t) \leq -K(\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})} + t^4\|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}) + C \int f_t^2 G^{-(1+\delta)}.$$

(2) For the case $f_t = S_{\mathcal{B}^*}(t)f_0$, the only change is that the last three terms changes to

$$\begin{aligned} & (Cat^2 - 8ct^3) \int |\nabla_v f_t| |\nabla_x f_t| G^{-(1+\delta)} \\ & - (at^2 \sum_{i=1}^d \int |\nabla_v(\partial_{v_i} f_t G^{-1})|^2 G^{1-\delta} + bt^6 \sum_{i=1}^d \int |\nabla_v(\partial_{x_i} f_t G^{-1})|^2 G^{1-\delta} \\ & - 2ct^4 \sum_{i=1}^d \int 2\nabla_v(\partial_{v_i} f_t G^{-1}) \cdot \nabla_v(\partial_{x_i} f_t G^{-1}) G^{1-\delta} - \frac{\delta(1-\delta)}{2} (at^2 \int v^2 (\nabla_v f)^2 G^{-(1+\delta)} \\ & + bt^6 \int v^2 (\nabla_x f)^2 G^{-(1+\delta)} - 2ct^4 \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}), \end{aligned}$$

which will not change the result. The proof of Lemma 5.3 is ended. \square

The following proof of this section is true for both cases.

To prove Lemma 5.1 we still need some other lemmas.

Lemma 5.4. *For any $0 \leq \delta < 1$ we have*

$$\int |\nabla_{x,v}(fG^{-1/2(1+\delta)})|^2 \leq \int |\nabla_{x,v}f|^2 G^{-(1+\delta)} + C \int f^2 G^{-(1+\delta)},$$

Prove of Lemma 5.4 : For any weight function m we have

$$\begin{aligned} \int |\nabla_x(fm)|^2 &= \int |\nabla_x fm + \nabla_x m f|^2 \\ &= \int |\nabla_x f|^2 m^2 + \int |\nabla_x m|^2 f^2 + \int 2f \nabla_x f m \nabla_x m \\ &= \int |\nabla_x f|^2 m^2 + \int |\nabla_x m|^2 f^2 - \int \frac{1}{2} f^2 \Delta_x(m^2) \\ &= \int |\nabla_x f|^2 m^2 + \int (|\nabla_x m|^2 - \frac{1}{2} \Delta_x(m^2)) f^2, \end{aligned}$$

taking $m = G^{-1/2(1+\delta)}$ we have

$$\begin{aligned} & \int |\nabla_x(fG^{-1/2(1+\delta)})|^2 \\ &= \int |\nabla_x f|^2 G^{-(1+\delta)} + \int (|\nabla_x G^{-1/2(1+\delta)}|^2 - \frac{1}{2} \Delta_x G^{-(1+\delta)}) f^2 \\ &= \int |\nabla_x f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4} |\nabla_x V(x)|^2 + \frac{1+\delta}{2} \Delta_x V(x)) f^2 G^{-(1+\delta)} \\ &\leq \int |\nabla_x f|^2 G^{-(1+\delta)} + C \int f^2 G^{-(1+\delta)}, \end{aligned}$$

similarly

$$\begin{aligned}
& \int |\nabla_v(fG^{-1/2(1+\delta)})|^2 \\
&= \int |\nabla_v f|^2 G^{-(1+\delta)} + \int (|\nabla_v G^{-1/2(1+\delta)}|^2 - \frac{1}{2}\Delta_v G^{-(1+\delta)}) f^2 \\
&= \int |\nabla_v f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4}v^2 + \frac{1+\delta}{2}d) f^2 G^{-(1+\delta)} \\
&\leq \int |\nabla_v f|^2 G^{-(1+\delta)}.
\end{aligned}$$

Putting together the two inequalities we obtain the result. \square

Lemma 5.5. *Nash's inequality: for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, there exist a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}} \leq C_d \|f\|_{L^1}^{2/d} \|\nabla_v f\|_{L^2},$$

Proof of Nash's inequality: For any $R > 0$ we have

$$\begin{aligned}
\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 &= \int_{|\xi| \leq R} |\hat{f}|^2 + \int_{|\xi| \geq R} |\hat{f}|^2 \\
&\leq c_d R^d \|\hat{f}\|_{L^\infty}^2 + \frac{1}{R^2} \int_{|\xi| \leq R} \xi^2 |\hat{f}|^2 \\
&\leq c_d R^d \|f\|_{L^1}^2 + \frac{1}{R^2} \|\nabla_v f\|_{L^2}^2,
\end{aligned}$$

taking an optimal R by setting $R = (\|f\|_{L^2}^2 / c_d \|f\|_{L^1}^2)^{\frac{1}{d+2}}$, we are done the proof. \square

Lemma 5.6. *For any $0 \leq \delta < 1$ we have*

$$\frac{d}{dt} \int |f| G^{-1/2(1+\delta)} \leq d \int |f| G^{-1/2(1+\delta)},$$

which implies

$$\int |f_t| G^{-1/2(1+\delta)} \leq C e^{dt} \int |f_0| G^{-1/2(1+\delta)},$$

in particular we have

$$\int |f_t| G^{-1/2(1+\delta)} \leq C \int |f_0| G^{-1/2(1+\delta)}, \quad \forall t \in [0, \eta],$$

Proof of Lemma 5.6: For the case $f_t = S_{\mathcal{B}}(t)f_0$, by Lemma 4.2, we have

$$\begin{aligned} & \frac{d}{dt} \int |f| G^{-1/2(1+\delta)} \\ &= \int |f| (\Delta_v G^{-1/2(1+\delta)} - v \cdot \nabla_v G^{-1/2(1+\delta)} \\ & \quad + v \cdot \nabla_x G^{-1/2(1+\delta)} - \nabla V(x) \cdot \nabla_v G^{-1/2(1+\delta)} - M \chi_R G^{-1/2(1+\delta)}) \\ & \leq \int |f| \left(\frac{1+\delta}{2} d - \frac{(1+\delta)(1-\delta)}{4} v^2 \right) G^{-1/2(1+\delta)} \leq \int |f| d G^{-1/2(1+\delta)}, \end{aligned}$$

as $\mathcal{T}G^{-1/2(1+\delta)} = 0$, the result is still true for the case $f_t = S_{\mathcal{B}^*}(t)f_0$ \square

Now we come to the proof of our original lemma.

Proof of Lemma 5.1: We define

$$\mathcal{G}(t, f_t) = B \|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + t^Z \mathcal{F}(t, f_t),$$

with $Z > 0$ to be fixed and \mathcal{F} defined in Lemma 5.3. It's important to recall here that

$$\begin{aligned} \mathcal{F}(t, f_t) &:= A \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + at^2 \|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \\ & \quad + 2ct^4 \langle \nabla_v f_t, \nabla_x f_t \rangle_{L^2(G^{-1/2(1+\delta)})}^2 + bt^6 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2. \end{aligned}$$

We choose $t \in [0, \eta]$, η small such that $(a+b+c)Zt^{Z+1} \leq \frac{1}{2}Kt^Z$ (K is also in Lemma 5.3). Then by Lemma 5.6 and Lemma 5.3

$$\begin{aligned} \frac{d}{dt} \mathcal{G}(t, f_t) &\leq dB \|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + Zt^{Z-1} \mathcal{F}(t, f_t) \\ & \quad - Kt^Z (\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + t^4 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2) \\ & \quad + Ct^Z \int f_t^2 G^{-(1+\delta)} \\ &\leq dB \|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + Ct^{Z-1} \int f_t^2 G^{-(1+\delta)} \\ & \quad - \frac{K}{2} t^Z (\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + t^4 \|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}^2), \end{aligned}$$

Nash's inequality and Lemma 5.4 implies

$$\begin{aligned} \int f_t^2 G^{-(1+\delta)} &\leq \left(\int |f_t| G^{-1/2(1+\delta)} \right)^{\frac{4}{d+2}} \left(\int |\nabla_{x,v}(f_t G^{-1/2(1+\delta)})|^2 \right)^{\frac{d}{d+2}} \\ &\leq \left(\int |f_t| G^{-1/2(1+\delta)} \right)^{\frac{4}{d+2}} \left(\int |\nabla_{x,v} f_t|^2 G^{-(1+\delta)} + C \int f_t^2 G^{-(1+\delta)} \right)^{\frac{d}{d+2}}, \end{aligned}$$

using Young's inequality we have

$$\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \leq C_\epsilon t^{-5d} \|f\|_{L^1(G^{-1/2(1+\delta)})}^2 + \epsilon t^5 (\|\nabla_{x,v} f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 + C \|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2),$$

we deduce

$$\|f_t\|_{L^2(G^{-1/2(1+\delta)})}^2 \leq 2C_\epsilon t^{-5d} \|f\|_{L^1(G^{-1/2(1+\delta)})}^2 + 2\epsilon t^5 \|\nabla_{x,v} f_t\|_{L^2(G^{-1/2(1+\delta)})}^2,$$

taking ϵ small we have

$$\frac{d}{dt}\mathcal{G}(t, f_t) \leq dB\|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2 + C_1 t^{Z-1-5d}\|f_t\|_{L^1(G^{-1/2(1+\delta)})}^2,$$

finally choose $Z = 1 + 5d$, and using Lemma 5.6 then we deduce

$$\forall t \in [0, \eta], \quad \mathcal{G}(t, f_t) \leq \mathcal{G}(0, f_0) + C_1\|f_0\|_{L^1(G^{-1/2(1+\delta)})}^2 \leq C_2\|f_0\|_{L^1(G^{-1/2(1+\delta)})}^2,$$

the proof is ended. \square

6. L^p CONVERGENCE FOR THE KFP MODEL

To reach the final convergence we first need:

Lemma 6.1. *Recall $\mathcal{A} = M\chi_R$, $\mathcal{B} = \mathcal{L} - M\chi_R$, using the estimates in former sections, for any $\epsilon > 0$ small, we can have some estimates on $\mathcal{AS}_{\mathcal{B}}(t)$*

$$\|\mathcal{AS}_{\mathcal{B}}(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

and

$$\|\mathcal{AS}_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

for some $a > 0$, and for any $0 < b < \frac{\gamma}{2-\gamma}$ we have

$$\|\mathcal{AS}_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} e^{-at^b},$$

for $\alpha = \frac{5d+1}{2}$ and some $a > 0$.

Proof of Lemma 6.1 : (1) Using the result in Lemma 3.3 we have

$$\|\mathcal{S}_{\mathcal{B}}(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

recall $\mathcal{A} = M\chi_R$, so

$$\|\mathcal{A}\|_{L^2(G^{-\frac{1}{2}}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim 1,$$

combining the above 2 inequalities we are done.

(2) Also by Lemma 3.3 we have

$$\|\mathcal{S}_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon}) \rightarrow L^1} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

and obviously

$$\|\mathcal{A}\|_{L^1 \rightarrow L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim 1,$$

so combine the 2 inequalities we are done.

(3) For the third inequality, we split it into two parts. When $t \in [0, \eta]$, $e^{-at^{\frac{\gamma}{2-\gamma}}} \geq e^{-a\eta^{\frac{\gamma}{2-\gamma}}}$, and $\mathcal{A} \leq MI$, using Lemma 5.1, we have

$$\|\mathcal{AS}_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} \lesssim t^{-\alpha} e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad a > 0, \forall t \in [0, \eta].$$

When $t \geq \eta$, by Lemma 5.1 we have

$$\|\mathcal{S}_B(\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim \eta^\alpha \lesssim 1,$$

using Lemma 3.3, we have

$$\|\mathcal{S}_B(t - \eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-\frac{1}{2}})} \leq C e^{-a(t-\eta)^{\frac{\gamma}{2-\gamma}}} \leq C e^{-at^{\frac{\gamma}{2-\gamma}}},$$

for \mathcal{A} obviously

$$\|\mathcal{A}\|_{L^2(G^{-\frac{1}{2}}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim 1,$$

when $t > \eta$, gathering all the inequalities above, we have

$$\|\mathcal{A}\mathcal{S}_B(t)\|_{L^1(G^{-1/2(1+2\epsilon)}) \rightarrow L^2(G^{-1/2(1+2\epsilon)})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}} \lesssim t^{-\alpha} e^{-at^b}.$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, combining the two cases, the proof is ended. \square

Lemma 6.2. *similarly as Lemma 6.1, for any $p \in (2, \infty)$, we have*

$$\|\mathcal{S}_B(t)\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^2(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

and

$$\|\mathcal{S}_B(t)\mathcal{A}\|_{L^p(G^{-1/2}) \rightarrow L^p(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

for some $a > 0$, and for any $0 < b < \frac{\gamma}{2-\gamma}$ we have

$$\|\mathcal{S}_B(t)\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^p(G^{-1/2})} \lesssim t^{-\beta} e^{-at^b},$$

for some $\beta > 0$ and some $a > 0$.

Proof of Lemma 6.2 : (1) Using the result in Lemma 3.3 we have

$$\|\mathcal{S}_B(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

recall $\mathcal{A} = M\chi_R$, so

$$\|\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim 1,$$

combining the above 2 inequalities we are done.

(2) Also by Lemma 3.3 we have

$$\|\mathcal{S}_B(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)}) \rightarrow L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad a > 0,$$

and obviously

$$\|\mathcal{A}\|_{L^p(G^{-1/2}) \rightarrow L^p(G^{-(\frac{p-1}{p}+\epsilon)})} \lesssim 1,$$

so combine the 2 inequalities we are done.

(3) For the third inequality, we split it into two parts. Interpolate between Lemma 3.3 and Lemma 5.1 we have

$$\|\mathcal{S}_B(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^p(G^{-1/2})} \lesssim t^{-\beta},$$

for some $\beta > 0$.

When $t \in [0, \eta]$, $e^{-at\frac{\gamma}{2-\gamma}} \geq e^{-a\eta\frac{\gamma}{2-\gamma}}$, and $\mathcal{A} \leq MI$, we have

$$\|\mathcal{S}_B(t)\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^p(G^{-1/2})} \lesssim t^{-\beta} \lesssim t^{-\beta} e^{-at\frac{\gamma}{2-\gamma}}, a > 0, \forall t \in [0, \eta].$$

When $t \geq \eta$, interpolate between Lemma 3.3 and Lemma 5.1 and take $2\delta = \gamma$ we have

$$\|\mathcal{S}_B(\eta)\|_{L^2(e^{\epsilon H^\delta} G^{-1/2}) \rightarrow L^p(G^{-1/2})} \lesssim \eta^{-\beta} \lesssim 1,$$

using Lemma 3.3, we have

$$\|\mathcal{S}_B(t-\eta)\|_{L^2(e^{2\epsilon H^\delta} G^{-1/2}) \rightarrow L^2(e^{\epsilon H^\delta} G^{-1/2})} \lesssim e^{-a(t-\eta)\frac{\gamma}{2-\gamma}} \lesssim e^{-at\frac{\gamma}{2-\gamma}},$$

for \mathcal{A} obviously

$$\|\mathcal{A}\|_{L^2(G^{-1/2}) \rightarrow L^2(e^{2\epsilon H^\delta} G^{-1/2})} \lesssim 1,$$

when $t > \eta$, gathering all the inequalities above, we have

$$\|\mathcal{A}\mathcal{S}_B(t)\|_{L^1(G^{-1/2(1+2\epsilon)}) \rightarrow L^2(G^{-1/2(1+2\epsilon)})} \lesssim e^{-at\frac{\gamma}{2-\gamma}} \lesssim t^{-\beta} e^{-at^b}.$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, combining the two cases, the proof is ended. \square

Then we need an abstract lemma like Lemma 2.4 in [3].

Lemma 6.3. *let X, Y be two Banach spaces, $S(t)$ a semigroup such that for all $t \geq 0$ and some $0 < a, 0 < b < 1$ we have*

$$\|S(t)\|_{X \rightarrow X} \leq C_X e^{-at^b}, \quad \|S(t)\|_{Y \rightarrow Y} \leq C_Y e^{-at^b},$$

and for some $0 < \alpha$, we have

$$\|S(t)\|_{X \rightarrow Y} \leq C_{X,Y} t^{-\alpha} e^{-at^b}.$$

Then we can have that for all integer $n > 0$

$$\|S^{(*n)}(t)\|_{X \rightarrow X} \leq C_{X,n} t^{n-1} e^{-at^b},$$

similarly

$$\|S^{(*n)}(t)\|_{Y \rightarrow Y} \leq C_{Y,n} t^{n-1} e^{-at^b},$$

and

$$\|S^{(*n)}(t)\|_{X \rightarrow Y} \leq C_{X,Y,n} t^{n-\alpha-1} e^{-at^b},$$

in particular for $\alpha + 1 < n$, and $\forall b^* < b$

$$\|S^{(*n)}(t)\|_{X \rightarrow Y} \leq C_{X,Y,n} e^{-at^{b^*}}.$$

Proof of Lemma 6.3: We use the fact that $t^b \leq s^b + (t-s)^b$ for any $0 \leq s \leq t, 0 < b < 1$, by induction

$$\begin{aligned} \|S^{*(n+1)}(t)\|_{X \rightarrow X} &\leq \int_0^t \|S(t-s)S^{*n}(s)\|_{X \rightarrow X} ds \leq C_n \int_0^t e^{-a(t-s)^b} e^{-as^b} s^{n-1} ds \\ &\leq C_n e^{-at^b} \int_0^t s^{n-1} ds = C_{n+1} e^{-at^b} t^n, \end{aligned}$$

and we compute

$$\begin{aligned}
 \|S^{*(n+1)}(t)\|_{X \rightarrow Y} &\leq \int_0^{t/2} \|S^{*(n)}(t-s)\|_{X \rightarrow Y} \|S(s)\|_{X \rightarrow X} ds \\
 &+ \int_{t/2}^t \|S^{*(n)}(t-s)\|_{Y \rightarrow Y} \|S(s)\|_{X \rightarrow Y} \\
 &\leq C_n C_1 \int_0^{t/2} (t-s)^{n-\alpha-1} e^{-a(t-s)^b} e^{-as^b} ds \\
 &+ C_n C_1 \int_{t/2}^t (t-s)^{n-1} e^{-a(t-s)^b} s^{-\alpha} e^{-as^b} ds \\
 &\leq C_n C_1 e^{-at^b} t^{n-\alpha} \left(\int_0^{1/2} (1-\tau)^{n-\alpha-1} d\tau + \int_{1/2}^1 (1-\tau)^{n-1} \tau^{-\alpha} d\tau \right) \\
 &\leq C_{n+1} t^{n-\alpha} e^{-at^b},
 \end{aligned}$$

the proof of the lemma is ended. □

Then we come to the final proof.

Proof of Theorem 1.1: We only prove the case when $m = H^k$, the case $m = G^{\frac{p-1}{p}}(1+\epsilon)$ is similar. By Duhamel formula, we have

$$\begin{aligned}
 S_{\mathcal{L}} &= S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{L}}) = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}} \\
 &= S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}) = S_{\mathcal{B}} + (S_{\mathcal{L}}\mathcal{A}) * S_{\mathcal{B}},
 \end{aligned}$$

iterate the formula in this way

$$\begin{aligned}
 S_{\mathcal{L}} &= S_{\mathcal{B}} + (S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})) * (\mathcal{A}S_{\mathcal{B}}) \\
 &= S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}}) + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*2)} \\
 &= S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*2)},
 \end{aligned}$$

we can get

$$\begin{aligned}
 S_{\mathcal{L}}(I - \Pi) &= (I - \Pi) \left\{ S_{\mathcal{B}} + \sum_{l=1}^{n-1} (S_{\mathcal{B}}\mathcal{A})^{(*l)} * (S_{\mathcal{B}}) \right\} \\
 &+ \{(I - \Pi)S_{\mathcal{L}}\} * (\mathcal{A}S_{\mathcal{B}}(t))^{*n}.
 \end{aligned}$$

For the first term we have from Theorem 4.1

$$\|S_{\mathcal{B}}(t)\|_{L^1(m) \rightarrow L^1(m^\theta)} \lesssim (1+t)^{-a}.$$

For the second term, we have by Lemma 3.3

$$\|(S_{\mathcal{B}}(t)\mathcal{A})\|_{L^1(m) \rightarrow L^1(m)} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

so by Lemma 6.3

$$\|(S_{\mathcal{B}}(t)\mathcal{A})^{*l}\|_{L^1(m) \rightarrow L^1(m)} \lesssim t^{l-1} e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

For the last term we have from Theorem 4.1

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(m) \rightarrow L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

by Lemma 6.1, for any $0 < b < \frac{\gamma}{2-\gamma}$, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n-1)}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{n-\alpha-2} e^{-at^b},$$

by Theorem 3.1

$$\|S_{\mathcal{L}}(t)(I - \Pi)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \rightarrow L^2(G^{-1/2})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

and for the identity operator I , obviously we have

$$\|I\|_{L^2(G^{-\frac{1}{2}}) \rightarrow L^1(m^\theta)} \lesssim 1,$$

so taking $n > \alpha + 2$ and gathering all the terms together, and use this inequality for any $\epsilon > 0$

$$\begin{aligned} \int_0^t (1+t-s)^{-a} e^{-Cs^b} ds &\leq \int_0^t \left(\frac{1+s}{1+t}\right)^a e^{-Cs^b} ds = (1+t)^{-a} \int_0^t (1+s)^{-a} e^{-Cs^b} ds \\ &= (1+t)^{-a} \left(\int_0^{t^\epsilon} + \int_{t^\epsilon}^t \right) (1+s)^{-a} e^{-Cs^b} ds \lesssim (1+t)^{-a+\epsilon}, \end{aligned}$$

we deduce

$$\|S_{\mathcal{L}}(t)(I - \Pi)\|_{L^1(m) \rightarrow L^1(m^\theta)} \lesssim (1+t)^{-a+\epsilon},$$

for the case $m = G^{-(\frac{p-1}{p}+\epsilon)}$, the only change is this time we have

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)}) \rightarrow L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

using similar argument we can get

$$\|S_{\mathcal{L}}(t)(I - \Pi)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)}) \rightarrow L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

which implies Theorem 1.1 □

REFERENCES

- [1] DOLBEAULT, J., MOUHOT, C., AND SCHMEISER, C. Hypocoercivity for kinetic equations conserving mass. *Trans. Amer. Math. Soc.* 367 (2015), no. 6, 3807-3828
- [2] DOLBEAULT, J., MOUHOT, C., AND SCHMEISER, C. Hypocoercivity for kinetic equations with linear relaxation terms. *C. R. Math. Acad. Sci. Paris* 347, 9-10 (2009), 511-516.
- [3] KAVIAN, O., MISCHLER, S. The Fokker Planck equation with subcritical confinement force (arXiv 2015)
- [4] MISCHLER, S., MOUHOT, C. Exponential stability of slowing decaying solutions to the Kinetic-Fokker-Planck equation *Arch. Ration. Mech. Anal.* 221 (2016), no. 2, 677-723.
- [5] VILLANI, C. Hypocoercivity. *Mem. Amer. Math Soc.* 202(2009), no. 950
- [6] DUAN, R. Hypocoercivity of linear degenerately dissipative kinetic equations. *Nonlinearity* 24, 8 (2011), 2165-2189.

- [7] ECKMANN, J.-P., AND HAIRER, M. Spectral properties of hypoelliptic operators. *Comm. Math. Phys.* 235, 2 (2003), 233-253
- [8] MOUHOT, C., AND NEUMANN, L. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. *Nonlinearity* 19, 4 (2006), 969-998.
- [9] LIEB, E. AND LOSS, M. *Analysis* 2nd. American Mathematical Society.
- [10] RÖCKNER, M., AND WANG, F.-Y Weak Poincare inequalities and L^2 -convergence rates of Markov semigroups. *J. Funct. Anal.* 185, 2 (2001), 564-603.
- [11] MOUHOT, C. Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials. *Comm. Math. Phys.* 261, 3 (2006), 629-672.
- [12] GUALDANI, M. P., MISCHLER, S., AND MOUHOT, C. Factorization of non-symmetric operators and exponential H-Theorem. hal-00495786.