THE KINETIC FOKKER-PLANCK EQUATION WITH WEAK CONFINEMENT FORCE

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Abstract. We consider the kinetic Fokker-Planck equation with weak confinement force which do not have an exponential convergence to the equilibrium. We proved some (polynomial and sub-exponential) rate of convergence to the equilibrium (depending on the space to which the initial datum belongs). Our results generalized the result in [5] [1] [2] to weak confinement case.

Contents

1. Introduction

In this paper, we consider the weak hypocoercivity issue for the kinetic Fokker-Planck(KFP for short) equation

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + div_v(vf), \tag{1.1}$$

for a density function $f = f(t, x, v)$, with $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$,. The evolution equations are complemented with an initial data

$$f(0,) = f_0 \ \ on\ \mathbb{R}^{2d}.$$

We make the fundamental assumption on the confinement potential V

$$V(x) = \langle x \rangle^\gamma, \quad \gamma \in (0, 1],$$

where $\langle x \rangle^2 := 1 + |x|^2$.

We make some fundamental observations. One is mass conservative, $M(f_0) = M(f(t,.))$, where we define

$$M(f) = \int_{R^d \times R^d} f dx dv.$$

We also observe the existence of steady state G for the KFP models $\mathcal{L}G = 0$, given by $G = Ce^{-W}, W = \frac{v^2}{2} + V(x)$, C is the normalizing constant such that

$$M(G) = 1.$$

Also observe that contrary to the case $\gamma \geq 1$, a Poincaré inequality of the type

$$\exists c > 0, \qquad \int_{\mathbb{R}^d} |f(x)|^2 exp(-V(x)) dx \leq c \int_{\mathbb{R}^d} |\nabla f(x)|^2 exp(-V(x)) dx,$$

for f such that

$$\int_{\mathbb{R}^d} f(x) exp(-V(x)) dx = 0,$$

does not hold, but only a weaker version of this inequality remains true (see [10], or below section 2). In particular, there is no spectral gap for the associated operator $\mathcal{L}$, nor is there an exponential trend to the equilibrium for the associated semigroup.

In order to state our main result, we introduce some notations.

We will denote $H = 1 + J + K + Q$, where $J = x^2, K = 2v \cdot x, Q = 3v^2$. Observe that $\frac{1}{2}(x^2 + v^2 + 1) \leq H \leq 4(x^2 + v^2 + 1)$, which means H is equivalent to $x^2 + v^2 + 1$

For a given weight function m, we will denote $L^p(m) = \{f | fm \in L^p\}$ and $\|f\|_{L^p(m)} = \|fm\|_{L^p}$.

With these notations we can introduce the main result of this paper:

Theorem 1.1. *(1) For any initial data $f_0 \in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p \in [1, \infty)$, $\epsilon > 0$ small, the associated solution $f(t,.)$ of the kinetic Fokker-Planck equation (1.1) satisfies*

$$\|f(t,.) - M(f_0)G\|_{L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-Ct^b} \|f_0 - M(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})}.$$

$$\forall 0 < b < \frac{\gamma}{2-\gamma}.$$

(2)Let $m = H^k$, $0 < k$, For any initial data $f_0 \in L^1(m)$, the associated solution $f(t,.)$ of the kinetic Fokker-Planck equation (1.1) satisfies

$$\|f(t,.) - M(f_0)G\|_{L^1(m^\theta)} \lesssim (1+t)^{-a} \quad \|f_0 - M(f_0)G\|_{L^1(m)},$$

$$\forall 0 < \theta < 1, \ \forall 0 < a < \frac{k(1-\theta)}{1-\frac{\gamma}{2}}.$$

Remark 1.2. The constant in the theorem does not depend on f_0, only depends on $\gamma, d, \epsilon, \theta$

Remark 1.3. The Theorem 1.1 also true when V behaves like $\langle x\rangle^\gamma$, that is for any V satisfying

$$C_1\langle x\rangle^\gamma \le V(x) \le C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \le x\cdot\nabla_x V(x) \le C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

and

$$|D_x^2 V(x)| \le C_5\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

for some constant $C_i > 0$, $R > 0$.

Remark 1.4. There are many classical results on the case $1 \le \gamma$. In this case there is an exponentially decay proved by Villani [5] and Dolbeault, Mouhot, Schmeiser [1] [2]

For any initial data $f_0 \in L^2(G^{-1/2})$, $f(t,.)$ is the solution of (1.1) correspond to this f_0 , we have:

$$\|f(t,.) - M(f_0)G\|_{L^2(G^{-1/2})} \lesssim e^{-Ct}\|f_0 - M(f_0)G\|_{L^2(G^{-1/2})} \quad .$$

Let us briefly explain the main ideas behind our method of proof.

We first introduce four spaces $E_1 = L^2(G^{-1/2})$,$E_2 = L^2(G^{-1/2}e^{\epsilon_1 V(x)})$, $E_3 = L^2(G^{-(1+\epsilon_2)/2})$ and $E_0 = L^2(G^{-1/2}\langle x\rangle^{\gamma-1})$ (remember $\gamma \in (0,1)$), take $\epsilon_1 > 0$ and $\epsilon_2 > 0$ small such that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset L^2$, and E_1 is an exponentially interpolation between E_0 and E_2. We first use an argument as in [1][2] to prove, for any $f_0 \in E_3$, the solution to the KFP equation 1.1 satisfies

$$\frac{d}{dt}\|f(t)\|_{E_1} \le -\lambda\|f(t)\|_{E_0},$$

for some constant $\lambda > 0$. We use this and Duhamel formula to prove

$$\|f(t)\|_{E_2} \lesssim \|f_0\|_{E_3}.$$

Combine the two inequalities and using a exponentially interpolation argument as in [3], we can have

$$\|f(t)\|_{E_1} \lesssim e^{-at^b}\|f_0\|_{E_3}, \tag{1.2}$$

for some $a > 0, b \in (0,1)$.

We then generalize the decay estimate to a wider class of Banach spaces by adapting the extension theory introduced in [11] and developed in [4] [12]. We first introduce a notation.

If $\mathcal{T}_i$ with $i = 1,2$ are two given operator valued measurable functions defined on $(0,\infty)$, we denote by

$$(\mathcal{T}_1 * \mathcal{T}_2)(t) = \int_0^t \mathcal{T}_1(s)\mathcal{T}_2(t-s)ds,$$

their convolution on $\mathbb{R}_+$. We then denote $\mathcal{T}^{(*0)} := 1$, $\mathcal{T}^{(*1)} := \mathcal{T}$ and for any $k \ge 2$, $\mathcal{T}^{(*k)} := \mathcal{T}^{*(k-1)} * \mathcal{T}$.

Then we consider two Banach spaces $E_4 = L^1(m)$ and $E_5 = L^1(m^\theta)$, $\theta \in (0,1)$. We introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is an appropriately defined bounded operator so that $\mathcal{B}$ becomes a dissipative operator. Then we can show that, for any integer $k \geq 0$, we have

$$\|((S_{\mathcal{B}}\mathcal{A})^{*k} * S_{\mathcal{B}})(t)\|_{E_4 \to E_5} \in L^1(\mathbb{R}^+), \tag{1.3}$$

and for some n large, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{*n}(t)\|_{E_4 \to E_3} \in L^1(\mathbb{R}_+), \tag{1.4}$$

and our weight m is small such that

$$\|I\|_{E_1 \to E_5} \lesssim 1. \tag{1.5}$$

Using the semigroup version of Duhamel formula

$$\begin{aligned} S_{\mathcal{L}} \quad &= S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{L}}) = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}} \\ &= S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}) = S_{\mathcal{B}} + (S_{\mathcal{L}}\mathcal{A}) * S_{\mathcal{B}} \end{aligned}$$

Iterating this formula we get

$$S_{\mathcal{L}} \quad = S_{\mathcal{B}} + \sum_{l=1}^{n-1} (S_{\mathcal{B}}) * (\mathcal{A}S_{\mathcal{B}})^{(*l)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}(t))^{*n}, \tag{1.6}$$

using the above formula (1.6) and the estimates (1.3) (1.4) (1.5) we can conclude $\|S_{\mathcal{L}}\|_{E_4 \to E_5} \in L^1(\mathbb{R}_+)$, which is a rough version of the estimates presented in Theorem 1.1.

The organization of the following paper is as follows:

Section 2 we will prove a L^2 estimate for the KFP model.

In section 3 we will use the result in Section 2 to prove the L^2 convergence for the KFP equation, which is the (1) $p = 2$ of the Theorem 1.1

Section 4 we introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and prove some L^1 estimate on semigroup $S_{\mathcal{B}}$.

Section 5 is the proof of a regularization estimate on $S_{\mathcal{B}}$ to go from L^2 to L^1 and L^∞

Section 6 use the regularization lemma to prove L^p convergence for KFP equation, which is Theorem 1.1.

2. L^2 framework: Dirichlet form and rate of convergence estimate

For later discussion we introduce some notations for the whole paper. The first is the splitting of the KFP operator $\mathcal{L} = \mathcal{T} + \mathcal{S}$, where $\mathcal{T}$ stand for the transport part:

$$\mathcal{T}f = -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f,$$

and $\mathcal{S}$ stand for the collision operator

$$\mathcal{S}f = \Delta_v f + div_v(vf),$$

We will denote the cut-off function χ such that $\chi(x,v) \in [0,1]$, $\chi(x,v) \in C^\infty$, $\chi(x,v) = 1$ when $x^2+v^2 \leq 1$, $\chi(x,v) = 0$ when $x^2+v^2 \geq 2$, and denote $\chi_R = \chi(x/R, v/R)$.

We introduce the splitting of the KFP operator $\mathcal{L}$ and denote it by

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} = M\chi_R(x,v).$$

We will denote $S_{\mathcal{L}}(t)$ the semigroup associated with operator $\mathcal{L}$, similarly $S_{\mathcal{B}}(t)$ the semigroup associated with operator $\mathcal{B}$. We use $\int f$ in place of $\int_{\mathbb{R}^d \times R^d} f dxdv$ for short, similarly $\int f dx$ means $\int_{\mathbb{R}^d} f dx$, $\int f dv$ means $\int_{\mathbb{R}^d} f dv$. $B_{|x|\leq\rho}$ is used to denote the ball such that $\{x \in \mathbb{R}^d || x| \leq \rho\}$,similarly B_ρ means the ball such that $\{x, v \in \mathbb{R}^d || x|^2 + v^2 \leq \rho\}$.

For $V(x) = \langle x\rangle^\gamma, 0 < \gamma \leq 1$, when we write the term $\langle \nabla V\rangle^{-1}$ and $\langle \nabla V\rangle$, $\langle \nabla V\rangle^{-1}$ means $\langle x\rangle^{1-\gamma}$, $\langle \nabla V\rangle$ means $\langle x\rangle^{\gamma-1}$.

And we denote the projection operator $\Pi f = M(f)G$.

With these notations we introduce the Dirichlet form adapted to our problem. We define

$$\rho_f = \rho[f] = \int f dv, \quad j_f = j[f] = \int v f dv.$$

$$M = Ce^{-v^2/2}, \ \int M dv = 1, \ \pi f = M\rho_f, \ \pi^\perp = 1 - \pi, \ f^\perp = \pi^\perp f.$$

Recall for the kinetic Fokker-Planck model

$$\mathcal{S}f = \Delta_v f + div_v(vf), \mathcal{T}f = -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f,$$

define an elliptic operator and its dual

$$\Delta_V u := div_x(\nabla_x u + \nabla_x V u), \quad \Delta_V^* u = \Delta_x u - \nabla_x V \cdot \nabla_x u,$$

we define $u = (\Delta_V^*)^{-1}\xi$ the solution to the above elliptic equation (note that u can differ by a constant)

$$\Delta_V^* u = \xi \ on \ \mathbb{R}^d, \ with \ 0 = \langle ue^{-V}\langle \nabla V\rangle^{-2}\rangle := \int ue^{-V}\langle \nabla V\rangle^{-2} dx,$$

we use this notation to define a scalar product by

$$\begin{aligned}((f,g)) \quad &:= (f,g)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j_f, (\rho_g e^V \langle \nabla V\rangle^2))_{L^2} \\ &+\epsilon((\rho_f e^V \langle \nabla V\rangle^2), \Delta_V^{-1}\nabla_x j_g)_{L^2} \\ &= (f,g)_{\mathcal{H}} + \epsilon(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V\rangle^2))_{L^2} \\ &+\epsilon((\nabla_x(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V\rangle^2), j_g)_{L^2},\end{aligned}$$

and we define

$$\begin{aligned}D[f] \quad &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], (\rho_f e^V \langle \nabla V\rangle^2))_{L^2} \\ &+\epsilon((\rho[-\mathcal{L}f] e^V \langle \nabla V\rangle^2), \Delta_V^{-1}\nabla_x j_f)_{L^2}.\end{aligned}$$

With these notations we can come to our first theorem.

Theorem 2.1. *For the kinetic Fokker-Planck model with $V(x) = \langle x\rangle^\gamma, 0 < \gamma < 1$, $\mathcal{H} = L^2(G^{-1/2})$ $\mathcal{H}_1 = L^2(G^{-1/2}\langle\nabla V\rangle)$, we define $\mathcal{H}_0 = \{h \in \mathcal{H}, \int f dxdv = 0\}$. Then there exists $\epsilon > 0$ small such that on $\mathcal{H}_0$ the norm $((f,f))$ defined above is equivalent to the norm of $\mathcal{H}$, moreover there exist $\lambda > 0$, such that*

$$D[f] \geq \lambda \|f\|^2_{\mathcal{H}_1}, \quad \forall f \in \mathcal{H}_0$$

Remark 2.2. Note that

$$\langle \rho_f e^V \langle\nabla V\rangle^2 e^{-V}\langle\nabla V\rangle^{-2}\rangle dx = \int \rho_f dx = 0,$$

this is one of the core of the construction.

Remark 2.3. in Villani's paper [5], a H^1 version of our theorem is established for the kinetic Fokker-Planck operator for both the case $r \geq 1$ and $0 < r \leq 1$.

Remark 2.4. Our first statement is a generalization of [1] [2], and the method of our second estimate is based on [3].

Before proving the theorem, we need some lemmas.

First we need local Poincare inequality:

Lemma 2.5. *We say that W satisfies a local Poincare inequality on a bounded open set Ω if there exist some constant $\kappa_\Omega > 0$ such that:*

$$\int_\Omega h^2 W \leq k_\Omega \int_\Omega |\nabla h|^2 W + \frac{1}{W(\Omega)}(\int_\Omega hW)^2,$$

for any nice function $h : \mathbb{R}^d \to \mathbb{R}$ and where we denote $W(\Omega) := \langle W 1_\Omega\rangle$.

Under the assumption $W, W^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$, we can prove that the function W satisfies the local Poincare inequality.

Prove of Lemma 2.5: see [9]

And the weak Poincaré inequality.

Lemma 2.6. *For any $u \in \mathcal{D}(\mathbb{R}^d)$ such that $\langle u e^{-V}\langle\nabla V\rangle^{-2}\rangle = 0$, we have*

$$\|u\|_{L^2(\langle\nabla V\rangle e^{-V/2})} \lesssim \|\nabla u\|_{L^2(e^{-V/2})}$$

Proof of Lemma 2.6 : We need to prove for any $h \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} h e^{-V}\langle\nabla V\rangle^{-2} = 0,$$

we have

$$\int_{\mathbb{R}^d} |\nabla h|^2 e^{-V} \geq C \int_{\mathbb{R}^d} h^2 e^{-V}\langle x\rangle^{2(\gamma-1)}.$$

Taking $g = he^{-1/2V}$, we can have $\nabla g = \nabla h e^{-\frac{1}{2}V} - \frac{1}{2}\nabla V h e^{-\frac{1}{2}V}$, so

$$\begin{aligned}\int |\nabla g|^2 &= \int |\nabla h|^2 e^{-V} + \int h^2 \frac{1}{4}|\nabla V|^2 e^{-V} - \int \frac{1}{2}\nabla(h^2)\cdot\nabla V e^{-V} \\ &= \int |\nabla h|^2 e^{-V} + \int h^2(\frac{1}{2}\Delta V - \frac{1}{4}|\nabla V|^2)e^{-V},\end{aligned}$$

from this we can deduce for some K and $R_0 > 0$

$$\int |\nabla h|^2 e^{-V} \geq \int \frac{1}{8} h^2 \langle \nabla V \rangle^2 e^{-V} - K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2}.$$

Define

$$\epsilon_R := \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6},$$

and

$$Z_R := \int_{B_R} e^{-V} \langle \nabla V \rangle^{-2},$$

then by

$$\int_{\mathbb{R}^d} h e^{-V} \langle \nabla V \rangle^{-2} = 0,$$

we can get

$$\begin{aligned} (\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2})^2 \quad &= (\int_{B_R^c} h e^{-V} \langle \nabla V \rangle^{-2})^2 \\ &\leq \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2 \int_{B_R^c} e^{-V} \langle \nabla V \rangle^{-6} \\ &\leq \epsilon_R \int_{B_R^c} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Using Lemma 2.5 we have

$$\begin{aligned} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^{-2} \quad &\leq C_R \int_{B_R} |\nabla h|^2 e^{-V} \langle \nabla V \rangle^{-2} + \frac{1}{Z_R} (\int_{B_R} h e^{-V} \langle \nabla V \rangle^{-2})^2 \\ &\leq C_R' \int_{B_R} |\nabla h|^2 e^{-V} + \frac{\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2. \end{aligned}$$

Putting all the inequalities together we have

$$\begin{aligned} \int h^2 e^{-V} \langle \nabla V \rangle^2 \quad &\leq 8 \int |\nabla h|^2 e^{-V} + 8K \int_{B_{R_0}} h^2 e^{-V} \langle \nabla V \rangle^{-2} \\ &\leq 8(1 + K C_R') \int_{B_R} |\nabla h|^2 e^{-V} + \frac{8K\epsilon_R}{Z_R} \int_{B_R} h^2 e^{-V} \langle \nabla V \rangle^2, \end{aligned}$$

by taking R large such that: $\frac{8K\epsilon_R}{Z_R} \leq \frac{1}{2}$, we are done. □

Lemma 2.7. *(Elliptic Estimate) For any $\xi_1 \in L^2(\langle \nabla V \rangle^{-1} e^{-V/2})$ and $\xi_2 \in L^2(e^{-V/2})$, the solution $u \in L^2(e^{-V/2})$ to the elliptic equation*

$$-\Delta_V^* u = \xi_1 + \nabla \xi_2, \ \langle u e^{-V} \langle \nabla V \rangle^{-2} \rangle = 0,$$

satisfies

$$\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})} \lesssim A := \|\xi_1\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})} + \|\xi_2\|_{L^2(e^{-V/2})}.$$

Similarly for any $\xi \in L^2(e^{-V/2})$, the solution $u \in L(e^{-V/2})$ to the elliptic problem

$$-\Delta_V^* u = \xi,\ \langle u e^{-V} \langle \nabla V \rangle^{-2} \rangle = 0,$$

satisfies

(2.1)
$$\|u\|_{L^2(\langle \nabla V \rangle^2 e^{-V/2})} + \|\nabla u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|D^2 u\|_{L^2(e^{-V/2})} \lesssim \|\xi\|_{L^2(e^{-V/2}\langle \nabla V \rangle^{-1})}.$$

Remark 2.8. I believe in (2.1) we can improve the last term from $\|\xi\|_{L^2(e^{-V/2}\langle \nabla V \rangle^{-1})}$ to $\|\xi\|_{L^2(e^{-V/2})}$, but this result is enough to continue our proof.

Proof of Lemma 2.7: First observe

$$e^V div_x[e^{-V}\nabla_x u] = \Delta_x u - \nabla_x V \cdot \nabla_x u = \Delta_V^* u,$$

integrate the equation by ue^{-V} and use this observation we get

$$-\int e^V div_x[e^{-V}\nabla_x u] u e^{-V} = \int (\xi_1 + \nabla \cdot \xi_2) u e^{-V}.$$

Performing integration by parts we have

$$\int e^{-V} |\nabla_x u|^2 = \int (\xi_1 u + \xi_2 \cdot \nabla u - \xi_2 \cdot \nabla V u) e^{-V},$$

by Lemma 2.6

$$\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} \lesssim \|\nabla u\|_{L^2(e^{-V/2})},$$

we deduce

$$\|u\|^2_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|\nabla u\|^2_{L^2(e^{-V/2})} \lesssim A \times (\|u\|_{L^2(\langle \nabla V \rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})}),$$

so the proof for the first inequality is done.

For the second inequality, since $\langle \nabla V \rangle \leq 1$ so the only thing need to be proved is the $\|D^2 u\|_{L^2(e^{-V/2})}$ term, by integration by parts

$$\begin{aligned}
\int |D^2 u|^2 e^{-V} &= \sum_{i,j=1}^d \int (\partial_{ij}^2 u)^2 e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_i u (\partial_{ij}^2 u \partial_j V - \partial_{ijj}^3 u) e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_{jj} u \partial_i (\partial_i u e^{-V}) - \frac{1}{2} \int (\partial_i u)^2 \partial_j (\partial_j V e^{-V}) \\
&= \int (\Delta u)(-\Delta_V^* u) e^{-V} + \int |\nabla u|^2 (|\nabla V|^2 - \Delta V) e^{-V} \\
&\lesssim \|D^2 u\|_{L^2(e^{-V/2})} \|\xi\|_{L^2(e^{-V/2})} + \|\langle \nabla V \rangle \nabla u\|_{L^2(e^{-V/2})},
\end{aligned}$$

where in the third equality we have used

$$\begin{aligned}\int \partial_{ij}^2 u\partial_i u\partial_j V e^{-V} &= -\int \partial_i u\partial_j(\partial_i u\partial_j V e^{-V})\\ &= -\int \partial_{ij}^2 u\partial_i u\partial_j V e^{-V} - \int (\partial_i u)^2\partial_j(\partial_j V e^{-V}),\end{aligned}$$

which implies

$$\int \partial_{ij}^2 u\partial_i u\partial_j V e^{-V} = -\frac{1}{2}\int (\partial_i u)^2\partial_j(\partial_j V e^{-V}),$$

and in the forth equality we have used

$$e^V \sum_{i=1}^{d} \partial_i(e^{-V}\partial_i u) = e^V div_x[e^{-V}\nabla_x u] = u = \Delta_V^* u,$$

so our proof is done. □

Now we can turn to the proof of Theorem 2.1.

Proof of Theorem 2.1 :First, we prove the equivalence of the norms $((\ , \ ))$ and $(\ , \ )_{\mathcal{H}}$. by Cauchy-Schwarz inequality and Lemma 2.7, we have

$$(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V\langle\nabla V\rangle^2))_{L^2} \le \|j_f\|_{L^2(e^{V/2})}\|\rho_g e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})},$$

and obviously

$$\|\rho_g e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})} = \|\rho_g\|_{L^2(\langle\nabla V\rangle e^{V/2})} \le \|\rho_g\|_{L^2(e^{V/2})} \lesssim \|g\|_{\mathcal{H}}.$$

Using elementary observation

$$|j_f| \lesssim \|f\|_{L^2(e^{v^2/4})} \quad |\rho_f| \lesssim \|f\|_{L^2(e^{v^2/4})},$$

we deduce

$$(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V\langle\nabla V\rangle^2))_{L^2} \lesssim \|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}}.$$

The third term in the definition of $((\ , \ ))$ can be estimated in the same way and that ends the proof of equivalence of norms.

□

Now we turn to prove the main estimate of the theorem. We split the $D[f]$ into 3 parts

$$D[f] = T_1 + \epsilon T_2 + \epsilon T_3$$

with

$$\begin{aligned}T_1 &= (\mathcal{L}f, f)_{\mathcal{H}}\\ T_2 &= (\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle\nabla V\rangle)}\\ T_3 &= ((\Delta_V)^{-1}\nabla_x j_f, \rho[-\mathcal{L}f])_{L^2(e^{V/2}\langle\nabla V\rangle)} \ ,\end{aligned}$$

and compute them separately.
For the T_1 term, we have

$$\begin{aligned} T_1 \quad &:= (-\mathcal{T}f + \mathcal{S}f, f)_{\mathcal{H}} = (-\mathcal{S}f, f)_{\mathcal{H}} \\ &= -\int [\Delta_v f + div_v(vf)] f M^{-1} e^V = \int |\nabla_v (f/M)|^2 M e^V \\ &\geq k_p \int |f/M - \rho_f|^2 M e^V = k_p \|f - \rho_f M\|_{\mathcal{H}}^2 = k_p \|f^{\perp}\|_{\mathcal{H}}^2, \end{aligned}$$

note in the last line we use the classical Poincare inequality.
For the T_2 term, we split it as

$$\begin{aligned} T_2 \quad &:= (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V \rangle)} \\ &= (\Delta_V^{-1} \nabla_x j[-\mathcal{T}\pi f], \rho_f)_{L^2(e^{V/2}\langle \nabla V \rangle)} \\ &+ (\Delta_V^{-1} \nabla_x j[-\mathcal{T}f^{\perp}], \rho_f)_{L^2(e^{V/2}\langle \nabla V \rangle)} \\ &+ (\Delta_V^{-1} \nabla_x j[-\mathcal{S}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V \rangle)} \\ &:= T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned}$$

We first observe

$$\mathcal{T}\pi f = -v \cdot \nabla_x \rho_f M - \nabla_x V \cdot v \rho_f M = -e^{-V} M v \cdot \nabla_x (\rho_f / e^{-V}),$$

so we can have

$$j[-\mathcal{T}\pi f] = \langle v v_k M \rangle e^{-V} \partial_{x_k} (\rho_f / e^{-V}) = e^{-V} \nabla_x (\rho_f / e^{-V}),$$

next observe that

$$e^V div_x [e^{-V} \nabla_x u] = \Delta_V^* u,$$

then we have

$$\begin{aligned} T_{2,1} \quad &= (j[-\mathcal{T}\pi f], \nabla (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= (\rho_f, [e^V div_x (e^{-V} \nabla)][(\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2)])_{L^2} \\ &= \|\rho_f e^{V/2} \langle \nabla V \rangle\|_{L^2}^2 = \|\pi f\|_{\mathcal{H}_1}^2. \end{aligned}$$

Using the notation $\eta_1 = \langle v \otimes v f^{\perp} \rangle$ and $\eta_{2,\alpha\beta} = \langle v_\alpha \partial_{v_\beta} f^{\perp} \rangle$, observe that

$$|\eta_1| \lesssim \|f^{\perp}\|_{L^2(e^{v^2/4})}, |\eta_2| \lesssim \|f^{\perp}\|_{L^2(e^{v^2/4})},$$

we compute

$$\begin{aligned} T_{2,2} \quad &= (j[-\mathcal{T}f^{\perp}], \nabla (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= (D\eta_1 + \eta_2 \nabla V, \nabla (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= (\eta_1, D^2 (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} + (\eta_2, \nabla V \nabla (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &= \|\eta_1\|_{L^2(e^{V/2})} \|D^2 (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})} \\ &+ \|\eta_2\|_{L^2(e^{V/2})} \|\nabla V \nabla (\Delta_V^*)^{-1} (\rho_f e^V \langle \nabla V \rangle^2)\|_{L^2(e^{-V/2})}, \end{aligned}$$

and by Lemma 2.7

$$\begin{aligned} T_{2,2} &\lesssim \|\eta_1\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} + \|\eta_2\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

For $T_{2,3}$ using

$$\begin{aligned} j[-\mathcal{S}f] = j[-\mathcal{S}f^\perp] \qquad &= -\int v[\Delta_v f^\perp + div_v(vf^\perp)]dv \\ &= d\int f^\perp v dv \lesssim \|f^\perp\|_{L^2(e^{v^2/4})}, \end{aligned}$$

we have by Lemma 2.7

$$\begin{aligned} T_{2,3} \qquad &= (j[-Sf], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &\le \|j[-Sf]\|_{L^2(e^{V/2})}\|\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})} \\ &= \|f^\perp\|_{\mathcal{H}}\|\rho_f\|_{L^2(\langle\nabla V\rangle e^{V/2})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

Finally we come to the T_3 term using

$$\rho[-Sf] = \int \nabla_v\cdot(\nabla_v f + vf)dv = 0,$$

and

$$\begin{aligned} \rho[-Tf] \qquad &= \rho[v\nabla_x f - \nabla_x V(x)\nabla_v f] \\ &= \int v\nabla_x f - \nabla_x V(x)\nabla_v f dv \\ &= \nabla_x\int v f dv \\ &= \nabla_x j[f], \end{aligned}$$

we have by $\nabla(\langle\nabla V\rangle^2) \lesssim \langle\nabla V\rangle^2$ and $\langle\nabla V\rangle^2 \lesssim \langle\nabla V\rangle$ we get

$$\begin{aligned} T_3 \qquad &= ((\Delta_V)^{-1}\nabla_x j_f, \rho[-\mathcal{L}f])_{L^2(e^{V/2}\langle\nabla V\rangle)} \\ &= ((\Delta_V)^{-1}\nabla_x j[f^\perp], \rho[-\mathcal{T}f])_{L^2(e^{V/2}\langle\nabla V\rangle)} \\ &= (j[-f^\perp], \nabla(\Delta_V^*)^{-1}(\rho[-\mathcal{T}f]e^V\langle\nabla V\rangle^2)_{L^2} \\ &= (j[-f^\perp], \nabla(\Delta_V^*)^{-1}(\nabla_x j[f]e^V\langle\nabla V\rangle^2)_{L^2} \\ &= \|j[f^\perp]\|_{L^2(e^{V/2})}\|\nabla(\Delta_V^*)^{-1}[\nabla_x(j_f e^V\langle\nabla V\rangle^2) - \nabla V j_f e^V\langle\nabla V\rangle^2 - \nabla(\langle\nabla V\rangle^2)j_f e^V]\|_{L^2(e^{-V/2})}, \end{aligned}$$

again by Lemma 2.7 we have

$$\begin{aligned} T_3 \qquad &\lesssim \|j[f^\perp]\|_{L^2(e^{V/2})}(\|j_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} + \|j_f e^V\nabla(\langle\nabla V\rangle^2)\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})}) \\ &\lesssim \|f^\perp\|_{\mathcal{H}}\|f\|_{\mathcal{H}_1}. \end{aligned}$$

Gathering all the term together we have for $\epsilon > 0$ small enough

$$\begin{aligned} D[f] \quad &\geq \|f^{\perp}\|_{\mathcal{H}}^2 + \epsilon\|\pi f\|_{\mathcal{H}_1}^2 - \epsilon 2K\|f^{\perp}\|_{\mathcal{H}}\|f\|_{\mathcal{H}_1} - \epsilon 2K\|f^{\perp}\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1} \\ &\geq \|f^{\perp}\|_{\mathcal{H}}^2 + \epsilon\|\pi f\|_{\mathcal{H}_1}^2 - (2\epsilon + 4\epsilon^{1/2})K\|f^{\perp}\|_{\mathcal{H}}^2 - \epsilon^{3/2}4K\|\pi f\|_{\mathcal{H}_1}^2 \\ &\geq \frac{1}{2}(\|f^{\perp}\|_{\mathcal{H}}^2 + \epsilon\|\pi f\|_{\mathcal{H}_1}^2) \geq \frac{\epsilon}{M}\|f\|_{\mathcal{H}_1}, \end{aligned}$$

so the proof is ended. □

3. L^2 sub-exponential decay for the kinetic Fokker-Planck equation based on a splitting

Theorem 3.1. *Using the notation and results in Theorem 2.1, we have*

$$\|S_{\mathcal{L}}(t)f_0\|_{L^2(G^{-\frac{1}{2}})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}}\|f_0\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})},$$

for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)}) \cap \mathcal{H}_0$, $\epsilon > 0$ small or equivalently

$$\|S_{\mathcal{L}}(t)(I-\Pi)f_0\|_{L^2(G^{-\frac{1}{2}})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}}\|(I-\Pi)f_0\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})},$$

for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)})$, $\epsilon > 0$ small.

During the proof of Theorem 3.1 we will make use of the following Gronwall's Lemma.

Lemma 3.2. *(Gronwall's Lemma) Suppose $0 < b,\ 0 < c,$ and $0 \leq u(t) \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ satisfying:*

$$u^{'}(t) \leq -cu(t) + b,$$

in the sense of distribution $\mathcal{D}'((0,\infty))$ we have:

$$u(t) \leq u(0)e^{-ct} + \frac{b}{c}.$$

Since the result is well-known, the proof is omitted. □

Recall the splitting of $\mathcal{L}$

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} = M\chi_R(x,v),$$

we first prove some convergence on the semigroup associated with $\mathcal{B}$ denoted by $S_{\mathcal{B}}(t)$.

Lemma 3.3. *For any $p \in [1,\infty)$ we have*
(1)For a given weight function m, we have

$$\int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m \leq \frac{1}{p}\int f^p G^{-(p-1)}\tilde{m}$$

with

$$\tilde{m} = \Delta_v m - \nabla_v m \cdot v - \nabla V(x)\cdot\nabla_v m + v\cdot\nabla_x m,$$

(2)Taking $m = e^{\epsilon H^\delta}$, $\epsilon > 0$ *if* $0 < \delta < \frac{\gamma}{2}$, ϵ *small enough if* $\delta = \frac{\gamma}{2}$, $H = 3v^2 + 2x \cdot v + x^2 + 1$, *we have*

$$\int f^{p-1}(\mathcal{B}f)G^{-(p-1)}e^{\epsilon H^\delta} \leq -C\int f^p G^{-(p-1)}e^{\epsilon H^\delta}H^{\frac{\delta}{2}+\gamma-1},$$

for some M *and* R *large.*

(3)Using the results in (2), we can prove

$$\|S_{\mathcal{B}}(t)\|_{L^p(e^{2\epsilon H^\delta}G^{-\frac{p-1}{p}})\to L^p(e^{\epsilon H^\delta}G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

for some $a > 0$. *In particular, this implies*

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})\to L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Prove of Lemma 3.3: Step 1: Recall $G^{-1} = Ce^{v^2/2+V(x)}$. We write

$$\int f^{p-1}(\mathcal{L}f)G^{-(p-1)}m = \int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m + \int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m.$$

We first compute the contribution of the term with operator $\mathcal{T}$

$$\begin{aligned}\int f^{p-1}(\mathcal{T}f)G^{-(p-1)}m \quad &= \frac{1}{p}\int \mathcal{T}(f^p)G^{-(p-1)}m \\ &= -\frac{1}{p}\int f^p\mathcal{T}(G^{-(p-1)}m) \\ &= \frac{1}{p}\int f^p G^{-(p-1)}(v\cdot\nabla_x m - \nabla V(x)\cdot\nabla_v m),\end{aligned}$$

for the term with operator $\mathcal{S}$, using integration by parts and we get

$$\begin{aligned}&\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ &= \int f^{p-1}(\Delta_v f + div_v(vf))G^{-(p-1)}m \\ &= -\int \nabla_v((fG^{-1})^{p-1}m)\cdot(\nabla_v f + vf) \\ &= -\int ((fG^{-1})^{p-1}\nabla_v m + (p-1)(fG^{-1})^{p-2}\nabla_v(fG^{-1})m)\cdot\nabla_v(fG^{-1})G \\ &= -\int (p-1)|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm - \frac{1}{p}\nabla_v((fG^{-1})^p)\cdot(\nabla_v m)G,\end{aligned}$$

performing another integration by parts on the latter term we have

$$\begin{aligned}&\int f^{p-1}(\mathcal{S}f)G^{-(p-1)}m \\ &= \int -C|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm + \frac{1}{p}\nabla_v\cdot(G\nabla_v m)(fG^{-1})^p \\ &= \int -C|\nabla_v(fG^{-1})|^2(fG^{-1})^{p-2}Gm + \frac{1}{p}(\Delta_v m - v\cdot\nabla_v m)f^p G^{-(p-1)},\end{aligned}$$

putting together the two identities, we obtain the result. □

Step 2: We particular use $m = e^{\epsilon H^\delta}$ now, we easily compute:

$$\frac{\nabla_v m}{m} = \delta\epsilon \frac{\nabla_v H}{H^{1-\delta}}, \frac{\nabla_x m}{m} = \delta\epsilon \frac{\nabla_x H}{H^{1-\delta}},$$

$$\frac{\Delta_v m}{m} \leq \delta\epsilon \frac{\Delta_v H}{H^{1-\delta}} + (\delta\epsilon)^2 \frac{|\nabla_v H|^2}{H^{2(1-\delta)}}.$$

We deduce that $\phi = \frac{\tilde{m}}{m}$ satisfies

$$\frac{\phi H^{1-\delta}}{\epsilon\delta} \leq \Delta_v H + \epsilon\delta \frac{|\nabla_v H|^2}{H^{1-\delta}} - v \cdot \nabla_v H + v \cdot \nabla_x H - \nabla_x V(x) \cdot \nabla_v H,$$

recall $H = 3v^2 + 2v \cdot x + x^2 + 1$, we have

$$\nabla_v H = 6v + 2x, \nabla_x H = 2v + 2x, \Delta_v H = 6,$$

so let $\epsilon > 0$ arbitrary if $0 < 2\delta < \gamma$, ϵ small if $2\delta = \gamma$,we have

$$\begin{aligned}
&\Delta_v H + 2\epsilon\delta \frac{|\nabla_v H|^2}{H^{1-\delta}} + v \cdot \nabla_x H - v \cdot \nabla_v H - \nabla_x V(x) \cdot \nabla_v H \\
&= 6 + \epsilon\delta \frac{(6v + 2x)^2}{H^{1-\delta}} + 2v^2 + 2x \cdot v - 6v^2 - 2x \cdot v - 6v \cdot \nabla_x V(x) - 2x \cdot \nabla_x V(x) \\
&< (2v^2 + C_1 v + C_2 v^{2\delta} - 6v^2) + (C_3 \epsilon\delta x^{2\delta} - 2x \cdot \nabla_x V(x)) + C \\
&< -C_4 v^2 - C_5 x \cdot \nabla_x V(x) + C_6 \\
&< -C_7 H^{\frac{\gamma}{2}} + M\chi_R,
\end{aligned}$$

for some $C_i > 0$. Then we get $\phi \leq \frac{-C}{H^{1-\delta-\frac{\gamma}{2}}} + M\chi_R$, the proof of (2) is ended. □

Step 3: During the proof of this part, we will denote $f_t = S_{\mathcal{B}}(t) f_0$ the solution to $\partial_t f = \mathcal{B} f, f(0) = f_0$. First we have by (2)

$$\int f^{p-1} (\mathcal{B} f) G^{-(p-1)} e^{2\epsilon H^\delta} \leq 0,$$

which implies

$$\int f_t^p G^{-(p-1)} e^{2\epsilon H^\delta} \leq \int f_0^p G^{-(p-1)} e^{2\epsilon H^\delta} := Y_1,$$

define

$$Y := \int f_t^p G^{-(p-1)} e^{\epsilon H^\delta},$$

and making use of the results in (2), for any $\rho > 0$ we have

$$\begin{aligned}\frac{d}{dt}Y &= 2\int f_t^{p-1}\mathcal{B}f_t G^{-(p-1)}e^{\epsilon H^\delta}\\ &\le -a\int f_t^p G^{-(p-1)}e^{\epsilon H^\delta}H^{\delta+\frac{\gamma}{2}-1}\\ &\le -a\int f_t^p G^{-(p-1)}e^{\epsilon H^\delta}\langle x\rangle^{2\delta+\gamma-2}\\ &\le -a\int_{B_{|x|\le\rho}} f_t^p G^{-(p-1)}e^{\epsilon H^\delta}\langle x\rangle^{2\delta+\gamma-2}.\end{aligned}$$

As $2\delta+\gamma<2$, so $0\le|x|\le\rho$ can imply $\langle x\rangle^{2\delta+\gamma-2}\ge\langle\rho\rangle^{2\delta+\gamma-2}$, we get

$$\begin{aligned}\frac{d}{dt}Y &\le -a\langle\rho\rangle^{2\delta+\gamma-2}\int_{B_{|x|\le\rho}} f_t^p G^{-(p-1)}e^{\epsilon H^\delta}\\ &\le -a\langle\rho\rangle^{2\delta+\gamma-2}Y + a\langle\rho\rangle^{2\delta+\gamma-2}\int_{B_{|x|\ge\rho}} f_t^p G^{-(p-1)}e^{\epsilon H^\delta},\end{aligned}$$

and $|x|\ge\rho$ can imply $e^{\epsilon\langle x\rangle^{2\delta}}\ge e^{\epsilon\langle\rho\rangle^{2\delta}}$, we have

$$\begin{aligned}\frac{d}{dt}Y &\le -a\langle\rho\rangle^{2\delta+\gamma-2}Y + a\langle\rho\rangle^{2\delta+\gamma-2}e^{-\epsilon\langle\rho\rangle^{2\delta}}\int_{B_{|x|\ge\rho}} f_t^p G^{-(p-1)}e^{\epsilon H^\delta}e^{\epsilon\langle x\rangle^{2\delta}}\\ &\le -a\langle\rho\rangle^{2\delta+\gamma-2}Y + a\langle\rho\rangle^{2\delta+\gamma-2}e^{-\epsilon\langle\rho\rangle^{2\delta}}\int f_t^p G^{-(p-1)}e^{\epsilon H^\delta}e^{\epsilon\langle x\rangle^{2\delta}}\\ &\le -a\langle\rho\rangle^{2\delta+\gamma-2}Y + a\langle\rho\rangle^{2\delta+\gamma-2}e^{-\epsilon\langle\rho\rangle^{2\delta}}CY_1.\end{aligned}$$

We can deduce from Gronwall's Lemma

$$\begin{aligned}Y(t) &\le e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t}Y(0) + Ce^{-\epsilon\langle\rho\rangle^{2\delta}}Y_1\\ &\lesssim (e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t} + e^{-\epsilon\langle\rho\rangle^{2\delta}})Y_1,\end{aligned}$$

choosing ρ such that $a\langle\rho\rangle^{2\delta+\gamma-2}t = \epsilon\langle\rho\rangle^{2\delta}$, that is $\langle\rho\rangle^{2-\gamma} = Ct$, then

$$Y(t)\le C_1 e^{-C_2 t^{\frac{2\delta}{2-\gamma}}}Y_2,$$

for some $C_i>0$, then the proof of Lemma 3.3 is complete. □

Now we come to prove Theorem 3.1.

Proof of Theorem 3.1: By Theorem 2.1, we have

$$\|S_{\mathcal{L}}(t)\|_{L^2(G^{-1/2})\to L^2(G^{-1/2})}\lesssim 1,$$

recall $\mathcal{A} = M\chi_R$, so we have

$$\|\mathcal{A}\|_{L^2(G^{-1/2})\to L^2(e^{2\epsilon H^\delta}G^{-1/2})}\lesssim 1,$$

and by Lemma 3.3 when $p=2$

$$\|S_{\mathcal{B}}(t)\|_{L^2(e^{2\epsilon H^\delta}G^{-1/2})\to L^2(e^{\epsilon H^\delta}G^{-1/2})}\lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

gathering the three estimates and using Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}\mathcal{A}} * S_{\mathcal{L}},$$

we deduce

$$\|S_{\mathcal{L}}(t)\|_{L^2(e^{2\epsilon H^\delta} G^{-1/2}) \to L^2(e^{\epsilon H^\delta} G^{-1/2})} \lesssim 1,$$

taking $2\delta = \gamma$, ϵ small enough we have

$$\int f_t^2 G^{-1} e^{\epsilon H^{\frac{\gamma}{2}}} \leq C \int f_0^2 G^{-1} e^{2\epsilon H^{\frac{\gamma}{2}}} =: Y_3,$$

and we define

$$Y_2(t) := \int f_t^2 G^{-1}.$$

Using the result from Theorem 2.1, we have

$$\begin{aligned} \frac{d}{dt} Y_2 &= 2 \int f_t \mathcal{L} f_t G^{-1} \\ &\leq -a \int f_t^2 G^{-1} \langle x \rangle^{2(\gamma-1)} \\ &\leq -a \int_{B_{|x| \leq \rho}} f_t^2 G^{-1} \langle x \rangle^{2(\gamma-1)}, \end{aligned}$$

as $2\gamma < 2$, so $0 \leq |x| \leq \rho$ can imply $\langle x \rangle^{2(\gamma-1)} \geq \langle \rho \rangle^{2(\gamma-1)}$, we have

$$\begin{aligned} \frac{d}{dt} Y_2 &\leq -a \langle \rho \rangle^{2(\gamma-1)} \int_{B_{|x| \leq \rho}} f_t^2 G^{-1} \\ &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} \int_{B_{|x| \geq \rho}} f_t^2 G^{-1}, \end{aligned}$$

for any $\epsilon > 0$ small, we can find $\epsilon_2 > 0$ such that :$\epsilon_2 V(x) \leq \epsilon H^{\frac{\gamma}{2}}$, and $|x| \geq \rho$ can imply $e^{\epsilon_2 \langle x \rangle^\gamma} \geq e^{\epsilon_2 \langle \rho \rangle^\gamma}$, so

$$\begin{aligned} \frac{d}{dt} Y_2 &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} e^{-\epsilon_2 \langle \rho \rangle^\gamma} \int_{B_{|x| \geq \rho}} f_t^2 G^{-1} e^{\epsilon_2 V(x)} \\ &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} e^{-\epsilon_2 \langle \rho \rangle^\gamma} \int f_t^2 G^{-1} e^{\epsilon H^{\frac{\gamma}{2}}} \\ &\leq -a \langle \rho \rangle^{2(\gamma-1)} Y_2 + a \langle \rho \rangle^{2(\gamma-1)} e^{-\epsilon_2 \langle \rho \rangle^\gamma} C Y_3, \end{aligned}$$

we can deduce thanks to Gronwall's Lemma

$$\begin{aligned} Y_2(t) &\leq e^{-a \langle \rho \rangle^{2(\gamma-1)} t} Y_2(0) + C e^{-\epsilon_2 \langle \rho \rangle^\gamma} Y_3. \\ &\lesssim (e^{-a \langle \rho \rangle^{2(\gamma-1)} t} + e^{-\epsilon_2 \langle \rho \rangle^\gamma}) Y_3, \end{aligned}$$

choosing ρ such that $a \langle \rho \rangle^{2(\gamma-1)} t = \epsilon_2 \langle \rho \rangle^\gamma$, that is $\langle \rho \rangle^{2-\gamma} = Ct$, then

$$Y_2(t) \leq C_1 e^{-C_2 t^{\gamma/(2-\gamma)}} Y_3.$$

As $H^{\frac{\gamma}{2}} \lesssim C(\frac{v^2}{2} + V(x))$, we have:

$$e^{\epsilon H^{\frac{\gamma}{2}}} \leq G^{-C\epsilon},$$

so taking ϵ small the proof of Theorem 3.1 is done. □

4. L^1 convergence on $S_{\mathcal{B}}$

This section we prove the decay rate for $S_{\mathcal{B}}$.

Theorem 4.1. *Let $H = 1 + x^2 + 2\langle v, x\rangle + 3v^2$, for any $\theta \in (0,1)$ and for $\forall l > 0$, we have*

$$\|\mathcal{S}_{\mathcal{B}}(t)\|_{L^1(H^l)\to L^1(H^{l\theta})} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l(1-\theta)}{1-\frac{\gamma}{2}},$$

To continue our proof we first need a lemma.

Lemma 4.2. *For the kinetic Fokker Planck operator $\mathcal{L}$, let m be a weight function, for any $p \in [1,\infty]$ we have*

$$\int (\mathcal{L}f) f^{p-1} m^p = -(p-1)\int |\nabla_v (mf)|^2 (mf)^{p-2} + \int f^p m^p \phi,$$

with

$$\phi = \frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + (\frac{2}{p}-1)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m},$$

in particular when $p = 1$

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

Proof of Lemma 4.2:First we have

$$\int (\mathcal{L}f) f^{p-1} m^p = \int f^{p-1}\mathcal{S}f m^p + \int f^{p-1}\mathcal{T}f m^p,$$

we first compute the contribution of the term with operator $\mathcal{T}$, we have

$$\int f^{p-1}\mathcal{T}f m^p = \frac{1}{p}\int \mathcal{T}(f^p) m^p = -\int f^p m^{p-1}\mathcal{T}m = -\int f^p m^p \frac{\mathcal{T}m}{m},$$

then we come to the term with operator $\mathcal{S}$

$$\int (\mathcal{S}f) f^{p-1} m^p = \int f^{p-1} m^p (\Delta_v f + div_v(vf)) := C_1 + C_2,$$

we first compute the C_2 term

$$\begin{aligned} C_2 &= \int f^{p-1} m^p (div_v v f + v\cdot\nabla_v f) \\ &= \int f^p (div_v v) m^p - \frac{1}{p}\int f^p div_v(v m^p) \\ &= \int f^p [(1-\frac{1}{p})d - v\cdot\frac{\nabla_v m}{m}] m^p, \end{aligned}$$

and we turn to the C_1 term

$$\begin{aligned} C_1 &= \int f^{p-1}m^p\Delta_v f = -\int \nabla_v(f^{p-1}m^p)\cdot\nabla_v f \\ &= \int -(p-1)|\nabla_v f|^2 f^{p-2}m^p - \frac{1}{p}\int \nabla_v f^p\cdot\nabla_v m^p, \end{aligned}$$

using $\nabla_v(mf) = m\nabla_v f + f\nabla_v m$, we have

$$\begin{aligned} C_1 &= -(p-1)\int |\nabla_v(mf)|^2 f^p m^{p-2} + (p-1)\int |\nabla_v m|^2 f^p m^{p-2} \\ &+\frac{2(p-1)}{p^2}\int \nabla_v(f^p)\cdot\nabla_v(m^p) - \frac{1}{p}\int \nabla_v(f^p)\cdot\nabla_v(m^p) \\ &= -(p-1)\int |\nabla_v(mf)|^2 f^{p-2}m^p + (p-1)\int |\nabla_v m|^2 f^p m^{p-2} \\ &+\frac{p-2}{p^2}\int f^p\Delta_v m^p, \end{aligned}$$

using $\Delta_v m^p = p\Delta_v m\ m^{p-1} + p(p-1)|\nabla_v m|^2 m^{p-2}$, we can conclude

$$C_1 = -(p-1)\int |\nabla_v(mf)|^2 f^{p-2}m^{p-2} + \int f^p m^p[(\frac{2}{p}-1)\frac{\Delta_v m}{m} + 2(1-\frac{1}{p})\frac{|\nabla_v m|^2}{m^2}].$$

Combining the equalities above we are done. □

Proof of Theorem 4.1 : From Lemma 4.2 we have

$$\begin{aligned} \int(\mathcal{B}f)f^{p-1}m^p &= \int(\mathcal{L} - M\chi_R)f^{p-1}m^p \\ &= -(p-1)\int |\nabla_v(mf)|^2(mf)^{p-2} + \int f^p m^p\phi, \end{aligned} \tag{4.1}$$

with

$$\phi = [\frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + (\frac{2}{p}-1)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R],$$

for $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - M\chi_R.$$

(1) Let $m = H^k$, we have

$$\frac{\nabla_v m}{m} = k\frac{\nabla_v H}{H}, \frac{\nabla_x m}{m} = k\frac{\nabla_x H}{H},$$

and

$$\frac{\Delta_v m}{m} = \frac{k\Delta_v H}{H} + \frac{k(k-1)|\nabla_v H|^2}{H^2},$$

in sum we have for ϕ

$$\frac{\phi H}{k} = \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} - v\cdot\nabla_v H + v\cdot\nabla_x H - \nabla_x V(x)\cdot\nabla_v H - M\chi_R,$$

Recall $H = 3v^2 + 2\langle v, x\rangle + x^2$, we have

$$\nabla_v H = 6v + 2x, \nabla_x H = 2v + 2x, \Delta_v H = 6,$$

we compute

$$\begin{aligned}
&\Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} + v\cdot\nabla_x H - v\cdot\nabla_v H - \nabla_x V(x)\cdot\nabla_v H \\
&= 6 + (k-1)\frac{(6v+2x)^2}{H} + 2v^2 + 2x\cdot v - 6v^2 \\
&-2x\cdot v - 6v\cdot\nabla_x V(x) - 2x\cdot\nabla_x V(x) \\
&< (2v^2 + Cv - 6v^2) - 2x\cdot\nabla_x V(x) + C \\
&< -C_1 v^2 - C_2 x\cdot\nabla_x V(x) + C_3 \\
&< -C_4 H^{\frac{\gamma}{2}} + M_1\chi_{R_1},
\end{aligned}$$

so taking M and R very large, we have $\phi \le -CH^{\frac{\gamma}{2}-1}$, taking this result into equation (4.1), we have:

(4.2)

$$\frac{d}{dt}Y_4(t) := \frac{d}{dt}\int |f_{\mathcal{B}}(t)|H^k = \int sign(f_{\mathcal{B}}(t))\mathcal{B}f_{\mathcal{B}}(t)H^k \le -C\int |f_B(t)|H^{k-1+\frac{\gamma}{2}}, k > 1,$$

using this result, first we have for any l large fixed, we can take M and R large such that

$$\frac{d}{dt}\int |f_{\mathcal{B}}(t)|H^l \le 0,$$

which implies

$$\int |f_{\mathcal{B}}(t)|H^l \le \int |f_0|H^l := Y_5,$$

and using Holder's inequality:

$$\int |f_B(t)|H^k \le (\int |f_B(t)|H^{k-1+\frac{\gamma}{2}})^{\alpha}(\int |f_B(t)|H^l)^{1-\alpha},$$

with

$$\alpha = \frac{l-k}{l-k+1-\frac{\gamma}{2}} \in [0,1],$$

which means

$$(\int |f_B(t)|H^k)^{\frac{1}{\alpha}}(\int |f_B(t)|H^l)^{\frac{\alpha-1}{\alpha}} \le \int |f_B(t)|H^{k-1+\frac{\gamma}{2}},$$

with

$$\alpha = \frac{l-k}{l-k+1-\frac{\gamma}{2}} \in [0,1],$$

taking the Holder's inequality into 4.2 we have

$$\frac{d}{dt}Y_4(t) \le -C(Y_4(t))^{\frac{1}{\alpha}}Y_5^{\frac{\alpha-1}{\alpha}},$$

using $Y_4(0) \leq Y_5$, we deduce by a simple argument

$$Y_4(t) \leq C_\alpha \frac{1}{(1+t)^{\frac{\alpha}{1-\alpha}}} Y_5,$$

which means we get a polynomial decay

$$\|\mathcal{S}_B(t)\|_{L^p(H^l)\to L^p(H^k)} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l-k}{1-\frac{\gamma}{2}}, \quad \forall 0 < k < l,$$

we can write it in this way

$$\|\mathcal{S}_B(t)\|_{L^p(H^l)\to L^p(H^{l\theta})} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l(1-\theta)}{1-\frac{\gamma}{2}}, \quad \forall 0 < \theta < 1, 0 < l,$$

so the proof is ended □

5. Regularization Lemma

This section we prove a regularization lemma.

In this section we will denote $\mathcal{L}^* = \mathcal{L}^*_{G^{-1/2}} = \mathcal{S} - \mathcal{T}$ be the dual operator of $\mathcal{L}$ on $L^2(G^{-1/2})$ such that

$$\int (\mathcal{L}f) g G^{-1} = \int (\mathcal{L}^* g) f G^{-1}.$$

for any smooth function f, g, and denote $\mathcal{B}^* = \mathcal{L}^* - M\chi_R$

Lemma 5.1. *For any $0 \leq \delta < 1$, there exist $\eta > 0$ such that*

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

and for $\mathcal{B}^*$ we have similar result

$$\|\mathcal{S}_{\mathcal{B}^*}(t) f\|_{L^2(G^{-1/2(1+\delta)})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(G^{-1/2(1+\delta)})}, \quad \forall t \in [0, \eta].$$

in particular by duality this implies

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^\infty(G^{-1/2})} \lesssim \frac{1}{t^{\frac{5d+1}{2}}} \|f\|_{L^2(G^{-1/2})}, \quad \forall t \in [0, \eta].$$

We start with some elementary lemmas.

Lemma 5.2. *For any $0 \leq \delta < 1$, we have*

$$\begin{aligned} \int f(\mathcal{L}g) G^{-(1+\delta)} + \int g(\mathcal{L}f) G^{-(1+\delta)} \quad &= -2 \int \nabla_v (f G^{-1}) \cdot \nabla_v (g G^{-1}) G^{1-\delta} \\ &+ \delta d \int f g G^{-(1+\delta)} - \delta(1-\delta) \int v^2 f g G^{-(1+\delta)}, \end{aligned}$$

in particular

$$\begin{aligned}\int f(\mathcal{L}f)G^{-(1+\delta)} \quad &= -\int |\nabla_v(fG^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2}\int f^2 G^{-(1+\delta)} \\ &\quad -\frac{\delta(1-\delta)}{2}\int v^2 f^2 G^{-(1+\delta)},\end{aligned}$$

we also have another version of $\int f(\mathcal{L}f)G^{-(1+\delta)}$

$$\begin{aligned}\int f(\mathcal{L}f)G^{-(1+\delta)} \quad &= -\int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2}\int v^2 f^2 G^{-(1+\delta)} \\ &\quad +\frac{(2+\delta)d}{2}\int f^2 G^{-(1+\delta)},\end{aligned}$$

and all the results above remain true when $\mathcal{L}$ *is replaced by* $\mathcal{L}^*$.

Proof of Lemma 5.2: First remember $\mathcal{T}(G^{-(1+\delta)}) = 0$, so we have

$$\int f(\mathcal{T}g)G^{-(1+\delta)} = \int \mathcal{T}(fG^{-(1+\delta)})g = -\int (\mathcal{T}f)gG^{-(1+\delta)},$$

which implies

$$\int f(\mathcal{T}g)G^{-(1+\delta)} + \int (\mathcal{T}f)gG^{-(1+\delta)} = 0,$$

As $\mathcal{L} = \mathcal{S} + \mathcal{T}, \mathcal{L}^* = \mathcal{S} - \mathcal{T}$, as the $\mathcal{T}$ term is 0, so the computation result of $\mathcal{L}$ and $\mathcal{L}^*$ is the same, for the term with operator $\mathcal{S}$ we have

$$\begin{aligned}\int f(\mathcal{S}g)G^{-(1+\delta)} \quad &= -\int \nabla_v(fG^{-(1+\delta)})\cdot(\nabla_v g + vg) \\ &= -\int (\nabla_v f + (1+\delta)vf)\cdot(\nabla_v g + vg)G^{-(1+\delta)} \\ &= -\int \nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta} - \int \delta v^2 fgG^{-(1+\delta)} - \int \delta f v\cdot\nabla_v gG^{-(1+\delta)},\end{aligned}$$

using integration by parts, we have

$$\begin{aligned}\int \delta v f\nabla_v gG^{-(1+\delta)} \quad &= -\int \delta g\nabla_v\cdot(vfG^{-(1+\delta)}) \\ &= -\int \delta v\cdot\nabla_v fgG^{-(1+\delta)} - \int \delta d fgG^{-(1+\delta)} - \int \delta(1+\delta)v^2 fgG^{-(1+\delta)},\end{aligned}$$

so

$$\begin{aligned}&\int (f(\mathcal{S}g) + g(\mathcal{S}f))G^{-(1+\delta)} \\ &= -2\int \nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta} + \delta d\int fgG^{-(1+\delta)} - \delta(1-\delta)\int v^2 fgG^{-(1+\delta)},\end{aligned}$$

Gathering the two terms, the proof of the first two equalities are done. □
For the last equality, we have:

$$\begin{aligned}\int f\mathcal{S}fG^{-(1+\delta)} &= -\int(\nabla_v f+(1+\delta)vf)\cdot(\nabla_v f+vf)G^{-(1+\delta)}\\ &= -\int|\nabla_v f|^2G^{-(1+\delta)}-\int(1+\delta)v^2f^2G^{-(1+\delta)}-\int(2+\delta)fv\cdot\nabla_v fG^{-(1+\delta)}\\ &= -\int|\nabla_v f|^2G^{-(1+\delta)}-\int(1+\delta)v^2f^2G^{-(1+\delta)}+\frac{2+\delta}{2}\int\nabla_v\cdot(vG^{-(1+\delta)})f^2\\ &= -\int|\nabla_v f|^2G^{-(1+\delta)}+\frac{\delta(1+\delta)}{2}\int v^2f^2G^{-(1+\delta)}+\frac{(2+\delta)d}{2}\int f^2G^{-(1+\delta)}.\end{aligned}$$

Putting together the above equality with $\int f\mathcal{T}fG^{-(1+\delta)}=0$ the proof is done. □

Note in the following of this section we will split into two cases, and denote f_t in two ways, $f_t=S_{\mathcal{B}}(t)f_0$ or $f_t=S_{\mathcal{B}^*}(t)f_0$.

Lemma 5.3. *When $f_t=S_{\mathcal{B}}(t)f_0$, we define an energy functional*

$$\begin{aligned}\mathcal{F}(t,f_t) &:= A\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})}+at^2\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta))})}\\ &\quad+2ct^4\langle\nabla_v f_t,\nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})}+bt^6\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta))})},\end{aligned}$$

when $f_t=S_{\mathcal{B}^*}(t)f_0$, we define an energy functional

$$\begin{aligned}\mathcal{F}(t,f_t) &:= A\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})}+at^2\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta))})}\\ &\quad-2ct^4\langle\nabla_v f_t,\nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})}+bt^6\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta))})},\end{aligned}$$

with $a,b,c>0, c\le\sqrt{ab}$ and A large enough.Then for both cases we have

$$\frac{d}{dt}F(t,f_t)\le -K(\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})}+t^4\|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})})+C\int f_t^2G^{-(1+\delta)},$$

for all $t\in[0,\eta]$ and for some $K>0, C>0$.

Proof of Lemma 5.3
We split the computation into several parts and then put them together.
Step 1: (1)For the case $f_t=S_{\mathcal{B}}(t)f_0$, first using both two equalities in Lemma 5.2 we have

$$\begin{aligned}\frac{d}{dt}\int f^2G^{-(1+\delta)} &= \int f(\mathcal{L}-M\chi_R)fG^{-(1+\delta)}\\ &= \frac{1-\delta}{2}\int f\mathcal{L}fG^{-(1+\delta)}+\frac{1+\delta}{2}\int f\mathcal{L}fG^{-(1+\delta)}-\int M\chi_R f^2G^{-(1+\delta)}\\ &\le -\frac{1-\delta}{2}\int|\nabla_v f|^2G^{-(1+\delta)}-\frac{1+\delta}{2}\int|\nabla_v(fG^{-1})|^2G^{1-\delta}+C\int f^2G^{-(1+\delta)}\\ &\le -\frac{1-\delta}{2}\int|\nabla_v f|^2G^{-(1+\delta)}+C\int f^2G^{-(1+\delta)}.\end{aligned}$$

(2)As Lemma 5.2 is also true for $\mathcal{L}^*$, so the above inequality is also true for $\mathcal{L}^*$.
Step 2 :(1)For the case $f_t = S_{\mathcal{B}}(t)f_0$, we easily compute

$$\partial_{x_i}\mathcal{L}f = \mathcal{L}\partial_{x_i}f + \partial_{x_i}(\nabla_x V(x)\cdot\nabla_v f) = \mathcal{L}\partial_{x_i}f + \sum_{j=1}^{d}\partial^2_{x_ix_j}V\partial_{v_j}f,$$

using this and Lemma 5.2 we have

$$\begin{aligned}\frac{d}{dt}\int(\partial_{x_i}f)^2G^{-(1+\delta)} &= \int\partial_{x_i}f\partial_{x_i}(\mathcal{L}-M\chi_R)fG^{-(1+\delta)}\\ &= -\int|\nabla_v(\partial_{x_i}fG^{-1})|^2G^{1-\delta} + \frac{\delta d}{2}\int(\partial_{x_i}f)^2G^{-(1+\delta)}\\ &-\frac{\delta(1-\delta)}{2}\int v^2(\partial_{x_i}f)^2G^{-(1+\delta)} + \int\partial_{x_i}f\sum_{j=1}^{d}\partial^2_{x_ix_j}V\partial_{v_j}fG^{-(1+\delta)}\\ &-\int M\chi_R|\partial_{x_i}f|^2G^{-(1+\delta)} + \int M\partial_{x_i}f\partial_{x_i}\chi_R fG^{-(1+\delta)},\end{aligned}$$

using Cauchy-Schwarz inequality and summing up by i, we get

$$\begin{aligned}&\frac{d}{dt}\int|\nabla_x f|^2G^{-(1+\delta)}\\ &\le -\sum_{i=1}^{d}\int|\nabla_v(\partial_{x_i}fG^{-1})|^2G^{1-\delta} - \frac{\delta(1-\delta)}{2}\int v^2(\nabla_x f)^2G^{-(1+\delta)}\\ &+C\int|\nabla_v f|^2G^{-(1+\delta)} + C\int|\nabla_x f|^2G^{-(1+\delta)} + C\int|f|^2G^{-(1+\delta)},\end{aligned}$$

for some $C>0$.
(2)For the case $f_t = S_{\mathcal{B}^*}(t)f_0$, we have

$$\partial_{x_i}\mathcal{L}^*f = \mathcal{L}^*\partial_{x_i}f - \partial_{x_i}(\nabla_x V(x)\cdot\nabla_v f) = \mathcal{L}^*\partial_{x_i}f - \sum_{j=1}^{d}\partial^2_{x_ix_j}V\partial_{v_j}f,$$

using the same argument we still have the same result as above.
Step 3:(1)For the case $f_t = S_{\mathcal{B}}(t)f_0$, we similarly compute

$$\partial_{v_i}\mathcal{L}f = \mathcal{L}\partial_{v_i}f - \partial_{x_i}f + \partial_{v_i}f,$$

using this inequality and Lemma 5.2, we have

$$\begin{aligned}\frac{d}{dt}\int(\partial_{v_i}f)^2G^{-(1+\delta)} &= \int \partial_{v_i}f\partial_{v_i}(\mathcal{L}-M\chi_R)fG^{-(1+\delta)}\\ &= -\int|\nabla_v(\partial_{v_i}fG^{-1})|^2G^{1-\delta}+\frac{\delta d}{2}\int(\partial_{v_i}f)^2G^{-(1+\delta)}\\ &-\frac{\delta(1-\delta)}{2}\int v^2(\partial_{v_i}f)^2G^{-(1+\delta)}-\int\partial_{x_i}f\partial_{v_i}fG^{-(1+\delta)}\\ &+\int|\partial_{v_i}f|^2G^{-(1+\delta)}-\int M\chi_R|\partial_{v_i}f|^2G^{-(1+\delta)}+M\partial_{v_i}f\partial_{v_i}\chi_RfG^{-(1+\delta)},\end{aligned}$$

using Cauchy-Schwarz inequality and summing up by i we get

$$\begin{aligned}&\frac{d}{dt}\int|\nabla_vf|^2G^{-(1+\delta)}\\ &\leq-\sum_{i=1}^d\int|\nabla_v(\partial_{v_i}fG^{-1})|^2G^{1-\delta}+C\int|\nabla_xf||\nabla_vf|G^{-(1+\delta)}\\ &+C\int|\nabla_vf|^2G^{-(1+\delta)}+C\int|f|^2G^{-(1+\delta)}-\frac{\delta(1-\delta)}{2}\int v^2(\nabla_vf)^2G^{-(1+\delta)}.\end{aligned}$$

For the case $f_t=S_{\mathcal{B}^*}(t)f_0$, we have

$$\partial_{v_i}\mathcal{L}^*f=\mathcal{L}^*\partial_{v_i}f+\partial_{x_i}f+\partial_{v_i}f,$$

using the same argument we still have the same result as above.
Step 4:(1)For the case $f_t=S_{\mathcal{B}}(t)f_0$, for the crossing term we have

$$\begin{aligned}\frac{d}{dt}\int 2\partial_{v_i}f\partial_{x_i}fG^{-(1+\delta)} &= (\int\partial_{v_i}f\partial_{x_i}\mathcal{L}fG^{-(1+\delta)}+\int\partial_{v_i}\mathcal{L}f\partial_{x_i}fG^{-(1+\delta)})\\ &-(\int\partial_{v_i}f\partial_{x_i}(M\chi_Rf)G^{-(1+\delta)}+\int\partial_{x_i}(M\chi_Rf)\partial_{v_i}fG^{-(1+\delta)})\\ &:=W_1+W_2,\end{aligned}$$

we split it into two parts, for the first part still using

$$\partial_{x_i}\mathcal{L}f=\mathcal{L}\partial_{x_i}f+\partial_{x_i}(\nabla_xV(x)\cdot\nabla_vf)=\mathcal{L}\partial_{x_i}f+\sum_{j=1}^d\partial^2_{x_ix_j}V\partial_{v_j}f,$$

and

$$\partial_{v_i}\mathcal{L}f=\mathcal{L}\partial_{v_i}f-\partial_{x_i}f+\partial_{v_i}f,$$

we have

$$\begin{aligned}W_1 &= \int\partial_{v_i}f\mathcal{L}(\partial_{x_i}f)G^{-(1+\delta)}+\int\mathcal{L}(\partial_{v_i}f)\partial_{x_i}fG^{-(1+\delta)}\\ &+\int\partial_{v_i}f\sum_{j=1}^d\partial_{x_ix_j}V(x)\partial_{v_j}fG^{-(1+\delta)}-\int|\partial_{x_i}f|^2G^{-(1+\delta)}+\int\partial_{x_i}f\partial_{v_i}fG^{-(1+\delta)},\end{aligned}$$

using Lemma 5.2, we have

$$\begin{aligned} W_1 \quad &= -\int 2\nabla_v(\partial_{v_i} f G^{-1})\cdot\nabla_v(\partial_{x_i} f G^{-1})G^{1-\delta} + \delta d\int \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} \\ &-\delta(1-\delta)\int v^2 \partial_{v_i} f \partial_{x_i} f G^{-(1+\delta)} + \int \partial_{v_i} f \sum_{j=1}^{d} \partial_{x_i x_j} V(x) \partial_{v_j} f G^{-(1+\delta)} \\ &-\int |\partial_{x_i} f|^2 G^{-(1+\delta)} + \int \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)}. \end{aligned}$$

For the W_2 term we have

$$\begin{aligned} W_2 = \quad &-\int \partial_{v_i} f \partial_{x_i}(M\chi_R f) G^{-(1+\delta)} - \int \partial_{x_i}(M\chi_R f)\partial_{v_i} f G^{-(1+\delta)} \\ &= -\int 2M\chi_R \partial_{x_i} f \partial_{v_i} f G^{-(1+\delta)} + \int \partial_{v_i} f M \partial_{x_i}\chi_R f G^{-(1+\delta)} + \int M\partial_{v_i}\chi_R f \partial_{x_i} f G^{-(1+\delta)} \\ &\le C\int |\partial_{x_i} f||\partial_{v_i} f| G^{-(1+\delta)} + C\int |\partial_{v_i} f||f| G^{-(1+\delta)} + C\int |f||\partial_{x_i} f| G^{-(1+\delta)}, \end{aligned}$$

combining the two parts ,using Cauchy-Schwarz inequality and summing up by i we get

$$\begin{aligned} &\frac{d}{dt}\int 2\nabla_x f\cdot\nabla_v f G^{-(1+\delta)} \\ &\le -\sum_{i=1}^{d}\int 2\nabla_v(\partial_{v_i} f G^{-1})\cdot\nabla_v(\partial_{x_i} f G^{-1})G^{1-\delta} - \frac{1}{2}\int |\nabla_x f|^2 G^{-(1+\delta)} \\ &+C\int |\nabla_v f|^2 G^{-(1+\delta)} + C\int |f|^2 G^{-(1+\delta)} - \delta(1-\delta)\int v^2 \nabla_v f\cdot\nabla_x f G^{-(1+\delta)}. \end{aligned}$$

(2For the case $f_t = S_{\mathcal{B}^*}(t) f_0$ using

$$\partial_{x_i}\mathcal{L}^* f = \mathcal{L}^*\partial_{x_i} f - \partial_{x_i}(\nabla_x V(x)\cdot\nabla_v f) = \mathcal{L}^*\partial_{x_i} f - \sum_{j=1}^{d}\partial^2_{x_i x_j} V \partial_{v_j} f,$$

and

$$\partial_{v_i}\mathcal{L}^* f = \mathcal{L}^*\partial_{v_i} f + \partial_{x_i} f + \partial_{v_i} f,$$

we will get a slightly different result

$$\begin{aligned} &-\frac{d}{dt}\int 2\nabla_x f\cdot\nabla_v f G^{-(1+\delta)} \\ &\le \sum_{i=1}^{d}\int 2\nabla_v(\partial_{v_i} f G^{-1})\cdot\nabla_v(\partial_{x_i} f G^{-1})G^{1-\delta} - \frac{1}{2}\int |\nabla_x f|^2 G^{-(1+\delta)} \\ &+C\int |\nabla_v f|^2 G^{-(1+\delta)} + C\int |f|^2 G^{-(1+\delta)} + \delta(1-\delta)\int v^2 \nabla_v f\cdot\nabla_x f G^{-(1+\delta)}. \end{aligned}$$

Step 5: (1)For the case $f_t = S_{\mathcal{B}}(t)f_0$, gathering all the terms together, recall

$$\begin{aligned}\mathcal{F}(t,f_t) \quad &:= A\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + at^2\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta))})} \\ &+2ct^4\langle\nabla_v f_t, \nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})} + bt^6\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta))})},\end{aligned}$$

with $a,b,c>0, c\leq\sqrt{ab}$ and A large enough. We easily compute

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}(t,f_t) \\ &= A\frac{d}{dt}\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + at^2\frac{d}{dt}\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \\ &+2ct^4\frac{d}{dt}\langle\nabla_v f_t,\nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})} + bt^6\frac{d}{dt}\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \\ &+2at\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + 8ct^3\langle\nabla_v f_t,\nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})} + 6bt^5\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})}.\end{aligned}$$

Using the results in Step 1-4 and gathering all the terms together we have

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}(t,f_t) \\ &\leq (2at - \frac{A(1-\delta)}{2} + Cat^2 + 2Ct^4c + Cbt^6)\int|\nabla_v f_t|^2G^{-(1+\delta)} \\ &+(6bt^5 - ct^4 + Cbt^6)\int|\nabla_x f_t|^2G^{-(1+\delta)} + C\int f_t^2G^{-(1+\delta)} \\ &+(8ct^3 + Cat^2)\int|\nabla_v f_t||\nabla_x f_t|G^{-(1+\delta)} \\ &-(at^2\sum_{i=1}^d\int|\nabla_v(\partial_{v_i}f_tG^{-1})|^2G^{1-\delta} + bt^6\sum_{i=1}^d\int|\nabla_v(\partial_{x_i}f_tG^{-1})|^2G^{1-\delta} \\ &+2ct^4\sum_{i=1}^d\int 2\nabla_v(\partial_{v_i}f_tG^{-1})\cdot\nabla_v(\partial_{x_i}f_tG^{-1})G^{1-\delta}) - \frac{\delta(1-\delta)}{2}(at^2\int v^2(\nabla_v f)^2G^{-(1+\delta)} \\ &+bt^6\int v^2(\nabla_x f)^2G^{-(1+\delta)} + 2ct^4\int v^2\nabla_v f\cdot\nabla_x fG^{-(1+\delta)}),\end{aligned}$$

by our taking on a,b,c we can have

$$|2ct^4\int v^2\nabla_v f\cdot\nabla_x fG^{-(1+\delta)}| \leq at^2\int v^2(\nabla_v f)^2G^{-(1+\delta)} + bt^6\int v^2(\nabla_x f)^2G^{-(1+\delta)},$$

and

$$\begin{aligned}&|2ct^4\sum_{i=1}^d\int 2\nabla_v(\partial_{v_i}f_tG^{-1})\cdot\nabla_v(\partial_{x_i}f_tG^{-1})G^{1-\delta}| \\ \leq\quad &at^2\sum_{i=1}^d\int|\nabla_v(\partial_{v_i}f_tG^{-1})|^2G^{1-\delta} + bt^6\sum_{i=1}^d\int|\nabla_v(\partial_{x_i}f_tG^{-1})|^2G^{1-\delta},\end{aligned}$$

by taking A large and $0 < \eta$ small ($t \in [0, \eta]$), we have for some $K > 0$:

$$\frac{d}{dt}\mathcal{F}(t, f_t) \quad \leq -K(\|\nabla_v f_t\|_{L^2(G^{-1/2(1+\delta)})} + t^4\|\nabla_x f_t\|_{L^2(G^{-1/2(1+\delta)})}) + C\int f_t^2 G^{-(1+\delta)}.$$

(2)For the case $f_t = S_{\mathcal{B}^*}(t)f_0$, the only change is that the last three terms changes to

$$\begin{aligned}
&(Cat^2 - 8ct^3)\int |\nabla_v f_t||\nabla_x f_t| G^{-(1+\delta)} \\
&-(at^2\sum_{i=1}^d \int |\nabla_v(\partial_{v_i} f_t G^{-1})|^2 G^{1-\delta} + bt^6 \sum_{i=1}^d \int |\nabla_v(\partial_{x_i} f_t G^{-1})|^2 G^{1-\delta} \\
&-2ct^4 \sum_{i=1}^d \int 2\nabla_v(\partial_{v_i} f_t G^{-1}) \cdot \nabla_v(\partial_{x_i} f_t G^{-1}) G^{1-\delta}) - \frac{\delta(1-\delta)}{2}(at^2 \int v^2 (\nabla_v f)^2 G^{-(1+\delta)} \\
&+bt^6 \int v^2 (\nabla_x f)^2 G^{-(1+\delta)} - 2ct^4 \int v^2 \nabla_v f \cdot \nabla_x f G^{-(1+\delta)}),
\end{aligned}$$

which will not change the result. The proof of Lemma 5.3 is ended. □

The following proof of this section is true for both cases.

To prove Lemma 5.1 we still need some other lemmas.

Lemma 5.4. *For any $0 \leq \delta < 1$ we have*

$$\int |\nabla_{x,v}(fG^{-1/2(1+\delta)})|^2 \leq \int |\nabla_{x,v} f|^2 G^{-(1+\delta)} + C\int f^2 G^{-(1+\delta)},$$

Prove of Lemma 5.4 : For any weight function m we have

$$\begin{aligned}
\int |\nabla_x(fm)|^2 \quad &= \int |\nabla_x f m + \nabla_x m f|^2 \\
&= \int |\nabla_x f|^2 m^2 + \int |\nabla_x m|^2 f^2 + \int 2f\nabla_x f m \nabla_x m \\
&= \int |\nabla_x f|^2 m^2 + \int |\nabla_x m|^2 f^2 - \int \frac{1}{2} f^2 \Delta_x(m^2) \\
&= \int |\nabla_x f|^2 m^2 + \int (|\nabla_x m|^2 - \frac{1}{2}\Delta_x(m^2)) f^2,
\end{aligned}$$

taking $m = G^{-1/2(1+\delta)}$ we have

$$\begin{aligned}
&\int |\nabla_x(fG^{-1/2(1+\delta)})|^2 \\
&= \int |\nabla_x f|^2 G^{-(1+\delta)} + \int (|\nabla_x G^{-1/2(1+\delta)}|^2 - \frac{1}{2}\Delta_x G^{-(1+\delta)}) f^2 \\
&= \int |\nabla_x f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4}|\nabla_x V(x)|^2 + \frac{1+\delta}{2}\Delta_x V(x)) f^2 G^{-(1+\delta)} \\
&\leq \int |\nabla_x f|^2 G^{-(1+\delta)} + C\int f^2 G^{-(1+\delta)},
\end{aligned}$$

similarly

$$\begin{aligned}
&\int |\nabla_v (fG^{-1/2(1+\delta)})|^2 \\
&= \int |\nabla_v f|^2 G^{-(1+\delta)} + \int (|\nabla_v G^{-1/2(1+\delta)}|^2 - \frac{1}{2}\Delta_v G^{-(1+\delta)}) f^2 \\
&= \int |\nabla_v f|^2 G^{-(1+\delta)} + \int -(\frac{(1+\delta)^2}{4} v^2 + \frac{1+\delta}{2} d) f^2 G^{-(1+\delta)} \\
&\leq \int |\nabla_v f|^2 G^{-(1+\delta)}.
\end{aligned}$$

Putting together the two inequalities we obtain the result. □

Lemma 5.5. *Nash's inequality: for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$,there exist a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}} \leq C_d \|f\|_{L^1}^{2/d} \|\nabla_v f\|_{L^2},$$

Proof of Nash's inequality: For any $R > 0$ we have

$$\begin{aligned}
\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 \qquad &= \int_{|\xi| \leq R} |\hat{f}|^2 + \int_{|\xi| \geq R} |\hat{f}|^2 \\
&\leq c_d R^d \|\hat{f}\|_{L^\infty}^2 + \frac{1}{R^2} \int_{|\xi| \leq R} \xi^2 |\hat{f}|^2 \\
&\leq c_d R^d \|f\|_{L^1}^2 + \frac{1}{R^2} \|\nabla_v f\|_{L^2}^2,
\end{aligned}$$

taking an optimal R by setting $R = (\|f\|_{L^2}^2 / c_d \|f\|_{L^1}^2)^{\frac{1}{d+2}}$, we are done the proof. □

Lemma 5.6. *For any $0 \leq \delta < 1$ we have*

$$\frac{d}{dt} \int |f| G^{-1/2(1+\delta)} \leq d \int |f| G^{-1/2(1+\delta)},$$

which implies

$$\int |f_t| G^{-1/2(1+\delta)} \leq C e^{dt} \int |f_0| G^{-1/2(1+\delta)},$$

in particular we have

$$\int |f_t| G^{-1/2(1+\delta)} \leq C \int |f_0| G^{-1/2(1+\delta)}, \quad \forall t \in [0, \eta],$$

Proof of Lemma 5.6:For the case $f_t = S_{\mathcal{B}}(t)f_0$, by Lemma 4.2, we have

$$\begin{aligned}
&\frac{d}{dt}\int |f|G^{-1/2(1+\delta)}\\
&= \int |f|(\Delta_v G^{-1/2(1+\delta)} - v\cdot\nabla_v G^{-1/2(1+\delta)}\\
&+v\cdot\nabla_x G^{-1/2(1+\delta)} - \nabla V(x)\cdot\nabla_v G^{-1/2(1+\delta)} - M\chi_R G^{-1/2(1+\delta)})\\
&\leq \int |f|(\frac{1+\delta}{2}d - \frac{(1+\delta)(1-\delta)}{4}v^2)G^{-1/2(1+\delta)} \leq \int |f|dG^{-1/2(1+\delta)},
\end{aligned}$$

as $\mathcal{T}G^{-1/2(1+\delta)} = 0$, the result is still true for the case $f_t = S_{\mathcal{B}^*}(t)f_0$ □

Now we come to the proof of our original lemma.

Proof of Lemma 5.1: We define

$$\mathcal{G}(t,f_t) = B\|f_t\|^2_{L^1(G^{-1/2(1+\delta)})} + t^Z\mathcal{F}(t,f_t),$$

with $Z>0$ to be fixed and $\mathcal{F}$ defined in Lemma 5.3. It's important to recall here that

$$\begin{aligned}
\mathcal{F}(t,f_t) \quad &:= A\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + at^2\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)}))}\\
&+2ct^4\langle\nabla_v f_t, \nabla_x f_t\rangle^2_{L^2(G^{-1/2(1+\delta)})} + bt^6\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)}))}.
\end{aligned}$$

We choose $t\in[0,\eta]$,η small such that $(a+b+c)Zt^{Z+1} \leq \frac{1}{2}Kt^Z$ (K is also in Lemma 5.3). Then by Lemma 5.6 and Lemma 5.3

$$\begin{aligned}
\frac{d}{dt}\mathcal{G}(t,f_t) \quad &\leq dB\|f_t\|^2_{L^1(G^{-1/2(1+\delta)})} + Zt^{Z-1}\mathcal{F}(t,f_t)\\
&-Kt^Z(\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + t^4\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})})\\
&+Ct^Z\int f_t^2 G^{-(1+\delta)}\\
&\leq dB\|f_t\|^2_{L^1(G^{-1/2(1+\delta)})} + Ct^{Z-1}\int f_t^2 G^{-(1+\delta)}\\
&-\frac{K}{2}t^Z(\|\nabla_v f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + t^4\|\nabla_x f_t\|^2_{L^2(G^{-1/2(1+\delta)})}),
\end{aligned}$$

Nash's inequality and Lemma 5.4 implies

$$\begin{aligned}
\int f_t^2 G^{-(1+\delta)} \quad &\leq (\int |f_t|G^{-1/2(1+\delta)})^{\frac{4}{d+2}}(\int |\nabla_{x,v}(f_t G^{-1/2(1+\delta)})|^2)^{\frac{d}{d+2}}\\
&\leq (\int |f_t|G^{-1/2(1+\delta)})^{\frac{4}{d+2}}(\int |\nabla_{x,v} f_t|^2 G^{-(1+\delta)} + C\int f_t^2 G^{-(1+\delta)})^{\frac{d}{d+2}},
\end{aligned}$$

using Young's inequality we have

$$\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \leq C_\epsilon t^{-5d}\|f\|^2_{L^1(G^{-1/2(1+\delta)})} + \epsilon t^5(\|\nabla_{x,v} f_t\|^2_{L^2(G^{-1/2(1+\delta)})} + C\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})}),$$

we deduce

$$\|f_t\|^2_{L^2(G^{-1/2(1+\delta)})} \leq 2C_\epsilon t^{-5d}\|f\|^2_{L^1(G^{-1/2(1+\delta)})} + 2\epsilon t^5\|\nabla_{x,v} f_t\|^2_{L^2(G^{-1/2(1+\delta)})},$$

taking ϵ small we have

$$\frac{d}{dt}\mathcal{G}(t,f_t)\leq dB\|f_t\|^2_{L^1(G^{-1/2(1+\delta)})}+C_1t^{Z-1-5d}\|f_t\|^2_{L^1(G^{-1/2(1+\delta)})},$$

finally choose $Z=1+5d$,and using Lemma 5.6 then we deduce

$$\forall t\in[0,\eta],\quad \mathcal{G}(t,f_t)\leq \mathcal{G}(0,f_0)+C_1\|f_0\|^2_{L^1(G^{-1/2(1+\delta)})}\leq C_2\|f_0\|^2_{L^1(G^{-1/2(1+\delta)})},$$

the proof is ended. □

6. L^p convergence for the KFP model

To reach the final convergence we first need:

Lemma 6.1. *Recall $\mathcal{A}=M\chi_R$, $\mathcal{B}=\mathcal{L}-M\chi_R$, using the estimates in former sections, for any $\epsilon>0$ small, we can have some estimates on $\mathcal{A}S_{\mathcal{B}}(t)$*

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

and

$$\|\mathcal{A}\mathcal{S}_B(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^1(G^{-(\frac{1}{2}+\epsilon)})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

for some $a>0$, and for any $0<b<\frac{\gamma}{2-\gamma}$ we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim t^{-\alpha}e^{-at^b},$$

for $\alpha=\frac{5d+1}{2}$ and some $a>0$.

Proof of Lemma 6.1 : (1) Using the result in Lemma 3.3 we have

$$\|\mathcal{S}_B(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

recall $\mathcal{A}=M\chi_R$, so

$$\|\mathcal{A}\|_{L^2(G^{-\frac{1}{2}})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim 1,$$

combining the above 2 inequalities we are done.
(2) Also by Lemma 3.3 we have

$$\|\mathcal{S}_B(t)\|_{L^1(G^{-\epsilon})\to L^1}\lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

and obviously

$$\|\mathcal{A}\|_{L^1\to L^1(G^{-(\frac{1}{2}+\epsilon)})}\lesssim 1,$$

so combine the 2 inequalities we are done.
(3) For the third inequality, we split it into two parts. When $t\in[0,\eta]$, $e^{-at^{\frac{\gamma}{2-\gamma}}}\geq e^{-a\eta^{\frac{\gamma}{2-\gamma}}}$,and $\mathcal{A}\leq MI$, using Lemma 5.1, we have

$$\|\mathcal{A}\mathcal{S}_B(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim t^{-\alpha}\lesssim t^{-\alpha}e^{-at^{\frac{\gamma}{2-\gamma}}},a>0,\forall t\in[0,\eta].$$

When $t \geq \eta$, by Lemma 5.1 we have

$$\|S_{\mathcal{B}}(\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim \eta^{\alpha} \lesssim 1,$$

using Lemma 3.3,we have

$$\|\mathcal{S}_{\mathcal{B}}(t-\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})} \leq Ce^{-a(t-\eta)^{\frac{\gamma}{2-\gamma}}} \leq Ce^{-at^{\frac{\gamma}{2-\gamma}}},$$

for $\mathcal{A}$ obviously

$$\|\mathcal{A}\|_{L^2(G^{-\frac{1}{2}})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim 1,$$

when $t > \eta$, gathering all the inequalities above, we have

$$\|\mathcal{A}\mathcal{S}_B(t)\|_{L^1(G^{-1/2(1+2\epsilon)})\to L^2(G^{-1/2(1+2\epsilon)})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}} \lesssim t^{-\alpha}e^{-at^b}.$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, combining the two cases, the proof is ended. □

Lemma 6.2. *similarly as Lemma 6.1, for any $p \in (2,\infty)$, we have*

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-1/2})\to L^2(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

and

$$\|\mathcal{S}_B(t)\mathcal{A}\|_{L^p(G^{-1/2})\to L^p(G^{-1/2})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

for some $a > 0$, and for any $0 < b < \frac{\gamma}{2-\gamma}$ we have

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-1/2})\to L^p(G^{-1/2})} \lesssim t^{-\beta}e^{-at^b},$$

for some $\beta > 0$ and some $a > 0$.

Proof of Lemma 6.2 : (1) Using the result in Lemma 3.3 we have

$$\|\mathcal{S}_B(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

recall $\mathcal{A} = M\chi_R$, so

$$\|\mathcal{A}\|_{L^2(G^{-1/2})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim 1,$$

combining the above 2 inequalities we are done.
(2) Also by Lemma 3.3 we have

$$\|\mathcal{S}_B(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})\to L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, a > 0,$$

and obviously

$$\|\mathcal{A}\|_{L^p(G^{-1/2})\to L^p(G^{-(\frac{p-1}{p}+\epsilon)})} \lesssim 1,$$

so combine the 2 inequalities we are done.
(3) For the third inequality, we split it into two parts. Interpolate between Lemma 3.3 and Lemma 5.1 we have

$$\|\mathcal{S}_B(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^p(G^{-1/2})} \lesssim t^{-\beta},$$

for some $\beta > 0$.

When $t \in [0, \eta]$, $e^{-at^{\frac{\gamma}{2-\gamma}}} \geq e^{-a\eta^{\frac{\gamma}{2-\gamma}}}$,and $\mathcal{A} \leq MI$, we have

$$\|\mathcal{S}_B(t)\mathcal{A}\|_{L^2(G^{-1/2})\to L^p(G^{-1/2})} \lesssim t^{-\beta} \lesssim t^{-\beta}e^{-at^{\frac{\gamma}{2-\gamma}}}, a > 0, \forall t \in [0, \eta].$$

When $t \geq \eta$, interpolate between Lemma 3.3 and Lemma 5.1 and take $2\delta = \gamma$ we have

$$\|S_{\mathcal{B}}(\eta)\|_{L^2(e^{\epsilon H^\delta}G^{-1/2})\to L^p(G^{-1/2})} \lesssim \eta^{-\beta} \lesssim 1,$$

using Lemma 3.3,we have

$$\|\mathcal{S}_{\mathcal{B}}(t-\eta)\|_{L^2(e^{2\epsilon H^\delta}G^{-1/2})\to L^2(e^{\epsilon H^\delta}G^{-1/2})} \lesssim e^{-a(t-\eta)^{\frac{\gamma}{2-\gamma}}} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

for $\mathcal{A}$ obviously

$$\|\mathcal{A}\|_{L^2(G^{-1/2})\to L^2(e^{2\epsilon H^\delta}G^{-1/2})} \lesssim 1,$$

when $t > \eta$, gathering all the inequalities above, we have

$$\|\mathcal{A}\mathcal{S}_B(t)\|_{L^1(G^{-1/2(1+2\epsilon)})\to L^2(G^{-1/2(1+2\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}} \lesssim t^{-\beta}e^{-at^b}.$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, combining the two cases, the proof is ended. □

Then we need an abstract lemma like Lemma 2.4 in [3].

Lemma 6.3. *let X, Y be two Banach spaces, $S(t)$ a semigroup such that for all $t \geq 0$and some $0 < a, 0 < b < 1$ we have*

$$\|S(t)\|_{X\to X} \leq C_X e^{-at^b}, \quad \|S(t)\|_{Y\to Y} \leq C_Y e^{-at^b},$$

and for some $0 < \alpha$, we have

$$\|S(t)\|_{X\to Y} \leq C_{X,Y}t^{-\alpha}e^{-at^b}.$$

Then we can have that for all integer $n > 0$

$$\|S^{(*n)}(t)\|_{X\to X} \leq C_{X,n}t^{n-1}e^{-at^b},$$

similarly

$$\|S^{(*n)}(t)\|_{Y\to Y} \leq C_{Y,n}t^{n-1}e^{-at^b},$$

and

$$\|S^{(*n)}(t)\|_{X\to Y} \leq C_{X,Y,n}t^{n-\alpha-1}e^{-at^b},$$

in particular for $\alpha + 1 < n$, and $\forall b^* < b$

$$\|S^{(*n)}(t)\|_{X\to Y} \leq C_{X,Y,n}e^{-at^{b^*}}.$$

Proof of Lemma 6.3: We use the fact that $t^b \leq s^b + (t-s)^b$ for any $0 \leq s \leq t, 0 < b < 1$, by induction

$$\begin{aligned}\|S^{*(n+1)}(t)\|_{X\to X} &\leq \int_0^t \|S(t-s)S^{*n}(s)\|_{X\to X} \leq C_n \int_0^t e^{-a(t-s)^b}e^{-as^b}s^{n-1}ds \\ &\leq C_n e^{-at^b}\int_0^t s^{n-1}ds = C_{n+1}e^{-at^b}t^n,\end{aligned}$$

and we compute

$$\begin{aligned}\|S^{*(n+1)}(t)\|_{X\to Y} &\le \int_0^{t/2}\|S^{*(n)}(t-s)\|_{X\to Y}\|S(s)\|_{X\to X}ds\\ &+\int_{t/2}^{t}\|S^{*(n)}(t-s)\|_{Y\to Y}\|S(s)\|_{X\to Y}\\ &\le C_nC_1\int_0^{t/2}(t-s)^{n-\alpha-1}e^{-a(t-s)^b}e^{-as^b}ds\\ &+C_nC_1\int_{t/2}^{t}(t-s)^{n-1}e^{-a(t-s)^b}s^{-\alpha}e^{-as^b}ds\\ &\le C_nC_1e^{-at^b}t^{n-\alpha}(\int_0^{1/2}(1-\tau)^{n-\alpha-1}d\tau+\int_{1/2}^{1}(1-\tau)^{n-1}\tau^{-\alpha}d\tau)\\ &\le C_{n+1}t^{n-\alpha}e^{-at^b},\end{aligned}$$

the proof of the lemma is ended. □

Then we come to the final proof.

Proof of Theorem 1.1: We only prove the case when $m = H^k$, the case $m = G^{\frac{p-1}{p}}(1+\epsilon)$ is similar. By Duhamel formula, we have

$$\begin{aligned}S_{\mathcal{L}} &= S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{L}}) = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}}\\ &= S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}) = S_{\mathcal{B}} + (S_{\mathcal{L}}\mathcal{A}) * S_{\mathcal{B}},\end{aligned}$$

iterate the formula in this way

$$\begin{aligned}S_{\mathcal{L}} &= S_{\mathcal{B}} + (S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})) * (\mathcal{A}S_{\mathcal{B}})\\ &= S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}}) + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*2)}\\ &= S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*2)},\end{aligned}$$

we can get

$$\begin{aligned}S_{\mathcal{L}}(I-\Pi) &= (I-\Pi)\{S_{\mathcal{B}} + \sum_{l=1}^{n-1}(S_{\mathcal{B}}\mathcal{A})^{(*l)} * (S_{\mathcal{B}})\}\\ &\quad +\{(I-\Pi)S_{\mathcal{L}}\} * (\mathcal{A}S_{\mathcal{B}}(t))^{*n}.\end{aligned}$$

For the first term we have from Theorem 4.1

$$\|S_{\mathcal{B}}(t)\|_{L^1(m)\to L^1(m^\theta)} \lesssim (1+t)^{-a}.$$

For the second term, we have by Lemma 3.3

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^1(m)\to L^1(m)} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

so by Lemma 6.3

$$\|(S_{\mathcal{B}}(t)\mathcal{A})^{*l}\|_{L^1(m)\to L^1(m)} \lesssim t^{l-1}e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

For the last term we have from Theorem 4.1

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(m)\to L^1(G^{-(\frac{1}{2}+\epsilon)})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

by Lemma 6.1, for any $0<b<\frac{\gamma}{2-\gamma}$, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n-1)}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim t^{n-\alpha-2}e^{-at^b},$$

by Theorem 3.1

$$\|S_{\mathcal{L}}(t)(I-\Pi)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-1/2})}\lesssim e^{-at^{\frac{2\delta}{2-\gamma}}},$$

and for the identity operator I, obviously we have

$$\|I\|_{L^2(G^{-\frac{1}{2}})\to L^1(m^\theta)}\lesssim 1,$$

so taking $n>\alpha+2$ and gathering all the terms together, and use this inequality for any $\epsilon>0$

$$\begin{aligned}\int_0^t(1+t-s)^{-a}e^{-Cs^b}ds&\le\int_0^t(\frac{1+s}{1+t})^a e^{-Cs^b}ds=(1+t)^{-a}\int_0^t(1+s)^{-a}e^{-Cs^b}ds\\&=(1+t)^{-a}(\int_0^{t^\epsilon}+\int_{t^\epsilon}^t)(1+s)^{-a}e^{-Cs^b}ds\lesssim(1+t)^{-a+\epsilon},\end{aligned}$$

we deduce

$$\|S_{\mathcal{L}}(t)(I-\Pi)\|_{L^1(m)\to L^1(m^\theta)}\lesssim(1+t)^{-a+\epsilon},$$

for the case $m=G^{-(\frac{p-1}{p}+\epsilon)}$, the only change is this time we have

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})\to L^p(G^{-\frac{p-1}{p}})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

using similar argument we can get

$$\|S_{\mathcal{L}}(t)(I-\Pi)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})\to L^p(G^{-\frac{p-1}{p}})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

which implies Theorem 1.1 □

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