



**HAL**  
open science

# MIXING TIME OF A KINETICALLY CONSTRAINED SPIN MODEL ON TREES: POWER LAW SCALING AT CRITICALITY

N. Cancrini, F. Martinelli, C. Roberto, C. Toninelli

► **To cite this version:**

N. Cancrini, F. Martinelli, C. Roberto, C. Toninelli. MIXING TIME OF A KINETICALLY CONSTRAINED SPIN MODEL ON TREES: POWER LAW SCALING AT CRITICALITY. *Probability Theory and Related Fields*, 2015, 161 (1-2), pp.247 - 266. 10.1007/s00440-014-0548-x . hal-01696432

**HAL Id: hal-01696432**

**<https://hal.science/hal-01696432>**

Submitted on 30 Jan 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MIXING TIME OF A KINETICALLY CONSTRAINED SPIN MODEL ON TREES: POWER LAW SCALING AT CRITICALITY

N. CANCRINI, F. MARTINELLI, C. ROBERTO, AND C. TONINELLI

ABSTRACT. On the rooted  $k$ -ary tree we consider a 0-1 kinetically constrained spin model in which the occupancy variable at each node is re-sampled with rate one from the Bernoulli( $p$ ) measure iff all its children are empty. For this process the following picture was conjectured to hold. As long as  $p$  is below the percolation threshold  $p_c = 1/k$  the process is ergodic with a finite relaxation time while, for  $p > p_c$ , the process on the infinite tree is no longer ergodic and the relaxation time on a finite regular sub-tree becomes exponentially large in the depth of the tree. At the critical point  $p = p_c$  the process on the infinite tree is still ergodic but with an infinite relaxation time. Moreover, on finite sub-trees, the relaxation time grows polynomially in the depth of the tree.

The conjecture was recently proved by the second and fourth author except at criticality. Here we analyse the critical and quasi-critical case and prove for the relevant time scales: (i) power law behaviour in the depth of the tree at  $p = p_c$  and (ii) power law scaling in  $(p_c - p)^{-1}$  when  $p$  approaches  $p_c$  from below. Our results, which are very close to those obtained recently for the Ising model at the spin glass critical point, represent the first rigorous analysis of a kinetically constrained model at criticality.

## 1. INTRODUCTION

On the state space  $\{0, 1\}^{\mathbb{T}^k}$ , where  $\mathbb{T}^k$  is the regular rooted tree with  $k \geq 2$  children for each node, we consider a constrained spin model in which each spin, with rate one and iff all its children are zero, chooses a new value in  $\{0, 1\}$  with probability  $1 - p$  and  $p$  respectively. This model belongs to the class of *kinetically constrained spin models* which have been introduced in physics literature to model liquid/glass transition or, more generally, glassy dynamics (see [10, 17] for physical background and [3] for related mathematical work). As for most of the kinetically constrained models, the Bernoulli( $p$ ) product measure  $\mu$  is a reversible measure for the process.

When  $k = 1$  the model coincides with the well known East model [12] (see also [1, 3, 4, 8, 9] for rigorous analysis). As soon as  $k \geq 2$ , the model shares some of the key features of another well known kinetically constrained system, namely the North East model [3, 13]. More specifically, since above the critical density  $p_c = 1/k$  the occupied vertices begin to percolate (under the reversible measure  $\mu$ ), blocked clusters appear and time ergodicity is lost. It is therefore particularly interesting to study the relaxation to equilibrium in e.g. finite sub-trees of  $\mathbb{T}^k$ , when the density  $p$  is below, equal or above the critical density  $p_c = 1/k$ .

In [16] it was recently proved that, as long as  $p < p_c$ , the process on the infinite tree is exponentially ergodic with a finite relaxation time  $T_{\text{rel}}$ . Under the same assumption,

---

Work supported by the European Research Council through the ‘‘Advanced Grant’’ PTRELSS 228032. The first author was also partially supported by PRIN 2009 protocollo n.2009TA2595.02 and the fourth author by the French Ministry of Education through the ANR BLAN07-2184264.

on a finite tree with suitable boundary conditions on the leaves the mixing time was also shown to be linear in the depth of the tree. When instead  $p > p_c$  the ergodicity on the infinite tree is lost and both the relaxation and the mixing times for finite trees diverge exponentially fast in the depth of the tree.

In this paper we tackle for the first time the critical case  $p = p_c$ . Our main results, answering a question of Aldous-Diaconis [1], can be formulated as follows.

- *Critical case.* Assume  $p = p_c$  and let  $\mathbb{T}$  be a finite  $k$ -ary rooted tree of depth<sup>1</sup>  $L$ . Denote by  $T_{\text{rel}}(\mathbb{T})$  and  $T_{\text{mix}}(\mathbb{T})$  the relaxation time of the process on  $\mathbb{T}$  with no constraints for the spins at the leaves (cf definitions 1.3 and 1.4). Then (cf Theorem 1)  $T_{\text{rel}} = \Omega(L^2)$  and  $T_{\text{rel}} = O(L^{2+\beta})$  for some  $0 \leq \beta < \infty$ .
- *Quasi-critical case.* Assume  $p = p_c - \epsilon$ ,  $0 < \epsilon \ll 1$ , and let  $T_{\text{rel}}$  be the relaxation time for the process on the infinite tree  $\mathbb{T}^k$ . Then (cf Theorem 2)  $T_{\text{rel}} = \Omega(\epsilon^{-2})$  and  $T_{\text{rel}} = O(\epsilon^{-2-\alpha})$  for some  $\alpha \geq 0$ .
- *Mixing time.* We basically show (cf Theorem 3) that the mixing time on a finite  $k$ -ary rooted tree of depth  $L$  behaves like  $L \times T_{\text{rel}}$ .

Our results, which are identical to those proved for the Ising model on trees at the spin glass critical point [6], represent the first rigorous analysis of a kinetically constrained model at criticality. As shown in [16], our approach has a good chance to apply also to other models with an ergodicity phase transition, notably the North-East model on  $\mathbb{Z}^2$  for which the critical density  $p_c$  coincides with the oriented percolation threshold [3]. .

### 1.1. Model, notation and background.

*The graph.* The model we consider is defined on the infinite rooted  $k$ -ary tree  $\mathbb{T}^k$  with root  $r$  and vertex set  $V$ . For each  $x \in V$ ,  $\mathcal{K}_x$  will denote the set of its  $k$  children and  $d_x$  its *depth*, i.e. the graph distance between  $x$  and the root  $r$ . The finite  $k$ -ary subtree of  $\mathbb{T}^k$  with  $n$  levels is the set  $\mathbb{T}_n^k = \{x \in \mathbb{T}^k : d_x \leq n\}$ . For  $x \in \mathbb{T}_n^k$ ,  $\mathbb{T}_{x,n}^k$  will denote the  $k$ -ary sub-tree of  $\mathbb{T}_n^k$  rooted at  $x$  with depth  $n - d_x$ , where  $d_x$  is the depth of  $x$ . In other words the leaves of  $\mathbb{T}_{x,n}^k$  are a subset of the leaves of  $\mathbb{T}_n^k$ . We also set  $\hat{\mathbb{T}}_{x,n}^k = \mathbb{T}_{x,n}^k \setminus \{x\}$  (See Figure 1 below). In the sequel, whenever no confusion arises, we will drop the superscripts  $k, n$  from  $\mathbb{T}_n^k$  and  $\mathbb{T}_{x,n}^k$ .

*The configuration spaces.* We choose as configuration space the set  $\Omega = \{0, 1\}^V$  whose elements will usually be assigned Greek letters. We will often write  $\eta_x$  for the value at  $x$  of the element  $\eta \in \Omega$ . We will also write  $\Omega_A$  for the set  $\{0, 1\}^A$ ,  $A \subseteq V$ . With a slight abuse of notation, for any  $A \subseteq V$  and any  $\eta, \omega \in \Omega$ , we let  $\eta_A$  be the restriction of  $\eta$  to the set  $A$  and  $\eta_A \cdot \omega_{A^c}$  be the configuration which equals  $\eta$  on  $A$  and  $\omega$  on  $V \setminus A$ .

*Probability measures.* For any  $A \subseteq V$  we denote by  $\mu_A$  the product measure  $\otimes_{x \in A} \mu_x$  where each factor  $\mu_x$  is the Bernoulli measure on  $\{0, 1\}$  with  $\mu_x(1) = p$  and  $\mu_x(0) = q$  with  $q = 1 - p$ . If  $A = V$  we abbreviate  $\mu_V$  to  $\mu$ . Also, with a slight abuse of notation, for any finite  $A \subset V$ , we will write  $\mu(\eta_A) = \mu_A(\eta_A)$ .

<sup>1</sup>We use here the convention that the depth is the graph distance between the root and the leaves.

*Conditional expectations and conditional variances.* Given  $A \subset V$  and a function  $f : \Omega \rightarrow \mathbb{R}$  depending on finitely many variables, in the sequel referred to as *local function*, we define the function  $\eta_{A^c} \mapsto \mu_A(f)(\eta_{A^c})$  by the formula:

$$\mu_A(f)(\eta_{A^c}) := \sum_{\sigma \in \Omega_A} \mu_A(\sigma) f(\sigma_A \cdot \eta_{A^c}).$$

Clearly  $\mu_A(f)$  coincides with the *conditional expectation* of  $f$  given the configuration outside  $A$ . Similarly we write  $\text{Var}_A(f) = \mu_A(f^2) - \mu_A(f)^2$  for the *conditional variance* of  $f$  given  $\eta_{A^c}$ . Note that  $\text{Var}_A(f) = 0$  iff  $f$  does not depend on the configuration inside  $A$ . When  $A = V$ , respectively  $A = \{x\}$  for some  $x \in V$ , we abbreviate  $\text{Var}_V(f)$  to  $\text{Var}(f)$ , respectively  $\text{Var}_{\{x\}}(f)$  to  $\text{Var}_x(f)$ .

**Definition 1.1** (OFA-kf model). *The OFA-kf (Oriented Fredrickson-Andersen  $k$ -facilitated) model at density  $p$  is a continuous time Glauber type Markov process on  $\Omega$ , reversible w.r.t.  $\mu$ , with Markov semigroup  $P_t = e^{t\mathcal{L}}$  whose infinitesimal generator  $\mathcal{L}$  acts on local functions  $f : \Omega \mapsto \mathbb{R}$  as follows:*

$$\mathcal{L}f(\omega) = \sum_{x \in \mathbb{T}^k} c_x(\omega) [\mu_x(f)(\omega) - f(\omega)]. \quad (1.1)$$

The function  $c_x$ , in the sequel referred to as the *constraint* at  $x$ , is defined by

$$c_x(\omega) = \begin{cases} 1 & \text{if } \omega_y = 0 \forall y \in \mathcal{K}_x \\ 0 & \text{otherwise.} \end{cases} \quad (1.2)$$

It is easy to check by standard methods (see e.g. [14]) that the process is well defined and that its generator can be extended to non-positive self-adjoint operators on  $L^2(\mathbb{T}^k, \mu)$ .

The OFA-kf process can of course be defined also on finite rooted trees. In this case and in order to ensure irreducibility of the Markov chain the constraints  $c_x$  must be suitably modified.

**Definition 1.2** (Finite volume dynamics). *Let  $\mathbb{T}$  be a finite subtree of  $\mathbb{T}^k$  and let, for any  $\eta \in \Omega_{\mathbb{T}}$ ,  $\eta^0 \in \Omega$  denote the extension of  $\eta$  in  $\Omega$  given by*

$$\eta_x^0 = \begin{cases} \eta_x & \text{if } x \in \mathbb{T} \\ 0 & \text{if } x \in \mathbb{T}^k \setminus \mathbb{T}. \end{cases}$$

For any  $x \in \mathbb{T}$  define the finite constraints  $c_{\mathbb{T},x}$  by

$$c_{\mathbb{T},x}(\eta) = c_x(\eta^0). \quad (1.3)$$

We will then consider the irreducible, continuous time Markov chains on  $\Omega_{\mathbb{T}}$  with generator

$$\mathcal{L}_{\mathbb{T}}f = \sum_{x \in \mathbb{T}} c_{\mathbb{T},x} [\mu_x(f) - f] \quad \eta \in \Omega_{\mathbb{T}}. \quad (1.4)$$

Note that irreducibility of the above defined finite volume dynamics is guaranteed by the fact that starting from the empty leaves one can empty any configuration via allowed spin flips. It is natural to define (see [3]) the critical density for the model by:

$$p_c = \sup\{p \in [0, 1] : 0 \text{ is simple eigenvalue of } \mathcal{L}\} \quad (1.5)$$

The regime  $p < p_c$  is called the *ergodic region* and we say that an *ergodicity breaking transition* occurs at the critical density. In [16] it has been established that  $p_c$  coincides with the percolation threshold  $1/k$  and that for all  $p < p_c$  the value 0 is a simple eigenvalue of the generator  $\mathcal{L}$ . Actually much more is known but first we need to introduce some relevant time scales.

**Definition 1.3** (The relaxation time). *Let  $\mathcal{D}(f) := \mu(f, -\mathcal{L}f)$  be the Dirichlet form corresponding to the generator  $\mathcal{L}$ . We define the spectral gap of the process as*

$$\text{gap}(\mathcal{L}) := \inf_{\substack{f \in \text{Dom}(\mathcal{L}) \\ f \neq \text{const}}} \frac{\mathcal{D}(f)}{\text{Var}(f)} \quad (1.6)$$

We also define the relaxation time by  $T_{\text{rel}} := \text{gap}(\mathcal{L})^{-1}$ . Similarly, if  $\mathbb{T}$  is a finite rooted tree, we define  $T_{\text{rel}}(\mathbb{T}) := \text{gap}(\mathcal{L}_{\mathbb{T}})^{-1}$ .

**Definition 1.4** (Mixing times). *Let  $\mathbb{T}$  be a finite rooted sub-tree of  $\mathbb{T}^k$ . For any  $\eta \in \Omega_{\mathbb{T}}$  we denote by  $\nu_t^\eta$  the law at time  $t$  of the Markov chain with generator  $\mathcal{L}_{\mathbb{T}}$  and by  $h_t^\eta$  its relative density w.r.t.  $\mu_{\mathbb{T}}$ . Following [18], we define the family of mixing times  $\{T_a(\mathbb{T})\}_{a \geq 1}$  by*

$$T_a(\mathbb{T}) := \inf \left\{ t \geq 0 : \max_{\eta} \mu_{\mathbb{T}}(|h_t^\eta - 1|^a)^{1/a} \leq 1/4 \right\}.$$

Notice that  $T_1(\mathbb{T})$  coincides with the usual mixing time  $T_{\text{mix}}(\mathbb{T})$  of the chain (see e.g. [15]) and that, for any  $a \geq 1$ ,  $T_1 \leq T_a$ .

With the above notation it was proved in [16] that

- (i) for all  $p < p_c$ ,  $T_{\text{rel}} < +\infty$  and that the mixing time on a finite regular  $k$ -ary sub-tree of depth  $L$  grows linearly in  $L$ ;
- (ii) if  $p > p_c$ , then both the relaxation time and the mixing time on a finite regular  $k$ -ary sub-tree of depth  $L$  grow exponentially fast in  $L$ .

## 1.2. Main Results.

Our first contribution concerns the critical case  $p = p_c$ .

**Theorem 1.** *Fix  $k \geq 2$  and assume  $p = p_c$ . Then there exist constants  $c > 0$  and  $\beta \geq 0$ , with  $\beta$  independent of  $k$ , such that for each  $L$*

$$c^{-1}L^2 \leq T_{\text{rel}}(\mathbb{T}_L^k) \leq cL^{2+\beta}.$$

**Remark 1.5.** *The above result implies, in particular, that the relaxation time for the critical process on the infinite tree  $\mathbb{T}^k$  is infinite. However the process is still ergodic in the sense that 0 is a simple eigenvalue of the generator  $\mathcal{L}$ . This can be proven following the same lines of [3, Proposition 2.5] by using the key ingredient that, at  $p = p_c$ , there is no infinite percolation of occupied vertices a.s..*

Our second main result deals with the quasi-critical regime,  $p = p_c - \epsilon$  with  $0 < \epsilon \ll 1$ , on the infinite tree  $\mathbb{T}^k$ .

**Theorem 2.** Fix  $k \geq 2$  and assume  $p < p_c$ . Then there exist constants  $a > 0$  and  $\alpha \geq 0$ , with  $\alpha$  independent of  $k$ , such that

$$a^{-1}(p_c - p)^{-2} \leq T_{\text{rel}} \leq a(p_c - p)^{-(2+\alpha)}$$

The last result derives some consequences of the above theorems for the mixing time on a finite sub-tree.

**Theorem 3.** There exists  $c > 0$  such that, for all  $L$ ,

$$\frac{1}{c}L T_{\text{rel}}(\mathbb{T}_{\lfloor L/2 \rfloor}^k) \leq T_1(\mathbb{T}_L^k) \leq T_2(\mathbb{T}_L^k) \leq cL T_{\text{rel}}(\mathbb{T}_L^k). \quad (1.7)$$

In particular:

(i) if  $p = p_c$ , then

$$c^{-1}L^3 \leq T_1(\mathbb{T}_L^k) \leq cL^{3+\beta}.$$

(ii) If  $p < p_c$ ,

$$\frac{1}{c}(p_c - p)^{-2}L \leq T_1(\mathbb{T}_L^k) \leq cL(p_c - p)^{-(2+\alpha)}$$

for some constants  $\alpha, \beta \geq 0$  independent of  $L$ .

**1.3. Additional notation and technical preliminaries.** We first introduce the natural bootstrap map for the model.

**Definition 1.6.** The bootstrap map  $B : \{0, 1\}^{\mathbb{T}^k} \rightarrow \{0, 1\}^{\mathbb{T}^k}$  associated to the OFA-kf model is defined by

$$B(\eta)_x = \begin{cases} 0 & \text{if either } \eta_x = 0 \text{ or } c_x(\eta) = 1 \\ 1 & \text{otherwise} \end{cases} \quad (1.8)$$

with  $c_x$  defined in (1.2).

**Remark 1.7.** Notice that: (i) if after  $n$ -iterations of the bootstrap map  $c_x(B^n(\eta)) = 1$  then, even if  $\eta_x = 1$ , the percolation cluster of 1's attached to  $x$  is contained in the first  $n$ -levels below  $x$  and (ii) the bootstrap critical point (see e.g. [2]) coincides with the percolation threshold  $p_c = 1/k$ .

Secondly we formulate two technical results which will be useful in the sequel. Let  $E_x^{(n)} = \{\eta : B^n(\eta)_x = 1\}$  and define  $p_n := \mu(E_r^{(n)})$ . Notice that  $p_n$  is increasing in  $p$  and that  $p_n \leq p$  for all  $n$ .

**Lemma 1.8.**

(i) If  $p \leq p_c$  then  $p_n \leq \frac{2}{(k-1)n}$  for all  $n \geq 1$ .

(ii) Assume  $p = p_c - \epsilon$  with  $\epsilon \in [0, 1/k]$ . Then  $p_n \leq p(1 - \epsilon k)^n$  for all  $n \geq 1$ .

*Proof.*

(i) Using the monotonicity in  $p$  of the  $p_n$ 's it is enough to prove the statement for  $p = p_c$ . We start from

$$\mu(E_r^{(n+1)}) = p\mu\left(\bigcup_{x \in \mathcal{K}_r} E_x^{(n)}\right), \quad (1.9)$$

or, equivalently,

$$p_{n+1} = p(1 - (1 - p_n)^k).$$

Using inclusion-exclusion inequalities (1.9) implies (recall that  $p = 1/k$ )

$$\begin{aligned} p_{n+1} &\leq \frac{1}{k} \left[ kp_n - \binom{k}{2} p_n^2 + \binom{k}{3} p_n^3 \right] \\ &= p_n - \frac{(k-1)}{2} p_n^2 + \frac{(k-1)(k-2)}{6} p_n^3. \end{aligned} \quad (1.10)$$

One readily checks that the r.h.s. of (1.10) is increasing in  $p_n \in [0, 1/k]$ . Thus, if we assume inductively that  $p_n \leq \frac{2}{(k-1)n}$ ,  $n \geq 2$ , we obtain

$$p_{n+1} \leq \frac{2}{(k-1)} \left[ \frac{1}{n} - \frac{1}{n^2} + \frac{2(k-2)}{3(k-1)n^3} \right] \leq \frac{2}{(k-1)(n+1)} \quad n \geq 2.$$

The base case  $p_2$  follows from the trivial observation that  $p_2 \leq p_1 \leq \frac{1}{k} < \frac{1}{k-1}$ .

(ii) Boole inequality applied to (1.9) gives

$$p_{n+1} \leq pkp_n = (1 - \epsilon k)p_n \leq \dots \leq (1 - \epsilon k)^n p.$$

□

The second technical ingredient is the following monotonicity result for the spectral gap (see [3, Lemma 2.11] for a proof).

**Lemma 1.9.** *Let  $\mathbb{T}_1 \subset \mathbb{T}_2$  be two sub-trees of  $\mathbb{T}^k$ . Then,*

$$\text{gap}(\mathcal{L}_{\mathbb{T}_1}) \geq \text{gap}(\mathcal{L}_{\mathbb{T}_2}).$$

## 2. THE CRITICAL CASE: PROOF OF THEOREM 1

**2.1. Upper bound of the relaxation time.** Let  $\mathbb{T} \equiv \mathbb{T}_L^k$ ,  $\mathbb{T}_x \equiv T_{x,L}^k$  and  $\hat{\mathbb{T}}_x \equiv \hat{T}_{x,L}^k$ . We divide the proof of the upper bound of  $T_{\text{rel}}(\mathbb{T})$  in three steps.

**2.1.1. First step.** [Comparison with a long-range auxiliary dynamics]. Motivated by [16] we introduce auxiliary *long range* constraints as follows.

**Definition 2.1.** *For any integer  $\ell \geq 1$  we set*

$$c_x^{(\ell)}(\eta) = \begin{cases} 1 & \text{if } c_x(B^{\ell-1}(\eta)) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 2.2.** *One can use the functions  $c_x^{(\ell)}$  to define an auxiliary long range dynamics with generator given by (1.1) with  $c_x$  replaced by  $c_x^{(\ell)}$ . For this new constrained dynamics a vertex  $x$  is free to flip iff, by a sequence of at most  $\ell$  flips satisfying the original constraints (1.2) all the children of  $x$  can be made vacant.*

Fix now  $\delta \in (0, 1/9)$  and choose  $\ell = (1 - \delta)L$  (neglecting integer part). Let also  $c_{\mathbb{T},x}^{(\ell)}(\eta) := c_x^{(\ell)}(\eta^0)$  where  $\eta^0$  is given in Definition 1.2 respectively. Notice that  $c_{\mathbb{T},x}^{(\ell)}(\eta) \equiv 1$  iff  $d_x > L - \ell$ . We will establish the inequality

$$\text{Var}_{\mathbb{T}}(f) \leq \lambda \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(c_{\mathbb{T},x}^{(\ell)} f)) \right) \quad \forall f \quad (2.1)$$

with  $\lambda = 2(\frac{1-\delta}{1-9\delta})$ .

**Remark 2.3.** Inequality (2.1) will be proven following the strategy of [16]. Notice however that here we don't perform another Cauchy-Schwartz inequality to pull out the constraint  $c_{\mathbb{T},x}^{(\ell)}$  and get the Dirichlet form with long range constraints.

We start from

$$\text{Var}_{\mathbb{T}}(f) \leq \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f)) \right). \quad (2.2)$$

The above inequality follows easily from a repeated use of the formula for conditional variance and we refer to section 4.1 in [16] for a short proof. We now examine a generic term  $\mu \left( \text{Var}_x \left( \mu_{\hat{\mathbb{T}}_x}(f) \right) \right)$  in the r.h.s. of (2.2). We write

$$\mu_{\hat{\mathbb{T}}_x}(f) = \mu_{\hat{\mathbb{T}}_x} \left( c_{\mathbb{T},x}^{(\ell)} f \right) + \mu_{\hat{\mathbb{T}}_x} \left( [1 - c_{\mathbb{T},x}^{(\ell)}] f \right)$$

so that

$$\text{Var}_x \left( \mu_{\hat{\mathbb{T}}_x}(f) \right) \leq 2 \text{Var}_x \left( \mu_{\hat{\mathbb{T}}_x} \left( c_{\mathbb{T},x}^{(\ell)} f \right) \right) + 2 \text{Var}_x \left( \mu_{\hat{\mathbb{T}}_x} \left( (1 - c_{\mathbb{T},x}^{(\ell)}) f \right) \right). \quad (2.3)$$

We now consider the second term  $\text{Var}_x \left( \mu_{\hat{\mathbb{T}}_x} \left( (1 - c_{\mathbb{T},x}^{(\ell)}) f \right) \right)$ . Without loss of generality we can assume  $\mu_{\hat{\mathbb{T}}_x}(f) = 0$ . Recall that the constraint  $c_{\mathbb{T},x}^{(\ell)}$  depends only on the spin configuration in the first  $\ell$  levels below  $x$ , in the sequel denoted by  $\Delta_x$  (see Figure 1).

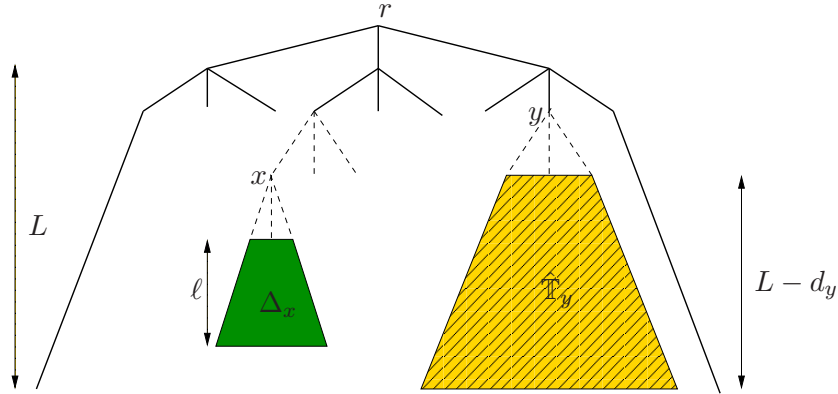


FIGURE 1. For  $k = 3$ , the tree  $\mathbb{T}$  rooted at  $r$ , of depth  $L$  (i.e. with  $L$  levels below  $r$ ), the set  $\Delta_x$  and the sub-set  $\hat{\mathbb{T}}_y$ .

Thus

$$\mu_{\hat{\mathbb{T}}_x} \left( (1 - c_{\mathbb{T},x}^{(\ell)}) f \right) = \mu_{\hat{\mathbb{T}}_x} \left( (1 - c_{\mathbb{T},x}^{(\ell)}) \mu_{\hat{\mathbb{T}}_x \setminus \Delta_x}(f) \right)$$



and

$$\begin{aligned}
\mathrm{Var}_x \left( \mu_{\hat{\mathbb{T}}_x} \left( (1 - c_{\mathbb{T},x}^{(\ell)}) f \right) \right) &\leq \mu_{\mathbb{T}_x} \left( \mu_{\hat{\mathbb{T}}_x} \left( (1 - c_{\mathbb{T},x}^{(\ell)}) \mu_{\hat{\mathbb{T}}_x \setminus \Delta_x} (f) \right)^2 \right) \\
&\leq \mu_{\mathbb{T}_x} (1 - c_{\mathbb{T},x}^{(\ell)}) \mu_{\mathbb{T}_x} \left( \mu_{\hat{\mathbb{T}}_x \setminus \Delta_x} (f)^2 \right) \\
&= \mu_{\mathbb{T}_x} (1 - c_{\mathbb{T},x}^{(\ell)}) \mathrm{Var}_{\mathbb{T}_x} \left( \mu_{\hat{\mathbb{T}}_x \setminus \Delta_x} (f) \right) \\
&\leq \mu_{\mathbb{T}_x} (1 - c_{\mathbb{T},x}^{(\ell)}) \sum_{y \in \Delta_x \cup x} \mu_{\mathbb{T}_x} \left( \mathrm{Var}_y (\mu_{\hat{\mathbb{T}}_y} (f)) \right) \tag{2.4}
\end{aligned}$$

where we used Cauchy-Schwartz inequality, the fact that  $c_{\mathbb{T},x}^{(\ell)}$  does not depend on  $\eta_x$  and (2.2) in the last inequality. From the definition of  $c_{\mathbb{T},x}^{(\ell)}$  on the finite tree  $\mathbb{T}$  it holds

$$\mu_{\mathbb{T}_x} (1 - c_{\mathbb{T},x}^{(\ell)}) = \begin{cases} 0 & \text{if } d_x > \delta L \\ p\ell/p & \text{otherwise} \end{cases} \tag{2.5}$$

In conclusion, using (2.3), (2.4) and (2.5),

$$\begin{aligned}
&\sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left[ \mathrm{Var}_x (\mu_{\hat{\mathbb{T}}_x} (f)) \right] \tag{2.6} \\
&\leq 2 \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left[ \mathrm{Var}_x (\mu_{\hat{\mathbb{T}}_x} (c_{\mathbb{T},x}^{(\ell)} f)) \right] + 2 \frac{p\ell}{p} \sum_{d_x \leq \delta L} \sum_{y \in \Delta_x \cup x} \mu_{\mathbb{T}} [\mathrm{Var}_y (\mu_{\hat{\mathbb{T}}_y} (f))] \\
&\leq 2 \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left[ \mathrm{Var}_x (\mu_{\hat{\mathbb{T}}_x} (c_{\mathbb{T},x}^{(\ell)} f)) \right] + 2 \frac{p\ell}{p} \left[ \max_z N_z \right] \sum_y \mu_{\mathbb{T}} [\mathrm{Var}_y (\mu_{\hat{\mathbb{T}}_y} (f))] \tag{2.7}
\end{aligned}$$

where

$$N_z := \#\{x : \Delta_x \ni z, d_x \leq \delta L\} \leq \min(\delta L, \ell + 1).$$

Part (i) of Lemma 1.8 implies that  $p\ell \leq \frac{2}{(k-1)\ell} = \frac{2}{(k-1)(1-\delta)L}$  so that

$$\begin{aligned}
&\sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \mathrm{Var}_x (\mu_{\hat{\mathbb{T}}_x} (f)) \right) \\
&\leq 2 \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left[ \mathrm{Var}_x (\mu_{\hat{\mathbb{T}}_x} (c_{\mathbb{T},x}^{(\ell)} f)) \right] + \frac{4\delta}{p(1-\delta)(k-1)} \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} [\mathrm{Var}_x (\mu_{\hat{\mathbb{T}}_x} (f))] \tag{2.8}
\end{aligned}$$

Since  $p = 1/k$  and  $k/(k-1) \leq 2$ , inequality (2.1) holds with  $\lambda = 2(1-\delta)/(1-9\delta)$  provided  $8\delta/(1-\delta) < 1$ .

2.1.2. *Second step.* [Analysis of the auxiliary dynamics]. Let  $h_i = \alpha^i$ ,  $\alpha > 1$  to be fixed later on, and let

$$T_i := T_{\mathrm{rel}}(\mathbb{T}_{h_i \wedge \ell}^k). \tag{2.9}$$

We shall now prove that

$$\sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \mathrm{Var}_x (\mu_{\hat{\mathbb{T}}_x} (c_{\mathbb{T},x}^{(\ell)} f)) \right) \leq \left[ 2 + \frac{4\alpha}{p(k-1)} \left( \sum_{i=1}^{n-1} \sqrt{T_i} \right)^2 \right] \mathcal{D}_{\mathbb{T}}(f), \tag{2.10}$$

with  $n$  such that  $h_{n-1} < \ell \leq h_n$ .

The starting point is (2.1). For any  $x \in \mathbb{T}$  we introduce a scale decomposition of the constraint  $c_{\mathbb{T},x}^{(\ell)}$  as follows  $c_{\mathbb{T},x}^{(\ell)} = \sum_{i=0}^{n-1} \chi_i + c_{\mathbb{T},x}$ , where  $\chi_i := c_{\mathbb{T},x}^{(h_{i+1} \wedge \ell)} - c_{\mathbb{T},x}^{(h_i \wedge \ell)}$ . Thus

$$\begin{aligned} & \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(c_{\mathbb{T},x}^{(\ell)} f)) \right) \\ & \leq 2 \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(c_{\mathbb{T},x} f)) \right) + 2 \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(\sum_{i=0}^{n-1} \chi_i f)) \right) \\ & \leq 2\mathcal{D}_{\mathbb{T}}(f) + 2 \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(\sum_{i=0}^{n-1} \chi_i f)) \right), \end{aligned}$$

where in the last inequality we used convexity to conclude that

$$\mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(c_{\mathbb{T},x} f)) \right) \leq \mu_{\mathbb{T}}(c_{\mathbb{T},x} \text{Var}_x(f)).$$

We now examine the key term  $\sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(\sum_{i=0}^{n-1} \chi_i f)) \right)$ .

Observe first that  $\chi_i = 0$  if  $h_i \geq \ell$  and that  $\chi_i = 1$  implies the number of iterations of the bootstrap map necessary to make the node  $x$  flippable is at least  $h_i$  but no more than  $h_{i+1} \wedge \ell$ . In particular, if  $\chi_i(\eta) = 1$ , there exists a “line” of zeros of  $\eta$  within  $h_{i+1} \wedge \ell$  levels below  $x$ . For such an  $\eta$  we denote by  $\Gamma(\eta)$  the “lowest” such line constructed as follows.

Consider the nodes in  $\mathbb{T}_x$  at distance  $h_{i+1} \wedge \ell$  from  $x$ . Let us order them from left to right as  $z_1, z_2, \dots$ ; start from  $z_1$  and find the first empty site on the branch leading to  $x$ . Call this vertex  $y_1$  and forget about all the  $z_i$ 's having  $y_1$  as ancestor. Say that the remaining nodes are  $z_{k_1}, z_{k_1+1}, \dots$ ; repeat the construction for  $z_{k_1}$  to get a new empty node  $y_2$  and so forth. At the end of this procedure some of the  $y_i$  may have some other  $y_k$  as ancestor. In this case we remove the former from our collection and we relabel accordingly. The line  $\Gamma(\eta)$  is then the final collection  $(y_1, y_2, \dots)$ .

We denote by  $\mathcal{G}_i$  the space of all possible realisations of  $\Gamma$ . Moreover, given  $\gamma \in \mathcal{G}_i$ , we denote by  $\hat{\mathbb{T}}_x^{\gamma,+}$  all the nodes in  $\hat{\mathbb{T}}_x$  which have no ancestor in  $\gamma$ , i.e. the part of the tree “above”  $\gamma$ . Note that the above construction of  $\Gamma$  is made without looking at the configuration above  $\Gamma$ . This observation together with the definition of the variance and Cauchy-Schwarz inequality gives

$$\begin{aligned} \text{Var}_x \left( \mu_{\hat{\mathbb{T}}_x} \left( \sum_{i=0}^{n-1} \chi_i f \right) \right) &= p(1-p) \left[ \sum_{i=0}^{n-1} \mu_{\hat{\mathbb{T}}_x}(\chi_i \nabla_x f) \right]^2 \\ &= p(1-p) \left[ \sum_{i=0}^{n-1} \sum_{\gamma \in \mathcal{G}_i} \mu_{\hat{\mathbb{T}}_x \setminus \hat{\mathbb{T}}_x^{\gamma,+}} \left( \mathbb{1}_{\Gamma=\gamma} \mu_{\hat{\mathbb{T}}_x^{\gamma,+}}(\chi_i \nabla_x f) \right) \right]^2 \quad (2.11) \\ &\leq p(1-p) \left[ \sum_{i=0}^{n-1} \sum_{\gamma \in \mathcal{G}_i} \mu_{\hat{\mathbb{T}}_x \setminus \hat{\mathbb{T}}_x^{\gamma,+}} \left( \mathbb{1}_{\Gamma=\gamma} \sqrt{\mu_{\hat{\mathbb{T}}_x^{\gamma,+}}(\chi_i) \mu_{\hat{\mathbb{T}}_x^{\gamma,+}}(|\nabla_x f|^2)} \right) \right]^2. \end{aligned}$$

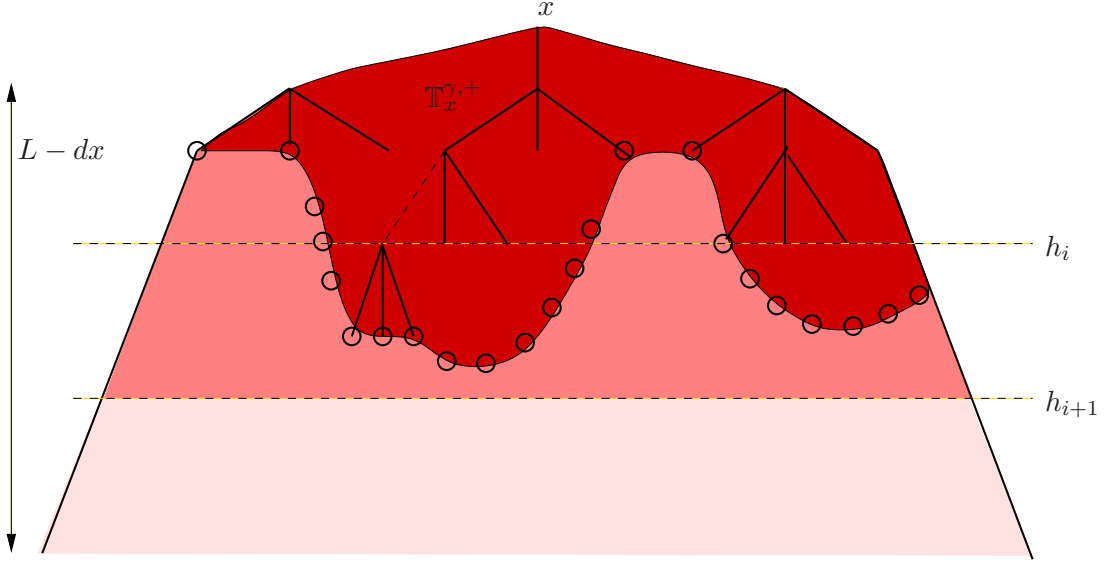


FIGURE 2. For  $k = 3$ , the sub-tree  $\mathbb{T}_x$  rooted at  $x$  and a configuration  $\eta$  such that  $\chi_i(\eta) = 1$ . The line of empty sites corresponds to a set  $\gamma \in \mathcal{G}_i$ .

where  $\nabla_x f(\eta) = f(\eta^x) - f(\eta)$  with  $\eta_y^x = \eta_y$  if  $y \neq x$  and  $\eta_x^x = 1 - \eta_x$ . Consider now the last factor inside the square root and multiply it by  $p(1 - p)$ . It holds

$$p(1 - p)\mu_{T_x^{\gamma,+}}(|\nabla_x f|^2) = \mu_{T_x^{\gamma,+}}(\text{Var}_x(f)) \leq \text{Var}_{\mathbb{T}_x^{\gamma,+}}(f) \leq T_{\text{rel}}(\mathbb{T}_x^{\gamma,+})\mathcal{D}_{\mathbb{T}_x^{\gamma,+}}(f)$$

where we used the convexity of the variance and the Poincaré inequality. Lemma 1.9 now gives  $T_{\text{rel}}(\mathbb{T}_x^{\gamma,+}) \leq T_{i+1}$ . In conclusion

$$p(1 - p)\mu_{T_x^{\gamma,+}}(|\nabla_x f|^2) \leq T_{i+1}\mathcal{D}_{\mathbb{T}_x^{\gamma,+}}(f).$$

To bound the first factor inside the square root of (2.1.2) we note that  $\mathbb{1}_{\Gamma=\gamma}c_{\mathbb{T},x}^{(h_i)} = \mathbb{1}_{\Gamma=\gamma}c_{\mathbb{T}_x^{\gamma,+},x}^{(h_i)}$ . Indeed the finite volume constraints  $c_{T_x^{\gamma,+},y}$  are defined with zeros on the set  $\gamma$  of the leaves of  $T_x^{\gamma,+}$  (see (1.3)) and in turn  $\mathbb{1}_{\Gamma(\eta)=\gamma}$  guarantees the presence of such zeros for the configuration  $\eta$ . Thus, using the monotonicity on the volume of the probability that the root  $x$  is connected to the level  $h_i$ ,

$$\begin{aligned} \mathbb{1}_{\Gamma=\gamma}\mu_{T_x^{\gamma,+}}(\chi_i) &\leq \mathbb{1}_{\Gamma=\gamma}\mu_{T_x^{\gamma,+}}(1 - c_x^{(h_i)}) \\ &= \mathbb{1}_{\Gamma=\gamma}\mu_{T_x^{\gamma,+}}(1 - c_{T_x^{\gamma,+},x}^{(h_i)}) \leq \mu(1 - c_x^{(h_i)}) = p_{h_i}/p. \end{aligned}$$

In conclusion, the r.h.s. of (2.1.2) is bounded from above by

$$\begin{aligned}
& \frac{1}{p} \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} p_{h_i} \mu_{\hat{\mathbb{T}}_x} \left( \sum_{\gamma \in \mathcal{G}_i} \mathbb{1}_{\Gamma=\gamma} \sqrt{\mathcal{D}_{\mathbb{T}_x^{\gamma,+}}(f)} \right) \right)^2 \\
& \leq \frac{1}{p} \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} p_{h_i} \sqrt{\mu_{\hat{\mathbb{T}}_x} \left( \sum_{\gamma \in \mathcal{G}_i} \mathbb{1}_{\Gamma=\gamma} \mathcal{D}_{\mathbb{T}_x^{\gamma,+}}(f) \right)} \right)^2 \\
& \leq \frac{1}{p} \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} \right) \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} p_{h_i} \mu_{\hat{\mathbb{T}}_x} \left( \sum_{\gamma \in \mathcal{G}_i} \mathbb{1}_{\Gamma=\gamma} \mathcal{D}_{\mathbb{T}_x^{\gamma,+}}(f) \right) \right) \\
& \leq \frac{1}{p} \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} \right) \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} p_{h_i} \sum_{\substack{y \in \hat{\mathbb{T}}_x \\ d_y \leq d_x + h_{i+1}}} \mu_{\hat{\mathbb{T}}_x}(c_y \text{Var}_y(f)) \right) \tag{2.12}
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the first and second inequality together with

$$\mu_{\hat{\mathbb{T}}_x} \left( \sum_{\gamma \in \mathcal{G}_i} \mathbb{1}_{\Gamma=\gamma} \mathcal{D}_{\mathbb{T}_x^{\gamma,+}}(f) \right) \leq \sum_{\substack{y \in \hat{\mathbb{T}}_x \\ d_y \leq d_x + h_{i+1}}} \mu_{\hat{\mathbb{T}}_x}(c_{\mathbb{T},y} \text{Var}_y(f))$$

because  $\mathbb{1}_{\Gamma(\eta)=\gamma} c_{T_x^{\gamma,+},y}(\eta) = \mathbb{1}_{\Gamma(\eta)=\gamma} c_{\mathbb{T},y}(\eta)$ . If we now average over  $\mu_{\mathbb{T}}$  and sum over  $x \in \mathbb{T}$  the above result we get that

$$\begin{aligned}
& \sum_{x \in \mathbb{T}} \mu_{\mathbb{T}} \left( \text{Var}_x \left( \mu_{\hat{\mathbb{T}}_x} \left( \sum_{i=0}^{n-1} \chi_i f \right) \right) \right) \\
& \leq \frac{1}{p} \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} \right) \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} p_{h_i} \sum_{x \in \mathbb{T}} \sum_{\substack{y \in \hat{\mathbb{T}}_x \\ d_y \leq d_x + h_{i+1}}} \mu_{\mathbb{T}}(c_{\mathbb{T},y} \text{Var}_y(f)) \right) \\
& \leq \frac{1}{p} \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} \right) \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} p_{h_i} h_{i+1} \right) \mathcal{D}_{\mathbb{T}}(f) \\
& \leq \frac{2\alpha}{p(k-1)} \left( \sum_{i=0}^{n-1} \sqrt{T_{i+1}} \right)^2 \mathcal{D}_{\mathbb{T}}(f)
\end{aligned}$$

and (2.10) follows. Above we used the exponential growth of the scales  $\{h_i\}_i$  together with (i) of Lemma 1.8 to obtain  $p_{h_i} h_{i+1} \leq 2\alpha/(k-1)$ .

2.1.3. *Third step.* [Recurrence]. With the above notation (2.1) and (2.10) yield the following key recursive inequality:

$$T_{\text{rel}}(\mathbb{T}) \leq \lambda \left[ 2 + \frac{4\alpha}{p(k-1)} \left( \sum_{i=0}^{n-1} \sqrt{T_i} \right)^2 \right]$$

with  $T_i$  given by (2.9) and  $\lambda = 2\frac{1-\delta}{1-9\delta}$ . Suppose now that  $L = \alpha^{N+1}$  and  $\ell = \alpha^N$  with  $\alpha = (1-\delta)^{-1}$ . Then  $T_{\text{rel}}(\mathbb{T}) = T_{N+1}$  and  $n = N$ . If we set  $a_i := \sqrt{T_i}$  then we get

$$a_{N+1} \leq c \sum_{i=0}^N a_i, \quad c = \lambda^{1/2} \left( 2 + \frac{4\alpha}{p(k-1)} \right)^{1/2},$$

which implies that  $b_n := \sum_{i=0}^n a_i$  satisfies  $b_{N+1} \leq (1+c)b_N$ . In conclusion

$$T_{\text{rel}}(\mathbb{T}) = a_{N+1}^2 \leq b_{N+1}^2 \leq (1+c)^{2N} b_1^2.$$

The proof of the upper bound of  $T_{\text{rel}}(\mathbb{T})$  in Theorem 1 is complete if the depth  $L$  is of the form  $\alpha^n$ ,  $n \in \mathbb{N}$ . The extension to general values of  $L$  follows at once from Lemma 1.9.

**2.2. Lower bound on the relaxation time  $T_{\text{rel}}$ .** Let us consider as test function to be inserted into the variational characterisation of  $T_{\text{rel}}(\mathbb{T})$  the cardinality  $N_r$  of the percolation cluster  $\mathcal{C}_r$  of occupied sites associated to the root  $r$ . More formally

$$N_r(\eta) := \#\{x \in \mathbb{T} : \eta_y = 1 \ \forall y \in \gamma_x\}$$

where  $\gamma_x$  is the unique path in  $\mathbb{T}$  joining  $x$  to the root  $r$ . Notice that  $N_r$  can be written as  $N_r(\eta) = \eta_r (\sum_{i=1}^k N_{x_i} + 1)$ , where  $\{x_i\}_{i=1}^k$  are the children of the root and  $N_{x_i}$  denotes the analogous of the quantity  $N_r$  with  $\mathbb{T}$  replaced by the sub-tree  $\mathbb{T}_{x_i}$  rooted at  $x_i$ .

We now compute the variance and Dirichlet form of  $N_r$ . Clearly

$$c^{-1} \sum_{x \in \mathbb{T}} \mu(x \text{ is a leaf of } \mathcal{C}_r) \leq \mathcal{D}_{\mathbb{T}}(N_r) \leq c \sum_{x \in \mathbb{T}} \mu(x \text{ is a leaf of } \mathcal{C}_r) \leq c\mu(N_r)$$

for some constant  $c = c(k)$ . Moreover  $\mu(N_r) = p(k\mu(N_{x_1}) + 1)$  which, for  $p = p_c = 1/k$ , implies that  $\mu(N_r) = (L+1)/k$ . To compute  $\text{Var}_{\mathbb{T}}(N_r)$  we use the above expression for  $N_r$  together with the formula for conditional variance to write

$$\begin{aligned} \text{Var}_{\mathbb{T}}(N_r) &= \mu(\text{Var}_{\mathbb{T}}(N_r | \eta_r)) + \text{Var}_{\mathbb{T}}(\mu(N_r | \eta_r)) \\ &= pk \text{Var}_{\mathbb{T}_{x_1}}(N_{x_1}) + \text{Var}_{\mathbb{T}}(\eta_r(k\mu(N_{x_1}) + 1)) \\ &= \text{Var}_{\mathbb{T}_{x_1}}(N_{x_1}) + p(1-p)(L+1)^2. \end{aligned} \tag{2.13}$$

Hence  $\text{Var}_{\mathbb{T}}(N_r) \geq c'L^3$  and

$$T_{\text{rel}}(\mathbb{T}) \geq \frac{\text{Var}_{\mathbb{T}}(N_r)}{\mathcal{D}_{\mathbb{T}}(N_r)} \geq c''L^2.$$

□

### 3. THE QUASI-CRITICAL CASE: PROOF OF THEOREM 2

Here we assume  $p = p_c - \epsilon$ ,  $\epsilon > 0$  and, without loss of generality, we assume that  $\epsilon k \ll 1$ . Recall that we work directly on the infinite tree  $\mathbb{T}^k$ .

**3.1. Upper bound on the relaxation time  $T_{\text{rel}}$ .** We first claim that, for any  $\ell$  such that  $2\ell(1 - \epsilon k)^\ell < 1$ , one has

$$\text{Var}(f) \leq \lambda \sum_{x \in \mathbb{T}^k} \mu \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(c_{\mathbb{T},x}^{(\ell)} f)) \right) \quad (3.1)$$

with  $\lambda = \frac{2}{1 - 2(\ell+1)(1-\epsilon k)^\ell}$ . The proof of (3.1) starts from inequality (2.4), whose derivation does not depend on the value of  $p$ . After that we proceed as follows. Since  $p = p_c - \epsilon$ , Lemma 1.8(ii) implies that

$$\mu_{T_x}(1 - c_{\mathbb{T},x}^{(\ell)}) = \frac{p\ell}{p} \leq (1 - \epsilon k)^\ell \quad \forall x \in \mathbb{T}^k.$$

Thus

$$\begin{aligned} \text{Var}(f) &\leq \sum_{x \in \mathbb{T}^k} \mu \left[ \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f)) \right] \\ &\leq 2 \sum_{x \in \mathbb{T}^k} \mu_{\mathbb{T}} \left[ \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(c_{\mathbb{T},x}^{(\ell)} f)) \right] + 2(\ell+1)(1 - \epsilon k)^\ell \sum_{x \in \mathbb{T}^k} \mu[\text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f))] \end{aligned}$$

and (3.1) follows.

Now choose  $\ell = -2 \frac{\log(\epsilon k)}{\epsilon k}$ , so that  $\lambda < 4$  in (3.1) for any  $\epsilon$  small enough, and define, for  $x \in \mathbb{T}^k$ ,  $\mathbb{T}_x$  as the finite  $k$ -ary tree rooted at  $x$  of depth  $\ell$ .

Exactly the same arguments leading to (2.12), but without the subtleties of the intermediate scales  $\{h_i\}_i$ , show that

$$\mu \left( \text{Var}_x(\mu_{\hat{\mathbb{T}}_x}(c_x^{(\ell)} f)) \right) \leq T_{\text{rel}}(\mathbb{T}) \sum_{y \in \mathbb{T}_x} \mu(c_y \text{Var}_y(f)). \quad (3.2)$$

If we now combine (3.2) together with (3.1) we get

$$\text{Var}(f) \leq 4\ell T_{\text{rel}}(\mathbb{T}) \mathcal{D}(f) \quad (3.3)$$

for all  $\epsilon$  small enough. Finally we claim that  $T_{\text{rel}}(\mathbb{T}) \leq c\ell^\beta$  for some appropriate constants  $c, \beta$ .

To prove the claim it is enough to observe that, in its proof for the case  $p = p_c$  given in section 2, only *upper bounds* on percolation probabilities played a role. By monotonicity these bounds hold for *any*  $p \leq p_c$ . Hence the claim. In conclusion

$$\text{Var}(f) \leq c\ell^{1+\beta} \mathcal{D}(f)$$

and  $T_{\text{rel}} \leq c\ell^{1+\beta} = c'\epsilon^{-(1+\beta)}$ .

**3.2. Lower bound of the relaxation time  $T_{\text{rel}}$ .** Thanks to Lemma 1.9,  $T_{\text{rel}} \geq T_{\text{rel}}(\mathbb{T})$  for any finite sub-tree  $\mathbb{T}$ . We now choose  $\mathbb{T}$  as the  $k$ -ary tree rooted at  $r$  with depth  $\ell = \lfloor 1/\epsilon \rfloor$  and proceed exactly as in the proof of Theorem 1. Using the notation of section 2.2 we have

$$\mathcal{D}_{\mathbb{T}}(N_r) \leq c\mu(N_r) \leq c'\ell$$

where we used the fact that the average of  $N_r$  at  $p < p_c$  is bounded from above by the same average computed at  $p = p_c$  since  $N_r$  is increasing (w.r.t. the natural partial order

in  $\Omega_{\mathbb{T}}$ ). To compute  $\text{Var}_{\mathbb{T}}(N_r)$  we proceed recursively starting from (cf (2.13))

$$\begin{aligned}\text{Var}_{\mathbb{T}}(N_r) &= (1 - k\epsilon) \text{Var}_{\mathbb{T}_{x_1}}(N_{x_1}) + \frac{1-p}{p} \mu(N_r)^2 \\ \mu(N_r) &= (1 - \epsilon k) \mu(N_{x_1}) + p\end{aligned}$$

Since the number of steps of the iteration is  $\lfloor 1/\epsilon \rfloor$  one immediately concludes that  $\mu(N_r) \geq c_k \ell$  and  $\text{Var}_{\mathbb{T}}(N_r) \geq c'_k \ell^3$  for some constant  $c_k$  depending only on  $k$ . Thus

$$T_{\text{rel}} \geq T_{\text{rel}}(\mathbb{T}) \geq \frac{\text{Var}_{\mathbb{T}}(N_r)}{\mathcal{D}_{\mathbb{T}}(N_r)} \geq c \ell^2 = c \epsilon^{-2},$$

for some constant  $c > 0$ .

#### 4. MIXING TIMES: PROOF OF THEOREM 3

The specific statement (i) and (ii) are a direct consequence of (1.7), Theorem 1 and Theorem 2. The upper bound  $T_1(\mathbb{T}) \leq T_2(\mathbb{T}) \leq c L T_{\text{rel}}(\mathbb{T})$  was proved in [16, Corollary 1]]. It remains to prove the lower bound and this is what we do now following an idea of [6].

Consider two probability measures  $\pi, \nu$  on  $\Omega_{\mathbb{T}}$  and recall their *Hellinger* distance

$$d_{\mathcal{H}}(\pi, \nu) := \sqrt{2 - 2I_{\mathcal{H}}(\pi, \nu)},$$

where

$$I_{\mathcal{H}}(\pi, \nu) := \sum_{\omega} \sqrt{\pi(\omega)\nu(\omega)}.$$

Clearly

$$I_{\mathcal{H}}(\pi, \nu) \geq \sum_{\eta \in \Omega_{\mathbb{T}}} \pi(\eta) \wedge \nu(\eta) \geq 1 - \|\pi - \nu\|_{TV}.$$

If we combine the above inequality with [7, Lemma 4.2 (i)] we get

$$\frac{1}{2} d_{\mathcal{H}}(\pi, \nu)^2 \leq \|\pi - \nu\|_{TV} \leq d_{\mathcal{H}}(\pi, \nu).$$

Assume now that  $\pi, \nu$  are product measures,  $\pi = \prod_{i=1}^n \pi_i$ ,  $\nu = \prod_{i=1}^n \nu_i$ , so that

$$I_{\mathcal{H}}(\pi, \nu) := \prod_{i=1}^n I_{\mathcal{H}}(\pi_i, \nu_i).$$

Therefore

$$\begin{aligned}\|\pi - \nu\|_{TV} &\geq 1 - I_{\mathcal{H}}(\pi, \nu) = 1 - \prod_{i=1}^n I_{\mathcal{H}}(\pi_i, \nu_i) \\ &= 1 - \prod_{i=1}^n \left(1 - \frac{1}{2} d_{\mathcal{H}}(\pi_i, \nu_i)^2\right) \\ &\geq 1 - \prod_{i=1}^n \left(1 - \frac{1}{2} \|\pi_i - \nu_i\|_{TV}^2\right) \\ &\geq 1 - e^{-\sum_i \frac{1}{2} \|\pi_i - \nu_i\|_{TV}^2}.\end{aligned}\tag{4.1}$$

Suppose now that, for each  $i \leq n$ ,  $\nu_i$  is the distribution at time  $t$  of some finite, ergodic, continuous time Markov chain  $X^{(i)}$ , reversible w.r.t.  $\pi_i$  and with initial state  $x_i$ . In this case the measure  $\nu$  is the distribution at time  $t$  of the product chain  $X = \otimes_i X_i$  started from  $x = (x_1, \dots, x_n)$  and  $\pi$  is the reversible measure.

Let  $\lambda_i$  be the spectral gap of the chain  $X^{(i)}$ , let  $f_i$  be the corresponding eigenvector and choose the starting state  $x_i$  in such a way that  $|f_i(x_i)| = \|f_i\|_\infty$ . Then

$$\begin{aligned} \|\pi_i - \nu_i\|_{TV} &\geq \frac{1}{2} \frac{1}{\|f_i\|_\infty} |\pi_i(f_i) - \nu_i(f_i)| = \frac{1}{2} \frac{|f(x_i)|}{\|f_i\|_\infty} e^{-\lambda_i t} \\ &= \frac{1}{2} e^{-\lambda_i t}, \end{aligned} \tag{4.2}$$

where we used  $\pi_i(f_i) = 0$  because  $f_i$  is orthogonal to the constant functions.

In conclusion, by combining together (4.1) and (4.2), we get

$$\|\pi - \nu\|_{TV} \geq 1 - e^{-\frac{1}{8} \sum_i e^{-2\lambda_i t}}.$$

Therefore, if  $t = t^*$  with

$$t^* = \frac{1}{2} \left[ \frac{1}{\max_i \lambda_i} \log n - \frac{1}{\min_i \lambda_i} \log 8 \right],$$

then  $\|\pi - \nu\|_{TV} \geq 1 - e^{-1}$ . Thus the mixing time of the product chain  $X$  is larger than  $t^*$ .

We now apply the above strategy to prove a lower bound on  $T_1(\mathbb{T})$ .

Let  $\mathbb{T}^{(i)}$  be the  $i^{\text{th}}$  (according to some arbitrary order)  $k$ -ary sub-tree of depth  $\lceil L/2 \rceil$  rooted at the  $\lfloor L/2 \rfloor$ -level of  $\mathbb{T}$  and consider the OFA-kf model on  $\cup_i \mathbb{T}^{(i)}$ . Clearly such a chain  $X$  is a product chain,  $X = \otimes_i X_i$ , where each of the individual chain is the OFA-kf model on  $\mathbb{T}^{(i)}$ . The key observation now is that, due to the oriented character of the constraints, the projection on  $\cup_i \mathbb{T}^{(i)}$  of the OFA-kf model on  $\mathbb{T}$  coincides with the chain  $X$ . Hence  $T_1(\mathbb{T}) \geq t_{\text{mix}}$  if  $t_{\text{mix}}$  denotes the mixing time of the product chain  $X$ . According to the previous discussion and with  $n = k^{\lfloor L/2 \rfloor}$  the number of sub-trees  $\mathbb{T}^{(i)}$  we get

$$\begin{aligned} T_1(\mathbb{T}) &\geq t_{\text{mix}} \geq \frac{1}{2} (\log n - \log 8) \text{gap}(\mathcal{L}_{\mathbb{T}'})^{-1} = \frac{1}{2} (\log n - \log 8) T_{\text{rel}}(\mathbb{T}') \\ &\geq \frac{1}{c} L T_{\text{rel}}(\mathbb{T}') \end{aligned}$$

for some constant  $c > 0$  where we used translation invariance to conclude that the spectral gap  $\lambda_i$  of the chain  $X_i$  coincides with  $\text{gap}(\mathcal{L}_{\mathbb{T}'})$  for any  $i$ ,  $\mathbb{T}'$  denoting a  $k$ -ary rooted tree of depth  $\lceil L/2 \rceil$ .

## 5. CONCLUDING REMARKS AND OPEN PROBLEMS

(i) It is a very interesting problem to determine exactly the critical exponents for the critical and quasi-critical case and in particular to verify whether the *lower bounds* in Theorems 1 and 3 give the correct growth of the corresponding time scales as a function of the depth of the tree.

(ii) A key ingredient of our analysis is the fact that the percolation transition on  $\mathbb{T}^k$  is *continuous*, i.e. with probability one there is no infinite cluster of occupied sites at



$p = p_c$  and the probability that the cluster of the root touches more than  $n$  levels decays polynomially in  $1/n$ . A very challenging open problem is the extension of the approach described in this work to models with a *discontinuous* (or first-order) phase transition for the corresponding bootstrap percolation problem.

The first instance of the above general question goes as follows. On  $\mathbb{T}^3$  consider the analog of the OFA-kf model in which the constraint at each vertex  $x$  requires now at least two of the three children of  $x$  to be empty. It can be shown [2] that the critical value of the corresponding bootstrap percolation problem is  $p_c = \frac{8}{9}$  and that after infinitely many iterations of the bootstrap map the root belongs to an infinite cluster of occupied sites with probability equal to  $\frac{3}{4}$ . In [16] it was proved that  $T_{\text{rel}} < +\infty$  for all  $p < p_c$ . At  $p_c$  the process is clearly no longer ergodic, contrary to what happens for the OFA-kf model, because of the presence of infinite bootstrap percolation clusters which are blocked under the dynamics. Finally, for  $p > p_c$ , the relaxation time on a finite sub-tree diverges exponentially fast in the depth of the tree.

The interesting challenge is to decide the behaviour of e.g. the relaxation time on a finite 3-ary rooted tree of finite depth  $L$  at  $p_c$ . On one hand, the fact that

$$\mathbb{P}(\text{the root belongs to a occupied cluster reaching the leaves}) \sim 3/4,$$

may suggest a scaling of  $T_{\text{rel}}$  in  $L$  much more rapid than for the critical OFA-kf, even faster than  $\text{Poly}(L)$ . On the other hand, the test function given by the indicator of the event that the root is still occupied after  $L$  iterations of the bootstrap map, which at  $p > p_c$  gives an exponential growth in  $L$  of  $T_{\text{rel}}$ , at  $p = p_c$  gives  $T_{\text{rel}} = \Omega(L^2)$ , exactly as in the OFA-k model. The same bound  $\Omega(L^2)$  is found using another test function closer to the one used in section 2.2.

Here we conjecture that  $T_{\text{rel}}$  is still  $\text{Poly}(L)$ . This conjecture is supported by numerical simulations for the unoriented version of the same model [19], namely the model on the unrooted tree with connectivity  $k+1 = 4$  in which the kinetic constraint requires at least two empty neighbours (actually these numerical results concern the relaxation time of the *persistence function* in the quasi-critical regime, a new time scale which can be bounded from above by  $T_{\text{rel}}$  [3]). Another element in favour of our guess is the fact that the phase transition occurring at  $p_c$  has really a mixed first-second order character as indicated by some non-rigorous work [5, 11].

## REFERENCES

- [1] D. Aldous and P. Diaconis, *The asymmetric one-dimensional constrained Ising model: rigorous results*, J. Statist. Phys. **107** (2002), no. 5-6, 945–975.
- [2] J. Balogh, Y. Peres, and G. Pete, *Bootstrap percolation on infinite trees and non-amenable groups*, Combin. Probab. Comput. **15** (2006), no. 5, 715–730.
- [3] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Kinetically constrained spin models*, Probability Theory and Related Fields **140** (2008), no. 3-4, 459–504.
- [4] N. Cancrini, F. Martinelli, R. Schonmann, and C. Toninelli, *Facilitated oriented spin models: some non equilibrium results*, J. Stat. Phys. **138** (2010), no. 6, 1109–1123.
- [5] J. Chalupa, P. L. Leath, and G. R. Reich, *Bootstrap percolation on a Bethe lattice*, J.Phys.C **12** (1979), L31-L37.
- [6] J. Ding, E. Lubetzky, and Y. Peres, *Mixing time of critical Ising model on trees is polynomial in the height*, Comm. Math. Phys. **295** (2010), no. 1, 161–207.
- [7] W. Evans, C. Kenyon, Y. Peres, and L. J. Schulman, *Broadcasting on trees and the Ising model*, Ann. Appl. Probab. **10** (2000), no. 2, 410–433.

- [8] A. Faggionato, F. Martinelli, C. Roberto, and C. Toninelli, *Aging through hierarchical coalescence in the east model*, Com. Math. Phys. **309** (2012), 459–495.
- [9] A. Faggionato, F. Martinelli, C. Roberto, and C. Toninelli, *The East model: recent results and new progresses*, 2012. preprint available at arXiv:1205.1607.
- [10] J. Garrahan, P. Sollich, and C. Toninelli, *Dynamical heterogeneities and kinetically constrained models*, Dynamical heterogeneities in glasses, colloids and granular media and jamming transitions, 2011, pp. 341–369.
- [11] A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, *k-core (bootstrap) percolation on complex networks: Critical phenomena and nonlocal effects*, Phys. Rev. E **73** (2006), 056101.
- [12] J. Jäckle and S. Eisinger, *A hierarchically constrained kinetic ising model*, Z. Phys. B: Condens. Matter **84** (1991), no. 1, 115–124.
- [13] G. Kordzakhia and S. P. Lalley, *Ergodicity and mixing properties of the northeast model*, J. Appl. Probab. **43** (2006), no. 3, 782-792.
- [14] T. M. Liggett, *Interacting particle systems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 276, Springer-Verlag, New York, 1985.
- [15] D. A. Levin, Y. Peres, and E. Wilmer, *Markov Chains and Mixing Times*, American Mathematical Society, 2008.
- [16] F. Martinelli and C. Toninelli, *Kinetically constrained spin models on trees*, 2012. Annals of Applied Probability, in press.
- [17] F. Ritort and P. Sollich, *Glassy dynamics of kinetically constrained models*, Adv Phys **52** (2003), no. 4, 219–342.
- [18] L. Saloff-Coste, *Lectures on finite Markov chains*, Lectures on probability theory and statistics (Saint-Flour, 1996), 1997, pp. 301–413.
- [19] S. Sellitto, G. Biroli, and C. Toninelli, *Facilitated spin models on Bethe lattice: bootstrap percolation, mode coupling transition and glassy dynamics*, Europhysics Lett. **69** (2005), 496-512.

*E-mail address:* nicoleтта.canocrini@roma1.infn.it

DIIE UNIV. L'AQUILA, 1-67100 L'AQUILA, ITALY

*E-mail address:* martin@mat.uniroma3.it

DIP. MATEMATICA, UNIV. ROMA TRE, LARGO S.L.MURIALDO 00146, ROMA, ITALY

*E-mail address:* cyril.roberto@math.cnrs.fr

UNIVERSITÉ PARIS OUEST NANTERRE LA DÉFENSE - MODAL'X, 200 AVENUE DE LA RÉPUBLIQUE 92000 NANTERRE, FRANCE

*E-mail address:* Cristina.Toninelli@lpt.ens.fr

LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES CNRS-UMR 7599 UNIVERSITÉS PARIS VI-VII 4, PLACE JUSSIEU F-75252 PARIS CEDEX 05 FRANCE