TOWARDS A UNIVERSALITY PICTURE FOR THE RELAXATION TO EQUILIBRIUM OF KINETICALLY CONSTRAINED MODELS

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ABSTRACT. Recent years have seen a great deal of progress in our understanding of bootstrap percolation models, a particular class of *monotone cellular automata*. In the two dimensional lattice $\mathbb{Z}^2$ there is now a quite satisfactory understanding of their evolution starting from a random initial condition, with a strikingly beautiful universality picture for their critical behavior. Much less is known for their non-monotone stochastic counterpart, namely *kinetically constrained models* (KCM). In KCM each vertex is resampled (independently) at rate one by tossing a p-coin iff it can be infected in the next step by the bootstrap model. In particular infection can also heal, hence the non-monotonicity. Besides the connection with bootstrap percolation, KCM have an interest in their own as they feature some of the most striking features of the liquid/glass transition, a major and still largely open problem in condensed matter physics. In this paper we pave the way towards proving universality results for KCM similar to those for bootstrap percolation. Our novel and general approach establishes a close connection between the critical scaling of characteristic time scales for KCM and the scaling of the critical length in critical bootstrap models. Although the full proof of universality for KCM is deferred to a forthcoming paper, here we apply our general method to the Friedrickson-Andersen k-facilitated models, amongst the most studied KCM, and to the Gravner-Griffeath model. In both cases our results are close to optimal.

1. INTRODUCTION

In recent years remarkable progress have been obtained in understanding the behaviour of a particular class of monotone cellular automata known as *bootstrap percolation*. A general bootstrap cellular automata [4] is specified by its *update family* $\mathcal{U} = \{U_1, \dots, U_m\}$ of finite subsets of $\mathbb{Z}^d \setminus 0$. Once $\mathcal{U}$ is given, the $\mathcal{U}$-bootstrap percolation process on e.g. the d dimensional torus of linear size n, $\mathbb{Z}_n^d$, is as follows. Given a set $A \subset \mathbb{Z}_n^d$ of initially infected vertices, set $A_0 = A$, and define recursively for each $t \in \mathbb{N}$

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}_n^d : \ x + U_k \subset A_t \ \text{ for some } k \in (1, \dots m)\}.$$

In other words a vertex x becomes infected at time $t+1$ if the translate by x of at least one element of the update family is already entirely infected at time t, and infected vertices remain infected forever. A much studied example is the classical r-neighbour model (see [2] and references therein) in which a vertex gets infected if *at least* r among its nearest neighbours are infected, namely the update family is formed by all the r-subsets of the set of the nearest neighbours of the origin.

A central problem for bootstrap models is their long time evolution when the initial infected set A_0 is q-random, *i.e.* each vertex of $\mathbb{Z}_n^d$, independently from the other vertices, is initially declared to be infected with probability $q \in (0,1)$. A key quantity is then the *critical percolation threshold* $q_c(n;\mathcal{U})$ defined as the smallest q such that the probability (over A_0) that eventually the whole torus becomes infected is at least $1/2$. Closely related quantities are the *critical length*

$$L_c(q,\mathcal{U}) := \min\{n : q_c(n,\mathcal{U}) = q\},$$

and the mean infection time of the origin $\mathbb{E}(\tau(A,\mathcal{U}))$, where

$$\tau(A,\mathcal{U}) := \min(t \geqslant 0 : 0 \in A_t).$$

In [3–5] beautiful universality results for general $\mathcal{U}$-bootstrap percolation processes in two dimension satisfying $\lim_{n\to\infty} q_c(n,\mathcal{U}) = 0$ have been established, yielding the sharp

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behavior of $q_c(n)$ as $n \to \infty$ and of $L_c(q,\mathcal{U})$ and $\tau(A,\mathcal{U})$ as $q \to 0$. For a nice review of these results we refer the reader to [20, Section 1].

A quite natural stochastic counterpart of bootstrap percolation models are particular interacting particle systems known as *kinetically constrained models* (KCM). Given a $\mathcal{U}$-bootstrap model, the associated KCM is the continuos time reversible Markov process on $\Omega = \{0,1\}^{\mathbb{Z}^d}$ constructed as follows. Call a vertex *infected* if it is in the zero state. Then each vertex x, with rate one and independently across $\mathbb{Z}^d$, is resampled by tossing a p-coin (Prob$(1) = p$) iff the update rule of the bootstrap process at x was fulfilled by the current configuration [9]. It is easy to check that such a process is reversible w.r.t. the Bernoulli(p) product measure on Ω. Notice that if $q := 1-p \ll 1$, it is very unlikely for a vertex to become infected (even if it would have been infected by the bootstrap process). Observe moreover that infected vertices may heal. The latter feature implies, in particular, that the KCM is not *monotone/attractive*, a fact that rules out several powerful tools from interacting particle systems theory like monotone coupling and censoring.

Besides the connection with cellular automata, KCM are of interest in their own. They have been in fact introduced in the physics literature in the '80's to model the liquid/glass transition, a major and still largely open problem in condensed matter physics [14]. Extensive numerical simulations indicate that they display a remarkable *glassy* behavior, including heterogeneous dynamics, the occurrence of ergodicity breaking transitions, multiple invariant measures and anomalously long time scales (see for example [14] and references therein).

It has been proved in [9] that a KCM undergoes an ergodicity breaking transition at $q_c := \liminf_{n\to\infty} q_c(n,\mathcal{U})$ and a major problem, both from the physical and mathematical point of view, is to determine the precise divergence of its characteristic time scales when $q \downarrow q_c$. A very natural time scale is the first time $\tau_0(\mathbb{Z}^d;\mathcal{U})$ at which the state of the origin is updated when the initial law is the reversible measure (*i.e.* the initial configuration consists of i.i.d. Bernoulli(p) variables). Via a general argument based on the finite-speed of propagation considerations, it is possible to prove [9] that $\mathbb{E}(\tau_0(\mathbb{Z}^d;\mathcal{U}))$ is lower bounded by the critical length $L_c(q;\mathcal{U})$ of the corresponding bootstrap percolation model, where $\mathbb{E}(\cdot)$ denotes the average w.r.t. the stationary process. Unfortunately a *general* upper bound on $\mathbb{E}(\tau_0(\mathbb{Z}^d;\mathcal{U}))$, related to the best constant in the Poincaré inequality for the KCM, is much poorer and of the form $\mathbb{E}(\tau_0(\mathbb{Z}^d;\mathcal{U})) \leqslant \exp(cL_c^d)$ [9]. Though this bound has been greatly improved for special choices of the update family $\mathcal{U}$, yielding in some cases the correct behavior (cf. [8, 10, 11]), for general KCM and contrary to the situation of bootstrap percolation in two dimensions there is yet no universality picture for the scaling of $\mathbb{E}(\tau_0(\mathbb{Z}^d;\mathcal{U}))$.

For all those KCM such that $\lim_{n\to\infty} q_c(n;\mathcal{U}) = 0$ it is possible to find in [20, Section 2] some conjectures, formulated jointly with the author R. Morris, on the classification of their universality classes and on the link between the scaling of $\mathbb{E}(\tau_0(\mathbb{Z}^d;\mathcal{U}))$ and that of $L_c(q;\mathcal{U})$. For such KCM it is necessary to introduce a more refined[1] classification of their universality classes w.r.t. bootstrap percolation models in order to take into account the effect of the presence of *energy barriers* on the scaling of $\mathbb{E}(\tau_0(\mathbb{Z}^d;\mathcal{U}))$ as $q \to 0$.

In this paper we develop a novel and general approach to pin down the dominant relaxation mechanism and obtain a much tighter upper bound for $\mathbb{E}_\mu(\tau_0(\mathbb{Z}^d;\mathcal{U}))$ in terms of the critical length $L_c(q;\mathcal{U})$. Our method is designed particularly for KCM such that $\lim_{n\to\infty} q_c(n;\mathcal{U}) = 0$. By applying our strategy it is possible to prove that, for a large class of KCM in two dimensions (the *critical* α*-unrooted* models in the language of [20]),

$$\mathbb{E}_\mu(\tau_0(\mathbb{Z}^d;\mathcal{U})) = O(L_c(q;\mathcal{U})^{\beta(q)}), \quad \beta(q) = \text{poly}(\log\log L_c(q;\mathcal{U})) \quad \text{as } q \to 0. \tag{1.1}$$

[1] It is necessary to distinguish between update families $\mathcal{U}$ for which the *critical droplet* (in bootstrap percolation jargon) is constrained to move inside a cone or not. Examples of the first instance are the d-dimensional East model and the Duarte model in $\mathbb{Z}^2$. In the first case $\mathcal{U}$ consists of the 1-subsets of $\cup_{i=1}^d\{-\vec{e}_i\}$ and in the second case of the 2-subsets of $\{\vec{e}_1, \pm\vec{e}_2\}$.

The proof of this result together with the analysis of the universality picture of KCM in two dimensions, including the proof of the conjectures in [20] for the supercritical models, is postponed to a forthcoming work [19]. Here we apply our technique to the KCM with update family corresponding to k-neighbour bootstrap percolation in any dimension, a much studied KCM known in the physics literature as the Friedrickson Andersen k-facilitated model [1]. We also consider the two dimensional Gravner-Griffeath model [16], in which the update family $\mathcal{U}$ are the 3-subsets of the set consisting of the nearest neighbours of the origin together with the vertices $(\pm 2, 0)$, a model featuring a striking anisotropy in its bootstrap evolution. In both cases our results (Theorem 4.3) establishes (1.1) in $d = 2$ and a tight connection between $\mathbb{E}(\tau_0)$ and $L_c(q;\mathcal{U})$ in $d \geq 3$.

1.1. **Plan of the paper.** In section 2, after introducing the relevant notation and motivated by the connection between $\mathbb{E}(\tau_0)$ and the Poincaré inequality, we prove a (constrained) Poincaré inequality for very general KCM (Theorem 1) satisfying a rather flexible condition involving the range of the update family $\mathcal{U}$ and the probability that an update is feasible. Constrained Poincaré inequality for KCM, implying positive spectral gap and exponential mixing, have already been established [9], mainly using the so-called halving method. Here, inspired by our previous analysis of KCM on trees [8, 18], we develop an alternative method which, besides being more natural and direct, applies as well to update families with a large (depending on q) or infinite number of elements. As an example, in section 2.2 we prove a Poincaré inequality for the KCM for which the constraint requires that the oriented neighbors of the to-be-update vertex belong to an infinite cluster of infected vertices.

Section 3, and its main outcomes summarised in Corollary 3.9, is somehow the core of the work. By applying Theorem 1 together with a renormalization argument and canonical paths arguments, we prove a sharp bound on the best constant in the Poincare' inequality for general KCM. This bound involves the probability of occurence of a *critical droplet* (in the bootstrap percolation language) together with certain *congestion constants* related to the cost of moving around the droplet. In this section we made an effort to keep the framework as general as possible, in order to construct a very flexible tool that can be applied to any choice of constraints in any dimensions.

In section 4 we introduce the Friedrickson-Andersen k-facilitated (FA-kf) and Gravner-Griffeath (GG) models and state our main result Theorem 4.3 for the scaling of $\mathbb{E}(\tau_0(\mathbb{Z}^d;\mathcal{U}))$ in these cases. Finally in section 5 we prove Theorem 4.3 by bounding (model by model) the congestion constants appearing in the key inequality of Corollary 3.9.

2. A constrained Poincaré inequality for product measures

2.0.1. *Notation.* For any integer n we will write $[n]$ for the set $\{1, 2, \dots, n\}$. Given $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ we denote its ℓ^1-norm by $\|x\|_1 = \sum_{i=1}^d |x_i|$ and by $d_1(\cdot,\cdot)$ the associated distance function. Given two vertices $x \neq y$ we will say that x precedes y and we will write $x \prec y$ if $x_i \leqslant y_i$ for all $i \in [d]$. The collection $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_d\}$ will denote the canonical basis of $\mathbb{Z}^d$. Given a set $\Lambda \subset \mathbb{Z}^d$ we define its *external boundary* as the set

$$\partial\Lambda = \{y \in \mathbb{Z}^d \setminus \Lambda \,:\, \exists\, x \in \Lambda \text{ with } \|x - y\|_1 = 1\}\,.$$

2.0.2. *The probability space.* Given a finite set $\mathcal{S}$ and $\Lambda \subseteq \mathbb{Z}^d$, we will denote by Ω_Λ the product space $\mathcal{S}^\Lambda$ endowed with the product topology. Given $V \subset \Lambda$ and $\omega \in \Omega_\Lambda$ we will write ω_V for the restriction of ω to V. Finally we will denote by μ_Λ the product measure $\mu_\Lambda = \otimes_{x\in\Lambda}\, \hat{\mu}_x$ on Ω_Λ, where $\hat{\mu}_x = \hat{\mu}\ \forall x \in \mathbb{Z}^d$ and $\hat{\mu}$ is a positive probability measure on $\mathcal{S}$. Expectation and variance w.r.t. μ_Λ are denoted by $\mathbb{E}_\Lambda(\cdot)$, $\mathrm{Var}_\Lambda(\cdot)$ respectively. If $\Lambda = \mathbb{Z}^d$ the subscript Λ will be dropped from the notation.

In several applications the probability space $(S, \hat{\mu})$ will be the "particle space" $S = \{0,1\}^V$ where V is a finite subset (a "block" as it is sometimes called) of $\mathbb{Z}^d$ and $\hat{\mu} = \otimes_{x\in V} B(p)$, $B(p)$ being the p-Bernoulli measure.

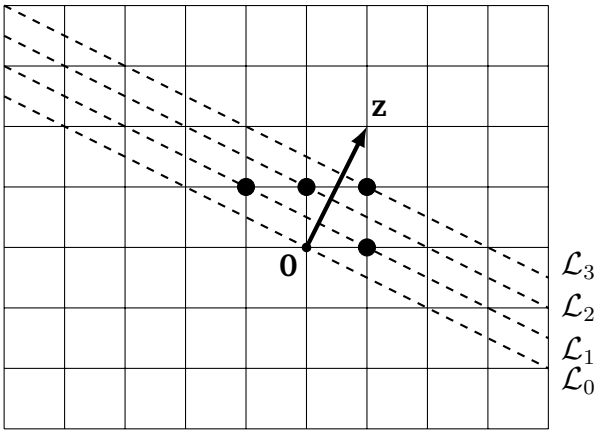

FIGURE 1. An example in two dimensions of a constraint satisfying the exterior condition w.r.t. a sequence of increasing half-spaces. Only the slices $\{\mathcal{L}_n\}_{n=0}^3$ are drawn. The constraint c_0 requires that the restriction of the configuration ω to each one of the four vertices around the origin (black dots) belongs to a certain subset $G \subset S$.

2.0.3. *The constraints.* For each $x \in \mathbb{Z}^d$ let $\Delta_x \subset \mathbb{Z}^d \setminus \{x\}$ be a finite set, let $\mathcal{A}_x$ be an event depending on the variables $\{\omega_y\}_{y\in\Delta_x}$ and let c_x be its indicator function. By construction c_x does not depend on ω_x. In the sequel we will refer to c_x as the *constraint at* x and to $\varepsilon_x := \mu(1-c_x) = \mu(\mathcal{A}_x^c)$ and Δ_x as its *failure probability* and *support* respectively. In our approach based on a martingale decomposition of the variance $\mathrm{Var}(f)$ of any local function $f : \Omega \mapsto \mathbb{R}$, a key role is played by constraints satisfying the following *exterior condition*.

Definition 2.1 (Exterior condition)**.** *Given an exhausting*[2] *collection of subsets* $\{V_n\}_{n=-\infty}^{\infty}$ *of* $\mathbb{Z}^d$*, let* $\mathcal{L}_n := V_n \setminus V_{n-1}$ *be the* n^{th}*-shell and, for any* $x \in \mathcal{L}_n$*, let the* exterior *of* x *be the set* $Ext_x := \cup_{j=n+1}^{\infty}\mathcal{L}_j$*. We then say that the family of constraints* $\{c_x\}_{x\in\mathbb{Z}^d}$ *satisfies the exterior condition w.r.t.* $\{V_n\}_{n=-\infty}^{\infty}$ *if* $\Delta_x \subset Ext_x$ *for all* x*.*

Example 1. *A concrete example of a class of constraints satisfying the exterior condition and entering in the applications to kinetically constrained models is as follows. Fix a vertex* $z \succ 0$ *and let* $\mathcal{L}_0 = \{x \in \mathbb{Z}^d : \langle x, z\rangle = 0\}$*, where* $\langle\cdot,\cdot\rangle$ *is the usual scalar product and* x, z *are treated as vectors in* $\mathbb{R}^d$*. For* $n \geq 1$ *let* $\mathcal{L}_n = \mathcal{L}_0 + n\delta\vec{z}$ *where* $\delta = \sup\{\delta' > 0 : (\mathcal{L}_0 + \delta'\vec{z}) \cap \mathbb{Z}^d = \emptyset\}$*. Similarly for* $n \leqslant -1$*. Finally set* $V_n := \cup_{j=-\infty}^{n}\mathcal{L}_j$ *(cf. Figure 1). Then the constraints are defined as follows. Let* $G \subset S$ *be an single site event and let* $\mathcal{U} = (U_1, \ldots, U_m)$ *be a finite family of subsets of the half-space* $\{x \in \mathbb{Z}^d : \langle x, z\rangle > 0\} = \cup_{i=1}^{\infty}\mathcal{L}_i$*. Then* $c_0(\omega)$ *is the indicator of the event that there exists* $U \in \mathcal{U}$ *such that* $\omega_x \in G$ *for all* $x \in U$*. The constraint* c_x *at any other vertex* x *is obtained by translating the above construction by* x*. For example in* $d = 2$ *one could take* $S = \{0, 1\}$*,* $G = \{0\}$*,* $z = (1, 1)$*,* $m = 1$ *and* $U = \{(0, 1), (1, 0)\}$*, a case known as the North-East model (cf. e.g. [9]). In all the applications discussed in this paper* $z = (1, \ldots, 1)$ *but in order to prove the universality results discussed in the introduction more general choices of* z *will be necessary.*

2.1. **Poincaré inequality.** For simplicity we state our main result directly for the infinite lattice $\mathbb{Z}^d$. There is also a finite volume version in a box $\Lambda \subset \mathbb{Z}^d$ which is proved exactly in the same way.

Let $\{c_x^{(i)}\}_{x\in\mathbb{Z}^d}$, $i = 1, \ldots, k$ be a family of constraints with supports $\Delta_x^{(i)}$ and failure probabilities $\varepsilon_x^{(i)}$. For any non-empty $I \subset [k]$ let $\lambda_I \in (0, +\infty)$ be a positive weight, let $\varepsilon_x^{(I)} = \mu(\prod_{i\in I}(1 - c_x^{(i)}))$ and let $\Delta_x^{(I)} = \cup_{i\in I}\Delta_x^{(i)}$.

[2] That is $V_n \subset V_{n+1}$ for all n and $\cup_n V_n = \mathbb{Z}^d$.

Theorem 1. *Assume that there exists a choice of $\{\lambda_I\}_{I\subset[k]}$ such that*

$$2\Big(\sum_{\substack{I\subset[k]\\ I\neq\emptyset}}\lambda_I\Big)\sup_z\sum_{\substack{I\subset[k]\\ I\neq\emptyset}}\sum_{\substack{x\in\mathbb{Z}^d\\ x\cup\Delta_x^{(I)}\ni z}}\lambda_I^{-1}\varepsilon_x^{(I)}<1/4. \tag{2.1}$$

Suppose in addition that there exists an exhausting family $\{V_n\}_{n=-\infty}^{\infty}$ of sets of $\mathbb{Z}^d$ such that, for any $i\in[k]$, the constraints $\{c_x^{(i)}\}_{x\in\mathbb{Z}^d}$ satisfy the exterior condition w.r.t. $\{V_n\}_{n=-\infty}^{\infty}$. Then, for any local (i.e. depending on finitely many variables) function $f:\Omega\mapsto\mathbb{R}$,

$$\mathrm{Var}(f)\leqslant 4\sum_x\mu\Big(\big[\prod_{i=1}^k c_x^{(i)}\big]\,\mathrm{Var}_x(f)\Big). \tag{2.2}$$

Remark 2.2. *As it is well known, the Poincaré inequality (2.2) is equivalent to say that the spectral gap of the reversible Markov process on Ω with Dirichlet form given by $\mathcal{D}(f)=\sum_x\mu\Big(\big[\prod_{i=1}^k c_x^{(i)}\big]\,\mathrm{Var}_x(f)\Big)$ is greater than $1/4$. Such a process, a kind of generalised KCM, can be informally described as follows. With rate one and independently across $\mathbb{Z}^d$ each variable $\omega_x(t)\in S$, $x\in\mathbb{Z}^d$, is resampled from $\hat\mu_x$ iff $\prod_{i=1}^k c_x^{(i)}(\omega(t))=1$.*

Remark 2.3. *It is easy to construct examples of constraints for which the exterior condition is violated and the r.h.s. of (2.2) is zero for a suitable local function f. Take for instance $S=\{0,1\}$, $d=2$ and c_x the indicator of the event that at least three nearest neighbors of x are in the zero state. If $f(\omega)=\omega_0\omega_{\vec e_1}\omega_{\vec e_1+\vec e_2}\omega_{\vec e_2}$ then $c_x(\omega)\,\mathrm{Var}_x(f)=0$ for all ω and all $x\in\mathbb{Z}^d$ while $\mathrm{Var}(f)>0$. In this case there does not exist an exhausting family $\{V_n\}_{n=1}^{\infty}$ such that the constraints satisfy the exterior condition w.r.t. $\{V_n\}_{n=1}^{\infty}$.*

Remark 2.4. *For certain applications the following monotonicity property turns out to be useful. Suppose that $\{c_x^{(i)}\}_{x\in\mathbb{Z}^d,\,i\in[k]}$ satisfy the condition of the theorem and let $\{\hat c_x^{(i)}\}_{x\in\mathbb{Z}^d,\,i\in[k]}$ be another family of constraints which are dominated by the first ones in the sense that $c_x^{(i)}\leqslant\hat c_x^{(i)}$ for all i,x. Then clearly (2.2) holds with $c_x^{(i)}$ replaced bt $\hat c_x^{(i)}$ even if the latter does not satisfy the exterior condition. As an example take $S=\{0,1\}$, $k=1$ and $\hat c_x$ the constraint that at least one neighbor of x is in the zero state and c_x the same but restricted to the neighbors of the form $x+\vec e_i$, $i\in[d]$.*

Proof of Theorem 1. We first treat the case of a single constraint $k=1$. After that we will explain how to generalize the argument to $k>1$ constraints. We begin with a simple result.

Lemma 2.5. *For any local function f*

$$\mathrm{Var}(f)\leqslant\sum_x\mu\big(\mathrm{Var}_x\big(\mu_{Ext_x}(f)\big)\big). \tag{2.3}$$

Proof of the Lemma. Let $\{V_i\}_{i=-\infty}^{\infty}$ be the exhausting family of sets w.r.t. which all the constraints satisfy the exterior condition, let $\mathcal{L}_i=V_i\setminus V_{i-1}$ be the corresponding i^{th}-shell and assume w.l.o.g. that the support of f is contained in $\cup_{i=0}^n\mathcal{L}_i$. Let finally $\Lambda_j=\cup_{i=n-j}^n\mathcal{L}_i$, $j\leqslant n$. Using the formula for conditional variance together with the fact that μ is a product measure we get:

$$\begin{aligned}\mathrm{Var}(f) &= \mu\big(\mathrm{Var}_{\Lambda_0}(f)\big)+\mathrm{Var}\big(\mu_{\Lambda_0}(f)\big)\\ &= \mu\big(\mathrm{Var}_{\Lambda_0}(f)\big)+\mu\big(\mathrm{Var}_{\Lambda_1}\big[\mu_{\Lambda_0}(f)\big]\big)+\mathrm{Var}\big(\mu_{\Lambda_1}\big[\mu_{\Lambda_0}(f)\big]\big)\\ &\vdots\\ &= \mu\big(\mathrm{Var}_{\Lambda_0}[f]\big)+\sum_{j=0}^{n-1}\mu\big(\mathrm{Var}_{\Lambda_{j+1}}\big[\mu_{\Lambda_j}(f)\big]\big).\end{aligned}$$

Recall now the standard inequality valid for any product probability measure $\nu=\nu_1\otimes\nu_2$:

$$\mathrm{Var}_\nu(f)\leqslant\nu(\mathrm{Var}_{\nu_1}(f))+\nu(\mathrm{Var}_{\nu_2}(f)).$$

If we apply the inequality to $\mathrm{Var}_{\Lambda_{j+1}}\big[\mu_{\Lambda_j}(f)\big]$ and observe that $\mu_{\Lambda_j}(f)$ does not depend on the variables in Λ_j, we get immediately

$$\mu(\mathrm{Var}_{\Lambda_{j+1}}\big[\mu_{\Lambda_j}(f)\big]) \leqslant \sum_{x\in\Lambda_{j+1}\setminus\Lambda_j}\mu\big(\mathrm{Var}_x\big(\mu_{\Lambda_j}(f)\big)\big) = \sum_{x\in\Lambda_{j+1}\setminus\Lambda_j}\mu\big(\mathrm{Var}_x\big(\mu_{Ext_x}(f)\big)\big).$$

Analogously,

$$\mu\big(\mathrm{Var}_{\Lambda_0}[f]\big) \leqslant \sum_{x\in\Lambda_0}\mu\big(\mathrm{Var}_x(f)\big) = \sum_{x\in\Lambda_0}\mu\big(\mathrm{Var}_x(\mu_{Ext_x}(f))\big),$$

because $\mu_{Ext_x}(f) = f$ for any $x \in \Lambda_0$. The proof of the claim is complete. □

We can now prove the theorem for $k = 1$ and the starting point is (2.3). We begin by examining a generic term $\mu\big(\mathrm{Var}_x(\mu_{Ext_x}(f))\big)$ for which we write

$$\mu_{Ext_x}(f) = \mu_{Ext_x}\big(c_x f\big) + \mu_{Ext_x}\big(\big[1-c_x\big]f\big),$$

so that

$$\mathrm{Var}_x\big(\mu_{Ext_x}(f)\big) \leqslant 2\,\mathrm{Var}_x\big(\mu_{Ext_x}\big(c_x f\big)\big) + 2\,\mathrm{Var}_x\big(\mu_{Ext_x}\big(\big[1-c_x\big]f\big)\big). \tag{2.4}$$

Since $c_x(\omega)$ does not depend on ω_x, the convexity of the variance implies that the first term in the above r.h.s. satisfies

$$\mathrm{Var}_x\big(\mu_{Ext_x}\big(c_x f\big)\big) \leqslant \mu_{Ext_x}\big(\mathrm{Var}_x\big(c_x f\big)\big) = \mu_{Ext_x}\big(c_x\,\mathrm{Var}_x(f)\big).$$

We now turn to the analysis of the more complicated second term in the r.h.s. of (2.4).

$$\begin{aligned}\mathrm{Var}_x\big(\mu_{Ext_x}\big(\big[1-c_x\big]f\big)\big) &= \mathrm{Var}_x\big(\mu_{Ext_x}\big(\big[1-c_x\big]\big(f-\mu_{Ext_x\cup\{x\}}(f)+\mu_{Ext_x\cup\{x\}}(f)\big)\big)\big)\\ &= \mathrm{Var}_x\big(\mu_{Ext_x}\big(\big[1-c_x\big]g\big)\big),\end{aligned}$$

where $g := f-\mu_{Ext_x\cup\{x\}}(f)$ and we used the fact that $\mathrm{Var}_x\big(\mu_{Ext_x}([1-c_x]\mu_{Ext_x\cup\{x\}}(f))\big) = 0$.

Recall now that the constraint c_x depends only on $\{\omega_y\}_{y\in\Delta_x}$ with $\Delta_x \subset Ext_x$. Thus

$$\mu_{Ext_x}\big(\big[1-c_x\big]g\big) = \mu_{Ext_x}\left([1-c_x]\mu_{Ext_x\setminus\Delta_x}(g)\right)$$

and a Schwarz-inequality then gives:

$$\begin{aligned}\mathrm{Var}_x\big(\mu_{Ext_x}\big(\big[1-c_x\big]g\big)\big) &\leqslant \mu_x\left(\big(\mu_{Ext_x}\big((1-c_x)\mu_{Ext_x\setminus\Delta_x}g\big)\big)^2\right)\\ &\leqslant \varepsilon_x\mu_{Ext_x\cup\{x\}}\Big(\big[\mu_{Ext_x\setminus\Delta_x}(g)\big]^2\Big).\end{aligned} \tag{2.5}$$

Next we note that

$$\mu_{Ext_x\cup\{x\}}\Big(\big[\mu_{Ext_x\setminus\Delta_x}(g)\big]^2\Big) = \mu_{x\cup\Delta_x}\big(\mu_{Ext_x\setminus\Delta_x}(g)^2\big) = \mathrm{Var}_{x\cup\Delta_x}\left(\mu_{Ext_x\setminus\Delta_x}(g)\right), \tag{2.6}$$

where we used the fact that $\mu_{x\cup\Delta_x}\left(\mu_{Ext_x\setminus\Delta_x}(g)\right) = \mu_{Ext_x\cup\{x\}}(g) = 0$ by the definition of g. Then by using (2.3), (2.5) and (2.6) we get

$$\begin{aligned}\mathrm{Var}_x\big(\mu_{Ext_x}\big(\big[1-c_x\big]g\big)\big) &\leqslant \varepsilon_x\sum_{z\in x\cup\Delta_x}\mu_{x\cup\Delta_x}\left(\mathrm{Var}_z\left(\mu_{Ext_z}\left[\mu_{Ext_x\setminus\Delta_x}(g)\right]\right)\right)\\ &\leqslant \varepsilon_x\sum_{z\in x\cup\Delta_x}\mu_{Ext_x\cup\{x\}}\left(\mathrm{Var}_z(\mu_{Ext_z}(g)\right)\\ &= \varepsilon_x\sum_{z\in x\cup\Delta_x}\mu_{Ext_x\cup\{x\}}\left(\mathrm{Var}_z(\mu_{Ext_z}(f)\right),\end{aligned} \tag{2.7}$$

where we use the convexity of the variance to obtain the second inequality.

In conclusion,

$$\begin{aligned}&\sum_x\mu\left(\mathrm{Var}_x(\mu_{Ext_x}(f))\right)\\ &\leqslant 2\sum_x\mu\big(c_x\,\mathrm{Var}_x(f)\big) + 2\sum_x\varepsilon_x\sum_{z\in x\cup\Delta_x}\mu\left(\mathrm{Var}_z\big(\mu_{Ext_z}(f)\big)\right)\\ &\leqslant 2\sum_x\mu\big(c_x\,\mathrm{Var}_x(f)\big) + 2\left[\sup_z\sum_{x:\,x\cup\Delta_x\ni z}\varepsilon_x\right]\sum_z\mu\left(\mathrm{Var}_z\big(\mu_{Ext_z}(f)\big)\right).\end{aligned} \tag{2.8}$$

If $\sup_z \sum_{x:\ x\cup\Delta_x\ni z} \varepsilon_x \leqslant 1/4$ we get

$$\sum_x \mu\left(\mathrm{Var}_x\big(\mu_{Ext_x}(f)\big)\right) \leqslant 4\sum_x \mu\big(c_x\,\mathrm{Var}_x(f)\big).$$

We now turn to the general case $k > 1$. Let $c_x = \prod_i c_x^{(i)}$ and recall the definition of $\varepsilon_x^{(I)}$ and of $\Delta_x^{(I)}$ for any non-empty $I \subset [k]$. Let also $d_x^{(I)} = \prod_{i\in I}(1-c_x^{(i)})$ so that $\varepsilon_x^{(I)} = \mu(d_x^{(I)})$. Notice that (inclusion/exclusion formula)

$$1 - c_x = \sum_{\substack{I\subset[k]\\ I\neq\emptyset}} (-1)^{\mathrm{Parity}(I)+1} d_x^{(I)} = \sum_{\substack{I\subset[k]\\ I\neq\emptyset}} (-1)^{1+\mathrm{Parity}(I)} \sqrt{\lambda_I}\, d_x^{(I)}/\sqrt{\lambda_I}.$$

Thus the delicate term $\mathrm{Var}_x\big(\mu_{Ext_x}\big(\big([1-c_x]f\big)\big)$ in (2.4) can be bounded from above using the Schwartz inequality by

$$\Big(\sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \lambda_I\Big) \sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \lambda_I^{-1}\,\mathrm{Var}_x\big(\mu_{Ext_x}\big(d_x^{(I)}\,f\big)\big).$$

At this stage we apply the steps leading to (2.7) to each term $\mathrm{Var}_x\big(\mu_{Ext_x}\big(d_x^{(I)}\,f\big)\big)$ to get

$$\mathrm{Var}_x\big(\mu_{Ext_x}\big(\big([1-c_x]f\big)\big) \leqslant \Big(\sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \lambda_I\Big) \sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \lambda_I^{-1}\varepsilon_x^{(I)} \sum_{z\in x\cup\Delta_x^{(I)}} \mu_{Ext_x\cup\{x\}}\left(\mathrm{Var}_z(\mu_{Ext_z}(f)\right).$$

As in (2.8) we conclude that

$$\sum_x \mu\left(\mathrm{Var}_x\big(\mu_{Ext_x}(f)\big)\right)$$
$$\leqslant 2\sum_x \mu\big(c_x\,\mathrm{Var}_x(f)\big) + 2\Big(\sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \lambda_I\Big)\Big(\sup_z \sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \sum_{\substack{x\\ x\cup\Delta_x^{(I)}\ni z}} \lambda_I^{-1}\varepsilon_x^{(I)}\Big) \sum_z \mu\left(\mathrm{Var}_z\big(\mu_{Ext_z}(f)\big)\right),$$

which proves the theorem if

$$\Big(\sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \lambda_I\Big)\Big(\sup_z \sum_{\substack{I\subset[k]\\ I\neq\emptyset}} \sum_{\substack{x\\ x\cup\Delta_x^{(I)}\ni z}} \lambda_I^{-1}\varepsilon_x^{(I)}\Big) \leqslant 1/4.$$

□

2.2. **An application within supercritical percolation in two dimensions.** In this section we restrict ourselves to the case in which the single site probability space $(S,\hat{\mu})$ coincides with $(\{0,1\},B(p))$ and the lattice dimension is equal to two. Given $\omega\in\Omega := \Omega_{\mathbb{Z}^2}$ we will say that $x\in\mathcal{C}(\omega) := \{x\in\mathbb{Z}^2 :\ \omega_x = 0\}$ belongs to an *infinite cluster of zeros* if the connected (w.r.t. to the graph structure of $\mathbb{Z}^2$) component of $\mathcal{C}(\omega)$ containing x is unbounded. It is well known that there exists $p_c\in(0,1)$[3] such that

$$\theta(p) := \mu(\text{the origin belongs to an infinite cluster})$$

is positive iff $p < p_c$ and that moreover there exists μ-a.s. a unique unbounded component of $\mathcal{C}(\omega)$. Let c_x^∞ be the indicator function of the event that at least two nearest neighbors of x belong to an infinite cluster of zeros.

Theorem 2.6. *There exists $p_0\in(0,p_c)$ such that for any $p\leqslant p_0$ and any local function f*

$$\mathrm{Var}(f) \leqslant 4\sum_x \mu\Big(c_x^\infty\,\mathrm{Var}_x(f)\Big). \tag{2.9}$$

Remark 2.7. *It follows in particular that, for all p sufficiently small, the kinetically constrained model (cf. Section 4 for a detailed definition) with the above constraints is exponentially ergodic in $L^2(\mu)$ with relaxation time bounded by 4.*

[3]The conjectured treshold p_c is approximately $1-p_c\approx 0.59$ [15]

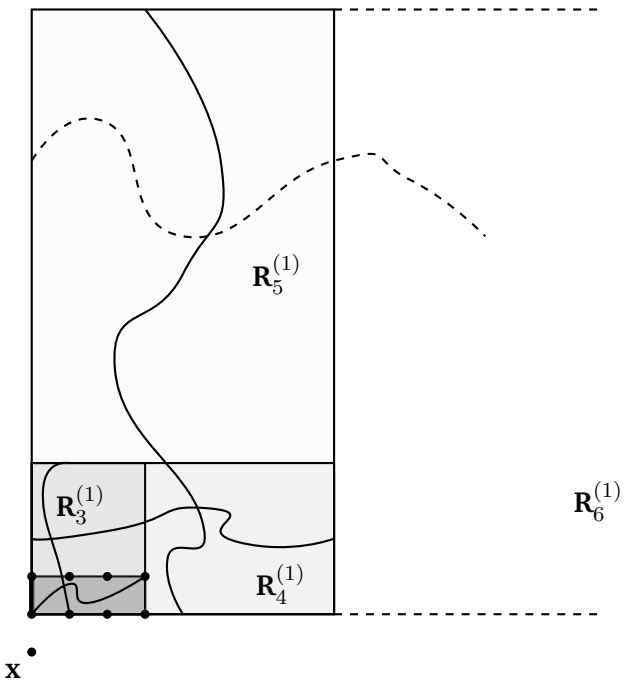

FIGURE 2. A drawing of the first five rectangles $\{R_n^{(1)} + x\}_{n=1}^5$ together with a pictorial representation of the hard crossings of zeros (the solid lines) required by the auxiliary contraint $c_x^{(n,1)}$. The dashed curved line represents a piece of the hard crossing for the next rectangle $R_6^{(1)} + x$ (the two horizontal dashed lines). Notice that each rectangle has its leftmost lowermost vertex always at $x + \vec{e}_2$ and that the first rectangle $R_1^{(1)}$ consists of only two vertices, $x + \vec{e}_2$ and $x + 2\vec{e}_2$.

Proof. We will make use of the following standard construction for super-critical percolation [7]. Let $\ell_n = 2^n$ and define R_n to be a rectangle of the form either $[\ell_n] \times [\ell_{n-1}]$ or $[\ell_{n-1}] \times [\ell_n]$ according to whether n is even or odd. We will also denote by $R_n^{(1)}(R_n^{(2)})$ the rectangle obtained by translating R_n by the vector $-\vec{e}_1(-\vec{e}_2)$ (see Figure 2). With the help of the families $\{R_n^{(1)}, R_n^{(2)}\}_{n\in\mathbb{N}}$ we finally introduce a new family of constraints as follows.

For $i = 1, 2$ let $c_x^{(n,i)}$ be the indicator function of the event that inside the rectangle $R_n^{(i)} + x$ there exists a path $\gamma = (x^{(1)}, \dots, x^{(m)})$ joining the two opposite shortest sides such that $\omega_{x^{(j)}} = 0$ for all $j \in [m]$[4]. Let also $c_x^{(0)}$ be the indicator of the event that $\omega_{x+\vec{e}_1} = \omega_{x+\vec{e}_2} = 0$. Notice that, by construction, the above constraints satisfy the exterior condition 2.1 w.r.t. to the half-spaces defined in Example 1 with $z = (1, 1)$. Moreover it is easy to check that

$$c_x^{(0)} \prod_{n=1}^{\infty} c_x^{(n,1)} c_x^{(n,2)} \leqslant c_x^{\infty} \quad \forall x, \tag{2.10}$$

so that it is enough to prove the constrained Poincaré inequality (2.9) with c_x^{∞} replaced by $c_x^{(0)} \prod_{n=1}^{\infty} c_x^{(n,1)} c_x^{(n,2)}$. More precisely we will prove that, for any $k \in \mathbb{N}$ and any local function f,

$$\mathrm{Var}(f) \leqslant 4 \sum_x \mu\Big(c_x^{(0)} \prod_{n=1}^{k} c_x^{(n,1)} c_x^{(n,2)} \, \mathrm{Var}_x(f) \Big). \tag{2.11}$$

[4]A path γ in $\mathbb{Z}^d$ of length $|\gamma| := k$ is a ordered sequence of k vertices of $\mathbb{Z}^2$ such that two consecutive sites are nearest neighbors of each other. A path with the properties described in the text is usually referred to as a *hard crossing*.

The theorem will then follow by taking the limit $k \to +\infty$ and using (2.10). In order to prove (2.11) we want to apply Theorem 1 which in turn requires finding a family of weights $\{\lambda_I\}_{I\subset[k]\cup\{0\}}$ satisfying (2.1). For this purpose we first recall a standard estimates from super-critical site percolation [15] valid for all p small enough:

$$\mu(1-c_x^{(n,i)}) \leqslant e^{-m(p)\ell_n},$$
$$\mu(1-c_x^{(0)}) \leqslant 2p,$$

with $\lim_{p\to 0} m(p) = +\infty$. In particular, recalling the definition of $\varepsilon_x^{(I)}$ and $\Delta_x^{(I)}$ from Section 2.1, we have the following bounds:

$$\begin{aligned} &\varepsilon_x^{(I)} \leqslant e^{-m(p)\ell_{n(I)}}, && \Delta_x^{(I)} \leqslant 3\ell_{n(I)}^2 && \text{if } n(I) := \max\{i \in I\} > 0,\\ &\varepsilon_x^{(I)} \leqslant 2p, && \Delta_x^{(I)} \leqslant 2 && \text{otherwise.}\end{aligned}$$

Let now $\lambda_I = e^{-\frac{m(p)}{2}\ell_{n(I)}}$ if $I \neq \{0\}$ and $\lambda_I = \sqrt{p}$ if $I = \{0\}$. With this choice it is easy to check that there exists p_0 independent of k such that for $p < p_0$

$$\sum_{I\subset[k]\cup\{0\}} \lambda_I \leqslant 1$$

and

$$\begin{aligned}\sup_z \sum_{\substack{I\subset[k]\cup\{0\}\\ I\neq\emptyset}} \sum_{\substack{x\in\mathbb{Z}^d\\ x\cup\Delta_x^{(I)}\ni z}} \lambda_I^{-1}\varepsilon_x^{(I)} &\leqslant 3 \sum_{\substack{I\subset[k]\cup\{0\}\\ I\neq\emptyset,\ I\neq\{0\}}} \ell_{n(I)}^2 e^{-\frac{m(p)}{2}\ell_{n(I)}} + 4\sqrt{p}\\ &\leqslant 3\sum_{n=1}^{\infty} 2^n 4^n e^{-\frac{m(p)}{2}\ell_n} + 4\sqrt{p} \leqslant 1/4.\end{aligned}$$

In conclusion (2.1) holds for all small enough p independent of k and the theorem follows. □

3. A general approach to prove a Poincaré inequality for kinetically constrained spin models

In this section we start from the general constrained Poincaré inequality proved in Theorem 1 to develop a quite robust and general scheme proving a special kind of Poincaré constrained inequality (cf. Theorem 3.2 and Corollary 3.9) inspired by kinetically constrained models. The approach developed below will allow us, in particular, to relate the scaling of the persistence time of certain kinetically constrained models near the ergodicity threshold to the scaling of the critical length scale of the corresponding bootstrap percolation model. The application of the techniques developed here to the whole class of two dimensional critical models is deferred to a future work [10]. Concrete and succesful applications to basic kinetically constrained models (cf. Theorem 4.3) will be given in the next section. The starting point of our approach is the definition of good and super-good single site events.

Given two events G_1, G_2 in the probability space $(S, \hat\mu)$ let $p_1 := \hat\mu(G_1)$ and $p_2 := \hat\mu(G_2)$. We will assume that G_1 is very likely while G_2 is very unlikely. In the sequel we will refer to G_1 and G_2 as the *good* and *super-good* events respectively.

Definition 3.1 (Good and super-good paths)**.** *Given $\omega \in \Omega = S^{\mathbb{Z}^d}$ we will say that a vertex x is good if $\omega_x \in G_1$ and super-good if $\omega_x \in G_2$. We will say that a path $\gamma = (x^{(1)}, \dots, x^{(k)})$ is a* good path for ω *if each vertex in γ is good. A path will be called super-good if it is good and it contains at least one super-good vertex.*

Before stating the main result we need a last notion. For any mapping $G_1 \overset{\Phi}{\mapsto} G_2$ let

$$\lambda_\Phi = \max_{\sigma\in G_2} \sum_{\sigma'\in G_1:\ \Phi(\sigma')=\sigma} \frac{\hat\mu(\sigma')}{\hat\mu(\sigma)}, \tag{3.1}$$

and, for any ω such that $\omega_x \in G_1$, let $\Phi^{(x)} : \Omega \mapsto \Omega$ be given by

$$\Phi^{(x)}(\omega)_z := \begin{cases} \Phi(\omega_x) & \text{if } z = x \\ \omega_z & \text{otherwise.} \end{cases} \tag{3.2}$$

Theorem 3.2. *There exist $\delta \ll 1$ and $c > 0$ such that, for any $G_1 \overset{\Phi}{\mapsto} G_2$ and all p_1, p_2 with $\max(p_2, (1-p_1)\log(1/p_2)^2) \leqslant \delta$, the following holds:*

$$\begin{aligned} \mathrm{Var}(f) \leqslant c\,(\lambda_\Phi p_2^{-4})^d \Big[\sum_x \mu\Big(\big[\prod_{i\in[d]} \mathbb{1}_{\{\omega_{x+\vec{e}_i}\in G_2\}}\big] \mathrm{Var}_x(f)\Big) \\ + \sum_{x,y:\ d_1(x,y)=1} \mu\Big(\mathbb{1}_{\{\omega_x\in G_1, \omega_y\in G_2\}} \Big[f(\Phi^{(x)}(\omega)) - f(\omega)\Big]^2\Big)\Big]. \end{aligned} \tag{3.3}$$

Remark 3.3. *We could have stated Theorem 3.2 in a more general form in which the constraint $\prod_{i\in[d]} \mathbb{1}_{\{\omega_{x+\vec{e}_i}\in G_2\}}$, appearing in the first term in the r.h.s. of (3.3), is replaced by $\prod_{y\in A+x} \mathbb{1}_{\{\omega_y\in G_2\}}$, where $A \subset Ext_0$ is some finite set whose cardinality is independent of p_1, p_2. For example in two dimensions A could be $\{\vec{e}_1\}\cup\{\vec{e}_2+\vec{e}_1\}\cup\{\vec{e}_2\}\cup\cdots\cup\{\vec{e}_2-m\vec{e}_1\}$. For future applications [19] the freedom given by the choice of the set A will be quite crucial. The proof in this slightly more general case is identical to the one given below. The same applies for the developments discussed in Section 3.1.*

The first term in the r.h.s of (3.3) is a constrained Dirichlet form $\mathcal{D}(f)$ as in the r.h.s. of (2.2), with constraints $c_x := \prod_{i\in[d]} \mathbb{1}_{\{\omega_{x+\vec{e}_i}\in G_2\}}$. These constraints satisfy the exterior condition w.r.t. the half-spaces defined in Example 1 with $z = (1, \dots, 1)$ but, at the same time, they are very unlikely (recall that $\hat{\mu}(G_2) \ll 1$) so that we cannot apply directly Theorem 1 to our setting. Moreover the fact that the $\{c_x\}$ are unlikely implies that a Poincaré inequality of the form $\mathrm{Var}(f) \leqslant C\mathcal{D}(f)$ for all local f and some finite constant C cannot hold. To see that take for instance $\{f_n\}_{n=1}^\infty$ to be a sequence of local functions approximating the indicator of the event that the origin belongs to an infinite oriented cluster of not super-good vertices [5]. Thus the second term in the r.h.s. of (3.3) plays an important role.

Our approach is first to prove a different kind of constrained Poincaré inequality (cf. Proposition 3.4) in which the term in (3.3) involving Φ is missing and the constraint c_x above is replaced by the weaker (and very likely) constraint that for all $i \in [d]$ there exists a super-good path $\gamma^{(i)}$ in $\mathbb{Z}^2 \setminus \{x\}$ starting at $x + \vec{e}_i$ and of length not larger than $1/p_2^2$. Secondly (cf. Lemma 3.5), using repeatedly the mapping Φ^x for each $x \in \gamma^{(i)}$ starting at the super-good vertex of $\gamma^{(i)}$, we "bring" the super-good vertex of $\gamma^{(i)}$ at $x + \vec{e}_i$. In doing that we pay a cost which is embodied in the second term in the r.h.s. of (3.3).

Proof. In what follows we assume that we have fixed some mapping $G_1 \overset{\Phi}{\mapsto} G_2$. We begin by proving the first step of the roadmap just described.

Proposition 3.4. *There exists $\delta \ll 1$ such that, for all p_1, p_2 satisfying $\max(p_2, (1-p_1)\log(1/p_2)^2) \leqslant \delta$, the following holds. Let $\mathbb{1}_x$ be the indicator of the event that $\forall i \in [d]$ there exists a super-good path $\gamma^{(i)}$ of length at most $1/p_2^2$ starting at $x + \vec{e}_i$. Then, for any local f,*

$$\mathrm{Var}(f) \leqslant 4 \sum_x \mu\left(\mathbb{1}_x \mathrm{Var}_x(f)\right). \tag{3.4}$$

Proof of the proposition. In what follows all the auxiliary constraints that we will need to introduce will satisfy the exterior condition w.r.t. the exhausting family of half-spaces defined in Example 1 with $z = (1, \dots, 1)$.

[5] In other words there exists a infinite path $\gamma = (x^{(1)}, \dots, x^{(k)}, \dots)$ starting at the origin such that $x^{(i)} \prec x^{(i+1)}$ and $\omega_{x^{(i)}} \notin G_2$ for all i.

Let $\ell = \lfloor 2\log(1/p_2) \rfloor$, $L = \lfloor e^\ell \rfloor$ and let us define two family of constraints $\{c_x^{(1)}, c_x^{(2)}\}_{x\in\mathbb{Z}^d}$ as follows:

$$c_x^{(1)} = \begin{cases} 1 & \text{if for all } i \in [d] \text{ and all } k \in [\ell] \text{ the vertex } x + k\vec{e}_i \text{ is good,} \\ 0 & \text{otherwise,} \end{cases}$$

$$c_x^{(2)} = \begin{cases} 1 & \text{if for all } i \in [d] \ \exists \text{ a super-good path in } Ext_x \text{ of length at most } L \\ & \text{ starting in the set } \{x + \vec{e}_i, \dots, x + \ell\vec{e}_i\} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $c_x^{(1)} c_x^{(2)} \leqslant \mathbb{1}_x$. In order to apply theorem 1 to the above constraints we need to verify the key condition (2.1). For this purpose we begin to observe that the corresponding supports satisfy $\Delta_x^{(1)} \subset \cup_{i=1}^d \{x + \vec{e}_i, \dots, x + \ell\vec{e}_i\}$ and $\Delta_x^{(2)} \subset \{y \in \mathbb{Z}^d : d_1(x, y) \leqslant \ell + L\}$. In particular there exists a numerical constant $\hat{\delta}$ such that the condition for the validity of Theorem 1 holds if

$$d\ell\mu(1 - c_x^{(1)}) + (\ell + L)^d\big(\mu((1 - c_x^{(2)})) + \mu((1 - c_x^{(1)})(1 - c_x^{(2)}))\big) \leqslant \hat{\delta}. \tag{3.5}$$

A simple union bound proves that $\mu(1-c_x^{(1)}) \leqslant d\ell(1-p_1)$, while standard super-critical percolation bounds [6] valid for large enough values of p_1 prove that

$$\mu((1 - c_x^{(1)})(1 - c_x^{(2)})) \leqslant \mu(1 - c_x^{(2)}) \leqslant d\left(e^{-c\log(1/(1-p_1))\ell} + (1 - p_2)^L\right)$$

for some constant $c > 0$. It is now immediate to verify that given $\hat{\delta} > 0$ there exists $\delta > 0$ small enough such that $\max(p_2, (1 - p_1)\log(1/p_2)^2) \leqslant \delta$ implies (3.5). □

Notice that so far the mapping Φ played no role. We will now use it in order to bound a generic term $\mu\left(\mathbb{1}_x \operatorname{Var}_x(f)\right)$ appearing in (3.4). Without loss of generality we only treat the case $x = 0$.

Lemma 3.5. *In the same setting of Theorem 3.2 there exists $c > 0$ independent of p_1, p_2, Φ such that*

$$\begin{aligned}\mu\Big(\mathbb{1}_0 \operatorname{Var}_0(f)\Big) \leqslant c\,(\lambda_\Phi p_2^{-2})^d \Big[\mu\Big(\Big[\prod_{i\in[d]} \mathbb{1}_{\{\omega_{\vec{e}_i}\in G_2\}}\Big] \operatorname{Var}_0(f)\Big) \\ + \sum_{\substack{x,y:\in\Lambda\setminus\{0\} \\ d_1(x,y)=1}} \mu\Big(\mathbb{1}_{\{\omega_x\in G_1, \omega_y\in G_2\}} \Big[f(\Phi^{(x)}(\omega)) - f(\omega)\Big]^2\Big)\Big],\end{aligned} \tag{3.6}$$

where Λ is the box centered at the origin of side $2\lfloor 1/p_2^2 \rfloor$.

By combining together Lemma 3.5 and Proposition 3.4 we get the statement of the theorem. □

Proof of Lemma 3.5. Recall that $\mathbb{1}_0$ is the indicator of the event, call it SG_i, that there exists a super-good path $\gamma^{(i)}$ in $\mathbb{Z}^2 \setminus \{0\}$ of length at most $L \equiv 1/p_2^2$ starting at $x + \vec{e}_i$. Clearly SG_i is identical to the event that there exists $\gamma = (x^{(1)}, \dots, x^{(L)}) \subset \mathbb{Z}^2 \setminus \{0\}$, such that:

- each vertex $x^{(j)}$ appears exactly once (*i.e.* the path is simple) and $x^{(1)} = \vec{e}_i$,
- there exist $n \leqslant L$ such that $x^{(n)}$ is super-good,
- all the vertices $x^{(j)}$ with $j \leqslant n$ are good.

Fix $i = 1$ and let us order in some way the set $\mathcal{P}$ of simple paths in $\mathbb{Z}^d \setminus \{0\}$ of length L starting at $\vec{e}_1$. For any $\omega \in \cap_{i\in[d]} SG_i$ let γ^* be the smallest path in $\mathcal{P}$ satisfying the

[6] Fix e.g. the first direction. The probability that none of the vertices $x + \vec{e}_1, \dots, x + \ell\vec{e}_1$ belong to an infinite good path in Ext_x is exponentially small in ℓ while the probability that a given path of length L is super-good conditionally on being good is at least $1 - (1 - p_2)^L$.

above set of conditions and let $\nu = \nu(\omega)$ be the index of the first super-good vertex in γ^*. Thus

$$\mu\Big(\mathbb{1}_0 \operatorname{Var}_0(f)\Big) = \sum_{\gamma\in\mathcal{P}}\sum_{n=1}^{L} \mu\Big(\mathbb{1}_{\{\gamma^*=\gamma\}}\mathbb{1}_{\{\nu=n\}}\prod_{j=2}^{d}\mathbb{1}_{\{SG_j\}}\, F\Big), \tag{3.7}$$

where

$$F(\omega) := \operatorname{Var}_0(f)(\omega) = \frac{1}{2}\sum_{\sigma,\sigma'\in S}\hat\mu(\sigma)\hat\mu(\sigma')\left(f(\omega\otimes\sigma)-f(\omega\otimes\sigma')\right)^2,$$

where the notation $\omega\otimes\sigma$ denotes the configuration equal to σ at $x=0$ and equal to ω elsewhere.

Given $\gamma = (x^{(1)},\dots,x^{(L)})\in\mathcal{P}$ and $n\leqslant L$ together with $\omega\in\prod_{j\in[d]}SG_j$ such that $\gamma^*(\omega)=\gamma$ and $\nu(\omega)=n$, let $\Phi^{(i)}(\omega)$ be given by (recall (3.2))

$$\Phi^{(i)}(\omega)_x = \begin{cases}\Phi^{(x^{(i)})}(\omega_x) & \text{if } i\leqslant n-1\\ \omega_x & \text{otherwise.}\end{cases}$$

Thus the mapping $\phi^{(i)}$, $i\leqslant n-1$, makes the configuration ω super-good in $x^{(i)}$ and leaves it unchanged elsewhere. For $i=n$ the mapping $\Phi^{(n)}$ is the identity. With the above notation and using the Cauchy-Schwartz inequality we get

$$F(\omega)\leqslant 2F(\Phi^{(1)}(\omega)) + 4\sum_{\sigma\in S}\hat\mu(\sigma)\Big(f(\Phi^{(1)}(\omega)\otimes\sigma)-f(\omega\otimes\sigma)\Big)^2. \tag{3.8}$$

The first term in the r.h.s. of (3.8) gives a contribution to the r.h.s of (3.7) not larger than

$$2\lambda_\Phi\mu\Big(\mathbb{1}_{\{\omega_{\vec e_1}\in G_2\}}\prod_{j=2}^{d}\mathbb{1}_{\{SG_j\}}\operatorname{Var}_0(f)\Big). \tag{3.9}$$

Above, after the change of variable $\eta := \Phi^{(1)}(\omega)$, we used (3.1) together with the obvious facts that η is super-good at $\vec e_1$ and it belongs to $\prod_{j=2}^d\mathbb{1}_{SG_j}$.

In order to bound from above the contribution of the second term in the r.h.s. of (3.8) we write

$$\begin{aligned}\Big(f(\Phi^{(1)}(\omega)\otimes\sigma)-f(\omega\otimes\sigma)\Big)^2 &= \left(\sum_{i=1}^{n-1}\Big[f(\Phi^{(i+1)}(\omega)\otimes\sigma)-f(\Phi^{(i)}(\omega)\otimes\sigma)\Big]\right)^2\\ &\leqslant (n-1)\sum_{i=1}^{n-1}\Big[f(\Phi^{(i+1)}(\omega)\otimes\sigma)-f(\Phi^{(i)}(\omega)\otimes\sigma)\Big]^2\\ &\leqslant L\sum_{i=1}^{n-1}\Big[f(\Phi^{(i+1)}(\omega)\otimes\sigma)-f(\Phi^{(i)}(\omega)\otimes\sigma)\Big]^2. \end{aligned}\tag{3.10}$$

In turn each summand is bounded from above by

$$2\Big[f(\Phi^{(i+1)}(\Phi^{(i)}(\omega))\otimes\sigma)-f(\Phi^{(i)}(\omega)\otimes\sigma)\Big]^2 + 2\Big[f(\Phi^{(i+1)}(\omega)\otimes\sigma)-f(\Phi^{(i+1)}(\Phi^{(i)}(\omega))\otimes\sigma)\Big]^2.$$

Using the fact that $\Phi^{(i+1)}(\Phi^{(i)}(\omega)) = \Phi^{(i)}(\Phi^{(i+1)}(\omega))$, we see that both terms in the r.h.s. above have a similar structure. We will therefore treat explicitly only the first

one. Recalling that Λ is the box centered at the origin with side $2\lfloor 1/p_2^2 \rfloor$, we get

$$2L\mu\Big(\sum_{\gamma\in\mathcal{P}}\sum_{n=1}^{L}\sum_{i=1}^{n-1}\mathbb{1}_{\{\gamma^*=\gamma\}}\mathbb{1}_{\{\nu=n\}}\prod_{j=2}^{d}\mathbb{1}_{\{SG_j\}}\times$$
$$\times\sum_{\sigma\in S}\hat{\mu}(\sigma)\Big[f(\Phi^{(i+1)}(\Phi^{(i)}(\omega))\otimes\sigma)-f(\Phi^{(i)}(\omega)\otimes\sigma)\Big]^2\Big)$$
$$=2L\mu\Big(\mathbb{1}_{SG_1}\prod_{j=2}^{d}\mathbb{1}_{\{SG_j\}}\sum_{i=1}^{\nu-1}\Big[f(\Phi^{(i+1)}(\Phi^{(i)}(\omega))\otimes\sigma)-f(\Phi^{(i)}(\omega)\otimes\sigma)\Big]^2\Big)$$
$$\leqslant 2L\sum_{\substack{x,y\in\Lambda\setminus\{0\}\\ d_1(x,y)=1}}\mu\Big(\mathbb{1}_{\{\omega_x\in G_1,\,\omega_y\in G_1\}}\Big[f(\Phi^{(x)}(\Phi^{(y)}(\omega))\otimes\sigma)-f(\Phi^{(y)}(\omega)\otimes\sigma)\Big]^2\Big).$$

After the change of variable $\eta\equiv\Phi^{(y)}(\omega)$ inside the expectation, the above quantity can be bounded from above by

$$2L\lambda_\Phi\sum_{\substack{x,y\in\Lambda\setminus\{0\}\\ d_1(x,y)=1}}\mu\Big(\mathbb{1}_{\{\eta_x\in G_1,\eta_y\in G_2\}}\Big[f(\Phi^{(x)}(\eta))\otimes\sigma)-f(\eta\otimes\sigma)\Big]^2\Big).$$

Putting all together we get that there exist a constant $c>0$ such that

$$\mu\Big(\mathbb{1}_0\,\mathrm{Var}_0(f)\Big)\leqslant c\lambda_\Phi p_2^{-2}\Big[\mu\Big(\mathbb{1}_{\{\omega_{\vec{e}_1}\in G_2\}}\prod_{j=2}^{d}\mathbb{1}_{\{SG_j\}}\,\mathrm{Var}_0(f)\Big)$$
$$+\sum_{\substack{x,y\in\Lambda\setminus\{0\}\\ d_1(x,y)=1}}\mu\Big(\mathbb{1}_{\{\eta_x\in G_1,\eta_y\in G_2\}}\Big[f(\Phi^{(x)}(\eta))\otimes\sigma)-f(\eta\otimes\sigma)\Big]^2\Big)\Big].$$

We can now analyse the first term inside the above square bracket by repeating the above analysis for the second direction. In $d-1$ steps the proof is complete. □

3.1. **A canonical paths bound of the r.h.s. of (3.3).** In this section we proceed further by analysing the r.h.s. of (3.3) in the special case in which $S=\{0,1\}^V$, $V=\prod_{i=1}^d[n_i]$ for some integers $\{n_i\}_{i=1}^d$, and $\hat{\mu}$ is the Bernoulli(p) product measure. We will write $|V|$ for the cardinality of V. In this setting the probability space $(S^{\mathbb{Z}^d},\mu)$ becomes isomorphic to (Ω,μ) where $\Omega=\{0,1\}^{\mathbb{Z}^d}$ and μ is the Bernoulli(p) product measure. It is therefore convenient to do a relabelling of the variables $\omega\in S^{\mathbb{Z}^d}$ as follows.

Let $\mathbb{Z}^d(\vec{n})$ be the renormalised lattice $\otimes_{i=1}^d(n_i\mathbb{Z})$ and let, for $x\in\mathbb{Z}^d(\vec{n})$, $V_x:=V+x$. We will write $x\sim y$ iff x,y are nearest neighbor in the renormalised lattice $\mathbb{Z}^d(\vec{n})$. The old "block" variable $\omega_x\in S$ associated to V_x is renamed as $\omega_{V_x}=\{\omega_y\}_{y\in V_x}$ with now $\omega_y\in\{0,1\}$ for all y's. In particular the local variance term $\mathrm{Var}_x(f)$ appearing in the r.h.s. of (3.3) becomes $\mathrm{Var}_{V_x}(f)$. Accordingly we rewrite the mapping $\Phi^{(x)}$, $x\in\mathbb{Z}^d(\vec{n})$, as $\Phi^{(V_x)}$.

In order to formulate our bounds we need to define the *canonical paths* (cf. e.g. [22]).

Definition 3.6 (Canonical paths). *Let $\omega,\omega'\in\Omega$ be two configurations which differ in finitely many vertices. We say that $\Gamma_{\omega,\omega'}\equiv(\omega^{(1)},\omega^{(2)},\dots,\omega^{(k)})$ is a canonical path between ω,ω' if (i) $\omega^{(1)}=\omega$, $\omega^{(k)}=\omega'$, (ii) $\omega^{(i)}\neq\omega^{(j)}$ for all $i\neq j$ (no loops) and (iii) for any $i\in[k-1]$ the configuration $\omega^{(i+1)}$ is obtained from $\omega^{(i)}$ by a single spin flip. The integer k will be referred to as the length of the path.*

The bounds on the individual terms in the r.h.s. of (3.3) are then as follows.

Lemma 3.7. *We assume that, for any $x\in\mathbb{Z}^d(\vec{n})$, any $z\in V_x$ and any ω such that $\omega_{V_x+\vec{e}_i}\in G_2$ for all $i\in[d]$, a canonical path Γ_{ω,ω^z} has been defined such that a generic*

transition in the path consists of a spin flip in $V_x \cup (\cup_{i=1}^d \{V_x + \vec{e}_i\})$. *Let*

$$\rho_A = \sup_{x\in\mathbb{Z}^d(\vec{n})} \max_{z\in V_x} \sup_{\omega'} \sum_{\substack{\omega:\ \omega_{V_x+\vec{e}_i}\in G_2,\,\forall i\in[d] \\ \omega'\in\Gamma_{\omega,\omega^z}}} \frac{\mu(\omega)}{\mu(\omega')}.$$

be the congestion constant *of the family of canonical paths and let* N_A *be their maximal length. Then*

$$\sum_{x\in\mathbb{Z}^d(\vec{n})} \mu\Big(\Big[\prod_{i\in[d]} \mathbb{1}_{\{\omega_{V_x+\vec{e}_i}\in G_2\}}\Big] \mathrm{Var}_{V_x}(f)\Big) \leqslant c\rho_A N_A |V|^2 \sum_{y\in\mathbb{Z}^d} \mu\big(\mathbb{1}_y^A(\omega)\,\mathrm{Var}_y(f)\big),$$

for a numerical constant $c > 0$, *where* $\mathbb{1}_y^A(\omega)$ *is the indicator of the event that there exists* $x \in \mathbb{Z}^d(\vec{n})$, $z \in V_x$ *and* $\bar{\omega}$ *such that* $\bar{\omega}_{V_x+\vec{e}_i} \in G_2$, $\forall i \in [d]$ *and the pair* (ω, ω^y) *form a transition of the canonical path between* $\bar{\omega}$ *and* $\bar{\omega}^z$.

Lemma 3.8. *We assume that, for any* $x \sim y$ *and any* $\omega \in \Omega$ *such that* $\omega_{V_x} \in G_1$ *and* $\omega_{V_y} \in G_2$, *a canonical path between* ω *and* $\Phi^{(V_x)}(\omega)$ *has been defined such that a generic transition in the path consists of a spin flip in* $V_x \cup V_y$. *Let*

$$\rho_B = \sup_{\omega'} \sup_{x\sim y} \sum_{\substack{\omega:\ \omega_{V_x}\in G_1,\,\omega_{V_y}\in G_2 \\ \omega'\in\Gamma_{\omega,\Phi^{(V_x)}(\omega)}}} \frac{\mu(\omega)}{\mu(\omega')}$$

and let N_B *be the maximal length of the paths. Then*

$$\sum_{x\sim y} \mu\Big(\mathbb{1}_{\{\omega_{V_x}\in G_1,\omega_{V_y}\in G_2\}} \Big[f(\Phi^{(V_x)}(\omega)) - f(\omega)\Big]^2\Big)\Big] \leqslant c\rho_B N_B\,|V| \sum_{z\in\mathbb{Z}^d} \mu\Big(\mathbb{1}_z^B(\omega)\,\mathrm{Var}_z(f)\Big)$$

for a numerical constant $c > 0$, *where* $\mathbb{1}_z^B(\omega)$ *is the indicator of the event that there exists* $x \sim y$ *and* ω' *such that* $\omega'_{V_x} \in G_1,\ \omega'_{V_y} \in G_2$ *and the pair* (ω, ω^z) *form the transition of the canonical path between* ω' *and* $\Phi^{(V_x)}(\omega')$.

The proof of the above two lemmas is practically identical so we only prove the first one.

Proof of Lemma 3.7. The starting inequality is

$$\mathrm{Var}_{V_x}(f) \leqslant \sum_{z\in V_x} \mu(\mathrm{Var}_z(f)).$$

For simplicity in the sequel we assume $x = 0$. Given ω such that $\omega_{V+\vec{e}_i} \in G_2\,\forall i \in [d]$ and $z \in V$, let $\Gamma_{\omega,\omega^z} = (\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(k)})$ be the corresponding canonical path. Then

$$\mathrm{Var}_z(f)(\omega) = p(1-p)[f(\omega^z) - f(\omega)]^2 \leqslant p(1-p)k \sum_{j=1}^k [f(\omega^{(i+1)}) - f(\omega^{(i)})]^2,$$

so that

$$\mu\Big(\mathbb{1}_{\{\omega_{V+\vec{e}_i}\in G_2\,\forall i\in[d]\}}\,\mathrm{Var}_z(f)\Big) \leqslant N_A p(1-p)\mu\Big(\sum_{i=1}^{k-1}\Big[f(\omega^{(i+1)}) - f(\omega^{(i)})\Big]^2\Big) \leqslant c\rho_A N_A \sum_{y\in V\cup(\cup_{i=1}^d V+\vec{e}_i)} \mu\Big(\mathbb{1}_y^A(\omega)\,\mathrm{Var}_y(f)\Big),$$

where $\mathbb{1}_y^A(\omega)$ is as in the statement and, after the change of variables $\omega = \omega^{(i)}$, we used the definition of ρ_A to bound the relative density between $\omega^{(i)}$ and ω. The statement of the lemma now follows at once. □

For future purpose we summarise the conclusion of our bounds.

Corollary 3.9. *In the same assumptions of Lemmas 3.7 and 3.8*

$$\mathrm{Var}(f) \leqslant c\,(\lambda_\Phi p_2^{-4})^d \Big[\rho_A N_A |V|^2 \sum_z \mu\Big(\mathbb{1}_z^A(\omega)\,\mathrm{Var}_z(f)\Big) + \rho_B N_B |V| \sum_z \mu\Big(\mathbb{1}_z^B(\omega)\,\mathrm{Var}_z(f)\Big)\Big]$$

Remark 3.10. *In the application to KCM the choice of the canonical paths entering in the above corollary will always be such that* $\max\big(\mathbb{1}_z^A(\omega), \mathbb{1}_z^B(\omega)\big) \leqslant c_z(\omega)$, *where* c_z *is the constraint of the KCM at* $z \in \mathbb{Z}^d$. *Thus in this case the conclusion of the Corollary implies a Poincaré inequality* $\mathrm{Var}(f) \leqslant C\mathcal{D}(f)$, *where* $\mathcal{D}(f) = \sum_z \mu(c_z\,\mathrm{Var}_z(f))$ *is the Dirichlet form of the KCM (cf. Remark 2.2) and* C *satisfies*

$$C \leqslant c\,(\lambda_\Phi p_2^{-4})^d \max\Big(\rho_A N_A |V|^2, \rho_B N_B |V|\Big).$$

4. Application to specific KCM models

In this section we begin by recalling the definition of the Fredrickson-Andersen constrained spin models with k-facilitation (FA-kf in the sequel) introduced by H.C. Andersen and G.H. Friedrikson in [1] and of the GG constrained spin model. As it will clear in a moment, the FA-kf models are closely related to the so-called k-neighbor model in bootstrap percolation, while the GG model is related to the anisotropic bootstrap percolation model introduced by Gravner-Griffeath [16]. As such, the dynamical properties of both models near the ergodicity threshold are intimately related to the scaling properties of the corresponding bootstrap percolation models in the same regime. Finally we state our main result relating the persistence time with the critical bootstrap percolation length. This will be proven in section 5 using Corollary 3.9. The key step will consist in finding suitable (*i.e.* depending on the specific choice of the constraints) good and super-good events G_1, G_2, map ϕ and canonical paths.

4.0.1. *The models.* We will work with the probability space (Ω, μ) where $\Omega = \{0,1\}^{\mathbb{Z}^d}$ and μ is the product Bernoulli(p) and we will be interested in the asymptotic regime $q \downarrow 0$ where $q = 1 - p$. A generic *kinetically constrained model* (KCM in the sequel) is a particular interacting particle system, *i.e.* a Markov process on Ω, described by the Markov generator

$$(\mathcal{L}f)(\omega) = \sum_{x \in \mathbb{Z}^d} c_x(\omega)\big(\mu_x(f) - f\big)(\omega),$$

where $\mu_x(f)$ is the Bernoulli(p)-average of $f(\omega)$ w.r.t. to the variable ω_x. The constraints $\{c_x\}_{x\in\mathbb{Z}^d}$ are defined as follows. Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite collection of finite subsets of $\mathbb{Z}^d \setminus \{0\}$. We call $\mathcal{U}$ the *update family* of the process and each $X \in \mathcal{U}$ an *update rule*. Then c_x is the indicator function of the event that there exists an update rule $X \in \mathcal{U}$ such that $\omega_y = 0\ \forall y \in X + x$. We emphasize that we do not assume that the constraints satisfy the exterior property of Section 2.0.3. Using these assumptions it is easy to check (cf. [9] for a detailed analysis) that $\mathcal{L}$ becomes the generator of a reversible Markov process on Ω, with reversible measure μ.

In the FA-kf model one takes as $\mathcal{U}$ the family of k-subsets of the set of nearest neighbors of the origin. In the GG model in two dimensions one takes $\mathcal{U}$ as the family of 3-subsets of the set of nearest neighbors of the origin together with the vertices $\{\pm 2\vec{e}_1\}$. In the terminology of bootstrap percolation (see e.g. [3] and the recent survey [20]) the FA-kf models belong to the family of *critical balanced models* while the GG model is critical and *unbalanced*. Such a difference will appear clearly in the sequel.

We now define the two main quantities characterising the dynamics of the KCMs. The first one is the relaxation time $T_{\mathrm{rel}}(q;\mathcal{U})$ of the generator $\mathcal{L}$, defined as the best constant C in the Poincaré inequality

$$\mathrm{Var}(f) \leqslant C\mathcal{D}(f) \quad \text{for all local } f, \tag{4.1}$$

where $\mathcal{D}(f) = \frac{1}{2}\sum_x \mu\big(c_x\,\mathrm{Var}_x(f)\big)$ is the Dirichlet form associated to $\mathcal{L}$. A finite relaxation time implies that the reversible measure μ is mixing for the semigroup P_t with

exponentially decaying time auto-correlations,

$$\mathrm{Var}\left(e^{t\mathcal{L}}f\right) \leqslant e^{-t/T_{\mathrm{rel}}}\,\mathrm{Var}(f), \qquad f\in L^2(\mu).$$

The second (random) quantity is the first time the spin at the origin reaches the zero state:

$$\tau_0 = \inf\{t\geq 0:\ \omega_0(t)=0\}.$$

In the physics literature the hitting time τ_0 is usually referred to as the *persistence time*, while, in the bootstrap percolation framework, it would be more conveniently dubbed *infection time*.

It is well known (cf. [9]) that for the FA-kf models T_{rel} and $\mathbb{E}_\mu(\tau_0)$ are finite for any $q>0$, where $\mathbb{E}_\mu(\cdot)$ denotes the average w.r.t. the law of the stationary KCM. The methods of [9] together with the results of [3] also prove this result for GG model. Our aim is to compute the rate at which T_{rel} and τ diverge (the latter either in mean or with high probability w.r.t. the stationary KCM) as $q\to 0$. In order to compare our results to similar divergences found in bootstrap percolation models on the finite torus $\mathbb{Z}_n^d$ of side n, we first formally define, following [4], these processes and their critical behaviour.

Definition 4.1 (The bootstrap process on $\mathbb{Z}_n^d$)**.** *Given an update family $\mathcal{U}$, a set $A\subset\mathbb{Z}_n^d$ and $\omega\in\{0,1\}^{\mathbb{Z}_n^d}$ such that $\omega_x=0$ iff $x\in A$, one sets recursively for $t\in\mathbb{N}$,*

$$A_{t+1} = A_t\cup\{x\in\mathbb{Z}_n^d:\ x+U_k\subset A_t \ \text{ for some } k\in[m]\},\quad A_0=A.$$

*We then define the $\mathcal{U}$-*update closure *of A the set*

$$[A]_{\mathcal{U}} = \cup_{t=0}^{\infty} A_t.$$

Definition 4.2. *We say that $A\subset\mathbb{Z}_n^d$ is q-random and we will write $\mathbb{P}_q$ for its law, if A coincides with the set $\{x\in\mathbb{Z}_n^d:\ \omega_x=0\}$, $\omega\sim\mu$. We then define the* critical probability *$q_c(n;\mathcal{U})$ and the* critical length *$L_c(q;\mathcal{U})$ of the $\mathcal{U}$-process as*

$$q_c(n;\mathcal{U}) = \inf\{q:\ \mathbb{P}_q([A]_{\mathcal{U}}=\mathbb{Z}_n^d)\geq 1/2\},$$
$$L_c(q;\mathcal{U}) = \min\{n:\ q_c(n,\mathcal{U})=q\}.$$

4.1. **Main result.** We begin to recall what is known on the asymptotic scaling of the critical length $L_c(q;\mathcal{U})$, relaxation time $T_{\mathrm{rel}}(q;\mathcal{U})$ and hitting time τ_0 as $q\to 0$ for the FA-kf and GG models.

For the FA-kf model in $\mathbb{Z}^d$ it was proved in [2] (cf. the introduction there for a short account of previous relevant results) that for any d,k with $d\geq k\geq 2$ there exists an explicit constant $\lambda(d,k)$ such that

$$L_c(q;\mathcal{U}) = \exp_{(k-1)}\Big(\frac{\lambda(d,k)+o(1)}{q^{1/(d-k+1)}}\Big), \tag{4.2}$$

where $\exp_{(r)}$ denotes the r-times iterated exponential, $\exp_{(r+1)}(x)=\exp(\exp_{(r)}(x))$. For the GG model it was established [12] (see also [13] for a detailed analysis of the $o(1)$ term below) that instead

$$L_c(q;\mathcal{U}) = \exp\Big(\frac{(\log(1/q))^2}{12q}(1+o(1))\Big).$$

As far as the asymptotic behaviour $T_{\mathrm{rel}}(q;\mathcal{U})$ as $q\to 0$ is concerned, only the FA-kf model has been considered so far and the following bounds have been proved in [9]. There exists $c>0$ such that

$$L_c(q;\mathcal{U})^{1-o(1)} \leqslant T_{\mathrm{rel}}(q;\mathcal{U}) \leqslant \exp\big(c/q^5\big) \qquad d=k=2,$$
$$L_c(q;\mathcal{U})^{1-o(1)} \leqslant T_{\mathrm{rel}}(q;\mathcal{U}) \leqslant \exp_{(d-1)}\big(c/q\big) \qquad d\geq 3,\ k\leqslant d.$$

Notice that the above upper bounds are very far from $L_c(q;\mathcal{U})$. In [9, Theorem 3.6] it was also proved that the large deviations of τ_0 can be controlled in terms of $T_{\mathrm{rel}}(q;\mathcal{U})$. More precisely it holds in great generality that

$$\mathbb{P}_\mu(\tau_0\geq t) \leqslant \exp\big(-cq\,t/T_{\mathrm{rel}}(q;\mathcal{U})\big)$$

for some $c>0$ independent of q. In particular $\mathbb{E}_\mu(\tau_0)=O(T_{\mathrm{rel}}(q;\mathcal{U})/q)$. A matching lower bound in terms of $T_{\mathrm{rel}}(q;\mathcal{U})$ was missing. Instead in [9, Section 6.3] a rather

general and simple argument, based on the so-called "finite speed of propagation", proved that, for all models considered here,

$$\mathbb{E}_\mu(\tau_0) \geq L_c(q;\mathcal{U})^{1-o(1)}.$$

In conclusion, while the control of the critical length $L_c(q;\mathcal{U})$ is rather sharp, the relaxation time $T_{\rm rel}(q;\mathcal{U})$ and the mean hitting time $\mathbb{E}_\mu(\tau_0)$ are still poorly controlled. The main outcome of the theorem below is a much tighter connection between $T_{\rm rel}(q;\mathcal{U})$, and therefore $\mathbb{E}_\mu(\tau_0)$, and $L_c(q;\mathcal{U})$.

Theorem 4.3. *For the FA-2f model in $\mathbb{Z}^d$ and the GG model there exists $\alpha > 0$ such that*

$$T_{\rm rel}(q;\mathcal{U}) = O\Big(L_c(q;\mathcal{U})^{\log(1/q)^\alpha}\Big). \tag{4.3}$$

For FA-kf model in $\mathbb{Z}^d$ with $3 \leqslant k \leqslant d$ there exists $c > \lambda(d,k)$ such that

$$T_{\rm rel}(q;\mathcal{U}) \leqslant \exp_{(k-1)}\big(c/q^{1/(d-k+1)}\big). \tag{4.4}$$

5. Proof of theorem 4.3

5.0.1. *Reader's guide and notation.* The proof of the theorem uses all the machinery which was developed in the previous sections. Therefore, for all the above models, the coarse-grained probability space $(S,\hat\mu)$ (cf. e.g. the beginning of Section 3.1) will be of the form $S = \{0,1\}^V$, with $V = \prod_{i=1}^d [n_i]$ and $\hat\mu$ the product Bernoulli(p) measure.

The starting point of the proof is to make an appropriate choice for the value of $\vec n = (n_1,\dots,n_d)$ together with a working definition of the good and super-good events $G_1, G_2 \subset S$ and of the mapping $G_1 \overset{\Phi}{\mapsto} G_2$ (cf. Section 3) for each model. Clearly, in order to apply Theorem 3.2 and Corollary 3.9, our choice of $(\vec n, G_1, G_2)$ must ensure that the probabilities $p_1 = \hat\mu(G_1)$ and $p_2 = \hat\mu(G_2)$ satisfy the basic condition $\lim_{q\to 0}(1-p_1)\big(\log(1/p_2)\big)^2 = 0$ of Theorem 3.2. In the FA-kf models no direction plays a special role (it is a balanced model in the language of [20]) and therefore we choose $n_i = n$ for all $i \in [d]$. In the GG the above symmetry is broken and we will need to distinguish between the two directions. This part of the proof is carried out in Part I (see below). The second part of the proof (cf. Part II below) involves defining appropriately the canonical paths appearing in Lemma 3.7 and 3.8 (see also Corollary 3.9) and bounding the corresponding length and congestion constants.

Carrying out the above program could become particularly heavy from a notational point of view. Therefore we will sometimes adopt a more descriptive and informal approach. More specifically, given a configuration $\omega \in \{0,1\}^{\mathbb{Z}^d}$ and a region $\Lambda \subset \mathbb{Z}^d$, we will declare Λ *empty (occupied)* if $\omega \restriction \Lambda = 0$ (1). While constructing the canonical paths appearing in Lemmas 3.7 and 3.8 we will say that we *empty (fill)* Λ if we flip to $0\,(1)$, one by one according to some preassigned schedule (*i.e.* an ordering of the to-do flips), all the occupied/empty sites of Λ. It is important to emphasize that the schedules involved in the operations of emptying or filling a region will *always* be such that each spin flip dictated by the schedule will occur while fulfilling the specific constraint of each model. Schedules with this property will be dubbed *legal schedules*. A closely related notion is that of *legal canonical path*.

Definition 5.1. *Given a KCM let $\{c_x\}_{x\in\mathbb{Z}^d}$ be the corresponding family of constraints. A* legal canonical path *between two configurations ω,ω' is a canonical path $\Gamma_{\omega,\omega'} \equiv (\omega^{(1)},\omega^{(2)},\dots,\omega^{(m)})$ with the additional property that $c_{x^{(i)}}(\omega^{(i)}) = 1$ $\forall i \in [m-1]$, where ω^x denotes the configuration obtained from ω by flipping the value ω_x and $x^{(i)}$ is the vertex such that $\omega^{(i+1)} = (\omega^{(i)})^{x^{(i)}}$. We say that the canonical path is* decreasing (increasing) *if for any $i \in [m-1]$ and any $x \in \mathbb{Z}^d$ $\omega_x^{(i+1)} \leqslant \omega_x^{(i)}$ ($\omega_x^{(i+1)} \geq \omega_x^{(i)}$).*

We now recall the notion of an *internally spanned set* which will play a crucial role in the definition of the good and super-good events.

Definition 5.2 (Internally spanned)**.** *Consider a KCM with updating family $\mathcal{U}$. Given $\Lambda \subset \mathbb{Z}^d$ and $\omega \in \{0,1\}^{\mathbb{Z}^d}$, we say that Λ is $\mathcal{U}$-*internally spanned *(for ω), and write $\omega \in I(\mathcal{U},\Lambda)$, iff $[\{x\in\Lambda:\ \omega_x = 0\}]_{\mathcal{U}} = \Lambda$. When the KCM is the FA-kf model in d dimensions*

we will sometimes write $I(d,k,\Lambda)$ instead of $I(\mathcal{U},\Lambda)$ and we will say that Λ is k-internally spanned.

Remark 5.3. *For the FA-kf model it is known that* [6] *for $L \geq CL_c(q;\mathcal{U})$, $L \in \mathbb{N}$ and C a large enough numerical constant,*

$$\hat{\mu}(I(d,k,[L]^d) \geq 1 - \exp(-L/L_c(q;\mathcal{U})). \tag{5.1}$$

Clearly for any KCM the following holds. If ω is such that the region Λ is $\mathcal{U}$-internally spanned by ω and ω' is the configuration equal to zero in Λ and equal to ω elsewhere, then there exists a legal decreasing canonical path $\Gamma_{\omega,\omega'}$ which only uses flips inside Λ. In particular the length of $\Gamma_{\omega,\omega'}$ is at most $|\Lambda|$. By reversing the path we get a legal increasing path between ω' and ω.

Before starting the actual proof, it will be useful to fix some additional notation. Given the hypercube $\Lambda = [n]^d$ and $i \in [d]$, we set $E_i(\Lambda) = \{x \in \Lambda : \ x_j = 1, \ j \neq i\}$ and we call it the i^{th}-*edge* of Λ[7] . Any $(d-1)$-dimensional set of the form $\Lambda \cap \{x : x_i = j\}$, $j \in [n]$, will be called an i-*slice* and it will be denoted by $Sl_j(\Lambda;i)$. A generic i-*frame* $F_j(\Lambda;i)$, $j \in [n]$, is the $(d-2)$-dimensional subset of $Sl_j(\Lambda;i)$ consisting of the vertices x such that $x_k = 1$ for some $k \neq i$. If $\Lambda' = x + \Lambda$ then $E_i(\Lambda') = E_i(\Lambda) + x$ etc. If clear from the context we will drop the specification Λ from the notation.

5.1. **Part I.** Here we define the blocks of the coarse-grained analysis together with the good and super-good events and the mapping Φ.

5.1.1. *The FA-kf model with $k \geq 3$.* Let ℓ be the critical length for the FA-(k-1)f model in $\mathbb{Z}^{d-1}$ given by (4.2) with $d \to d-1$ and $k \to k-1$, and fix $n = A\ell \log \ell$ with $A > 2(d-1)+1$ for all $i \in [d]$.

Definition 5.4 (G_1, G_2, Φ)**.** *The good event G_1 consists of all $\omega \in S$ such that for all $i \in [d]$ every i-slice of V is $(k-1)$-internally spanned. The super-good event G_2 consists of all $\omega \in G_1$ such that the first slice in any direction is empty. The mapping $G_1 \overset{\Phi}{\mapsto} G_2$ is defined by $\Phi(\omega)_x = 0$ if $x \in \cup_{i=1}^d Sl_1(V;i)$ and $\Phi(\omega)_x = \omega_x$ otherwise.*

With the triple (G_1, G_2, Φ) we get immediately that

$$\begin{aligned} (1-p_1) &\leqslant dn(1 - \hat{\mu}(I(d-1,k-1,[n]^{d-1}))), \\ p_2 = \hat{\mu}(G_2) &\geq (1-p_1)q^{dn^{d-1}}, \\ \lambda_\Phi &\leqslant \left(\frac{2}{q}\right)^{dn^{d-1}}. \end{aligned}$$

Using (5.1) together with the definition of n, we get immediately that $1-p_1 \leqslant A\ell^{-(A-1)} \log \ell$ so that $\lim_{q\to 0}(1-p_1)\big(\log(1/p_2)\big)^2 = 0$ for all $A > 2d-1$.

5.1.2. *The FA-kf model with $k = 2$.* In this case we choose $V = \prod_{i\in[d]}[n_i]$ with $n_i = \big(\frac{A}{q}\log(1/q)\big)^{1/(d-1)}$ with $A > 3/(d-1)$.

Definition 5.5 (G_1, G_2, Φ)**.** *The good event G_1 consists of all $\omega \in S$ such that, for all $i \in [d]$ every i-slice of V contains at least one empty vertex. The super-good event G_2 consists of all $\omega \in G_1$ such that any i-edge of V is empty. The mapping $G_1 \overset{\Phi}{\mapsto} G_2$ is defined by $\Phi(\omega)_x = 0$ if $x \in \cup_{j=1}^d E_j$ and $\Phi(\omega)_x = \omega_x$ otherwise.*

As before we easily get

$$1 - p_1 = \hat{\mu}(G_1^c) \leqslant dn(1-q)^{n^{d-1}} \leqslant dnq^A, \quad p_2 = \hat{\mu}(G_2) \geq q^{nd}, \quad \lambda_\Phi \leqslant \frac{2^{nd}}{q^{nd}},$$

where 2^{nd} is the number of possible configurations $\omega' \in \{0,1\}^{\cup_i E_i}$. In particular, for all $A > 3/(d-1)$, $\lim_{q\to 0}(1-p_1)\big(\log(1/p_2)\big)^2 = 0$.

[7] Strictly speaking an edge of V is a set of the form $\{x \in V : \ x_j \in \{1,n\} \ \forall j \neq i\}$. Here we will only need edges with one end-point at the vertex $(1,\dots,1)$.

5.1.3. *The GG model.* Here we choose $n_1 = \lfloor \frac{A\log(1/q)}{q^2} \rfloor$ and $n_2 = \lfloor \frac{A\log(1/q)}{q} \rfloor$, $A > 6$.

Definition 5.6. *We say that $\omega \in G_1$ if all columns of $V = [n_1] \times [n_2]$ contain at least one empty vertex and all rows contain at least one pair of adjacent empty vertices (x, x'). We say that $\omega \in G_2$ if $\omega \in G_1$ and the first two adjacent columns of V are empty. The mapping Φ is the one which empties the first two columns of V.*

Again we easily obtain that

$$1 - p_1 = O\Big(q^{(A-2)/2}\log(1/q)\Big), \quad p_2 = O\Big(\exp\big[-\frac{2A}{q}\log(1/q)^2\big]\Big), \quad \lambda_\Phi = O\big(2^{2n_2}/q^{2n_2}\big).$$

so that $\lim_{q\to 0}(1-p_1)\big(\log(1/p_2)\big)^2 = 0$ for $A > 6$.

Notice that for all models the factor $\big(\lambda_\Phi/p_2^4\big)^d|V|$ appearing in Corollary 3.9 is bounded from above by the r.h.s. of (4.3) and (4.4).

5.2. **Part II.** Here we complete the proof of Theorem 4.3 by defining the canonical paths appearing in Lemmas 3.7 and 3.8 in such a way that:

(a) they are legal canonical paths;
(b) the congestion constants ρ_A, ρ_B and the maximum length of the paths N_A, N_B are such that $\max(\rho_A N_a, \rho_B N_B)$ is bounded from above by r.h.s. of (4.3) for the FA-2f and the GG models and by the r.h.s. of (4.4) for the FA-kf model, $k \geq 3$.

A very useful strategy to carry out this program is based on the following simple result.

Lemma 5.7. *Fix ω and let $\Lambda_1, \Lambda_2, \dots, \Lambda_N$ be N regions with the property that, for any j and $k = j \pm 1$, if we empty Λ_j then we can also empty Λ_k by means of a legal schedule using only flips in Λ_k. Assume that ω is such that Λ_1 is empty and let ω' be obtained from ω by emptying Λ_N. Then there exists a legal canonical path $\Gamma_{\omega,\omega'} = (\omega^{(1)}, \dots, \omega^{(m)})$, $m \leqslant 2\sum_i |\Lambda_i|$, such that for any $j \in [m]$ the following holds. If the configuration $\omega^{(j+1)}$ is obtained from $\omega^{(j)}$ by flipping a vertex in Λ_{k_j} then all the discrepancies (i.e. the vertices where they differ) between ω and $\omega^{(j)}$ are contained in $\Lambda_{k_j-1} \cup \Lambda_{k_j} \cup \Lambda_{k_j+1}$ if $k_j < N$ and in $\Lambda_{N-1} \cup \Lambda_N$ if $k_j = N$.*

Proof. By assumption we can first empty Λ_2 and then Λ_3 by using flips first in Λ_2 and then in Λ_3. Let η be the new configuration and let σ be the configuration obtained from ω by emptying Λ_3. We can then restore the original values of ω in Λ_2 by reversing the legal canonical path $\Gamma_{\sigma,\eta}$. Starting from σ we can iteratively repeat the above procedure and get a final legal canonical path $\Gamma_{\omega,\omega'}$ with the required property. □

Remark 5.8. *The fact that the discrepancies between an intermediate step of the path $\omega^{(j)}$ and the starting configuration ω are contained in a triple of consecutive Λ_i's allows us to easily upper bound the congestion constant $\rho_\Gamma := \sup_{\tilde\omega} \sum_{\omega:\, \Gamma_{\omega,\omega'} \ni \tilde\omega} \frac{\mu(\omega)}{\mu(\tilde\omega)}$ of the family $\{\Gamma_{\omega,\omega'}\}_{\omega\in S}$ by $(2/q)^{\max_i(|\Lambda_{i-2}|+|\Lambda_{i-1}|+|\Lambda_i|)}$. This observation will be the main tool to bound the congestion constants ρ_A, ρ_B appearing in Corollary 3.9.*

5.2.1. *The FA-kf model with $k \geq 3$.* As before set $V = [n]^d$ with n as in Section 5.1.1. The proof is based on a series of simple observations which, under certain natural assumptions, ensure the existence of legal canonical paths with some prescribed properties.

Claim 5.9. *Let ω be a configuration such that the i-slice $Sl_j(V;i)$ is empty and the i-slice $Sl_{j-1}(V;i)$ is $(k-1)$-internally spanned. Let ω' be such that $\omega' \restriction Sl_{j-1}(V;i) = 0$ and ω' coincides with ω elsewhere. Then there is a legal decreasing canonical path $\Gamma_{\omega,\omega'}$ which uses only flips inside $S_{j-1}(V;i)$. Similarly if we replace $S_{j-1}(V;i)$ with $S_{j+1}(V;i)$.*

Proof. The result can be immediately proven by noticing that each site in $Sl_{j-1}(V;i)$ has an empty neighbour in $Sl_j(V;i)$. Since $Sl_{j-1}(V;i)$ is $(k-1)$-internally spanned, the legal (w.r.t. to the FA-(k-1)f constraint) monotone path which empties it is also legal w.r.t. the FA-kf constraint. □

Claim 5.10. *Fix $i \in [d]$, $m \in [n]$ and let (ω, ω') be a pair of configurations satisfying at least one of the following conditions:*

(a) ω *is such that the first* i*-slice is empty and all the others are* $(k-1)$*-internally spanned and* ω' *is obtained from* ω *by emptying the* m^{th} i*-slice and the first* $m-1$ i*-frames.*
(b) ω *is such that* $\cup_{i=1}^d Sl_1(V;i)$ *is empty and* ω' *is obtained from* ω *by emptying* $Sl_m(V;i)$.

Then there exists a legal canonical path $\Gamma_{\omega,\omega'} = (\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(N)})$ *with* $N \leqslant 2n^d$ *such that the only discrepancies between* ω *and* $\omega^{(j)}$, $j \in [N]$, *belong to the set*

$$Sl_{k_j-1}(V,i) \cup Sl_{k_j}(V,i) \cup Sl_{k_j+1}(V;i) \cup \left(\cup_{\ell=1}^{k_j} F_\ell(V;i)\right),$$

where k_j *is such that the flip connecting* $\omega^{(j)}$ *to* $\omega^{(j+1)}$ *occurs in the* k_j^{th} i*-slice.*

Proof. Case (a). In this case we simply apply Lemma 5.7 and Claim 5.9 to the first m i-slices with a twist. After emptying the j^{th} i-slice, $j = 1, 2, \dots, m$, instead of reconstructing the original values of ω in the previous slice we do so only in $Sl_{j-1}(V;i) \setminus F_{j-1}(V;i)$. In such a way the i-frames once emptied remain so and we get to the final configuration ω' by a legal canonical path satisfying the required property.

Case (b). We use again Lemma 5.7 and Claim 5.9. The base case $k = 2, d = 2$ follows by observing that the i-slices, $i = 1, 2$, are 1-internally spanned since they all contain an empty site. The case $k = 2$ and $d > 2$ follows by induction. In fact $Sl_2(V;i)$ is of the form $\Lambda \times \{x_i = 2\}$ with Λ isomorphic to $[n]^{d-1}$. Moreover $\cup_{i=1}^{d-1} Sl_1(\Lambda; j) \times \{x_i = 2\} \subset \cup_{j=1}^d Sl_1(V;j)$ and therefore it is empty by assumption. By the inductive hypothesis for $k = 2, d-1$ we can empty $Sl_2(V;i)$ using only flips inside $Sl_2(V;i)$. This concludes the proof for $k = 2$ and any $d \geq 2$. We thus assume the result true for $(k-1, d-1)$ and prove it for (k,d), $d \geq k$. In this case we apply Lemma 5.7 to the regions $\Lambda_j := Sl_j(V;i) \cup \left(\cup_{i=1}^d Sl_1(V;i)\right)$. For simplicity and w.l.o.g we only verify the assumption of the lemma for the pair Λ_1, Λ_2. In this case we aim at constructing a legal canonical path that empties $Sl_2(V;i)$ using only flips there.

Thus, using the inductive hypothesis and the fact that each site on $Sl_2(V;i)$ has an additional empty neighbour in $Sl_1(V;i)$, we can empty $Sl_2(V;i)$ by a legal canonical path which uses flips only in $Sl_2(V;i)$. □

We are now ready to state the main result for the case under consideration.

Proposition 5.11. *In the above setting there exists a choice of the canonical paths occurring in Lemmas 3.7 and 3.8 such that, for a suitable positive constant* c*,*

- *each path is a legal canonical path and* $\max(N_A, N_B) \leqslant cn^d$*;*
- $\max(\rho_A, \rho_B) \leqslant (1/q)^{cn^{d-1}}$.

Using that $n = A\ell \log \ell$, ℓ being the critical length for the FA-(k-1)f model in $\mathbb{Z}^{d-1}$ given by (cf. (4.2))

$$\ell = \exp_{(k-2)}\Big(\frac{\lambda(d-1,k-1) + o(1)}{q^{1/(d-k+1)}}\Big),$$

the proposition implies that

$$\max(\rho_A N_A, \rho_B N_B) \leqslant \text{r.h.s. of (4.4)},$$

so that the conclusion of Theorem 4.3 for the case $k \geq 3$ follows from Corollary 3.9. □

Proof of the proposition. We begin by examining the choice of the canonical paths appearing in Lemma 3.8. Using the definition of the good and super-good events G_1, G_2 given in Section 5.1.1, our choice for the canonical paths is the one dictated by (a) of Claim 5.10. In this case, using Remark 5.8, $N_B \leqslant cn^d$ and $\rho_B \leqslant (1/q)^{n^{d-1}}$ for some constant $c > 0$.

We now turn to the canonical paths appearing in Lemma 3.7. Fix ω and z as in the lemma and observe that, using (b) of claim 5.10, we can empty all the slices $S_{z_i+1}(V;i)$, $i \in [d]$, via a legal schedule. Call ω' the configuration obtained in this way. In ω' we can make a flip at z since z has at least d empty neighbors. We can finally reverse the path from ω to ω' to obtain our final legal canonical path between ω and ω^z. Claim 5.10 again implies that $N_A \rho_A \leqslant cn^{2d} 1/q^{cn^{d-1}}$. □

5.2.2. *The FA-kf model with $k = 2$.* As before set $V = [n]^d$ with n as in Section 5.1.2. For any $x \in V$ we define the *cross at* x as the set $\mathcal{C}_x(V) := \cup_{i=1}^d \mathcal{C}_x(V; i)$ with

$$\mathcal{C}_x(V; i) := \{x' \in V : \ x'_j = x_j \ \forall j \neq i\}.$$

Notice that the cross of the vertex $(1, 1, \dots, 1) \in V$ is the union of the edges $E_i(V)$.

Claim 5.12. *Given $x, y \in V$ such that $y = x \pm \vec{e}_i$ for some $i \in [d]$, let ω be such that $\mathcal{C}_x(V)$ is empty and let ω' be the configuration obtained from ω by emptying the cross at y. Then there exists a legal decreasing canonical path $\Gamma_{\omega,\omega'} = (\omega^{(1)}, \dots, \omega^{(m)})$, $m \leqslant 2dn$, using only flips in $\mathcal{C}_x(V) \cup \mathcal{C}_y(V)$.*

Proof. Since $y = x + \pm\vec{e}_i$ then necessarily $\mathcal{C}_y(V; i) = \mathcal{C}_x(V; i)$. Consider now the vertex $z = y \pm \vec{e}_j$ with $j \neq i$. This vertex has two empty neighbors: one is y and another belongs to $\mathcal{C}_x(V)$. Therefore z can be emptied. We can iterate until we empty the j^{th} arm of the cross $\mathcal{C}_y(V)$ and then repeat the procedure for all the remaining direction but the i^{th}-one. □

As for the case $k \geq 3$ we have:

Proposition 5.13. *In the above setting there exists a choice of the canonical paths occurring in Lemmas 3.7 and 3.8 such that, for a suitable positive constant c,*

- *each path is a legal canonical path and* $\max(N_A, N_B) \leqslant cn^2$*;*
- $\max(\rho_A, \rho_B) \leqslant (1/q)^{cn}$.

Using that $n = \left(\frac{A}{q}\log(1/q)\right)^{1/(d-1)}$, the proposition implies that

$$\max(\rho_A N_A, \rho_B N_B) \leqslant \text{r.h.s. of (4.3)},$$

so that the conclusion of Theorem 4.3 for the case $k = 2$ follows from Corollary 3.9. □

Proof of Proposition 5.13. We begin by examining the choice of the canonical paths appearing in Lemma 3.8. Fix ω and suppose that we have two hypercubes $V = [n]^d$ and $V' = V + (n+1)\vec{e}_1$ such that $\omega \restriction V$ is good and $\omega \restriction V'$ is super-good. Let also ω' be obtained from ω by emptying the cross of the vertex $(1, 1, \dots, 1) \in V$ so that $\omega' \restriction V$ is super-good. Let now $z^{(i)}$ be the first (according to some apriori order) vertex in the $(n-i+1)^{th}$ 1-slice $Sl_{n-i+1}(V; 1)$ which is empty and let $\bar{z}^{(i)} = z^{(i)} + \vec{e}_1$. Observe that the vertex $\bar{z}^{(i)}$ belong to the same 1-slice of V as the vertex $z^{(i-1)}$ and that the vertex $z^{(i)}$ exists for all $i \in [n]$ because $\omega \restriction V$ is good. Finally let $\gamma = (x^{(1)}, \dots x^{(m)})$, $m \leqslant n^2$, be the geometric path connecting $x^{(1)} := (1, \dots, 1) + n\vec{e}_1 \in V'$ with $x^{(m)} := (1, \dots, 1) \in V$ constructed according to the following schedule:

(a) join $x^{(1)}$ with $\bar{z}^{(1)}$ by first adjusting the second coordinate, then the third one etc;
(b) join $\bar{z}^{(1)}$ to $z^{(1)}$;
(c) repeat the above steps with $x^{(1)}$ replaced by $z^{(1)}$ and $\bar{z}^{(1)}$ by $\bar{z}^{(2)}$ etc.

Next, for $i \in [m]$, let Λ_i be the cross $\mathcal{C}_{x^{(i)}}(V^{(i)})$ where $V^{(i)}$ is the hypercube $V + (x_1^{(i)} - 1)\vec{e}_1$. Notice that $x^{(i)} \in Sl_1(V^{(i)}; 1)$. We claim that the above sets satisfy the assumption of Lemma 5.7. If the hypercubes $V^{(i)}, V^{(i+1)}$ are the same then the claim follows immediately from Claim 5.12. If $V^{(i+1)} = V^{(i)} - \vec{e}_1$ then necessarily the pair $(x^{(i)}, x^{(i+1)})$ must be of the form $(\bar{z}^{(j)}, z^{(j)})$ for some j and having the cross $\mathcal{C}_{x^{(i)}}(V^{(i)})$ empty implies that also the cross $\mathcal{C}_{x^{(i)}}(V^{(i+1)})$ is empty because, by assumption, $\omega_{z^{(j)}} = 0$. Thus we can apply again Claim 5.12, this time in the hypercube $V^{(i+1)}$, and empty Λ_{i+1}. It is now a simple check to verify that the path defined in this way satisfy $N_B \leqslant cn^2$ and $\rho_B \leqslant e^{cn}$ for some constant $c > 0$.

We now examine the canonical paths entering in Lemma 3.7. Let ω be such that all the hypercubes $V + \vec{e}_i$, $i \in [d]$, are super-good, let $z \in V$ and let ω' be obtained from ω by flipping ω_z. W.l.o.g. we assume in the sequel that $z = (1, \dots, 1)$.

Let $\tilde{\omega}$ be the intermediate configuration obtained from ω by emptying the cross (in V) of the vertex $x^{(1)} := (n, \dots, n)$. Using Lemma 5.12 it is easy to check that there exists a legal canonical path $\Gamma_{\omega,\tilde{\omega}}$ with a congestion constant $\rho_\Gamma \leqslant (1/q)^{cn}$ for some constant $c > 0$. Next let $\gamma = (x^{(1)}, \dots, x^{(m)})$ be a geometric path connecting $x^{(1)}$ with

the vertex $z+\sum_{i=1}^d \vec{e}_i$ and define $\Lambda_i = \mathcal{C}_{x^{(i)}}(V)$. Using Claim 5.12 and the definition of $\tilde{\omega}$ the sets $\{\Lambda_i\}_{i=1}^m$ satisfy the assumption of Lemma 5.7. In conclusion we have proved the existence of a legal canonical path $\Gamma_{\omega,\hat{\omega}}$ where $\hat{\omega}$ is obtained from ω by emptying the cross of $x^{(m)}$. Now we can legally flip z and then reverse the path $\Gamma_{\omega,\hat{\omega}}$ to finally get to $\omega' = \omega^z$. In conclusion we have obtained a legal canonical path $\Gamma_{\omega,\omega'}$ and the claimed properties of N_A and ρ_A follow at once from its explicit construction. □

5.2.3. *The GG model.* Recall that in this case the basic block V is the $[n_1]\times[n_2]$ rectangle, with n_1, n_2 as in Section 5.1.3. Moreover, given $\omega \in \{0,1\}^V$, the block V is good if every column contains an empty site and every row contains a pair of adjacent empty sites. It is super-good if it is good and the first two columns are empty.

In this setting two basic observations will be at the basis of our definition of the canonical paths appearing in Lemmas 3.7 and 3.8. Fix an integer n together with $\omega \in \{0,1\}^{[4]\times[n+1]}$ and consider four consecutive columns $C_i = \{x=(i,j),\ j\in[n]\}$, $i\in[4]$.

(1) If C_1, C_2 are empty and C_3 contains an empty site, then C_3 can be emptied by a legal decreasing canonical path using only flips in C_3. Similarly if the role of C_1 and C_3 is interchanged.
(2) If C_1, C_2 are empty and the two vertices $x=(3,n+1)$ and $y=(4,n+1)$ above the 3^{th} and 4^{th} column are also empty, then C_3 and C_4 can be emptied by a legal decreasing canonical path using only flips in $C_3\cup C_4$. Similarly if the role of the pair (C_1,C_2) and (C_3,C_4) is interchanged and the sites x,y are replaced by $x'=(1,n+1), y'=(2,n+1)$.

Using the above we can prove our final proposition.

Proposition 5.14. *For the GG model there exists a choice of the canonical paths occurring in Lemmas 3.7 and 3.8 such that, for a suitable positive constant c,*

- *each path is a legal canonical path and* $\max(N_A, N_B) \leqslant cn_1n_2$;
- $\max(\rho_A,\rho_B) \leqslant (1/q)^{cn_2}$.

Proof. We begin with the definition of the canonical paths appearing in Lemma 3.8 with, for simplicity, $V_x = V$ and $V_y = V'$ where V' is either $V+(n_1+1)\vec{e}_1$ or $V+(n_2+1)\vec{e}_2$. For simplicity we will not make any attempt to optimize our construction, *i.e.* to improve over the constant c above.

In the first case, $V' = V+(n_1+1)\vec{e}_1$, let $\omega\in\{0,1\}^{V\cup V'}$ be such that V is good and V' is super-good and let ω' be obtained from ω by emptying the first two columns of V. Then we can use observation (1) above together with Lemma 5.7 to get that there exists a legal canonical path $\Gamma_{\omega,\omega'}$ of maximal length cn_1n_2 and congestion constant $\rho_B \leqslant (1/q)^{cn_2}$ for some constant $c>0$. Notice that in this case we didn't use the fact that if V is good then every row contains a pair of adjacent empty sites (cf. Figure 3).

In the second case, $V' = V+(n_2+1)\vec{e}_2$, for $i\in[n]$ define a_i as the smallest integer $j\in[n-1]$ such that $x=(j,n-i+1)$ and $y=(j+1,n-i+1)$ are both empty. Using that V is good the integer a_i is well defined. Let also Λ_i denotes the two semi-columns in $V\cup V'$ above the vertices $(a_i,n-i+1)$ and $(a_i+1,n-i+1)$ (cf. Figure 4).

Using observation (1) together with Lemma 5.7 we can then obtain a legal canonical path between ω and ω', whose length is at most cn_1n_2 and whose congestion constant is bounded from above by $(1/q)^{cn_2}$ for some $c>0$ independent of i, as follows:

(a) starting from the first two empty columns in V', we begin to empty Λ_1. Then, starting from the two empty semi-columns $\Lambda_1\cup\{a_1,n\}\cup\{a_1+1,n\}$, we empty the two sites $x=(1,n), x'=(2,n)$ while restoring the original values of ω in all the other sites of $V\cup V'$.
(b) We now repeat the same procedure with Λ_1 replaced by Λ_2 and (x,x') replaced by $\hat{x}=(1,n-1), \hat{x}'=(2,n-1)$, starting from the two empty semi-columns obtained by adding to the first two columns of V' the empty sites $(1,n),(2,n)$.
(c) We iterate until reaching ω'.

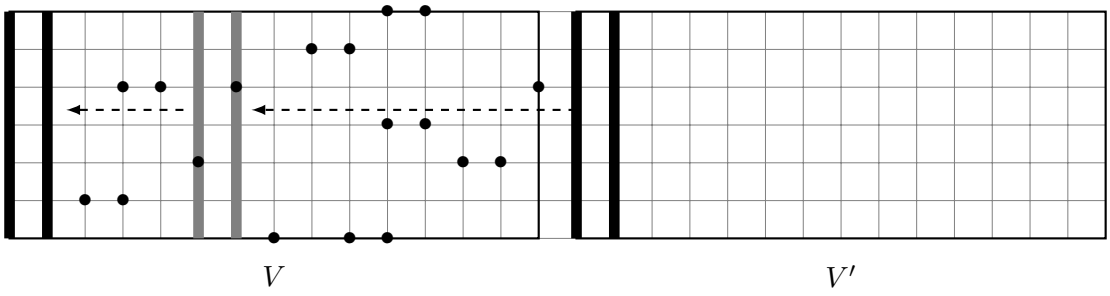

FIGURE 3. A sketch of the canonical path $\Gamma_{\omega,\omega'}$ appearing in Lemma 3.8 for two horizontally adjacent blocks. Only the 1^{st} and 2^{nd} empty columns of the right super-good block are drawn (black). The black dots in the left block denote the empty sites, while the gray columns denote the different positions of the pair of adjacent columns inside the path. Notice the pair of adjacent empty sites on each row.

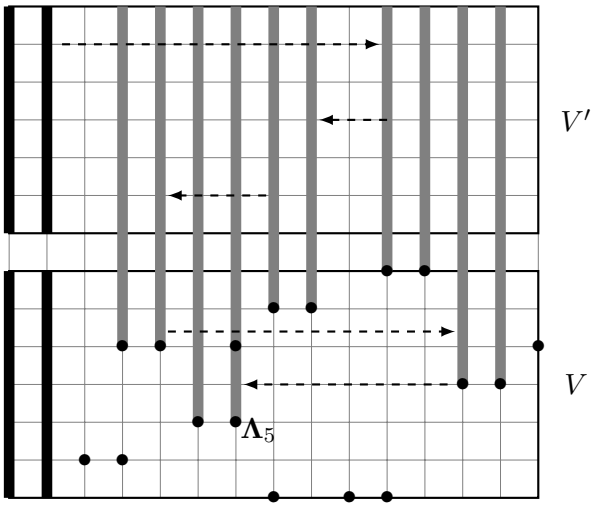

FIGURE 4. A sketch of the canonical path $\Gamma_{\omega,\omega'}$ for two vertically adjacent blocks. The sequence of the dashed arrows must be read from top to bottom. Initially the 1^{st} and 2^{nd} empty columns of the top block (drawn in thick black) travel until they sit above the first pair of adjacent empty sites on the top row of the bottom block. At this time their height grows by one unit. Later in the path this new pair of empty columns is moved above the first pair of adjacent empty sites on the next to top row of the bottom block and so forth until the 1^{st} and 2^{nd} columns of the bottom block become empty.

It remains to consider the construction of the canonical paths appearing in Lemma 3.7 and for that we use both (1) and (2) above.

Fix ω such that $V_1 := V + (n_1+1)\vec{e}_1$ and $V_2 := V + (n_2+1)\vec{e}_2$ are super-good, let $z \in V$ and let $\omega' = \omega^z$. For simplicity and w.l.o.g. we assume $z = (1,1)$. We can then obtain a legal canonical path between ω and ω' with the required properties as follows:

(a) by combining observation (1) with Lemma 5.7 we first empty the last two columns of V_2 without doing any flip inside $V \cup V_1$;
(b) at this stage the last two columns of V_2 are empty because of (a) and the first two columns of V_1 are also empty because V_1 was super-good. Thus, using observation (2), we empty the last two columns of V;
(c) finally we restore the original configuration in V_2 by reverting the path in the first step.
(d) We repeat the above three steps with a twist: we first empty the 4^{th} and 3^{rd} last column of V_2, then the 4^{th} and 3^{rd} last column of V. We then restore the original

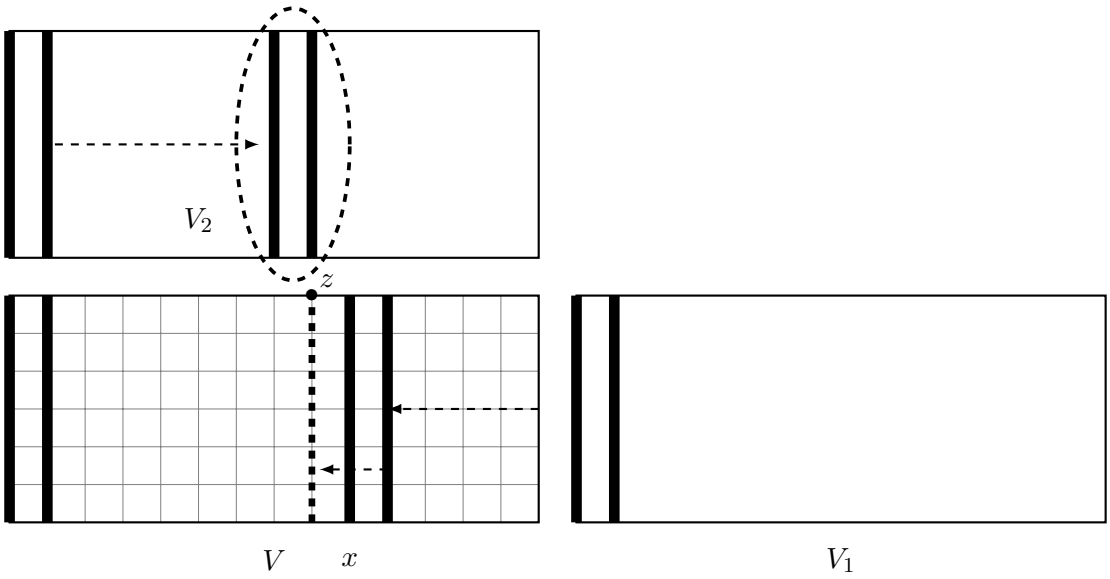

FIGURE 5. A sketch of the canonical path $\Gamma_{\omega,\omega'}$ appearing in Lemma 3.7. Assuming that the path has been able to empty the two black columns of V, then it is possible to move these two columns one step further to the left as follows. First move the initial pair of double empty columns in V_2 to the new position encircled by the dashed ellipse, then, starting with the vertex z, empty the dashed black column in V and finally restore the original values of ω to the right of x and then in V_2.

configuration ω in the last two columns of V and, subsequently, we finally restore ω in V_2. We have now reached the intermediate configuration obtained from ω by emptying the 4^{th} and 3^{rd} last column of V.

(e) We iterate the above step until reaching the configuration obtained from ω by emptying the 2^{nd} and 3^{rd} column of V.

(f) Finally, using again (2) above and Lemma 5.7, we empty the vertex $(1,2)$. At this stage we can do a flip in the corner $(1,1)$ since the vertices $(1,2),(2,1)$ and $(3,1)$ are all empty.

(g) The final step is to retrace the steps of the path which emptied $(1,2)$ and then those of the path which emptied the 2^{nd} and 3^{rd} column of V in such a way that we end up in the configuration ω'.

□

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