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Abstract. Homogenization in linear elliptic problems usually assumes coercivity of the accompanying Dirichlet form. In linear elasticity, coercivity is not ensured through mere (strong) ellipticity so that the usual estimates that render homogenization meaningful break down unless stronger assumptions, like very strong ellipticity, are put into place. Here, we demonstrate that a L^2-type homogenization process can still be performed, very strong ellipticity notwithstanding, for a specific two-phase two dimensional problem whose significance derives from prior work establishing that one can lose strong ellipticity in such a setting, *provided* that homogenization turns out to be meaningful.

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1. Introduction

This paper may be viewed as a sequel to both [2] and [6]. Those, in turn, were a two-dimensional revisiting of [7] in the light of [8]. The issue at stake was whether one could lose strict strong ellipticity when performing a homogenization process on a periodic mixture of two isotropic elastic materials, one being (strictly) very strongly elliptic while the other is only (strictly) strongly elliptic. We start this introduction with a brief overview of the problem that had been addressed in those papers, restricting all considerations to the two-dimensional case.

We consider throughout an elasticity tensor (Hooke's law) of the form

$$\mathbb{L} \in L^\infty\big(\mathscr{T}_2; \mathscr{L}_s(\mathbb{R}^{2\times 2}_s)\big),$$

where $\mathscr{T}_2$ is the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathscr{L}_s(\mathbb{R}^{2\times 2}_s)$ denotes the set of symmetric mappings from the set of 2×2 symmetric matrices onto itself. Note that there is a canonical identification $\mathcal{I}$ between $\mathscr{T}_2$ and the unit cell $Y_2 := [0,1)^2$; for simplicity, we will denote by y both an element of $\mathscr{T}_2$ and its image under the mapping $\mathcal{I}$.

The tensor-valued function $\mathbb{L}$ defined in $\mathscr{T}_2$ is extended by Y_2-periodicity to $\mathbb{R}^2$ as

$$\mathbb{L}(y+\kappa) = \mathbb{L}(y), \quad \text{a.e. in } \mathbb{R}^2,\ \forall\, \kappa \in \mathbb{Z}^2,$$

so that the rescaled function $\mathbb{L}(x/\varepsilon)$ is εY_2-periodic.

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We then consider the Dirichlet boundary value problem on a bounded open domain $\Omega \subset \mathbb{R}^2$

$$\left\{ \begin{array}{rcl} -\operatorname{div}\left(\mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\right) & = & f \text{ in } \Omega \\ u^\varepsilon & = & 0 \text{ on } \partial\Omega, \end{array} \right. \tag{1.1}$$

with $f \in H^{-1}(\Omega;\mathbb{R}^2)$. We could impose a very strong ellipticity condition on $\mathbb{L}$, namely

$$\alpha_{\rm vse}(\mathbb{L}) := \operatorname*{ess\text{-}inf}_{y\in\mathscr{T}_2} \left(\min\left\{\mathbb{L}(y)M\cdot M : M \in \mathbb{R}^{2\times 2}_s, |M| = 1\right\}\right) > 0. \tag{1.2}$$

In such a setting, homogenization is straightforward; see e.g. the remarks in [11, Ch. 6, Sec. 11].

Instead, we will merely impose (strict) strong ellipticity, that is

$$\alpha_{\rm se}(\mathbb{L}) := \operatorname*{ess\text{-}inf}_{y\in\mathscr{T}_2} \left(\min\left\{\mathbb{L}(y)(a\otimes b)\cdot(a\otimes b) : a,b \in \mathbb{R}^2, |a| = |b| = 1\right\}\right) > 0, \tag{1.3}$$

and this throughout.

Remark 1.1 (Ellipticity and isotropy)**.** Whenever $\mathbb{L}$ is isotropic, that is

$$\mathbb{L}(y)M = \lambda(y)\,\mathrm{tr}\,(M)\,I_2 + 2\mu(y)\,M, \quad \text{for } y \in \mathscr{T}_2,\ M \in \mathbb{R}^{2\times 2}_s,$$

then (1.2) reads as

$$\operatorname*{ess\text{-}inf}_{y\in\mathscr{T}_2} \left(\min\left\{\mu(y), \lambda(y)+\mu(y)\right\}\right) > 0$$

while (1.3) reads as

$$\operatorname*{ess\text{-}inf}_{y\in\mathscr{T}_2} \left(\min\left\{\mu(y), \lambda(y)+2\mu(y)\right\}\right) > 0. \qquad \P$$

The strong ellipticity condition (1.3) is the starting point of the study of homogenization performed in [7] from a variational standpoint, that of Γ-convergence. Under that condition, the authors investigate the Γ-convergence, for the weak topology of $H^1_0(\Omega;\mathbb{R}^2)$ on bounded sets (a metrizable topology), of the Dirichlet integral

$$\int_\Omega \mathbb{L}(x/\varepsilon)\nabla v\cdot\nabla v\,dx.$$

Then, under certain conditions that will be recalled in Section 2, the Γ-limit is given through the expected homogenization formula

$$\mathbb{L}^0 M\cdot M := \min\left\{\int_{Y_2} \mathbb{L}(y)(M+\nabla v)\cdot(M+\nabla v)\,dy : v \in H^1(\mathscr{T}_2;\mathbb{R}^2)\right\} \tag{1.4}$$

in spite of the lack of very strong ellipticity.

In [8, 9], the viewpoint is somewhat different. The author, S. Gutiérrez, looks at a two-phase layering of a very strongly elliptic isotropic material with a strongly elliptic isotropic material. Assuming that the homogenization process makes sense, he shows that strict strong ellipticity can be lost through that process for a very specific combination of Lamé coefficients (see (2.6) below) and for a volume fraction 1/2 of each phase.

Our goal in the previous study [2] was to reconcile those two sets of results, or more precisely, to demonstrate that Gutiérrez' viewpoint expounded in [8, 9] fit within the variational framework set forth in [7] and that the example produced in those papers is the only possible one within the class of laminate-like microstructures. Then, it is shown in [6] that the Gutiérrez pathology is in essence canonical, that is that inclusion-type microstructures never give rise to such a pathology.

The concatenation of those results may be seen as an indictment of linear elasticity, especially when confronted with its scalar analogue where ellipticity cannot be weakened through a homogenization process. However, our results, hence those of Gutiérrez, had to be tempered by the realization that Γ-convergence *a priori* assumes convergence of the relevant sequences in the *ad hoc* topology (here the weak-topology on bounded sets of H^1_0). The derivation of a bound that allows for such an assumption not to be vacuous is not part of the Γ-convergence process, yet it is essential lest that process become a gratuitous mathematical exercise.

This is the task that we propose to undertake in this study. To this end we add to the Dirichlet integral a zeroth-order term of the form $\int_\Omega |v|^2\,dx$ which will immediately provide compactness in the weak topology of $L^2(\Omega;\mathbb{R}^2)$. We are then led to an investigation, for the *weak topology on bounded sets of* $L^2(\Omega;\mathbb{R}^2)$, of the Γ-limit of the Dirichlet integral

$$\int_\Omega \mathbb{L}(x/\varepsilon)\nabla v\cdot\nabla v\,dx.$$

Our results, detailed in Theorems 3.3, 3.4, essentially state that, at least for periodic mixtures of two isotropic materials that satisfy the constraints imposed in [8], the ensuing Γ-limit is identical to that which had been previously obtained for the *weak topology on bounded sets of* $H^1_0(\Omega;\mathbb{R}^2)$. An immediate consequence is that Gutiérrez' example does provide a *bona fide* loss of strict strong ellipticity in two-phase two-dimensional periodic homogenization, and not only one that would be conditioned upon some otherwordly bound on minimizing sequences; see Remark 4.1.

In Section 2, we provide a quick review of the results that are relevant to our investigation. Then Section 3 details the precise assumptions under which we obtain Theorems 3.3, 3.4 and present the proofs of those theorems. Finally, Section 4 briefly details the impact of our results on those previously obtained in [2, 6, 8].

Throughout the paper, the following remark will play a decisive role. Since, for $v\in H^1_0(\Omega;\mathbb{R}^2)$, the mapping $v\mapsto \det(\nabla v)$ is a null Lagrangian, we are at liberty to replace the Dirichlet integral under investigation by

$$\int_\Omega \left\{\mathbb{L}(x/\varepsilon)\nabla v\cdot\nabla v + c\det(\nabla v)\right\}dx,$$

for any $c\in\mathbb{R}$, thereby replacing

$$M\mapsto \mathbb{L}(y)M,\ M\in\mathbb{R}^{2\times 2},$$

by

$$M\mapsto \mathbb{L}(y)M+\frac{c}{2}\operatorname{cof}(M),\ M\in\mathbb{R}^{2\times 2}.$$

Notationwise,

- I_2 is the unit matrix of $\mathbb{R}^{2\times 2}$; $R^\perp$ is the $\pi/2$-rotation matrix $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$;
- $A\cdot B$ is the Frobenius inner product between two elements of $A,B\in\mathbb{R}^{2\times 2}$, that is $A\cdot B:=\operatorname{tr}(A^TB)$;
- If $A:=\left(\begin{smallmatrix}a & c\\ b & d\end{smallmatrix}\right)\in\mathbb{R}^{2\times 2}$, the cofactor matrix of A is $\operatorname{cof}(A):=\left(\begin{smallmatrix}d & -b\\ -c & a\end{smallmatrix}\right)$;
- If $\mathbb{K}:\mathbb{R}^p\to\mathbb{R}^p$ is a linear mapping, the *pseudo-inverse* of $\mathbb{K}$, denoted by $\mathbb{K}^{-1}$, is defined on its range $\operatorname{Im}(\mathbb{K})$ as follows: for any $\xi\in\mathbb{R}^p$, $\mathbb{K}^{-1}(\mathbb{K}\,\xi)$ is the orthogonal projection of ξ onto the orthogonal space $[\operatorname{Ker}(\mathbb{K})]^\perp$, so that $\mathbb{K}\big(\mathbb{K}^{-1}(\mathbb{K}\,\xi)\big)=\mathbb{K}\,\xi$;

- If u is a distribution (an element of $\mathscr{D}'(\mathbb{R}^2;\mathbb{R}^2)$), then
$$\text{curl } u := \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}$$
while
$$\mathbf{E}(u) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} \end{pmatrix};$$
- $H^1_{\text{per}}(Y_2;\mathbb{R}^p)$ (resp. $L^2_{\text{per}}(Y_2;\mathbb{R}^p), C^p_{\text{per}}(Y_2;\mathbb{R}^p)$) is the space of the functions in $H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^p)$ (resp. $L^2_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^p), C^p(\mathbb{R}^2;\mathbb{R}^p)$) which are Y_2-periodic;
- For any subset $\mathscr{Z} \in \mathscr{T}_2$, we agree to denote by Z its representative in Y_2 through the canonical representation $\mathcal{I}$ introduced earlier, and by $Z^\#$ its representative in $\mathbb{R}^2$, that is the open "periodic" set
$$Z^\# := \overbrace{\bigcup_{k\in\mathbb{Z}^2} (k + \overline{Z})}^{\circ}.$$
- Throughout, the variable x will refer to a running point in a bounded open domain $\Omega \subset \mathbb{R}^2$, while the variable y will refer to a running point in Y_2 (or $\mathscr{T}_2$, or $k + Y_2,\ k \in \mathbb{Z}^2$);
- If $\mathscr{I}^\varepsilon$ is an ε-indexed sequence of functionals with
$$\mathscr{I}^\varepsilon : X \to \mathbb{R},$$
(X reflexive Banach space), we will write that $\mathscr{I}^\varepsilon \overset{\Gamma(X)}{\rightharpoonup} \mathscr{I}^0$, with
$$\mathscr{I}^0 : X \to \mathbb{R},$$
if $\mathscr{I}^\varepsilon$ Γ-converges to $\mathscr{I}^0$ for the weak topology on bounded sets of X (see *e.g.* [4] for the appropriate definition); and
- $u^\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup u^0$ where $u^\varepsilon \in L^2(\Omega;\mathbb{R}^2)$ and $u^0 \in L^2(\Omega \times \mathscr{T}_2;\mathbb{R}^2)$ iff u^ε two-scale converges to u^0 in the sense of Nguetseng; see *e.g.* [10, 1].

2. Known results

As previously announced, this short section recalls the relevant results obtained in [7], [3]. For vector-valued (linear) problems, a successful application of Lax-Milgram's lemma to a Dirichlet problem of the type (1.1) hinges on the positivity of the following functional coercivity constant:

$$\Lambda(\mathbb{L}) := \inf\left\{\int_{\mathbb{R}^2} \mathbb{L}(y)\nabla v \cdot \nabla v \, dy : v \in C^\infty_c(\Omega;\mathbb{R}^2),\ \int_{\mathbb{R}^2} |\nabla v|^2 \, dy = 1\right\}. \tag{2.1}$$

As long as $\Lambda(\mathbb{L}) > 0$, existence and uniqueness of the solution to (1.1) is guaranteed by Lax-Milgram's lemma.

Further, according to classical results in the theory of homogenization, under condition (1.2) the solution $u^\varepsilon \in H^1_0(\Omega;\mathbb{R}^2)$ of (1.1) satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup u, & \text{weakly in } H^1_0(\Omega;\mathbb{R}^2) \\ \mathbb{L}(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup \mathbb{L}^0\nabla u, & \text{weakly in } L^2(\Omega;\mathbb{R}^{2\times 2}) \\ -\operatorname{div}\left(\mathbb{L}^0\nabla u\right) = f, \end{cases} \tag{2.2}$$

with $\mathbb{L}^0$ given by (1.4). The same result holds true when (1.2) is replaced by the condition that $\Lambda(\mathbb{L}) > 0$; see [5].

When $\Lambda(\mathbb{L}) = 0$, the situation is more intricate. A first result was obtained in [7, Theorem 3.4(i)], namely

Theorem 2.1. *If* $\Lambda(\mathbb{L}) \geq 0$ *and*

(2.3)
$$\Lambda_{\rm per}(\mathbb{L}) := \inf \left\{ \int_{Y_2} \mathbb{L}(y)\nabla v \cdot \nabla v \; dy : v \in H^1_{\rm per}(Y_2;\mathbb{R}^2), \; \int_{Y_2} |\nabla v|^2 dy = 1 \right\} > 0,$$

then, $\mathscr{I}^\varepsilon \overset{\Gamma(H^1_0)}{\rightharpoonup} \mathscr{I}^0$, *with* $\mathbb{L}^0$ *given by* (1.4).

This was very recently improved by A. Braides & M. Briane as reported in [3, Theorem 2.4]. The result is as follows:

Theorem 2.2. *If* $\Lambda(\mathbb{L}) \geq 0$, *then,* $\mathscr{I}^\varepsilon \overset{\Gamma(H^1_0)}{\rightharpoonup} \mathscr{I}^0$, *with* $\mathbb{L}^0$ *given by*

$$\mathbb{L}^0 M \cdot M := \inf \left\{ \int_{Y_2} \mathbb{L}(y)(M + \nabla v) \cdot (M + \nabla v)\, dy : v \in H^1_{\rm per}(Y_2;\mathbb{R}^2) \right\}. \tag{2.4}$$

Note that dropping the restriction that $\Lambda_{\rm per}(\mathbb{L})$ (which is always above $\Lambda(\mathbb{L})$) be positive changes the minimum in (1.4) into an infimum in (2.4).

As announced in the introduction, we are only interested in the kind of two-phase mixture that can lead, in the layering case, to the degeneracy first observed in [8]. Specifically, we assume the existence of 2 isotropic phases $\mathscr{Z}_1, \mathscr{Z}_2$ of $\mathscr{T}_2$ – and of the associated subsets Z_1 and Z_2 of Y_2, or still $Z_1^\#$ and $Z_2^\#$ of $\mathbb{R}^2$ (see notation) – such that

$$\begin{cases} \mathscr{Z}_1, \mathscr{Z}_2 \text{ are open, } C^2 \text{ subsets of } \mathscr{T}_2; \\ \mathscr{Z}_1 \cap \mathscr{Z}_2 = \emptyset \quad \text{and} \quad \bar{\mathscr{Z}_1} \cup \bar{\mathscr{Z}_2} = \mathscr{T}_2; \\ Z_2^\# \text{ is connected;} \\ \mathscr{Z}_1 \text{ has a finite number of connected components in } \mathscr{T}_2. \end{cases} \tag{2.5}$$

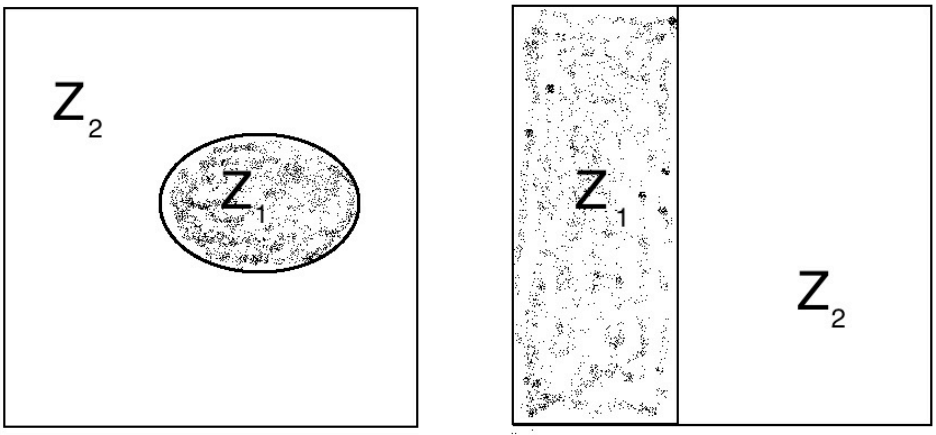

FIGURE 1. Typical allowed micro-geometries: inclusion of the good material or layering.

We then define

$$\left\{\begin{array}{cc} \mathbb{L}(y)M = \lambda(y)\,\mathrm{tr}\,(M)\,I_2 + 2\mu(y)M, & y \in \mathscr{T}_2,\ M \in \mathbb{R}^{2\times 2} \\ \lambda(y) = \lambda_i,\ \mu(y) = \mu_i, & \text{in } \mathscr{Z}_i,\ i = 1,2 \\ 0 < -\lambda_2 - \mu_2 = \mu_1 < \mu_2,\ \lambda_1 + \mu_1 > 0. & \end{array}\right. \tag{2.6}$$

which implies in particular that

$$\lambda_2 + 2\mu_2 > 0,$$

that is that phase 2 is only strongly elliptic ($\lambda_2 + \mu_2 < 0$) while phase 1 is very strongly elliptic ($\lambda_1 + \mu_1 > 0$).

Then the following result, which brings together [2, Theorem 2.2] and [6, Theorem 2.1], holds true:

Theorem 2.3. *Under assumptions,* (2.5), (2.6), $\Lambda(\mathbb{L}) \geq 0$ *and* $\Lambda_{\rm per}(\mathbb{L}) > 0$.

Consequently, Theorem 2.1 can be applied to the setting at hand and we obtain the following

Corollary 2.4. *Set, under assumptions* (2.5), (2.6),

$$\mathscr{J}^{\,\varepsilon}(v) := \int_\Omega \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v\, dx$$

with $\mathbb{L}^0$ *given by* (1.4) *and*

$$\mathscr{J}^{\,0}(v) := \int_\Omega \mathbb{L}^0\nabla v \cdot \nabla v\, dx.$$

Then $\mathscr{J}^{\,\varepsilon} \overset{\Gamma(H^1_0)}{\rightharpoonup} \mathscr{J}^{\,0}$.

Our goal in the next section is to prove that the Corollary remains true when adding to $\mathscr{J}^{\,\varepsilon}$ a zeroth order term of the form

$$\int_\Omega v^2\, dx$$

and replacing the weak topology on bounded sets of $H^1_0(\Omega;\mathbb{R}^2)$ by that on bounded sets of $L^2(\Omega;\mathbb{R}^2)$.

3. The main results

Consider $\mathbb{L}(y)$ given by (2.6) and $\mathbb{L}^0$ given by (2.4). Set, for $v \in L^2(\Omega;\mathbb{R}^2)$,

$$\mathscr{I}^{\,\varepsilon}(v) := \left\{\begin{array}{cc} \displaystyle\int_\Omega \left\{\mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v + v^2\right\} dx, , & v \in H^1_0(\Omega;\mathbb{R}^2) \\ \infty & \text{else.} \end{array}\right.$$

Also define the following two functionals:

$$\mathscr{I}^{\,0}(v) := \left\{\begin{array}{cc} \displaystyle\int_\Omega \left\{\mathbb{L}^0\nabla v \cdot \nabla v + v^2\right\} dx, & v \in H^1_0(\Omega;\mathbb{R}^2) \\ \infty & \text{else;} \end{array}\right. \tag{3.1}$$

and, under the additional assumption that

$$\mathbb{L}^0_{2222} = 0, \tag{3.2}$$

$$\mathscr{I}^{1/2}(v) := \begin{cases} \displaystyle\int_\Omega \left\{\mathbb{L}^0\nabla v\cdot\nabla v + v^2\right\} dx, & v\in\mathscr{X} \\ \infty & \text{else,} \end{cases} \tag{3.3}$$

where, if ν is the exterior normal on $\partial\Omega$,

$$\mathscr{X} := \Big\{v\in L^2(\Omega;\mathbb{R}^2) : v_1\in H^1_0(\Omega),\ v_2\in L^2(\Omega), \\ \frac{\partial v_2}{\partial x_1}\in L^2(\Omega) \text{ and } v_2\nu_1 = 0 \text{ on } \partial\Omega\Big\}. \tag{3.4}$$

Remark 3.1. In (3.3) the cross terms

$$\int_\Omega \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\ dx$$

must be replaced by

$$\int_\Omega \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\ dx$$

so that, provided that $\mathbb{L}^0_{2222} = 0$, which is the case in the specific setting at hand, the expression $\int_\Omega \mathbb{L}^0\nabla u\cdot\nabla u\ dx$ has a meaning for $u\in\mathscr{X}$ and boils down to the classical one when $u\in H^1_0(\Omega;\mathbb{R}^2)$.

Remark 3.2. It is immediately checked that $\mathscr{X}$ is a Hilbert space when endowed with the following inner product:

$$\langle u,v\rangle_{\mathscr{X}} := \int_\Omega u\cdot v\ dx + \int_\Omega \nabla u_1\cdot\nabla v_1\ dx + \int_\Omega \frac{\partial u_2}{\partial x_1}\frac{\partial v_2}{\partial x_1}\ dx.$$

Furthermore, $C^\infty_c(\Omega;\mathbb{R}^2)$ is a dense subspace of $\mathscr{X}$, provided that Ω is C^1. Indeed, take $u\in\mathscr{X}$. The first component u_1 is in $H^1_0(\Omega;\mathbb{R}^2)$. Defining

$$\check{u}_2 := \begin{cases} u_2, & x\in\Omega \\ 0, & \text{else} \end{cases}$$

we have, thanks to the boundary condition in the definition (3.4) of $\mathscr{X}$,

$$\int_{\mathbb{R}^2} \check{u}_2\frac{\partial\varphi}{\partial x_1}\ dx + \int_{\mathbb{R}^2}\frac{\partial u_2}{\partial x_1}\varphi\ dx = 0.$$

for any $\varphi\in C^\infty_c(\mathbb{R}^2)$, that is

$$\check{u}_2, \frac{\partial\check{u}_2}{\partial x_1}\in L^2(\mathbb{R}^2) \quad\text{with}\quad \frac{\partial\check{u}_2}{\partial x_1} = \begin{cases} \dfrac{\partial u_2}{\partial x_1}, & x\in\Omega \\ 0, & \text{else.} \end{cases}$$

Because Ω has a C^1-boundary, we can always assume, thanks to the implicit function theorem, that, at each point $x^0\in\partial\Omega$, there exists a ball $B(x^0,r_{x_0})$ and a C^1-function $f:\mathbb{R}\to\mathbb{R}$ such that

$$\Omega\cap B(x^0,r_{x_0}) = \{(x_1,x_2)\in B(x^0,r_{x_0}) : x_2 > f(x_1)\}$$

or

$$\Omega\cap B(x^0,r_{x_0}) = \{(x_1,x_2)\in B(x^0,r_{x_0}) : x_1 > f(x_2)\}.$$

In the first case, we translate $\check{u}$ in the direction x_2, thereby setting $\check{u}^t_2(x_1,x_2) := \check{u}_2(x_1,x_2-t)$, $t>0$, while, in the second case, we translate $\check{u}_2$ in the direction x_1, thereby setting $\check{u}^t_2(x_1,x_2) := \check{u}_2(x_1-t,x_2)$, $t>0$. This has the effect of creating a new function $\check{u}_2$ which is identically null near $\Omega\cap B(x^0,r_{x_0})$. We then mollify this

function with a mollifier φ^t, with support depending on t, thereby creating yet a new function $\bar{u}_2^t = \varphi^t * \check{u}_2^t \in C_c^\infty(\Omega \cap B(x^0, r_{x_0})$ which will be such that

$$\lim_t \left\{ \|\check{u}_2^t - u\|_{L^2(\Omega\cap B(x^0,r_{x_0}))} + \Big\|\frac{\partial \check{u}_2^t}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\Big\|_{L^2(\Omega\cap B(x^0,r_{x_0}))} \right\} = 0.$$

A partition of unity of the boundary and a diagonalization argument then allow one to construct a sequence of $C_c^\infty(\Omega)$-functions such that the same convergences take place over $L^2(\Omega)$. ¶

We propose to investigate the (sequential) Γ-convergence properties of $\mathscr{I}^\varepsilon$ to $\mathscr{I}^0$ or $\mathscr{I}^{1/2}$ for the weak topology of $L^2(\Omega;\mathbb{R}^2)$.

We will prove the following theorems which address both the case of a laminate and that of a matrix-inclusion type mixture. The first theorem does not completely characterize the Γ-limit to the extent that it is assumed *a priori* that the target field u lies in $H_0^1(\Omega;\mathbb{R}^2)$. By contrast, the second theorem is a complete characterization of the Γ-limit but it does restrict the geometry of laminate-like mixtures to be that made of *bona fide* layers, *i.e.,* straight strips of material.

Theorem 3.3 ("Smooth targets"). *Under assumptions* (2.5), (2.6)*, there exists a subsequence of* $\{\varepsilon\}$ *(not relabeled) such that*

$$\mathscr{I}^\varepsilon \overset{\Gamma(L^2)}{\rightharpoonup} \mathscr{I},$$

where, for $u \in H_0^1(\Omega;\mathbb{R}^2)$, $\mathscr{I}(u) = \mathscr{I}^0(u)$ *given by* (3.1) *and* $\mathbb{L}^0$ *given by* (2.4) *(and, even better, by* (1.4)*).*

Theorem 3.4 ("General targets"). *Under assumptions* (2.5), (2.6)*, then the following holds true:*

(i) *If* $\bar{Z}_1 \subset Y_2$ *(the inclusion case) and if* Ω *is a bounded open Lipschitz domain in* $\mathbb{R}^2$*, then*

$$\mathscr{I}^\varepsilon \overset{\Gamma(L^2)}{\rightharpoonup} \mathscr{I}^0$$

given by (3.1) *and* $\mathbb{L}^0$ *given by* (2.4) *(and, even better, by* (1.4)*);*

(ii) *If* $Z_1 = (0,\theta)\times(0,1)$ *(or* $(0,1)\times(0,\theta)$*) for some* $\theta \in (0,1)$ *(the straight layer case) and if* Ω *is a bounded open* C^1 *domain in* $\mathbb{R}^2$*, then*

$$\mathscr{I}^\varepsilon \overset{\Gamma(L^2)}{\rightharpoonup} \begin{cases} \mathscr{I}^0 & \text{if } \theta \neq 1/2 \\ \mathscr{I}^{1/2} & \text{if } \theta = 1/2 \ \text{ (the Gutiérrez case)} \end{cases}$$

which are given by (3.1), (3.3)*, respectively, and with* $\mathbb{L}^0$ *given by* (2.4) *(and, even better, by* (1.4)*).*

Remark 3.5. In strict parallel with Remark 2.6 in [2], we do not know whether the result of those Theorems still hold true when $H_0^1(\Omega;\mathbb{R}^2)$ is replaced by $H^1(\Omega;\mathbb{R}^2)$ in the Γ-convergence statement. ¶

Remark 3.6. Our geometric assumptions are designed to comprise two cases, that of one, or several inclusions of the good material (phase 1), and that of a layer-like microstructure where phase 1 reaches two parallel sides of the unit cell Y_2, or maybe a combination of both. We made no effort to check that our assumptions are optimal although we conjecture that the statement of Theorem 3.4 will remain true for all two-phase geometries, the degeneracy occurring *only* in the Gutiérrez case. ¶

The rest of this section is devoted to the proofs of Theorem 3.3, 3.4.

3.1. **Proof of Theorem 3.3.** First, because of the compactness of the injection mapping from $H^1_0(\Omega;\mathbb{R}^2)$ into $L^2(\Omega;\mathbb{R}^2)$ and in view of Corollary 2.4,

$$\mathscr{I}^\varepsilon \overset{\Gamma(H^1_0)}{\rightarrow} \mathscr{I}^0$$

with $\mathbb{L}^0$ actually given by (1.4) (a min in lieu of an inf). We want to prove that the same result holds for the weak L^2-topology, at least for a subsequence of $\{\varepsilon\}$. By a classical compactness result we can assert the existence of a subsequence of $\{\varepsilon\}$ such that the Γ- lim exists. Our goal is to show that that limit, denoted by $\mathscr{I}(u)$, is precisely $\mathscr{I}^0(u)$ when $u \in H^1_0(\Omega;\mathbb{R}^2)$. Clearly, the Γ- lim sup will a fortiori hold in that topology, *provided that the target field* $u \in H^1_0(\Omega;\mathbb{R}^2)$. It thus remains to address the proof of the Γ- lim inf which is what the rest of this subsection is about.

To that end and in the spirit of [8], we add an integrated null Lagrangian to the energy so as to render the energy density pointwise nonnegative. Thus we set, for any $M \in \mathbb{R}^{2\times 2}$,

$$\mathbb{K}_j M := \mathbb{L}_j M + 2\mu_1 \mathrm{cof}\,(M) = \lambda_j \mathrm{tr}\,(M) I_2 + \mu_j (M + M^T) + 2\mu_1 \mathrm{cof}\,(M)\,, \quad j = 1, 2$$

(thereby taking c at the end of the introduction to be $2\mu_1$) so that

$$\mathbb{K}_j M \cdot M = \mathbb{L}_j M \cdot M + 4\mu_1 \det(M) \geq 0, \quad j = 1, 2$$

and define

$$\mathbb{K}(y) \equiv \mathbb{K}_j \text{ in } \mathscr{Z}_j, \quad j = 1, 2.$$

Because the determinant is a null Lagrangian, for $v \in H^1_0(\Omega;\mathbb{R}^2)$,

$$\mathscr{I}^\varepsilon(v) = \int_\Omega \left\{ \mathbb{K}(x/\varepsilon) \nabla v \cdot \nabla v + v^2 \right\} dx$$

Consider a sequence $\{u^\varepsilon\}_\varepsilon$ converging weakly in $L^2(\Omega;\mathbb{R}^2)$ to $u \in L^2(\Omega;\mathbb{R}^2)$. Then, for a subsequence (still indexed by ε), we are at liberty to assume that $\liminf \mathscr{I}^\varepsilon(u^\varepsilon)$ is actually a limit. The Γ- lim inf inequality is trivial if that limit is ∞ so that we can assume henceforth that, for some $\infty > C > 0$,

$$\mathscr{I}^\varepsilon(u^\varepsilon) \leq C. \tag{3.5}$$

Further, according to *e.g.* [1, Theorem 1.2], a subsequence (still indexed by ε) of that sequence two-scale converges to some $u^0(x,y) \in L^2(\Omega \times \mathscr{T}_2;\mathbb{R}^2)$. In other words,

$$u^\varepsilon \rightrightarrows u^0. \tag{3.6}$$

Also, in view of (3.5) and because it is easily shown that

$$\mathbb{K}(y) \text{ is a nonnegative as a quadratic form} \tag{3.7}$$

while clearly all its components are bounded, for yet another subsequence (not relabeled),

$$\mathbb{K}(x/\varepsilon)\nabla u^\varepsilon \rightrightarrows H(x,y) \quad \text{with } H \in L^2(\Omega \times Y_2;\mathbb{R}^{2\times 2}), \tag{3.8}$$

and also, for future use,

$$\mathbb{K}^{\frac{1}{2}}(x/\varepsilon)\nabla u^\varepsilon \rightrightarrows S(x,y) \quad \text{with } S \in L^2(\Omega \times Y_2;\mathbb{R}^{2\times 2}). \tag{3.9}$$

In particular,

$$\varepsilon\,\mathbb{K}(x/\varepsilon)\nabla u^\varepsilon \rightleftharpoons 0. \tag{3.10}$$

Take $\Phi(x,y)\in C_c^\infty(\Omega\times\mathscr{T}_2;\mathbb{R}^{2\times 2})$ with compact support in $\Omega\times\mathscr{Z}_1$. From (3.10) we get, with obvious notation, that

$$0=-\lim_{\varepsilon\to 0}\int_\Omega \varepsilon\,\mathbb{K}(x/\varepsilon)\nabla u^\varepsilon\cdot\Phi(x,x/\varepsilon)\,dx=$$

$$\lim_{\varepsilon\to 0}\sum_{ijkh}\int_\Omega (\mathbb{K})_{ijkh}(x/\varepsilon)\,u_k^\varepsilon\,\frac{\partial\Phi_{ij}}{\partial y_h}(x,x/\varepsilon)\,dx=$$

$$\sum_{ijkh}\int_{\Omega\times Z_1}(\mathbb{K}_1)_{ijkh}\,u_k^0(x,y)\,\frac{\partial\Phi_{ij}}{\partial y_h}\,dx\,dy=-\int_{\Omega\times Z_1}\big(\mathbb{K}_1\nabla_y u^0(x,y)\big)\cdot\Phi(x,y)\,dx\,dy,$$

so that

$$\mathbb{K}_1\nabla_y u^0(x,y)\equiv 0\quad\text{in }\Omega\times Z_1^\#, \tag{3.11}$$

and similarly

$$\mathbb{K}_2\nabla_y u^0(x,y)\equiv 0\quad\text{in }\Omega\times Z_2^\#. \tag{3.12}$$

In view of the explicit expressions for $\mathbb{K}_i$ (3.11), (3.12) imply that

$$\lambda_i\Big(\frac{\partial u_1^0}{\partial y_1}+\frac{\partial u_2^0}{\partial y_2}\Big)+2\mu_i\frac{\partial u_1^0}{\partial y_1}+2\mu_1\frac{\partial u_2^0}{\partial y_2}=0$$

$$\lambda_i\Big(\frac{\partial u_1^0}{\partial y_1}+\frac{\partial u_2^0}{\partial y_2}\Big)+2\mu_i\frac{\partial u_2^0}{\partial y_2}+2\mu_1\frac{\partial u_1^0}{\partial y_1}=0$$

$$\mu_i\Big(\frac{\partial u_1^0}{\partial y_2}+\frac{\partial u_2^0}{\partial y_1}\Big)-2\mu_1\frac{\partial u_1^0}{\partial y_2}=0$$

$$\mu_i\Big(\frac{\partial u_1^0}{\partial y_2}+\frac{\partial u_2^0}{\partial y_1}\Big)-2\mu_1\frac{\partial u_2^0}{\partial y_1}=0.$$

So, in phase 1, that is on $Z_1^\#$, using (2.6) we get

$$\frac{\partial u_1^0}{\partial y_1}+\frac{\partial u_2^0}{\partial y_2}=0,\quad \frac{\partial u_2^0}{\partial y_1}-\frac{\partial u_1^0}{\partial y_2}=0 \tag{3.13}$$

while in phase 2, that is on $Z_2^\#$, still using (2.6) we get

$$\frac{\partial u_1^0}{\partial y_1}=\frac{\partial u_2^0}{\partial y_2},\quad \frac{\partial u_2^0}{\partial y_1}=\frac{\partial u_1^0}{\partial y_2}=0. \tag{3.14}$$

From (3.13) we conclude that, in phase 1,

$$\triangle_y u_1^0=\triangle_y u_2^0=0. \tag{3.15}$$

Step 1 – u^0 does not oscillate. We now exploit the two previous set of relations under the micro-geometric assumptions of Theorem 3.3 to demonstrate that

$$u^0(x,y)=u(x)\text{ is independent of }y, \tag{3.16}$$

where, thanks to (3.6),

$$u^\varepsilon\rightharpoonup u,\ \text{weakly in } L^2(\Omega;\mathbb{R}^2). \tag{3.17}$$

We first notice that, in view of (3.14) and of the connectedness of phase 2,

$$u_1^0 = \alpha(x)\, y_1 + \beta(x), \quad u_2^0 = \alpha(x)\, y_2 + \gamma(x), \quad y \in Z_2^{\#},$$

for some functions $\alpha(x), \beta(x), \gamma(x)$. By Y_2-periodicity of u_i^0, $\alpha(x) \equiv 0$. Thus $\nabla_y u^0 = 0$, or equivalently,

$$u^0(x,y) = u(x), \ y \in Z_2^{\#}, \tag{3.18}$$

for some $u \in L^2(\Omega;\mathbb{R}^2)$.

Consider $\Phi \in C^1_{\rm per}(Y_2;\mathbb{R}^{2\times 2})$ with

$$\sum_{ijh} (\mathbb{K}_1 - \mathbb{K}_2)_{ijkh} \Phi_{ij}(y) \nu_h(y) = 0 \quad \text{on } \partial \mathscr{Z}_1, \tag{3.19}$$

that condition being necessary for $\mathrm{div}_y(\mathbb{K}^T(y)\Phi(y))$ to be an admissible test function for two-scale convergence. In (3.19) $\nu(y)$ denotes the exterior normal to Z_2 at y.

In view of (3.10), (3.18), (3.19), we get that, for any $\varphi \in C_c^\infty(\Omega; C^\infty_{\rm per}(Y_2))$,

$$\begin{aligned}
0 = &- \lim_{\varepsilon\to 0} \int_\Omega \varepsilon\, \mathbb{K}(x/\varepsilon) \nabla u_\varepsilon \cdot \varphi(x, x/\varepsilon) \Phi(x/\varepsilon)\, dx \\
&= \lim_{\varepsilon\to 0} \int_\Omega \Big[\frac{\partial}{\partial y_h} \big\{ (\mathbb{K}(y))_{ijkh} \varphi(x,y) \Phi_{ij}(y) \big\} \Big] (x, x/\varepsilon)\, (u_\varepsilon)_k\, dx = \\
&\quad \sum_{ijkh} \int_{\Omega\times Z_1} \frac{\partial}{\partial y_h} \big\{ (\mathbb{K}_1)_{ijkh} \varphi(x,y) \Phi_{ij}(y) \big\} u_k^0(x,y)\, dx\, dy + \\
&\quad \sum_{ijkh} \int_{\Omega\times Z_2} \frac{\partial}{\partial y_h} \big\{ (\mathbb{K}_2)_{ijkh} \varphi(x,y) \Phi_{ij}(y) \big\} u_k(x)\, dx\, dy = \\
&\quad \sum_{ijkh} \int_{\Omega\times Z_1} \frac{\partial}{\partial y_h} \big\{ (\mathbb{K}_1)_{ijkh} \varphi(x,y) \Phi_{ij}(y) \big\} u_k^0(x,y)\, dx\, dy + \\
&\qquad \sum_{ijkh} \int_{\Omega\times(\partial Z_1 \cap Y_2)} (\mathbb{K}_1)_{ijkh} u_k(x) \nu_h(y) \varphi(x,y) \Phi_{ij}(y)\, dx\, d\mathscr{H}^1_y.
\end{aligned}$$

Set $v^0(x,y) := u^0(x,y) - u(x)$. Then,

$$\sum_{ijkh} \int_{\Omega\times Z_1} \frac{\partial}{\partial y_h} \big\{ (\mathbb{K}_1)_{ijkh} \varphi(x,y) \Phi_{ij}(y) \big\} v_k^0(x,y)\, dx\, dy = 0,$$

while, thanks to (3.11), $\mathbb{K}_1 \nabla_y v^0 = 0$. This implies that $\sum_{kh} (\mathbb{K}_1)_{ijkh} v_k^0(x,y) \nu_h(y)$ – which is meaningful as an element of $L^2(\Omega; H^{-1/2}(\partial Z_1))$ – is such that, for any $\varphi \in C_c^\infty(\Omega; C^\infty_{\rm per}(Y_2))$ and any Φ satisfying (3.19),

$$\sum_{ij} \int_\Omega \Big\langle \sum_{kh} (\mathbb{K}_1)_{ijkh} v_k^0(x,y) \nu_h(y), \varphi(x,y) \Phi_{ij}(y) \Big\rangle_{H^{-1/2}(\partial Z_1), H^{1/2}(\partial Z_1)} dx = 0. \tag{3.20}$$

Now simple algebra using the explicit expression for $\mathbb{K}_1, \mathbb{K}_2$ as well as (2.6) would show that, for any $\xi \in \mathbb{R}^2$ and any $\nu \in S^1$, one can choose a matrix Φ such that

$$\sum_{ijh} (\mathbb{K}_1)_{ijkh} \Phi_{ij} \nu_h = \sum_{ijh} (\mathbb{K}_2)_{ijkh} \Phi_{ij} \nu_h = \xi_k, \quad k = 1, 2, \tag{3.21}$$

so that, in particular, (3.19) can always be met, provided that each connected component of $\mathscr{Z}_1$ has a C^2 boundary because the normal $\nu(y)$ is then a C^1-function of $y \in \partial\mathscr{Z}_1$ so that one can define $\Phi(y)$ satisfying (3.21) as a C^1 function on $\partial\mathscr{Z}_1$, hence by *e.g.* Whitney's extension theorem as a C^1 function on $\mathscr{T}_2$.

Consider a connected component $Z^\#$ of phase 1 in $\mathbb{R}^2$. Then, for almost every $x \in \Omega$,

$$v^0(x,y) \text{ has a well defined trace in } H^{-1/2}(\partial Z^\#;\mathbb{R}^2).$$

In view of (3.20), (3.21), of the arbitrariness of φ and of that of ξ, we conclude that that trace satisfies

$$v^0(x,\cdot) = 0 \text{ on } \partial Z^\#. \tag{3.22}$$

Fix x. According to (3.13), there exists a potential $\zeta_x \in H^1(Z \cap (-R,R)^2)$ for any $R > 0$, such that

$$v^0(x,y) = R^\perp \nabla \zeta_x(y)$$

and

$$\triangle_y \zeta_x = 0 \text{ in } Z^\#.$$

Further, in view of (3.22),

$$R^\perp \nabla \zeta_x \cdot \nu = \nabla \zeta_x \cdot \nu^\perp = 0 \text{ on } \partial Z^\#$$

so that ζ_x is constant on each connected component of $\partial Z^\#$. Thus, by elliptic regularity $\zeta_x \in H^2(Z \cap (-R,R)^2)$ for any $R > 0$, hence $v^0 \in H^1(Z \cap (-R,R)^2)$ for any $R > 0$. In view of (3.15), (3.22), we conclude that $v^0 \equiv 0$, hence (3.16).

Step 2 – Identification of the Γ- $\liminf$. Consider $\Phi \in L^2_{\rm per}(Y_2;\mathbb{R}^{2\times 2})$ such that

$$\operatorname{div}\big(\mathbb{K}^{\frac{1}{2}}(y)\Phi(y)\big) = 0 \quad \text{in } \mathbb{R}^2, \tag{3.23}$$

or equivalently,

$$\int_{Y_2} \mathbb{K}^{\frac{1}{2}}(y)\Phi(y) \cdot \nabla\psi(y)\, dy = 0 \quad \forall\, \psi \in H^1_{\rm per}(Y_2;\mathbb{R}^2),$$

and also consider $\varphi \in C^\infty(\bar\Omega)$.

Then, since $u^\varepsilon \in H^1_0(\Omega;\mathbb{R}^2)$ and in view of (3.23),

$$\int_\Omega \varphi(x)\, \mathbb{K}^{\frac{1}{2}}(x/\varepsilon)\nabla u^\varepsilon \cdot \Phi(x/\varepsilon)\, dx = -\sum_{ijkh} \int_\Omega u^\varepsilon_k \cdot \mathbb{K}^{\frac{1}{2}}_{ijkh}(x/\varepsilon)\Phi_{ij}(x/\varepsilon)\frac{\partial\varphi}{\partial x_h}(x)\, dx.$$

Recalling (3.6), (3.9), we can pass to the two-scale limit in the previous expression and obtain, thanks to (3.16),

$$\int_{\Omega\times Y_2} \varphi(x) S(x,y) \cdot \Phi(y)\, dx\, dy = -\sum_{ijkh} \int_{\Omega\times Y_2} u_k(x) \cdot \mathbb{K}^{\frac{1}{2}}_{ijkh}(y)\Phi_{ij}(y)\frac{\partial\varphi}{\partial x_h}(x)\, dx\, dy. \tag{3.24}$$

Assuming henceforth that $u \in H^1(\Omega;\mathbb{R}^2)$, we obtain from (3.24) that

$$\int_{\Omega\times Y_2} S(x,y) \cdot \Phi(y)\, \varphi(x)\, dx\, dy = \int_{\Omega\times Y_2} \mathbb{K}^{\frac{1}{2}}(y)\nabla_x u(x) \cdot \Phi(y)\, \varphi(x)\, dx\, dy.$$

By density, the result still holds with the test functions $\varphi(x)\Phi(y)$ replaced by the set of $\Psi(x,y) \in L^2(\Omega; L^2_{\rm per}(Y_2;\mathbb{R}^{2\times 2}))$ such that

$$\operatorname{div}_y\big(\mathbb{K}^{\frac{1}{2}}(y)\Psi(x,y)\big) = 0 \quad \text{in } \mathbb{R}^2,$$

or equivalently, due to the symmetry of $\mathbb{K}(y)$,

$$\int_{\Omega\times Y_2} \Psi(x,y)\cdot \mathbb{K}^{\frac{1}{2}}(y)\nabla_y v(x,y)\,dy = 0 \quad \forall\, v\in L^2(\Omega; H^1_{\rm per}(Y_2;\mathbb{R}^2)).$$

Hence the $L^2(\Omega; L^2_{\rm per}(Y_2;\mathbb{R}^{2\times 2}))$-orthogonal to that set is the L^2-closure of

$$\mathscr{K}_\nabla := \left\{ \mathbb{K}^{\frac{1}{2}}(y)\nabla_y v(x,y) : v\in L^2(\Omega; H^1_{\rm per}(Y_2;\mathbb{R}^2))\right\}.$$

Thus,

$$S(x,y) = \mathbb{K}^{\frac{1}{2}}(y)\nabla_x u(x) + \xi(x,y)$$

for some ξ in the closure of $\mathscr{K}_\nabla$. Thus, there exists a sequence

$$v_n \in L^2(\Omega; H^1_{\rm per}(Y_2;\mathbb{R}^2))$$

such that $\mathbb{K}^{\frac{1}{2}}(y)\nabla_y v_n \to \xi$ strongly in $L^2(\Omega; L^2_{\rm per}(Y_2;\mathbb{R}^{2\times 2}))$.

We now appeal to [1, Proposition 1.6] which yields

$$\liminf_{\varepsilon\to 0} \|\mathbb{K}^{\frac{1}{2}}(x/\varepsilon)\nabla u^\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2}))} \geq \|S\|^2_{L^2(\Omega\times Y_2;\mathbb{R}^{2\times 2})} = \lim_n \|\mathbb{K}^{\frac{1}{2}}(y)\nabla_x u(x) + \mathbb{K}^{\frac{1}{2}}(y)\nabla_y v_n\|^2_{L^2(\Omega\times Y_2;\mathbb{R}^2)}. \tag{3.25}$$

But recall that

$$\|\mathbb{K}^{\frac{1}{2}}(x/\varepsilon)\nabla u^\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2}))} = \int_\Omega \mathbb{K}(x/\varepsilon)\nabla u^\varepsilon\cdot\nabla u^\varepsilon\,dx = \int_\Omega \mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\cdot\nabla u^\varepsilon\,dx$$

because the determinant is a null Lagrangian.

Thus, from (3.25) and by weak L^2-lower semi-continuity of $\|u^\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)}$ we conclude that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathscr{I}^\varepsilon(u^\varepsilon) \geq \\ \lim_n \int_{\Omega\times Y_2} \mathbb{K}(y)(\nabla_x u(x) + \nabla_y v_n(x,y))\cdot(\nabla_x u(x) + \nabla_y v_n(x,y))\,dx\,dy + \int_\Omega |u|^2\,dx \geq \\ \inf\left\{ \int_{\Omega\times Y_2} \mathbb{K}(y)(\nabla_x u(x) + \nabla_y v(y))\cdot(\nabla_x u(x) + \nabla_y v(y))\,dx\,dy : v\in H^1(\mathscr{T}_2;\mathbb{R}^2)\right\} \\ + \int_\Omega |u|^2\,dx.\end{aligned} \tag{3.26}$$

In the light of the definition (2.4) for $\mathbb{L}^0$, we finally get

$$\liminf_{\varepsilon\to 0} \mathscr{I}^\varepsilon(u^\varepsilon) \geq \int_\Omega \left\{ \mathbb{L}^0\nabla_x u\cdot\nabla_x u + |u|^2\right\} dx$$

provided that $u\in H^1(\Omega;\mathbb{R}^2)$, hence, *a fortiori* provided that $u\in H^1_0(\Omega;\mathbb{R}^2)$.

The proof of Theorem 3.3 is complete.

3.2. **Proof of Theorem 3.4.** Recall that, in the proof of Theorem 3.3 we were at liberty to assume that

$$\mathscr{I}^\varepsilon(u^\varepsilon) \leq C < \infty,$$

otherwise the $\Gamma-\liminf$ inequality is trivially verified. Consequently, if we can show that, under that condition, the target function u is in $H^1_0(\Omega;\mathbb{R}^2)$, then we will be done as remarked at the onset of Subsection 3.1. Such will be the case except when dealing with straight layers (case (ii)) under the condition that $\theta = 1/2$. In that case we will have to show that, for those target fields u that are not in $H^1_0(\Omega;\mathbb{R}^2)$ a recovery sequence for the Γ- $\limsup$ (in)equality can be obtained by density.

Returning to (3.24), setting

$$N_{kh} := \sum_{ij} \int_{Y_2} \mathbb{K}^{\frac{1}{2}}_{ijkh}(y)\Phi_{ij}(y)\,dy \tag{3.27}$$

and varying φ in $C^\infty_c(\Omega)$, we conclude that

$$N \cdot \nabla u \in L^2(\Omega). \tag{3.28}$$

We now remark that $\mathbb{K}(y)$, a symmetric mapping on $\mathbb{R}^{2\times 2}$, has for eigenvalues $2(\lambda(y)+\mu(y)+\mu_1)$, $-2\mu_1$ and $2(\mu(y)-\mu_1)$ with eigenspaces respectively generated by

$$I_2, \quad R^\perp$$

and, for the last eigenvalue, by

$$G := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad H := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

Consequently, its kernel for $y \in Z_2$ is

$$\mathrm{Ker}\left(\mathbb{K}(y)\right) = \mathrm{Ker}\left(\mathbb{K}_2\right) := \{\gamma I_2,\ \gamma \text{ arbitrary in } \mathbb{R}\}, \tag{3.29}$$

while its kernel for $y \in Z_1$ is

$$\mathrm{Ker}\left(\mathbb{K}(y)\right) = \mathrm{Ker}\left(\mathbb{K}_1\right) := \left\{\left(\begin{smallmatrix} \alpha & \beta \\ \beta & -\alpha \end{smallmatrix}\right) : \alpha, \beta \text{ arbitrary in } \mathbb{R}\right\}. \tag{3.30}$$

Step 1 – Case(i). First assume that $\bar{Z}_1 \subset Y_2$ and that $M \in \mathbb{R}^{2\times 2}_s$. We then define

$$\Psi(y) := \frac{1}{2}\left\{My\cdot y - \varphi(y-\kappa)M(y-\kappa)\cdot(y-\kappa)\right\} \quad \text{for any } y \in Y_2+\kappa,\ \kappa \in \mathbb{Z}^2,$$

with $\varphi \in C^2_c(Y_2)$, $\varphi \equiv 1$ in Z_1. Then clearly, $\nabla\Psi - My \in H^1_{\mathrm{per}}(Y_2;\mathbb{R}^2)$. Further, $\nabla\Psi = M\kappa$ in $Z_1+\kappa$ hence $\nabla^2\Psi \equiv 0$ in $Z_1+\kappa$ while $\nabla^2\Psi R^\perp \in \mathrm{Ker}^\perp\left(\mathbb{K}_2\right)$ thus belongs to the range of $\mathbb{K}_2$ in $Z_2+\kappa$.

It is thus meaningful to define $\Phi(y) := \mathbb{K}^{-\frac{1}{2}}(y)\big(\nabla^2\Psi(y)\,R^\perp\big)$ where $\mathbb{K}^{-\frac{1}{2}}$ is the pseudo-inverse of $\mathbb{K}^{\frac{1}{2}}$ (see the notation at the close of the introduction).

We get

$$\int_{Y_2} \mathbb{K}^{\frac{1}{2}}(y)\Phi(y)\,dy = \int_{Y_2} \nabla^2\Psi(y)\,R^\perp\,dy = MR^\perp,$$

while Φ satisfies (3.23) since for any $v \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2)$ with periodic gradient,

$$\mathrm{div}\left(\nabla v\,R^\perp\right) = 0 \quad \text{in } \mathbb{R}^2,$$

or equivalently,

$$\int_{Y_2} \nabla v(y)\,R^\perp \cdot \nabla\psi(y)\,dy = 0 \quad \forall\,\psi \in H^1_{\mathrm{per}}(Y_2;\mathbb{R}^2).$$

We finally obtain that $MR^\perp \cdot \nabla u \in L^2(\Omega)$. Since, when M spans $\mathbb{R}^{2\times 2}_s$, $N := MR^\perp$ spans the set of all 2×2 matrices with 0-trace, we infer from (3.28) that

$$\frac{\partial u_1}{\partial x_2},\ \frac{\partial u_2}{\partial x_1},\ \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \text{ are in } L^2(\Omega).$$

This is equivalent to stating that $\mathbf{E}(R^\perp u) \in L^2(\Omega;\mathbb{R}^{2\times 2}_s)$. Since Ω is C^1 – Lipschitz would suffice here – Korn's inequality allows us to conclude that $R^\perp u \in H^1(\Omega;\mathbb{R}^2)$, hence that

$$u \in H^1(\Omega;\mathbb{R}^2) \tag{3.31}$$

in that case.

Since, for an arbitrary trace-free matrix N, we can choose Φ constrained by (3.23) so that (3.27) is satisfied, then actually

$$u \in H^1_0(\Omega;\mathbb{R}^2). \tag{3.32}$$

Indeed, take $x_0 \in \partial\Omega$ to be a Lebesgue point for $u\lfloor\partial\Omega$ – which lies in particular in $L^2_{\mathscr{H}^1}(\partial\Omega;\mathbb{R}^2)$ – as well as for $\nu(x_0)$, the exterior normal to Ω at x_0. Then take an arbitrary trace-free N and the associated Φ.

By (3.31) we already know that $u \in H^1(\Omega;\mathbb{R}^2)$, so that (3.24) reads as

$$\int_{\Omega\times Y_2} \varphi(x) S(x,y)\cdot \Phi_{x_0}(y)\,dx\,dy = \int_\Omega N\cdot\nabla u(x)\,\varphi(x)\,dx - \int_{\partial\Omega} N\nu(x)\cdot u(x)\,\varphi(x)\,d\mathscr{H}^1.$$

But, taking first $\varphi \in C^\infty_c(\Omega)$ and remarking that, in such a case, the first two integrals are equal and bounded by a constant times $\|\varphi\|_{L^2(\Omega)}$, we immediately conclude that, for any $\varphi \in C^\infty(\bar\Omega)$,

$$\int_{\partial\Omega} N\nu(x)\cdot u(x)\,\varphi(x)\,d\mathscr{H}^1 = 0.$$

Thus, $N\nu(x)\cdot u(x) = 0$ $\mathscr{H}^1$-a.e. on $\partial\Omega$, hence, since x_0 is a Lebesgue point, $N\nu(x_0)\cdot u(x_0) = 0$ from which it is immediately concluded that $u_i(x_0) = 0,\ i = 1,2$, hence (3.32).

But, in such a case we can apply Theorem 3.3 which thus delivers the Γ-limit.

Step 2 – Case (ii). Assume now that Z_1 is a straight layer, that is that there exists $0 < \theta < 1$ such that $(0,\theta)\times(0,1) = Z_1 \cap \mathring{Y}$.

For an arbitrary matrix $M \in \mathbb{R}^{2\times 2}$ define $v(y)$ as

$$v(y) := My + \left(\int_0^{y_1} [\chi(t) - \theta]\,dt\right)\xi,\ \xi \in \mathbb{R}^2,$$

where $\chi(y)$ is the characteristic function of phase 1 with volume fraction

$$\int_{Y_2} \chi(y)\ dy := \theta.$$

Then, $v(y) - My$ is Y_2-periodic and

$$\nabla v(y) = \chi(y_1)(M + (1-\theta)\xi\otimes e_1) + (1-\chi(y_1))(M - \theta\xi\otimes e_1).$$

According to (3.30), (3.29), for $\nabla v(y)R^\perp$ to be in the image of $\mathbb{K}(y)$ we must have both

$$(M + (1-\theta)\xi\otimes e_1)R^\perp\cdot G = (M + (1-\theta)\xi\otimes e_1)R^\perp\cdot H = 0$$

i.e., $(1-\theta)\xi_1 = M_{22} - M_{11}$, $(1-\theta)\xi_2 = -M_{12} - M_{21}$ and

$$(M - \theta\xi \otimes e_1)R^\perp \cdot I_2 = 0,$$

i.e., $\theta\xi_2 = -M_{12} + M_{21}$.

Since ξ can be arbitrary, this imposes as sole condition on M that

$$(2\theta - 1)M_{12} + M_{21} = 0.$$

Then,

$$\int_{Y_2} \nabla v(y) R^\perp \, dy = MR^\perp = \begin{pmatrix} M_{12} & -M_{11} \\ M_{22} & (2\theta-1)\, M_{12} \end{pmatrix} = N_\theta$$

where

$$N_\theta = \begin{pmatrix} a & c \\ b & (2\theta-1)a \end{pmatrix}, \quad \text{with} \quad \begin{cases} a, b, c \text{ arbitrary, if } \theta \neq 1/2 \\ d = 0, \text{ if } \theta = 1/2. \end{cases}$$

In view of (3.28), we obtain that

$$\frac{\partial u_1}{\partial x_2}, \ \frac{\partial u_2}{\partial x_2}, \ \frac{\partial u_1}{\partial x_1} + (1-2\theta)\frac{\partial u_2}{\partial x_2} \text{ are in } L^2(\Omega),$$

or equivalently,

$$\mathbf{E}(P_\theta u) \in L^2(\Omega; \mathbb{R}^{2\times 2}) \quad \text{with} \quad P_\theta := \begin{pmatrix} 0 & 1-2\theta \\ 1 & 0 \end{pmatrix}.$$

Using Korn's inequality once again, we thus conclude that $u \in H^1(\Omega; \mathbb{R}^2)$, except when $\theta = 1/2$ in which case $\dfrac{\partial u_2}{\partial x_2}$ might not be in $L^2(\Omega)$.

In view of (3.28), we thus conclude that all partial derivatives of u are in $L^2(\Omega)$ except when $\theta = 1/2$ in which case $\dfrac{\partial u_2}{\partial x_2}$ might not be.

Remark 3.7. Actually, when $\theta = 1/2$, then all Φ's that are such that (3.23) is satisfied produce, through (3.27), a matrix M with $M_{21} = 0$, hence a matrix $N := MR^\perp$ such that $N_{22} = 0$.

Indeed, the existence of Φ is equivalent to that of $v(y) = My + w(y)$ with $w \in H^1_{\rm per}(Y_2; \mathbb{R}^2)$ such that

$$\nabla v(y) R^\perp = \mathbb{K}^{\frac{1}{2}}(y_1)\Phi(y)$$

which implies that, for a.e. $y_1 \in (0,1)$,

$$\mathbb{K}^{\frac{1}{2}}(y_1) \int_0^1 \Phi(y_1, t) \, dt = \int_0^1 \nabla v(y_1, t) R^\perp \, dt.$$

In view of (3.30), (3.29), the last relation yield in particular that

$$\begin{cases} v_1(y_1, 1) - v_1(y_1, 0) = -\displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2) \, dy_2, \ 0 \le y_1 \le 1/2 \\ v_1(y_1, 1) - v_1(y_1, 0) = +\displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2) \, dy_2, \ 1/2 \le y_1 \le 1, \end{cases}$$

or still, since $v(y) = My + w(y)$ with w Y_2-periodic,

$$\begin{cases} M_{12} = -\displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2)\ dy_2,\ 0 \le y_1 \le 1/2 \\ M_{12} = +\displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2)\ dy_2,\ 1/2 \le y_1 \le 1, \end{cases}$$

But then

$$M_{21} = \int_{Y_2} \frac{\partial v_2}{\partial y_1}(y_1, y_2)\ dy_1 dy_2 = 0.$$

¶

Then, through an argument identical to that used in case (i), we find that, for x_0 Lebesgue point for $u_1 \lfloor \partial\Omega$ (and for $u_2 \lfloor \partial\Omega$ as well if $\theta \neq 1/2$), $u_1(x_0) = 0$ (and $u_2(x_0) = 0$ if $\theta \neq 1/2$) while, if $\theta = 1/2$, $u_2\nu_1$, which is well defined as an element of $H^{-\frac{1}{2}}(\Omega)$, satisfies $u_2\nu_1 = 0$.

So, here again, we can apply Theorem 3.3 provided that $\theta \neq 1/2$. It thus remains to compute the Γ-limit in case (ii) when $\theta = 1/2$. This is the object of the last step below.

Step 3 – Identification of the Γ-limit – case (ii) – $\theta = 1/2$; the Gutiérrez case. As far as the Γ- lim sup is concerned there is nothing to prove once again, because, as already stated at the onset of Subsection 3.1 we know the existence of a recovery sequence for any target field $u \in H^1_0(\Omega; \mathbb{R}^2)$. But, according to Remark 3.2, $H^1_0(\Omega; \mathbb{R}^2)$ is a fortiori dense in $\mathscr{X}$. So any element $u \in \mathscr{X}$ can be in turn viewed as the limit in the topology induced by the inner product $\langle , \rangle_{\mathscr{X}}$ of a sequence $u^p \in H^1_0(\Omega; \mathbb{R}^2)$. Since, as noted in Remark 3.1, $\partial u^p_2/\partial x_2$ does not enter the expression

$$\int_\Omega \mathbb{L}^0 \nabla u^p \cdot \nabla u^p\ dx,$$

we immediately get that

$$\lim_p \int_\Omega \mathbb{L}^0 \nabla u^p \cdot \nabla u^p\ dx = \int_\Omega \mathbb{L}^0 \nabla u \cdot \nabla u\ dx.$$

A diagonalization process concludes the argument.

Consider now, for $u \in \mathscr{X}$, a sequence $u^\varepsilon \in H^1_0(\Omega; \mathbb{R}^2)$ such that $u^\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega; \mathbb{R}^2)$. We revisit Step 2 in the proof of Theorem 3.3 in Subsection 3.1, taking into account Remark 3.7. Since, because of that remark $N_{22} = 0$, (3.24) now reads as

(3.33)
$$\int_{\Omega\times Y_2} \varphi(x) S(x,y) \cdot \Phi(y)\, dx\, dy = -\sum_{\substack{ijkh \\ (k,h) \neq (2,2)}} \int_{\Omega\times Y_2} u_k(x)\, \mathbb{K}^{\frac{1}{2}}_{ijkh}(y) \Phi_{ij}(y) \frac{\partial \varphi}{\partial x_h}(x)\, dx\, dy.$$

The rest of the argument goes through exactly as in Step 2, yielding, in lieu of (3.26),

$$\begin{aligned}(3.34)\quad \liminf_{\varepsilon\to 0}\mathscr{I}^\varepsilon(u^\varepsilon)\geq\\ \inf\left\{\int_{\Omega\times Y_2}\mathbb{K}(y)(\nabla'_x u(x)+\nabla_y v(y))\cdot(\nabla'_x u(x)+\nabla_y v(y))\,dx\,dy : v\in H^1(\mathscr{T}_2;\mathbb{R}^2)\right\}\\ +\int_\Omega |u|^2\,dx\\ =\inf\left\{\int_{\Omega\times Y_2}\mathbb{L}(y)(\nabla'_x u(x)+\nabla_y v(y))\cdot(\nabla'_x u(x)+\nabla_y v(y))\,dx\,dy : v\in H^1(\mathscr{T}_2;\mathbb{R}^2)\right\}\\ +4\mu_1\int_\Omega \det\nabla'_x u\;dx+\int_\Omega|u|^2\,dx\\ =\int_\Omega \mathbb{L}^0\nabla'_x u\cdot\nabla'_x u\;dx+4\mu_1\int_\Omega\det\nabla'_x u\;dx+\int_\Omega|u|^2\,dx\end{aligned}$$

with $\nabla'_x u:=\nabla_x u-\dfrac{\partial u_2}{\partial x_2}e_2\otimes e_2$.

It now suffices to remark that, in this specific setting and because $-\lambda_2-\mu_2=\mu_1$, the precise expression for $\mathbb{L}^0$ in the basis (e_1,e_2) is as follows (see [8]):

(3.35)

$$\mathbb{L}^0_{1111}=\frac{2}{\dfrac{1}{\lambda_1+2\mu_1}+\dfrac{1}{\lambda_2+2\mu_2}},\quad \mathbb{L}^0_{1122}=\mathbb{L}^0_{2211}=\frac{\dfrac{\lambda_1}{\lambda_1+2\mu_1}+\dfrac{\lambda_2}{\lambda_2+2\mu_2}}{\dfrac{1}{\lambda_1+2\mu_1}+\dfrac{1}{\lambda_2+2\mu_2}}=-2\mu_1,$$

(3.36)

$$\mathbb{L}^0_{1212}=\mathbb{L}^0_{1221}=\mathbb{L}^0_{2112}=\mathbb{L}^0_{2121}=\frac{2\mu_1\mu_2}{\mu_1+\mu_2},\quad \mathbb{L}^0_{1112}=\mathbb{L}^0_{1121}=\mathbb{L}^0_{2111}=\mathbb{L}^0_{1211}=0,$$

$$(3.37)\qquad \mathbb{L}^0_{1222}=\mathbb{L}^0_{2122}=\mathbb{L}^0_{2212}=\mathbb{L}^0_{2221}=0\quad\text{and}\quad \mathbb{L}^0_{2222}=0.$$

Consequently, recalling Remark 3.1,

$$\begin{aligned}\int_\Omega\mathbb{L}^0\nabla_x u\cdot\nabla_x u\;dx&=\int_\Omega\mathbb{L}^0\nabla'_x u\cdot\nabla'_x u\;dx+2\,\mathbb{L}^0_{1122}\int_\Omega\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\;dx\\ &=\int_\Omega\mathbb{L}^0\nabla'_x u\cdot\nabla'_x u\;dx+4\mu_1\int_\Omega\det\nabla'_x u\;dx.\end{aligned}$$

which, in view of (3.34), proves the Γ- lim inf inequality in the Gutiérrez case.

4. Concluding remarks

Remark 4.1. Thanks to Theorem 3.4, we can now consider the Gutiérrez case. Take *e.g.* $f\in L^2(\Omega;\mathbb{R}^2)$, and consider an almost minimizing sequence $\{u^\varepsilon\in H^1_0(\Omega;\mathbb{R}^2)\}_\varepsilon$ for the functionals

$$\mathscr{I}^\varepsilon(v)-2\int_\Omega f\cdot v\,dx$$

.

It is clearly bounded in $L^2(\Omega;\mathbb{R}^2)$ and we conclude in particular to the L^2-weak convergence of (a subsequence of) this sequence of almost minimizers to a minimizer $u^0 \in \mathscr{X}$ of the Γ-limit functional

$$\mathscr{I}^0(v) - 2\int_\Omega f\cdot v\,dx$$

which, in view of the precise values of $\mathbb{L}^0$ in (3.35), can be easily checked to satisfy

$$-\mathrm{div}\,(\mathbb{L}^0\nabla u^0) + u^0 = f \quad \text{in } \Omega,$$

where $\mathbb{L}^0\nabla u^0$ is *a priori* a distribution since $u^0 \in \mathscr{X}$. ¶

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