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Dynamics of beams on a periodical foundation: an analytical approach

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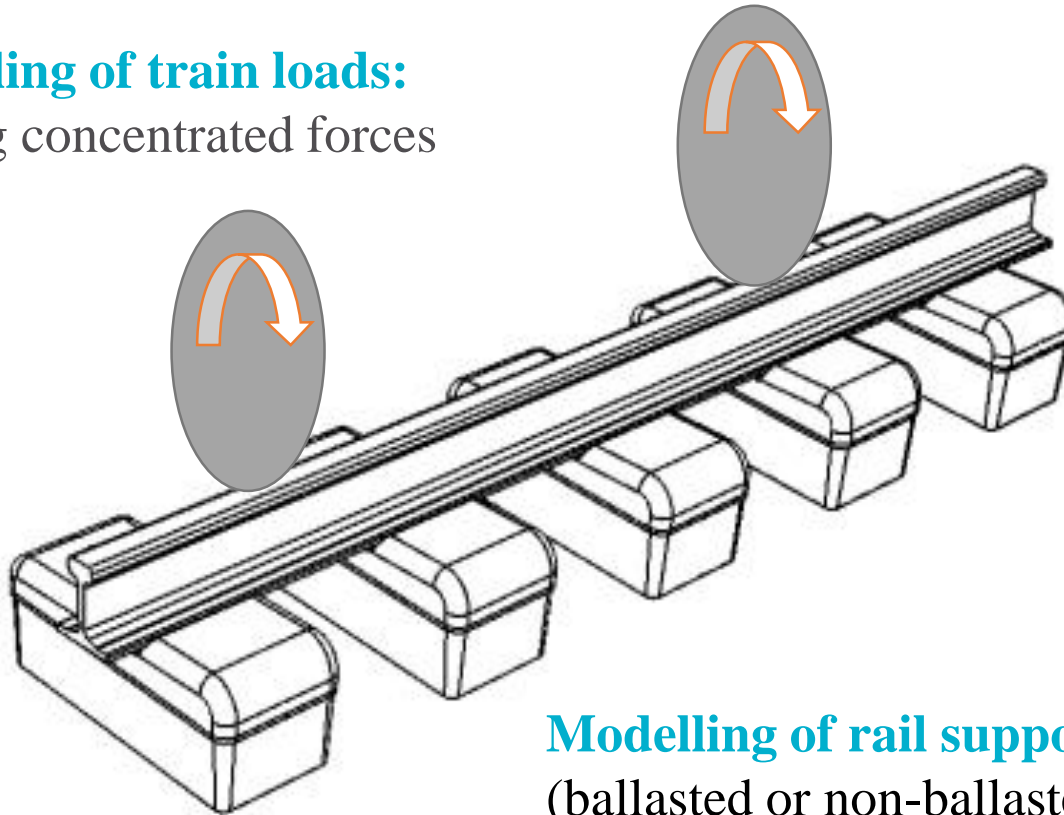
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Metz, 27th October 2016

Dynamics of railway track

Modelling of train loads:
moving concentrated forces

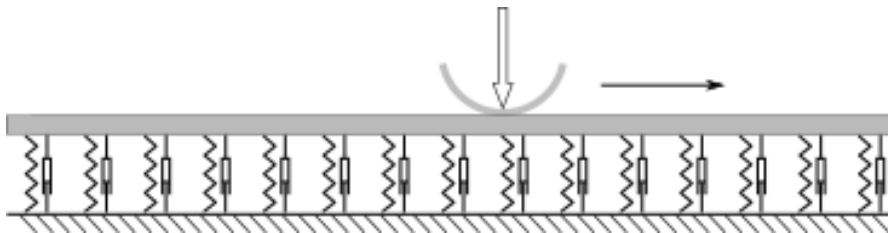


Modelling of rail:
beam models (Euler-Bernoulli or Timoshenko)

Modelling of rail supports
(ballasted or non-ballasted track):
linear or non-linear behavior

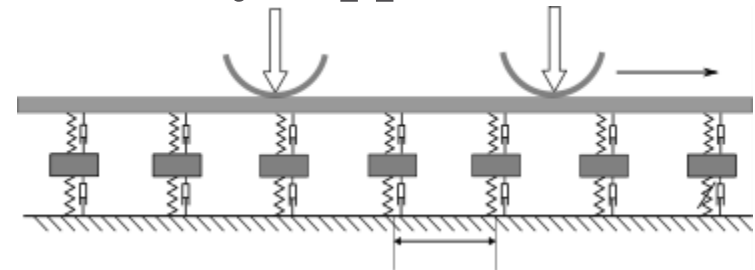
Modelling of rail supports

Beam on a continuous foundation



- ❖ The beam is supported by a homogenous foundation¹
- ❖ The reaction force of the foundation applies to all the beam length

Periodically supported beam



- ❖ The beam is supported by supports at periodical distances²
- ❖ The reaction forces of supports are concentrated forces

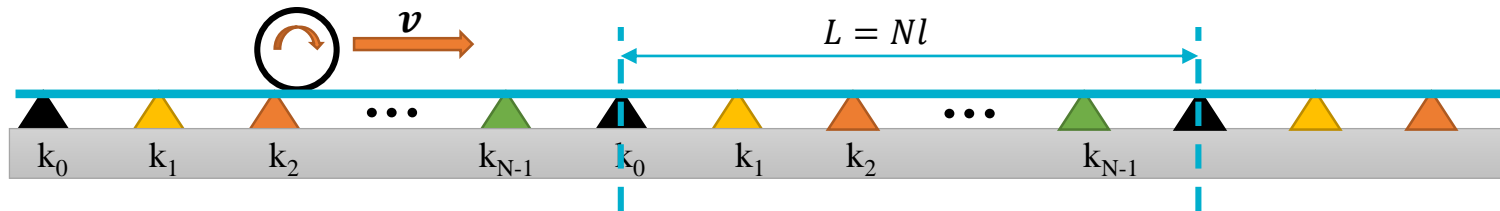


- Influence of non-homogeneous foundation
- Influence of support width

¹ Fryba, L. Vibration of Solids and Structures Under Moving Loads. Noordhoff International Publishing, Leyden (1972)

² Mead, D. J. Free wave propagation in periodically supported, infinite beams, Journal of Sound and Vibration 11 (2) (1970)

Periodic supports



- ❖ N different types of supports which are distributed periodically along the beam

$$k_{p+N} = k_p \quad (0 \leq p < N)$$

- ❖ Behavior law of the supports

$$\underline{\hat{R}} = -\underline{\mathbf{K}} \underline{\hat{w}} \quad (1)$$

- ❖ In the steady-state, the responses of supports are repeated when the moving loads cover the length $L = Nl$

$$R_{p+N}(t) = R_p\left(t - \frac{L}{v}\right)$$

where L/v is the time for the moving loads cover the distance L .

Euler-Bernoulli equation

$$EI \frac{\partial^4 w}{\partial x^4} + \rho S \frac{\partial^2 w}{\partial t^2} + Q \delta(x - vt) = \sum_n R_n(t) \delta(x - nl)$$

Fourier transforms

- with respect to t , then with respect to x

Dirac comb properties

Inverse Fourier transform

- with respect to x

General relation

- between reaction forces R_n ($0 \leq n < N$) and beam displacement w

Relation between reaction forces – displacements

- ❖ Matrix form

$$\underline{\underline{\mathbf{D}}} \underline{\hat{R}} = \eta_e Q_e \underline{I} + \underline{\hat{W}} \quad (2)$$

where $\underline{\underline{\mathbf{D}}}$, η_e , Q_e depend only on the parameters of the beams and the loads.

- ❖ Discrete Fourier form

$$\mathcal{F}_q(\underline{\hat{W}}) = \chi_q \mathcal{F}_q(\underline{\hat{R}}) - N Q_e \delta_{1q} \quad (3)$$

- ❖ When the supports are identical, this relation becomes the relation of the equivalent system of a periodically supported beam³

$$\hat{w}(x, \omega) = \hat{R}(\omega) \eta(x, \omega) + Q_E(\omega) e^{-i \frac{\omega}{v} x}$$

³ Hoang et al., Response of a periodically supported beam on a nonlinear foundation subjected to moving loads, Nonlinear Dynamics (2016)

- ❖ **Direct method:** By combining directly the general relation (2) with the behaviour law (1) of the supports, we can obtain the responses

$$\underline{\hat{W}} = \eta_e Q_e \left(\underline{\underline{\mathbf{D}}} \underline{\underline{\mathbf{K}}} + \underline{\underline{\mathbf{I}}} \right)^{-1} \underline{I}$$

- ❖ **Iterative method:** by using equation (3) to calculate the responses by iteration through 3 steps:

- Take the responses with the uniform supports as the value of the initial iteration
- Calculate the variation of forces by using the behavior law
- Recalculate the displacements by using the relation (3) with FFT.

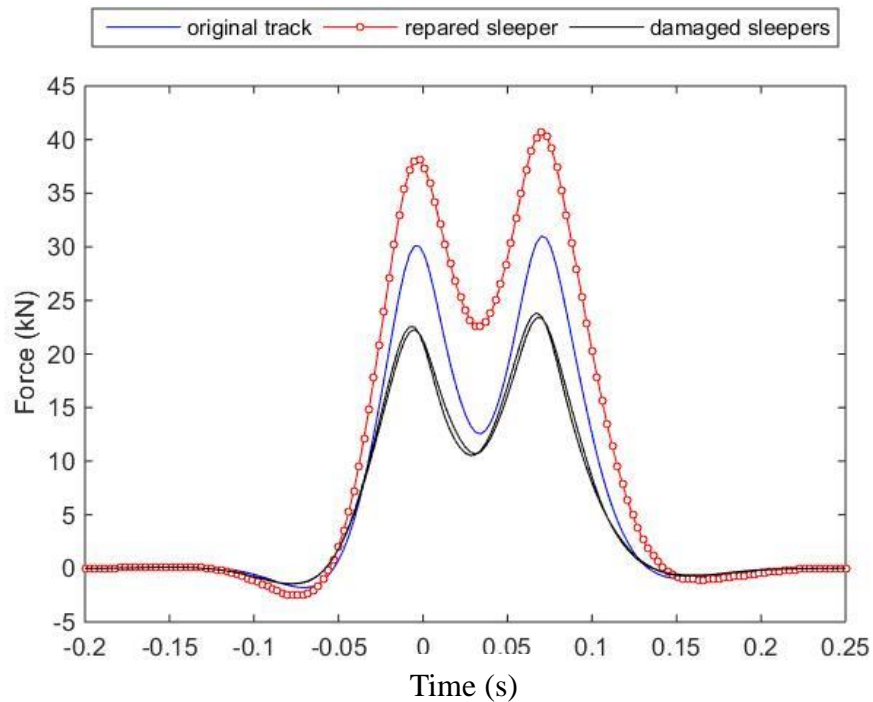
Numerical example

- ❖ When railway sleepers are damaged, we can do the replacement 1 by 3 in order to reduce the maintenance time. This process creates periodical supports of period $N = 3$

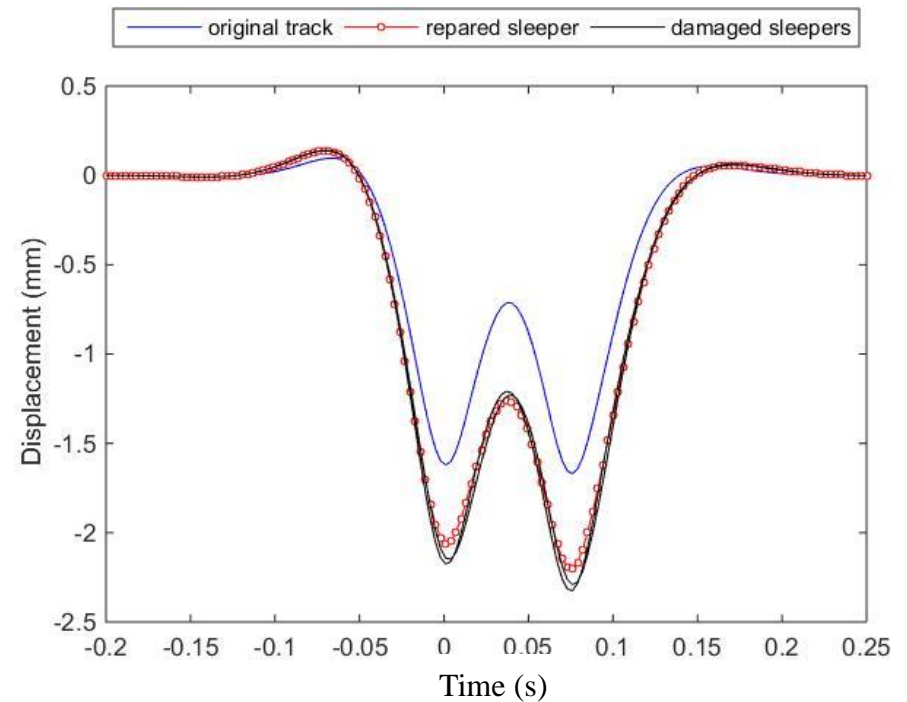
Parameter	Notation	Unit	Value
Rail mass	ρS	kg/m	60
Rail stiffness	EI	MNm ²	6,3
Train speed	v	km/h	160
Charge per wheel	Q	kN	100
Wheels distance	H	m	3
Block mass	m	kg	90
Sleeper distance	l	m	0,6
Stiffness of rail pad	k_p	MN/m	200
Damping coefficient of rail pad	η_p	MNs/m	1,0
Stiffness of sleeper	k_f	MN/m	20
Damping coefficient of sleeper	η_f	MNs/m	0,2
Stiffness of damaged sleeper	k_d	MN/m	10

Numerical example: results

Reaction forces of the supports



Displacement of the supports



Beam on a periodical foundation

- ❖ Consider an Euler-Bernoulli beam on a periodical foundation submitted moving forces

- ❖ Constitutive law of the foundation:

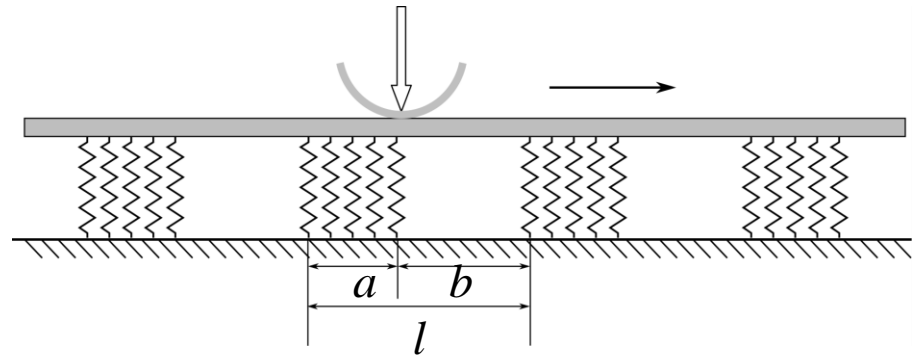
$$R_n(x, t) = \begin{cases} k_a w(x, t) & \text{if } nl < x < nl + a \\ k_b w(x, t) & \text{if } nl + a < x < (n+1)l \end{cases}$$

- ❖ Reaction forces of the foundation described by

$$F(x, t) = \sum_{n=-\infty}^{\infty} R_n(x, t)$$

- ❖ Dynamical equation of the beam

$$EI \frac{\partial^4 w(x, t)}{\partial x^4} + \rho S \frac{\partial^2 w(x, t)}{\partial t^2} + Q \delta(x - vt) = F(x, t)$$



- ❖ **Condition of periodicity:** In the steady state, the reaction forces are repeated when the moving forces cover a distance equal to the period length of the foundation

$$R_n(x, t) = R_0\left(x - nl, t - \frac{nl}{v}\right)$$

where $R_0(x, t)$ is the reaction force of the foundation in an interval $[0, l]$

Green function

- ❖ Let \mathcal{L} be the linear operation on the right hand side of the dynamic equation. The Green function of \mathcal{L} is defined by

$$\mathcal{L}G(x, s) = \sum_n \delta(x - nl - s) \quad (4)$$

- ❖ We can write

$$\int \mathcal{L}G(x, s)\hat{R}_0(s, \omega)ds = \sum_n \hat{R}_0(x - nl, \omega) = \mathcal{L}\hat{w}(x, \omega)$$

- ❖ Thus, the response of the beam is calculated by

$$\hat{w}(x, \omega) = \int G(x, s)\hat{R}_0(s, \omega)ds$$

- ❖ Equation (4) is exactly the dynamic equation of a periodically supported beam with $R(\omega) \equiv 1$, which is characterized by the relation³

$$\hat{w}_r(x, \omega) = \hat{R}(\omega)\eta(x, \omega) + Q_E(\omega)e^{-i\frac{\omega}{v}x}$$

- ❖ Thus, the Green function is given by

$$G(x, s) = \eta(x - s, \omega) + Q_E(\omega)e^{-i\frac{\omega}{v}(x-s)}$$

Conclusion

- ❖ The model of beams on a foundation can be extended for modelling the dynamics of railway tracks on a nonhomogeneous foundation
- ❖ When the reaction forces of supports are concentrated, the responses of the railway track on a periodical foundation can be calculated by an analytical method
- ❖ Moreover, the supports with large width (the reaction forces can not be considered as point forces), the Green function is demonstrated and it can be used to compute the responses by an iterative method.

Thank you for your attention!

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