

Topology-preserving transformations of discrete images in the cubic grid.

Recent results and open problems.

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Introduction

Who did the work?

Results obtained thanks to (the hard work of):

- Gilles Bertrand, professor, A3SI, Paris
- Michel Couprie, professor, A3SI, Paris
- Sylvain Faisan, assistant professor, LSIIT, Strasbourg
- Loïc Mazo, PhD student, LSIIT (+ A3SI), Strasbourg
- Vincent Noblet, CNRS research engineer, LSIIT, Strasbourg
- Nicolas Passat, assistant professor, LSIIT, Strasbourg

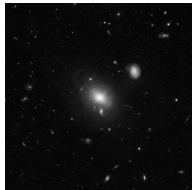
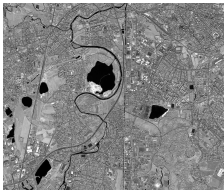
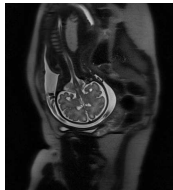


What is a digital image?

Digital imaging:

- (bio)medical data (MRI, CT, ...)
- remote sensing data (satellite images, ...)
- astronomical data (telescope images, ...)
- multimedia data (photos, videos, ...)
- etc.

An image \sim a function defined on a part $E \subset \mathbb{Z}^n$ (with, $n = 2$ or 3)



Digital image processing

Processing/analysing a digital (colour/grey-level/binary) image:

- segmentation
- filtering/correction
- registration
- classification/clustering
- knowledge extraction
- etc.

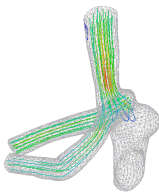
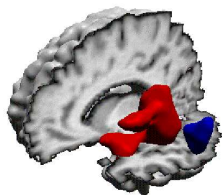
Often have to use and/or preserve correct properties related to:

- radiometry (colour, intensity, ...)
- morphology (shape)
- geometry (size, orientation, curvature, ...)
- **topology (connected components, holes, tunnels, ...)**

Why is it (sometimes) important to consider topology?

Non exhaustive list of (hopefully good) reasons:

- “landmarks” for improving segmentation, registration, etc.
- correct results for object analysis (ex: medical diagnosis, road extraction)
- correct results for high level post-processing (ex: blood flow simulation)
- correct input for image synthesis (ex: 3D modelling of art objects)
- etc.



How to consider topology?

Several ways to consider topology:

- 1 Topological properties extraction:
“what are the topological properties of an object?”
- 2 Topological properties handling:
“how to control the topological properties of an object?”
- 3 **Topological properties preservation:**
“how to guarantee that the topological properties of an object will not change?”

Topology preservation: what does it mean?

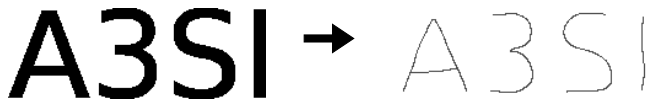
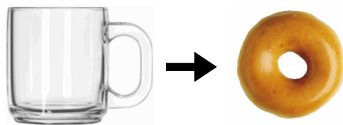
“Topology preservation”: abuse of language for “topological properties preservation”.

Topological properties (“from the weakest to the strongest”):

- Euler characteristic
- Betti numbers
- homology/co-homology groups
- homotopy type (homotopy equivalence)
- homeomorphism
- fundamental group

(Quite) informally: two objects X and Y share the same homotopy type if one can “pass continuously from X to Y and from Y to X ” (possibly with a decrease in the object dimension).

Homotopy-type: examples and justification



Homotopy-type has several virtues:

- sufficiently strong to correctly model the “topology preservation” between two objects
- sufficiently weak to be conveniently considered in \mathbb{Z}^n
- defined via a notion of deformation which can be algorithmically handled

Purpose and problems to solve

Purpose of this talk (1/2)

Question

How can we transform a digital (binary) object without altering its homotopy type?

Applications related to this question:

- segmentation by monotonic deformation: reduction, region-growing (Mangin, JMIV 1995)
- atlas-based segmentation (Bazin, TMI 2007)
- (digital) deformable models (Han, PAMI 2003)
- skeletonisation
- object warping based on image registration
- topology correction (Aktouf, PRL 2002)

Purpose of this talk (2/2)

Talk divided into 3 parts:

- 1 links between continuous and digital topology
- 2 (non?) convergence of topology-preserving transformations
- 3 (if enough time. . .) what happens if binary images become n-ary?

which deal with several issues related to the above question, and:

- 1 present recent works related to these issues
- 2 states some (still open) problems

Continuous topology vs. digital topology

Digital topology concepts

In order to deal with topology in digital spaces (\mathbb{Z}^n), several ad hoc concepts have been defined:

- adjacencies: 6, 18, 26, and adjacency couples: (6, 26), (26, 6), etc. (Rosenfeld, JACM 1966)
- paths
- connected components
- (digital) fundamental group (Kong, C&G 1989)

These notions are used to deal with topological properties of objects in \mathbb{Z}^n .

Digital/continuous equivalence (1/2)

Postulate

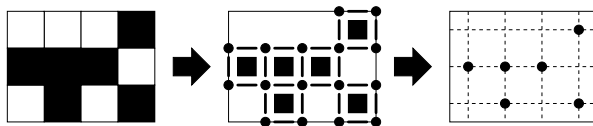
These ad hoc notions defined in \mathbb{Z}^n actually model the topological properties defined in \mathbb{R}^n (by assuming that \mathbb{Z}^n divides \mathbb{R}^n into unit n -cubes).

Proposition (Mazo et al., research report 2010)

This postulate (generally admitted, but not so obvious!) is true.

It has been proved in two steps:

From \mathbb{R}^n to the space of complexes (\mathbb{F}^n) and then from \mathbb{F}^n to \mathbb{Z}^n



Digital/continuous equivalence (2/2)

In particular, the notion of **simple point** leads to a “satisfactory” way to transform an image in a topology preserving fashion.

Definition (unformal...)

A simple point is a point $x \in X \subset \mathbb{Z}^n$ which can be removed from X without altering its homotopy type.

N.B.: several characterisations of simple points (in \mathbb{Z}^2 , \mathbb{Z}^3 , \mathbb{Z}^4), which formally define them de facto.

Theorem (Barmak, Advances in Mathematics 2008 + Mazo et al.)

The homotopy groups of a digital object are not changed by the deletion of a simple point. In particular, this deletion corresponds to a deformation retract on the continuous analogue.

(Intermediate) conclusion

Two good news:

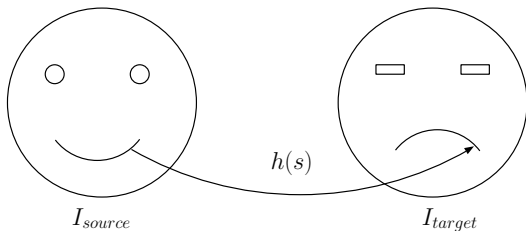
- “digital topology” defined since the 60’s, correctly matches the “continuous reality”
- simple points actually enable to transform digital objects without homotopy type modification

The way to use simple points will be considered in the next part of this talk (Issue 2).

But before that, just a question about continuous topology and image registration. . .

Image registration

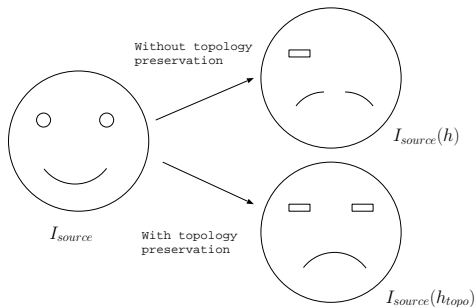
Image registration aims at estimating consistently a mapping between two images.



$$h = \arg \min E(I_{source}, I_{target}, h)$$

Topology-preserving nonrigid image registration (1/2)

An important issue in (nonrigid) image registration is to enforce the estimated transformation to preserve the **continuous topology** (and in particular, the homotopy type)



Topology-preserving nonrigid image registration (2/2)

Conditions for enforcing h to be an homeomorphism:

- h is continuous
- the determinant of the Jacobian of h is strictly positive
- h is invariant on the boundaries of the image

Requires to solve the optimisation problem (Noblet, TIP 2005):

$$h = \arg \min_{\substack{\forall s \in \Omega, J_h(s) > 0 \\ \forall s \in \partial\Omega, h(s) = s}} E(I_{source}, I_{target}, h)$$

Image registration and atlas-based segmentation

Image registration is generally used to defined accurate deformation fields involved in the guidance of image warping, especially for atlas-based segmentation.

Definition (Atlas-based segmentation)

If X is an image, S_X is its (binary) segmentation, Y is another image (with the same semantics as X), and h the deformation field computed from the registration of X onto Y , then, $S_Y = h^{-1}(S_X)$ is a (binary) segmentation of Y .

- \oplus Image registration provides **continuous topology-preserving** deformation fields
- \ominus **topology preservation** of digital objects deformed by such fields **is not considered!**

Fusing continuous topology and discrete topology

Proposition: develop a digital topology preserving warping method for discrete images, guided by deformation fields generated by continuous topology preserving registration method (Faisan, MICCAI 2008, TIP 2011).

Only one similar (but not equivalent) work in the literature: Bazin, IPMI 2007, MICCAI 2007.

Method already described in CTIC 2008 (for those who remember).

Image warping method - Overview

Input:

- $M \subset \mathbb{Z}^3$ (topological model);
- $D : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ (deformation field: “ $K = D(M)$ ”).

Process: deformation of M guided by a cost function $f(D, M)$.

Output:

- $X \subset \mathbb{Z}^3$:
 - topologically equivalent to M
 - geometrically “as close as possible” to K

Image warping method - Overview

Constrained optimisation problem:

$$X = \arg \min_{Y \sim M} d(Y, M, D)$$

Critical points:

- the distance d (not discussed here)
- the way to preserve topology in Y (classical approach: iterative removal/addition of simple points)
- the optimisation strategy.

Image warping method - Optimisation strategy (1/2)

Purpose

Minimise the cost function by topology-preserving modification of points, i.e., converge onto a model topologically equivalent and geometrically similar to the continuous deformed image $D(M)$.

But: large displacements \Rightarrow possible convergence onto local minima.

A solution: deformation performed in a “smooth” way by estimating *intermediate* deformation fields $D^{(i)}$ computed from D .

$$D^{(0)} = Id, D^{(n)} = D$$

$$\forall j \in [0, n - 1], \forall x, \|D^{(j+1)}(x) - D^{(j)}(x)\| < 1$$

Image warping method - Optimisation strategy (1/2)

Remark

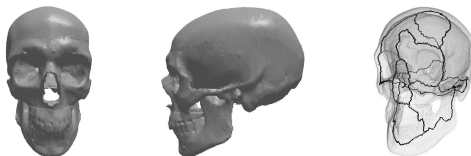
Lower bound for n : $\max_{x \in M} \|D(x) - x\|$.

The deformation fields $D^{(i)}$ ($0 < i < n$) are finally defined by:

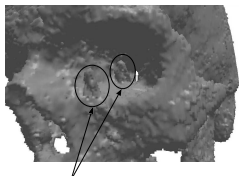
$$\forall x, (D^{(i)}(x) - x) = \frac{i}{n}(D(x) - x)$$

Starting from $S = M$, the dynamical process is iteratively carried out with $D^{(1)}, \dots, D^{(n)}$ using the currently deformed image S .

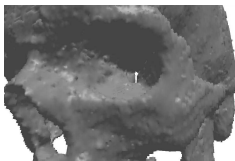
Image warping method - Application



Conclusion: it works correctly on each tested image (this is not a joke!).



deadlocks caused by topological constraints

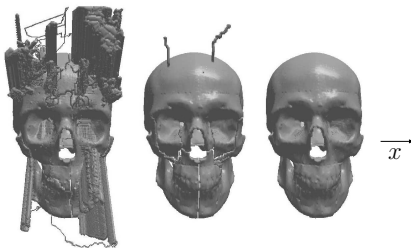
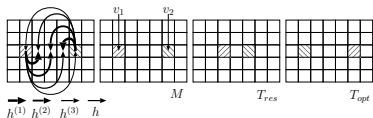


erroneous hole

Image warping method - Perspectives

But several remaining weaknesses:

- depends on the registration quality!
- possible topological deadlocks caused by:
 - the image resolution
 - the deformation field
 - the notions of topology themselves. . .



Topological deadlocks

Motivation: Homotopic skeletonisation

Homotopic skeletonisation (in \mathbb{Z}^2 , \mathbb{Z}^3 , ...):

- used to transform an object without topology modification
- generally defined thanks to the notion of “simple point”

Algorithm

Input: $X \subseteq \mathbb{Z}^n$, $Y \subseteq X$

Output: S topologically equivalent to X ($Y \subseteq S \subseteq X$)

$S \leftarrow X$

while $\exists x \in S \setminus Y$, *simple*(x, S) **do**

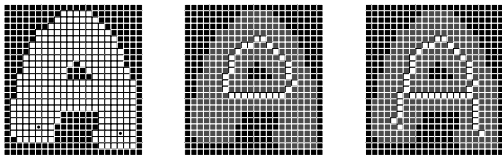
Choose $x \in S \setminus Y$, *simple*(x, S) *according to some criterion*

$S \leftarrow S \setminus \{x\}$

end while

N.B.: sometimes, use of alternative concept: cubical complexes (Cointepas, ICIP 1998), orders (Daragon, PhD thesis 2005)

Motivation: skeleton minimality



Question

Is S always a "minimal" result?

Answer

In \mathbb{Z}^2 : Yes (Ronse, DAM 1986)

Answer

In \mathbb{Z}^3 (and a fortiori in \mathbb{Z}^n , $n \geq 3$): No (e.g. Bing's House)

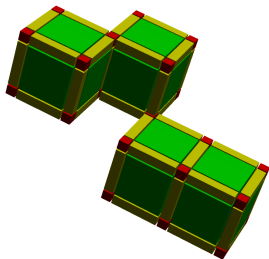
Question 1: what happens between 2D and 3D?

Question

What happens for spaces between the \mathbb{Z}^2 and \mathbb{Z}^2 ?

→ Case of the 2-D (pseudo)manifolds in n -D spaces ($n \geq 3$).

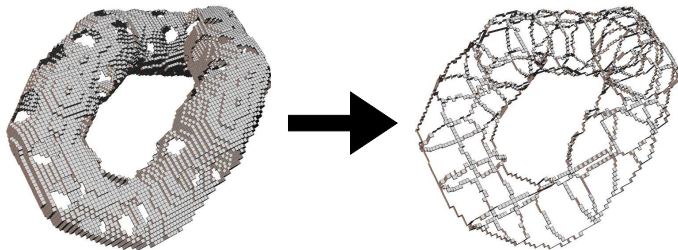
This question has been studied by in the context of cubical complexes, \mathbb{F}^n (Kovalevski, CVGIP 1989), which remains compliant with topology in \mathbb{Z}^n .



Question 1: between 2D and 3D

Answer (Passat et al., JMIV 2010)

In the space of 2-D (pseudo)manifolds in n -D spaces ($n \geq 3$), S is minimal.



Remark

Moreover, as in \mathbb{Z}^2 , S can be computed in a greedy fashion!

Question 2: and in higher dimensions?

Question

Then, what happens in \mathbb{Z}^3 (and in higher dimensions)?

→ **topological monsters**, which may lead to topological deadlocks in image transformation.

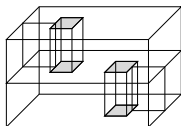
Consequence:

- for methods based on monotonic transforms (reduction, skeletonisation, region-growing), **possible convergence on a “local minimum”**
- for methods based on non-monotonic transforms (deformable models, atlas-based segmentation), **possible convergence on a “local minimum”, or no guarantee of termination** if a global minimum has to be found

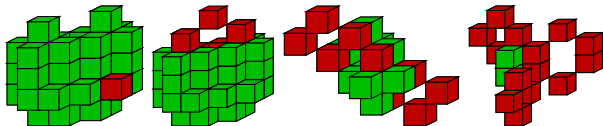
Topological monsters

Two kinds of topological monsters

- Naughty monsters (ex. Bing's house): do not really know what to do with them... (open problem)



- Less naughty monsters: can get rid of them (in a monotonic transformation).



Topological monsters and (minimal) simple sets

When a topological deadlock happens, **there is no simple point to remove**. But, there may exist **simple sets** (\sim sets of non-simple points), whose removal does not modify the topology of the object.

Definition (Unformal and partial)

A simple set S for an object $X \subset \mathbb{Z}^n$ is a set whose removal from X is a deformation retract on the continuous analogue of X .

Remark

Simple points are “singular” simple sets.

We focus in particular on the **minimal simple sets**, i.e., simple sets which do not strictly include another simple set.

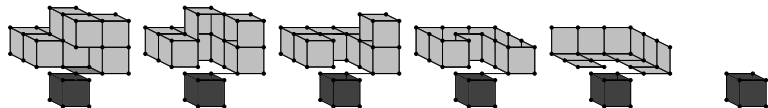
→ easier to detect, and hopefully sufficient to solve most of the problems.

2D simple sets in nD spaces

The notion of simple set can be extended from \mathbb{Z}^n to \mathbb{F}^n .
(It is actually specialised from \mathbb{F}^n to $\mathbb{Z}^n \dots$).

Proposition (Mazo et al., DCG 2010)

In \mathbb{F}^n ($n \geq 2$), any minimal 2D simple set can be (easily) characterised.



Proposition (Mazo et al., DCG 2010)

A simple set can be fully removed by iterative removal of minimal simple sets in a greedy fashion.

Topological monsters and simple sets: back to \mathbb{Z}^3

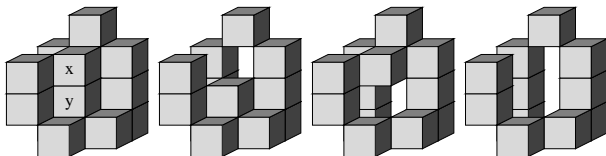
The problem of simple sets is solved for spaces of dimension $n \leq 2$ or simple sets of dimension $k \leq 2$.

→ The challenge is now to try to deal with spaces of dimension $n \geq 3$, and simple sets of dimension $3 \leq k \leq n$.

Partial result in the case of \mathbb{Z}^3 ("3D simple sets in 3D space"), for (minimal) simple pairs (\sim simple sets composed of two points)

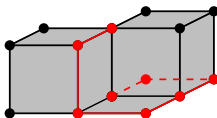
Proposition (Passat et al., JMIV 2008)

In \mathbb{Z}^3 any minimal 3D simple pair can be (easily) characterised.



Simple sets: open problems

- in \mathbb{Z}^n ($n \geq 3$), minimal simple sets can be of arbitrary size.
- except simple points and 3D minimal simple pairs, any other simple set is not necessarily pure (degenerated parts)



→ Which ones can be characterised, and how?

What about non-binary images?

Context

Digital images may require a segmentation/classification into several labels.

Examples: segmentation of the brain into $n \geq 3$ structures.



Remark

Models for binary image topology are not sufficient in this context.

Question

How can we handle the topology of a label image?

Some (partial) solutions (1/2)

Solution

1 - Never mind, let us use the binary model!

→ Poupon, MICCAI 1998: “surprising configurations can arise... these strange configurations do not really perturb the segmentation process”.

Solution

2 - Only consider “good” label images. (version 1)

→ Latecki, CVIU 1995: notion of well-composed sets (~ 6-adjacency everywhere).

→ Siqueira, JMIV 2008: modify the image until obtaining a well-composed one.

Some (partial) solutions (2/2)

Solution

3 - Only consider "good" label images. (version 2)

→ Mangin, JMIV 1995: use hypotheses which induce a "binary compliant" structure (ex. nested structures).

Solution

4 - Consider a binary-like model only where you can.

→ Bazin, TMI 2007: "some voxels may be left unclassified because of conflicting topological constraints".

Solution

5 - Use sufficient conditions to guarantee that a point is simple.

→ Dupas, DGCI 2009: notion of multi-label simple points ("If v is an ML-Simple point, then v is a simple point").

Towards a general solution

- 1, 2, 4: not acceptable with general hypotheses
- 3: correct but convergence (and termination?) problems
- 3, 5 based on a sound theory, but algorithmic restrictions

⇒ Need for a more general solution.

Work in progress: PhD thesis of Loïc Mazo, on:

- a complete model for topology in label images
- a modus operandi for topology preserving image transformation

based on cubical complexes (and compliant with digital imaging)

With results soon (hopefully. . .)

Conclusions and perspectives

What has been done / what remains to do (1/3)

Issue 1: Continuous topology vs. digital topology

- Done:
 - A proof of correctness of the topological definitions in \mathbb{Z}^n w.r.t. \mathbb{R}^n .
 - A practical way to fuse digital topology and continuous ones in transformation methods.
- To do:
 - Improve the way to guide such methods in case of really complex deformation fields.
 - Propose solutions to topological deadlocks due to image resolution.
 - Consider the issue of topology control instead of topology preservation.
 - ad lib. . .

What has been done / what remains to do (2/3)

Issue 2: Topological deadlocks

- Done:
 - Characterisation of “easy” minimal simple sets: 2D sets, 2D space, small 3D sets
 - Proof of convergence of greedy algorithms based on 2D minimal simple sets.
- To do:
 - How to characterise the other (minimal) simple sets?
 - How to guarantee the correct convergence/termination of methods on 3D spaces (and higher ones)?
 - How to deal with naughty monsters?
 - ad lib. . .

What has been done / what remains to do (3/3)

Issue 3: What about non-binary images?

- Done:
 - not so much
- To do:
 - nearly all

Related publications

Issue 1: Continuous topology vs. digital topology

- S. Faisan et al. **Topology-preserving warping of binary images according to one-to-one mappings.** IEEE TIP 2011
- L. Mazo et al. **Paths, homotopy and reduction in digital images.** Research report (on HAL)
- L. Mazo et al. **Digital imaging: a unified topological framework.** Research report (on HAL)

Issue 2: Topological deadlocks

- N. Passat et al. **Minimal simple pairs in the 3-D cubic grid.** JMIV 2008
- N. Passat et al. **An introduction to simple sets.** PRL 2009
- N. Passat et al. **Topological properties of thinning in 2-D pseudomanifolds.** JMIV 2010
- L. Mazo et al. **On 2-dimensional simple sets in n-dimensional cubic grids.** DCG 2010.

Thank you for your attention.

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