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Extinction and coming down from infinity of CB-processes with competition in a Lévy environment

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Abstract

In this note, we are interested on the event of extinction and the property of coming down from infinity of continuous state branching (or CB for short) processes with competition in a Lévy environment. In particular we prove, under the so-called Grey's condition together with the assumption that the Lévy environment does not drift towards infinity, that for any starting point the process becomes extinct in finite time a.s. Moreover if we replace the condition on the Lévy environment by an integrability condition on the competition mechanism, then the expectation of the extinction time is finite. Finally, if we assume that the competition mechanism is convex and that the jump structure associated to the environment possesses some positive exponential positive moments then we deduce that the process comes down from infinity.

KEY WORDS AND PHRASES: Continuous state branching processes in random environment, competition, extinction, coming down from infinity.

MSC 2000 subject classifications: 60J80, 60J75, 60J85.

1 Introduction and main results.

Continuous state branching (or CB) processes with competition have been studied by several authors, see for instance Ba and Pardoux [1], Berestycki et al. [2], Lambert [8], Ma [15] and Pardoux [17] and the references therein. A well known example of this family of processes is the so called logistic Feller diffusion which can be constructed as scaling limits of Bienaymé-Galton-Watson processes with competition, see for instance Lambert [8].

CB-processes with competition can be defined as the unique strong solution of an SDE and they are determined by two components; a branching mechanism ψ and a competition mechanism g . The competition mechanism g is a non-decreasing continuous function on $[0, \infty)$ with $g(0) = 0$ and the branching mechanism satisfies the so-called Lévy-Khintchine formula

$$\psi(u) = -bu + \gamma^2 u^2 + \int_{(0, \infty)} \left(e^{-uz} - 1 + uz \right) \mu(dz), \quad (1.1)$$

where $b \in \mathbb{R}$, $\gamma \geq 0$ and μ is a Radon measure concentrated on $(0, \infty)$ satisfying

$$\int_{(0, \infty)} (z \wedge z^2) \mu(dz) < \infty. \quad (1.2)$$

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According to Ma [15] (see also Berestycki et al. [2]) a CB-process with competition $Y = (Y_t, t \geq 0)$ can be defined as the unique strong solution of the following SDE

$$Y_t = Y_0 + \int_0^t bY_s ds - \int_0^t g(Y_s) ds + \int_0^t \sqrt{2\gamma^2 Y_s} dB_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z \tilde{N}^{(b)}(ds, dz, du), \quad (1.3)$$

where $B^{(b)}$ is a standard Brownian motion and $N^{(b)}$ is a Poisson random measure which is defined on \mathbb{R}_+^3 , with intensity measure $ds\mu(dz)du$ such that (1.2Introduction and main resultsequation.1.2) is satisfied, and $\tilde{N}^{(b)}$ denotes its compensated version.

Lambert [8] studied the long term behaviour of the logistic case i.e. $g(x) = cx^2$, for $x \geq 0$ and $c > 0$, using a Lamperti-type representation (random time change). Under the following log-moment condition

$$\int_1^\infty \log(u)\mu(du) < \infty,$$

Lambert [8] proved that either the process converges to a specified distribution, or, it goes extinct a.s. In the latter case, the expectation and the Laplace transform of the absorption time is also determined and the property of coming down from infinity is also established.

More general competition mechanisms were considered by Ba and Pardoux [1] in the case where the branching mechanism is of the form $\psi(u) = \gamma^2 u^2$, for $u \geq 0$, see also Chapter 8 in the monograph of Pardoux [17]. Actually, they allow the competition mechanism g to be a continuous function and not necessarily monotone and provided a necessary and sufficient condition for the process to become extinct. In this setting, if g is negative and non-increasing then the competition mechanism can be interpreted as cooperation in the sense of Gonzalez-Casanova et al. [5].

Our aim is to provide sufficient conditions under which a generalised version of the previous family of processes, that we call CB-processes with competition in a Lévy environment and will be formally introduced below, become extinct with positive probability or a.s., as well as determine under which conditions such family of processes comes down from infinity.

Here, we assume that the Lévy measure μ satisfies (1.2Introduction and main resultsequation.1.2), which allows us to define ψ as in (1.1Introduction and main resultsequation.1.1). Moreover, we observe that the previous assumption implies that $\psi'(0+) = -b$.

Recently, CB-processes with competition in a Lévy environment have been introduced by Palau and Pardo [16] as the unique strong solution of the following SDE

$$\begin{aligned} Z_t = Z_0 + b \int_0^t Z_s ds - \int_0^t g(Z_s) ds + \int_0^t \sqrt{2\gamma^2 Z_s} dB_s^{(b)} \\ + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z \tilde{N}^{(b)}(ds, dz, du) + \int_0^t Z_{s-} dS_s, \end{aligned} \quad (1.4)$$

where g is a non-decreasing continuous function on $[0, \infty)$ with $g(0) = 0$, $B^{(b)}$ and $N^{(b)}$ are defined as before and S is a Lévy process independent of $B^{(b)}$ and $N^{(b)}$ which can be written as follows

$$S_t = \mathbf{d}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} (e^z - 1) N^{(e)}(ds, dz) + \int_0^t \int_{(-1,1)} (e^z - 1) \tilde{N}^{(e)}(ds, dz), \quad (1.5)$$

with $\mathbf{d} \in \mathbb{R}$, $\sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion and $N^{(e)}$ is a Poisson random measure taking values on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $ds\pi(dz)$ satisfying

$$\int_{\mathbb{R} \setminus \{0\}} (1 \wedge z^2) \pi(dz) < \infty. \quad (1.6)$$

For our purposes, we also introduce the auxiliary Lévy process which is a modification of the process S ,

$$K_t = \beta t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} z N^{(e)}(ds, dz) + \int_0^t \int_{(-1,1)} z \tilde{N}^{(e)}(ds, dz), \quad t \geq 0,$$

where

$$\beta = \mathbf{d} + \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z) \pi(dz).$$

We denote by \mathbb{P}_x the law of Z starting from $x > 0$, and we define by $T_0 = \inf\{t \geq 0, Z_t = 0\}$ the first hitting time to 0 of Z , with the convention that $\inf\{\emptyset\} = \infty$. In the case without competition i.e. $g \equiv 0$, He et al. [6] proved that the so-called Grey's condition

$$\int^\infty \frac{du}{\psi(u)} < \infty \tag{1.7}$$

is necessary and sufficient for CB-processes in a Lévy environment to become extinct with positive probability (see Theorem 4.1 in [6]). Moreover, if the auxiliary process K does not drift to ∞ or equivalently

$$\liminf_{t \rightarrow \infty} K_t = -\infty, \tag{1.8}$$

and Grey's condition (1.7) holds, then its associated CB-process in a Lévy environment becomes extinct at finite time a.s., see Corollary 4.4 in [6].

Our first result, which follows from a comparison criteria for CB-processes with competition in a Lévy environment (see Lemma 2.1 below), gives a necessary condition under which they become extinct. Before stating it, we introduce the CB-process in a Lévy environment $Z^\sharp = (Z_t^\sharp, t \geq 0)$ as the unique strong solution of (1.4) but with $g \equiv 0$. For simplicity, we denote its law starting from $x > 0$ by \mathbb{P}_x^\sharp .

Proposition 1.1. *Assume that the Lévy measure μ associated to the branching mechanism ψ satisfies (1.2). For $y \geq x \geq 0$, we have that (Z, \mathbb{P}_x) is stochastically dominated by (Z, \mathbb{P}_y) .*

Moreover, the process (Z, \mathbb{P}_x) is stochastically dominated by $(Z^\sharp, \mathbb{P}_y^\sharp)$ and, in particular, if the branching mechanism ψ satisfies Grey's condition (1.7), then (Z, \mathbb{P}_x) becomes extinct with positive probability. Furthermore if K does not drift to ∞ or equivalently satisfies (1.8) then (Z, \mathbb{P}_x) becomes extinct at finite time a.s.

We now state our first main result which provides a sufficient condition on the competition mechanism for CB-processes with competition in a Lévy environment to become extinct a.s. even for favorable environments, i.e. when K drifts to $+\infty$. To this aim, we assume the following integral condition on the competition mechanism g : assume that there exists $z_0 > 0$ such that $g(z_0) > 0$ and

$$\int_{z_0}^\infty \frac{dy}{g(y)} < \infty. \tag{1.9}$$

Actually, the above condition is necessary for the expected time to extinction to be finite and when there is no environment or the environment has no negative jumps, the CB-process with competition in a Lévy random environment *comes down from infinity*. This phenomenon has been observed and studied by several authors in branching processes with interactions, see for instance González-Casanova et al. [5], Lambert [8], Li [13], Li et al. [14] and Pardoux [17] and also for stable jump diffusions [4]. Formally, we define the property of *coming down from infinity* in the sense that ∞ is a *continuous entrance point*, i.e.

$$\lim_{M \rightarrow \infty} \lim_{x \rightarrow \infty} \mathbb{P}_x(T_M < t) = 1 \quad \text{for all } t > 0,$$

where $T_M = \inf\{t \geq 0 : Z_t \leq M\}$ and the original process can be extended into a Feller process on $[0, \infty]$ (see for instance Theorem 20.13 in Kallenberg [7] for the diffusion case or Definition 2.2 for Feller processes in [4]).

Theorem 1.2. *Assume that the Lévy measure μ associated to the branching mechanism satisfies (1.2Introduction and 1.7Introduction and main result) and (1.9Introduction and main result) and Grey's condition (1.7Introduction and main result) hold, then*

$$\sup_{x \geq 0} \mathbb{E}_x [T_0] < \infty.$$

In particular, the point ∞ is a continuous entrance point and if the environment has no negative jumps then the process Z comes down from infinity.

We point out that the finiteness of the expected value of the extinction time when there is no random environment was first considered by Le in [9].

In Li et al. [14], the property of *coming down from infinity* is stated as follows: for any real-valued Markov process X , it comes down from infinity if ∞ is a continuous entrance point. The extension to a Feller process on $(-\infty, \infty]$ in this definition is not considered. However, as it is noted in Proposition 2.12 in [14], and when the process X has no negative jumps, the property that ∞ is a continuous entrance point is equivalent to

$$\lim_{b \rightarrow \infty} \lim_{x \rightarrow \infty} \mathbb{E}_x [\tau_b^-] = 0,$$

where $\tau_b^- = \inf\{t \geq 0 : X_t \leq b\}$. From the proof of Theorem 20.13 in Kallenberg, with the latter condition we can show that X can be extended into a Feller process on $(-\infty, \infty]$.

In the presence of negative jumps the property of *coming down from infinity* needs further knowledge of how the process X crosses any boundary from below. To be more precise, some conditions on

$$\lim_{x \rightarrow \infty} \mathbb{P}_x (X_{\tau_b^-} \in \cdot),$$

for all $b > 0$, are also needed. An interesting example is the case for stable jump diffusions with no negative jumps where under some technical conditions they come down from infinity and with the presence of negative jumps they don't, see Theorem 2.2 in Döring and Kyprianou [4]. Actually in the latter case we can find paths that jump instantaneously from ∞ to $-\infty$.

In our case, it seems complicated to determine

$$\lim_{x \rightarrow \infty} \mathbb{P}_x (Z_{T_M} \in \cdot),$$

for any $M > 0$. Nonetheless, if we assume that g is convex and existence of some positive exponential moments on the jump structure of the environment we can avoid such technicality. Indeed, our last result claims that the associated CB-process with competition in a Lévy environment *comes down from infinity*.

Theorem 1.3. *Assume that the Lévy measure μ associated to the branching mechanism satisfies (1.2Introduction and 1.7Introduction and main result) and g is convex, that the Lévy measure of the Lévy environment π satisfies*

$$\int_{(1, \infty)} e^z \pi(dz) < \infty,$$

and Grey's condition (1.7Introduction and main result) and (1.9Introduction and main result) hold, then for any $t \geq 0$,

$$\sup_{x \geq 0} \mathbb{E}_x [Z_t] < \infty, \tag{1.10}$$

and in particular the process comes down from infinity.

The previous result can be applied to the particular case when the competition mechanism is logistic (i.e. $g(x) = cx^2$) and the random environment is driven by a Brownian motion. Actually, further explicit computations can be carried out for the Laplace transform of the extinction time under \mathbb{P}_∞ , the law of the process starting from ∞ , as it is stated in Leman and Pardo [11] where part of this paper was initially presented. The results about the logistic branching process in a Brownian environment that appeared in [11] will be presented in Leman and Pardo [12] where some new material is also included.

The remainder of this note is devoted to the proofs.

2 Proofs

In order to prove Proposition 1.1theorem.1.1, we introduce the following stochastic processes as unique strong solutions of the SDE's. For $i = 1, 2$, we let

$$Z_t^{(i)} = Z_0^{(i)} + \int_0^t g_i(Z_s^{(i)})ds + \int_0^t \sqrt{2\gamma^2 Z_s^{(i)}} dB_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}^{(i)}} z \tilde{N}^{(b)}(ds, dz, du) + \int_0^t Z_{s-}^{(i)} dS_s^{(i)},$$

where

$$S_t^{(i)} = dt + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} b_i(z) N^{(e)}(ds, dz) + \int_0^t \int_{(-1,1)} (e^z - 1) \tilde{N}^{(e)}(ds, dz),$$

with $g_1(z) \geq g_2(z)$, for $z \geq 0$, and $b_1(z) \geq b_2(z)$ for $z \in \mathbb{R}$ such that, for $i = 1, 2$

$$b_i(z) + 1 \geq 0, \quad \text{for } z \in \mathbb{R}.$$

We also assume that for each $m \geq 0$, there is a non-decreasing concave function $z \mapsto r_m(z)$ on \mathbb{R}_+ satisfying $\int_{0+} r_m(z) dz = \infty$ and

$$|g_i(x) - g_i(y)| + \mathbf{d}|x - y| + |x - y| \int_{(-1,1)^c} (|b_i(z)| \wedge m) \pi(dz) \leq r_m(|x - y|), \quad \text{for } i = 1, 2, \quad (2.1)$$

for every $0 \leq x, y \leq m$. According to Proposition 1 in Palau and Pardo [16], the previous SDE's possess unique positive strong solutions that we denote by $Z^{(i)}$ for $i = 1, 2$.

Lemma 2.1. *If $Z_0^{(1)} \geq Z_0^{(2)}$, a.s. then*

$$\mathbb{P}\left(Z_t^{(2)} \leq Z_t^{(1)} \text{ for all } t \geq 0\right) = 1.$$

Proof. Let $\tau_m = \inf\{t \geq 0 : Z_s^{(1)} \geq m \text{ or } Z_s^{(2)} \geq m\}$ for $m \geq 1$. According to the proof of Proposition 1 in Palau and Pardo [16], for $i = 1, 2$, we have $Z_t^{(i)} = Z_t^{(i,m)}$ for $t < \tau_m$, where $Z^{(i,m)}$ is the unique strong solution to

$$\begin{aligned} Z_t^{(i,m)} &= Z_0^{(i)} + \int_0^t g_i(Z_s^{(i,m)} \wedge m) ds + \int_0^t \sqrt{2\gamma^2 Z_s^{(i,m)} \wedge m} dB_s^{(b)} \\ &\quad + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}^{(i,m)} \wedge m} (z \wedge m) \tilde{N}^{(b)}(ds, dz, du) + \int_0^t (Z_{s-}^{(i,m)} \wedge m) dS_s^{(i,m)}, \end{aligned}$$

where

$$S_t^{(i)} = dt + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} (b_i(z) \wedge m) N^{(e)}(ds, dz) + \int_0^t \int_{(-1,1)} ((e^z - 1) \wedge m) \tilde{N}^{(e)}(ds, dz).$$

In other words for $m \geq 1$, we have

$$\mathbb{P}\left(Z_t^{(1)} \geq Z_t^{(2)}, \text{ for all } t < \tau_m\right) = \mathbb{P}\left(Z_t^{(1,m)} \geq Z_t^{(2,m)}, \text{ for all } t < \tau_m\right).$$

Then, a direct application of Theorem 2.2 in [3] implies that the latter probability equals one (a detailed proof, which follows arguments of Theorem 2.2 in [3], can also be found in the arXiv version v2 of the present article [11]). This ends the proof of Lemma 2.1theorem.2.1. \square

Proof of Proposition 1.1theorem.1.1. The first statement follows directly from Lemma 2.1theorem.2.1 by taking

$$g_1(z) = g_2(z) = (\mathbf{d} + b)z - g(z) \quad \text{for } z \geq 0 \quad \text{and} \quad b_1(z) = b_2(z) = e^z - 1 \quad \text{for } z \in \mathbb{R}.$$

For the second statement, we recall that the competition mechanism g is positive and non-decreasing implying that we can take $g_1(z) = (\mathbf{d} + b)z$, $g_2(z) = (\mathbf{d} + b)z - g(z)$ and $b_1(z) = b_2(z) = e^z - 1$. Again from Lemma 2.1theorem.2.1, we deduce that the process (Z, \mathbb{P}_x) is stochastically dominated by $(Z^\sharp, \mathbb{P}_y^\sharp)$ for $y \geq x$.

The last part of the statement follows from Theorem 4.1 and Corollary 4.4 in [6] applied to $(Z^\sharp, \mathbb{P}_y^\sharp)$ and the fact that the latter stochastically dominates (Z, \mathbb{P}_x) . \square

We now prove Theorem 1.2theorem.1.2.

Proof of Theorem 1.2theorem.1.2. First of all from Lemma 2.1theorem.2.1, it is enough to prove our result for a process with a random environment which has no downward jumps larger than $1 - e^{-1}$. Hence, we assume in all this proof that Z is solution to (1.4Introduction and main resultsequation.1.4) with

$$S_t = \mathbf{d}t + \sigma B_t^{(e)} + \int_0^t \int_{(1,\infty)} (e^z - 1)N^{(e)}(ds, dz) + \int_0^t \int_{(-1,1)} (e^z - 1)\tilde{N}^{(e)}(ds, dz), \quad (2.2)$$

Recall that T_M denotes the first passage time for the process Z below a level $M > 0$, i.e.

$$T_M := \inf\{t \geq 0, Z_t \leq M\}.$$

As we will see below, the finiteness of the first moment of such random times will be useful for deducing our result. Hence, we first show that there exist $M > 0$ such that

$$\sup_{x \geq 0} \mathbb{E}_x[T_M] = \sup_{x \geq M} \mathbb{E}_x[T_M] < \infty. \quad (2.3)$$

With this goal in mind, we observe from assumption (1.9Introduction and main resultsequation.1.9) that

$$\lim_{y \rightarrow +\infty} \frac{g(y)}{y} = \infty \quad \text{and} \quad \lim_{y \rightarrow +\infty} \frac{\theta y - g(y)}{y} = -\infty, \quad (2.4)$$

for $\theta := \max\{b + \mathbf{d}, 0\}$. In addition from Lemma 2.3 in Le and Pardoux [10], we deduce that there exists $a_0 > 0$ such that $g(y) - \theta y > 0$ for any $y \geq a_0$ and

$$\int_{a_0}^{\infty} \frac{dy}{g(y) - \theta y} < \infty. \quad (2.5)$$

We then introduce $A > \theta(e - 1)$ in such a way that the inequality below holds

$$C(A) := 1 - \left(\frac{\theta(2\gamma^2 + \sigma^2)}{2A^2} + \frac{\theta}{A(A - \theta)} \int_{(0,1)} z^2 \mu(dz) + \frac{1}{A} \left(\int_{(1,\infty)} z \mu(dz) + \bar{\pi}(1) \right) \right. \\ \left. + \left(\frac{\theta}{A^2} + \frac{\theta}{A(A - \theta(e^1 - 1))} \right) \int_{(-1,1)} z^2 \pi(dz) \right) > 0, \quad (2.6)$$

where $\bar{\pi}(x) = \pi((x, \infty))$, $x \geq 0$. From the way we choose A and from (2.4Proofsequeation.2.4) and (2.5Proofsequeation.2.5), it is clear that there exists a constant $M > (a_0 + 1)e$ such that

$$\int_{Me^{-1}}^{\infty} \frac{dw}{g(w) - \theta w} \leq \frac{1}{A} \quad \text{and} \quad g(y) - \theta y \geq Ay \geq A, \quad \text{for all } y \geq Me^{-1}. \quad (2.7)$$

Such constant M will be our threshold. For our purposes, we define the function G in $C^2(\mathbb{R})$ as follows

$$G(y) = \begin{cases} \int_{a_0}^y \frac{dw}{g(w) - \theta w} & \text{if } y \geq a_0 + 1, \\ 0 & \text{if } y \leq a_0, \end{cases}$$

and such that G is non-negative and non-decreasing. Thus applying Itô's formula to $G(Z_{t \wedge T_M})$, we find

$$\begin{aligned} G(Z_{t \wedge T_M}) - G(Z_0) &= -t \wedge T_M - \int_0^{t \wedge T_M} \frac{g'(Z_s) - \theta}{(g(Z_s) - \theta Z_s)^2} \left(\gamma^2 Z_s + \frac{\sigma^2}{2} Z_s^2 \right) ds \\ &\quad + \int_0^{t \wedge T_M} \frac{\sqrt{2\gamma^2 Z_s}}{g(Z_s) - \theta Z_s} dB_s^{(b)} + \int_0^{t \wedge T_M} \frac{\sigma Z_s}{g(Z_s) - \theta Z_s} dB_s^{(e)} \\ &\quad + \int_0^{t \wedge T_M} \int_{(0, \infty)} Z_s \left(G(Z_s + z) - G(Z_s) - \frac{z}{g(Z_s) - \theta Z_s} \right) \mu(dz) ds \\ &\quad + \int_0^{t \wedge T_M} \int_{(0, \infty)} \int_0^{Z_{s-}} [G(Z_{s-} + z) - G(Z_{s-})] \tilde{N}^{(b)}(ds, dz, du) \\ &\quad + \int_0^{t \wedge T_M} \int_{(1, \infty)} [G(e^z Z_{s-}) - G(Z_{s-})] N^{(e)}(ds, dz) \\ &\quad + \int_0^{t \wedge T_M} \int_{(-1, 1)} [G(e^z Z_{s-}) - G(Z_{s-})] \tilde{N}^{(e)}(ds, dz) \\ &\quad + \int_0^{t \wedge T_M} \int_{(-1, 1)} \left(G(e^z Z_s) - G(Z_s) - \frac{(e^z - 1)Z_s}{g(Z_s) - \theta Z_s} \right) \pi(dz) ds. \end{aligned} \quad (2.8)$$

Firstly, note that $Z_s \geq Me^{-1} > a_0 + 1$ for any $s \leq t \wedge T_M$, in other words, we have an explicit formula for $G(Z_s)$. Next, we take expectations under the assumption that the process Z starts at $x \geq M$, in both sides of the previous identity and we study separately each term of the right-hand side. Our aim is to show that each expectation can be bounded from above using $\mathbb{E}_x[t \wedge T_M]$. For simplicity, we enumerate the lines in order of appearance.

(1) For the first integral of the right hand side of (2.8Proofsequeation.2.8), we recall that $Z_s \geq Me^{-1}$ for $s \leq t \wedge T_M$, that g is non-decreasing and we use the second formula in (2.7Proofsequeation.2.7) to deduce

$$\mathbb{E}_x \left[\int_0^{t \wedge T_M} \frac{\theta - g'(Z_s)}{(g(Z_s) - \theta Z_s)^2} \left(\gamma^2 Z_s + \frac{\sigma^2}{2} Z_s^2 \right) ds \right] \leq \frac{\theta(2\gamma^2 + \sigma^2)}{2A^2} \mathbb{E}_x[t \wedge T_M]. \quad (2.9)$$

(2) For the two Itô integrals of the second line of the right-hand side of (2.8Proofsequeation.2.8), we first observe that both are continuous local martingales. Since their quadratic variations satisfy

$$\mathbb{E}_x \left[\int_0^{t \wedge T_M} \left| \frac{\sqrt{2\gamma^2 Z_s}}{\theta Z_s - g(Z_s)} \right|^2 ds \right] \leq \frac{2\gamma^2 t}{A^2} \quad \text{and} \quad \mathbb{E}_x \left[\int_0^{t \wedge T_M} \left| \frac{\sigma Z_s}{\theta Z_s - g(Z_s)} \right|^2 ds \right] \leq \frac{\sigma^2 t}{A^2},$$

then both processes are martingales and therefore their expectations are equal to 0.

(3) We study the integral that appears in the third line in (2.8Proofsequeation.2.8) by separating $(0, \infty)$ into two parts $(0, 1]$ and $(1, \infty)$ and we first deal with the integral restricted to $(0, 1)$. Since g is non-decreasing, we bound $G(Z_s + z) - G(Z_s)$ from above by $z(g(Z_s) - \theta(Z_s + z))^{-1}$. In addition with the second formula in (2.7Proofsequeation.2.7), we obtain the following upper bound

$$\begin{aligned} \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(0,1)} Z_s \left[G(Z_s + z) - G(Z_s) - \frac{z}{g(Z_s) - \theta Z_s} \right] \mu(dz) ds \right] \\ \leq \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(0,1)} \frac{\theta z^2 Z_s}{(g(Z_s) - \theta(Z_s + z))(g(Z_s) - \theta Z_s)} \mu(dz) ds \right] \\ \leq \mathbb{E}_x [t \wedge T_M] \frac{\theta}{A(A - \theta)} \int_{(0,1)} z^2 \mu(dz) < \infty. \end{aligned} \quad (2.10)$$

Concerning the integral restricted to $(1, \infty)$, we drop the last term, which is negative, and we use again the second formula in (2.7Proofsequeation.2.7) to bound $G(Z_s + z) - G(Z_s)$ and find

$$\begin{aligned} \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(1,\infty)} Z_s \left[G(Z_s + z) - G(Z_s) - \frac{z}{g(Z_s) - \theta Z_s} \right] \mu(dz) ds \right] \\ \leq \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(1,\infty)} \int_0^z \frac{Z_s}{A(Z_s + w)} dw \mu(dz) ds \right] \leq \frac{\int_{(1,\infty)} z \mu(dz)}{A} \mathbb{E}_x [t \wedge T_M]. \end{aligned} \quad (2.11)$$

(4) We prove that the integral in the fourth line is a martingale and thus has expectation equals 0. Again, we split the interval $(0, \infty)$ into $(0, 1]$ and $(1, \infty)$ and use similar computations as in part (3) in order to deduce that the integral restricted to $(0, 1]$ is a square integrable martingale and the integral restricted to $(1, \infty)$ is a martingale. In other words, we manipulate

$$\mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(0,\infty)} Z_s f(G(Z_s + z) - G(Z_s), z) \mu(dz) ds \right],$$

with $f(x, z) = x^2 \mathbf{1}_{(0,1]}(z)$ and $f(x, z) = |x| \mathbf{1}_{(1,\infty)}(z)$ respectively.

(5) The stochastic integral in the fifth line in (2.8Proofsequeation.2.8) can be studied using first Fubini's Theorem and then the first inequality in (2.7Proofsequeation.2.7). We obtain

$$\mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(1,\infty)} |G(e^z Z_{s-}) - G(Z_{s-})| \pi(dz) ds \right] \leq \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{Z_s}^{\infty} \frac{dw}{g(w) - \theta w} \int_1^{\infty} \pi(dz) ds \right] \leq \frac{t}{A} \bar{\pi}(1).$$

This ensures that the stochastic integral can be written as the sum of a martingale and a finite variation process. Hence its expectation is bounded from above by $\mathbb{E}[t \wedge T_M] \bar{\pi}(1)/A$.

(6) Observe that the integral term of the sixth line is a square integrable martingale and that its expectation is thus equal to 0. Indeed, similarly as for the fourth line case, using that $g(w) - \theta w \geq Aw$, we find

$$\mathbb{E} \left[\int_0^{t \wedge T_M} \int_{(-1,1)} \left(\int_{Z_s}^{e^z Z_s} \frac{dw}{g(w) - \theta w} \right)^2 \pi(dz) ds \right] \leq \frac{\int_{(-1,1)} z^2 \pi(dz)}{A^2} t < \infty. \quad (2.12)$$

(7) Finally, we study the last line in (2.8Proofsequeation.2.8) by splitting again the integral into two parts, i.e. we split $(-1, 1)$ into $(-1, 0]$ and $(0, 1)$. Thus, using again the second inequality of

(2.7Proofsequeation.2.7) and the fact that $A > \theta(e - 1)$, we deduce that for any $w \in [0, y(e^z - 1)]$, $y \geq 1$ and $z \in (-1, 1)$,

$$g(y + w) - \theta(y + w) \geq g(y) - \theta ye^z \geq Ay - \theta(e - 1)y > 0.$$

Hence,

$$\begin{aligned} \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(0,1)} \left(G(e^z Z_s) - G(Z_s) - \frac{(e^z - 1)Z_s}{g(Z_s) - \theta Z_s} \right) \pi(dz) ds \right] \\ \leq \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(0,1)} \left(\frac{(e^z - 1)Z_s}{g(Z_s) - \theta Z_s e^z} - \frac{(e^z - 1)Z_s}{g(Z_s) - \theta Z_s} \right) \pi(dz) ds \right] \\ \leq \mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(0,1)} \frac{\theta(e^z - 1)^2 (Z_s)^2}{(AZ_s - \theta(e^1 - 1)Z_s)(g(Z_s) - \theta Z_s)} \pi(dz) ds \right] \\ \leq \mathbb{E}_x \left[t \wedge T_M \right] \frac{\theta \int_{(0,1)} z^2 \pi(dz)}{A(A - \theta(e^1 - 1))}. \end{aligned} \quad (2.13)$$

Similarly, we deal with the second part of the integral and deduce

$$\mathbb{E}_x \left[\int_0^{t \wedge T_M} \int_{(-1,0)} \left(G(e^z Z_s) - G(Z_s) - \frac{(e^z - 1)Z_s}{g(Z_s) - \theta Z_s} \right) \pi(dz) ds \right] \leq \mathbb{E}_x \left[t \wedge T_M \right] \frac{\theta \int_{(-1,0)} z^2 \pi(dz)}{A^2}. \quad (2.14)$$

Thus putting all pieces together (i.e. inequalities (2.9Proofsequeation.2.9), (2.10Proofsequeation.2.10), (2.11Proofsequeation.2.11), (2.13Proofsequeation.2.13), (2.14Proofsequeation.2.14) and the bound found in (5) together with (2.8Proofsequeation.2.8), (2.6Proofsequeation.2.6) and the three null-expectations), we deduce

$$\mathbb{E}_x \left[\int_x^{Z_{t \wedge T_M}} \frac{dw}{g(w) - \theta w} \right] \leq -C(A) \mathbb{E}[t \wedge T_M],$$

with $C(A) > 0$. In other words, for any $x, t \geq 0$,

$$\mathbb{E}_x \left[t \wedge T_M \right] \leq \frac{1}{C(A)} \mathbb{E}_x \left[\int_{Z_{t \wedge T_M}}^x \frac{dw}{g(w) - \theta w} \right] \leq \frac{1}{C(A)} \int_{Me^{-1}}^\infty \frac{dw}{g(w) - \theta w}.$$

Hence using the Monotone Convergence Theorem, as t goes to ∞ , we deduce (2.3Proofsequeation.2.3).

In order to finish the proof of the first statement, we first show that the time to extinction for the process Z starting from M is not almost surely infinite. We recall that we assumed that the environment has no negative jumps larger than $1 - e^{-1}$. Using Proposition 1.1theorem.1.1 (both processes with the same restriction on the negative jumps of the environment), we observe that for any $x \leq M$, the process (Z, \mathbb{P}_x) is stochastically dominated by $(Z^\sharp, \mathbb{P}_x^\sharp)$. The process Z^\sharp is a CB-process in a Lévy random environment (without competition) which is characterized by the branching mechanism $\psi(\lambda)$. Since ψ satisfies the Grey's condition, Theorem 4.1 of [6] ensures that there is $t_0 > 0$ for which

$$0 < \mathbb{P}_M^\sharp \left(Z_{t_0}^\sharp = 0 \right) \leq \inf_{x \leq M} \mathbb{P}_x \left(Z_{t_0} = 0 \right) := p.$$

Then we denote by T_M^x for the stopping time T_M under \mathbb{P}_x . Reasoning by recurrence and using the Markov property, we prove that the extinction time of Z is stochastically dominated from above by the random variable $\sum_{i=1}^\xi (\bar{\tau}_i + t_0)$, where ξ is a geometric random variable that counts the number of random steps before Z becomes extinct and $\{\bar{\tau}_i\}_{i \geq 0}$ are i.i.d., independent of ξ and have the same distribution as $\sup_{x \geq 0} T_M^x$. To be more precise, the algorithm is as follows: we start from x , we wait a random time

$\tau_1 \leq \bar{\tau}_1$ until the process is below the level M and then the process becomes extinct before an time interval of size t_0 with probability p . If the process is not extinct after the time $\tau_1 + t_0$, we start again the procedure thanks to the Markov property. Hence,

$$\sup_{x \geq 0} \mathbb{E}_x[T_0] \leq \frac{1}{p} \left(t_0 + \mathbb{E} \left[\sup_{x \geq 0} T_M^x \right] \right) = \frac{1}{p} \left(t_0 + \sup_{x \geq 0} \mathbb{E}_x[T_M] \right) < \infty,$$

which ends the first part of the proof of Theorem 1.2theorem.1.2.

It remains to prove that the point ∞ is a continuous entrance point and that under the absence of negative jumps on the jump structure of the environment the process can be extended to a Feller process on $[0, \infty]$. Proposition 1.1theorem.1.1 guarantees that the sequence of random variables (T_m, \mathbb{P}_x) is increasing with respect to x . Thus, it converges almost surely to a random variable here denoted by T_m^∞ . Then, from the first part of this proof, for any $m \geq M$,

$$\sup_{x \geq m} \mathbb{E}_x[T_m] \leq \int_{me^{-1}}^{\infty} \frac{dw}{g(w) - \theta w} \xrightarrow{m \rightarrow \infty} 0. \quad (2.15)$$

From Chebyshev's inequality, we deduce that for any $t > 0$

$$\lim_{m \rightarrow \infty} \lim_{x \rightarrow \infty} \mathbb{P}_x(T_m > t) \leq \lim_{m \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{\mathbb{E}_x[T_m]}{t} = 0.$$

In other words the point ∞ is a continuous entrance point.

For the extension to a Feller process in the absence of negative jumps, let us introduce the semigroup $(\mathbb{P}_t, t \geq 0)$ associated to the process Z as follows: for any continuous function f , we have $\mathbb{P}_t f(x) = \mathbb{E}_x[f(Z_t)]$ which is well defined on $[0, \infty)$. Our aim is to prove that it can be extended on $[0, \infty]$. With this aim in mind, we set $t \geq 0$ and a continuous function f and we prove that $(\mathbb{P}_t f(x), x \geq 0)$ is a Cauchy sequence in \mathbb{R} when x goes to ∞ .

We fix $\varepsilon > 0$ and we use the sequence of stopping times $\{T_m, m \geq 0\}$ together with the Markov property, to find

$$\mathbb{P}_t f(x) = \mathbb{E}_x [\mathbf{1}_{\{T_m > t\}} f(Z_t)] + \mathbb{E}_x [\mathbf{1}_{\{T_m \leq t\}} \mathbb{P}_{t-T_m} f(Z_{T_m})].$$

Since $Z_{T_m} = m$ a.s., from the absence of negative jumps, we observe, for any $x, y \in [m, \infty)$,

$$|\mathbb{P}_t f(x) - \mathbb{P}_t f(y)| \leq \left| \mathbb{E}_x [h(m, T_m)] - \mathbb{E}_y [h(m, T_m)] \right| + 2\|f\| \left(\mathbb{P}_x(T_m > t) + \mathbb{P}_y(T_m > t) \right), \quad (2.16)$$

where $h(m, T_m) = \mathbf{1}_{\{T_m \leq t\}} \mathbb{P}_{t-T_m} f(m) + \mathbf{1}_{\{T_m > t\}} f(m)$. We end the proof by using similar arguments to those of the proof of Theorem 20.13 in [7]. Indeed, since the sequence of random variables (T_m, \mathbb{P}_x) converges towards T_m^∞ and (2.15Proofsequeation.2.15) holds. Together with the fact that $(T_m, \mathbb{P}_x) \leq (T_{m'}, \mathbb{P}_x)$ for any $m \leq m'$, we deduce that the sequence $\{T_m^\infty, m \geq 0\}$ is decreasing and converges to 0 a.s. Then we can fix $m > M$ such that for any x, y sufficiently large,

$$2\|f\| \left(\mathbb{P}_x(T_m > t_0) + \mathbb{P}_y(T_m > t_0) \right) \leq \varepsilon. \quad (2.17)$$

Moreover, (2.15Proofsequeation.2.15) implies that $\mathbb{P}_x \circ (T_m)^{-1}$ converges weakly when x goes to ∞ . Since the mapping $s \mapsto h(m, s)$ is continuous, for x, y sufficiently large, we have

$$\left| \mathbb{E}_x [h(m, T_m)] - \mathbb{E}_y [h(m, T_m)] \right| \leq \varepsilon. \quad (2.18)$$

Hence $(\mathbb{P}_t f(x), x \geq 0)$ is a Cauchy sequence in \mathbb{R} when x goes to ∞ , whose limit is denoted by $\mathbb{P}_t f(\infty)$. To complete the proof of Theorem 1.2, it is sufficient to note that for any x sufficiently large and any t sufficiently small,

$$|\mathbb{P}_t f(\infty) - f(\infty)| \leq |\mathbb{P}_t f(\infty) - \mathbb{P}_t f(x)| + \|\mathbb{P}_t f - f\|_{[0, \infty)} + |f(x) - f(\infty)| \leq \varepsilon.$$

The second term of the right-hand side is small since f is continuous on $[0, \infty)$ and Z is càdlàg. \square

Proof of Theorem 1.3. We first prove, under our assumptions, that for any fixed $t \geq 0$, the expectation of Z_t is finite. To this aim, we consider the stopping time

$$\tau_m = \inf\{t \geq 0, Z_t \geq m\},$$

which is a localizing sequence of all local martingales in (1.4). Then for any $s, t \geq 0$,

$$\mathbb{E}_x[Z_{t \wedge \tau_m}] = \mathbb{E}_x[Z_s] + \left(b + \mathbf{d} + \int_{(-1, 1)^c} (e^z - 1)\pi(dz) \right) \mathbb{E}_x \left[\int_s^{t \wedge \tau_m} Z_u du \right] - \mathbb{E}_x \left[\int_s^{t \wedge \tau_m} g(Z_u) du \right].$$

Let us denote $\bar{\theta} := \max\left\{ b + \mathbf{d} + \int_{(-1, 1)^c} (e^z - 1)\pi(dz), 0 \right\}$. Then using Fatou's lemma, the monotone convergence theorem, the positivity of g and Jensen's inequality, we obtain that for any $s, t \geq 0$,

$$\lim_{m \rightarrow \infty} \mathbb{E}_x[Z_{t \wedge \tau_m}] \leq \mathbb{E}_x[Z_t] \leq \mathbb{E}_x[Z_s] + \bar{\theta} \int_s^t \mathbb{E}_x[Z_u] du - \int_s^t g(\mathbb{E}_x[Z_u]) du.$$

Thus, let us study the unique solution to

$$\begin{cases} v'_x(t) = \bar{\theta}v_x(t) - g(v_x(t)), \quad \forall t \in [0, +\infty), \\ v_x(0) = x. \end{cases} \quad (2.19)$$

As g is convex and satisfies (1.9), there exists $z_0 > 0$ such that $z \mapsto \bar{\theta}z - g(z)$ is negative and decreasing on $(z_0, +\infty)$ and $\bar{\theta}z_0 - g(z_0) = 0$. Thus, as soon as $x > z_0$, $v_x(t) > z_0$ for any $t \geq 0$ and

$$v_x(t) = I^{-1}(t + I(x)), \quad \text{where} \quad I(z) = \int_z^\infty \frac{du}{g(u) - \bar{\theta}u}.$$

Moreover, using an argument by contradiction, we prove that for any $t \geq 0$, $\mathbb{E}_x[Z_t] \leq v_x(t)$, as soon as $x > z_0$. As I is non-decreasing, we deduce that for any $t \geq 0$ and for any $x \geq z_0$,

$$\mathbb{E}_x[Z_t] \leq I^{-1}(t) < \infty.$$

This is sufficient to obtain (1.10).

Finally, we prove that the process Z comes down from infinity. First, we observe from Theorem 1.2 that ∞ is a continuous entrance point. Then recall that from Proposition 1.1, (Z, \mathbb{P}_x) is stochastically dominated by (Z, \mathbb{P}_y) for any $x \leq y$. Hence for any $t \geq 0$, $\lim_{x \rightarrow +\infty} (Z_t, \mathbb{P}_x)$ exists a.s., here denoted by Z_t^∞ , and such limit has finite expectation. To end the proof, it suffices to prove that the semi-group $(\mathbb{P}_t, t \geq 0)$ associated to the process Z can be extended on $[0, \infty]$. To this aim, we set $t \geq 0$ and a continuous function f on $[0, \infty]$, and we define $\mathbb{P}_t f(\infty)$ by $\mathbb{E}[f(Z_t^\infty)]$. From the a.s. convergence of (Z_t, \mathbb{P}_x) , it is straightforward that $\lim_{x \rightarrow \infty} \mathbb{P}_t f(x) = \mathbb{P}_t f(\infty)$. We end the proof similarly as the one of Theorem 1.2 by proving that $\mathbb{P}_t f(\infty)$ converges to $f(\infty)$ when t goes to 0. \square

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