Extinction time of CB-processes with competition in a Lévy random environment

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January 25, 2018

Abstract

In this paper, we are interested on the extinction time of continuous state branching processes with competition in a Lévy random environment. In particular we prove, under the so-called Grey's condition together with the assumption that the Lévy random environment does not drift towards infinity, that for any starting point the process gets extinct in finite time a.s. Moreover if we replace the condition on the Lévy random environment by a technical integrability condition on the competition mechanism, then the process also gets extinct in finite time a.s. and it comes down from infinity under the condition that the negative jumps associated to the environment are driven by a compound Poisson process.

Then the logistic case in a Brownian random environment is treated. Our arguments are base on a Lamperti-type representation where the driven process turns out to be a perturbed Feller diffusion by an independent spectrally positive Lévy process. If the independent random perturbation is a subordinator then the process converges to a specified distribution; otherwise, it goes extinct a.s. In the latter case and following a similar approach to Lambert [11], we provide the expectation and the Laplace transform of the absorption time, as a functional of the solution to a Riccati differential equation.

Key words and phrases: Continuous state branching processes in random environment, competition, population dynamics, logistic process, extinction, Continuous state branching processes with immigration, Ricatti differential equations

MSC 2000 subject classifications: 60J80, 60J70, 60J85.

1 Introduction and main results.

The prototypical example of continuous state branching processes (or CB-processes) with competition is the so-called logistic Feller diffusion which is defined as the unique strong solution of the following stochastic differential equation (SDE),

$$Y_t = Y_0 + b\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)} - c\int_0^t Y_s^2 \mathrm{d}s, \qquad t \geq 0,$$

where $b \in \mathbb{R}, c > 0$ and $B^{(b)} = (B_t^{(b)}; t \geq 0)$ is a standard Brownian motion. Such family of processes has been studied by several authors, see for instance [2, 11, 14] and the references therein. An important

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feature of the logistic Feller diffusion is that it can also be constructed as scaling limits of Bienaymé-Galton-Watson processes with competition which are continuous time Markov chains where time steps are the non overlapping generations with individuals behaving independently from one another and each giving birth to a random number of offspring (belonging to the next generation) but also considering competition pressure, in other words each pair of individuals interact at a fixed rate and one of them is killed as result of such interaction. For further details of such convergence we refer to section 2.4 in Lambert [11]

Using a Lamperti-type random time change representation, Lambert [11] generalised the logistic Feller diffusion by replacing the diffusion term by a general CB-process. More precisely, Lambert considered the following generalised Ornstein-Uhlenbeck process starting from $x > 0$, which is described as the unique strong solution of the SDE

$$\mathrm{d}R_t = \mathrm{d}X_t - cR_t\mathrm{d}t,$$

where $c > 0$ and $X = (X_t, t \geq 0)$ denotes a spectrally positive Lévy process, that is to say a càdlàg stochastic process with independent and stationary increments with no negative jumps. We denote by $\mathbf{P}_x$ for the law of X started from $x \in \mathbb{R}$. For simplicity, we let $\mathbf{P} = \mathbf{P}_0$. It is known that the law of any spectrally positive Lévy process X is completely characterized by its Laplace exponent ψ which is defined as $\psi(\lambda) = \log \mathbf{E}[e^{-\lambda X_1}]$ for $\lambda \geq 0$, and satisfies the so-called Lévy-Khintchine formula

$$\psi(u) = -bu + \gamma^2 u^2 + \int_{(0,\infty)} \Big(e^{-ux} - 1 + ux\mathbf{1}_{\{x<1\}}\Big)\mu(\mathrm{d}x), \tag{1.1}$$

where $b \in \mathbb{R}$, $\gamma \geq 0$ and μ is a Radon measure concentrated on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge z^2)\mu(\mathrm{d}z) < \infty. \tag{1.2}$$

It is well known that the triplet (b, γ, μ) characterises the law of X. According to Theorem 17.5 in Sato [20], the following log-moment condition

$$\mathbf{E}\Big[\log^+ X_1\Big] < \infty,$$

is necessary and sufficient for the process R to possess an invariant distribution. It is also important to note that this log-moment condition is equivalent to

$$\int_1^\infty \log(u)\mu(\mathrm{d}u) < \infty, \tag{1.3}$$

see for instance Theorem 25.3 in [20]. For further details on Lévy and generalised Ornstein-Uhlenbeck processes, we refer the monograph of Sato [20].

Let T_0^R denotes the first hitting time of 0 of the generalised Ornstein-Uhlenbeck process R, i.e. $T_0^R = \inf\{s : R_s = 0\}$, and consider the random clock

$$\eta_t = \int_0^{t\wedge T_0^R} \frac{\mathrm{d}s}{R_s}, \qquad \text{for} \quad t > 0.$$

Let C denotes the right-continuous inverse of the clock η. According to Lambert [11] the logistic branching process is defined as follows

$$Y_t = \begin{cases} R_{C_t} & \text{if } 0 \leq t < \eta_\infty \\ 0 & \text{if } \eta_\infty < \infty \text{ and } t \geq \eta_\infty. \end{cases}$$

It is important to note that the above definition is inconsistent with the fact that the process R is positive, drifts to ∞ and $\eta_\infty < \infty$ a.s. The latter may occur when X is a subordinator and the log-moment condition (1.3) is not satisfied. Actually, the process Y does not explode if the log-moment condition (1.3) holds.

The function ψ is also known as the branching mechanism of the logistic branching process Y. We also note that when $c = 0$, the process Y is a CB-process and the latter time change relationship is the so-called Lamperti transform which was established by Lamperti [12]. Some interesting path properties of the logistic branching processes were derived by Lambert [11] as consequence of this path transformation. For instance, in the case when the process X is a subordinator i.e. its Laplace exponent is of the form

$$\psi(u) = -\delta u - \int_{(0,\infty)} (1 - e^{-ux})\mu(\mathrm{d}x), \quad u \geq 0,$$

with $\delta \geq 0$, the log-moment condition (1.3) is satisfied and one of the following conditions $\delta \neq 0$, $\mu(0,\infty) = \infty$ or $c < \mu(0,\infty) < \infty$ holds then the process Y is positive recurrent on $(\delta/c, \infty)$ and possesses a stationary distribution which can be computed explicitly. Moreover if (1.3) holds but none of the latter conditions are satisfied, then the process Y is null recurrent in $(0,\infty)$ and converges to 0 in probability (see Theorem 3.4 in [11]).

When X is not a subordinator and condition (1.3) is satisfied, then the process Y goes to 0 a.s. Moreover, the process Y gets extinct in finite time a.s. accordingly as

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty, \tag{1.4}$$

which is the so-called Grey's condition. Let T_0^Y denotes the time to extinction of the process Y, i.e $T_0^Y = \inf\{t \geq 0 : Y_t = 0\}$. In [11], under Grey's condition, the Laplace transform of T_0^Y was computed explicitly and the law of the process coming down from infinity was also determined.

More general competition mechanisms were considered by Ba and Pardoux [1] in the case when the branching mechanism is of the form $\psi(u) = \gamma^2 u^2$, for $u \geq 0$, see also Chapter 8 in the monograph of Pardoux [16]. In this case, the CB-process with competition can be written as the unique strong solution of the following SDE

$$Y_t = Y_0 + \int_0^t h(Y_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)},$$

where h is a continuous function satisfying $h(0) = 0$ and such that

$$h(x+y) - h(x) \leq Ky, \qquad x, y \geq 0,$$

for some positive constant K. According to Ba and Pardoux, the process Y gets extinct in finite time if and only if

$$\int_1^{\infty} \exp\left\{-\frac{1}{2}\int_1^u \frac{h(r)}{r}\mathrm{d}r\right\}\mathrm{d}u = \infty.$$

Recently, for a general branching mechanism ψ satisfying (1.1) with

$$\int_{(0,\infty)} (z \wedge z^2)\mu(\mathrm{d}z) < \infty, \tag{1.5}$$

or equivalently $-\psi'(0+) < \infty$, Ma [14] (see also Berestycki et al. [2]) considered a general competition mechanism g which is a non-decreasing continuous function on $[0,\infty)$ with $g(0) = 0$ and proved that the associated branching process with competition satisfies the following SDE

$$Y_t = Y_0 + \int_0^t bY_s\mathrm{d}s - \int_0^t g(Y_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $B^{(b)}$ is a standard Brownian motion which is independent of the Poisson random measure $N^{(b)}$ which is defined on $\mathbb{R}^3_+$, with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ such that (1.5) is satisfied, and $\widetilde{N}^{(b)}$ denotes its compensated version.

Our aim is to study the time to extinction of a generalized version of the previous family of processes that we call CB-processes with competition in a Lévy random environment. Such family of processes has been introduced recently by Palau and Pardo [15] as the unique strong solution of the following SDE

$$\begin{aligned} Z_t = Z_0 + b\int_0^t Z_s\mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B^{(b)}_s + \int_0^t Z_{s-}\mathrm{d}S_s \\ + \int_0^t\int_{[1,\infty)}\int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t\int_{(0,1)}\int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u), \end{aligned} \tag{1.6}$$

where g is a non-decreasing continuous function on $[0,\infty)$ with $g(0)=0$, $B^{(b)}$ and $N^{(b)}$ are defined as before but with the difference that the measure μ satisfies integral condition (1.2) and S is a Lévy process independent of $B^{(b)}$ and $N^{(b)}$ which can be written as follows

$$S_t = \mathtt{d}t + \sigma B^{(e)}_t + \int_0^t\int_{(-1,1)^c}(e^z-1)N^{(e)}(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{(-1,1)}(e^z-1)\widetilde{N}^{(e)}(\mathrm{d}s,\mathrm{d}z), \tag{1.7}$$

with $\mathtt{d}\in\mathbb{R}, \sigma\geq 0$, $B^{(e)}=(B^{(e)}_t, t\geq 0)$ is a standard Brownian motion and $N^{(e)}$ is a Poisson random measure taking values on $\mathbb{R}_+\times\mathbb{R}$ independent of $B^{(e)}$ and with intensity $\mathrm{d}s\pi(\mathrm{d}z)$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}}(1\wedge z^2)\pi(\mathrm{d}z)<\infty. \tag{1.8}$$

For our purposes, we also introduce the auxiliary Lévy process

$$K_t = \beta t + \sigma B^{(e)}_t + \int_0^t\int_{(-1,1)^c} zN^{(e)}(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{(-1,1)} z\widetilde{N}^{(e)}(\mathrm{d}s,\mathrm{d}z), \qquad t\geq 0,$$

where

$$\beta = \mathtt{d} - \frac{\sigma^2}{2} - \int_{(-1,1)}(e^z-1-z)\pi(\mathrm{d}z).$$

It is important to note that according to Palau and Pardo [15], we can take the competition mechanism g to be a real-valued continuous function satisfying some technical conditions and not necessarily positive and non-decreasing. In other words, if we consider g to be negative and non-increasing then the competition mechanism can be interpreted as cooperation in the sense of Gonzalez-Casanova et al. [6]. We treat more general competition mechanisms in Section 2 only in the diffusion case, that is to say, when there are no jumps coming neither from the branching mechanism or the random environment.

We denote by $\mathbb{P}_x$ the law of Z starting from $x>0$, and define the first hitting time to 0 of Z as follows

$$T_0 = \inf\{t\geq 0, Z_t = 0\},$$

with the convention that $\inf\{\emptyset\}=\infty$. If there is no competition i.e. $g\equiv 0$, He et al. [19] proved that Grey's condition (1.4) is a necessary and sufficient condition for CB-processes in a Lévy random environment to become extinct with positive probability, see Theorem 4.1 in [19]. Moreover, if the auxiliary process K does not drift to ∞ or equivalently

$$\liminf_{t\to\infty} K_t = -\infty,$$

and Grey's condition (1.4) holds, then its associated CB-process in a Lévy random environment becomes extinct at finite time a.s., see Corollary 4.4 in [19].

For simplicity of exposition, we first present our results for the case when the random environment is driven by a general Lévy process and the competition mechanism is a non-decreasing positive continuous function. Then we deal with the logistic case, i.e. $g(x) = cx^2$ for $c > 0$, in a Brownian random environment where more details can be provided about the extinction time and the asymptotic behaviour for large times.

1.1 Lévy random environment case.

Here, we assume that the Lévy measure μ satisfies (1.5) or equivalently $|\psi'(0+)| < \infty$. Hence, the SDE (1.6) can be simplified as follows

$$\begin{aligned} Z_t = Z_0 - \psi'(0+)\int_0^t Z_s \mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} \\ + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}\mathrm{d}S_s, \end{aligned} \tag{1.9}$$

where

$$\psi'(0+) = -b - \int_{[1,\infty)} z\mu(\mathrm{d}z),$$

S is defined by (1.7) and its associated Lévy measure satisfies (1.8), and g is a non-decreasing continuous function with $g(0) = 0$.

Our first result provides a comparison criteria for CB-processes with competition in a Lévy random environment and implicitly gives a necessary condition under which they become extinct. Before we state our results, we introduce the CB-process in a Lévy random environment $Z^\sharp = (Z_t^\sharp, t \geq 0)$ as the unique strong solution of the following SDE

$$\begin{aligned} Z_t^\sharp =& Z_0^\sharp - \psi(0+)\int_0^t Z_s^\sharp \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s^\sharp}\mathrm{d}B_s^{(b)} + \int_0^t Z_{s-}^\sharp \mathrm{d}S_s \\ &+ \int_0^t \int_{(0,1)} \int_0^{Z_{s-}^\sharp} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}^\sharp} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{1.10}$$

For simplicity, we denote its law starting from $x \geq 0$ by $\mathbb{P}_x^\sharp$.

Theorem 1.1. *Assume that the Lévy measure μ associated to the branching mechanism ψ satisfies* (1.5). *For $y \geq x \geq 0$, we have that $(Z, \mathbb{P}_x)$ is stochastically dominated by $(Z, \mathbb{P}_y)$. Moreover, the process $(Z, \mathbb{P}_x)$ is stochastically dominated by $(Z^\sharp, \mathbb{P}_y^\sharp)$.*

In particular if the branching mechanism ψ satisfies Grey's condition (1.4), *then $(Z, \mathbb{P}_x)$ becomes extinct with positive probability. Furthermore if K does not drift to ∞, i.e.*

$$\liminf_{t\to\infty} K_t = -\infty,$$

then $(Z, \mathbb{P}_x)$ becomes extinct at finite time a.s.

For our next result, we assume the following integral condtion on the competition mechanism g. Assume that there exists $z_0 > 0$ such that $g(z_0) > 0$ and

$$\int_{z_0}^\infty \frac{\mathrm{d}y}{g(y)} < \infty. \tag{1.11}$$

Under an integral condition on the negative jumps of the random environment, the following result says that a CB-process with competition in a Lévy random environment comes down from infinity. This phenomena has been observed and studied by several authors in branching processes with interactions, see for instance González-Casanova et al. [6], Lambert [11] and Pardoux [16]. Formally, we define the law $\mathbb{P}_\infty$ starting from infinity with values in $\mathbb{R}_+ \cup \{\infty\}$ as the limits of the laws $\mathbb{P}_x$ of the process issued from x. When the limiting process is non-degenerate, it hits finite values in finite time with positive probability. This behaviour is captured by the notion of *coming down from infinity*.

We now state the main result of this part.

Theorem 1.2. *Assume that the Lévy measure μ associated to the branching mechanism satisfies* (1.5). *We also suppose that Grey's condition* (1.4) *and the integral condition on the competition mechanism* (1.11) *hold, then*

$$\sup_{x\geq 0} \mathbb{E}_x\big[T_0\big] < \infty.$$

In addition, if $\int_{(-1,0)} \pi(\mathrm{d}z) < \infty$, the process comes down from infinity.

The integrability condition in the previous Theorem implies that the negative jumps of the random environment are driven by a compound Poisson process. The intuition of the necessity of this condition comes from the fact that a negative jump of the process Z is proportional to its size and if the random environment has infinite activity and the process Z starts from a very large value then immediately after, the process Z may have a massive negative jump even if the jump size of the random environment is very small. In other words, any large threshold from below may be cross by Z by a very large jump even if the size of the jumps of the environment are very small. This situation does not seem very easy to control and it is crucial to take the starting point of Z to go to infinity. The integrability condition that we impose intuitively implies that negative jumps of Z are allowed after an exponential holding time meaning that the process has enough time to "come down from infinity" and then produces negative jumps of finite size.

We point out that the finiteness of the expected value of the extinction time when there is no random environment was first considered by Le [17].

1.2 Logistic case in a Brownian random environment.

In the sequel, we assume that the random environment is driven by a Brownian motion and that the competition mechanism is logistic, that is to say $g(x) = cx^2$, for $x \geq 0$ and $c > 0$. Since the structure of the random environment allow us to have a better understanding of the process, as we will see in Section 4, here we allow to the Lévy measure, associated to the branching mechanism, to satisfy (1.2), in other words $\psi'(0+)$ may take the value $-\infty$.

For this particular case, the SDE (1.9) can be rewritten as follows

$$\begin{aligned} Z_t = Z_0 + b\int_0^t Z_s\mathrm{d}s - c\int_0^t Z_s^2\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \sigma\int_0^t Z_s\mathrm{d}B_s^{(e)} \\ + \int_0^t\int_{[1,\infty)}\int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t\int_{(0,1)}\int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u), \end{aligned} \tag{1.12}$$

with $c > 0$ and $\sigma \geq 0$. Observe that when there is no environment, i.e. $\sigma = 0$, the process Z corresponds to the so-called logistic branching process which was already described at the beginning of the introduction and deeply studied by Lambert [11] . We also observe that the linear drift case, i.e $\psi(u) = -bu$ for $u \geq 0$, corresponds to the monomorphic model of a single population living in a patchy environment which was studied recently in Evans et al. [5].

It is also important to note that in this case a Lamperti-type representation is satisfied for Z and it is very useful for the development of the next results. For simplicity, we provide such time-change representation in Section 4 which is established for more general competition mechanisms than the logistic case, and we continue here with the exposition of our results.

We first deal with the case when the branching mechanism is associated to the Laplace transform of a subordinator, that is to say

$$\psi(z) = -\delta z - \int_{(0,\infty)} (1 - e^{-zu})\mu(\mathrm{d}u), \tag{1.13}$$

where

$$\int_{(0,\infty)} (1 \wedge u)\mu(\mathrm{d}u) < \infty \qquad \text{and} \qquad \delta = b - \int_{(0,1)} u\mu(\mathrm{d}u) \geq 0.$$

Under this assumption, we deduce the following identity for the total population size of process Z up to time T_a, the first hitting time of process Z at a, which was defined in (1.20). Let us define

$$\omega(x) = cx + \frac{\sigma^2 x^2}{2} \quad \text{and} \quad f_\lambda(x) = \int_0^\infty \frac{\mathrm{d}z}{\omega(z)} \exp\left\{-xz + \int_\ell^z \frac{\lambda - \psi(u)}{\omega(u)}\mathrm{d}u\right\}, \qquad x \geq 0, \tag{1.14}$$

where ℓ is an arbitrary constant larger than 0.

Proposition 1.3. *Let Z be the unique strong solution of* (1.12) *with branching mechanism given by* (1.13)*. For every $\lambda > 0$ and $x \geq a \geq 0$, we have*

$$\mathbb{E}_x\left[\exp\left\{-\lambda \int_0^{T_a} Z_s \mathrm{d}s\right\}\right] = \frac{f_\lambda(x)}{f_\lambda(a)}. \tag{1.15}$$

The following Lemma is needed for the description of the invariant distribution of Z whenever it exists. We point out that the following two results focus on the case $\sigma^2 > 0$ since the case $\sigma^2 = 0$ has been already treated by Lambert [11]. Before we continue with our exposition, we introduce the following notation. Let

$$m(\lambda) := \int_0^\lambda \frac{\psi(u)}{\omega(u)}\mathrm{d}u, \qquad \text{for} \quad \lambda \geq 0, \tag{1.16}$$

which is well defined under the log-moment condition (1.3).

Lemma 1.4. *Assume that $\sigma^2 > 0$ and that the branching mechanism is given by* (1.13) *and satisfies the log-moment condition* (1.3)*. Then the following identity holds*

$$-m(\lambda) = \frac{2}{\sigma^2} \int_0^\infty \left(1 - e^{-\lambda z}\right) \frac{e^{-\frac{2c}{\sigma^2}z}}{z} \left(\delta + \int_0^z e^{\frac{2c}{\sigma^2}u} \bar{\mu}(u)\mathrm{d}u\right) \mathrm{d}z, \tag{1.17}$$

where $\bar{\mu}(x) = \mu(x, \infty)$, and

$$\int_{(0,\infty)} e^{-\lambda z} \nu(\mathrm{d}z) = e^{m(\lambda)}, \qquad \lambda \geq 0,$$

defines a unique probability measure ν on $(0, \infty)$ which is infinitely divisible. In addition, it is self-decomposable whenever $\bar{\mu}(0) \leq \delta$.

We recall that self-decomposable distributions on $(0, \infty)$ is a subclass of infinitely divisible distributions whose Lévy measures have densities which are decreasing on $(0, \infty)$. We refer to Sato [20] for further details of self-decomposable distributions. It is also important to note that condition (1.3) guarantees that $m(\lambda)$ is well defined (see Corollary 3.21 in Li [13]).

In order to introduce the limiting distribution associated to Z, when it exists, we first provide conditions under which $\int_{(0,\infty)} s^{-1}\nu(\mathrm{d}s)$ is finite. For any z sufficiently small, we define two sequences of functions as follows

$$l^{(1)}(z) = |\ln(z)| \qquad \text{and} \qquad l^{(k)}(z) = \ln(l^{(k-1)}(z)), \qquad k \in \mathbb{N}, k \geq 2,$$
$$I^{(1)}(z) = l^{(1)}(z)\int_0^z \bar{\mu}(w)\mathrm{d}w \quad \text{and} \quad I^{(k)}(z) = l^{(k)}(z)\left(I^{(k-1)}(z) - \frac{\sigma^2}{2}\right), \quad k \in \mathbb{N}, k \geq 2.$$

Observe that for any $k \in \mathbb{N}$, $I^{(k)}(z)$ is well defined for z sufficiently small. On the other hand $l^{(k)}(z)$ is well defined for both, z sufficiently small and large. Then, for any continuous function f taking values in $\mathbb{R}$, we set

$$\mathtt{Adh}(f) = \left[\liminf_{z\to 0} f(z), \limsup_{z\to 0} f(z)\right] \subset \mathbb{R}.$$

We are now ready to establish the following two conditions:

(∂) There exists $n \in \mathbb{N}$ such that $\inf(\mathtt{Adh}(I^{(n)})) > \dfrac{\sigma^2}{2}$ and $\mathtt{Adh}(I^{(k)}) = \left\{\dfrac{\sigma^2}{2}\right\}$, for all $k \in \{1, .., n-1\}$,

($\eth$) There exists $n \in \mathbb{N}$ such that $\sup(\mathtt{Adh}(I^{(n)})) < \dfrac{\sigma^2}{2}$ and $\mathtt{Adh}(I^{(k)}) = \left\{\dfrac{\sigma^2}{2}\right\}$, for all $k \in \{1, .., n-1\}$.

For instance if $\bar{\mu}(0) < \infty$, that is to say ψ is the Laplace exponent of a compound Poisson process, then condition ($\eth$) holds.

Proposition 1.5. *Assume that $2\delta \geq \sigma^2 > 0$, $c > 0$ and that the branching mechanism is given by* (1.13). *Then the point* 0 *is polar, that is to say $\mathbb{P}_x(T_0 < \infty) = 0$ for all $x > 0$. Moreover if*

$$\int_0^1 \frac{\mathrm{d}z}{z}\exp\left\{-\int_z^1\int_0^\infty \frac{(1-e^{-us})}{\omega(u)}\mu(\mathrm{d}s)\mathrm{d}u\right\} = \infty \tag{1.18}$$

Z is recurrent. Additionally,

a) If $2\delta > \sigma^2$ then the process Z is positive recurrent in $(0,\infty)$. Its invariant distribution ρ has a finite expected value if and only if (1.3) *holds. If the latter holds, then ρ is the size-biased distribution of ν, in other words*

$$\rho(\mathrm{d}z) = \left(\int_{(0,\infty)} s^{-1}\nu(\mathrm{d}s)\right)^{-1} z^{-1}\nu(\mathrm{d}z), \qquad z > 0. \tag{1.19}$$

b) Assume that $2\delta = \sigma^2$ and (1.3) *holds,*

b.1) if condition (∂) is also satisfied, then Z is positive recurrent in $(0,\infty)$ and its invariant probability is defined by (1.19),

b.2) or if condition ($\eth$) is satisfied, then the process Z is null recurrent and converges to 0 *in probability.*

Finally, if (1.18) *is not satisfied, then Z is transient and for any $x \geq a > 0$,*

$$\mathbb{P}_x\left(\lim_{t\geq 0} Z_t = \infty\right) = 1 \qquad \textit{and} \qquad \mathbb{P}_x\left(\inf_{t\geq 0} Z_t < a\right) = \frac{f_0(x)}{f_0(a)}.$$

It is important to note that (1.18) is satisfied as soon as (1.3) holds.

Proposition 1.6. *Assume that $\sigma^2 > 2\delta$ but $\sigma^2 > 0$, $c > 0$ and X is a subordinator, then the process converges to 0 with positive probability, in other words $\mathbb{P}_x(\lim_{t\to 0} Z_t = 0) > 0$, for all $x > 0$.*

Finally, we consider the case when the process X is not a subordinator, in other words the branching mechanism ψ satisfies that there exist $\vartheta \geq 0$ such that $\psi(z) > 0$ for any $z \geq \vartheta$. For simplicity if the branching mechanism ψ satisfies the latter property together with the log-moment condition (1.3), we say that it is general.

Our last result provides a complete characterization of the Laplace transform of the stopping time

$$T_a = \inf\{t \geq 0 : Z_t \leq a\}, \tag{1.20}$$

for $a \leq 0$. Our results strengthen those of Lambert [11] in the following way, it treats the time before hitting any level, and not only for the extinction time, and also it considers the presence of the Brownian environment.

Recall the definition of m from (1.16) and introduce the functional

$$\mathtt{I}(\lambda) := \int_0^\lambda e^{m(u)}\mathrm{d}u, \qquad \text{for} \quad \lambda \geq 0. \tag{1.21}$$

Observe from our assumptions that m is positive on (ϑ, ∞) implying that $\mathtt{I}(z)$ is a bijection from $\mathbb{R}_+$ into itself. We denote its inverse by φ and a simple computation provides

$$\varphi'(z) = \exp(-m \circ \varphi(z)). \tag{1.22}$$

The formulation of the Laplace transform of T_a will be written in terms of the solution to a Ricatti equation. Similarly to Lemma 2.1 in [11], we deduce the following Lemma on the Ricatti equation of our interest.

Lemma 1.7. *For any $\lambda > 0$, there exists a unique non-negative solution y_λ to the equation*

$$y' = y^2 - \lambda r^2, \tag{1.23}$$

where $r(z) = \frac{\varphi'(z)}{\sqrt{\omega(\varphi(z))}}$ such that it vanishes at ∞. Moreover, y_λ is positive on $(0, \infty)$, and for any z sufficiently small or large, $y_\lambda(z) \leq \sqrt{\lambda} r(z)$. As a consequence, y_λ is integrable at 0, and it decreases initially and ultimately.

We now state our last result. Recall that the infinitesimal generator $\mathcal{U}$ of the process Z satisfies that for any $f \in C^2$

$$\mathcal{U}f(x) = (bx - cx)f'(x) + \left(\gamma^2 x + \frac{\sigma^2}{2}x^2\right) f''(x) + x\int_{(0,\infty)} \left(f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{z<1\}}\right)\mu(\mathrm{d}z), \tag{1.24}$$

see for instance Theorem 1 in Palau and Pardo [15].

Theorem 1.8. *Assume that the branching mechanism ψ is general and that condition (1.5) holds. Hence the function*

$$h_\lambda(x) := 1 + \lambda \int_0^\infty \frac{e^{-xz - m(z)}}{\omega(z)} \exp\left\{-\int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v\right\} \int_0^z \exp\left\{m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v\right\} \mathrm{d}u\mathrm{d}z \tag{1.25}$$

is well defined and positive for any $x > 0$ and $\lambda > 0$ and it is a non-increasing C^2-function on $(0, \infty)$. Moreover it solves

$$\mathcal{U}h_\lambda(x) = \lambda h_\lambda(x), \text{ for any } x > 0. \tag{1.26}$$

Furthermore, if ψ satisfies the Grey's condition (1.4)*, h_λ is also defined at* 0*,*

$$h_\lambda(0) = \exp\left\{\int_0^\infty y_\lambda(v)dv\right\} < \infty,$$

and, for any $x \geq a \geq 0$,

$$\mathbb{E}_x\left[e^{-\lambda T_a}\right] = \frac{h_\lambda(x)}{h_\lambda(a)}, \tag{1.27}$$

and for any $x > 0$,

$$\mathbb{E}_x[T_0] = \int_0^\infty \mathrm{d}u\, e^{m(u)} \int_u^\infty \frac{e^{-m(z)}}{\omega(z)}(1 - e^{-zx})\mathrm{d}z. \tag{1.28}$$

Moreover under our assumptions, the process comes down from infinity.

The remainder of this paper is organised as follows. Section 2 is devoted to branching diffusions with interactions in a Brownian random environment. We decide to treat this case separately since the competition mechanism g may take negative and positive values and the techniques we use here are different from the rest of the paper. Our methodology are based on the theory of scale functions for diffusions. This allow us to provide a necessary and sufficient condition for extinction and moreover, the Laplace transform of hitting times is computed explicitly in terms of a Ricatti equation. Such results seems complicated to obtain with the presence of jumps coming from the branching mechanism or the random environment. In Section 3, the proofs Theorems 1.1 and 1.2 are provided. Sections 4 and 5 are devoted to the case when the random environment is driven by a Brownian motion. In particular, Section 4 treats the Lamperti-type representation and finally, Section 5 deals with the logistic case.

2 Branching diffusion with interactions in a Brownian random environment

Here, we focus on the Feller diffusion case and general competition mechanism where more explicit functionals of the process can be computed. In this particular case, the SDE (1.9) is simplified as follows

$$Z_t = Z_0 + b\int_0^t Z_s\mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t \sigma Z_s \mathrm{d}B_s^{(e)}. \tag{2.1}$$

It is important to note that in this case g is a real-valued continuous function satisfying the conditions in Proposition 1 in [15] (see also (3.1)).

Proposition 2.1. *Assume that Z is the unique strong solution of* (2.1)*, then*

$$\mathbb{P}_x\Big(T_0 < \infty\Big) = 1 \qquad \textit{if and only if} \qquad \int^\infty \exp\left\{2\int_1^u \frac{g(z) - bz}{2\gamma^2 z + \sigma^2 z^2}\mathrm{d}z\right\}\mathrm{d}u = \infty. \tag{2.2}$$

Moreover

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z_t = \infty\right) = 1 - \mathbb{P}_x\Big(T_0 < \infty\Big).$$

In particular, we may have the following situations

i) If there exist $z_0 > 0$ and $w < b - \frac{\sigma^2}{2}$ such that for any $z \geq z_0$, $g(z) \leq wz$, then $\mathbb{P}_x(T_0 < \infty) < 1$. An example of this situation is the cooperative case, that is to say when $g(z)$ is decreasing and $b > \frac{\sigma^2}{2}$.

ii) If there exist $z_0 > 0$ and $w > b - \frac{\sigma^2}{2}$ such that for any $z \geq z_0$, $g(z) \geq wz$, then $\mathbb{P}_x(T_0 < \infty) = 1$. An example of this situation are large competition mechanisms, that is to say for $g(z) \geq bz$ for any z large enough. For instance, the latter holds for the so-called logistic case i.e. $g(z) = cz^2$.

Proof of Proposition 2.1. We first observe from Dubins-Schwarz Theorem, that the law of Z is equal to the law of the following diffusion

$$\mathrm{d}Y_t = (bY_t - g(Y_t))\mathrm{d}t - \sqrt{2\gamma^2 Y_t + \sigma^2 Y_t^2}\mathrm{d}W_t,$$

where W is a standard Brownian motion. Associated to Y, we introduce for any $z \in \mathbb{R}$,

$$\mathtt{b}(z) := g(z) - bz, \qquad \mathtt{d}(z) := \frac{1}{2}\Big(2\gamma^2 z + \sigma^2 z^2\Big),$$

as well as the following functions related with the scale function of Y, for any $x, l \in \mathbb{R}_+$

$$s(l) = \exp\left\{\int_1^l \frac{\mathtt{b}(z)}{\mathtt{d}(z)}\mathrm{d}z\right\}, \quad S(l,x) = \int_l^x s(u)\mathrm{d}u \quad \text{and} \quad \Sigma(l,x) = \int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)s(\eta)}\mathrm{d}\eta\right) s(u)\mathrm{d}u.$$

Observe that for any $x \in \mathbb{R}_+$,

$$S(0,x) = \int_0^x \exp\left\{2\int_1^u \frac{g(z) - bz}{2\gamma^2 z + \sigma^2 z^2}\mathrm{d}z\right\}\mathrm{d}u. \tag{2.3}$$

For simplicity, we denote $S(x) = S(0,x)$.

In order to prove the first statement of this proposition, we follow the approach of Chapter 15 in Karlin and Taylor [10] which ensures that the equivalence (2.2) follows from the study of $\lim_{l\to 0}\Sigma(l,x)$. Indeed, According to Lemma 15.6.3 in [10], the finiteness of $\lim_{l\to 0}\Sigma(l,x)$ for an $x > 0$ implies the finiteness of $\lim_{l\to 0} S(l,x) = S(0,x)$ for all $x \geq 0$. Thus Lemma 15.6.2 in [10] guarantees that for any $y \geq x$, $T_0 \wedge T_y < \infty$, a.s., and Section 3 of Chapter 15 provides the following formulation

$$\mathbb{P}_x(T_0 < T_y) = \frac{S(x) - S(y)}{S(0) - S(y)}. \tag{2.4}$$

By making y tend to ∞, we find the equivalence (2.2) as required.

Hence let us show that $\lim_{l\to 0}\Sigma(l,x)$ is finite. In order to do so, we fix $\varepsilon > 0$ and $x \in (0,1)$ in such a way that for any $z \leq x$, $|\mathtt{b}(z)| \leq \varepsilon$. Therefore

$$\begin{aligned}
\Sigma(l,x) &= \int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)}\exp\left\{\int_\eta^1 \frac{\mathtt{b}(z)}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}\eta\right)\exp\left\{-\int_u^1 \frac{\mathtt{b}(z)}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}u \\
&\leq C_1(x)\int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)}\exp\left\{\int_\eta^x \frac{\varepsilon}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}\eta\right)\exp\left\{\int_u^x \frac{\varepsilon}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}u \\
&\leq C_2(x)\int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)}\left(\frac{1+\frac{\sigma^2}{2\gamma^2}\eta}{\eta}\right)^{\varepsilon/\gamma^2}\mathrm{d}\eta\right)\left(\frac{1+\frac{\sigma^2}{2\gamma^2}u}{u}\right)^{\varepsilon/\gamma^2}\mathrm{d}u,
\end{aligned} \tag{2.5}$$

where $C_1(x)$ and $C_2(x)$ are positive constants that only depend on x. Moreover, in a neighbourhood of 0, we have

$$\frac{1}{\mathtt{d}(\eta)}\left(\frac{1+\frac{\sigma^2}{2\gamma^2}\eta}{\eta}\right)^{\varepsilon/\gamma^2} \underset{\eta\to 0}{\sim} \frac{1}{2\gamma^2}\frac{1}{\eta^{1+\varepsilon/\gamma^2}},$$

which is not integrable at 0. Hence,

$$\int_u^x \frac{1}{\mathtt{d}(\eta)} \left(\frac{1 + \frac{\sigma^2}{2\gamma^2}\eta}{\eta} \right)^{\varepsilon/\gamma^2} \mathrm{d}\eta \underset{u\to 0}{\sim} C_3(x) \frac{1}{u^{\varepsilon/\gamma^2}},$$

where $C_3(x)$ is a positive constant that only depends on x. This implies that the integrand on the right-hand side of the last inequality in (2.5) is equivalent to $u^{-2\varepsilon/\gamma^2}$ which is integrable at 0 as soon as ε is chosen small enough. The latter implies that $\lim_{l\to 0} \Sigma(l, x) < \infty$ which completes the first statement of this proposition.

In order to finish the proof, note that for any $y > x$,

$$\mathbb{P}_x \left(\lim_{t\to\infty} Z(t) = \infty \right) \geq \mathbb{P}_x(T_y < T_0) = \frac{S(0) - S(x)}{S(y) - S(0)}.$$

Since it holds for any $y \geq x$, we can take y goes to ∞. By writing $S(\infty) := \lim_{y\to\infty} S(y) \in (0, \infty]$, we deduce

$$\mathbb{P}_x \left(\lim_{t\to\infty} Z(t) = \infty \right) \geq \frac{S(0) - S(x)}{S(\infty) - S(0)},$$

and the right-hand side is equal to $1 - \mathbb{P}_x(T_0 < \infty)$ according to (2.4), whenever $S(\infty)$ is finite or not. This ends the proof. □

Furthermore, we are able to compute the Laplace transform of the first passage time T_a, for $a \geq 0$, defined in (1.20), by using the solution to the Ricatti equation described in the next Lemma. With this aim in mind, we recall the definition of the scale function S from (2.3) and observe that the proof of Proposition 2.1 guarantees that it is well-defined. Moreover, it is clear that the function $S : \mathbb{R}_+ \to (0, S(\infty))$ is continuous and bijective, and under condition (2.2), $S(\infty)$ equals ∞. We denote by $\bar{\varphi}(x)$ the inverse of S on $(0, S(\infty))$. Following similar arguments to those provided in the proof of Lemma 2.1 in Lambert [11], we deduce the following properties on the solution to the Ricatti equation that we are interested in.

Lemma 2.2. *For any $\lambda > 0$, there exists a unique non-negative solution $\bar{y}_\lambda$ on $(0, S(\infty))$ to the equation*

$$y' = y^2 - \lambda \bar{r}^2,$$

where

$$\bar{r}(z) = \frac{\bar{\varphi}'(z)}{\sqrt{\gamma^2 \bar{\varphi}(z) + \frac{\sigma^2}{2}(\bar{\varphi}(z))^2}},$$

such that it vanishes at $S(\infty)$. Moreover, $\bar{y}_\lambda$ is positive on $(0, S(\infty))$, and for any z sufficiently small or close to $S(\infty)$, $\bar{y}_\lambda(z) \leq \sqrt{\lambda}\bar{r}(z)$. In particular, $\bar{y}_\lambda$ is integrable at 0 if $\gamma \neq 0$, and it decreases initially and ultimately.

From Lambert [11], it is enough to study the behaviour of r in order to deduce Lemma 2.2. Using $\kappa(z) = \bar{r}^2(S(z))$, we prove that $\bar{r}(z)$ goes to ∞ or to 0 accordingly as z goes to 0 or $S(\infty)$ and that it is integrable at 0. Moreover r decreases initially and ultimately.

Our next result provides explicitly the Laplace transform of T_a in terms of the function $\bar{y}_\lambda$.

Proposition 2.3. *Assume that $\gamma > 0$. Then, for any $x \geq a \geq 0$, and for any $\lambda > 0$,*

$$\mathbb{E}_x \left[e^{-\lambda T_a} \right] = \exp \left\{ - \int_{S(a)}^{S(x)} \bar{y}_\lambda(u) \mathrm{d}u \right\}. \tag{2.6}$$

Note that if (2.2) is satisfied, then $T_a < \infty$ a.s.

Proof. Let $x \geq a > 0$, then $(Z_{t\wedge T_a}, t \geq 0)$, under $\mathbb{P}_x$, is a process with values in $[a, \infty)$. For any $y \geq a$, we define

$$f_{\lambda,a}(y) = \exp\left\{ - \int_{S(a)}^{S(x)} \bar{y}_\lambda(u) \mathrm{d}u \right\}.$$

A direct computation ensures that $f_{\lambda,a}$ is a C^2-function on $[a, \infty)$, bounded by 1, $f_{\lambda,a}(a) = 1$ and such that it solves

$$\mathtt{d}(y) f''(y) - \bar{b}(y) f'(y) - \lambda f(y) = 0. \tag{2.7}$$

Applying Itô Formula to the function $F(t, y) = e^{-\lambda t} f_{\lambda,a}(y)$ and the process $(Z_{t\wedge T_a}, t \geq 0)$, we obtain by means of (2.7)

$$e^{-\lambda t} f_{\lambda,a}(Z_{t\wedge T_a}) = f_{\lambda,a}(x) + \int_0^{t\wedge T_a} f'_{\lambda,a}(Z_s) \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s^{(b)} + \sigma \int_0^{t\wedge T_a} f'_{\lambda,a}(Z_s) Z_s \mathrm{d}B_s^{(e)}.$$

We then use a sequence of stopping time $(T_n, n \geq 1)$ that reduces the two local martingales of the right-hand side and from the optimal stopping theorem, we obtain for any $n \geq 1$

$$\mathbb{E}_x \left[e^{-\lambda T_n \wedge T_a} f_{\lambda,a}(Z_{T_n \wedge T_a}) \right] = f_{\lambda,a}(x).$$

Letting n goes to ∞ gives (2.6) for any $x \geq a > 0$. We finally let a goes to 0 to deduce the result for $a = 0$ and conclude the proof. □

3 Proofs of Theorems 1.1 & 1.2

In order to prove Theorem 1.1, we introduce the following stochastic processes as unique strong solutions of the SDE's. For $i = 1, 2$, we let

$$Z_t^{(i)} = Z_0^{(i)} + \int_0^t g_i(Z_s^{(i)}) \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s^{(i)}} \mathrm{d}B_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}^{(i)}} z \widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}^{(i)} \mathrm{d}S_s^{(i)},$$

where

$$S_t^{(i)} = \mathtt{d}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} b_i(z) N^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)} (e^z - 1) \widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z),$$

with $g_1(z) \geq g_2(z)$, for $z \geq 0$, and $b_1(z) \geq b_2(z)$ for $z \in \mathbb{R}$ such that, for $i = 1, 2$

$$b_i(z) + 1 \geq 0, \qquad \text{for} \quad z \in \mathbb{R}.$$

We also assume that for each $m \geq 0$, there is a non-decreasing concave function $z \mapsto r_m(z)$ on $\mathbb{R}_+$ satisfying $\int_{0+} r_m(z) \mathrm{d}z = \infty$ and

$$|g_i(x) - g_i(y)| + \mathtt{d}|x - y| + |x - y| \int_{(-1,1)^c} \Big(|b_i(z)| \wedge m \Big) \pi(\mathrm{d}z) \leq r_m(|x - y|), \qquad \text{for} \quad i = 1, 2, \tag{3.1}$$

for every $0 \leq x, y \leq m$. According to Proposition 1 in Palau and Pardo [15], the previous SDE's possess unique positive strong solutions that we denote by $Z^{(i)}$ for $i = 1, 2$.

Lemma 3.1. *If $Z_0^{(1)} \geq Z_0^{(2)}$, a.s. then*

$$\mathbb{P}\Big(Z_t^{(2)} \leq Z_t^{(1)} \ \ \textit{for all} \ \ t \geq 0 \Big) = 1.$$

Proof. Our arguments follow similar reasonings as those used in Theorem 2.2 in [3]. For each integer $n \geq 1$, define $a_n = \exp\{-n(n+1)/2\}$. Then a_n decreases to 0 as n increases and $\int_{a_n}^{a_{n-1}} z^{-1}\mathrm{d}z = n$, for any $n \geq 1$. Let $x \mapsto h_n(x)$ be a positive continuous function with support on (a_n, a_{n-1}) such that

$$\int_{a_n}^{a_{n-1}} h_n(z)\mathrm{d}z = 1,$$

and $h_n(x) \leq 2(nx)^{-1}$, for $x > 0$. For $n \geq 0$, we introduce

$$f_n(z) = \int_0^z \mathrm{d}y \int_0^y h_n(x)\mathrm{d}x, \qquad z \in \mathbb{R}.$$

Observe that $0 \leq f_n'(z) \leq 1$ and $0 \leq z f_n''(z) \leq \frac{2}{n}$, for any $z > 0$. It is also clear that for any $y > 0$ and $x \in \mathbb{R}$

$$|f_n(x+y) - f_n(y)| \leq |x|,$$

in addition, from Taylor expansion formulas, we have for any $y > 0$ and $x \in \mathbb{R}$ such that $x + y > 0$,

$$|f_n(x+y) - f_n(y) - x f_n'(y)| \leq 2\left(|x| \wedge \frac{x^2}{n}\int_0^1 \frac{(1-u)}{y+ux}\mathrm{d}u\right), \tag{3.2}$$

in particular $f_n(x+y) - f_n(y) - x f_n'(y)$ converges to 0 when n goes to ∞. Moreover, we have that $f_n(z)$ converge towards $z^+ = 0 \vee z$ non-decreasingly as n increases.

Let $\tau_m = \inf\{t \geq 0 : Z_s^{(1)} \geq m \text{ or } Z_s^{(2)} \geq m\}$ for $m \geq 1$. According to the proof of Proposition 1 in Palau and Pardo [15], for $i = 1, 2$, we have $Z_t^{(i)} = Z_t^{(i,m)}$ for $t < \tau_m$, where $Z^{(i,m)}$ is the unique strong solution to

$$\begin{aligned} Z_t^{(i,m)} &= Z_0^{(i)} + \int_0^t g_i(Z_s^{(i,m)} \wedge m)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s^{(i,m)} \wedge m}\ \mathrm{d}B_s^{(b)} \\ &\quad + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}^{(i,m)} \wedge m} (z \wedge m)\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \Big(Z_{s-}^{(i,m)} \wedge m\Big)\mathrm{d}S_s^{(i,m)}, \end{aligned}$$

where

$$S_t^{(i)} = \mathtt{d}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} \Big(b_i(z) \wedge m\Big)N^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)} \Big((e^z - 1) \wedge m\Big)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z).$$

In other words for $m \geq 1$, we have

$$\mathbb{P}\left(Z_t^{(1)} \geq Z_t^{(2)}, \text{ for all } t < \tau_m\right) = \mathbb{P}\left(Z_t^{(1,m)} \geq Z_t^{(2,m)}, \text{ for all } \ t < \tau_m\right).$$

We now prove that the latter probability equals one. For $t \geq 0$, we let $\zeta(t) = Z_t^{(2,m)} - Z_t^{(1,m)}$ and

$$\zeta_m(t) = Z_t^{(2,m)} \wedge m - Z_t^{(1,m)} \wedge m.$$

For $\zeta(s) \leq 0$, we observe $f_n(\zeta(s)) = f_n'(\zeta(s)) = f_n''(\zeta(s)) = 0$ and

$$\zeta(s) + Z_s^{(2,m)} g_2(z) \wedge m - Z_s^{(1,m)} g_1(z) \wedge m \leq \zeta(s) + \zeta(s) g_2(z) \wedge m \leq \zeta(s)\big(1 + g_2(z) \wedge m\big) \leq 0.$$

We also note that if $\zeta(s) \leq 0$, then $\zeta_m(s) \leq 0$. Hence by Itô's formula, we have

$$\begin{aligned}
f_n(\zeta(t)) = f_n(\zeta(0)) &+ \int_0^t f_n'(\zeta(s)) \left(\left(g_2(Z_s^{(2,m)} \wedge m) - g_1(Z_s^{(1,m)} \wedge m) \right) + \mathtt{d}\zeta_m(s) \right) \mathbf{1}_{\{\zeta(s)>0\}} \mathrm{d}s \\
&+ \frac{1}{2} \int_0^t f_n''(\zeta(s)) \left(2\gamma^2 \left(\sqrt{Z_s^{(2,m)} \wedge m} - \sqrt{Z_s^{(1,m)} \wedge m} \right)^2 + \sigma^2 \zeta_m^2(s) \right) \mathbf{1}_{\{\zeta(s)>0\}} \mathrm{d}s \\
&+ \int_0^t \int_{(0,\infty)} \zeta_m(s) \left(f_n(\zeta(s) + z \wedge m) - f_n(\zeta(s)) - (z \wedge m) f_n'(\zeta(s)) \right) \mathbf{1}_{\{\zeta(s)>0\}} \mu(\mathrm{d}z) \mathrm{d}s \\
&+ \int_0^t \int_{(-1,1)} \left(f_n(\zeta(s) + \ell_1(z,m)\zeta_m(s)) - f_n(\zeta(s)) - \ell_1(z,m)\zeta_m(s) f_n'(\zeta(s)) \right) \mathbf{1}_{\{\zeta(s)>0\}} \pi(\mathrm{d}z) \mathrm{d}s \\
&+ \int_0^t \int_{(-1,1)^c} \left(f_n(\zeta(s) + \ell_2(s,z,m)) - f_n(\zeta(s)) \right) \mathbf{1}_{\{\zeta(s)>0\}} \pi(\mathrm{d}z) \mathrm{d}s + M_n(t),
\end{aligned} \tag{3.3}$$

where $\ell_1(z,m) = (e^z - 1) \wedge m$,

$$\ell_2(s,z,m) = (Z_s^{(2,m)} \wedge m)(b_2(z) \wedge m) - (Z_s^{(1,m)} \wedge m)(b_1(z) \wedge m),$$

and

$$\begin{aligned}
M_n(t) = &\sqrt{2\gamma^2} \int_0^t f_n'(\zeta(s)) \left(\sqrt{Z_s^{(2,m)} \wedge m} - \sqrt{Z_s^{(1,m)} \wedge m} \right) \mathrm{d}B_s^{(b)} + \sigma \int_0^t f_n'(\zeta(s)) \zeta_m(s) \mathrm{d}B_s^{(e)} \\
&+ \int_0^t \int_{(0,\infty)} \int_{Z_{s-}^{(1,m)} \wedge m}^{Z_{s-}^{(2,m)} \wedge m} \left(f_n(\zeta(s-) + z \wedge m) - f_n(\zeta(s-)) \right) \widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\
&+ \int_0^t \int_{(-1,1)} \left(f_n(\zeta(s-) + \ell_1(z,m)\zeta_m(s)) - f_n(\zeta(s-)) \right) \widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) \\
&+ \int_0^t \int_{(-1,1)^c} \left(f_n(\zeta(s-) + \ell_2(s,z,m)) - f_n(\zeta(s-)) \right) \widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Hence the process $(M_n(t \wedge \tau_m), t \geq 0)$ is a martingale. If $Z^{(1)}(0) \geq Z^{(2)}(0)$ a.s., we take expectations in both sides of (3.3) at time $t \wedge \tau_m$ and let n goes to ∞. Let us prove that the expectation of the second line of (3.3) converges to 0 as n increases. Indeed, since $\zeta(t) = \zeta_m(t)$ for $t < \tau_m$, we find the following upper bounds on $\{s \leq \tau_m\}$ whose last is independent from n,

$$\begin{aligned}
f_n''(\zeta(s)) &\left(2\gamma^2 \left(\sqrt{Z_s^{(2,m)} \wedge m} - \sqrt{Z_s^{(1,m)} \wedge m} \right)^2 + \sigma^2 \zeta_m^2(s) \right) \mathbf{1}_{\{\zeta(s)>0\}} \\
&\leq (2\gamma^2 + m\sigma^2) f_n''(\zeta_m(s)) \zeta_m(s) \leq \frac{2(2\gamma^2 + m\sigma^2)}{n} \leq 2(2\gamma^2 + m\sigma^2).
\end{aligned}$$

Hence, the dominated converge theorem implies the convergence of the expectation of the second term in (3.3) towards 0 as n goes to ∞. Similarly the expectation of the third and fourth terms in (3.3) converge also weakly to 0. Indeed, from (3.2) we obtain the following upper bound on $\{s \leq \tau_m\}$, for any $z \geq 0$,

$$\begin{aligned}
&\zeta_m(s) \left(f_n(\zeta(s) + z \wedge m) - f_n(\zeta(s)) - (z \wedge m) f_n'(\zeta(s)) \right) \mathbf{1}_{\{\zeta(s)>0\}} \\
&\qquad \leq 2 \left(\zeta_m(s)(z \wedge m) \wedge \frac{(z \wedge m)^2}{n} \int_0^1 \frac{\zeta_m(s)(1-u)}{\zeta_m(s) + u(z \wedge m)} \mathrm{d}u \right) \mathbf{1}_{\{\zeta(s)>0\}} \leq 2 \left(mz \wedge \frac{z^2}{n} \right),
\end{aligned}$$

and since $\ell_1(z,m) \geq -1$ and $|\ell_1(z,m)| \leq e|z|$ for any $z \in [-1,1]$, we deduce

$$\begin{aligned}
&\Big(f_n(\zeta(s) + \ell_1(z,m)\zeta_m(s)) - f_n(\zeta(s)) - \ell_1(z,m)\zeta_m(s) f_n'(\zeta(s))\Big)\mathbf{1}_{\{\zeta(s)>0\}} \\
&\qquad \leq 2|\ell_1(z,m)\zeta_m(s)| \wedge \left(\frac{|\ell_1(z,m)\zeta_m(s)|^2}{n} \int_0^1 \frac{(1-u)}{\zeta_m(s) + u\ell_1(z,m)\zeta_m(s)} \mathrm{d}u \right) \mathbf{1}_{\{\zeta(s)>0\}} \\
&\qquad \leq 2me \left(|z| \wedge e\frac{|z|^2}{n} \right).
\end{aligned}$$

Putting all the pieces together, we obtain

$$\begin{aligned}
\mathbb{E}[\zeta(t\wedge\tau_m)^+] \leq \mathbb{E}\left[\int_0^{t\wedge\tau_m} \Big(\Big(g_2(Z_s^{(2,m)} \wedge m) - g_2(Z_s^{(1,m)} \wedge m) \Big) + \mathtt{d}\zeta_m(s) \Big) \mathbf{1}_{\{\zeta(s)>0\}} \mathrm{d}s \right] \\
+ \mathbb{E}\left[\int_0^{t\wedge\tau_m} \zeta_m(s) \int_{(-1,1)^c} (|b_2(z)| \wedge m) \pi(\mathrm{d}z) \mathbf{1}_{\{\zeta(s)>0\}} \mathrm{d}s \right].
\end{aligned}$$

In other words from the inequality (3.1), we have

$$\mathbb{E}[\zeta(t\wedge\tau_m)^+] \leq \mathbb{E}\left[\int_0^{t\wedge\tau_m} r_m(\zeta_m(s)) \mathbf{1}_{\{\zeta(s)>0\}} \mathrm{d}s \right],$$

and since $\zeta(t) = \zeta_m(t)$ for $t < \tau_m$ and r_m is non-decreasing, concave with $r_m(0) = 0$, we finally get

$$\begin{aligned}
\mathbb{E}[\zeta(t\wedge\tau_m)^+] &\leq \mathbb{E}\left[\int_0^{t\wedge\tau_m} r_m(\zeta(s)) \mathbf{1}_{\{\zeta(s)>0\}} \mathrm{d}s \right] \\
&\leq \mathbb{E}\left[\int_0^{t} r_m(\zeta(s) \mathbf{1}_{\{\zeta(s)>0, s\leq\tau_m\}}) \mathrm{d}s \right] \\
&\leq \int_0^t r_m\left(\mathbb{E}\left[\zeta(t\wedge\tau_m)^+ \right] \right) \mathrm{d}s.
\end{aligned}$$

Then $\mathbb{E}[\zeta(t\wedge\tau_m)^+] = 0$ for all $t \geq 0$. In other words,

$$\mathbb{P}\left(Z_t^{(1)} \geq Z_t^{(2)}, \text{ for all } t < \tau_m \right) = 1,$$

which implies our result since the latter holds for any $m \geq 1$. □

Proof of Theorem 1.1. The first statement follows directly from Lemma 3.1 by taking

$$g_1(z) = g_2(z) = (\mathtt{d} - \psi(0+))z - g(z) \quad \text{for } z \geq 0 \qquad \text{and} \qquad b_1(z) = b_2(z) = e^z - 1 \quad \text{for } z \in \mathbb{R}.$$

For the second statement, we recall that the competition mechanism g is positive and non-decreasing implying that we can take $g_1(z) = (\mathtt{d} - \psi(0+))z$, $g_2(z) = (\mathtt{d} - \psi'(0+))z - g(z)$ and $b_1(z) = b_2(z) = e^z - 1$. Again from Lemma 3.1, we deduce that the process $(Z, \mathbb{P}_x)$ is stochastically dominated by $(Z^\sharp, \mathbb{P}_y^\sharp)$ for $y \geq x$.

The last part of the statement follows from Theorem 4.1 and Corollary 4.4 in [19] applied to $(Z^\sharp, \mathbb{P}_y^\sharp)$ and the fact that the latter stochastically dominates $(Z, \mathbb{P}_x)$. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. First of all from Lemma 3.1, it is enough to prove our result for a process without downward jumps larger than $1-e^{-1}$. Hence, we assume in all this proof that Z is solution to (1.9) with

$$S_t = \mathtt{d}t + \sigma B_t^{(e)} + \int_0^t \int_{(1,\infty)} (e^z - 1) N^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)} (e^z - 1) \widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z), \tag{3.4}$$

Let us denote by T_M for the first passage time for the process Z below a level $M > 0$ defined in (1.20). As we will see below, the finiteness of the first moment of such random times will be useful for deducing our result. Hence, we first show

$$\sup_{x \geq 0} \mathbb{E}_x\big[T_M\big] < \infty. \tag{3.5}$$

With this goal in mind, we observe from assumption (1.11) that

$$\lim_{y \to +\infty} \frac{g(y)}{y} = \infty \quad \text{and} \quad \lim_{y \to +\infty} \frac{\theta y - g(y)}{y} = -\infty, \tag{3.6}$$

for $\theta := -\psi'(0+) + \mathtt{d}$. In addition from Lemma 2.3 in Le and Pardoux [18], we deduce that there exists $a_0 > 0$ such that $g(y) - \theta y > 0$ for any $y \geq a_0$ and

$$\int_{a_0}^{\infty} \frac{\mathrm{d}y}{g(y) - \theta y} < \infty. \tag{3.7}$$

We then introduce $A > \theta(e-1)$ in such a way that the inequality below holds

$$\begin{aligned} C(A) := 1 - \Bigg(& \frac{\theta(2\gamma^2 + \sigma^2)}{2A^2} + \frac{\theta}{A(A-\theta)} \int_{(0,1)} z^2 \mu(\mathrm{d}z) \\ & + \frac{1}{A} \left(\int_{(1,\infty)} z \mu(\mathrm{d}z) + \overline{\pi}(1) \right) + \left(\frac{\theta}{A^2} + \frac{\theta}{A(A - \theta(e^1 - 1))} \right) \int_{(-1,1)} z^2 \pi(\mathrm{d}z) \Bigg) > 0, \end{aligned} \tag{3.8}$$

where $\overline{\pi}(x) = \pi((x,\infty))$, $x \geq 0$. From the way we choose A and from (3.6) and (3.7), it is clear that there exists a constant $M > (a_0 + 1)e$ such that

$$\int_{Me^{-1}}^{\infty} \frac{\mathrm{d}w}{g(w) - \theta w} \leq \frac{1}{A} \qquad \text{and} \qquad g(y) - \theta y \geq Ay, \qquad \text{for all} \quad y \geq Me^{-1}. \tag{3.9}$$

Such constant M will be our threshold. We also observe that (3.9) implies that for any $y \geq Me^{-1}$,

$$0 \leq \frac{1}{g(y) - \theta y} \leq \frac{y}{g(y) - \theta y} \leq \frac{1}{A} \quad \text{and} \quad A \leq g(y) - \theta y. \tag{3.10}$$

For our purposes, we define the function G in $C^2(\mathbb{R})$ as follows

$$G(y) = \begin{cases} \displaystyle\int_{a_0}^{y} \frac{\mathrm{d}w}{g(w) - \theta w} & \text{if } y \geq a_0 + 1, \\ 0 & \text{if } y \leq a_0, \end{cases}$$

and such that G is non-negative and non-decreasing. Thus applying Itô's formula to $G(Z_{t\wedge T_M})$, we find

$$\begin{aligned}
G(Z_{t\wedge T_M}) - G(Z_0) = -t\wedge T_M &- \int_0^{t\wedge T_M} \frac{g'(Z_s)-\theta}{(g(Z_s)-\theta Z_s)^2}\left(\gamma^2 Z_s + \frac{\sigma^2}{2}Z_s^2\right)\mathrm{d}s \\
&+ \int_0^{t\wedge T_M} \frac{\sqrt{2\gamma^2 Z_s}}{g(Z_s)-\theta Z_s}\mathrm{d}B_s^{(b)} + \int_0^{t\wedge T_M} \frac{\sigma Z_s}{g(Z_s)-\theta Z_s}\mathrm{d}B_s^{(e)} \\
&+ \int_0^{t\wedge T_M}\int_{(0,\infty)} Z_s\left(G(Z_s+z)-G(Z_s)-\frac{z}{g(Z_s)-\theta Z_s}\right)\mu(\mathrm{d}z)\mathrm{d}s \\
&+ \int_0^{t\wedge T_M}\int_{(0,\infty)}\int_0^{Z_{s-}} [G(Z_{s-}+z)-G(Z_{s-})]\widetilde{N}^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\
&+ \int_0^{t\wedge T_M}\int_{(1,\infty)} [G(e^z Z_{s-})-G(Z_{s-})]N^{(e)}(\mathrm{d}s,\mathrm{d}z) \\
&+ \int_0^{t\wedge T_M}\int_{(-1,1)} [G(e^z Z_{s-})-G(Z_{s-})]\widetilde{N}^{(e)}(\mathrm{d}s,\mathrm{d}z) \\
&+ \int_0^{t\wedge T_M}\int_{(-1,1)}\left(G(e^z Z_s)-G(Z_s)-\frac{(e^z-1)Z_s}{g(Z_s)-\theta Z_s}\right)\pi(\mathrm{d}z)\mathrm{d}s.
\end{aligned} \tag{3.11}$$

Next, we take expectations under the assumption that the process Z starts at $x \geq M$, in both sides of the previous identity and we study separately each term of the right-hand side. For simplicity, we enumerate the lines in order of appearance.

(1) For the first integral of the right hand side of (3.11), we recall that $Z_s \geq Me^{-1}$ for $s \leq t\wedge T_M$, that g is non-decreasing and use (3.10) to deduce

$$\mathbb{E}_x\left[\int_0^{t\wedge T_M} \frac{\theta - g'(Z_s)}{(g(Z_s)-\theta Z_s)^2}\left(\gamma^2 Z_s + \frac{\sigma^2}{2}Z_s^2\right)\mathrm{d}s\right] \leq \frac{\theta(2\gamma^2+\sigma^2)}{2A^2}\mathbb{E}_x\Big[t\wedge T_M\Big]. \tag{3.12}$$

(2) For the two Itô integrals of the second line of the right-hand side of (3.11), we first observe that both are continuous local martingales. Since their quadratic variations satisfy

$$\mathbb{E}_x\left[\int_0^{t\wedge T_M}\left|\frac{\sqrt{2\gamma^2 Z_s}}{\theta Z_s - g(Z_s)}\right|^2 \mathrm{d}s\right] \leq \frac{2\gamma^2 t}{A^2} \quad \text{and} \quad \mathbb{E}_x\left[\int_0^{t\wedge T_M}\left|\frac{\sigma Z_s}{\theta Z_s - g(Z_s)}\right|^2 \mathrm{d}s\right] \leq \frac{\sigma^2 t}{A^2},$$

then both processes are martingales and therefore their expectations are equal to 0.

(3) We study the integral that appears in the third line in (3.11) by separating $(0,\infty)$ into two parts $(0,1]$ and $(1,\infty)$. Note that $Z_s \geq Me^{-1} > a_0 + 1$ for any $s \leq t\wedge T_M$, in other words have an explicit formula for $G(Z_s)$. Now we deal with the integral restricted to $(0,1)$. Since g is non-decreasing and from the second inequality in (3.9), we obtain the following upper bound

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(0,1)} Z_s\left[G(Z_s+z)-G(Z_s)-\frac{z}{g(Z_s)-\theta Z_s}\right]\mu(\mathrm{d}z)\mathrm{d}s\right] \\
&\qquad \leq \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(0,1)} \frac{\theta z^2 Z_s}{(g(Z_s)-\theta(Z_s+z))(g(Z_s)-\theta Z_s)}\mu(\mathrm{d}z)\mathrm{d}s\right] \\
&\qquad \leq \mathbb{E}_x\Big[t\wedge T_M\Big]\frac{\theta}{A(A-\theta)}\int_{(0,1)} z^2\mu(\mathrm{d}z) < \infty.
\end{aligned} \tag{3.13}$$

where the last inequality follows from (3.10). Concerning the integral restricted to $(1,\infty)$, we have

$$\begin{aligned}\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(1,\infty)} Z_s\left[G(Z_s+z)-G(Z_s)-\frac{z}{g(Z_s)-\theta Z_s}\right]\mu(\mathrm{d}z)\mathrm{d}s\right]\\ \leq \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(1,\infty)} Z_s\left[G(Z_s+z)-G(Z_s)\right]\mu(\mathrm{d}z)\mathrm{d}s\right]\\ \leq \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(1,\infty)}\int_0^z \frac{Z_s}{A(Z_s+w)}\mathrm{d}w\mu(\mathrm{d}z)\mathrm{d}s\right]\\ \leq \frac{\mathbb{E}_x\Big[t\wedge T_M\Big]}{A}\int_{(1,\infty)} z\mu(\mathrm{d}z).\end{aligned} \tag{3.14}$$

(4) For the integral in the fourth line of the right-hand side of (3.11), we prove that it is a martingale and thus has expectation equals 0. Again, we split the interval $(0,\infty)$ into $(0,1]$ and $(1,\infty)$ and use similar computations as in part (3) in order to deduce that the integral restricted to $(0,1]$ is a square integrable martingale and the integral restricted to $(1,\infty)$ is a martingale. In other words, we manipulate

$$\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(0,\infty)} Z_s f(G(Z_s+z)-G(Z_s),z)\mu(\mathrm{d}z)\mathrm{d}s\right],$$

with $f(x,z)=x^2\mathbf{1}_{(0,1]}(z)$ and $f(x,z)=|x|\mathbf{1}_{(1,\infty)}(z)$ respectively.

(5) The stochastic integral in the fifth line in (3.11) can be studied using Fubini's Theorem. Indeed, using the first inequality in (3.9), we deduce

$$\begin{aligned}\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(1,\infty)}|G(e^zZ_{s-})-G(Z_{s-})|\pi(\mathrm{d}z)\mathrm{d}s\right]\\ \leq \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(1,\infty)}\int_{Z_s}^{e^zZ_s}\frac{\mathrm{d}w}{g(w)-\theta w}\pi(\mathrm{d}z)\mathrm{d}s\right]\\ \leq \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{Z_s}^{\infty}\frac{\mathrm{d}w}{g(w)-\theta w}\left(\int_1^\infty\pi(\mathrm{d}z)\right)\mathrm{d}s\right]\\ \leq t\overline{\pi}(1)\left(\int_{Me^{-1}}^\infty\frac{\mathrm{d}w}{g(w)-\theta w}\right)\leq\frac{t}{A}\overline{\pi}(1).\end{aligned}$$

This ensures that the stochastic integral can be written as the sum of a martingale and a finite variation process. In other words, we have

$$\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(1,\infty)}\big(G(e^zZ_{s-})-G(Z_{s-})\big)N^{(e)}(\mathrm{d}s,\mathrm{d}z)\right]\leq\mathbb{E}[t\wedge T_M]\frac{\overline{\pi}(1)}{A}. \tag{3.15}$$

(6) Observe that the integral term of the sixth line is a square integrable martingale. Indeed, similarly as for the fourth line case, we get

$$\begin{aligned}\mathbb{E}\left[\int_0^{t\wedge T_M}\int_{(-1,1)}\left(\int_{Z_s}^{e^zZ_s}\frac{\mathrm{d}w}{g(w)-\theta w}\right)^2\pi(\mathrm{d}z)\mathrm{d}s\right]&\leq\mathbb{E}\left[\int_0^{t\wedge T_M}\int_{(-1,1)}\left(\int_{Z_s}^{e^zZ_s}\frac{\mathrm{d}w}{Aw}\right)^2\pi(\mathrm{d}z)\mathrm{d}s\right]\\ &\leq\frac{t}{A^2}\int_{(-1,1)}z^2\pi(\mathrm{d}z)<\infty,\end{aligned} \tag{3.16}$$

which implies that its expectation equals 0.

(7) Finally, we study the last line in (3.11) by splitting again the integral into two parts, i.e. we split $(-1,1)$ into $(-1,0]$ and $(0,1)$. Thus, using again the second inequality of (3.9) and the fact that $A > \theta(e-1)$, we deduce that for any $w \in [0, y(e^z-1)]$, $y \geq 1$ and $z \in (-1,1)$,

$$g(y+w) - \theta(y+w) \geq g(y) - \theta y e^z \geq Ay - \theta(e-1)y > 0.$$

Hence,

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(0,1)}\left(G(e^zZ_s) - G(Z_s) - \frac{(e^z-1)Z_s}{g(Z_s)-\theta Z_s}\right)\pi(\mathrm{d}z)\mathrm{d}s\right] \\
&\quad= \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(0,1)}\left(\int_0^{Z_s(e^z-1)}\frac{\mathrm{d}w}{g(Z_s+w)-\theta(Z_s+w)} - \frac{(e^z-1)Z_s}{g(Z_s)-\theta Z_s}\right)\pi(\mathrm{d}z)\mathrm{d}s\right] \\
&\quad\leq \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(0,1)}\left(\frac{(e^z-1)Z_s}{g(Z_s)-\theta Z_s e^z} - \frac{(e^z-1)Z_s}{g(Z_s)-\theta Z_s}\right)\pi(\mathrm{d}z)\mathrm{d}s\right] \\
&\quad\leq \mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(0,1)}\frac{\theta(e^z-1)^2(Z_s)^2}{(AZ_s-\theta(e^1-1)Z_s)(g(Z_s)-\theta Z_s)}\pi(\mathrm{d}z)\mathrm{d}s\right] \\
&\quad\leq \mathbb{E}_x\Big[t\wedge T_M\Big]\frac{\theta}{A(A-\theta(e^1-1))}\int_{(0,1)}z^2\pi(\mathrm{d}z) < \infty.
\end{aligned}$$

Similarly, we deal with the second part of the integral and deduce

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(-1,0)}\left(G(e^zZ_s) - G(Z_s) - \frac{(e^z-1)Z_s}{g(Z_s)-\theta Z_s}\right)\pi(\mathrm{d}z)\mathrm{d}s\right] \\
&\qquad\leq \mathbb{E}_x\Big[t\wedge T_M\Big]\frac{\theta}{A^2}\int_{(-1,0)}z^2\pi(\mathrm{d}z) < \infty.
\end{aligned}$$

In other words, the expectation of the last line in (3.11) is bounded by

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge T_M}\int_{(-1,1)}\left(G(e^zZ_s) - G(Z_s) - \frac{(e^z-1)Z_s}{g(Z_s)-\theta Z_s}\right)\pi(\mathrm{d}z)\mathrm{d}s\right] \\
&\qquad\qquad\leq \mathbb{E}_x\Big[t\wedge T_M\Big]\left(\frac{\theta}{A^2} + \frac{\theta}{A(A-\theta(e^1-1))}\right)\int_{(-1,1)}z^2\pi(\mathrm{d}z).
\end{aligned} \tag{3.17}$$

Thus putting all pieces together (i.e. inequalities (3.12), (3.13), (3.14), (3.15) and (3.17) together with (3.11), (3.8) and the three null-expectations), we deduce

$$\mathbb{E}_x\left[\int_x^{Z_{t\wedge T_M}}\frac{\mathrm{d}w}{g(w)-\theta w}\right] \leq -C(A)\mathbb{E}[t\wedge T_M],$$

with $C(A) > 0$. In other words, for any $x, t \geq 0$,

$$\mathbb{E}_x\Big[t\wedge T_M\Big] \leq \frac{1}{C(A)}\mathbb{E}_x\left[\int_{Z_{t\wedge T_M}}^{x}\frac{\mathrm{d}w}{g(w)-\theta w}\right] \leq \frac{1}{C(A)}\int_{Me^{-1}}^{\infty}\frac{\mathrm{d}w}{g(w)-\theta w}.$$

Hence using the Monotone Convergence Theorem, as t goes to ∞, we obtain

$$\sup_{x\geq M}\mathbb{E}_x\big[T_M\Big] \leq \frac{1}{C(A)}\int_{Me^{-1}}^{+\infty}\frac{\mathrm{d}w}{g(w)-\theta w}.$$

Finally we deduce (3.5) by noting that $T_M = 0$ whenever the process Z starts at $x \leq M$.

In order to finish the proof of the first statement, we first show that the time to extinction for the process Z starting from M is not almost surely infinite. We recall that we assumed that the environment has no negative jumps larger than $1 - e^{-1}$. Using Theorem 1.1 (both processes with the same restriction on the negative jumps of the environment), we observe that for any $x \leq M$, the process $(Z, \mathbb{P}_x)$ is stochastically dominated by $(Z^\sharp, \mathbb{P}^\sharp_x)$. The process $Z^\sharp$ is a CB-process in a Lévy random environment which is characterized by the branching mechanism $\psi^\sharp(\lambda) = \psi(\lambda) - \psi'(0+)\lambda$. Since ψ satisfies the Grey's condition and is positive for any z sufficiently large, Lemma 2.3 in Le and Pardoux [18] guarantees that $\psi^\sharp$ satisfies also Grey's condition. Then Theorem 4.1 of [19] ensures that there is $t_0 > 0$ for which

$$0 < \mathbb{P}^\sharp_M\Big(Z^\sharp_{t_0} = 0\Big) \leq \inf_{x \leq M} \mathbb{P}_x\Big(Z_{t_0} = 0\Big) := p.$$

Next, we introduce an independent geometric random variable ξ with parameter p that counts the number of random steps needed in order that Z becomes extinct. For simplicity, we denote by $\mathcal{T}^x_M$ for the stopping time T_M under $\mathbb{P}_x$. The length of the random steps are bounded from above by a random variable distributed as $t_0 + \sup_{x\geq 0} \mathcal{T}^x_M$. To be more precise, the algorithm is as follows: we start from x and then we wait a random time τ_1 until the process is below the level M (note that if $x \leq M$, $\tau_1 = 0$). Hence τ_1 is stochastically dominated by a random variable $\bar{\tau}_1$ with the same distribution as $\mathcal{T}^x_M$. Then either the shifted process $Z \circ \theta_{\tau_1}$ become extinct before time t_0, with probability larger than p, and we stop the algorithm or it survives and we start again the procedure for the shifted process $Z \circ \theta_{t_0+\tau_1}$ which has the same law as the process Z starting from $Z_{t_0+\tau_1}$ thanks to the Markov property. In other words, the extinction time of Z is stochastically dominated from above by the random variable

$$\sum_{i=1}^{\xi} (t_0 + \bar{\tau}_i),$$

where $\{\bar{\tau}_i\}_{i\geq 0}$ are i.i.d. and independent of ξ. Hence,

$$\sup_{x\geq 0} \mathbb{E}_x[T_0] \leq \frac{1}{p}\left(t_0 + \sup_{x\geq 0} \mathbb{E}_x\big[T_M\big]\right) < \infty,$$

which ends the first part of the proof of Theorem 1.2.

It remains to prove that the process comes down from infinity under the assumption that $\int_{(-1,0)} \pi(\mathrm{d}z) < \infty$. Let us introduce the semigroup $(\mathtt{P}_t, t \geq 0)$ associated to the process Z as follows: for any continuous function f, we have $\mathtt{P}_t f(x) = \mathbb{E}_x[f(Z_t)]$ which is well defined on $[0, \infty)$. Our aim is to prove that it can be extended on $[0, \infty]$. With this aim in mind, we set $t \geq 0$ and a continuous function f and we prove that $(\mathtt{P}_t f(x), x \geq 0)$ is a Cauchy sequence in $\mathbb{R}$ when x goes to ∞.

First, we fix $\varepsilon > 0$ and take $t_0 \leq t \wedge \varepsilon(2\|f\| \int_{(-1,0)} \pi(\mathrm{d}u))^{-1}$. Then we use the sequence of stopping times $\{T_m\}_{m\geq 0}$ together with the Markov property, to find

$$\begin{aligned}
\mathtt{P}_t f(x) &= \mathbb{E}_x\left[\mathbf{1}_{\{T_m > t_0\}} f(Z_t)\right] + \mathbb{E}_x\left[\mathbf{1}_{\{T_m \leq t_0\}} \mathbb{E}_{Z_{T_m}}[f(Z_{t-T_m})]\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{\{T_m > t_0\}} f(Z_t)\right] + \mathbb{E}_x\left[\mathbf{1}_{\{T_m \leq t_0\}} \mathtt{P}_{t-T_m} f(Z_{T_m})\right].
\end{aligned}$$

For simplicity, we define the function h as follows

$$h(x, s) := \mathbf{1}_{\{s\leq t_0\}} \mathtt{P}_{t-s} f(x) + \mathbf{1}_{\{s>t_0\}} \mathtt{P}_{t-t_0} f(x),$$

which is a continuous with respect to s and deduce

$$\begin{aligned}
\mathtt{P}_t f(x) = \mathbb{E}_x\Big[\mathbf{1}_{\{T_m > t_0\}}\Big(f(Z_t) - \mathtt{P}_{t-t_0} f(m)\Big)\Big] + \mathbb{E}_x\left[h(m, T_m)\right] \\
+ \mathbb{E}_x\Big[\mathbf{1}_{\{T_m \leq t_0\}}\big(h(Z_{T_m}, T_m) - h(m, T_m)\big)\Big].
\end{aligned}$$

Hence for any $x, y \in [m, \infty)$, we observe

$$\begin{aligned}|\mathtt{P}_t f(x) - \mathtt{P}_t f(y)| \leq \Big|\mathbb{E}_x\Big[h(m, T_m)\Big] - \mathbb{E}_y\Big[h(m, T_m)\Big]\Big| + 2\|f\|\Big(\mathbb{P}_x(T_m > t_0) + \mathbb{P}_y(T_m > t_0)\Big) \\ + \Big|\mathbb{E}_x\Big[\mathbf{1}_{\{T_m \leq t_0\}}\big(h(Z_{T_m}, T_m) - h(m, T_m)\big)\Big]\Big| + \Big|\mathbb{E}_y\Big[\mathbf{1}_{\{T_m \leq t_0\}}\big(h(Z_{T_m}, T_m) - h(m, T_m)\big)\Big]\Big|.\end{aligned} \tag{3.18}$$

We deal with the first two terms of the right hand side of the previous inequality by using similar arguments to those of the proof of Theorem 20.13 in [8]. Indeed, Theorem 1.1 ensures that the sequence of random variables $(T_m, \mathbb{P}_x)$ is increasing with respect to x. Thus, it converges almost surely to a random variable here denoted by T_m^∞. Then, from the first part of this proof, for any $m \geq M$,

$$\sup_{x \geq m} \mathbb{E}_x[T_m] \leq \int_{me^{-1}}^{\infty} \frac{\mathrm{d}w}{g(w) - \theta w} \underset{m \to \infty}{\longrightarrow} 0. \tag{3.19}$$

Together with the fact that $(T_m, \mathbb{P}_x) \leq (T_{m'}, \mathbb{P}_x)$ for any $m \leq m'$, we deduce that the sequence $\{T_m^\infty\}_{m \geq 0}$ is decreasing and converges to 0 a.s. Then we can fix $m > M$ such that there exist $m_0 > m$ satisfying for any $x, y \geq m_0$,

$$2\|f\|\Big(\mathbb{P}_x(T_m > t_0) + \mathbb{P}_y(T_m > t_0)\Big) \leq \varepsilon. \tag{3.20}$$

Moreover, (3.19) implies that $\mathbb{P}_x \circ (T_m)^{-1}$ converges weakly when x goes to ∞. Since the mapping $s \mapsto h(m, s)$ is continuous, for m_0 is sufficiently large we have for any $x, y \geq m_0$,

$$\Big|\mathbb{E}_x\Big[h(m, T_m)\Big] - \mathbb{E}_y\Big[h(m, T_m)\Big]\Big| \leq \varepsilon. \tag{3.21}$$

It remains to treat the two last terms of (3.18). Note that both terms are different from 0 if and only if the process has a negative jump at time T_m. Moreover, recall that Z is solution to (1.9) where S is the random environment defined by (3.4), in other words there are no negative jumps for S smaller than $(1 - e^{-1})$. Hence, under the assumption $\int_{(-1,0)} \pi(\mathrm{d}u) < \infty$, we find

$$\begin{aligned}\Big|\mathbb{E}_x\Big[\mathbf{1}_{\{T_m \leq t_0\}}\Big(h(Z_{T_m}, T_m) - h(m, T_m)\Big)\Big]\Big| &\leq 2\|f\|\mathbb{P}_x\Big(T_m \leq t_0, Z_{T_m} < Z_{T_m -}\Big) \\ &\leq 2\|f\|\mathbb{P}_x\Big(\exists s \in [0, t_0], Z_s < Z_{s-}\Big) \\ &\leq 2\|f\|\mathbb{P}_x\left(\int_0^{t_0}\int_{(-1,0)} N^{(e)}(\mathrm{d}s, \mathrm{d}u) \geq 1\right) \\ &\leq 2\|f\|\mathbb{E}_x\left[\int_0^{t_0}\int_{(-1,0)} N^{(e)}(\mathrm{d}s, \mathrm{d}u)\right] \\ &= 2\|f\| t_0 \int_{(-1,0)} \pi(\mathrm{d}u) \leq \varepsilon.\end{aligned}$$

This completes the proof of Theorem 1.2. □

4 Lamperti-type transform for CB-processes with competition in a Brownian random environment.

Here, we take g to be a continuous function on $[0, \infty)$ with $g(0) = 0$. Under this assumption, the SDE (1.6) can be written as follows

$$\begin{aligned}Z_t = Z_0 + b\int_0^t Z_s \mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \sigma\int_0^t Z_s \mathrm{d}B_s^{(e)} \\ + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} z N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z \widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),\end{aligned} \tag{4.1}$$

with $\sigma \geq 0$. It is important to note that Proposition 1 in Palau and Pardo [15] guarantees that the above SDE has a unique strong positive solution.

The main result in this section is the Lamperti-type representation of a CB-process with competition in a Brownian random environment. Such random time change representation will be very useful to study path properties of the so-called logistic case. In order to state the Lamperti-type representation, we introduce the family of processes which are involved in the time change.

Let $X = (X_t, t \geq 0)$ be a spectrally positive Lévy process with characteristics $(-b, \gamma, \mu)$ and such that its Lévy measure μ satisfies (1.2). We also consider $W = (W_t, t \geq 0)$ a standard Brownian motion independent of X and assume that g is a continuous function on $[0, \infty)$ with $g(0) = 0$ and such that $\lim_{x\to 0} x^{-1}g(x)$ exists. According to Proposition 1 in Palau and Pardo [15] for each $x > 0$, there is a unique strong solution to

$$\mathrm{d}R_t = \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\mathrm{d}X_t - \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\frac{g(R_t)}{R_t}\mathrm{d}t + \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\sigma\sqrt{R_t}\mathrm{d}W_t, \tag{4.2}$$

with $R_0 = x$. The assumption that $\lim_{x\to 0} x^{-1}g(x)$ exists, is not necessary but it implies that we can use directly Proposition 1 of Palau and Pardo [15]. We can relax this assumption but further explanations are needed. Indeed a similar approach to Theorems 2.1 and 2.3 in Ma [14] will guarantee that the SDE defined above for a more general competition mechanism g has a unique strong solution.

It is important to note that in the logistic-case i.e. $g(x) = cx^2$, for $x \geq 0$ and some constant $c > 0$, the process R is a Feller diffusion which is perturbed by the Lévy process X. Moreover if the Lévy process X is a subordinator, then the process R turns out to be a CB-process with immigration.

We now state the Lamperti-type representation of CB-processes with competition in a Brownian random environment.

Theorem 4.1. *Let $R = (R_t, t \geq 0)$ be the unique strong solution of (4.2) and $T_0^R = \sup\{s : R_s = 0\}$. We also let C be the right-continuous inverse of η, where*

$$\eta_t = \int_0^{t\wedge T_0^R} \frac{\mathrm{d}s}{R_s}, \qquad t > 0.$$

Hence the process defined by

$$Z_t = \begin{cases} R_{C_t}, & \text{if } 0 \leq t < \eta_\infty \\ 0, & \text{if } \eta_\infty < \infty, T_0^R < \infty \text{ and } t \geq \eta_\infty, \\ \infty, & \text{if } \eta_\infty < \infty, T_0^R = \infty \text{ and } t \geq \eta_\infty, \end{cases}$$

satisfies the SDE (4.1).

Reciprocally, let Z be the unique strong solution to (4.1) with $Z_0 = x$ and let

$$C_t = \int_0^t Z_s\mathrm{d}s, \qquad t > 0.$$

If η denotes the right-continuous inverse of C, then the process defined by

$$R_t = Z_{\eta_t\wedge T_0} \qquad \text{for} \quad t \geq 0.$$

satisfies the SDE (4.2).

Proof of Theorem 4.1. Since X is a spectrally positive Lévy process and $R_{t-} = 0$ implies $R_t = 0$, we get $R_{t-} > 0$ if and only if $t \in [0, T_0^R)$. We also observe that X can be written as follows

$$X_t = bt + \sqrt{2}\gamma B_t + \int_0^t\int_{(0,1)} z\widetilde{M}(\mathrm{d}s, \mathrm{d}z) + \int_0^t\int_{[1,\infty)} zM(\mathrm{d}s, \mathrm{d}z),$$

where B is a standard Brownian motion and M is a Poisson random measure with intensity $\mathrm{d}s\mu(\mathrm{d}z)$ and $\widetilde{M}$ denotes its compensated version. Then from the latter identity and (4.2), we have

$$Z_t = x + b\int_0^{C_t\wedge T_0^R}\mathrm{d}s - \int_0^{C_t\wedge T_0^R}\frac{g(R_s)}{R_s}\mathrm{d}s + \sqrt{2}\gamma\int_0^{C_t\wedge T_0^R}\mathrm{d}B_s + \int_0^{C_t\wedge T_0^R}\sigma\sqrt{R_s}\mathrm{d}W_s$$
$$+\int_0^{C_t\wedge T_0^R}\int_{(0,1)} z\mathbf{1}_{\{R_{s-}>0\}}\widetilde{M}(\mathrm{d}s,\mathrm{d}z) + \int_0^{C_t\wedge T_0^R}\int_{[1,\infty)} z\mathbf{1}_{\{R_{s-}>0\}}M(\mathrm{d}s,\mathrm{d}z), \quad t\geq 0.$$

On the one hand, by straightforward computations we deduce

$$C_t\wedge T_0^R = \int_0^t Z_s\mathrm{d}s,$$

implying that

$$\int_0^{C_t\wedge T_0^R}\frac{g(R_s)}{R_s}\mathrm{d}s = \int_0^t g(Z_s)\mathrm{d}s,$$

and

$$L_t^{(1)} = \sqrt{2}\gamma\int_0^{C_t\wedge T_0^R}\mathrm{d}B_s \qquad \text{and} \qquad L_t^{(2)} = \sigma\int_0^{C_t\wedge T_0^R}\sqrt{R_s}\mathrm{d}W_s,$$

are independent continuous local martingales with increasing processes

$$\langle L^{(1)}\rangle_t = 2\gamma^2\int_0^t Z_s\mathrm{d}s \qquad \text{and} \qquad \langle L^{(2)}\rangle_t = \sigma^2\int_0^t Z_s^2\mathrm{d}s.$$

On the other hand, we define the random measure $N(\mathrm{d}s,\mathrm{d}z)$ on $(0,\infty)^2$ as follows

$$N((0,t]\times\Lambda) = \int_0^{C_t\wedge T_0^R}\int_{(0,\infty)}\mathbf{1}_\Lambda(z)\mathbf{1}_{\{R_{s-}>0\}}M(\mathrm{d}s,\mathrm{d}z).$$

Then $N(\mathrm{d}s,\mathrm{d}z)$ has predictable compensator

$$Z_{s-}\mathrm{d}s\mu(\mathrm{d}z).$$

By Theorems 7.1 and 7.4 in Ikeda and Watanabe [7], on an extension of the original probability space there exist two independent Brownian motions, $B^{(1)}$ and $B^{(2)}$, and a Poisson random measure $N(\mathrm{d}s,\mathrm{d}u,\mathrm{d}z)$ on $(0,\infty)^3$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ such that for any $t\geq 0$,

$$\int_0^{C_t\wedge T_0^R}\int_{[1,\infty)} z\mathbf{1}_{\{R_{s-}>0\}}M(\mathrm{d}s,\mathrm{d}z) = \int_0^t\int_{[1,\infty)}\int_0^{Z_{s-}} zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),$$

$$\int_0^{C_t\wedge T_0^R}\int_{(0,1)} z\mathbf{1}_{\{R_{s-}>0\}}\widetilde{M}(\mathrm{d}s,\mathrm{d}z) = \int_0^t\int_{(0,1)}\int_0^{Z_{s-}} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),$$

$$L_t^{(1)} = \int_0^t\sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(1)} \qquad \text{and} \qquad L_t^{(2)} = \sigma\int_0^t Z_s\mathrm{d}B_s^{(2)}.$$

Putting all the pieces together, we deduce that $(Z_t, t\geq 0)$ is a solution of (4.1).

For the reciprocal, we first observe that since Z has no negative jumps and $Z_{t-}=0$ implies $Z_t=0$, we get $Z_{t-}>0$ if and only if $Z_t>0$ for $t\in[0,T_0)$. Thus $R_{t-}>0$ if and only if $R_t>0$ for $t\in[0,\eta_{T_0})$, then for any $t\in[0,\eta_{T_0})$, the equation (4.2) is equivalent to

$$R_t = \mathrm{d}X_t - \frac{g(R_t)}{R_t}\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}W_t. \tag{4.3}$$

Since the process Z satisfies the SDE (1.6) and $R_t = Z_{\eta_t}$, we have

$$\begin{aligned} R_t =& Z_0 + b\int_0^{\eta_t\wedge T_0} Z_s\mathrm{d}s + \int_0^{\eta_t\wedge T_0}\sqrt{2\gamma^2 Z_s}\mathrm{d}B_s + \sigma\int_0^{\eta_t\wedge T_0} Z_s\mathrm{d}B_s^{(e)} \\ &+ \int_0^{\eta_t\wedge T_0}\int_{[1,\infty)}\int_0^{Z_{s-}} zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^{\eta_t\wedge T_0}\int_{(0,1)}\int_0^{Z_{s-}} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) - \int_0^{\eta_t\wedge T_0} g(Z_s)\mathrm{d}s. \end{aligned} \tag{4.4}$$

On the one hand, by straightforward computations we deduce

$$\int_0^{\eta_t\wedge T_0} Z_s\mathrm{d}s = t\wedge\eta_{T_0}, \qquad \text{and} \qquad \int_0^{\eta_t\wedge T_0} g(Z_s)\mathrm{d}s = \int_0^{t\wedge\eta_{T_0}}\frac{g(R_s)}{R_s}\mathrm{d}s.$$

The latter identities imply

$$M_t^{(1)} = \int_0^{\eta_t\wedge T_0}\sqrt{2\gamma^2 Z_s}\mathrm{d}B_s \qquad \text{and} \qquad M_t^{(2)} = \sigma\int_0^{\eta_t\wedge T_0} Z_s\mathrm{d}B_s^{(e)},$$

are independent continuous local martingales with increasing processes

$$\langle M^{(1)}\rangle_t = 2\gamma^2\int_0^{\eta_t\wedge T_0} Z_s\mathrm{d}s = 2\gamma^2(t\wedge\eta_{T_0}) \qquad \text{and} \qquad \langle M^{(2)}\rangle_t = \sigma^2\int_0^{\eta_t\wedge T_0} Z_s^2\mathrm{d}s = \sigma^2\int_0^{t\wedge\eta_{T_0}} R_s\mathrm{d}s.$$

On the other hand, we define the random measure $M(\mathrm{d}s,\mathrm{d}z)$ on $(0,\infty)^2$ as follows

$$M((0,t]\times\Lambda) = \int_0^{\eta_t\wedge T_0}\int_{(0,\infty)}\int_0^{Z_{s-}} \mathbf{1}_\Lambda(z)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_{\eta_{T_0}}^t\int_{(0,\infty)}\int_{Z_{s-}}^{Z_{s-}+1}\mathbf{1}_\Lambda(z)\mathbf{1}_{\{t>\eta_{T_0}\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \tag{4.5}$$

Then $M(\mathrm{d}s,\mathrm{d}z)$ has predictable compensator $\mathrm{d}s\mu(\mathrm{d}z)$. By Theorems 7.1 and 7.4 in Ikeda and Watanabe [7], on an extension of the original probability space there exist two independent Brownian motions, $B^{(1)}$ and $B^{(2)}$, and a Poisson random measure $M(\mathrm{d}s,\mathrm{d}z)$ on $(0,\infty)^2$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)$ such that for any $t\geq 0$,

$$M_t^{(1)} = B_{t\wedge\eta_{T_0}}^{(1)} \qquad \text{and} \qquad M_t^{(2)} = \sigma\int_0^{t\wedge\eta_{T_0}}\sqrt{R_s}\mathrm{d}B_s^{(2)}. \tag{4.6}$$

Putting all the pieces together, we deduce that (4.3) holds for $t\in[0,\eta_{T_0})$. Recall that $Z_{T_0-} = Z_{T_0} = 0$. Then on $\{\eta_{T_0}<\infty\}$ by using (4.4)-(4.6), we deduce that the right hand side of (4.3) is equal to 0 for $t=\eta_{T_0}$ and then for all $t\geq\eta_{T_0}$. □

5 Proofs of the logistic case

We first present the proofs of the case when the branching mechanism is associated with a subordinator.

5.1 Subordinator case

In this part, we provide the proofs of Lemma 1.4 and Proposition 1.5. The proof of Proposition 1.3 will follow directly from the Lamperti-type representation and the discussion below.

In the particular case when the spectrally positive Lévy process X is a subordinator in the Lamperti-type representation (see Theorem 4.1), the process R turns out to be a Feller diffusion with immigration. In other words, it is the unique positive strong solution of the following SDE

$$R_t = R_0 + X_t - c\int_0^t R_s\mathrm{d}s + \int_0^t\sqrt{\sigma^2 R_s}\mathrm{d}W_s. \tag{5.1}$$

The branching mechanism ω and the immigration mechanism ϕ associated to the process R, are given by

$$\omega(z) = cz + \frac{\sigma^2 z^2}{2} \quad \text{and} \quad \phi(z) = -\psi(z) = \delta z + \int_{(0,\infty)} (1 - e^{-zu})\mu(\mathrm{d}u),$$

where

$$\int_{(0,\infty)} (1 \wedge u)\mu(\mathrm{d}u) < \infty \qquad \text{and} \qquad \delta = b - \int_{(0,1)} u\mu(\mathrm{d}u) \geq 0.$$

This type of processes have been studied recently by many authors, see for instance the papers of Keller-Ressel and Mijatovic [9] and Duhalde et al. [4] and the references therein. In [9], the authors were interested in the invariant distribution associated to the process R and Duhalde et al. [4] studied first passage times problems and provide necessary and sufficient conditions for polarity and recurrence.

We now proceed with the proofs of Propositions 1.3, 1.5 and Lemma 1.4.

Proof of Proposition 1.3. The proof of this result is a direct consequence of the Lamperti-type representation (Theorem 4.1) and Theorem 1 in Duhalde et al. [4]. □

Proof of Lemma 1.4. First of all, we recall that m introduced as in (1.16) is well defined under the log-moment condition (1.3) since a similar computation as in Corollary 3.21 in Li [13] guarantees the integrability of ϕ/ω at 0. Then, from (1.14), we have

$$-m(\lambda) = \int_0^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z = \frac{2}{\sigma^2}\int_0^\lambda \left(\frac{\delta z}{\frac{2c}{\sigma^2}z + z^2} + \frac{1}{\frac{2c}{\sigma^2}z + z^2}\int_0^\infty (1 - e^{-zu})\mu(\mathrm{d}u) \right) \mathrm{d}z. \tag{5.2}$$

For simplicity, we define $K := 2c/\sigma^2$. Since all terms in (5.2) are positive, we can separate the above integral into two terms and study each of them independently. For the first term, we observe

$$\int_0^\lambda \frac{\delta z}{Kz + z^2}\mathrm{d}z = \delta \int_0^\lambda \int_0^\infty e^{-v(z+K)}\mathrm{d}v\mathrm{d}z = \int_0^\infty (1 - e^{-\lambda v})\frac{\delta e^{-Kv}}{v}\mathrm{d}v,$$

where the last equality follows from an application of Fubini-Tonelli's theorem. For the second term, we use again Fubini-Tonelli's theorem, to deduce

$$\int_0^\lambda \frac{1}{Kz + z^2}\left(\int_0^\infty (1 - e^{-zu})\mu(\mathrm{d}u) \right) \mathrm{d}z = \frac{1}{K}\int_0^\infty \left(\int_0^\lambda \frac{K(1 - e^{-zu})}{Kz + z^2}\mathrm{d}z \right) \mu(\mathrm{d}u).$$

Now, we fix $u > 0$ and study the integral inside the brackets. Since the map $z \mapsto (1 - e^{-zu})/z$ is integrable

at 0, we have

$$\begin{aligned}
\int_0^\lambda \frac{K(1-e^{-zu})}{Kz+z^2}\mathrm{d}z &= \int_0^\lambda \left(\frac{1-e^{-zu}}{z} - \frac{1-e^{-zu}}{K+z}\right)\mathrm{d}z \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - \int_K^{K+\lambda} \frac{1-e^{Ku}e^{-zu}}{z}\mathrm{d}z \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - (1-e^{Ku})\int_K^{K+\lambda}\frac{1}{z}\mathrm{d}z - e^{Ku}\int_K^{K+\lambda}\frac{1-e^{-zu}}{z}\mathrm{d}z \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - (1-e^{Ku})\int_K^{K+\lambda}\int_0^\infty e^{-zv}\mathrm{d}v\mathrm{d}z \\
&\qquad + e^{Ku}\left(\int_0^K \frac{1-e^{-zu}}{z}\mathrm{d}z - \int_0^{K+\lambda}\frac{1-e^{-zu}}{z}\mathrm{d}z\right) \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - (1-e^{Ku})\int_0^\infty \frac{e^{-Kv}}{v}(1-e^{-\lambda v})\mathrm{d}v - e^{Ku}\int_0^u \frac{e^{-Kv}}{v}(1-e^{-\lambda v})\mathrm{d}v \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}(1-e^{-Kv})\mathrm{d}v + (e^{Ku}-1)\int_u^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\mathrm{d}v
\end{aligned}$$

where the second identity follows from the change of variables $zu = \lambda v$, the third identity is obtained by adding and subtracting e^{Ku}, the fifth identity follows from Fubini-Tonelli's Theorem and the change of variables $Kv = zu$ and $(K+\lambda)v = zu$ and finally, the last identity follows by adding and subtracting

$$\int_0^u \frac{1-e^{-\lambda v}}{v}e^{-Kv}\mathrm{d}v.$$

In other words, we get

$$\int_0^\lambda \frac{K(1-e^{-zu})}{Kz+z^2}\mathrm{d}z = \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}(e^{K(v\wedge u)}-1)\mathrm{d}v = \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\left(\int_0^{v\wedge u} Ke^{Kz}\mathrm{d}z\right)\mathrm{d}v.$$

Putting all pieces together and using twice Fubini-Tonelli's theorem, we obtain

$$\begin{aligned}
\int_0^\lambda \frac{1}{Kz+z^2}\left(\int_0^\infty (1-e^{-zu})\mu(\mathrm{d}u)\right)\mathrm{d}z &= \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\left(\int_0^\infty \left(\int_0^{v\wedge u} e^{Kz}\mathrm{d}z\right)\mu(\mathrm{d}u)\right)\mathrm{d}v \\
&= \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\left(\int_0^v e^{Kz}\bar{\mu}(z)\mathrm{d}z\right)\mathrm{d}v.
\end{aligned}$$

Finally from identity (5.2) and the previous computations, we find (1.17).

Next, we define the positive measure $\Pi(\mathrm{d}z)$ as follows

$$\Pi(\mathrm{d}z) = \frac{2e^{-Kz}}{\sigma^2 z}\left(\delta + \int_0^z e^{Kv}\bar{\mu}(v)\mathrm{d}v\right)\mathrm{d}z, \tag{5.3}$$

and prove that $\int_{(0,\infty)}(1\wedge z)\Pi(\mathrm{d}z)$ is finite. From Fubini-Tonelli's theorem, we observe

$$\int_0^1 z\Pi(\mathrm{d}z) \le \frac{2}{\sigma^2}\left(\delta + e^K\int_0^1 \bar{\mu}(v)\mathrm{d}v\right) = \frac{2}{\sigma^2}\left(\delta + e^K\int_0^{+\infty}(1\wedge u)\mu(\mathrm{d}u)\right) < \infty.$$

Moreover,

$$\begin{aligned}\int_1^\infty \Pi(\mathrm{d}z) &= \frac{2}{\sigma^2}\left(\delta\int_1^\infty \frac{e^{-Kz}}{z}\mathrm{d}z + \int_1^\infty \frac{e^{-Kz}}{z}\int_0^z e^{Kv}\bar{\mu}(v)\mathrm{d}v\mathrm{d}z\right)\\ &\leq \frac{2}{\sigma^2}\left(\frac{\delta}{K}e^{-K} + \int_1^\infty e^{Kv}\bar{\mu}(v)\int_v^\infty \frac{e^{-Kz}}{z}\mathrm{d}z\mathrm{d}v\right)\\ &\leq \frac{2}{\sigma^2}\left(\frac{\delta}{K}e^{-K} + \int_1^\infty e^{Kv}\bar{\mu}(v)\int_v^\infty e^{-Kz}\left(\frac{1}{z}+\frac{1}{Kz^2}\right)\mathrm{d}z\mathrm{d}v\right)\\ &\leq \frac{2}{\sigma^2}\left(\frac{\delta}{K}e^{-K} + \frac{1}{K}\int_1^\infty \frac{\bar{\mu}(v)}{v}\mathrm{d}v\right),\end{aligned}$$

which is finite since the right-hand side of the last integral is equal to $\int_1^\infty \ln(u)\mu(\mathrm{d}u)$ which is finite under (1.3). In other words, the probability measure ν is infinitely divisible with support on $(0,\infty)$ and with Laplace exponent $-m$. Finally, if $\bar{\mu}(0)\leq b$, a simple computation ensures that k defined by

$$k(z) = \frac{2e^{-Kz}}{\sigma^2}\left(\delta + \int_0^z e^{Kv}\bar{\mu}(v)\mathrm{d}v\right),$$

is non-increasing and Theorem 15.10 in Sato [20] implies the self-decomposability of ν. □

Proof of Proposition 1.5. Recall that the process R, which is the unique strong solution to (5.1), is a CB-process with immigration. According to Theorem 2 in Duhalde et al. [4], the point 0 is polar for R, i.e. $T_0^R = \infty$ almost surely, accordingly as

$$\int_1^\infty \frac{\mathrm{d}\lambda}{\omega(\lambda)}\exp\left\{\int_1^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z\right\} = \infty. \tag{5.4}$$

Here, we show that the above integral condition is equivalent to $2\delta\geq\sigma^2$. Recall that $K = 2c/\sigma^2$, then for any $\lambda > 1$, we have

$$\begin{aligned}\int_1^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z &= \frac{2\delta}{\sigma^2}\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\int_0^\infty \mu(\mathrm{d}u)\int_1^\lambda \frac{1-e^{-zu}}{Kz+z^2}\mathrm{d}z\\ &\leq \frac{2\delta}{\sigma^2}\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\int_0^{x_0}\mu(\mathrm{d}u)\int_1^\lambda \frac{zu}{Kz+z^2}\mathrm{d}z + \frac{2}{\sigma^2}\int_{x_0}^\infty \mu(\mathrm{d}u)\int_1^\lambda \frac{1}{z^2}\mathrm{d}z\\ &\leq \frac{2\delta}{\sigma^2}\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\left(\int_0^{x_0} u\mu(\mathrm{d}u)\right)\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\bar{\mu}(x_0).\end{aligned}$$

The above inequality holds for any $x_0 > 0$, hence for any $\varepsilon > 0$, we can choose $x_0 > 0$ such that

$$\int_0^{x_0} u\mu(\mathrm{d}u) \leq \frac{\sigma^2}{2}\varepsilon.$$

Then for any $\lambda > 1$, the following inequalities hold

$$K_1(x_0)\frac{(K+\lambda)^{\frac{2\delta}{\sigma^2}}}{\lambda^2} \leq \frac{1}{\omega(\lambda)}\exp\left\{\int_1^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z\right\} \leq K_2(x_0)\frac{(K+\lambda)^{\frac{2\delta}{\sigma^2}+\varepsilon}}{\lambda^2},$$

where $K_1(x_0)$ and $K_2(x_0)$ are positive constants which are independent from λ. Therefore we conclude that (5.4) holds if and only if $2\delta\geq\sigma^2$.

Now, we deduce under which condition the process Z is recurrent. Recall from Theorem 3 in [4] that the process R is recurrent if and only if

$$\int_0^1 \frac{\mathrm{d}\lambda}{\omega(\lambda)}\exp\left\{-\int_\lambda^1 \frac{\phi(z)}{\omega(z)}\mathrm{d}z\right\} = \infty. \tag{5.5}$$

From the definition of functions ϕ and ω and the fact that $2\delta \geq \sigma^2$, we deduce that (5.5) holds if and only if (1.18) is satisfied.

We recall that $2\delta \geq \sigma^2$ implies that $T_0^R = \infty$ a.s. and thus for any $t \geq 0$,

$$\eta_t = \int_0^t \frac{1}{R_s} \mathrm{d}s \to \eta_\infty, \qquad \text{as} \quad t \to \infty.$$

Hence if we also assume that (1.18) holds, then R is recurrent in $(0, \infty)$ and $\eta_\infty = \infty$ a.s. Indeed, if we define recursively the sequences of finite stopping times as follows $\tau_0^+ = 0$, and for any $k \geq 1$,

$$\tau_{k+1}^- = \inf\{t \geq \tau_k^+, R_s \leq 1\} \qquad \text{and} \qquad \tau_{k+1}^+ = \inf\{t \geq \tau_{k+1}^-, R_s \geq 2\},$$

we deduce that, since $\{\tau_k^+ - \tau_k^-, k \geq 1\}$ is a sequence of strictly positive i.i.d random variables,

$$\eta_\infty = \int_0^\infty \frac{1}{R_s} \mathrm{d}s \geq \sum_{k \geq 1} \frac{1}{2}(\tau_k^+ - \tau_k^-) = \infty, \qquad \text{a.s..}$$

This implies that C_t, the right inverse of η_t, is well defined on $(0, \infty)$ and that $Z_t = R_{C_t}$ for any $t \geq 0$. Finally Z is also recurrent in $(0, \infty)$ and $T_0 = \infty$ almost surely. Moreover Z has an invariant measure that we denoted by ρ.

In order to characterise the invariant measure ρ, we use the infinitesimal generator $\mathcal{U}$ of Z. In other words, we have that ρ is an invariant measure for Z if and only if

$$\int_0^\infty \mathcal{U}f(z)\rho(\mathrm{d}z) = 0,$$

for any f in the domain of $\mathcal{U}$. According to Palau and Pardo [15], the infinitesimal generator $\mathcal{U}$ satisfies for any $f \in C_b^2(\mathbb{R}_+)$,

$$\mathcal{U}f(x) = x\mathcal{A}f(x) - cx^2 f'(x) + \frac{\sigma^2}{2} x^2 f''(x),$$

where $\mathcal{A}$ represents the generator of the spectrally positive Lévy process associated to the branching mechanism ψ. For the particular choice of $f(x) = e^{-\lambda x}$, for $\lambda > 0$, we observe $\mathcal{A}f(x) = \psi(\lambda)e^{-\lambda x}$ implying that

$$0 = \int_0^\infty \mathcal{U}f(z)\rho(\mathrm{d}z) = \int_0^\infty (\psi(\lambda) + \omega(\lambda)z)\, z e^{-\lambda z} \rho(\mathrm{d}z).$$

Then, similarly as in [11], we denote the Laplace transform of $z\rho(\mathrm{d}z)$ by χ and performing the previous identity, we observe that χ satisfies the ordinary differential equation $\psi(\lambda)\chi(\lambda) - \omega(\lambda)\chi'(\lambda) = 0$ on $(0, \infty)$. Straightforward computations implies that χ satisfies

$$\chi(\lambda) = K_0 \exp\left\{ \int_a^\lambda \frac{\psi(u)}{\omega(u)} \mathrm{d}u \right\}, \tag{5.6}$$

for some constants $K_0 > 0$ and $a \geq 0$. We can now prove the cases (a) and (b).

Let us assume that (1.3) is satisfied which is equivalent to the integrability of ψ/ω at 0. We take $a = 0$ in (5.6) and deduce that $\chi(\lambda) = K_0 \exp(m(\lambda))$ for some constant $K_0 > 0$. In other words, we have for $z \geq 0$

$$\rho(\mathrm{d}z) = K_0 \frac{1}{z} \nu(\mathrm{d}z),$$

with a possibility Dirac mass at 0, and where ν is defined as in Lemma 1.4. We can conclude as soon as we prove that $\varrho := \int_0^\infty z^{-1}\nu(\mathrm{d}z)$ is finite if $2b > \sigma^2$ or if $2b = \sigma^2$ and condition (∂) holds and it is

infinite if $2b = \sigma^2$ and condition (ð) holds. Indeed, if $\varrho < \infty$, ρ defined by (1.19) is the unique invariant probability measure of Z and consequently Z is positive recurrent. If $\varrho = \infty$, then all invariant measures of Z are non-integrable at 0, so that Z_t converges to 0 in probability and since Z oscillates in $(0, \infty)$ then it is null-recurrent.

Therefore, it remains to prove whether ϱ is finite or not. Note that formally,

$$\varrho = \int_{(0,\infty)} z^{-1}\nu(\mathrm{d}z) = \int_0^\infty e^{m(\lambda)}\mathrm{d}\lambda.$$

Hence, ϱ is finite if and only if $e^{m(\lambda)}$ is integrable at ∞. From the proof of Lemma 1.4, we deduce

$$-m(\lambda) = \frac{2}{\sigma^2}\int_0^\lambda \frac{\delta}{K+z}\mathrm{d}z + \int_0^{+\infty} \frac{(1-e^{-\lambda z})}{z} h(z)\mathrm{d}z, \tag{5.7}$$

where we recall that $K = 2c/\sigma^2$, and

$$h(z) = \frac{2}{\sigma^2} e^{-Kz}\int_0^z e^{Kw}\bar{\mu}(w)\mathrm{d}w.$$

Similarly to the proof of Theorem 53.6 in Sato [20] or Theorem 3.4 in Lambert [11], we take $x > 0$ and $\lambda > 1$, an split the interval $(0, \infty)$ into $(0, x/\lambda]$, $(x/\lambda, x]$ and (x, ∞). Recall that in the proof of Lemma 1.4, we saw

$$\int^\infty \frac{\bar{\mu}(w)}{w}\mathrm{d}w < \infty, \quad \int_0 \bar{\mu}(w)\mathrm{d}w < \infty \quad \text{and} \quad \int_u^\infty \frac{e^{-Kz}}{z}\mathrm{d}z \le \frac{e^{-Ku}}{Ku}.$$

Hence, we deduce

$$\begin{aligned}\int_x^\infty \frac{h(z)}{z}\mathrm{d}z &= \frac{2}{\sigma^2}\int_0^\infty e^{Kw}\bar{\mu}(w)\left(\int_{x\vee w}^\infty \frac{e^{-Kz}}{z}\mathrm{d}z\right)\mathrm{d}w \\ &\le \frac{2}{K\sigma^2}\left(\frac{1}{x}\int_0^x \bar{\mu}(w)\mathrm{d}w + \int_x^\infty \frac{\bar{\mu}(w)}{w}\mathrm{d}w\right) < \infty,\end{aligned}$$

which guarantees, together with the Dominated Convergence Theorem, that

$$\int_x^\infty (1-e^{-\lambda z})\frac{h(z)}{z}\mathrm{d}z \quad \text{converges as} \quad \lambda \to \infty.$$

On the other hand, we observe

$$\begin{aligned}\int_0^{x/\lambda} (1-e^{-\lambda z})\frac{h(z)}{z}\mathrm{d}z &= \frac{\sigma^2}{2}\int_0^x \frac{(1-e^{-z})}{z} e^{-\frac{Kz}{\lambda}}\left(\int_0^{z/\lambda} e^{Kw}\bar{\mu}(w)\mathrm{d}w\right)\mathrm{d}z \\ &\le \frac{\sigma^2}{2}e^{Kx}\int_0^x \frac{(1-e^{-z})}{z}\mathrm{d}z\int_0^x \bar{\mu}(w)\mathrm{d}w < \infty,\end{aligned}$$

which implies the convergence of

$$\int_0^{x/\lambda} (1-e^{-\lambda z})\frac{h(z)}{z}\mathrm{d}z \qquad \text{when} \quad \lambda \to \infty.$$

Using a similar change of variables, we can deduce that

$$\int_x^{x/\lambda} e^{-\lambda z}\frac{h(z)}{z}\mathrm{d}z \qquad \text{converges when} \quad \lambda \to \infty.$$

Putting the pieces together in (5.7), we deduce that for any $x > 0$ and for λ large enough

$$-m(\lambda) = \frac{2\delta}{\sigma^2} \ln\left(1 + \frac{\lambda}{K}\right) + \int_x^{x/\lambda} \frac{h(z)}{z} \mathrm{d}z + K_1(x) + o(1),$$

where $K_1(x)$ is a non-negative constant. Hence, for λ large enough and for any $x > 0$,

$$e^{m(\lambda)} = \frac{K_2(x)}{(1+\lambda)^{2\delta/\sigma^2}} \exp\left\{ - \int_x^{x/\lambda} \frac{h(z)}{z} \mathrm{d}z + o(1) \right\}, \tag{5.8}$$

where $K_2(x)$ is a positive constant. It thus remains to study the integral term in (5.8). Since it is positive, we can find $K_3(x) > 0$ such that for any λ large enough, $e^{m(\lambda)} \leq e^{K_3(x)} \lambda^{-2\delta/\sigma^2}$, and we conclude as soon as $2\delta > \sigma^2$. This implies part (a).

Next, we prove part (b). Assume that $2\delta = \sigma^2$. We concentrate on the case (∂) since the case $(\eth)$ uses similar arguments. For the sake of brevity, we left the latter case to the interested readers. Under condition (∂), there exists $n \in \mathbb{N}$ such that $\inf(\mathtt{Adh}(I^{(n)})) > \sigma^2/2$ and $\mathtt{Adh}(I^{(k)}) = \{\sigma^2/2\}$, for any $k \in \{1, .., n-1\}$. Let us define by recurrence the collection of functions $\bar{I}$ such that

$$\bar{I}^{(1)}(z) = l^{(1)}(z) h(z) \qquad \text{and} \qquad \bar{I}^{(k)}(z) = l^{(k)}(z) \left[\bar{I}^{(k-1)}(z) - 1 \right], \quad k \in \mathbb{N},\ k \geq 2.$$

Note that the sequences $\{\bar{I}^{(k)}\}_{k \leq n}$ and $\{I^{(k)}\}_{k \leq n}$ satisfy the same recurrences but are initialized on different values. From the definition of h and a recurrence argument, it is straightforward to compute that for any $k \in \mathbb{N}$, and for z small enough,

$$\frac{2}{\sigma^2} e^{-Kz} I^{(k)}(z) + (e^{-Kz} - 1) \sum_{j=2}^{k} \prod_{i=j}^{k} l^{(i)}(z) \leq \bar{I}^{(k)}(z) \leq \frac{2}{\sigma^2} e^{-Kz} I^{(k)}(z).$$

Since $(e^{-Kz} - 1)$ behaves as $-Kz$, for z small enough, the second term of the left hand side converges to 0 when z converges to 0 and we deduce that the sequences of functions $\{\bar{I}^{(k)}\}_{k \leq n}$ and $\{I^{(k)}\}_{k \leq n}$ satisfy a similar assumption. Indeed, $\inf(\mathtt{Adh}(\bar{I}^{(n)})) = A > 1$ and $\mathtt{Adh}(\bar{I}^{(k)}) = \{1\}$, for any $k \in \{1, .., n-1\}$. Let us fix $\varepsilon > 0$ such that $A - \varepsilon > 1$ and $x > 0$ such that $\bar{I}^{(n)}(x) \geq A - \varepsilon$. Using the definition of $\{\bar{I}^{(k)}\}_{k \geq 0}$ and a recurrence argument, we obtain that for any $k \in \mathbb{N}$ and for any z sufficiently small,

$$h(z) = \frac{\bar{I}^{(k)}(z)}{\prod_{i=1}^{k} l^{(i)}(z)} + \sum_{j=1}^{k-1} \frac{1}{\prod_{i=1}^{j} l^{(i)}(z)}.$$

Hence,

$$\int_{x/\lambda}^{x} \frac{h(z)}{z} \mathrm{d}z \geq (A - \varepsilon) \int_{x/\lambda}^{x} \frac{\mathrm{d}z}{z \prod_{i=1}^{n} l^{(i)}(z)} + \sum_{j=1}^{n-1} \int_{x/\lambda}^{x} \frac{\mathrm{d}z}{z \prod_{i=1}^{j} l^{(i)}(z)}. \tag{5.9}$$

Moreover from the definition of $l^{(j)}$, we have for any $j \in \mathbb{N}$,

$$\begin{aligned} \int_{x/\lambda}^{x} \frac{\mathrm{d}z}{z \prod_{i=1}^{j} l^{(i)}(z)} &= l^{(j+1)}(x) - l^{(j+1)}\left(\frac{x}{\lambda}\right) \\ &= l^{(j+1)}(x) + l^{(j+1)}(\lambda) - R^{(j+1)}(x, \lambda) \\ &= l^{(j+1)}(\lambda) + l^{(j+1)}(x) + o(1), \qquad \text{as} \quad \lambda \to \infty, \end{aligned}$$

where the sequence $\{R^{(k)}\}_{k\geq 2}$ are also constructed by recurrence as follows, for any x small enough and λ large enough,

$$R^{(2)}(x,\lambda) = \ln\left(1+\frac{l^{(1)}(x)}{l^{(1)}(\lambda)}\right) \qquad \text{and} \qquad R^{(k)}(z) = \ln\left(1+\frac{R^{(k-1)}(x,\lambda)}{l^{(k-1)}(\lambda)}\right), \quad k\geq 3.$$

We can verify by recurrence that $R^{(j+1)}(x,\lambda) = l^{(j+1)}(x/\lambda) + l^{(j+1)}(\lambda)$ and that $R^{(j+1)}(x,\lambda)$ converges to 0 when λ increases to ∞. Finally from (5.9), as soon as λ is sufficiently large, we have

$$\int_{x/\lambda}^{x} \frac{h(z)}{z}\mathrm{d}z \geq (A-\varepsilon)l^{(n+1)}(\lambda) + \sum_{j=1}^{n-1} l^{(j+1)}(\lambda) + K_4(x),$$

where $K_4(x)$ is a finite constant. Hence using (5.8), we deduce that for λ sufficiently large there exist a finite constant $K_5(x)$ such that

$$e^{m(\lambda)} \leq \frac{e^{K_5(x)}}{\lambda\prod_{i=1}^{n-1} l^{(i)}(\lambda)(l^{(n)}(\lambda))^{A-\varepsilon}}. \tag{5.10}$$

Since $A-\varepsilon > 1$, the right hand side of (5.10) is integrable at ∞. Indeed, for any a, b sufficiently large such that $l^{(n)}(x) > 0$ and $l^{(n)}(z) > 0$, with the change of variables $u = l^n(\lambda)$, we have

$$\int_a^b \frac{1}{\lambda\prod_{i=1}^{n-1} l^{(i)}(\lambda)(l^{(n)}(\lambda))^{A-\varepsilon}}\mathrm{d}\lambda = \int_{l^{(n)}(a)}^{l^{(n)}(b)} \frac{1}{u^{A-\varepsilon}}\mathrm{d}u \underset{b\to\infty}{\longrightarrow} \int_{l^{(n)}(a)}^{\infty} \frac{1}{u^{A-\varepsilon}}\mathrm{d}u < \infty.$$

Finally, we have proved that under condition (∂),

$$\int^{\infty} e^{m(\lambda)}\mathrm{d}\lambda < \infty.$$

This completes the proof of part (b) and the case when condition (1.3) is satisfied.

Now, we deal with the case when the log-moment condition (1.3) does not hold and $2\delta > \sigma^2$. Under this assumption we show that Z is still positive recurrent but its invariant distribution has an infinite expected value. Recall that condition $2\delta > \sigma^2$ guarantees that Z is recurrent with an invariant distribution ρ satisfying (5.6). However in this case, ψ/ω is not integrable at 0 and we can not take $a = 0$ in (5.6), instead we let $a = 1$. Formally, the following identity still holds

$$\int_0^{\infty} \chi(\lambda)\mathrm{d}\lambda = \int_0^{\infty} \rho(\mathrm{d}z).$$

Our aim is thus to prove that the latter identity is finite but the expected value of ρ is infinite.

On the one hand, recalling that $K = 2c/\sigma^2$ and taking λ smaller than 1, we use the definition of ψ and Fubini-Tonnelli's Theorem to deduce

$$\begin{aligned}
-\int_\lambda^1 \frac{\psi(z)}{\omega(z)}\mathrm{d}z &= \frac{2\delta}{\sigma^2}\ln\left(\frac{K+1}{K+\lambda}\right) + \frac{2}{\sigma^2}\int_0^{\infty}\left(\int_\lambda^1 \frac{1-e^{-zu}}{Kz+z^2}\mathrm{d}z\right)\mu(\mathrm{d}u) \\
&\leq \frac{2\delta}{\sigma^2}\ln\left(1+\frac{1}{K}\right) + \frac{2}{\sigma^2}\int_0^{A}\left(\int_\lambda^1 \frac{zu}{Kz}\mathrm{d}z\right)\mu(\mathrm{d}u) + \frac{2}{\sigma^2}\int_A^{\infty}\left(\int_\lambda^1 \frac{1}{Kz}\mathrm{d}z\right)\mu(\mathrm{d}u) \\
&\leq \frac{2\delta}{\sigma^2}\ln\left(1+\frac{1}{K}\right) + \frac{2}{K\sigma^2}\int_0^{A} u\mu(\mathrm{d}u) - \frac{2}{K\sigma^2}\ln(\lambda)\bar{\mu}(A),
\end{aligned}$$

for any $A > 0$. Thus, we take $A > 0$ in such a way that $\bar{\mu}(A) \leq K\sigma^2/4$. Implying that for any $\lambda \leq 1$, we get

$$\chi(\lambda) \leq K_0 \frac{e^{K(A)}}{\lambda^{1/2}},$$

with K_0 and $K(A)$ two positive constants which are independent from λ. In other words, χ is integrable near 0. On the other hand, since

$$\int_1^\lambda \frac{\psi(z)}{\omega(z)} \mathrm{d}z \leq -\frac{2b}{\sigma^2} \ln\left(\frac{K+1}{K+\lambda}\right),$$

we also have

$$\chi(\lambda) \leq K_0 \left(\frac{K+1}{K+\lambda}\right)^{\frac{2b}{\sigma^2}},$$

implying that

$$\int_0^\infty \chi(\lambda) \mathrm{d}\lambda < \infty,$$

since $2b > \sigma^2$. In other words Z has a finite invariant measure and also is positive recurrent. Moreover, since the log-moment condition (1.3) does not hold, a straightforward computation gives

$$\int_0^\infty z\rho(\mathrm{d}z) = \lim_{\lambda \to 0} \int_0^\infty e^{-\lambda z} z\rho(\mathrm{d}z) = \lim_{\lambda \to 0} \chi(\lambda) = \infty.$$

It remains to treat the case when (1.18) does not hold and implicitly the log-moment condition (1.3) neither. Under these assumptions, Theorem 3 by Duhalde et al. [4] guarantees that R is transient and from Theorem 6 in [4] and Proposition 4.4 by Keller-Ressel [9], we deduce that

$$\limsup_{t \to \infty} R_t = \infty \qquad \text{a.s.},$$

(see also the comments after Remark 3 in [4]). Thus R_t goes to ∞ a.s., when t goes to ∞, and the same conclusion holds for the process Z regardless η_∞ is finite or not. Finally, note that since the process Z goes to ∞ almost surely, T_a is finite if and only if $T_a^R = \int_0^{T_a} Z_s \mathrm{d}s$ is finite. In addition with Proposition 10 in [4], we obtain the following identities, for any $x \geq a > 0$,

$$\mathbb{P}_x\left(\inf_{t \geq 0} Z_t < a\right) = \mathbb{P}_x\left(T_a < \infty\right) = \mathbb{P}_x\left(T_a^R < \infty\right) = \frac{f_0(x)}{f_0(a)}.$$

To conclude the proof of Proposition 1.5, note that in all cases of the proof, $T_0 = \infty$ a.s. and so 0 is polar. □

Proof of Proposition 1.6. Recall from Theorem 2 in Duhalde et al. [4] that 0 is polar for the process R accordingly as (5.4) holds. From the proof of Proposition 1.5, we know that (5.4) is equivalent to $2\delta \geq \sigma^2$. Thus if we assume that $2\delta < \sigma^2$, we have that 0 is not polar for R and implicitly this implies that the set $\{T_0^R < \infty\}$ has positive probability. Therefore, from the Lamperti-type representation, we deduce that Z_t converges to 0 when t increases to ∞ on $\{T_0^R < \infty\}$. □

5.2 Non-subordinator case

This last part is devoted to the proof of Theorem 1.8. We recall that the associated Lévy process X which appears in (5.1) is general, that is to say, there exist $\vartheta \geq 0$ such that $\psi(z) > 0$ for any $z \geq \vartheta$ and the log-moment condition (1.3) is satisfied.

Proof of Theorem 1.8. Let us fix $\lambda > 0$, and denote by Φ the function

$$\Phi(z) := \frac{e^{-m(z)}}{\omega(z)} \exp\left\{ - \int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v \right\} \int_0^z \exp\left\{ m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v \right\} \mathrm{d}u,$$

in other words, we have

$$h_\lambda(x) = 1 + \lambda \int_0^\infty e^{-xz}\Phi(z)\mathrm{d}z,$$

which was defined in 1.25. We first prove that h_λ is well defined on $(0,\infty)$ or equivalently, we need to prove that the mapping $z \mapsto e^{-xz}\Phi(z)$ is integrable on $(0,\infty)$ as soon as $x > 0$. With this aim in mind, we observe that from the definition of m and $\mathtt{I}$, (1.16) and (1.21), it is straightforward

$$\exp\left\{ m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v \right\} \longrightarrow 1, \qquad \text{as} \quad u \to 0, \tag{5.11}$$

implying

$$e^{-xz}\Phi(z) \sim \frac{z}{\omega(z)}, \qquad \text{as} \quad z \to 0, \tag{5.12}$$

which is integrable at 0 since $z^{-1}\omega(z)$ goes to $1/c$ as z goes to 0. Then, in a neighbourhood of ∞, we see from Lemma 1.7 that $y_\lambda(z) \le \sqrt{\lambda}\frac{\varphi'(z)}{\sqrt{\omega(\varphi(z))}}$ which is equivalent to $\sqrt{2\lambda}\frac{\varphi'(z)}{\sigma\varphi(z)}$. Hence,

$$\int_0^{\mathtt{I}(z)} y_\lambda(u)\mathrm{d}u = \mathrm{O}\left(\ln(z)\right) \qquad \text{and} \qquad \mathtt{I}'(z)y_\lambda(\mathtt{I}(z)) = e^{m(z)}y_\lambda(\mathtt{I}(z)) \to 0, \qquad \text{as} \quad z \to \infty. \tag{5.13}$$

Then, for any $x > 0$ and $u \ge \vartheta$, we have

$$\left| \frac{\exp\left\{ m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v \right\}}{\left(\frac{x}{2} + \frac{\psi(u)}{\omega(u)} + \mathtt{I}'(u)y_\lambda(\mathtt{I}(u)) \right) \exp\left\{ \frac{xu}{2} + m(u) + \int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v \right\}} \right| \le \frac{2\exp\left\{ -\frac{xu}{2} + \int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v \right\}}{x},$$

which goes to 0 as u goes to ∞. In other words,

$$\int_0^z \exp\left\{ m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v \right\} \mathrm{d}z = \mathrm{o}\left(\exp\left\{ \frac{xz}{2} + m(z) + \int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v \right\} \right), \quad \text{as} \quad z \to \infty. \tag{5.14}$$

Finally from the definition of Φ, we obtain

$$e^{-zx}\Phi(z) = \mathrm{o}\left(\frac{1}{\omega(z)} e^{-\frac{xz}{2}} \right), \qquad \text{as} \quad z \to \infty, \tag{5.15}$$

implying the integrability of $z \mapsto e^{-zx}\Phi(z)$ at ∞. It is important to note that (5.12) and (5.15), also imply that the mappings $z \mapsto ze^{-xz}\Phi(z)$ and $z \mapsto z^2e^{-zx}\Phi(z)$ are integrable on $(0,\infty)$ and that h_λ is a C^2-function on $(0,\infty)$.

Now, we prove (1.26). Recall that the infinitesimal generator of Z satisfies (1.24), i.e. for any $f \in C_b^2(\mathbb{R}_+)$

$$\mathcal{U}f(x) = x\mathcal{A}f(x) - cx^2f'(x) + \frac{\sigma^2}{2}x^2f''(x) \tag{5.16}$$

where $\mathcal{A}$ is the generator of the spectrally positive Lévy process associated to branching mechanism ψ. Since, for $f(x) = e^{-zx}$, $\mathcal{A}f(x) = \psi(z)e^{-zx}$ with $z \geq 0$, we deduce using twice integrations by parts that

$$\begin{aligned}\mathcal{U}h_\lambda(x) - \lambda h_\lambda(x) &= \lambda \int_0^\infty \Big(x\psi(z) + x^2\omega(z) - \lambda\Big)\Phi(z)e^{-zx}\mathrm{d}z - \lambda \\ &= \lambda\Bigg(\int_0^\infty \Big((\psi\Phi)'(z) + (\omega\Phi)''(z) - \lambda\Phi(z)\Big)e^{-xz}\mathrm{d}z - 1 \\ &\quad - xw(z)\Phi(z)e^{-xz}\Big|_{z=0}^{z=\infty} + \Big(\psi(z)\Phi(z) + (\omega\Phi)'(z)\Big)e^{-xz}\Big|_{z=0}^{z=\infty}\Bigg).\end{aligned} \tag{5.17}$$

Let us prove that the right-hand side of the latter expression equals 0. Since $m'(z) = \frac{\psi(z)}{\omega(z)}$ and $\mathtt{I}'(z) = e^{m(z)}$, we get

$$(\omega\Phi)'(z) = -\psi(z)\Phi(z) - y_\lambda(\mathtt{I}(z))e^{-\int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v}\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u + e^{\int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v}. \tag{5.18}$$

In addition with the fact that y_λ is solution to (1.23), we deduce that $(\omega\Phi)''(z) = -(\psi\Phi)'(z) + \lambda\Phi(z)$ for any $z \geq 0$. On the other hand, using (5.12) and (5.15), we have that

$$xw(z)\Phi(z)e^{-xz}\Big|_{z=0}^{z=\infty} = 0,$$

and from (5.18), together with (5.13) and (5.14), we deduce

$$\lim_{z\to\infty}(\psi(z)\Phi(z) + (\omega\Phi)'(z))e^{-xz} = 0,$$

as soon as $x > 0$. Therefore, it remains to study the previous limit but when z goes to 0. However, according to (5.18),

$$\lim_{z\to 0}(\psi(z)\Phi(z) + (\omega\Phi)'(z))e^{-xz} = 1 - \lim_{z\to 0} y_\lambda(\mathtt{I}(z))\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u. \tag{5.19}$$

By Lemma 1.7 and (5.11), we deduce

$$y_\lambda(\mathtt{I}(z))\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u \leq \sqrt{\frac{\lambda}{\omega(z)}}e^{-m(z)}\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u \sim \sqrt{\frac{\lambda}{cz}}z, \quad \text{as} \quad z \to 0,$$

which implies that the right-hand side of (5.19) equals 1. In other words, the right-hand side of (5.17) equals 0, meaning that $\mathcal{U}h_\lambda(x) = \lambda h_\lambda(x)$ for any $x > 0$.

The next step is to prove that $\int_0^\infty y_\lambda(v)\mathrm{d}v$ is finite under the Grey's condition, indeed Lemma 1.7 is not enought to conclude. With this goal in mind, we fix $x > 0$ and $\lambda \geq 0$ and set the function $G_{\lambda,x}$ as follows,

$$G_{\lambda,x}(v) := \int_0^\infty e^{-\lambda t}\mathbb{E}_x\Big[e^{-vZ_t}\Big]\mathrm{d}t, \qquad \text{for any} \quad v \geq 0.$$

This function is related with the Laplace transform of T_0, indeed

$$\lim_{v\to\infty} \lambda G_{\lambda,x}(v) = \mathbb{E}_x\Big[e^{-\lambda T_0}\Big].$$

From Theorem 1.2 the latter is positive since $T_0 < \infty$ a.s.

Now, we provide some properties of $G_{\lambda,x}$. We first note that for any h belonging to the domain of $\mathcal{U}$, the following identity holds

$$\lambda \int_0^\infty e^{-\lambda t}\mathbb{E}_x\Big[h(Z_t)\Big]\mathrm{d}t = h(x) + \int_0^\infty e^{-\lambda t}\mathbb{E}_x\Big[\mathcal{U}h(Z_t)\Big]\mathrm{d}t.$$

By taking $h(x) = e^{-vx}$ together with identity (5.16), we deduce

$$\begin{aligned}\lambda G_{\lambda,x}(v) &= e^{-vx} + \int_0^\infty e^{-\lambda t}\mathbb{E}_x\left[\psi(v)Z_s e^{-vZ_s} + \omega(v)Z_t^2 e^{-vZ_t}\right]\mathrm{d}t\\ &= e^{-vx} - \psi(v)G'_{\lambda,x}(v) + \omega(v)G''_{\lambda,x}(v).\end{aligned}$$

Moreover $\lambda G_{\lambda,x}(0) = 1$ and the dominated convergence theorem implies

$$G'_{\lambda,x}(v) = -\int_0^\infty e^{-\lambda t}\mathbb{E}_x[Z_t e^{-vZ_t}1_{\{Z_t>0\}}]\mathrm{d}t \longrightarrow 0, \quad \text{as} \quad v\to\infty.$$

We now prove that $G_{\lambda,x}$ is the unique solution to $\omega(v)y''(v) - \psi(v)y'(v) - \lambda y(v) = e^{-vx}$ with conditions $\lambda y(0) = 1$ and $\lim_{v\to\infty} y'(v) = 0$. In order to do so, we prove that the following function, for any $v \geq 0$,

$$k(v) := \frac{1}{\lambda}e^{-\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}\left(1 + \lambda\int_0^v\int_u^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s + m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}z\mathrm{d}u\right) \tag{5.20}$$

satisfies the same conditions as $G_{\lambda,x}$. We first observe that

$$\begin{aligned}&\int_0^v\int_u^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s + m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}z\mathrm{d}u\\ &\qquad = \int_0^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s}\left(\int_0^{v\wedge z} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}u\right)\mathrm{d}z\end{aligned} \tag{5.21}$$

is finite according to (5.11). In other words, k is well defined. Moreover, $\lambda k(0) = 1$ and since $\mathtt{I}'(z) = \exp(m(z))$, a straightforward computation gives

$$k'(v) = -e^{m(v)}y_\lambda(\mathtt{I}(v))k(v) + e^{m(v)+\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}\int_v^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s}\mathrm{d}z. \tag{5.22}$$

From (5.14) and (5.21), we deduce that k is bounded by some constant C on $\mathbb{R}$ and from Lemma 1.7, we also see that

$$\left|e^{m(v)}y_\lambda(\mathtt{I}(v))k(v)\right| \leq C\sqrt{\frac{\lambda}{\omega(v)}} \longrightarrow 0, \qquad \text{as} \quad v\to+\infty.$$

For the second term of the right-hand side of (5.22), we use a similar arguments to those used to deduce (5.14) which gives

$$\int_v^\infty \frac{e^{-xz}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s}\mathrm{d}z = o\left(e^{-m(v)-\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s - \frac{xv}{2}}\right), \qquad \text{as} \quad v\to\infty.$$

That is to say that $k'(v)$ converges to 0 when v goes to ∞. Finally, from (5.22), a straightforward computation provides

$$\omega(v)k''(v) = \psi(v)k'(v) + \lambda k(v) - e^{-vx}. \tag{5.23}$$

Putting all pieces together, we prove that k and $G_{\lambda,x}$ satisfy the same differential equation with conditions $\lambda k(0) = 1$ and $\lim_{v\to\infty} k'(v) = 0$. However, the set of functions that satisfy $\omega(v)y''(v) - \psi(v)y'(v) - \lambda y(v) = e^{-vx}$ with conditions $\lambda y(0) = 1$ is exactly $S := \{k_A, A \in \mathbb{R}\}$, with

$$k_A(v) := k(v) + Ae^{-\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s} \int_0^v e^{m(u)+2\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s},$$

Let us prove that $\lim_{v\to+\infty} k'_A(v) = 0$ if and only if $A = 0$. Indeed,

$$k'_A(v) = k'(v) + Ae^{m(v)+2\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}\left[1 - \frac{1}{\alpha(v)}\int_0^v e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}u\right],$$

where

$$\frac{1}{\alpha(v)} := y_\lambda(\mathtt{I}(v))e^{-2\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}.$$

Using Lemma 1.7, we have

$$\alpha'(v) = e^{m(v)+2\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}\left[-\frac{y'_\lambda(\mathtt{I}(v))}{y_\lambda(\mathtt{I}(v))^2} + 2\right] = e^{m(v)+2\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}\left[1 + \lambda\frac{e^{-2m(v)}}{\omega(v)y^2_\lambda(\mathtt{I}(v))}\right] \geq 2,$$

for any v large enough. In other words, there exist $v_0 > 0$ such that for any $v \geq v_0$,

$$\frac{1}{\alpha(v)}\int_0^v e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}u \leq \frac{1}{2} - \frac{\alpha(v_0)}{2\alpha(v)} + \frac{1}{\alpha(v)}\int_0^{v_0} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}u.$$

Since $\lim_{v\to\infty}\alpha(v) = \infty$, the latter inequality guarantees

$$\limsup_{v\to\infty}\frac{1}{\alpha(v)}\int_0^v e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}u \leq \frac{1}{2}.$$

In addition to the expression of k'_A, we deduce that $\lim_{v\to\infty} k'_A(v) = 0$ if and only if $A = 0$. Thus there exist a unique function in S that satisfies $\lim_{v\to\infty} k'_A(v) = 0$. Finally, since both k and $G_{\lambda,x}$ belong to S and satisfy that their respective derivatives go to 0 as v increases then both functions are equals on $\mathbb{R}$.

Furthermore, with a direct application of Fubini's theorem

$$\lim_{v\to\infty}\int_0^v\int_u^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s+m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}z\mathrm{d}u = \int_0^\infty e^{-zx}\Phi(z)\mathrm{d}z > 0.$$

In addition with (5.20), we get

$$e^{-\int_0^\infty y_\lambda(s)\mathrm{d}s}\left(1 + \lambda\int_0^\infty e^{-zx}\Phi(z)\mathrm{d}z\right) = \lim_{v\to\infty}\lambda k(v) = \lim_{v\to\infty}\lambda G_{\lambda,x}(v) = \mathbb{E}_x\left[e^{-\lambda T_0}\right] > 0.$$

We conclude that $\int_0^\infty y_\lambda(v)\mathrm{d}v$ is finite and

$$\mathbb{E}_x\left[e^{-\lambda T_0}\right] = e^{-\int_0^\infty y_\lambda(v)\mathrm{d}v}\left(1 + \lambda\int_0^\infty e^{-zx}\Phi(z)\mathrm{d}z\right). \tag{5.24}$$

We next prove that $h_\lambda(0) = \exp\{\int_0^\infty y_\lambda(v)\mathrm{d}v\}$. The main issue comes from the following fact: we can not make x tend to 0 directly in the formula of h_λ since we do not know the integrability of $\Phi(z)$ near ∞. However, from (5.11) we know that for any $u \in (0,\infty)$,

$$\lambda\int_0^\infty \frac{1}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s}\int_0^{z\wedge v} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}u\mathrm{d}z < \infty.$$

The goal is to take v near ∞. Using Fubini's theorem the following two changes of variables $z \mapsto \mathtt{I}(z)$ and $v \mapsto \mathtt{I}(v)$, we find

$$\lambda \int_0^\infty \frac{1}{\omega(z)} e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s} \int_0^{z\wedge v} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u\mathrm{d}z$$
$$= \int_0^{\mathtt{I}(v)} e^{2\int_0^u y_\lambda(s)\mathrm{d}s} \int_u^\infty \lambda \frac{e^{-2m(\varphi(z))}}{w(\varphi(z))} e^{-\int_0^z y_\lambda(s)\mathrm{d}s} \mathrm{d}z\mathrm{d}u.$$

Recalling that

$$\lambda \frac{e^{-2m(\varphi(z))}}{w(\varphi(z))} = \lambda \frac{\varphi'(z)^2}{w(\varphi(z))} = y_\lambda^2(z) - y_\lambda'(z),$$

and using integration by parts, we finally deduce

$$\lambda \int_0^\infty \frac{1}{\omega(z)} e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s} \int_0^{z\wedge v} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u\mathrm{d}z = e^{\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s} - 1.$$

Since the integrand is positive, we let v tend to ∞ to find

$$h_\lambda(0) = 1 + \lambda \int_0^\infty \frac{1}{\omega(z)} e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s} \int_0^{z} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u\mathrm{d}z = e^{\int_0^{\infty} y_\lambda(s)\mathrm{d}s} \tag{5.25}$$

which is finite according to the previous step.

We now prove identity (1.27). First, let us assume that $x \geq a > 0$. Recalling that $\mathcal{U}h_\lambda = \lambda h_\lambda$, we deduce from Itô's formula

$$e^{-\lambda t\wedge T_a} h_\lambda(Z_{t\wedge T_a}) = h_\lambda(x) + \int_0^{t\wedge T_a} e^{-\lambda s} h_\lambda'(Z_s)\sqrt{2\gamma^2 Z_s}\mathrm{d}B_s + \int_0^{t\wedge T_a} \sigma e^{-\lambda s} h_\lambda'(Z_s) Z_s \mathrm{d}B_s^{(e)}$$
$$+ \int_0^{t\wedge T_a}\int_0^\infty \int_0^{Z_{s-}} e^{-\lambda s}\left(h_\lambda(Z_{s-}+z) - h_\lambda(Z_{s-})\right)\widetilde{N}^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

Moreover h_λ is positive non-increasing, h_λ' is negative non-decreasing, and $(Z_s, s \leq t\wedge T_a)$ take values on $[a,+\infty)$. We then use a sequence of stopping time $\{T_n\}_{n\geq 1}$ that reduces the local martingales of the right-hand side of the previous identity and from the optimal stopping theorem, we obtain for any $n \geq 1$

$$\mathbb{E}_x\left[e^{-\lambda t\wedge T_a\wedge T_n} h_\lambda(Z_{t\wedge T_a\wedge T_n})\right] = h_\lambda(x).$$

Since h_λ is bounded by $h_\lambda(0) < \infty$, we use the dominated convergence theorem and take n goes to 0 and t goes to ∞. Since $T_a < \infty$ a.s. according to Theorem 1.2 and thus $Z_{T_a} = a$ a.s., we deduce (1.27) for $x \geq a > 0$. For $a = 0$, identityt (1.27) has already been obtained in (5.24) and (5.25).

Next, we handle the result on the expectation of T_0, i.e. identity (1.28), using similar arguments as in the proof of Theorem 3.9 in [11]. We denote $H(t,\lambda)$ for the Laplace transform $\mathbb{E}_x[e^{-\lambda Z_t}]$, and observe

$$\lim_{\lambda\to\infty} \int_0^\infty (1 - H(t,\lambda))\mathrm{d}t = \mathbb{E}_x\Big[T_0\Big].$$

On the other hand, from (5.16), for any $t \geq 0, \lambda > 0$,

$$\frac{\partial H}{\partial t}(t,\lambda) = -\psi(\lambda)\frac{\partial H}{\partial \lambda}(t,\lambda) + \omega(\lambda)\frac{\partial^2 H}{\partial \lambda^2}(t,\lambda) = \omega(\lambda)e^{m(\lambda)}\frac{\partial}{\partial\lambda}\left(\frac{\partial H}{\partial\lambda}(t,\lambda)e^{-m(\lambda)}\right),$$

which, by integrating $\frac{\partial H}{\partial \lambda} e^{-m(\lambda)}$ with respect to λ yields to

$$\frac{\partial H}{\partial \lambda}(t,\lambda) = -e^{m(\lambda)} \int_\lambda^\infty \frac{e^{-m(u)}}{\omega(u)} \frac{\partial H}{\partial t}(t,u) \mathrm{d}u$$

and then integrating again with respect to λ and t on $[0,\lambda] \times \mathbb{R}$, we obtain

$$\int_0^\infty (1 - H(t,\lambda)) \mathrm{d}t = \int_0^\lambda e^{m(u)} \int_u^\infty \frac{e^{-m(z)}}{\omega(z)} (1 - e^{-zx}) \mathrm{d}z \mathrm{d}u.$$

Letting λ go to ∞, we deduce (1.28).

Finally the process Z comes down from infinity since, under our assumptions, it satisfies the hypothesis of Theorem 1.2. The proof of Theorem 1.8 is now complete. □

Acknowledgements. Both authors acknowledge support from the Royal Society and CONACyT-MEXICO.

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