On Saint Venant-Kirchhoff imperfect interfaces

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Using matched asymptotic expansions with fractional exponents, we obtain original transmission conditions describing the limit behavior for soft, hard and rigid thin interphases obeying the Saint Venant-Kirchhoff material model. The novel transmission conditions, generalizing the classical linear imperfect interface model, are discussed and compared with existing models proposed in the literature for thin films undergoing finite strain. As an example of implementation of the proposed interface laws, the uniaxial tension and compression responses of butt joints with soft and hard interphases are given in closed form.

1. Introduction

Adhesive bonding technology is widely employed in engineering structural assembly and especially in aeronautics industry, where the use of composite materials is necessary to lighten structures. Due to the presence of the adhesive layer, adhesive bonding joints are subjected to a complex state of stress with high stress concentrations and, consequently, accurate analysis and modeling of adhesive materials and bonded joints are required.

Because the adhesive layer is usually soft and very thin when compared with the characteristic dimensions of the structure, a relatively large number of elements in the thickness direction is necessary to achieve sufficiently accurate calculations in standard existing finite element codes. This gives rise to a large number of degrees of freedom and high simulation costs. To successfully deal with this difficulty, interphase modeling has to precede the computation of the numerical solution. A classical modeling approach consists in describing the adhesive as a distinct lower-dimensional continuum, i.e. a material surface.

A simplified two-dimensional modeling can be achieved by introducing suitable assumptions concerning the displacement and the stress fields inside the adhesive [21,15,16] or by applying the asymptotic expansion method [23,24,20,29]. The asymptotic expansion method provides a systematic and rigorous approach to obtain interfacial laws describing the mechanical behavior of the limit material surface accounting for the elastic properties of an elastic thin adhesive. In the small deformation theory, interfacial laws appropriate for linear elastic adhesives have been obtained by many authors, for a not exhaustive list see the reference works of Klarbring [23,24], Caillerie [10], Geymonat [19,20], Licht [38], and also [1,6,25, 27–31,41,43–46].

The modeling of thin adhesives in finite elasticity has received much less attention than adhesives undergoing small displacements [4,16–18,26,37]. In [16], an elastic adhesive joint is considered, with an adhesive made of isotropic Saint Venant-Kirchhoff material and flexible as compared to the adherents. The large displacement-small deformation problem is addressed by introducing a displacement linearly varying through the thickness of the adhesive. A weak formulation is then obtained for a geometrically non linear two-dimensional description of the adhesive.

In [17], the asymptotic expansion method is applied to study the mechanical behavior of a thin nonlinear elastic adhesive made of a material much softer than those of the two adherents. The Saint Venant-Kirchhoff material model is assumed for both the adherents and the adhesive and a two dimensional simplified model for the adhesive is obtained. The convergence of a three-dimensional solution towards the limit solution is also given, together with error estimates.

In [18], a thin adhesive layer made of a nonlinear incompressible elastic material is considered. The three-dimensional equilibrium problem, posed in a mixed variational form, is analyzed by using the asymptotic expansion method. Several limit two-dimensional models are obtained for the adhesive, according to the values of a parameter representing its elastic properties. The existence and the uniqueness of the solution of the limit problems are established and Γ-convergence techniques are applied in order to prove the convergence of the asymptotic expansion.

In [37], an adhesive bonded joint made of nonlinear elastic materials with nonconvex energy density is studied by using Γ-convergence techniques. In the limit problem, the adhesive layer is

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replaced by a constraint condition in the form of a contact law depending on the relative behavior of the two small parameters, the thickness and the stiffness of the adhesive.

Within the framework of nonlinear elasticity, two stored energy functions for the adhesive, the material model of Saint Venant-Kirchhoff and the model of Ciarlet-Geymonat, are studied in [26]. Using the asymptotic expansions method, the limit energies associated to the two stored energy functions are computed and a rigorous mathematical analysis of the two limit models is presented.

A composite structure consisting of two nonlinearly elastic plates bonded by a thin and soft adhesive layer is studied in [4]. The materials of the plates are Saint Venant-Kirchhoff materials, while a more general nonlinear relation is used for the adhesive. A two-dimensional plate model for the compound structure is obtained, in which the adhesive is taken into account only through its material response to a pure shear load.

In the present paper, we consider a joint made of two adherents and a thin adhesive modeled as Saint Venant-Kirchhoff materials, the simplest hyperelastic material model extending the linear elastic material to the nonlinear regime [12]. The stored energy density of the Saint Venant-Kirchhoff model is

$$W(\mathbf{E}(\mathbf{u})) = \sum_{i,j=1}^{3} \mu(E_{ij}(\mathbf{u}))^2 + \frac{\lambda}{2} \sum_{i=1}^{3} (E_{ii}(\mathbf{u}))^2 \tag{1.1}$$

where λ, μ are positive elastic constant called the Lamé's constants, and

$$\mathbf{E}(\mathbf{u}) = 1/2(\nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T\nabla\mathbf{u}) \tag{1.2}$$

are the components of the Green-Lagrange strain tensor for a displacement field $\mathbf{u}$.

Three different adhesive types are studied: in the first model the adhesive is "soft", i.e. the elastic coefficients of the adhesive, λ and μ, rescale as its thickness ε; in the second model the adhesive is "hard", i.e. λ and μ are independent of ε; in the third model the adhesive is "rigid", i.e. λ and μ rescale as the inverse of ε.

To obtain transmission conditions mechanically equivalent to the behavior of the three types of adhesives, an asymptotic method is proposed, using classical expansions in the hard and rigid case and fractional power series in the soft case. This proposal is mainly motivated by the analysis of Licht and Michaille [37], which identifies ε^{p-1}, with p the exponent entering the growth conditions on the adhesive stored energy, as a critical size of the adhesive stiffness. In particular, above this critical size the stiffness is large enough to provide a limit model of perfect interface, below this critical size the stiffness is too small to maintain perfect adherence and at the critical size an imperfect (soft) interface model that allow displacement discontinuities in the adhesive is obtained. The choice of expansions with fractional powers in the soft case is also motivated by a simple one-dimensional example presented in Section 2. The example shows that for a soft adhesive the jump of the displacement rescales like $\varepsilon^{2/3}$, for the Saint Venant-Kirchhoff energy (1.1) being $p-1=3$. The example also gives insights into the types of transmission conditions arising from different rescaling of the adhesive elastic stiffness and it highlights the role of the load rescaling and the possibly occurrence of multiple solutions due to the failure of quasiconvexity of the energy (1.1).

The three-dimensional equilibrium problem of the adhesive joint is considered in Section 3, where the strong and weak formulations of the mixed boundary value problem are introduced.

Section 4 is devoted to the asymptotic analysis, which is based on fractional matched asymptotic expansions. In Section 5, the transmission conditions obtained via the asymptotic expansion method are summarized, rewritten as interface laws and discussed in light of the existing results for elastic adhesives undergoing small and finite strains [1,5,20,21,23,28,29,42].

In Section 6, the interface laws calculated for the soft interface are compared with the results obtained via Γ-convergence techniques by Licht and Michaille in [37]. We show that the interface laws calculated in the present paper at the order zero for the cases of soft and hard interphase are in agreement with the results of Licht and Michaille, provided the interphase elastic stiffness appropriately rescales with its thickness. In particular, if the stiffness rescales like ε^3, then the minimization of the Γ-limit gives the variational forms of the zero and higher order interface laws that we calculate for the soft interphase. The higher order laws of imperfect interface that we calculate for the two cases of hard and rigid interphases do not find counterparts in [37].

Section 7 proposes the analysis of uniaxial tension and compression of a butt joint as an example of implementation of our contact laws. The example generalizes some analogous results given in [42] for small strains. The macroscopic response of the joint is calculated and plotted for the two cases of soft and hard interface taking into account different ratios of the adhesive/adherent thickness and stiffness. The example could serve also as elasticity solution benchmark, the uniaxial tension and compression responses being given in closed form.

In the paper, the usual summation convention is used. Latin indices take the values 1, 2, 3 and Greek indexes the values 1, 2.

2. A one-dimensional example

Consider a thin adhesive layer occupying the reference configuration given by interval $(0,\varepsilon)$ in contact with an adherent occupying the reference configuration (ε, l), as depicted in Fig. 1. The materials of the adhesive and the adherent are both nonlinear and modeled as Saint Venant–Kirchhoff materials. The bar is fixed at one end, say $x=0$, and subjected to a force Q at the other end, $x=l$. The thickness $\varepsilon$is small, e.g. $\varepsilon/l \ll 1$. The equilibrium problem of the composite bar, stated on the reference configuration, takes the form

$$(s+su')'=0 \quad \text{in } (0,\varepsilon)\cup(\varepsilon,l), \tag{2.1}$$

$$s = E_\varepsilon\left(u' + \tfrac{1}{2}(u')^2\right) \quad \text{in } (0,\varepsilon), \tag{2.2}$$

$$s = E\left(u' + \tfrac{1}{2}(u')^2\right) \quad \text{in } (\varepsilon,l), \tag{2.3}$$

$$u=0 \quad x=0, \tag{2.4}$$

$$[s+su']=0 \quad x=\varepsilon, \tag{2.5}$$

$$[u]=0 \quad x=\varepsilon, \tag{2.6}$$

$$s+su'=Q \quad x=l, \tag{2.7}$$

where a prime denotes the first derivative, the symbol $[f]{:=}f(\varepsilon^+)-f(\varepsilon^-)$ denotes the jump of $f\colon(0,l)\to\mathbb{R}$ at the point $x=\varepsilon$ and E_ε, E denote the elastic moduli of the adhesive and of the adherent, respectively. Integrating (2.1) under the conditions (2.5) and (2.7) gives

$$s+su'=Q \quad \text{in } [0,l]. \tag{2.8}$$

Substituting (2.2) and (2.3) into (2.8) yields the differential equations

$$(u')^3+3(u')^2+2u' = 2\frac{Q}{E_\varepsilon} \quad \text{in } (0,\varepsilon), \tag{2.9}$$

$$(u')^3+3(u')^2+2u' = 2\frac{Q}{E} \quad \text{in } (\varepsilon,l). \tag{2.10}$$

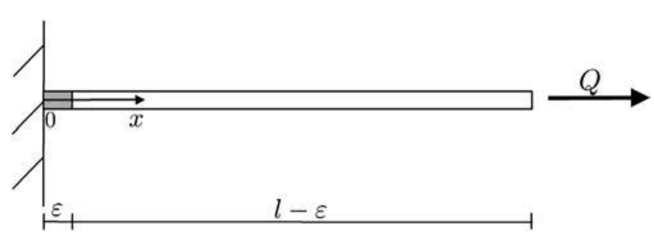

Fig. 1. Reference configuration of the one-dimensional composite bar made of a Saint Venant-Kirchhoff material.

These equations imply that u' is piecewise constant in the bar and its values are solutions of the above cubic equations which admit multiple solutions, depending on the values of Q/E_ε, Q/E. The multiplicity of solutions is due to the non monotonicity of the (cubic) stress-strain response function in the Saint Venant-Kirchhoff material model. When $Q>0$, it is immediately seen that Eqs. (2.9) and (2.10) have a unique solution $u'=\mathfrak{F}(z)>0$ with $z=Q/E_\varepsilon$ and $z=Q/E$, respectively. Integrating $u'=\mathfrak{F}(z)$ under the boundary conditions (2.3)–(2.7) gives the solution

$$u(x)=\begin{cases}\mathfrak{F}(Q/E_\varepsilon)x, & x\in[0,\varepsilon),\\ \mathfrak{F}(Q/E)(x-\varepsilon)+\mathfrak{F}(Q/E_\varepsilon)\varepsilon, & x\in[\varepsilon,l].\end{cases} \tag{2.11}$$

The solution (2.11) allows to evaluate the difference $u(\varepsilon)-u(0){=:}[\![u]\!]$ giving the jump of the displacement due to the presence of the deformable adhesive between the point $x=\varepsilon$ of the adherent and the constraint at $x=0$.

For a soft adhesive, e.g. $E_\varepsilon=\varepsilon\widehat{E}$, the solution (2.11) yields

$$[\![u]\!]_{\text{soft}}=\mathfrak{F}\left(\frac{Q}{\varepsilon\widehat{E}}\right)\varepsilon. \tag{2.12}$$

For a hard adhesive, e.g. $E_\varepsilon=\widehat{E}$, the solution (2.11) yields

$$[\![u]\!]_{\text{hard}}=\mathfrak{F}\left(\frac{Q}{\widehat{E}}\right)\varepsilon. \tag{2.13}$$

Finally, for a rigid adhesive, e.g. $E_\varepsilon=\widehat{E}\varepsilon^{-1}$, one has

$$[\![u]\!]_{\text{rigid}}=\mathfrak{F}\left(\frac{Q\varepsilon}{\widehat{E}}\right)\varepsilon. \tag{2.14}$$

Eq. (2.12) deserves some discussion. In view of the smallness of ε, one could use the asymptotic behavior $\mathfrak{F}(y)\sim(2y)^{1/3}$ as $y\to+\infty$ to conclude that

$$[\![u]\!]_{\text{soft}}=\mathfrak{F}\left(\frac{Q}{\varepsilon\widehat{E}}\right)\varepsilon\sim\left(2\frac{Q}{\varepsilon\widehat{E}}\right)^{1/3}\varepsilon=\left(2\frac{Q}{\widehat{E}}\right)^{1/3}\varepsilon^{2/3}\quad\text{as }\frac{Q}{\varepsilon\widehat{E}}\to+\infty. \tag{2.15}$$

Thus, this transmission condition can be interpreted as arising from a force scaling $Q\sim\varepsilon^q$ with $q<1$, which incorporates the special case $q=0$ of a force Q independent of ε. Notably, Eq. (2.15) cannot be used to recover the classical soft interface law of linear elasticity. The latter can however be reobtained from (2.12) by linearizing $\mathfrak{F}$ about the origin, $\mathfrak{F}(y)\sim y$ as $y\to0$, thus

$$[\![u]\!]_{\text{soft}}=\mathfrak{F}\left(\frac{Q}{\varepsilon\widehat{E}}\right)\varepsilon\sim\frac{Q}{\varepsilon\widehat{E}}\varepsilon=\frac{Q}{\widehat{E}}\quad\text{as }\frac{Q}{\varepsilon\widehat{E}}\to0. \tag{2.16}$$

Clearly, this last result relies on a force scaling $Q\sim\varepsilon^q$ with $q>1$.

In the remaining part of the paper, we will assume that the loads are independent of ε. This choice is simple and it brings to a non linear behavior of the adherents in the limit as $\varepsilon\to0$.

3. The three-dimensional problem

Let Ω be a composite body comprising two adherents, $\Omega^\varepsilon_\pm$, joined by an interphase, B^ε, as represented in Fig. 2. The interphase occupies a cylindrical region of height $\varepsilon$and cross-section $S\subset\mathbb{R}^2$, with ∂S a smooth boundary. An orthonormal Cartesian basis $(O,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is introduced, and let (x_1,x_2,x_3) be taken to denote the three coordinates of a particle in Ω. The origin lies at the center of the interphase midplane and the x_3-axis runs perpendicular to the interphase midplane, thus the domains $\Omega^\varepsilon_\pm$ and B^ε are defined by

$$\Omega^\varepsilon_\pm=\left\{(x_1,x_2,x_3)\in\Omega:\ \pm x_3>\frac{\varepsilon}{2}\right\}, \tag{3.1}$$

$$B^\varepsilon=\left\{(x_1,x_2,x_3)\in\Omega:\ |x_3|<\frac{\varepsilon}{2}\right\}. \tag{3.2}$$

Let $S^\varepsilon_\pm$ denote the interfaces between the adherents and the interphase:

$$S^\varepsilon_\pm=\left\{(x_1,x_2,x_3)\in\Omega:\ x_3=\pm\frac{\varepsilon}{2}\right\}. \tag{3.3}$$

On a part Γ_1 of $\partial\Omega$, an external load g is applied, and on a part Γ_0 of $\partial\Omega$ such that $\Gamma_0\cap\Gamma_1=\varnothing$ the displacement is imposed to vanish. It is also assumed that $\Gamma_0\cap B^\varepsilon=\varnothing$ and $\Gamma_1\cap B^\varepsilon=\varnothing$. A body force f is applied in $\Omega^\varepsilon_\pm$.

The equations governing the equilibrium problem of the composite structure are written as follows:

$$\begin{cases}(s^\varepsilon_{ij}+s^\varepsilon_{kj}u^\varepsilon_{i,k})_{,j}+f_i=0 & \text{in }\Omega^\varepsilon_\pm,\\ (s^\varepsilon_{ij}+s^\varepsilon_{kj}u^\varepsilon_{i,k})n_j=g_i & \text{on }\Gamma_1,\\ (s^\varepsilon_{ij}+s^\varepsilon_{kj}u^\varepsilon_{i,k})_{,j}=0 & \text{in }B^\varepsilon,\\ [\![s^\varepsilon_{i3}+s^\varepsilon_{k3}u^\varepsilon_{i,k}]\!]=0 & \text{on }S^\varepsilon_\pm,\\ [\![u^\varepsilon_i]\!]=0 & \text{on }S^\varepsilon_\pm,\\ u^\varepsilon_i=0 & \text{on }\Gamma_0,\\ s^\varepsilon_{ij}=A^\pm_{ijhk}E_{hk}(u^\varepsilon) & \text{in }\Omega^\varepsilon_\pm,\\ s^\varepsilon_{ij}=A^\varepsilon_{ijhk}E_{hk}(u^\varepsilon) & \text{in }B^\varepsilon,\end{cases} \tag{3.4}$$

where $\mathbf{s}^\varepsilon$ is the second Piola-Kirchhoff stress tensor, $\mathbf{E}(u^\varepsilon)$ is the Green-Lagrange strain tensor defined in (1.2), $\mathbf{A}^\pm$, $\mathbf{A}^\varepsilon$ are the elasticity tensors of the deformable adherents and the interphase, respectively. In the sequel, they will be considered isotropic, with Lamé's coefficients equal to $\lambda^\pm$, $\mu^\pm$ in the adherents and λ^ε, μ^ε in the interphase.

The existence and uniqueness of the problem (3.4) is not generally guaranteed. Indeed, the Saint-Venant Kirchhoff energy is not even rank-one convex [32], and thus the direct method of the calculus of variations does not apply. By using the implicit function theorem, an existence result can be established for the case when the pure displacement boundary value condition is considered and the body force is sufficiently small [14,39,36].

4. Asymptotic analysis

Because the thickness of the interphase is very small, it is natural to seek the solution of problem (3.4) by using asymptotic expansions with respect to the small parameter ε. The domain is rescaled using a classical procedure [12]. First, the following sets are introduced:

- $\Omega_\pm=\{(x_1,x_2,x_3)\in\Omega:\pm x_3>\frac{1}{2}\}$ (the rescaled adherents);
- $B=\{(x_1,x_2,x_3)\in\Omega:\ |x_3|<\frac{1}{2}\}$ (the rescaled interphase);
- $S_\pm=\{(x_1,x_2,x_3)\in\Omega:\ x_3=\pm\frac{1}{2}\}$.

Next, the interphase is rescaled into a domain of unit thickness (cf. Fig. 2). In particular,

- in the interphase, the following change of variable is introduced

$$(x_1,x_2,x_3)\in B^\varepsilon\to(z_1,z_2,z_3)\in B,\ \text{with }(z_1,z_2,z_3)=\left(x_1,x_2,\frac{x_3}{\varepsilon}\right)$$

and it is set

$$\hat{\mathbf{u}}^\varepsilon(z_1,z_2,z_3)=\mathbf{u}^\varepsilon(x_1,x_2,x_3), \tag{4.1}$$

$$\hat{\mathbf{s}}^\varepsilon(z_1,z_2,z_3)=\mathbf{s}^\varepsilon(x_1,x_2,x_3). \tag{4.2}$$

- In the adherents, the following change of variable is introduced

$$(x_1,x_2,x_3)\in\Omega^\varepsilon_\pm\to(z_1,z_2,z_3)\in\Omega_\pm, \tag{4.3}$$

with $(z_1,z_2,z_3)=(x_1,x_2,x_3\pm1/2\mp\varepsilon/2)$, and it is set

$$\bar{\mathbf{u}}^\varepsilon(z_1,z_2,z_3)=\mathbf{u}^\varepsilon(x_1,x_2,x_3), \tag{4.4}$$

$$\bar{\mathbf{s}}^\varepsilon(z_1,z_2,z_3)=\mathbf{s}^\varepsilon(x_1,x_2,x_3). \tag{4.5}$$

External forces are assumed to be independent of ε. As a consequence, it is set $\bar{f}(z_1,z_2,z_3)=f(x_1,x_2,x_3)$ and $\bar{g}(z_1,z_2,z_3)=g(x_1,x_2,x_3)$. Under the

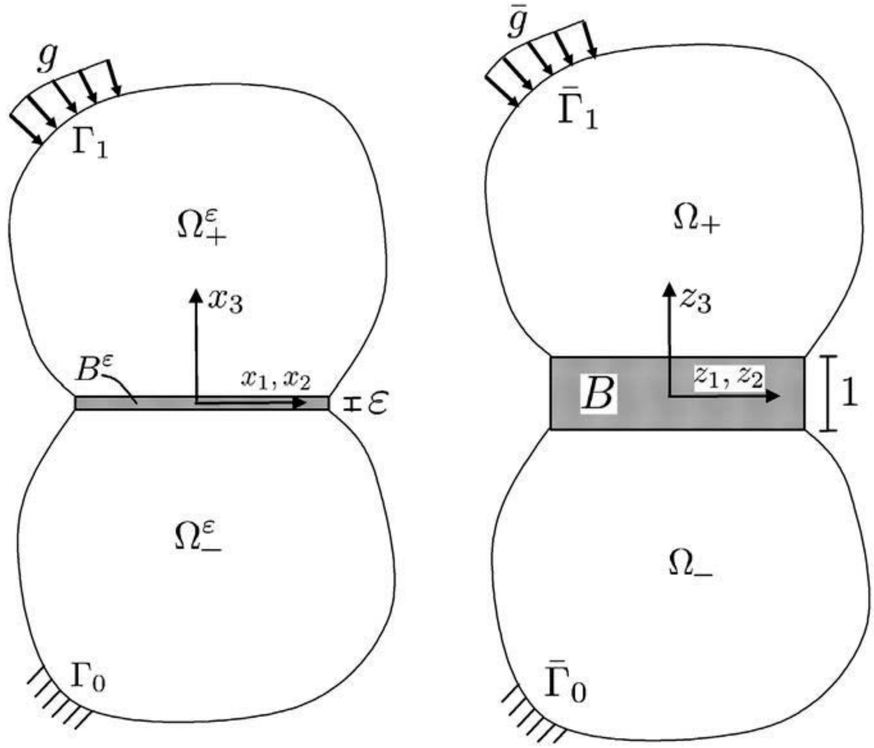

Fig. 2. Reference configuration of the joint viewed as a composite body made of two adherents in contact via a thin adhesive layer (left) and rescaled configuration of the joint (right).

change of variables, the domains Γ_0 and Γ_1 are transformed into the domains denoted by $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$, respectively.

The governing equations of the rescaled problem are as follows:

$$\begin{cases} (\bar{s}_{ij} + \bar{s}_{kj}\bar{u}_{i,k})_{,j} + \bar{f}_i = 0 & \text{in } \Omega_\pm, \\ (\bar{s}_{ij} + \bar{s}_{kj}\bar{u}_{i,k})n_j = \bar{g}_i & \text{on } \overline{\Gamma}_1, \\ (\hat{s}_{i\alpha} + \hat{u}_{i,\beta}\hat{s}_{\beta\alpha})_{,\alpha} + \frac{1}{\varepsilon}\{(\hat{u}_{i,3}\hat{s}_{3\alpha})_{,\alpha} + (\hat{s}_{i3} + \hat{u}_{i,\beta}\hat{s}_{\beta 3})_{,3}\} & \\ \quad + \frac{1}{\varepsilon^2}(\hat{u}_{i,3}\hat{s}_{33})_{,3} = 0 & \text{in } B, \\ \bar{s}_{i3} + \bar{s}_{k3}\bar{u}_{i,k} = \hat{s}_{i3} + \hat{s}_{\alpha 3}\hat{u}_{i,\alpha} + \frac{1}{\varepsilon}\hat{s}_{33}\hat{u}_{i,3} & \text{on } S_\pm,^{(\hat{\mathbf{u}})} \\ \bar{u}_i = \hat{u}_i & \text{on } S_\pm, \\ \bar{u}_i = 0 & \text{on } \overline{\Gamma}_0, \\ \bar{s}_{ij} = A^\pm_{ijhk}\overline{E}_{hk}(\bar{\bar{\mathbf{u}}}) & \text{in } \Omega_\pm, \\ \hat{s}_{ij} = A^\varepsilon_{ijhk}\widehat{E}_{hk}(\hat{\mathbf{u}}) & \text{in } B, \end{cases} \tag{4.6}$$

where $\overline{\mathbf{E}}, \widehat{\mathbf{E}}$ are the Green-Lagrange strain tensors in the rescaled adherents and interphase domains, respectively, having components:

$$\overline{E}_{ij}(\bar{\mathbf{u}}) = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i} + \bar{u}_{k,j}\bar{u}_{k,j}), \quad i, j = 1, 2, 3, \tag{4.7}$$

$$\widehat{E}_{\alpha\beta}(\hat{\mathbf{u}}) = \frac{1}{2}(\hat{u}_{\alpha,\beta} + \hat{u}_{\beta,\alpha} + \hat{u}_{k,\alpha}\bar{u}_{k,\beta}), \quad \alpha, \beta = 1, 2, \tag{4.8}$$

$$\widehat{E}_{\alpha 3}(\hat{\mathbf{u}}) = \frac{1}{2}\left(\frac{1}{\varepsilon}\hat{u}_{\alpha,3} + \hat{u}_{3,\alpha} + \frac{1}{\varepsilon}\hat{u}_{k,\alpha}\hat{u}_{k,3}\right), \quad \alpha = 1, 2, \tag{4.9}$$

$$\widehat{E}_{33}(\hat{\mathbf{u}}) = \frac{1}{2\varepsilon^2}(\hat{u}_{3,3} + \hat{u}_{k,3}\hat{u}_{k,3}). \tag{4.10}$$

If the interphase is isotropic, the constitutive equation of the rescaled interphase takes the form

$$\hat{\mathbf{s}}(\widehat{\mathbf{E}}(\hat{\mathbf{u}})) = 2\mu^\varepsilon\widehat{\mathbf{E}}(\hat{\mathbf{u}}) + \lambda^\varepsilon(\mathbf{I}\cdot\widehat{\mathbf{E}}(\hat{\mathbf{u}}))\mathbf{I}, \tag{4.11}$$

with $\mathbf{I}$ the identity tensor. In the following, to simplify the notation, we drop the dependence of $\hat{\mathbf{s}}$ on $\widehat{\mathbf{E}}(\hat{\mathbf{u}})$ and later on of $\bar{\mathbf{s}}$ on $\overline{\mathbf{E}}(\bar{\mathbf{u}})$). Substituting (4.8)–(4.10) into (4.11), one obtains

$$\hat{s}_{\alpha\beta} = \mu^\varepsilon(\hat{u}_{\alpha,\beta} + \hat{u}_{\beta,\alpha} + \hat{\mathbf{u}}_{,\alpha}\cdot\hat{\mathbf{u}}_{,\beta}) + \lambda^\varepsilon\left(\hat{u}_{\gamma,\gamma} + \frac{1}{2}\left|\hat{u}_{,\gamma}\right|^2 + \frac{1}{\varepsilon}\hat{u}_{3,3} + \frac{1}{2\varepsilon^2}\left|\hat{\mathbf{u}}_{,3}\right|^2\right) \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \tag{4.12}$$

$$\hat{s}_{\alpha 3} = \mu^\varepsilon\left(\hat{u}_{3,\alpha} + \frac{1}{\varepsilon}\hat{u}_{\alpha,3} + \frac{1}{\varepsilon}\hat{\mathbf{u}}_{,\alpha}\cdot\hat{\mathbf{u}}_{,3}\right), \quad \alpha = 1, 2, \tag{4.13}$$

$$\hat{s}_{33} = (2\mu^\varepsilon + \lambda^\varepsilon)\left(\frac{1}{\varepsilon}\hat{u}_{3,3} + \frac{1}{2\varepsilon^2}\left|\hat{\mathbf{u}}_{,3}\right|^2\right) + \lambda^\varepsilon\left(\hat{u}_{\gamma,\gamma} + \frac{1}{2}\left|\hat{u}_{,\gamma}\right|^2\right). \tag{4.14}$$

Motivated by (2.12), the following asymptotic series with fractional powers for the second Piola-Kirchhoff stress and the displacement in the original unrescaled domain are assumed:

$$s^\varepsilon_{ij} = s^0_{ij} + \varepsilon^{1/3}s^{1/3}_{ij} + \varepsilon^{2/3}s^{2/3}_{ij} + \varepsilon s^1_{ij} + \varepsilon^{4/3}s^{4/3}_{ij} + \varepsilon^{5/3}s^{5/3}_{ij} + \varepsilon^2\varepsilon s^2_{ij} + o(\varepsilon^2), \tag{4.15}$$

$$u^\varepsilon_i = u^0_i + \varepsilon^{1/3}u^{1/3}_i + \varepsilon^{2/3}u^{2/3}_i + \varepsilon u^1_i + \varepsilon^{4/3}u^{4/3}_i + \varepsilon^{5/3}u^{5/3}_i + \varepsilon^2 u^2_i + o(\varepsilon^2), \tag{4.16}$$

for $i, j = 1, 2, 3$. According to (4.15) and (4.16), the following asymptotic series in the rescaled interphase and adherents domains are obtained:

$$\hat{s}^\varepsilon_{ij} = \hat{s}^0_{ij} + \varepsilon^{1/3}\hat{s}^{1/3}_{ij} + \varepsilon^{2/3}\hat{s}^{2/3}_{ij} + \varepsilon\hat{s}^1_{ij} + \varepsilon^{4/3}\hat{s}^{4/3}_{ij} + \varepsilon^{5/3}\hat{s}^{5/3}_{ij} + \varepsilon^2\hat{s}^2_{ij} + o(\varepsilon^2), \tag{4.17}$$

$$\hat{u}^\varepsilon_i = \hat{u}^0_i + \varepsilon^{1/3}\hat{u}^{1/3}_i + \varepsilon^{2/3}\hat{u}^{2/3}_i + \varepsilon\hat{u}^1_i + \varepsilon^{4/3}\hat{u}^{4/3}_i + \varepsilon^{5/3}\hat{u}^{5/3}_i + \varepsilon^2\hat{u}^2_i + o(\varepsilon^2), \tag{4.18}$$

$$\bar{s}^\varepsilon_{ij} = \bar{s}^0_{ij} + \varepsilon^{1/3}\bar{s}^{1/3}_{ij} + \varepsilon^{2/3}\bar{s}^{2/3}_{ij} + \varepsilon\bar{s}^1_{ij} + \varepsilon^{4/3}\bar{s}^{4/3}_{ij} + \varepsilon^{5/3}\bar{s}^{5/3}_{ij} + \varepsilon^2\bar{s}^2_{ij} + o(\varepsilon^2), \tag{4.19}$$

$$\bar{u}^\varepsilon_i = \bar{u}^0_i + \varepsilon^{1/3}\bar{u}^{1/3}_i + \varepsilon^{2/3}\bar{u}^{2/3}_i + \varepsilon\bar{u}^1_i + \varepsilon^{4/3}\bar{u}^{4/3}_i + \varepsilon^{5/3}\bar{u}^{5/3}_i + \varepsilon^2\bar{u}^2_i + o(\varepsilon^2), \tag{4.20}$$

for $i, j = 1, 2, 3$.

4.1. *Expansions of the equilibrium equations in the interphase*

Substituting the expansions (4.17) and (4.18) into the equilibrium equations of the rescaled interphase (third equation in (4.6)), we obtain that the following conditions hold in B:

- Order -2:

$$(\hat{u}^0_{i,3}\hat{s}^0_{33})_{,3} = 0, \tag{4.21}$$

- Order $-5/3$:

$$(\hat{u}^0_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^0_{33})_{,3} = 0, \tag{4.22}$$

- Order $-4/3$:

$$(\hat{u}^0_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^0_{33})_{,3} = 0, \tag{4.23}$$

- Order -1:

$$(\hat{u}^0_{i,3}\hat{s}^0_{3\alpha})_{,\alpha} + (\hat{s}^0_{i3} + \hat{u}^0_{i,\beta}\hat{s}^0_{\beta 3})_{,3} \quad + (\hat{u}^0_{i,3}\hat{s}^1_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^1_{i,3}\hat{s}^0_{33})_{,3} = 0, \tag{4.24}$$

- Order $-2/3$:

$$\begin{aligned}&(\hat{u}^0_{i,3}\hat{s}^{1/3}_{3\alpha} + \hat{u}^{1/3}_{i,3}\hat{s}^0_{3\alpha})_{,\alpha} + (\hat{s}^{1/3}_{i3} + \hat{u}^0_{i,\beta}\hat{s}^{1/3}_{\beta 3} + \hat{u}^{1/3}_{i,\beta}\hat{s}^0_{\beta 3})_{,3}\\ &\quad + (\hat{u}^0_{i,3}\hat{s}^{4/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^1_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^1_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^{4/3}_{i,3}\hat{s}^0_{33})_{,3} = 0,\end{aligned} \tag{4.25}$$

- Order $-1/3$:

$$\begin{aligned}&(\hat{u}^0_{i,3}\hat{s}^{2/3}_{3\alpha} + \hat{u}^{1/3}_{i,3}\hat{s}^{1/3}_{3\alpha} + \hat{u}^{2/3}_{i,3}\hat{s}^0_{3\alpha})_{,\alpha} \quad + (\hat{s}^{2/3}_{i3} + \hat{u}^0_{i,\beta}\hat{s}^{2/3}_{\beta 3} + \hat{u}^{1/3}_{i,\beta}\hat{s}^{1/3}_{\beta 3} + \hat{u}^{2/3}_{i,\beta}\hat{s}^0_{\beta 3})_{,3}\\ &\quad + (\hat{u}^0_{i,3}\hat{s}^{5/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{4/3}_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^1_{33} + \hat{u}^1_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{4/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^{5/3}_{i,3}\hat{s}^0_{33})_{,3} = 0,\end{aligned} \tag{4.26}$$

- Order 0:

$$\begin{aligned}&(\hat{s}^0_{i\alpha} + \hat{u}^0_{i,\beta}\hat{s}^0_{\beta\alpha})_{,\alpha} + (\hat{u}^0_{i,3}\hat{s}^1_{3\alpha} + \hat{u}^{1/3}_{i,3}\hat{s}^{2/3}_{3\alpha} + \hat{u}^{2/3}_{i,3}\hat{s}^{1/3}_{3\alpha} + \hat{u}^1_{i,3}\hat{s}^0_{3\alpha})_{,\alpha}\\ &\quad + (\hat{s}^1_{i3} + \hat{u}^0_{i,\beta}\hat{s}^1_{\beta 3} + \hat{u}^{1/3}_{i,\beta}\hat{s}^{2/3}_{\beta 3} + \hat{u}^{2/3}_{i,\beta}\hat{s}^{1/3}_{\beta 3} + \hat{u}^1_{i,\beta}\hat{s}^0_{\beta 3})_{,3}\\ &\quad + (\hat{u}^0_{i,3}\hat{s}^2_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{5/3}_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^{4/3}_{33} + \hat{u}^1_{i,3}\hat{s}^1_{33} \quad + \hat{u}^{4/3}_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{5/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^2_{i,3}\hat{s}^0_{33})_{,3} = 0.\end{aligned} \tag{4.27}$$

4.2. *Expansions of the equilibrium equations in the adherents*

Substituting the expansions (4.19) and (4.20) into the equilibrium equations of the rescaled interphase (first equation in (4.6)), we obtain the following conditions:

$$\overline{P}^0_{ij,j} + \overline{f}_i = 0 \quad \text{in } \Omega_\pm, \tag{4.28}$$

$$\overline{P}^{1/3}_{ij,j} = 0 \quad \text{in } \Omega_\pm, \tag{4.29}$$

$$\overline{P}^{2/3}_{ij,j} = 0 \quad \text{in } \Omega_\pm, \tag{4.30}$$

$$\overline{P}^1_{ij,j} = 0 \quad \text{in } \Omega_\pm, \tag{4.31}$$

with $\overline{P}^l_{ij}$, $l = 0,\ 1/3,\ 2/3,\ 1$, the components of the first Piola-Kirchhoff stress tensor in the adherents:

$$\overline{P}^0_{ij} = \overline{s}^0_{ij} + \overline{u}^0_{i,k}\overline{s}^0_{kj}, \tag{4.32}$$

$$\overline{P}^{1/3}_{ij} = \overline{s}^{1/3}_{ij} + \overline{u}^0_{i,k}\overline{s}^{1/3}_{kj} + \overline{u}^{1/3}_{i,k}\overline{s}^0_{kj}, \tag{4.33}$$

$$\overline{P}^{2/3}_{ij} = \overline{s}^{2/3}_{ij} + \overline{u}^0_{i,k}\overline{s}^{2/3}_{kj} + \overline{u}^{1/3}_{i,k}\overline{s}^{1/3}_{kj} + \overline{u}^{2/3}_{i,k}\overline{s}^0_{kj}, \tag{4.34}$$

$$\overline{P}^1_{ij} = \overline{s}^1_{ij} + \overline{u}^0_{i,k}\overline{s}^1_{kj} + \overline{u}^{1/3}_{i,k}\overline{s}^{2/3}_{kj} + \overline{u}^{2/3}_{i,k}\overline{s}^{1/3}_{kj} + \overline{u}^1_{i,k}\overline{s}^0_{kj}. \tag{4.35}$$

Substituting the expansions (4.19) and (4.20) into the boundary conditions on $\overline{\Gamma}_1$ (second equation in (4.6)), we obtain the conditions

$$\overline{P}^0_{ij}n_j = \overline{g}_i \quad \text{on } \overline{\Gamma}_1, \tag{4.36}$$

$$\overline{P}^{1/3}_{ij}n_j = 0 \quad \text{on } \overline{\Gamma}_1, \tag{4.37}$$

$$\overline{P}^{2/3}_{ij}n_j = 0 \quad \text{on } \overline{\Gamma}_1, \tag{4.38}$$

$$\overline{P}^1_{ij}n_j = 0 \quad \text{on } \overline{\Gamma}_1. \tag{4.39}$$

4.3. *Expansions of the continuity condition of the traction at $S_\pm$*

Now we substitute the expansions (4.17) and (4.19) into the continuity condition of the traction at $S_\pm$ (fourth equation in (4.6)) and we obtain the following conditions which hold on $S_\pm$:

- Order -1:

$$0 = \hat{u}^0_{i,3}\hat{s}^0_{33}, \tag{4.40}$$

- Order $-2/3$:

$$0 = \hat{u}^0_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^0_{33}, \tag{4.41}$$

- Order $-1/3$:

$$0 = \hat{u}^0_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^0_{33}, \tag{4.42}$$

- Order 0:

$$\overline{P}^0_{i3} = \hat{s}^0_{i3} + \hat{u}^0_{i,\beta}\hat{s}^0_{\beta 3} + \hat{u}^0_{i,3}\hat{s}^1_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^1_{i,3}\hat{s}^0_{33}, \tag{4.43}$$

- Order 1/3:

$$\begin{aligned}\overline{P}^{1/3}_{i3} &= \hat{s}^{1/3}_{i3} + \hat{u}^0_{i,\beta}\hat{s}^{1/3}_{\beta 3} + \hat{u}^{1/3}_{i,\beta}\hat{s}^0_{\beta 3} + \hat{u}^0_{i,3}\hat{s}^{4/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^1_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^1_{i,3}\hat{s}^{1/3}_{33}\\ &\quad + \hat{u}^{4/3}_{i,3}\hat{s}^0_{33},\end{aligned} \tag{4.44}$$

- Order 2/3:

$$\begin{aligned}\overline{P}^{2/3}_{i3} &= \hat{s}^{2/3}_{i3} + \hat{u}^0_{i,\beta}\hat{s}^{2/3}_{\beta 3} + \hat{u}^{1/3}_{i,\beta}\hat{s}^{1/3}_{\beta 3} + \hat{u}^0_{i,\beta}\hat{s}^{2/3}_{\beta 3} + \hat{u}^0_{i,3}\hat{s}^{5/3}_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{4/3}_{33} + \hat{u}^{2/3}_{i,3}\hat{s}^1_{33}\\ &\quad + \hat{u}^1_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{4/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^{5/3}_{i,3}\hat{s}^0_{33},\end{aligned} \tag{4.45}$$

- Order 1:

$$\begin{aligned}\overline{P}^1_{i3} &= \hat{s}^1_{i3} + \hat{u}^0_{i,\beta}\hat{s}^1_{\beta 3} + \hat{u}^{2/3}_{i,\beta}\hat{s}^{2/3}_{\beta 3} + \hat{u}^{2/3}_{i,\beta}\hat{s}^{1/3}_{\beta 3} + \hat{u}^1_{i,\beta}\hat{s}^0_{\beta 3} + \hat{u}^0_{i,3}\hat{s}^2_{33} + \hat{u}^{1/3}_{i,3}\hat{s}^{5/3}_{33}\\ &\quad + \hat{u}^{2/3}_{i,3}\hat{s}^{4/3}_{33} + \hat{u}^1_{i,3}\hat{s}^1_{33} + \hat{u}^{4/3}_{i,3}\hat{s}^{2/3}_{33} + \hat{u}^{5/3}_{i,3}\hat{s}^{1/3}_{33} + \hat{u}^2_{i,3}\hat{s}^0_{33}.\end{aligned} \tag{4.46}$$

4.4. *Expansions of the constitutive equations of the interphase*

The equations written so far are general in the sense that they are independent of the constitutive behavior of the material. Three specific cases of elastic material are now considered for the interphase: a "soft" material, characterized by elastic moduli linearly rescaling with the thickness ε, a "hard" material, characterized by elastic moduli independent of the thickness ε, and a "rigid" material, characterized by elastic moduli linearly rescaling with ε^{-1}.

For the soft case, the full expansions (4.17)–(4.20) including the terms with fractional powers have been considered.

For the hard and the rigid cases, classical the terms with fractional powers have been eliminated when using (4.17)–(4.20), because they give rise to many useless equations. In other words, classical expansions have been used for the hard and the rigid cases.

4.4.1. *Interphase made of a "soft" material*

The Lamé's coefficients are assumed as follows:

$$\lambda^\varepsilon = \hat{\lambda}\,\varepsilon, \quad \mu^\varepsilon = \hat{\mu}\ \varepsilon. \tag{4.47}$$

Substituting the expansion (4.17) into the constitutive Eqs. (4.12)–(4.14) and using (4.47), one obtains the following conditions:

- Order -1:

$$0 = \hat{u}^0_{k,3}\hat{u}^0_{k,3}, \tag{4.48}$$

- Order $-2/3$:

$$0 = \hat{u}^0_{k,3}\hat{u}^{1/3}_{k,3}, \tag{4.49}$$

- Order $-1/3$:

$$0 = 2\hat{u}^0_{k,3}\hat{u}^{2/3}_{k,3} + \hat{u}^{1/3}_{k,3}\hat{u}^{1/3}_{k,3}, \tag{4.50}$$

- Order 0:

$$\hat{s}^0_{\alpha\beta} = \hat{\lambda}(\hat{u}^0_{3,3} + \hat{u}^0_{k,3}\hat{u}^1_{k,3} + \hat{u}^{1/3}_{k,3}\hat{u}^{2/3}_{k,3})\delta_{\alpha\beta}, \tag{4.51}$$

$$\hat{s}^0_{\alpha 3} = \hat{\mu}(\hat{u}^0_{\alpha,3} + \hat{u}^0_{k,\alpha}\hat{u}^0_{k,3}), \tag{4.52}$$

$$\hat{s}^0_{33} = (2\hat{\mu} + \hat{\lambda})(\hat{u}^0_{3,3} + \hat{u}^0_{k,3}\hat{u}^1_{k,3} + \hat{u}^{1/3}_{k,3}\hat{u}^{2/3}_{k,3}), \tag{4.53}$$

- Order $1/3$:

$$\hat{s}^1_{\alpha\beta} = \hat{\lambda}(\hat{u}^{1/3}_{3,3} + \hat{u}^0_{k,3}\hat{u}^{4/3}_{k,3} + \hat{u}^{1/3}_{k,3}\hat{u}^1_{k,3} + \tfrac{1}{2}\hat{u}^{2/3}_{k,3}\hat{u}^{2/3}_{k,3})\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \tag{4.54}$$

$$\hat{s}^{1/3}_{\alpha 3} = \hat{\mu}(\hat{u}^{1/3}_{\alpha,3} + \hat{u}^0_{k,\alpha}\hat{u}^{1/3}_{k,3} + \hat{u}^{1/3}_{k,\alpha}\hat{u}^0_{k,3}), \quad \alpha = 1, 2, \tag{4.55}$$

$$\hat{s}^{1/3}_{33} = (2\hat{\mu} + \hat{\lambda})(\hat{u}^{1/3}_{3,3} + \hat{u}^0_{k,3}\hat{u}^{4/3}_{k,3} + \hat{u}^{1/3}_{k,3}\hat{u}^1_{k,3} + \tfrac{1}{2}\hat{u}^{2/3}_{k,3}\hat{u}^{2/3}_{k,3}). \tag{4.56}$$

The conditions at the next orders are not considered here because they are expected to yield transmission conditions of higher orders.

From (4.48)–(4.55) it follows that

$$\hat{\mathbf{u}}^0_{,3} = 0 \quad \text{in } B \Rightarrow [\hat{\mathbf{u}}^0] = 0, \tag{4.57}$$

$$\hat{\mathbf{u}}^{1/3}_{,3} = 0 \quad \text{in } B \Rightarrow [\hat{\mathbf{u}}^{1/3}] = 0, \tag{4.58}$$

$$\hat{s}^0_{ij} = 0, \quad i, j = 1, 2, 3, \quad \text{in } B, \tag{4.59}$$

$$\hat{s}^{1/3}_{\alpha 3} = 0, \quad \alpha = 1, 2, \quad \text{in } B, \tag{4.60}$$

where, given any $f\colon B \mapsto R^3$, it has been set $[f](z_1, z_2) := f(z_1, z_2, (1/2)^-) - f(z_1, z_2, (-1/2)^+)$. Substituting (4.57)–(4.60) into the expansions of the equilibrium equations in the interphase and into the expansions of the continuity condition of the traction at $S_\pm$, we find that the Eqs. (4.21)–(4.23) and (4.40)–(4.42) are identically satisfied. Combining together the Eqs. (4.24) and (4.43), (4.25) and (4.44), (4.26) and (4.45), (4.27) and (4.46) and taking into account (4.57)–(4.60), we obtain the continuity of the first Piola-Kirchhoff stress vector

$$[\overline{\mathbf{P}}^l\mathbf{e}_3] = 0, \quad l = 0,\ 1/3,\ 2/3,\ 1. \tag{4.61}$$

Lastly, substituting (4.57)–(4.60) into (4.56) and substituting the result back into (4.24), we find that

$$(\hat{\mu} + \frac{1}{2}\hat{\lambda})(|\hat{\mathbf{u}}^{2/3}_{,3}|^2\hat{u}^{2/3}_{i,3})_{,3} = 0 \quad \text{in } B, \tag{4.62}$$

which implies that the vector $\hat{\mathbf{u}}^{1/3}_{,3}$ is independent of z_3. Solving with respect to $\hat{\mathbf{u}}^{1/3}_{,3}$ and integrating with respect to z_3 between 1/2 and $-1/2$ with the boundary condition $\overline{\mathbf{P}}^0\mathbf{e}_3 = \hat{\mathbf{u}}^{2/3}_{,3}\hat{s}^{1/3}_{33}$ (coming from (4.43)), it gives

$$[\hat{\mathbf{u}}^{2/3}] = \frac{1}{(\hat{\mu} + \frac{1}{2}\hat{\lambda})^{1/3}}\frac{1}{|\overline{\mathbf{P}}^0\mathbf{e}_3|^{2/3}}\overline{\mathbf{P}}^0\mathbf{e}_3. \tag{4.63}$$

4.4.2. *Interphase made of a "hard" material*

The Lamé's coefficients are now assumed to be independent of ε:

$$\lambda^\varepsilon = \hat{\lambda}, \quad \mu^\varepsilon = \hat{\mu}. \tag{4.64}$$

Substituting the expansion (4.17) deprived of the terms with fractional exponents into the constitutive Eqs. (4.12)–(4.14) and using (4.64), we obtain that the following conditions hold in B:

- Order -2:

$$0 = \hat{u}^0_{k,3}\hat{u}^0_{k,3}, \tag{4.65}$$

- Order -1:

$$0 = \hat{u}^0_{k,3}\hat{u}^1_{k,3} + \hat{u}^0_{3,3}, \tag{4.66}$$

$$0 = \hat{u}^0_{\alpha,3} + \hat{u}^0_{k,\alpha}\hat{u}^0_{k,3}, \tag{4.67}$$

- Order 0:

$$\hat{s}^0_{\alpha\beta} = \hat{\mu}(\hat{u}^0_{\alpha,\beta} + \hat{u}^0_{\beta,\alpha} + \hat{u}^0_{k,\alpha}\hat{u}^0_{k,\beta}) + \hat{\lambda}\left(\hat{u}^0_{\gamma,\gamma} + \frac{1}{2}(\hat{u}^0_{k,\gamma}\hat{u}^0_{k,\gamma})\right)\delta_{\alpha\beta} + \hat{\lambda}\left(\hat{u}^1_{3,3} + \frac{1}{2}\hat{u}^1_{k,3}\hat{u}^1_{k,3}\right)\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \tag{4.68}$$

$$\hat{s}^0_{\alpha 3} = \hat{\mu}(\hat{u}^0_{3,\alpha} + \hat{u}^1_{\alpha,3} + \hat{u}^0_{k,\alpha}\hat{u}^1_{k,3} + \hat{u}^1_{k,\alpha}\hat{u}^0_{k,3}), \quad \alpha = 1, 2, \tag{4.69}$$

$$\hat{s}^0_{33} = (2\hat{\mu} + \hat{\lambda})\left(\hat{u}^1_{3,3} + \frac{1}{2}\hat{u}^1_{k,3}\hat{u}^1_{k,3}\right) + \hat{\lambda}\left(\hat{u}^0_{\gamma,\gamma} + \frac{1}{2}(\hat{u}^0_{k,\gamma}\hat{u}^0_{k,\gamma})\right). \tag{4.70}$$

As before, conditions at the next orders are not considered because they are expected to yield transmission conditions of higher orders. Eq. (4.65) implies that $\hat{\mathbf{u}}^0_{,3} = 0$ in B, thus

$$[\hat{\mathbf{u}}^0] = 0. \tag{4.71}$$

Using this latter result, the remaining conditions (4.68)–(4.70) simplify as

$$\hat{\mathbf{s}}^0 = \hat{\mu}(\widehat{\mathbf{H}}_0 + \widehat{\mathbf{H}}_0^T + \widehat{\mathbf{H}}_0^T\widehat{\mathbf{H}}_0) + \hat{\lambda}\left(\mathbf{I}\cdot\widehat{\mathbf{H}}_0 + \frac{1}{2}\left|\widehat{\mathbf{H}}_0\right|^2\right)\mathbf{I} \tag{4.72}$$

with $\mathbf{I}$ the identity matrix and

$$\widehat{\mathbf{H}}_0 := \hat{\mathbf{u}}^0_{,1} \otimes \mathbf{e}_1 + \hat{\mathbf{u}}^0_{,2} \otimes \mathbf{e}_2 + \hat{\mathbf{u}}^1_{,3} \otimes \mathbf{e}_3. \tag{4.73}$$

For later use, we note in passing that Eq. (4.72) takes the component form

$$\hat{s}^0_{\alpha\beta} = \hat{\mu}(\hat{u}^0_{\alpha,\beta} + \hat{u}^0_{\beta,\alpha} + \hat{u}^0_{k,\alpha}\hat{u}^0_{k,\beta}) + \hat{\lambda}\left(\hat{u}^0_{\alpha,\alpha} + \frac{1}{2}(\hat{u}^0_{k,\alpha}\hat{u}^0_{k,\alpha})\right)\delta_{\alpha\beta} + \hat{\lambda}\left(\hat{u}^1_{3,3} + \frac{1}{2}(\hat{u}^1_{k,3}\hat{u}^1_{k,3})\right)\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \tag{4.74}$$

$$\hat{s}^0_{\alpha 3} = \hat{\mu}\left(\hat{u}^1_{\alpha,3} + \hat{u}^0_{3,\alpha} + \frac{1}{2}(\hat{u}^0_{k,\alpha}\hat{u}^1_{k,3})\right), \quad \alpha = 1, 2, \tag{4.75}$$

$$\hat{s}^0_{33} = (2\hat{\mu} + \hat{\lambda})\left(\hat{u}^1_{3,3} + \frac{1}{2}(\hat{u}^1_{k,3}\hat{u}^1_{k,3})\right) + \hat{\lambda}\left(\hat{u}^0_{\gamma,\gamma} + \frac{1}{2}(\hat{u}^0_{k,\gamma}\hat{u}^0_{k,\gamma})\right). \tag{4.76}$$

Substituting (4.71) into (4.21) makes (4.21) identically satisfied, while (4.24) reduces to

$$(\hat{s}^0_{i3} + \hat{u}^0_{i,\alpha}\hat{s}^0_{\alpha 3} + \hat{u}^1_{i,3}\hat{s}^0_{33})_{,3} = 0. \tag{4.77}$$

Integrating the latter equation with respect z_3 and using (4.32) and (4.43), one obtains the first contact conditions

$$[\overline{\mathbf{P}}^0\mathbf{e}_3] = 0. \tag{4.78}$$

implying the continuity of the traction vector at the order zero

To complete the analysis and obtain the remaining contact conditions, a first step is to prove that, the vectors $\hat{\mathbf{s}}^0\mathbf{e}_3$ and $\hat{\mathbf{u}}^1_{,3}$ are independent of z_3. The following Lemma, whose proof is postponed in Appendix, shows that this is true under suitable assumptions.

Lemma 1. *Let* $\mathbf{K}$ *be taken to denote the matrix*

$$\mathbf{K}:=\begin{pmatrix}\hat{\mu} & 0 & 0\\ 0 & \hat{\mu} & 0\\ 0 & 0 & 2\hat{\mu}+\hat{\lambda}\end{pmatrix}, \tag{4.79}$$

and let $\nabla\boldsymbol{\phi}:=\mathbf{I}+\widehat{\mathbf{H}}_0$. *If* $\nabla\boldsymbol{\phi}$ *and* $(\nabla\boldsymbol{\phi}\mathbf{K}(\nabla\boldsymbol{\phi})^T+\hat{s}^0_{33}\mathbf{I})$ *are invertible, then the vectors* $\hat{\mathbf{s}}^0\mathbf{e}_3$ *and* $\hat{\mathbf{u}}^1_{,3}$ *are independent of* z_3.

In the following, we assume that the hypotheses of the Lemma are satisfied. The condition that $\nabla\boldsymbol{\phi}$ be invertible is considered an acceptable hypothesis in view of the large-displacement small-strain situation depicted by the Saint Venant-Kirchhoff material model. The question of the invertibility of $(\nabla\boldsymbol{\phi}\mathbf{K}(\nabla\boldsymbol{\phi})^T+\hat{s}^0_{33}\mathbf{I})$ appears to be more complicated and it is not addressed here.

In view of the Lemma, the displacement vector field $\hat{u}^1$ can be represented in the form

$$\hat{\mathbf{u}}^1=[\hat{\mathbf{u}}^1]z_3+S(\hat{\mathbf{u}}^1), \tag{4.80}$$

where it has been set $S(f)(z_1,z_2):=1/2(f(z_1,z_2,1/2)+f(z_1,z_2,-1/2))$ for a given $f\colon B\mapsto R^3$.

Next, assuming that the hypotheses of the Lemma hold, we integrate (4.77) with respect to z_3 under the condition of continuity of the traction at $S_\pm$ and we use (4.80) and the definition (4.73) to get:

$$(\mathbf{I}+\widehat{\mathbf{H}}_0)\hat{\mathbf{s}}^0\mathbf{e}_3=\overline{\mathbf{P}}^0\mathbf{e}_3, \tag{4.81}$$

where now one has

$$\widehat{\mathbf{H}}_0=\hat{\mathbf{u}}^0_{,1}\otimes\mathbf{e}_1+\hat{\mathbf{u}}^0_{,2}\otimes\mathbf{e}_2+[\hat{\mathbf{u}}^3]\otimes\mathbf{e}_3. \tag{4.82}$$

After eliminating $\hat{s}^0_{\alpha3}$, $\alpha=1,2$, and $\hat{s}^0_{33}$ in (4.81) by using (4.75), (4.76) and (4.80), the following relation between the traction $\overline{\mathbf{P}}^0\mathbf{e}_3$ and the jump $[\hat{\mathbf{u}}^1]$ is deduced:

$$\overline{\mathbf{P}}^0\mathbf{e}_3=(\mathbf{I}+\widehat{\mathbf{H}}_0)\left\{\hat{\mu}(\widehat{\mathbf{H}}_0+\widehat{\mathbf{H}}_0^T+\widehat{\mathbf{H}}_0^T\widehat{\mathbf{H}}_0)+\hat{\lambda}\left(\mathbf{I}\cdot\widehat{\mathbf{H}}_0+\frac{1}{2}\left|\widehat{\mathbf{H}}_0\right|^2\right)\mathbf{I}\right\}\mathbf{e}_3 \tag{4.83}$$

Finally, integrating (4.27) with respect to z_3 and using (4.46) and the Lemma, we obtain

$$[\overline{P}^1_{i3}]=-(\hat{s}^0_{i\alpha}+\hat{u}^0_{i,\beta}\hat{s}^0_{\beta\alpha}+[\hat{u}^1_i]\hat{s}^0_{3\alpha})_{,\alpha}, \tag{4.84}$$

which, using (4.72), gives the condition

$$[\overline{\mathbf{P}}^1\mathbf{e}_3]=-\operatorname{div}_p\left((\mathbf{I}+\widehat{\mathbf{H}}_0)\left(\hat{\mu}(\widehat{\mathbf{H}}_0+\widehat{\mathbf{H}}_0^T+\widehat{\mathbf{H}}_0^T\widehat{\mathbf{H}}_0)+\hat{\lambda}\left(\mathbf{I}\cdot\widehat{\mathbf{H}}_0+\frac{1}{2}\left|\widehat{\mathbf{H}}_0\right|^2\right)\mathbf{I}\right)\right). \tag{4.85}$$

The notation div_p indicates the divergence in the plane of the interphase, e.g.

$$\operatorname{div}_p\mathbf{P}=(\mathbf{P}\mathbf{e}_1)_{,1}+(\mathbf{P}\mathbf{e}_2)_{,2}. \tag{4.86}$$

4.4.3. Interphase made of a "rigid" material

The Lamé's coefficients are now assumed as follows:

$$\lambda^\varepsilon=\frac{1}{\varepsilon}\hat{\lambda},\quad \mu^\varepsilon=\frac{1}{\varepsilon}\hat{\mu}. \tag{4.87}$$

Substituting the expansion (4.17) deprived of the terms with fractional exponents into the constitutive Eqs. (4.12)–(4.14) and using (4.87), the following conditions are found to hold in B:

- Order -3:

$$0=\hat{u}^0_{k,3}\hat{u}^0_{k,3}, \tag{4.88}$$

- Order -2

$$0=\hat{u}^0_{k,3}\hat{u}^1_{k,3}+\hat{u}^0_{3,3}, \tag{4.89}$$

$$0=\hat{u}^0_{\alpha,3}+\hat{u}^0_{k,\alpha}\hat{u}^0_{k,3}, \tag{4.90}$$

- Order -1

$$0=\hat{\mu}(\hat{u}^0_{\alpha,\beta}+\hat{u}^0_{\beta,\alpha}+\hat{u}^0_{k,\alpha}\hat{u}^0_{k,\beta})+\hat{\lambda}(\hat{u}^0_{\gamma,\gamma}+\tfrac{1}{2}(\hat{u}^0_{k,\gamma}\hat{u}^0_{k,\gamma}))\delta_{\alpha\beta} + \hat{\lambda}(\hat{u}^1_{3,3}+\hat{u}^0_{k,3}\hat{u}^2_{k,3}+\tfrac{1}{2}\hat{u}^1_{k,3}\hat{u}^1_{k,3})\delta_{\alpha\beta},\quad \alpha,\beta=1,2, \tag{4.91}$$

$$0=\hat{\mu}(\hat{u}^0_{3,\alpha}+\hat{u}^1_{\alpha,3}+\hat{u}^0_{k,\alpha}\hat{u}^1_{k,3}+\hat{u}^1_{k,\alpha}\hat{u}^0_{k,3}),\quad \alpha=1,2, \tag{4.92}$$

$$0=(2\hat{\mu}+\hat{\lambda})\left(\hat{u}^1_{3,3}+\hat{u}^0_{k,3}\hat{u}^2_{k,3}+\frac{1}{2}\hat{u}^1_{k,3}\hat{u}^1_{k,3}\right)+\hat{\lambda}\left(\hat{u}^0_{\gamma,\gamma}+\frac{1}{2}(\hat{u}^0_{k,\gamma}\hat{u}^0_{k,\gamma})\right). \tag{4.93}$$

Eq. (4.88) imply that $\hat{\mathbf{u}}^0_{,3}=0$, in B, thus (4.71) is reobtained. Introducing $\hat{\mathbf{u}}^0_{,3}=0$ in (4.91)–(4.93) gives

$$\hat{\mu}\widehat{\mathbf{E}}+\hat{\lambda}(\mathbf{I}\cdot\widehat{\mathbf{E}})\mathbf{I}=\mathbf{0}, \tag{4.94}$$

where the components of $\widehat{\mathbf{E}}$ are defined as follows:

$$\widehat{E}_{\alpha\beta}(\hat{\mathbf{u}}^0)=\tfrac{1}{2}(\hat{u}^0_{\alpha,\beta}+\hat{u}^0_{\beta,\alpha})+\tfrac{1}{2}\hat{u}^0_{k,\alpha}\hat{u}^0_{k,\beta},\quad \alpha,\beta=1,2, \tag{4.95}$$

$$\widehat{E}_{\alpha\beta}(\hat{\mathbf{u}}^0,\hat{\mathbf{u}}^1)=\frac{1}{2}(\hat{u}^0_{3,\alpha}+\hat{u}^1_{\alpha,3})+\tfrac{1}{2}\hat{u}^0_{k,\alpha}\hat{u}^1_{k,3}\quad \alpha=1,2, \tag{4.96}$$

$$\widehat{E}_{33}(\hat{\mathbf{u}}^1)=\hat{u}^1_{3,3}+\tfrac{1}{2}\hat{u}^1_{k,3}\hat{u}^1_{k,3}. \tag{4.97}$$

Eq. (4.94) implies

$$\widehat{E}_{\alpha\beta}(\hat{\mathbf{u}}^0)=0,\quad \alpha,\beta=1,2, \tag{4.98}$$

$$\widehat{E}_{\alpha3}(\hat{\mathbf{u}}^0,\hat{\mathbf{u}}^1)=0,\quad \alpha=1,2, \tag{4.99}$$

$$\widehat{E}_{33}(\hat{\mathbf{u}}^1)=0, \tag{4.100}$$

which are equivalent to the following conditions

$$\hat{u}^0_{1,1}+\tfrac{1}{2}|\hat{\mathbf{u}}^0_{,1}|^2=0, \tag{4.101}$$

$$\hat{u}^0_{2,2}+\tfrac{1}{2}|\hat{\mathbf{u}}^0_{,2}|^2=0, \tag{4.102}$$

$$\hat{u}^0_{1,2}+\hat{u}^0_{2,1}+\hat{\mathbf{u}}^0_{,1}\cdot\hat{\mathbf{u}}^0_{,2}=0, \tag{4.103}$$

$$(\hat{\mathbf{u}}^0_{,1}+\mathbf{e}_1)\cdot([\hat{\mathbf{u}}^1]+\mathbf{e}_3)=0, \tag{4.104}$$

$$(\hat{\mathbf{u}}^0_{,2}+\mathbf{e}_2)\cdot([\hat{\mathbf{u}}^1]+\mathbf{e}_3)=0, \tag{4.105}$$

$$|[\hat{\mathbf{u}}^1]+\mathbf{e}_3|^2=1. \tag{4.106}$$

Integrating (4.24)–(4.26) and using (4.43)–(4.45) the transmission conditions (4.78) are reobtained. Note that (4.104)–(4.106) imply that $[\hat{\mathbf{u}}^1]$ is independent of z_3, thus integrating (4.24) with respect to z_3 under the condition of continuity of the traction at $S_\pm$ and using (4.43), (4.71), (4.73) yields again (4.81). The latter allows to evaluate $\hat{\mathbf{s}}^0\mathbf{e}_3$ and shows that it is independent of z_3. Using these results together with (4.27) and (4.46), we reobtain Eq. (4.84), in which $\hat{s}_{\alpha,\beta}$, $\alpha,\beta=1,2$, are now undetermined.

4.5. Matching external and internal expansions

The transmission conditions obtained in Section 4.4 are appropriate for the rescaled equilibrium problem, prescribing the jump defined as $[\mathbf{f}](z_1,z_2):=\mathbf{f}(z_1,z_2,1/2)-\mathbf{f}(z_1,z_2,-1/2)$, with $\mathbf{f}\colon B\mapsto R^3$. In this Section, the transmission conditions are related to interface laws appropriate for the limit equilibrium problem, in which the interphase is replaced by the limit interface

$$S^0 = \{(x_1, x_2, x_3) \in \Omega\colon\ x_3 = 0\} \tag{4.107}$$

and the adherents by the domains

$$\Omega^0_\pm = \{(x_1, x_2, x_3) \in \Omega\colon\ \pm x_3 > 0\}. \tag{4.108}$$

Taking into account the asymptotic expansion (4.16) and assuming that the displacement in the adherent $\mathbf{u}^\varepsilon$ can be expanded in a Taylor series representation along the x_3– direction (external expansion), it results:

$$\mathbf{u}^\varepsilon\left(\bar{\mathbf{x}}, \pm\frac{\varepsilon}{2}\right) = \mathbf{u}^\varepsilon(\bar{\mathbf{x}}, 0^\pm) \pm \frac{\varepsilon}{2}\mathbf{u}^\varepsilon_{,3}(\bar{\mathbf{x}}, 0^\pm) + \cdots = \mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) + \varepsilon^{1/3}\mathbf{u}^{1/3}(\bar{\mathbf{x}}, 0^\pm) + \varepsilon^{2/3}\mathbf{u}^{2/3}(\bar{\mathbf{x}}, 0^\pm) + \varepsilon\left(\mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \frac{1}{2}\mathbf{u}^0_{,3}(\bar{\mathbf{x}}, 0^\pm)\right) + \cdots \quad (\ )$$

In view of the continuity of the displacements at the interfaces $S^\varepsilon_\pm$ and $S_\pm$ we also have

$$\mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) + \varepsilon^{1/3}\mathbf{u}^{1/3}(\bar{\mathbf{x}}, 0^\pm) + \varepsilon^{2/3}\mathbf{u}^{2/3}(\bar{\mathbf{x}}, 0^\pm) + \varepsilon(\mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \tfrac{1}{2}\mathbf{u}^0_{,3}(\bar{\mathbf{x}}, 0^\pm)) + \cdots \tag{4.110}$$

$$= \hat{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) + \varepsilon^{1/3}\hat{\mathbf{u}}^{1/3}\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \varepsilon^{2/3}\hat{\mathbf{u}}^{2/3}\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \varepsilon\hat{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) + \cdots \tag{4.111}$$

$$= \bar{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \varepsilon^{1/3}\bar{\mathbf{u}}^{1/3}\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \varepsilon^{2/3}\bar{\mathbf{u}}^{2/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) + \varepsilon\bar{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) + \cdots \tag{4.112}$$

Identifying the terms in the same powers of ε in the above external expansion and in the asymptotic expansions for $\hat{\mathbf{u}}^\varepsilon$ (internal expansion) and for $\bar{\mathbf{u}}^\varepsilon$, it is deduced that:

$$\begin{aligned}
\mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) &= \hat{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) = \bar{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right),\\
\mathbf{u}^{1/3}(\bar{\mathbf{x}}, 0^\pm) &= \hat{\mathbf{u}}^{1/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) = \bar{\mathbf{u}}^{1/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right),\\
\mathbf{u}^{2/3}(\bar{\mathbf{x}}, 0^\pm) &= \hat{\mathbf{u}}^{2/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) = \bar{\mathbf{u}}^{2/3}\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right),\\
\mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \tfrac{1}{2}\mathbf{u}^0_{,3}(\bar{\mathbf{x}}, 0^\pm) &= \hat{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right) = \bar{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right).
\end{aligned} \tag{4.113}$$

Analogous results can be obtained for the tractions:

$$\begin{aligned}
\mathbf{P}^0(\bar{\mathbf{x}}, 0^\pm)\mathbf{e}_3 &= \hat{\mathbf{P}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right)\mathbf{e}_3 = \bar{\mathbf{P}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right)\mathbf{e}_3,\\
\mathbf{P}^{1/3}(\bar{\mathbf{x}}, 0^\pm)\mathbf{e}_3 &= \hat{\mathbf{P}}^{1/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right)\mathbf{e}_3 = \bar{\mathbf{P}}^{1/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right)\mathbf{e}_3,\\
\mathbf{P}^{2/3}(\bar{\mathbf{x}}, 0^\pm)\mathbf{e}_3 &= \hat{\mathbf{P}}^{2/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right)\mathbf{e}_3 = \bar{\mathbf{P}}^{2/3}\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right)\mathbf{e}_3,\\
\mathbf{P}^1(\bar{\mathbf{x}}, 0^\pm)\mathbf{e}_3 \pm \tfrac{1}{2}\mathbf{P}^0_{,3}(\bar{\mathbf{x}}, 0^\pm)\mathbf{e}_3 &= \hat{\mathbf{P}}^1\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right)\mathbf{e}_3 = \bar{\mathbf{P}}^1\left(\bar{\mathbf{z}}, \pm\tfrac{1}{2}\right)\mathbf{e}_3.
\end{aligned} \tag{4.114}$$

Given now a function $\mathbf{f}\colon \Omega^0_+ \cup \Omega^0_- \mapsto \mathbb{R}^3$, we define

$$[\![\mathbf{f}]\!] := \mathbf{f}(\mathbf{x}, 0^+) - \mathbf{f}(\mathbf{x}, 0^-), \tag{4.115}$$

$$\mathcal{S}(\mathbf{f}) := \tfrac{1}{2}(\mathbf{f}(\mathbf{x}, 0^+) + \mathbf{f}(\mathbf{x}, 0^-)). \tag{4.116}$$

Then, the contact conditions appropriate for the limit equilibrium problem, i.e. expressed in terms of the fields defined on $\Omega^0_+ \cup \Omega^0_-$, can be obtained by substituting the following relations into the interphase laws:

$$[\bar{\mathbf{u}}^l] = [\![\mathbf{u}^l]\!] \quad l = 0,\ 1/3,\ 2/3, \tag{4.117}$$

$$[\bar{\mathbf{u}}^1] = [\![\mathbf{u}^1]\!] + \mathcal{S}(\mathbf{u}^0_{,3}), \tag{4.118}$$

$$[\bar{\mathbf{P}}^l\mathbf{e}_3] = [\![\mathbf{P}^l\mathbf{e}_3]\!] \quad l = 0,\ 1/3,\ 2/3, \tag{4.119}$$

$$[\bar{\mathbf{P}}^1\mathbf{e}_3] = [\![\mathbf{P}^1\mathbf{e}_3]\!] + \mathcal{S}(\mathbf{P}^0_{,3}\mathbf{e}_3). \tag{4.120}$$

4.6. Expansions of the constitutive equations of the adherents

Substituting the expansions (4.19) into the constitutive equation of the rescaled adherent (seventh equation in (4.6)) and taking into account that they have to be satisfied for any value of ε give the simple relations

$$\bar{s}^l_{ij} = A^\pm_{ijhk}\bar{E}_{hk}(\bar{u}^l) \quad l = 0,\ 1/3,\ 2/3,\ 1, \quad \text{in } \Omega_\pm. \tag{4.121}$$

5. Summary and discussion of the contact laws

This Section summarizes the transmission conditions obtained for the three material models of interphase considered in this paper: “soft”, “hard” and “rigid”. Next, using the matching conditions studied in the previous Section, interface conditions are obtained from the proposed transmission conditions. Lastly, these interface conditions are discussed in view of the laws calculated for soft and hard interphases in linear elasticity and modeling an imperfect contact because they allow for displacement discontinuities.

5.1. Interface laws for soft interphases

For the soft interphase, our asymptotic analysis yields the following results up to the second order:

$$[\bar{\mathbf{u}}^0] = 0, \quad [\bar{\mathbf{P}}^0\mathbf{e}_3] = 0, \tag{5.1}$$

$$[\bar{\mathbf{u}}^{1/3}] = 0, \quad [\bar{\mathbf{P}}^{1/3}\mathbf{e}_3] = 0, \tag{5.2}$$

$$[\bar{\mathbf{u}}^{2/3}] = \frac{1}{(\hat{\mu} + \frac{1}{2}\hat{\lambda})^{1/3}}\frac{1}{|\bar{\mathbf{P}}^0\mathbf{e}_3|^{2/3}}\bar{\mathbf{P}}^0\mathbf{e}_3, \quad [\bar{\mathbf{P}}^{2/3}\mathbf{e}_3] = 0. \tag{5.3}$$

Using the matching relations (4.117) and (4.119), the transmission conditions for the soft interphase can be rewritten in the final configuration $\Omega^0_+ \cup \Omega^0_- \cup S^0$ in a form involving only the fields in the adherents:

$$[\![\mathbf{u}^0]\!] = 0, \quad [\![\mathbf{P}^0\mathbf{e}_3]\!] = 0, \tag{5.4}$$

$$[\![\mathbf{u}^{1/3}]\!] = 0, \quad [\![\mathbf{P}^{1/3}\mathbf{e}_3]\!] = 0, \tag{5.5}$$

$$[\![\mathbf{u}^{2/3}]\!] = \frac{1}{(\hat{\mu} + \frac{1}{2}\hat{\lambda})^{1/3}}\frac{1}{|\mathbf{P}^0\mathbf{e}_3|^{2/3}}\mathbf{P}^0\mathbf{e}_3, \quad [\![\mathbf{P}^{2/3}\mathbf{e}_3]\!] = 0. \tag{5.6}$$

Recalling that for the soft interphase $\mu^\varepsilon = \hat{\mu}\,\varepsilon$, $\lambda^\varepsilon = \hat{\lambda}\,\varepsilon$, taking into account the expansions (4.15), (4.16) and the relations (4.32)–(4.34), we obtain

$$\mathbf{P}^\varepsilon\mathbf{e}_3 = \mathbf{P}^0\mathbf{e}_3 + O(\varepsilon^{1/3}), \tag{5.7}$$

$$[\![\mathbf{P}^\varepsilon\mathbf{e}_3]\!] = \varepsilon^{2/3}[\![\mathbf{P}^2\mathbf{e}_3]\!] + O(\varepsilon), \tag{5.8}$$

$$[\![\mathbf{u}^\varepsilon]\!] = \varepsilon^{2/3}[\![\mathbf{u}^2]\!] + O(\varepsilon), \tag{5.9}$$

which, substituted into (5.4)–(5.6), give

$$[\![\mathbf{P}^\varepsilon\mathbf{e}_3]\!] = 0 + o(\varepsilon), \tag{5.10}$$

$$\mathbf{P}^\varepsilon\mathbf{e}_3 = \frac{1}{2\varepsilon^3}(2\mu^\varepsilon + \lambda^\varepsilon)|[\![\mathbf{u}^\varepsilon]\!]|^2[\![\mathbf{u}^\varepsilon]\!] + o(\varepsilon^{1/3}). \tag{5.11}$$

These equations can be viewed as extending to the three-dimensional case the interface law (2.15) obtained in the one-dimensional case. Moreover, even though the Saint Venant-Kirchhoff material reduces to the linear elastic material under the approximation of small strains [12], a simple inspection of the transmission conditions (5.10), (5.11) shows that they do not reduce to the classical imperfect contact laws for soft interfaces in linear elasticity

$$[\![\mathbf{T}\mathbf{e}_3]\!] = 0, \quad \overline{\mathbf{T}}\mathbf{e}_3 = \mathbf{K}^{33}[\![\mathbf{u}]\!], \tag{5.12}$$

with $\mathbf{T}$ the Cauchy stress. This occurrence is completely analogous to the behavior of the one-dimensional interface law (2.15).

5.2. Interface laws for hard interphases

For the case of a hard interphase, our asymptotic analysis yields the following transmission conditions up to the third order:

$$[\overline{\mathbf{u}}^0] = 0, \quad [\overline{\mathbf{P}}^0\mathbf{e}_3] = 0, , \tag{5.13}$$

$$[\overline{\mathbf{u}}^1] = (\mathbf{I} + \overline{\mathbf{H}}_0)\left\{\hat{\mu}(\overline{\mathbf{H}}_0 + \overline{\mathbf{H}}_0^T + \overline{\mathbf{H}}_0^T\overline{\mathbf{H}}_0) + \hat{\lambda}\left(\mathbf{I}\cdot\overline{\mathbf{H}}_0 + \frac{1}{2}\left|\overline{\mathbf{H}}_0\right|^2\right)\mathbf{I}\right\}\mathbf{e}_3, \tag{5.14}$$

$$[\overline{\mathbf{P}}^1\mathbf{e}_3] = -\operatorname{div}_p\left((\mathbf{I} + \overline{\mathbf{H}}_0)\left(\hat{\mu}(\overline{\mathbf{H}}_0 + \overline{\mathbf{H}}_0^T + \overline{\mathbf{H}}_0^T\overline{\mathbf{H}}_0) + \hat{\lambda}\left(\mathbf{I}\cdot\overline{\mathbf{H}}_0 + \frac{1}{2}\left|\overline{\mathbf{H}}_0\right|^2\right)\mathbf{I}\right)\right), \tag{5.15}$$

with

$$\overline{\mathbf{H}}_0 = \overline{\mathbf{u}}^0_{,1} \otimes \mathbf{e}_1 + \overline{\mathbf{u}}^0_{,2} \otimes \mathbf{e}_2 + [\overline{\mathbf{u}}^1] \otimes \mathbf{e}_3. \tag{5.16}$$

In (5.13), the continuity conditions (4.113) and (4.114) have been taken into account. Using the matching conditions (4.117)–(4.120), the interface conditions in the final configuration can be rewritten in the following form:

$$[\![\mathbf{u}^0]\!] = 0, \quad [\![\mathbf{P}^0\mathbf{e}_3]\!] = 0, \tag{5.17}$$

$$\mathbf{P}^0\mathbf{e}_3 = (\mathbf{I} + \mathbf{H}_0)\left\{\hat{\mu}(\mathbf{H}_0 + \mathbf{H}_0^T + \mathbf{H}_0^T\mathbf{H}_0) + \hat{\lambda}\left(\mathbf{I}\cdot\mathbf{H}_0 + \frac{1}{2}\left|\mathbf{H}_0\right|^2\right)\mathbf{I}\right\}\mathbf{e}_3, \tag{5.18}$$

$$[\![\mathbf{P}^1\mathbf{e}_3]\!] = -\operatorname{div}_p\left((\mathbf{I} + \mathbf{H}_0)\left(\hat{\mu}(\mathbf{H}_0 + \mathbf{H}_0^T + \mathbf{H}_0^T\mathbf{H}_0) + \hat{\lambda}\left(\mathbf{I}\cdot\mathbf{H}_0 + \frac{1}{2}\left|\mathbf{H}_0\right|^2\right)\mathbf{I}\right)\right) - \mathcal{S}(\mathbf{P}^0_{,3}\mathbf{e}_3), \tag{5.19}$$

with

$$\mathbf{H}_0 = \mathbf{u}^0_{,1} \otimes \mathbf{e}_1 + \mathbf{u}^0_{,2} \otimes \mathbf{e}_2 + ([\![\mathbf{u}^1]\!] - \mathcal{S}(\mathbf{u}^0_{,3})) \otimes \mathbf{e}_3. \tag{5.20}$$

For the hard interphase one has $\mu^\varepsilon = \hat{\mu}$, $\lambda^\varepsilon = \hat{\lambda}$, and taking into account the expansions (4.15), (4.16)) and the relations (4.32)–(4.34), one finds

$$[\![\mathbf{P}^\varepsilon\mathbf{e}_3]\!] = \varepsilon[\![\mathbf{P}^3\mathbf{e}_3]\!] + O(\varepsilon), \tag{5.21}$$

$$[\![\mathbf{u}^\varepsilon]\!] = \varepsilon[\![\mathbf{u}^3]\!] + O(\varepsilon), \tag{5.22}$$

which, substituted into (5.17) and (5.15), allow to rewrite them in the form

$$\mathbf{P}^0\mathbf{e}_3 = (\mathbf{I} + \mathbf{H}_\varepsilon)\left\{\mu^\varepsilon(\mathbf{H}_\varepsilon + \mathbf{H}_\varepsilon^T + \mathbf{H}_\varepsilon^T\mathbf{H}_\varepsilon) + \lambda^\varepsilon\left(\mathbf{I}\cdot\mathbf{H}_\varepsilon + \frac{1}{2}\left|\mathbf{H}_\varepsilon\right|^2\right)\mathbf{I}\right\}\mathbf{e}_3 + O(\varepsilon), \tag{5.23}$$

$$[\![\mathbf{P}^\varepsilon\mathbf{e}_3]\!] = -\varepsilon\operatorname{div}_p\left((\mathbf{I} + \mathbf{H}_\varepsilon)\left(\mu^\varepsilon(\mathbf{H}_\varepsilon + \mathbf{H}_\varepsilon^T + \mathbf{H}_\varepsilon^T\mathbf{H}_\varepsilon) + \lambda^\varepsilon\left(\mathbf{I}\cdot\mathbf{H}_\varepsilon + \frac{1}{2}\left|\mathbf{H}_\varepsilon\right|^2\right)\mathbf{I}\right)\right) - \mathcal{S}(\mathbf{P}^0_{,3}\mathbf{e}_3) + O(\varepsilon), \tag{5.24}$$

with

$$\mathbf{H}_\varepsilon = \mathbf{u}^0_{,1} \otimes \mathbf{e}_1 + \mathbf{u}^0_{,2} \otimes \mathbf{e}_2 + \left(\frac{1}{\varepsilon}[\![\mathbf{u}^\varepsilon]\!] - \mathcal{S}(\mathbf{u}^0_{,3})\right) \otimes \mathbf{e}_3. \tag{5.25}$$

Eqs. (5.23), (5.24) have two peculiarities. First, they prescribe the jump of the displacement and of the traction vector implicitly, the jump of the displacement field entering both right-hand sides of the equations in a nonlinear way. Next, for "small" enough strains $|\mathbf{H}_\varepsilon| \ll 1$ and negligeable higher order terms in $\mathbf{H}_\varepsilon$, Eqs. (5.23) and (5.24) formally reduce to the higher order interface laws obtained for a hard, linear elastic interphase in [28,29,42]:

$$[u_1^\varepsilon] = \varepsilon\left(\frac{1}{\mu^\varepsilon}\sigma^0_{13} - u^0_{3,1} - \mathcal{S}(u^0_{1,3})\right) + O(\varepsilon) \tag{5.26}$$

$$[u_2^\varepsilon] = \varepsilon\left(\frac{1}{\mu^\varepsilon}\sigma^0_{23} - u^0_{3,2} - \mathcal{S}(u^0_{2,3})\right) + O(\varepsilon) \tag{5.27}$$

$$[u_3^\varepsilon] = \varepsilon\left(\frac{1}{(2\mu^\varepsilon + \lambda^\varepsilon)}\sigma^0_{33} - \frac{\lambda^\varepsilon}{(2\mu^\varepsilon + \lambda^\varepsilon)}(u^0_{1,1} + u^0_{2,2}) - \mathcal{S}(u^0_{3,3})\right) + O(\varepsilon) \tag{5.28}$$

$$[\sigma^\varepsilon_{13}] = -\varepsilon\left(\frac{4\mu^\varepsilon(\mu^\varepsilon + \lambda^\varepsilon)}{(2\mu^\varepsilon + \lambda^\varepsilon)}u^0_{1,11} - \mu^\varepsilon u^0_{1,22} - \frac{\mu^\varepsilon(2\mu^\varepsilon + 3\lambda^\varepsilon)}{(2\mu^\varepsilon + \lambda^\varepsilon)}u^0_{2,21} - \frac{\lambda^\varepsilon}{(\mu^\varepsilon + \lambda^\varepsilon)}\sigma^0_{33,1} - \mathcal{S}(\sigma^0_{13,3})\right) + O(\varepsilon), \tag{5.29}$$

$$[\sigma^\varepsilon_{23}] = -\varepsilon\left(\frac{4\mu^\varepsilon(\mu^\varepsilon + \lambda^\varepsilon)}{(2\mu^\varepsilon + \lambda^\varepsilon)}u^0_{2,22} - \mu^\varepsilon u^0_{2,11} - \frac{\mu^\varepsilon(2\mu^\varepsilon + 3\lambda^\varepsilon)}{(2\mu^\varepsilon + \lambda^\varepsilon)}u^0_{1,12} - \frac{\lambda^\varepsilon}{(2\mu^\varepsilon + \lambda^\varepsilon)}\sigma^0_{33,2} - \mathcal{S}(\sigma^0_{23,3})\right) + O(\varepsilon), \tag{5.30}$$

$$[\sigma^\varepsilon_{33}] = -\varepsilon(\sigma^0_{13,1} - \sigma^0_{23,2} - \mathcal{S}(\sigma^0_{33,3})) + O(\varepsilon), \tag{5.31}$$

with σ^ε_{ij} the components of the Cauchy stress in the adherents.

5.3. Interface laws for rigid interphases

For a rigid interphase, the asymptotic analysis yields the following transmission conditions up to the second order:

$$[\overline{\mathbf{u}}^0] = 0, \quad [\overline{\mathbf{P}}^0\mathbf{e}_3] = 0. \tag{5.32}$$

In addition to this equation, the two sets of conditions (4.101)–(4.103) and (4.104)–(4.106) have been obtained. Conditions (4.101)–(4.103) can be restated as

$$|\hat{\mathbf{u}}^0_{,1} + \mathbf{e}_1|^2 = 1, \tag{5.33}$$

$$|\hat{\mathbf{u}}^0_{,2} + \mathbf{e}_2|^2 = 1, \tag{5.34}$$

$$(\hat{\mathbf{u}}^0_{,1} + \mathbf{e}_1)\cdot(\hat{\mathbf{u}}^0_{,2} + \mathbf{e}_2) = 0, \tag{5.35}$$

implying that the deformation associated to $\hat{\mathbf{u}}^0(\cdot, \pm 1/2)$ is an isometric mapping of S into $\mathbb{R}^3$. Physically, the deformation associated to $\hat{\mathbf{u}}^0$ belongs to the class of "paper-folding" deformations, the deformations that a flat sheet of paper having the shape of S can undergo.

Conditions (4.104)–(4.106) give the additional restriction

$$[\hat{\mathbf{u}}^1] + \mathbf{e}_3 = \frac{(\hat{\mathbf{u}}^0_{,1} + \mathbf{e}_1) \wedge (\hat{\mathbf{u}}^0_{,2} + \mathbf{e}_2)}{|(\hat{\mathbf{u}}^0_{,1} + \mathbf{e}_1) \wedge (\hat{\mathbf{u}}^0_{,2} + \mathbf{e}_2)|}, \tag{5.36}$$

implying that the relative position vector (at the first order) between points on the top and the bottom of the interphase remains perpendicular to the deformed middle surface (at the zeroth order) without stretching.

The lowest order interface model corresponding to the rigid adhesive is thus the perfect interface model described by the classical

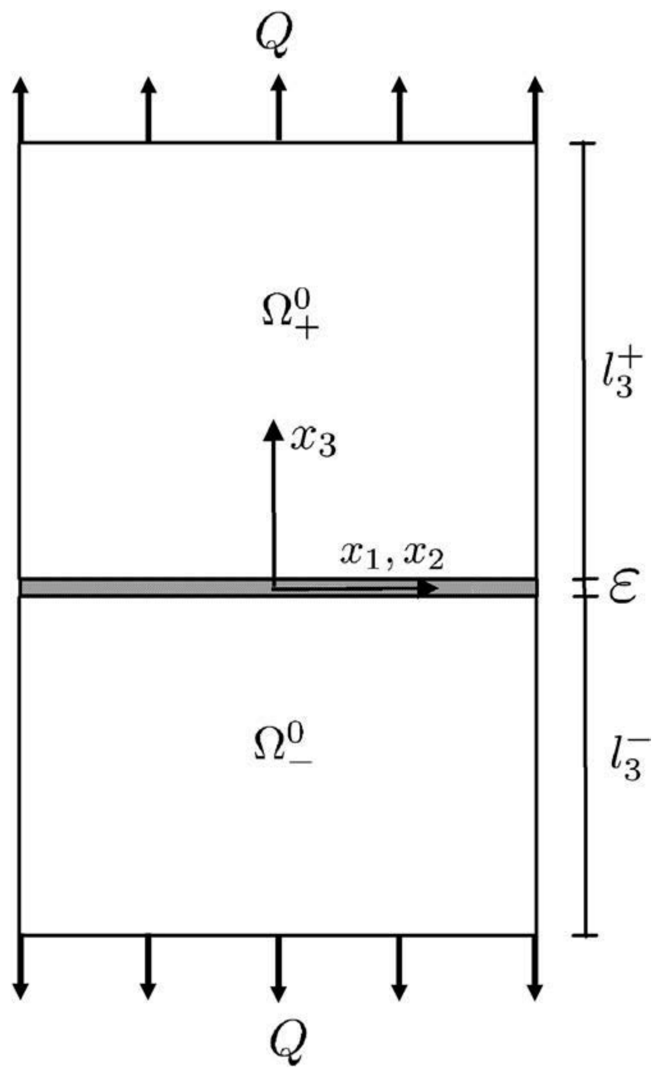

Fig. 3. Reference configuration of the butt joint studied in the example.

continuity conditions $[\![\mathbf{u}^0]\!] = 0$, $[\![\mathbf{P}^0\mathbf{e}_3]\!] = 0$, augmented by the further restriction that $\mathbf{u}^0$ corresponds to an isometric mapping.

6. Comparison of the soft interface laws with the limit model of Licht and Michaille

In [37], Licht and Michaille consider an elastic body constituted of adherent and interphase hyperelastic materials with nonconvex bulk energy density. In our notations and for the case of homogeneous materials, the total energy that they consider is

$$\mathcal{E}_{\varepsilon,\rho^\varepsilon}(\mathbf{u}):= \int_{\Omega^\varepsilon_\pm} h(\nabla\mathbf{u}(\mathbf{x}))dV_\mathbf{x} + \rho^\varepsilon \int_{B^\varepsilon} b(\nabla\mathbf{u}(\mathbf{x}))dV_\mathbf{x} - \mathcal{L}(\mathbf{u}), \tag{6.1}$$

where ρ^ε is a small parameter taking into account the low stiffness of the interphase and

$$\mathcal{L}(\mathbf{u}):= \int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})dV_\mathbf{x} + \int_{\Gamma_1} \mathbf{g}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})dA_\mathbf{x} \tag{6.2}$$

is the loading potential.

Licht and Michaille identify several limit problems depending on the relative order of magnitude of ρ^ε with respect to ε^3. In [37], three regimes are identified; in particular, the debonding phenomenon is characterized. For a Saint Venant-Kirchhoff material, [37] shows that

1. for $\rho^\varepsilon = \varepsilon^r$, $0 < r < 3$, the glue stiffness is sufficiently high to maintain adhesion. Licht and Michaille prove that in the limit problem the jump of the displacement at the interface vanishes (see space V_0 in [37]) and that the limit energy consists of the joint energy of the adhesives. In other words, the limit model of the thin adhesive layer is a perfect interface.
2. for $\rho^\varepsilon = \varepsilon^3$, the three-body limit problem obtained in [37] for a general energy contains the energy term

$$L\int_S Qb^{\infty,p}([\![\mathbf{u}(\mathbf{x})]\!] \otimes \mathbf{e}_3)dA_\mathbf{x}, \tag{6.3}$$

where $2L = \lim_{\varepsilon\to 0^+}\rho^\varepsilon/(\varepsilon)^{p-1} \in [0, +\infty)$, p is the growth exponent of b, $b^{\infty,p}$ is the density of the surface energy defined as follows:

$$b^{\infty,p}(\mathbf{F}):= \lim_{t\to+\infty}\frac{1}{t^p}b(t\mathbf{F}), \tag{6.4}$$

and $Qb^{\infty,p}$ is its quasiconvex envelope. For the Saint Venant- Kirchhoff energy density, one has $p = 4$ (cf. (1.1)). Indeed, by evaluating the energy (1.1) at $\mathbf{E} = \mathbf{E}(t\mathbf{F})$, with $\mathbf{F}$ the deformation gradient associated to $\mathbf{u}$, one can easily show that leading term of the energy $W\mathbf{E}(t\mathbf{F})$ when $t \to \infty$ is

$$\frac{t^4}{4}\left(\mu\left|\mathbf{F}^T\mathbf{F}\right|^2 + \frac{\lambda}{2}\left|\mathbf{F}\right|^4\right). \tag{6.5}$$

Now, using (6.6) and assuming to identify ρ^ε with the elastic constant μ^ε, one finds

$$b^{\infty,p}(\mathbf{F}) = \frac{1}{4}\left|\mathbf{F}^T\mathbf{F}\right|^2 + \frac{\hat{\lambda}}{8\hat{\mu}}\left|\mathbf{F}\right|^4, \tag{6.6}$$

where $\hat{\lambda}, \hat{\mu}$ are the rescaled Lam'e constants and the same rescaling with εhas been assumed for the two constants. The energy density (6.6) is clearly convex and thus $Qb^{\infty,p} = b^{\infty,p}$. Therefore, if $L \neq +\infty$ the limit surface energy (6.3) for $b^{\infty,p}$ as in (6.6) takes the form

$$L\int_S\left(\frac{1}{4}\left|(\mathbf{e}_3 \otimes [\![\mathbf{u}(\mathbf{x})]\!])([\![\mathbf{u}(\mathbf{x})]\!] \otimes \mathbf{e}_3)\right|^2 + \frac{\hat{\lambda}}{8\hat{\mu}}\left|[\![\mathbf{u}(\mathbf{x})]\!] \otimes \mathbf{e}_3\right|^4\right)dA_\mathbf{x} \tag{6.7}$$

which, after simplification, becomes

$$L\int_S \frac{1}{8\hat{\mu}}(2\hat{\mu} + \hat{\lambda})\left|[\![\mathbf{u}(\mathbf{x})]\!]\right|^4 dA_\mathbf{x}. \tag{6.8}$$

3. for $\rho^\varepsilon = \varepsilon^r$, $r > 3$, [37] shows that adhesion is lost. In the limit problem L = 0 and that there is no energy of the interphase left in the limit problem but the bodies can separate.

The case of a soft adhesive studied in the present paper is concerned with $\rho^\varepsilon = \varepsilon$ which is a subcase of the first case above studied by Licht and Michaille. This is a case where Licht and Michaille obtain perfect adhesion, so does the present paper (cf. (5.4)). The case of a hard adhesive $\rho^\varepsilon = 1$ and the case of a rigid adhesive $\rho^\varepsilon = \varepsilon$ are not *strictly sensu* studied in [37] since Licht and Michaille choose the glue stiffness to vanish (Fig. 3).

7. Uniaxial tension and compression of a butt joint

In this Section, two nonlinear elastic isotropic parallelepipeds Ω^0_- and Ω^0_+ are considered, having initial dimensions $l_1^+ \times l_2^+ \times l_3^+$ and $l_1^- \times l_2^- \times l_3^-$ respectively, and joined by an interface along a common face S^0. In the reference configuration, the composite structure is subjected to a tensile (compressive) load $Q > 0$ ($Q < 0$) aligned parallel to $\mathbf{e}_3$ and acting on the upper and the lower bases $\Gamma_1^+ = \{(x_1, x_2, x_3) \in \Omega: x_3 = l_3^+\}$ and $\Gamma_1^- = \{(x_1, x_2, x_3) \in \Omega: x_3 = -l_3^-\}$. On the remaining part of the boundary, $\partial_f\Omega^0$, the surface forces are taken to vanish. The load intensity Q is assumed to be independent of ε and body forces are null. The parallelepipeds are taken to be made of the same Saint Venant-Kirchhoff material, with Lamé constants λ, μ, but in the analysis, the related elastic constants

$$E = \mu\frac{(2\mu + 3\lambda)}{\lambda + \mu} \tag{7.1}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{7.2}$$

are used. In the following, λ^ε, μ^ε are taken to denote the (unrescaled) Lamé constants of the interface, and E^ε, ν^ε are taken to denote the related (unrescaled) elastic constants.

In the next Subsections, the equilibrium problem of the composite body made of two blocks joined by a non linear elastic interface is studied in the following two cases: (i) the interface behavior is soft and it is described by the interface laws (5.10) and (5.11); (ii) the interface behavior is hard and it is described by the interface laws (5.23) and (5.24).

7.1. Butt joint with soft interface behavior

Neglecting the higher order terms in ε, $\varepsilon^{1/3}$ in (5.10) and (5.11), the equilibrium problem of the joined structure is written as

$$\begin{cases} \mathrm{Div}\mathbf{P}^\varepsilon = 0 & \text{in } \Omega_+^0 \cup \Omega_-^0, \\ \mathbf{P}^\varepsilon = (\mathbf{I} + \nabla\mathbf{u}^\varepsilon)\left(\frac{E}{1+\nu}\mathbf{E}^\varepsilon + \frac{\nu E}{(1+\nu)(1-2\nu)}(\mathbf{I}\cdot\mathbf{E}^\varepsilon)\mathbf{I}\right) & \text{in } \Omega_+^0 \cup \Omega_-^0, \\ \mathbf{E}^\varepsilon = 1/2(\nabla\mathbf{u}^\varepsilon + (\nabla\mathbf{u}^\varepsilon)^T + (\nabla\mathbf{u}^\varepsilon)^T\nabla\mathbf{u}^\varepsilon) & \text{in } \Omega_+^0 \cup \Omega_-^0, \\ [\![\mathbf{P}^\varepsilon\mathbf{e}_3]\!] = \mathbf{0} & \text{on } S^0, \\ \mathbf{P}^\varepsilon\mathbf{e}_3 = \frac{1}{2\varepsilon^3}(2\mu^\varepsilon + \lambda^\varepsilon)\|[\![\mathbf{u}^\varepsilon]\!]\|^2[\![\mathbf{u}^\varepsilon]\!] & \text{on } S^0, \\ \mathbf{P}^\varepsilon\mathbf{e}_\alpha = \mathbf{0}, \quad \alpha = 1, 2, & \text{on } \partial_l\Omega^0, \\ \pm\mathbf{P}^\varepsilon\mathbf{e}_3 = \pm Q\mathbf{e}_3 & \text{on } \Gamma_1^\pm. \end{cases} \tag{7.3}$$

For the displacement field $\mathbf{u}^\varepsilon$: $\Omega_+^0 \cup \Omega_-^0 \mapsto \mathbb{R}^3$ we seek solutions of the form

$$\mathbf{u}^\varepsilon = (\lambda_1 - 1)(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \{(\lambda_3 - 1)x_3\}\mathbf{e}_3 \pm \frac{1}{2}[\![\mathbf{u}^\varepsilon]\!] \quad \text{in } \Omega_\pm^0, \tag{7.4}$$

with the λ_1, $\lambda_3 \in (0, +\infty)$, constants to be determined, representing the stretches parallel to the x_1 and x_3 axes, respectively, and the $[\![\mathbf{u}^\varepsilon]\!] \in \mathbb{R}^3$, a constant vector to be determined, representing the jump of the displacement at the interface S^0.

The Piola-Kirchhoff stress tensor corresponding to (7.4) is

$$\mathbf{P}^\varepsilon = P_{11}^\varepsilon(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + P_{33}^\varepsilon(\mathbf{e}_3 \otimes \mathbf{e}_3), \tag{7.5}$$

with

$$P_{11}^\varepsilon = \frac{E\lambda_1}{2(1+\nu)(1-2\nu)}(\lambda_1^2 - 1) + \frac{\nu E\lambda_1}{2(1+\nu)(1-2\nu)}(\lambda_3^2 - 1), \tag{7.6}$$

$$P_{33}^\varepsilon = \frac{E\lambda_3}{4(1+\nu)(1-2\nu)}(\lambda_3^2 - 1) + \frac{\nu E\lambda_3}{(1+\nu)(1-2\nu)}(\lambda_1^2 - 1). \tag{7.7}$$

Clearly, the divergence of $\mathbf{P}^\varepsilon$ vanishes together with the jump of $\mathbf{P}^\varepsilon$ at the interface S^0. To meet the natural boundary conditions on $\partial_l\Omega^0$, the vanishing of P_{11}^ε is imposed, which gives

$$\frac{\lambda_1^2}{(1+\nu)} + \frac{\nu\lambda_3^2}{(1+\nu)} = 1. \tag{7.8}$$

The above condition restricts the stretches λ_1 and λ_3 to take values in the intervals $(0, 1+\nu)$ and $(0, (1+\nu)/\nu)$, respectively. The occurrence of limit stretches can be interpreted as the failure of existence of solutions as in (7.4) for large strains, possibly related to the development of microstructure in the adherents [36]. Solving (7.8) with respect to λ_1 gives

$$\lambda_1 = \sqrt{1 + \nu - \nu\lambda_3^2}. \tag{7.9}$$

Substituting (7.9) back into (7.5) gives

$$\mathbf{P}^\varepsilon = \frac{E\lambda_3}{2}(\lambda_3^2 - 1)(\mathbf{e}_3 \otimes \mathbf{e}_3), \tag{7.10}$$

which, using the natural boundary condition on $\Gamma_1^\pm$, implies

$$Q = \frac{E\lambda_3}{2}(\lambda_3^2 - 1). \tag{7.11}$$

This equation determines λ_3 as a function of the load Q and, in view of the restriction $\lambda_3 \in (0, (1+\nu)/\nu)$, imposes the following restriction on the load

$$-\frac{\sqrt{3}}{9}E \le Q \le \frac{(1+\nu)(1+2\nu)}{2\nu^3}E. \tag{7.12}$$

The solution of (7.11) is

$$\lambda_3 = \begin{cases} \frac{2}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}\arccos\left(3\sqrt{3}\frac{Q}{E}\right)\right) & \text{if } -\frac{\sqrt{3}}{9} \le \frac{Q}{E} < \frac{\sqrt{3}}{9}, \\ \frac{2}{\sqrt{3}}\cosh\left(\frac{1}{\sqrt{3}}\mathrm{arccosh}\left(3\sqrt{3}\frac{Q}{E}\right)\right) & \text{if } \frac{\sqrt{3}}{9} \le \frac{Q}{E} < \frac{(1+\nu)(1+2\nu)}{2\nu^3}. \end{cases} \tag{7.13}$$

The jump $[\![\mathbf{u}^\varepsilon]\!]$ is determined through the interface condition in (7.3), which gives

$$[\![u_\alpha^\varepsilon]\!] = 0, \quad \alpha = 1, 2, \quad [\![u_3^\varepsilon]\!] = \frac{\sqrt{[3]2\,\varepsilon}}{\sqrt{[3]2\mu^\varepsilon + \lambda^\varepsilon}}\frac{Q}{|Q|^{2/3}}. \tag{7.14}$$

In view of (7.4), the macroscopic stretch along the x_3 axes, Λ, is

$$\Lambda{:=}1 + \frac{(\mathbf{u}(x_1, x_2, l_3^+) - \mathbf{u}(x_1, x_2, l_3^-))\cdot\mathbf{e}_3}{l} = \lambda_3 + \frac{[\![u_3^\varepsilon]\!]}{l}, \tag{7.15}$$

with $l{:=}l_3^+ + l_3^-$. The macroscopic response to uniaxial tension/compression is shown in Figs. 4, 5 in terms of applied surface force Q (divided by E) per unit area in the reference configuration and applied surface force $q = Q/\lambda_1^2$ (divided by E) per unit area in the deformed configuration versus the logaritmic strain $\epsilon = \ln\Lambda$. In the Figures, the Poisson's

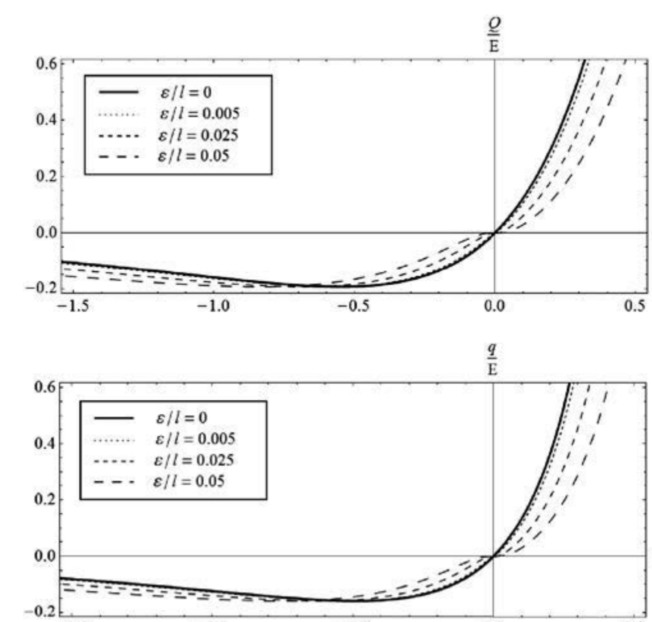

Fig. 4. Uniaxial tension and compression response of a butt joint with a soft thin interphase. Normalized applied surface force per unit area in the reference configuration, Q/E, and normalized applied surface force per unit area in the deformed configuration, q/E, versus the macroscopic logaritmic strain ϵ for the values $E/E^\varepsilon = 30$, $\nu = 0.33$, $\nu^\varepsilon = 0.4$ and different values of the interphase thickness. The thick solid line corresponds to the response without the thin interphase (i.e. vanishing thickness ε), the thin dashed curves to different adhesive/adherents thickness ratios $\varepsilon/l = \{0.005; 0.025; 0.05\}$, the dashing increasing with increasing ε/l.

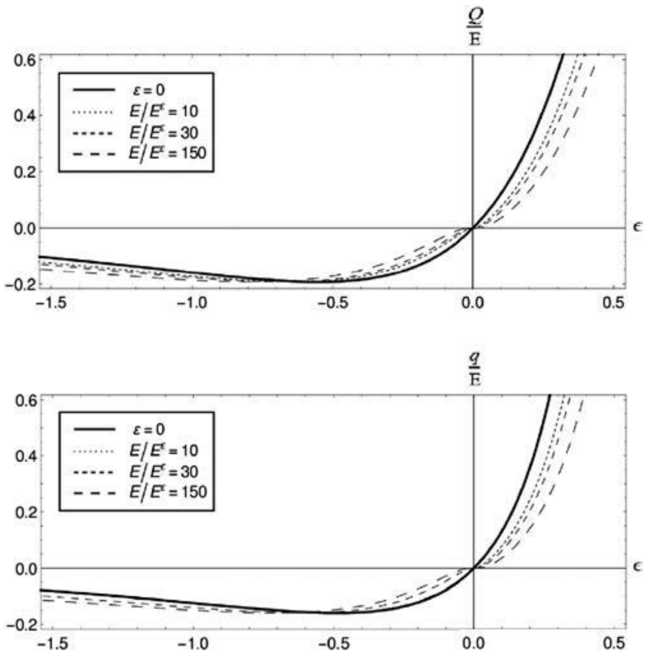

Fig. 5. Uniaxial tension and compression response of a butt joint with a soft thin interphase. Normalized applied surface force per unit area in the reference configuration, Q/E, and normalized applied surface force per unit area in the deformed configuration, q/E, versus the macroscopic logaritmic strain ϵ for the adhesive/adherents thickness ratio $\varepsilon/l = 0.025$ and Poisson's ratios $\nu = 0.33$, $\nu^{\varepsilon} = 0.4$. The thick solid line corresponds to the response without the thin interphase, the thin dashed curves to increasing ratios $E/E^{\varepsilon} = \{10, 30, 150\}$, the dashing space increasing with increasing E/E^{ε}.

ratios of the adherents and the adhesive are taken to be $\nu = 0.33$ and $\nu^{\varepsilon} = 0.4$, respectively, and the thick solid line corresponds to the macroscopic response calculated in absence of the soft interface.

In Fig. 4, the Young modulus of the adherents is 30 times greater than the Young modulus of the adhesive, $E/E^{\varepsilon} = 30$, and the thin dashed curves correspond to increasing values of the adhesive thickness, namely $\varepsilon/l = \{0.005; 0.025; 0.05\}$, the dashing space increasing with increasing ε. Fig. 4 shows that the (finely dashed) curve for $\varepsilon/l = 0.005$ and the (solid) curve for $\varepsilon = 0$ run very close to each other and that the adhesive thickness has a remarkable effect on the macroscopic response for $\varepsilon/l = 0.025, 0.05$.

In Fig. 5, the adhesive thickness is set $\varepsilon = 0.025/(l_3^+ + l_3^-)$ and the thin dashed curves correspond to increasing values of the Young modulus of the adherents, $E/E^{\varepsilon} = \{10; 30; 150\}$, the dashing increasing with increasing E/E^{ε}.

Both Figures also evidentiate a particular feature of the response curves taking into the presence of the adhesive near the origin: they fail to reproduce the behavior of the dashed curve at small Λ. This is related to the inability of (5.10), (5.11) to reduce to the classical imperfect contact laws for soft interfaces at small strains, as already remarked in Section 5.

7.2. Butt joint with hard interface behavior

Neglecting the higher order terms in εin the interface laws (5.23) and (5.24), the equilibrium problem of the structure made of two identical blocks joined by a hard interface is written as

$$\begin{cases} \mathrm{Div}\mathbf{P}^{\varepsilon} = 0 & \text{in } \Omega_+^0 \cup \Omega_-^0, \\ \mathbf{P}^{\varepsilon} = (\mathbf{I} + \nabla\mathbf{u}^{\varepsilon})\left(\frac{E}{1+\nu}\mathbf{E}^{\varepsilon} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\mathbf{I}\cdot\mathbf{E}^{\varepsilon})\mathbf{I}\right) & \text{in } \Omega_+^0 \cup \Omega_-^0, \\ \mathbf{E}^{\varepsilon} = 1/2(\nabla\mathbf{u}^{\varepsilon} + (\nabla\mathbf{u}^{\varepsilon})^T + (\nabla\mathbf{u}^{\varepsilon})^T\nabla\mathbf{u}^{\varepsilon}) & \text{in } \Omega_+^0 \cup \Omega_-^0, \\ [\![\mathbf{P}^{\varepsilon}\mathbf{e}_3]\!] = -\varepsilon\mathrm{div}_p((\mathbf{I} + \mathbf{H}_{\varepsilon})(\mu^{\varepsilon}(\mathbf{H}_{\varepsilon} + \mathbf{H}_{\varepsilon}^T + \mathbf{H}_{\varepsilon}^T\mathbf{H}_{\varepsilon}) & \\ \quad +\lambda^{\varepsilon}(\mathbf{I}\cdot\mathbf{H}_{\varepsilon} + \frac{1}{2}|\mathbf{H}_{\varepsilon}|^2)\mathbf{I})) - \mathcal{S}(\mathbf{P}^0_{,3}\mathbf{e}_3) & \text{on } S^0, \\ \mathbf{P}^0\mathbf{e}_3 = (\mathbf{I} + \mathbf{H}_{\varepsilon})\{\mu^{\varepsilon}(\mathbf{H}_{\varepsilon} + \mathbf{H}_{\varepsilon}^T + \mathbf{H}_{\varepsilon}^T\mathbf{H}_{\varepsilon}) & \\ \quad +\lambda^{\varepsilon}(\mathbf{I}\cdot\mathbf{H}_{\varepsilon} + \frac{1}{2}|\mathbf{H}_{\varepsilon}|^2)\mathbf{I}\}\mathbf{e}_3 & \text{on } S^0, \\ \mathbf{P}^{\varepsilon}\mathbf{e}_{\alpha} = \mathbf{0}, \ \ \alpha = 1, 2, & \text{on } \partial_l\Omega^0, \\ \pm\mathbf{P}^{\varepsilon}\mathbf{e}_3 = \pm Q\mathbf{e}_3 & \text{on } \Gamma_1^{\pm}, \end{cases} \tag{7.16}$$

with $\mathbf{H}_{\varepsilon}$ as in (5.25) and $\mathbf{u}^0$, $\mathbf{P}^0$ the displacement and the corresponding first Piola-Kirchhoff stress tensor solution of the equilibrium problem without the interface. In view of the results obtained in the previous subsection, the displacement $\mathbf{u}^0$ is given by (7.4), with $[\![\mathbf{u}^{\varepsilon}]\!] = 0$ and λ_1, λ_3 satisfying (7.9), (7.11). The tensor $\mathbf{P}^0$ is given by (7.10).

For the displacement field $\mathbf{u}^{\varepsilon}: \Omega_+^0 \cup \Omega_-^0 \mapsto \mathbb{R}^3$ we seek again a solution of the form (7.4). The constants λ_1, λ_3 are still chosen to satisfy (7.9) and (7.11), in order to match the constitutive equations of the adherents and the boundary conditions. Thus,

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^0 \pm \frac{1}{2}[\![\mathbf{u}^{\varepsilon}]\!] \quad \text{in } \Omega_{\pm}^0, \tag{7.17}$$

$$\mathbf{H}_{\varepsilon} = \lambda_1(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \left(\frac{1}{\varepsilon}[\![\mathbf{u}^{\varepsilon}]\!] - (\lambda_3 - 1)\mathbf{e}_3\right) \otimes \mathbf{e}_3. \tag{7.18}$$

The first Piola-Kirchhoff stress corresponding to (7.17) is still given by (7.10), i. e. $\mathbf{P}^{\varepsilon} = \mathbf{P}^0$. The (constant) jump $[\![\mathbf{u}^{\varepsilon}]\!]$ is determined in order to satisfy the interface laws in (7.16). Since $\mathbf{H}_{\varepsilon}$ and $\mathbf{P}^0$ are constant tensors, the first interface law reduces to $[\![\mathbf{P}^{\varepsilon}\mathbf{e}_3]\!] = 0$, which is identically satisfied. The second interface law gives the following conditions:

$$0 = \mu^{\varepsilon}\lambda_1[\![u_1^{\varepsilon}]\!]\left(2 + \frac{1}{\varepsilon}[\![u_3^{\varepsilon}]\!] - \lambda_3\right), \tag{7.19}$$

$$0 = \mu^{\varepsilon}\lambda_1[\![u_2^{\varepsilon}]\!]\left(2 + \frac{1}{\varepsilon}[\![u_3^{\varepsilon}]\!] - \lambda_3\right) \tag{7.20}$$

$$Q = \frac{(1-\nu^{\varepsilon})E^{\varepsilon}}{(1+\nu^{\varepsilon})(1-2\nu^{\varepsilon})}\left(2 + \frac{1}{\varepsilon}[\![u_3^{\varepsilon}]\!] - \lambda_3\right) \times \left(\left(2 + \frac{1}{\varepsilon}[\![u_3^{\varepsilon}]\!] - \lambda_3\right)^2 - 1 - \frac{4\nu^{\varepsilon}}{(1-\nu^{\varepsilon})}(1-\lambda_1)\right). \tag{7.21}$$

The first two of these conditions are satisfied by taking $[\![u_1^{\varepsilon}]\!] = 0 = [\![u_2^{\varepsilon}]\!]$.

In view of (7.11), the solution of the third (cubic) equation determines $[\![u_3^{\varepsilon}]\!]$ as a function of λ_3. In general, the solution to (7.21) is not unique and, as a selection criterion, the root of smallest modulus can be considered. This provides the continuity of the response curve Q/Λ through the origin, with Λ given again by (7.15).

Figs. 6 and 7 shown the macroscopic responses Q/ϵ and q/ϵ, with $q = Q/\lambda_1^2$ the load per unit area in the deformed configuration and $\epsilon = \ln\Lambda$ the logaritmic strain. The thick solid line corresponds to the macroscopic response calculated without taking into account the presence of the hard interface. To plot the Figures, the following values of the elastic constants have been assumed: $\nu = 0.33$, $\nu^{\varepsilon} = 0.4$. In Fig. 6, it has been set $E/E^{\varepsilon} = 1$ and an increasing dashing corresponds to increasing values of $\varepsilon/(l_3^+ + l_3^-)$ in the set $\{0.005, 0.025, 0.05\}$. In Fig. 7, it has been set $\varepsilon/(l_3^+ + l_3^-) = 0.025$ and an increasing dashing corresponds to increasing values of E/E^{ε} in the set $\{0.5, 10, 50\}$. From the Figures, it can be noted that the curves are almost overlapped, meaning that the presence of the hard interphase scarcely affects the macroscopic response for the given set of geometric and material parameters.

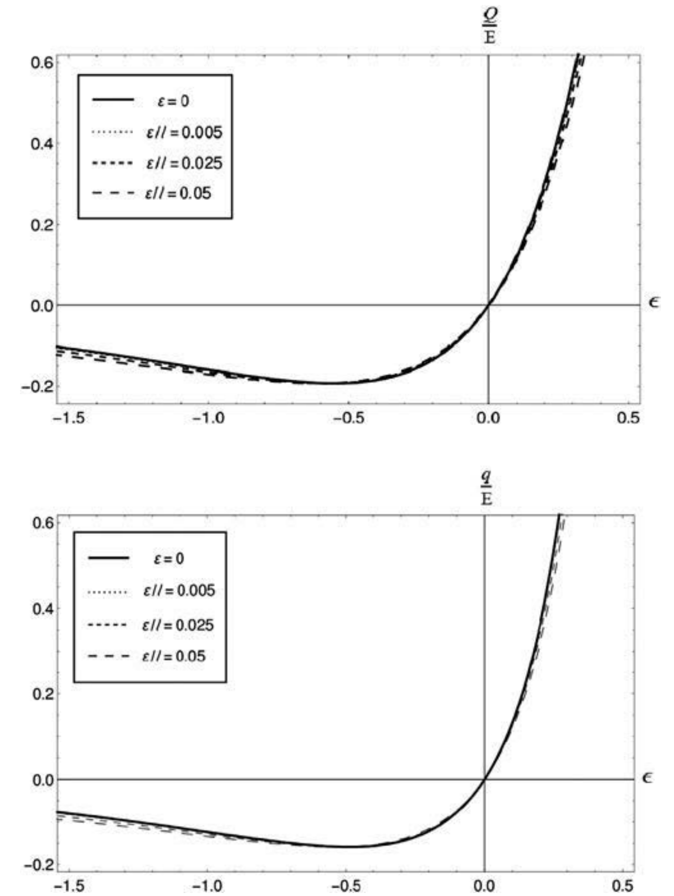

Fig. 6. Uniaxial tension and compression response of a butt joint with a hard thin interphase. Normalized applied surface force per unit area in the reference configuration, Q/E, and normalized applied surface force per unit area in the deformed configuration, q/E, versus the macroscopic logaritmic strain ϵ for the elastic constants $E/E^{\varepsilon} = 1$, $\nu = 0.33$, $\nu^{\varepsilon} = 0.4$. The thick solid line corresponds to the response without the thin interphase, the thin dashed curves to $\varepsilon/l = \{0.005;\ 0.025;\ 0.05\}$, the dashing increasing with increasing thickness ratio ε/l.

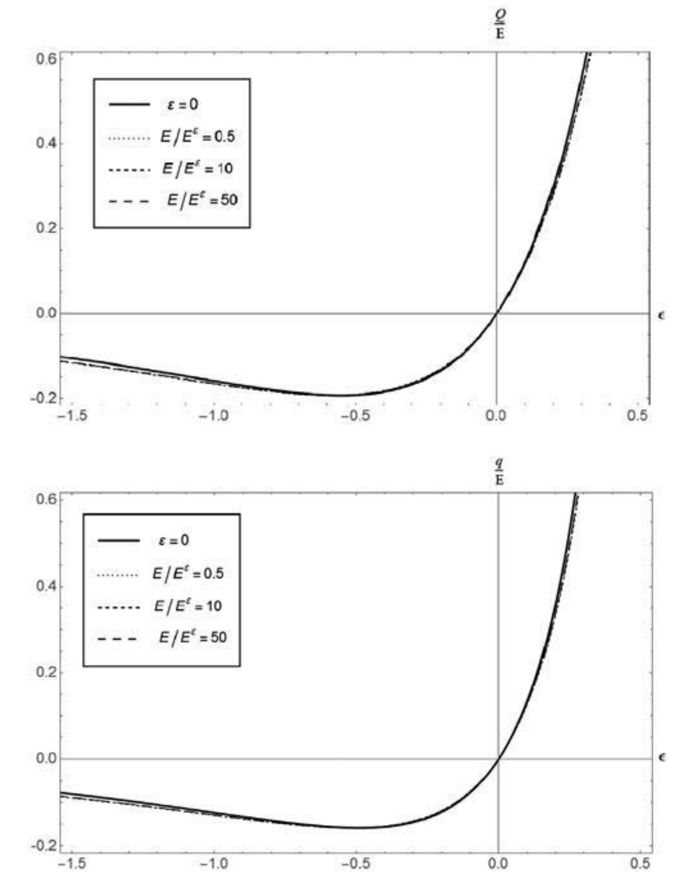

Fig. 7. Uniaxial tension and compression response of a butt joint with a hard thin interphase. Normalized applied surface force per unit area in the reference configuration, Q/E, and applied surface force per unit area in the deformed configuration, q/E, versus the macroscopic logaritmic strain ϵ for the thickness ratio $\varepsilon/l = 0.025$, and Poisson's ratios $\nu = 0.33$, $\nu^{\varepsilon} = 0.4$. The thick solid line corresponds to the response without the thin interphase, the thin dashed curves to increasing ratios $E/E^{\varepsilon} = \{0.5,\ 10,\ 50\}$, the dashing space increasing with increasing E/E^{ε}.

8. Conclusion

Using matched asymptotic expansions with fractional exponents, we have obtained original transmission conditions, appropriated for soft, hard and rigid adhesive materials obeying the Saint Venant-Kirchhoff model. The particular type of expansion chosen in the present paper, cf. (4.15) and (4.16), is strictly related to the exponent ($p = 4$) appearing in the growth conditions of the Saint Venant-Kirchhoff energy density. The same exponent enters the transmission conditions calculated for the soft adhesive, cf. (5.10), (5.11).

In the present paper, we restrict to a Saint Venant-Kirchhoff constitutive model, for which the exponent $1/3$ of the small parameter εappears, following the soft case of the one-dimensional example. In a more general situation, the fractional exponent of the asymptotic expansion is expected to depend on the exponent p appearing in the growth conditions of the energy density [37].

The transmission conditions proposed in the present paper find agreement with the results obtained via $\Gamma-$ convergence techniques by Licht and Michaille for the case of a soft adhesive [37]. Conditions (5.23) and (5.24) obtained for a hard interphase do not find analogous counterparts.

The nonlinear contact law calculated by Ganghoffer and Schultz in [17], which is similar to the one obtained by Edlund and Klarbring in [16], can not be compared with the interface laws obtained in the present paper for a soft interface. Indeed, an appropriate rescaling of the out-of-plane deformation component inside the interphase is assumed in [17], which is not used in our analysis.

Transmission conditions for a Saint Venant-Kirchhoff soft interface have been obtained also in [26]. Under the assumption of a linear scaling of the charges with the adhesive thickness, the limit behavior in the adherents is that of linear elasticity whereas it remains nonlinear in the adhesive. After linearization, the transmission conditions calculated in [26] provide the classical linear contact laws of spring-like type. The limit problem and the transmission conditions for a nonlinear Saint Venant-Kirchhoff soft interface obtained in this paper differ from the ones obtained in [26], having been obtained without applying any load scaling. In our approach, the limit behavior of the adherents remains nonlinear. As already remarked in Section 5.1, by linearizing the transmission conditions of the soft interface (Eqs. (5.10), (5.11)) for small strains, one cannot recover the classical contact laws of a linear elastic soft interface.

The situation is different for the case of a hard interface. Indeed, by linearizing the transmission conditions (5.23), (5.24) for small strains, one recovers the transmissions conditions calculated in [1,28,29,42] in the linearly elastic setting and generalizing the perfect interface case by taking into account higher order terms. Thus, the transmission conditions calculated in the present paper for a hard adhesive can be viewed a generalization of the transmission conditions for a hard interface in linear elasticity.

In [7–9,11], different cases of plate-like and shell-like linear elastic interphases are considered by scaling the intermediate layer stiffness with $1/\varepsilon$ (membrane interface) and $1/\varepsilon^3$ (inextensible flexural interface). The external loads, applied to the adherents, remain unscaled with respect to ε. Moreover, the asymptotic models are mathematically justified by virtue of a strong convergence argument. For an intermediate layer stiffness scaling with $1/\varepsilon$, in [8] it is found that the interphase behaves as an elastic membrane in the limit. This is different from the result obtained in the present paper, where it has been found that the rigid interface behaves as a perfect interface model at the zeroth order with the restriction that the deformation associated with the limit displacement is an isometric mapping. We believe that the difference may be due to our choice of the leading order in the inner expansion of the second Piola-Kirchhoff stress tensor (4.17), where the leading order has been simply chosen ε^0. A different choice of the leading order in (4.17) is expected to give rise to a completely different limit interface model for the rigid case.

The interface laws calculated in the present paper are expected to find significant applications in different contexts; definitely, they should be of importance in the analysis of adhesive joints, especially for all those applications requiring an accurate modeling of the nonlinear pre-peak behavior of the adhesive [2,3,13,22,33,40].

The proposed interface laws could also serve as generalization of the classical linear spring-type interface model in simulations of imperfect nonlinear bonding between the constitutive components of composites, in particular to study the influence of interfacial imperfections on the effective macroscopic behavior [34,35].

Acknowledgements

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Appendix

Lemma 1. *Let* $\mathbf{K}$ *be taken to denote the matrix*

$$\mathbf{K}:=\begin{pmatrix} \hat{\mu} & 0 & 0 \\ 0 & \hat{\mu} & 0 \\ 0 & 0 & 2\hat{\mu}+\hat{\lambda} \end{pmatrix}, \tag{A.1}$$

and let $\nabla\boldsymbol{\phi}:=\mathbf{I}+\widehat{\mathbf{H}}_0$. *If* $\nabla\boldsymbol{\phi}$ *and* $(\nabla\boldsymbol{\phi}\mathbf{K}(\nabla\boldsymbol{\phi})^T+\hat{s}^0_{,33}\mathbf{I})$ *are invertible, then the vectors* $\hat{\mathbf{s}}^0\mathbf{e}_3$ *and* $\hat{\mathbf{u}}^1_{,3}$ *are independent of* z_3.

Proof. From (4.77) one has $(\nabla\boldsymbol{\phi}\hat{\mathbf{s}}^0)\mathbf{e}_3$ independent of z_3. In other words,

$$\nabla\boldsymbol{\phi}(\hat{\mathbf{s}}^0\mathbf{e}_3)_{,3}+(\nabla\boldsymbol{\phi})_{,3}\hat{\mathbf{s}}^0=0. \tag{A.2}$$

From $\hat{\mathbf{u}}^0$ independent of z_3, we have

$$\nabla\boldsymbol{\phi}_{,3}=\hat{\mathbf{u}}^1_{,33}\otimes\mathbf{e}_3, \tag{A.3}$$

and

$$(\nabla\boldsymbol{\phi})_{,3}\hat{\mathbf{s}}^0\mathbf{e}_3=\hat{s}^0_{,33}\hat{\mathbf{u}}^1_{,33}. \tag{A.4}$$

From (4.69) and (4.70),

$$(\hat{\mathbf{s}}^0\mathbf{e}_3)_{,3}=\mathbf{K}(\nabla\boldsymbol{\phi})^T\hat{\mathbf{u}}^1_{,33}. \tag{A.5}$$

Inserting (A.4) and (A.5) in (A.2), one obtains

$$(\nabla\boldsymbol{\phi}\mathbf{K}(\nabla\boldsymbol{\phi})^T+\hat{s}^0_{,33}\mathbf{I})\hat{\mathbf{u}}^1_{,33}=\mathbf{0}. \tag{A.6}$$

If $(\nabla\boldsymbol{\phi}\mathbf{K}(\nabla\boldsymbol{\phi})^T+\hat{s}^0_{,33}\mathbf{I})$ is invertible, then the system (A.6) admits only the trivial solution $\hat{\mathbf{u}}^1_{,33}$, implying that $\hat{\mathbf{u}}^1_{,3}$ is independent of z_3. Therefore, in view of (A.5) and of the invertibility of $\nabla\boldsymbol{\phi}$, $\hat{\mathbf{s}}^0\mathbf{e}_3$ is also independent of z_3. □

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