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ON BLOWUP SOLUTIONS TO THE FOCUSING MASS-CRITICAL NONLINEAR FRACTIONAL SCHRÖDINGER EQUATION

VAN DUONG DINH

ABSTRACT. In this paper we study dynamical properties of blowup solutions to the focusing mass-critical nonlinear fractional Schrödinger equation. We establish a profile decomposition and a compactness lemma related to the equation. As a result, we obtain the L^2 -concentration and the limiting profile with minimal mass of blowup solutions.

1. INTRODUCTION

Consider the Cauchy problem for nonlinear fractional Schrödinger equations

$$\begin{cases} i\partial_t u - (-\Delta)^s u &= \mu |u|^\alpha u, & \text{on } [0, \infty) \times \mathbb{R}^d, \\ u(0) &= u_0, \end{cases} \quad (1.1)$$

where u is a complex valued function defined on $[0, \infty) \times \mathbb{R}^d$, $s \in (0, 1) \setminus \{1/2\}$ and $\alpha > 0$. The parameter $\mu = 1$ (or $\mu = -1$) corresponds to the defocusing (or focusing) case. The operator $(-\Delta)^s$ is the fractional Laplacian which is the Fourier multiplier by $|\xi|^{2s}$. The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics, which was discovered by Laskin [17] as a result of extending the Feynmann path integral, from the Brownian-like to Lévy-like quantum mechanical paths. The equation (1.1) enjoys the scaling invariance

$$u_\lambda(t, x) = \lambda^{\frac{2s}{\alpha}} u(\lambda^{2s}t, \lambda x), \quad \lambda > 0.$$

A computation shows

$$\|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\gamma + \frac{2s}{\alpha} - \frac{d}{2}} \|u_0\|_{\dot{H}^\gamma}.$$

We thus define the critical exponent

$$s_c := \frac{d}{2} - \frac{2s}{\alpha}. \quad (1.2)$$

The equation (1.1) also enjoys the formal conservation laws for the mass and the energy:

$$\begin{aligned} M(u(t)) &= \int |u(t, x)|^2 dx = M(u_0), \\ E(u(t)) &= \frac{1}{2} \int |(-\Delta)^{s/2} u(t, x)|^2 dx + \frac{\mu}{\alpha + 2} \int |u(t, x)|^{\alpha+2} dx = E(u_0). \end{aligned}$$

The local well-posedness for (1.1) in Sobolev spaces was studied in [13] (see also [5] for fractional Hartree equations). Note that the unitary group $e^{-it(-\Delta)^s}$ enjoys several types of Strichartz estimates (see e.g. [4] or [7] for Strichartz estimates with non-radial data; and [12], [16] or [2] for Strichartz estimates with radially symmetric data; and [8] or [3] for weighted Strichartz estimates). For non-radial data, these Strichartz estimates have a loss of derivatives. This makes the study of local well-posedness more difficult and leads to a weak local theory comparing to

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the standard nonlinear Schrödinger equation (see e.g. [13] or [7]). One can remove the loss of derivatives in Strichartz estimates by considering radially symmetric initial data. However, these Strichartz estimates without loss of derivatives require an restriction on the validity of s , that is $s \in \left[\frac{d}{2d-1}, 1 \right)$. We refer the reader to Section 2 for more details about Strichartz estimates and the local well-posedness in H^s for (1.1).

Recently, Boulenger-Himmelsbach-Lenzmann [1] proved blowup criteria for radial H^s solutions to the focusing (1.1). More precisely, they proved the following:

Theorem 1.1 (Blowup criteria [1]). *Let $d \geq 2$, $s \in (1/2, 1)$ and $\alpha > 0$. Let $u_0 \in H^s$ be radial such that the corresponding solution to the focusing (1.1) defines on the maximal time interval $[0, T)$.*

- **Mass-critical case**, i.e. $s_c = 0$ or $\alpha = \frac{4s}{d}$: *If $E(u_0) < 0$, then the solution u either blows up in finite time, i.e. $T < \infty$ or blows up in infinite time, i.e. $T = \infty$ and*

$$\|u(t)\|_{\dot{H}^s} \geq ct^s, \quad \forall t \geq t_*,$$

with some $C > 0$ and $t_ > 0$ that depend only on u_0, s and d .*

- **Mass and energy intercritical case**, i.e. $0 < s_c < s$ or $\frac{4s}{d} < \alpha < \frac{4s}{d-2s}$: *If $\alpha < 4s$ and*

$$E^{s_c}(u_0)M^{s-s_c}(u_0) < E^{s_c}(Q)M^{s-s_c}(Q), \quad \|u_0\|_{\dot{H}^s}^{s_c} \|u_0\|_{L^2}^{s-s_c} > \|Q\|_{\dot{H}^s}^{s_c} \|Q\|_{L^2}^{s-s_c},$$

where Q is the unique (modulo symmetries) positive radial solution to the elliptic equation

$$(-\Delta)^s Q + Q - |Q|^\alpha Q = 0,$$

then the solution blows up in finite time, i.e. $T < \infty$.

- **Energy-critical case**, i.e. $s_c = s$ or $\alpha = \frac{4s}{d-2s}$: *If $\alpha < 4s$ and*

$$E(u_0) < E(W), \quad \|u_0\|_{\dot{H}^s} > \|W\|_{\dot{H}^s},$$

where W is the unique (modulo symmetries) positive radial solution to the elliptic equation

$$(-\Delta)^s W - |W|^{\frac{4s}{d-2s}} W = 0,$$

the the solution blows up in finite time, i.e. $T < \infty$.

In this paper we are interested in dynamical properties of blowup solutions in H^s for the focusing mass-critical nonlinear fractional Schrödinger equation, i.e. $s \in (0, 1) \setminus \{1/2\}$, $\alpha = \frac{4s}{d}$ and $\mu = -1$ in (1.1). Before entering some details of our results, let us recall known results about blowup solutions in H^1 for the focusing mass-critical nonlinear Schrödinger equation

$$i\partial_t v + \Delta v = -|v|^{\frac{4}{d}} v, \quad v(0) = v_0 \in H^1. \quad (\text{mNLS})$$

The existence of blowup solutions in H^1 for (mNLS) was firstly proved by Glassey [11], where the author showed that for any negative initial data satisfying $|x|v_0 \in L^2$, the corresponding solution blows up in finite time. Ogawa-Tsutsumi [26, 27] showed the existence of blowup solutions for negative radial data in dimensions $d \geq 2$ and for any negative data (without radially symmetry) in the one dimensional case. The study of blowup H^1 solution to (mNLS) is connected to the notion of ground state which is the unique (up to symmetries) positive radial solution to the elliptic equation

$$\Delta R - R + |R|^{\frac{4}{d}} R = 0.$$

By the variational characteristic of the ground state, Weinstein [30] showed the structure and formation of singularity of the minimal mass blowup solution, i.e. $\|v_0\|_{L^2} = \|R\|_{L^2}$. He proved that the blowup solution remains close to the ground state R up to scaling and phase parameters, and also translation in the non-radial case. Merle-Tsutsumi [18], Tsutsumi [29] and Nava [25] proved

the L^2 -concentration of blowup solutions by using the variational characterization of ground state, that is, there exists $x(t) \in \mathbb{R}^d$ such that for all $r > 0$,

$$\liminf_{t \uparrow T} \int_{|x-x(t)| \leq r} |v(t, x)|^2 dx \geq \int |R(x)|^2 dx,$$

where T is the blowup time. Merle [19, 20] used the conformal invariance and compactness argument to characterize the finite time blowup solutions with minimal mass. More precisely, he proved that up to symmetries of the equation, the only finite time blowup solution with minimal mass is the pseudo-conformal transformation of the ground state. Hmidi-Keraani [14] gave a simplified proof of the characterization of blowup solutions with minimal mass of Merle by means of the profile decomposition and a refined compactness lemma. Merle-Raphaël [21, 22, 23] established sharp blowup rates, profiles of blowup solutions by the help of spectral properties.

As for (mNLS), the study of blowup solution to the focusing mass-critical nonlinear fractional Schrödinger equation is closely related to the notion of ground state which is the unique (modulo symmetries) positive radial solution of the elliptic equation

$$(-\Delta)^s Q + Q - |Q|^{\frac{4s}{d}} Q = 0. \quad (1.3)$$

The existence and uniqueness (up to symmetries) of ground state $Q \in H^s$ for (1.3) were recently shown in [9] and [10]. In [1, 10], the authors showed the sharp Gagliardo-Nirenberg inequality

$$\|f\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} \leq C_{\text{GN}} \|f\|_{\dot{H}^s}^2 \|f\|_{L^2}^{\frac{4s}{d}}, \quad (1.4)$$

where

$$C_{\text{GN}} = \frac{2s+d}{d} \|Q\|_{L^2}^{-\frac{4s}{d}}.$$

Using this sharp Gagliardo-Nirenberg inequality together with the conservation of mass and energy, it is easy to see that if $u_0 \in H^s$ satisfies

$$\|u_0\|_{L^2} < \|Q\|_{L^2},$$

then the corresponding solution exists globally in time. This implies that $\|Q\|_{L^2}$ is the critical mass for the formation of singularities.

To study blowup dynamics for data in H^s , we establish the profile decomposition for bounded sequences in H^s in the same spirit of [14]. With the help of this profile decomposition, we prove a compactness lemma related to the focusing mass-critical (NLFS).

Theorem 1.2 (Compactness lemma). *Let $d \geq 1$ and $0 < s < 1$. Let $(v_n)_{n \geq 1}$ be a bounded sequence in H^s such that*

$$\limsup_{n \rightarrow \infty} \|v_n\|_{\dot{H}^s} \leq M, \quad \limsup_{n \rightarrow \infty} \|v_n\|_{L^{\frac{4s}{d}+2}} \geq m.$$

Then there exists a sequence $(x_n)_{n \geq 1}$ in \mathbb{R}^d such that up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } H^s,$$

for some $V \in H^s$ satisfying

$$\|V\|_{L^2}^{\frac{4s}{d}} \geq \frac{d}{d+2s} \frac{m^{\frac{4s}{d}+2}}{M^2} \|Q\|_{L^2}^{\frac{4s}{d}}, \quad (1.5)$$

where Q is the unique solution to the elliptic equation (1.3).

Note that the lower bound on the L^2 -norm of V is optimal. Indeed, if we take $v_n = Q$, then we get the identity.

As a consequence of this compactness lemma, we show that the L^2 -norm of blowup solutions must concentrate by an amount which is bounded from below by $\|Q\|_{L^2}$ at the blowup time. Finally,

we show the limiting profile of blowup solutions with minimal mass $\|Q\|_{L^2}$. More precisely, we show that up to symmetries of the equation, the ground state Q is the profile for blowup solutions with minimal mass.

The paper is organized as follows. In Section 2, we recall Strichartz estimates for the fractional Schrödinger equation and the local well-posedness for (1.1) in non-radial and radial H^s initial data. In Section 3, we show the profile decomposition for bounded sequences in H^s and prove a compactness lemma related to the focusing mass-critical (1.1). The L^2 -concentration of blowup solutions is proved in Section 4. Finally, we show the limiting profile of blowup solutions with minimal mass in Section 5.

2. PRELIMINARIES

2.1. Strichartz estimates. In this subsection, we recall Strichartz estimates for the fractional Schrödinger equation. Let $I \subset \mathbb{R}$ and $p, q \in [1, \infty]$. We define the Strichartz norm

$$\|f\|_{L^p(I, L^q)} := \left(\int_I \left(\int_{\mathbb{R}^d} |f(t, x)|^q dx \right)^{\frac{1}{q}} dt \right)^{\frac{1}{p}},$$

with a usual modification when either p or q are infinity. We have three-types of Strichartz estimates for the fractional Schrödinger equation:

- For general data (see e.g. [4] or [7]): the following estimates hold for $d \geq 1$ and $s \in (0, 1) \setminus \{1/2\}$,

$$\|e^{-it(-\Delta)^s} \psi\|_{L^p(\mathbb{R}, L^q)} \lesssim \| |\nabla|^{\gamma_{p,q}} \psi \|_{L^2}, \quad (2.1)$$

$$\left\| \int e^{-i(t-\tau)(-\Delta)^s} f(\tau) d\tau \right\|_{L^p(\mathbb{R}, L^q)} \lesssim \| |\nabla|^{\gamma_{p,q} - \gamma_{a',b'} - 2s} f \|_{L^{a'}(\mathbb{R}, L^{b'})}, \quad (2.2)$$

where (p, q) and (a, b) are Schrödinger admissible, i.e.

$$p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2},$$

and

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{2s}{p},$$

similarly for $\gamma_{a',b'}$. Here (a, a') and (b, b') are conjugate pairs. It is worth noticing that for $s \in (0, 1) \setminus \{1/2\}$ the admissible condition $\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}$ implies $\gamma_{p,q} > 0$ for all admissible pairs (p, q) except $(p, q) = (\infty, 2)$. This means that the above Strichartz estimates have a loss of derivatives. In the local theory of the nonlinear fractional Schrödinger equation, this loss of derivatives makes the problem more difficult, and leads to a weak local well-posedness result comparing to the nonlinear Schrödinger equation (see Subsection 2.3).

- For radially symmetric data (see e.g. [16], [12] or [2]): the estimates (2.1) and (2.2) hold true for $d \geq 2$, $s \in (0, 1) \setminus \{1/2\}$ and $(p, q), (a, b)$ satisfy the radial Schrödinger admissible condition:

$$p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q) \neq \left(2, \frac{4d-2}{2d-3} \right), \quad \frac{2}{p} + \frac{2d-1}{q} \leq \frac{2d-1}{2}.$$

Note that the admissible condition $\frac{2}{p} + \frac{2d-1}{q} \leq \frac{2d-1}{2}$ allows us to choose (p, q) so that $\gamma_{p,q} = 0$. More precisely, we have for $d \geq 2$ and $\frac{d}{2d-1} \leq s < 1$ ¹,

$$\|e^{-it(-\Delta)^s} \psi\|_{L^p(\mathbb{R}, L^q)} \lesssim \|\psi\|_{L^2}, \quad (2.3)$$

$$\left\| \int e^{-i(t-\tau)(-\Delta)^s} f(\tau) d\tau \right\|_{L^p(\mathbb{R}, L^q)} \lesssim \|f\|_{L^{a'}(\mathbb{R}, L^{b'})}, \quad (2.4)$$

where ψ and f are radially symmetric and $(p, q), (a, b)$ satisfy the fractional admissible condition,

$$p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q) \neq \left(2, \frac{4d-2}{2d-3}\right), \quad \frac{2s}{p} + \frac{d}{q} = \frac{d}{2}. \quad (2.5)$$

These Strichartz estimates with no loss of derivatives allow us to give a similar local well-posedness result as for the nonlinear Schrödinger equation (see again Subsection 2.3).

- Weighted Strichartz estimates (see e.g. [8] or [3]): for $0 < \nu_1 < \frac{d-1}{2}$ and $\rho_1 \leq \frac{d-1}{2} - \nu_1$,

$$\| |x|^{\nu_1} |\nabla|^{\nu_1 - \frac{d}{2}} \langle \nabla_\omega \rangle^{\rho_1} e^{-it(-\Delta)^s} \psi \|_{L_t^\infty L_r^\infty L_\omega^2} \lesssim \|\psi\|_{L^2}, \quad (2.6)$$

and for $-\frac{d}{2} < \nu_2 < -\frac{1}{2}$ and $\rho_2 \leq -\frac{1}{2} - \nu_2$,

$$\| |x|^{\nu_2} |\nabla|^{s+\nu_2} \langle \nabla_\omega \rangle^{\rho_2} e^{-it(-\Delta)^s} \psi \|_{L^2(\mathbb{R}, L^2)} \lesssim \|\psi\|_{L^2}. \quad (2.7)$$

Here $\langle \nabla_\omega \rangle = \sqrt{1 - \Delta_\omega}$ with Δ_ω is the Laplace-Beltrami operator on the unit sphere \mathbb{S}^{d-1} . Here we use the notation

$$\|f\|_{L_r^p L_\omega^q} = \left(\int_0^\infty \left(\int_{\mathbb{S}^{d-1}} |f(r\omega)|^q d\omega \right)^{p/q} r^{d-1} dr \right)^{1/p}.$$

These weighted estimates are important to show the well-posedness below L^2 at least for the fractional Hartree equation (see [5]).

2.2. Nonlinear estimates. We recall the following fractional chain rule which is needed in the local well-posedness for (1.1).

Lemma 2.1 (Fractional chain rule [6, 15]). *Let $F \in C^1(\mathbb{C}, \mathbb{C})$ and $\gamma \in (0, 1)$. Then for $1 < q \leq q_2 < \infty$ and $1 < q_1 \leq \infty$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,*

$$\| |\nabla|^\gamma F(u) \|_{L^q} \lesssim \|F'(u)\|_{L^{q_1}} \| |\nabla|^\gamma u \|_{L^{q_2}}.$$

We refer the reader to [6, Proposition 3.1] for the proof of the above estimate when $1 < q_1 < \infty$ and to [15] for the proof when $q_1 = \infty$.

2.3. Local well-posedness in H^s . In this section, we recall the local well-posedness in the energy space H^s for (1.1). As mentioned in the introduction, we will separate two cases: non-radial initial data and radially symmetric initial data.

¹This condition follows by plugging $\gamma_{p,q} = 0$ to $\frac{2}{p} + \frac{2d-1}{q} \leq \frac{2d-1}{2}$.

Non-radial H^s initial data. We have the following result due to [13] (see also [7]).

Proposition 2.2 (Non-radial local theory [13, 7]). *Let $s \in (0, 1) \setminus \{1/2\}$ and $\alpha > 0$ be such that*

$$s > \begin{cases} \frac{1}{2} - \frac{2s}{\max(\alpha, 4)} & \text{if } d = 1, \\ \frac{d}{2} - \frac{2s}{\max(\alpha, 2)} & \text{if } d \geq 2. \end{cases} \quad (2.8)$$

Then for all $u_0 \in H^s$, there exist $T \in (0, \infty]$ and a unique solution to (1.1) satisfying

$$u \in C([0, T], H^s) \cap L_{\text{loc}}^p([0, T], L^\infty),$$

for some $p > \max(\alpha, 4)$ when $d = 1$ and some $p > \max(\alpha, 2)$ when $d \geq 2$. Moreover, the following properties hold:

- *If $T < \infty$, then $\|u(t)\|_{H^s} \rightarrow \infty$ as $t \uparrow T$.*
- *There is conservation of mass, i.e. $M(u(t)) = M(u_0)$ for all $t \in [0, T]$.*
- *There is conservation of energy, i.e. $E(u(t)) = E(u_0)$ for all $t \in [0, T]$.*

The proof of this result is based on Strichartz estimates and the contraction mapping argument. The loss of derivatives in Strichartz estimates can be compensated for by using the Sobolev embedding. We refer the reader to [13] or [7] for more details.

Remark 2.3. It follows from (2.8) and $s \in (0, 1) \setminus \{1/2\}$ that the local well-posedness for non radial data in H^s is available only for

$$\begin{cases} s \in (1/3, 1/2), & 0 < \alpha < \frac{4s}{1-2s} & \text{if } d = 1, \\ s \in (1/2, 1), & 0 < \alpha < \infty & \text{if } d = 1, \\ s \in (d/4, 1), & 0 < \alpha < \frac{4s}{d-2s} & \text{if } d = 2, 3. \end{cases} \quad (2.9)$$

In particular, in the mass-critical case $\alpha = \frac{4s}{d}$, the (1.1) is locally well-posed in H^s with

$$\begin{cases} s \in (1/3, 1/2) \cap (1/2, 1) & \text{if } d = 1, \\ s \in (d/4, 1) & \text{if } d = 2, 3. \end{cases}$$

Proposition 2.4 (Non-radial global existence [7]). *Let s, α and d be as in (2.9). Then for any $u_0 \in H^s$, the solution to (1.1) given in Proposition 2.2 can be extended to the whole \mathbb{R} if one of the following conditions is satisfied:*

- $\mu = 1$,
- $\mu = -1$ and $0 < \alpha < \frac{4s}{d}$,
- $\mu = -1$, $\alpha = \frac{4s}{d}$ and $\|u_0\|_{L^2}$ is small,
- $\mu = -1$ and $\|u_0\|_{H^s}$ is small.

Proof. The case $\mu = 1$ follows easily from the blowup alternative together with the conservation of mass and energy. The case $\mu = -1$ and $0 < \alpha < \frac{4s}{d}$ follows from the Gagliardo-Nirenberg inequality (see e.g. [28, Appendix]). Indeed, by Gagliardo-Nirenberg inequality and the mass conservation,

$$\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \lesssim \|u(t)\|_{\dot{H}^s}^{\frac{d\alpha}{2s}} \|u(t)\|_{L^2}^{\alpha+2-\frac{d\alpha}{2s}} = \|u(t)\|_{\dot{H}^s}^{\frac{d\alpha}{2s}} \|u_0\|_{L^2}^{\alpha+2-\frac{d\alpha}{2s}}.$$

The conservation of energy then implies

$$\frac{1}{2} \|u(t)\|_{\dot{H}^s}^2 = E(u(t)) + \frac{1}{\alpha+2} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \lesssim E(u_0) + \frac{1}{\alpha+2} \|u(t)\|_{\dot{H}^s}^{\frac{d\alpha}{2s}} \|u_0\|_{L^2}^{\alpha+2-\frac{d\alpha}{2s}}.$$

If $0 < \alpha < \frac{4s}{d}$, then $\frac{d\alpha}{2s} \in (0, 2)$ and hence $\|u(t)\|_{\dot{H}^s} \lesssim 1$. This combined with the conservation of mass yield the boundedness of $\|u(t)\|_{H^s}$ for any t belongs to the existence time. The blowup alternative gives the global existence. The case $\mu = -1, \alpha = \frac{4s}{d}$ and $\|u_0\|_{L^2}$ small is treated

similarly. It remains to treat the case $\mu = -1$ and $\|u_0\|_{\dot{H}^s}$ is small. Thanks to the Sobolev embedding with $\frac{1}{2} \leq \frac{1}{\alpha+2} + \frac{s}{d}$, we bound

$$\|f\|_{L^{\alpha+2}} \lesssim \|f\|_{H^s}.$$

This shows in particular that $E(u_0)$ is small if $\|u_0\|_{H^s}$ is small. Therefore,

$$\frac{1}{2}\|u(t)\|_{\dot{H}^s}^2 = E(u(t)) + \frac{1}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \leq E(u_0) + C\|u(t)\|_{\dot{H}^s}^{\alpha+2}.$$

This shows that $\|u(t)\|_{H^s}$ is bounded and the proof is complete. \square

Radial H^s initial data. Thanks to Strichartz estimates without loss of derivatives in the radial case, we have the following result.

Proposition 2.5 (Radial local theory). *Let $d \geq 2$ and $s \in \left[\frac{d}{2d-1}, 1\right)$ and $0 < \alpha < \frac{4s}{d-2s}$. Let*

$$p = \frac{4s(\alpha+2)}{\alpha(d-2s)}, \quad q = \frac{d(\alpha+2)}{d+\alpha s}. \quad (2.10)$$

Then for any $u_0 \in H^s$ radial, there exist $T \in (0, \infty]$ and a unique solution to (1.1) satisfying

$$u \in C([0, T], H^s) \cap L_{\text{loc}}^p([0, T], W^{s, q}).$$

Moreover, the following properties hold:

- *If $T < \infty$, then $\|u(t)\|_{\dot{H}^s} \rightarrow \infty$ as $t \uparrow T$.*
- *$u \in L_{\text{loc}}^a([0, T], W^{s, b})$ for any fractional admissible pair (a, b) .*
- *There is conservation of mass, i.e. $M(u(t)) = M(u_0)$ for all $t \in [0, T)$.*
- *There is conservation of energy, i.e. $E(u(t)) = E(u_0)$ for all $t \in [0, T)$.*

Proof. It is easy to check that (p, q) satisfies the fractional admissible condition (2.5). We choose (m, n) so that

$$\frac{1}{p'} = \frac{1}{p} + \frac{\alpha}{m}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{\alpha}{n}. \quad (2.11)$$

We see that

$$\frac{\alpha}{m} - \frac{\alpha}{p} = 1 - \frac{\alpha(d-2s)}{4s} =: \theta > 0, \quad q \leq n = \frac{dq}{d-sq}. \quad (2.12)$$

The later fact gives the Sobolev embedding $\dot{W}^{s, q} \hookrightarrow L^n$. Let us now consider

$$X := \left\{ C(I, H^s) \cap L^p(I, W^{s, q}) : \|u\|_{L^\infty(I, \dot{H}^s)} + \|u\|_{L^p(I, \dot{W}^{s, q})} \leq M \right\},$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^\infty(I, L^2)} + \|u - v\|_{L^p(I, L^q)},$$

where $I = [0, \zeta]$ and $M, \zeta > 0$ to be chosen later. By Duhamel's formula, it suffices to prove that the functional

$$\Phi(u)(t) := e^{-it(-\Delta)^s} u_0 - i\mu \int_0^t e^{-i(t-\tau)(-\Delta)^s} |u(\tau)|^\alpha u(\tau) d\tau$$

is a contraction on (X, d) . By radial Strichartz estimates (2.3) and (2.4),

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I, \dot{H}^s)} + \|\Phi(u)\|_{L^p(I, \dot{W}^{s, q})} &\lesssim \|u_0\|_{\dot{H}^s} + \| |u|^\alpha u \|_{L^{p'}(I, \dot{W}^{s, q'})}, \\ \|\Phi(u) - \Phi(v)\|_{L^\infty(I, L^2)} + \|\Phi(u) - \Phi(v)\|_{L^p(I, L^q)} &\lesssim \| |u|^\alpha u - |v|^\alpha v \|_{L^{p'}(I, L^{q'})}. \end{aligned}$$

The fractional chain rule given in Lemma 2.1 and the Hölder inequality give

$$\begin{aligned} \| |u|^\alpha u \|_{L^{p'}(I, \dot{W}^{s, q'})} &\lesssim \|u\|_{L^m(I, L^n)}^\alpha \|u\|_{L^p(I, \dot{W}^{s, q})}, \\ &\lesssim |I|^\theta \|u\|_{L^p(I, L^n)}^\alpha \|u\|_{L^p(I, \dot{W}^{s, q})} \\ &\lesssim |I|^\theta \|u\|_{L^p(I, \dot{W}^{s, q})}^{\alpha+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} \| |u|^\alpha - |v|^\alpha v \|_{L^{p'}(I, L^{q'})} &\lesssim \left(\|u\|_{L^m(I, L^n)}^\alpha + \|v\|_{L^m(I, L^n)}^\alpha \right) \|u - v\|_{L^p(I, L^q)} \\ &\lesssim |I|^\theta \left(\|u\|_{L^p(I, \dot{W}^{s, q})}^\alpha + \|v\|_{L^p(I, \dot{W}^{s, q})}^\alpha \right) \|u - v\|_{L^p(I, L^q)}. \end{aligned}$$

This shows that for all $u, v \in X$, there exists $C > 0$ independent of T and $u_0 \in H^s$ such that

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I, \dot{H}^s)} + \|\Phi(u)\|_{L^p(I, \dot{W}^{s, q})} &\leq C \|u_0\|_{\dot{H}^s} + C \zeta^\theta M^{\alpha+1}, \\ d(\Phi(u), \Phi(v)) &\leq C \zeta^\theta M^\alpha d(u, v). \end{aligned}$$

If we set $M = 2C \|u_0\|_{\dot{H}^s}$ and choose $\zeta > 0$ so that

$$C \zeta^\theta M^\alpha \leq \frac{1}{2},$$

then Φ is a strict contraction on (X, d) . This proves the existence of solution $u \in C(I, H^s) \cap L^p(I, W^{s, q})$. By radial Strichartz estimates, we see that $u \in L^a(I, W^{s, b})$ for any fractional admissible pairs (a, b) . The blowup alternative follows easily since the existence time depends only on the \dot{H}^s -norm of initial data. The proof is complete. \square

As in Proposition 2.4, we have the following criteria for global existence of radial solutions in H^s .

Proposition 2.6 (Radial global existence). *Let $d \geq 2, s \in \left[\frac{d}{2d-1}, 1 \right)$ and $0 < \alpha < \frac{4s}{d-2s}$. Then for any $u_0 \in H^s$ radial, the solution to (1.1) given in Proposition 2.5 can be extended to the whole \mathbb{R} if one of the following conditions is satisfied:*

- $\mu = 1$,
- $\mu = -1$ and $0 < \alpha < \frac{4s}{d}$,
- $\mu = -1$, $\alpha = \frac{4s}{d}$ and $\|u_0\|_{L^2}$ is small,
- $\mu = -1$ and $\|u_0\|_{H^s}$ is small.

Combining the local well-posedness for non-radial and radial initial data, we obtain the following summary.

	s	α	LWP
$d = 1$	$\frac{1}{3} < s < \frac{1}{2}$	$0 < \alpha < \frac{4s}{1-2s}$	u_0 non-radial
$d = 1$	$\frac{1}{2} < s < 1$	$0 < \alpha < \infty$	u_0 non-radial
$d = 2$	$\frac{1}{2} < s < 1$	$0 < \alpha < \frac{4s}{2-2s}$	u_0 non-radial
$d = 3$	$\frac{3}{5} \leq s \leq \frac{3}{4}$	$0 < \alpha < \frac{4s}{3-2s}$	u_0 radial
$d = 3$	$\frac{3}{4} < s < 1$	$0 < \alpha < \frac{4s}{3-2s}$	u_0 non-radial
$d \geq 4$	$\frac{d}{2d-1} \leq s < 1$	$0 < \alpha < \frac{4s}{d-2s}$	u_0 radial

TABLE 1. Local well-posedness (LWP) in H^s for NLS

Corollary 2.7 (Blowup rate). *Let d, α and $u_0 \in H^s$ be as in Table 1. Assume that the corresponding solution u to (1.1) given in Proposition 2.2 and Proposition 2.5 blows up at finite time $0 < T < \infty$. Then there exists $C > 0$ such that*

$$\|u(t)\|_{\dot{H}^s} > \frac{C}{(T-t)^{\frac{s-s_c}{2s}}}, \quad (2.13)$$

for all $0 < t < T$.

Proof. We follow the argument of Merle-Raphael [24]. Let $0 < t < T$ be fixed. We define

$$v_t(\tau, x) := \lambda^{\frac{2s}{\alpha}}(t)u(t + \lambda^{2s}(t)\tau, \lambda(t)x),$$

with $\lambda(t)$ to be chosen shortly. We see that v_t is well-defined for

$$t + \lambda^{2s}(t)\tau < T \quad \text{or} \quad \tau < \lambda^{-2s}(t)(T-t).$$

Moreover, v_t solves

$$i\partial_\tau v_t - (-\Delta)^s v_t = \mu|v_t|^\alpha v_t, \quad v_t(0) = \lambda^{\frac{2s}{\alpha}}(t)u(t, \lambda(t)x).$$

A direct computation shows

$$\|v_t(0)\|_{\dot{H}^s} = \lambda^{s-s_c}(t)\|u(t)\|_{\dot{H}^s}.$$

Since $s > s_c$, we choose $\lambda(t)$ so that $\|v_t(0)\|_{\dot{H}^s} = 1$. Thanks to the local theory, there exists $\tau_0 > 0$ such that v_t is defined on $[0, \tau_0]$. This shows that

$$\tau_0 < \lambda^{-2s}(t)(T-t) \quad \text{or} \quad \|u(t)\|_{\dot{H}^s} > \frac{\tau_0}{(T-t)^{\frac{s-s_c}{2s}}}.$$

The proof is complete. \square

3. PROFILE DECOMPOSITION

In this subsection, we use the profile decomposition for bounded consequences in H^s to show a compactness lemma related to the focusing mass-critical (1.1).

Theorem 3.1 (Profile decomposition). *Let $d \geq 1$ and $0 < s < 1$. Let $(v_n)_{n \geq 1}$ be a bounded sequence in H^s . Then there exist a subsequence of $(v_n)_{n \geq 1}$ (still denoted $(v_n)_{n \geq 1}$), a family $(x_n^j)_{j \geq 1}$ of sequences in \mathbb{R}^d and a sequence $(V^j)_{j \geq 1}$ of H^s functions such that*

- for every $k \neq j$,

$$|x_n^k - x_n^j| \rightarrow \infty, \quad \text{as } n \rightarrow \infty, \quad (3.1)$$

- for every $l \geq 1$ and every $x \in \mathbb{R}^d$,

$$v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x),$$

with

$$\limsup_{n \rightarrow \infty} \|v_n^l\|_{L^q} \rightarrow 0, \quad \text{as } l \rightarrow \infty, \quad (3.2)$$

for every $q \in (2, 2^*)$, where

$$2^* := \begin{cases} \frac{2}{1-2s} & \text{if } d = 1, s \in (0, \frac{1}{2}), \\ \infty & \text{if } d = 1, s \in [\frac{1}{2}, 1), \\ \frac{2d}{d-2s} & \text{if } d \geq 2, s \in (0, 1). \end{cases}$$

Moreover,

$$\|v_n\|_{L^2}^2 = \sum_{j=1}^l \|V^j\|_{L^2}^2 + \|v_n^l\|_{L^2}^2 + o_n(1), \quad (3.3)$$

$$\|v_n\|_{\dot{H}^s}^2 = \sum_{j=1}^l \|V^j\|_{\dot{H}^s}^2 + \|v_n^l\|_{\dot{H}^s}^2 + o_n(1), \quad (3.4)$$

as $n \rightarrow \infty$.

Proof. The proof is similar to the one given by Hmidi-Keraani [14, Proposition 3.1]. For reader's convenience, we recall some details. Since H^s is a Hilbert space, we denote $\Omega(v_n)$ the set of functions obtained as weak limits of sequences of the translated $v_n(\cdot + x_n)$ with $(x_n)_{n \geq 1}$ a sequence in \mathbb{R}^d . Denote

$$\eta(v_n) := \sup\{\|v\|_{L^2} + \|v\|_{\dot{H}^s} : v \in \Omega(v_n)\}.$$

Clearly,

$$\eta(v_n) \leq \limsup_{n \rightarrow \infty} \|v_n\|_{L^2} + \|v_n\|_{\dot{H}^s}.$$

We shall prove that there exist a sequence $(V^j)_{j \geq 1}$ of $\Omega(v_n)$ and a family $(x_n^j)_{j \geq 1}$ of sequences in \mathbb{R}^d such that for every $k \neq j$,

$$|x_n^k - x_n^j| \rightarrow \infty, \quad \text{as } n \rightarrow \infty,$$

and up to a subsequence, the sequence $(v_n)_{n \geq 1}$ can be written as for every $l \geq 1$ and every $x \in \mathbb{R}^d$,

$$v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x),$$

with $\eta(v_n^l) \rightarrow 0$ as $l \rightarrow \infty$. Moreover, the identities (3.3) and (3.4) hold as $n \rightarrow \infty$.

Indeed, if $\eta(v_n) = 0$, then we can take $V^j = 0$ for all $j \geq 1$. Otherwise we choose $V^1 \in \Omega(v_n)$ such that

$$\|V^1\|_{L^2} + \|V^1\|_{\dot{H}^s} \geq \frac{1}{2}\eta(v_n) > 0.$$

By the definition of $\Omega(v_n)$, there exists a sequence $(x_n^1)_{n \geq 1} \subset \mathbb{R}^d$ such that up to a subsequence,

$$v_n(\cdot + x_n^1) \rightharpoonup V^1 \text{ weakly in } H^s.$$

Set $v_n^1(x) := v_n(x) - V^1(x - x_n^1)$. We see that $v_n^1(\cdot + x_n^1) \rightharpoonup 0$ weakly in H^s and thus

$$\|v_n\|_{L^2}^2 = \|V^1\|_{L^2}^2 + \|v_n^1\|_{L^2}^2 + o_n(1),$$

$$\|v_n\|_{\dot{H}^s}^2 = \|V^1\|_{\dot{H}^s}^2 + \|v_n^1\|_{\dot{H}^s}^2 + o_n(1),$$

as $n \rightarrow \infty$. We now replace $(v_n)_{n \geq 1}$ by $(v_n^1)_{n \geq 1}$ and repeat the same process. If $\eta(v_n^1) = 0$, then we choose $V^j = 0$ for all $j \geq 2$. Otherwise there exist $V^2 \in \Omega(v_n^1)$ and a sequence $(x_n^2)_{n \geq 1} \subset \mathbb{R}^d$ such that

$$\|V^2\|_{L^2} + \|V^2\|_{\dot{H}^s} \geq \frac{1}{2}\eta(v_n^1) > 0,$$

and

$$v_n^1(\cdot + x_n^2) \rightharpoonup V^2 \text{ weakly in } H^s.$$

Set $v_n^2(x) := v_n^1(x) - V^2(x - x_n^2)$. We thus have $v_n^2(\cdot + x_n^2) \rightharpoonup 0$ weakly in H^s and

$$\|v_n^1\|_{L^2}^2 = \|V^2\|_{L^2}^2 + \|v_n^2\|_{L^2}^2 + o_n(1),$$

$$\|v_n^1\|_{\dot{H}^s}^2 = \|V^2\|_{\dot{H}^s}^2 + \|v_n^2\|_{\dot{H}^s}^2 + o_n(1),$$

as $n \rightarrow \infty$. We claim that

$$|x_n^1 - x_n^2| \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

In fact, if it is not true, then up to a subsequence, $x_n^1 - x_n^2 \rightarrow x_0$ as $n \rightarrow \infty$ for some $x_0 \in \mathbb{R}^d$. Since

$$v_n^1(x + x_n^2) = v_n^1(x + (x_n^2 - x_n^1) + x_n^1),$$

and $v_n^1(\cdot + x_n^1)$ converges weakly to 0, we see that $V^2 = 0$. This implies that $\eta(v_n^1) = 0$ and it is a contradiction. An argument of iteration and orthogonal extraction allows us to construct the family $(x_n^j)_{j \geq 1}$ of sequences in \mathbb{R}^d and the sequence $(V^j)_{j \geq 1}$ of H^s functions satisfying the claim above. Furthermore, the convergence of the series $\sum_{j \geq 1}^\infty \|V^j\|_{L^2}^2 + \|V^j\|_{\dot{H}^s}^2$ implies that

$$\|V^j\|_{L^2}^2 + \|V^j\|_{\dot{H}^s}^2 \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

By construction, we have

$$\eta(v_n^j) \leq 2 (\|V^{j+1}\|_{L^2} + \|V^{j+1}\|_{\dot{H}^s}),$$

which proves that $\eta(v_n^j) \rightarrow 0$ as $j \rightarrow \infty$. To complete the proof of Theorem 3.1, it remains to show (3.2). To do so, we introduce $\theta : \mathbb{R}^d \rightarrow [0, 1]$ satisfying $\theta(\xi) = 1$ for $|\xi| \leq 1$ and $\theta(\xi) = 0$ for $|\xi| \geq 2$. Given $R > 0$, define

$$\hat{\chi}_R(\xi) := \theta(\xi/R),$$

where $\hat{\cdot}$ is the Fourier transform of χ . In particular, we have $\hat{\chi}_R(\xi) = 1$ if $|\xi| \leq R$ and $\hat{\chi}_R(\xi) = 0$ if $|\xi| \geq 2R$. We write

$$v_n^l = \chi_R * v_n^l + (\delta - \chi_R) * v_n^l,$$

where $*$ is the convolution operator. Let $q \in (2, 2^*)$ be fixed. By Sobolev embedding and the Plancherel formula, we have

$$\begin{aligned} \|(\delta - \chi_R) * v_n^l\|_{L^q} &\lesssim \|(\delta - \chi_R) * v_n^l\|_{\dot{H}^\beta} \lesssim \left(\int |\xi|^{2\beta} |1 - \hat{\chi}_R(\xi)|^2 |\hat{v}_n^l(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim R^{\beta-s} \|v_n^l\|_{H^s}, \end{aligned}$$

where $\beta = \frac{d}{2} - \frac{d}{q} \in (0, s)$. On the other hand, the Hölder interpolation inequality implies

$$\begin{aligned} \|\chi_R * v_n^l\|_{L^q} &\lesssim \|\chi_R * v_n^l\|_{L^2}^{\frac{2}{q}} \|\chi_R * v_n^l\|_{L^\infty}^{1-\frac{2}{q}} \\ &\lesssim \|v_n^l\|_{L^2}^{\frac{2}{q}} \|\chi_R * v_n^l\|_{L^\infty}^{1-\frac{2}{q}}. \end{aligned}$$

Observe that

$$\limsup_{n \rightarrow \infty} \|\chi_R * v_n^l\|_{L^\infty} = \sup_{x_n} \limsup_{n \rightarrow \infty} |\chi_R * v_n^l(x_n)|.$$

Thus, by the definition of $\Omega(v_n^l)$, we infer that

$$\limsup_{n \rightarrow \infty} \|\chi_R * v_n^l\|_{L^\infty} \leq \sup \left\{ \left| \int \chi_R(-x)v(x)dx \right| : v \in \Omega(v_n^l) \right\}.$$

By the Plancherel formula, we have

$$\begin{aligned} \left| \int \chi_R(-x)v(x)dx \right| &= \left| \int \hat{\chi}_R(\xi)\hat{v}(\xi)d\xi \right| \lesssim \|\hat{\chi}_R\|_{L^2} \|v\|_{L^2} \\ &\lesssim R^{\frac{d}{2}} \|\theta\|_{L^2} \|v\|_{L^2} \lesssim R^{\frac{d}{2}} \eta(v_n^l). \end{aligned}$$

We thus obtain for every $l \geq 1$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|v_n^l\|_{L^q} &\lesssim \limsup_{n \rightarrow \infty} \|(\delta - \chi_R) * v_n^l\|_{L^q} + \limsup_{n \rightarrow \infty} \|\chi_R * v_n^l\|_{L^q} \\ &\lesssim R^{\beta-s} \|v_n^l\|_{H^s} + \|v_n^l\|_{L^2}^{\frac{2}{q}} \left[R^{\frac{d}{2}} \eta(v_n^l) \right]^{(1-\frac{2}{q})}. \end{aligned}$$

Choosing $R = [\eta(v_n^l)^{-1}]^{\frac{2}{d}-\epsilon}$ for some $\epsilon > 0$ small enough, we see that

$$\limsup_{n \rightarrow \infty} \|v_n^l\|_{L^q} \lesssim \eta(v_n^l)^{(s-\beta)(\frac{2}{d}-\epsilon)} \|v_n^l\|_{H^s} + \eta(v_n^l)^{\epsilon \frac{d}{2}(1-\frac{2}{q})} \|v_n^l\|_{L^2}^{\frac{2}{q}}.$$

Letting $l \rightarrow \infty$ and using the fact that $\eta(v_n^l) \rightarrow 0$ as $l \rightarrow \infty$ and the uniform boundedness in H^s of $(v_n^l)_{l \geq 1}$, we obtain

$$\limsup_{n \rightarrow \infty} \|v_n^l\|_{L^q} \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

The proof is complete. \square

We are now able to give the proof of the concentration compactness lemma given in Theorem 1.2.

Proof of Theorem 1.2. According to Theorem 3.1, there exist a sequence $(V^j)_{j \geq 1}$ of H^s functions and a family $(x_n^j)_{j \geq 1}$ of sequences in \mathbb{R}^d such that up to a subsequence, the sequence $(v_n)_{n \geq 1}$ can be written as

$$v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x),$$

and (3.2), (3.3), (3.4) hold. This implies that

$$\begin{aligned} m^{\frac{4s}{d}+2} &\leq \limsup_{n \rightarrow \infty} \|v_n\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} = \limsup_{n \rightarrow \infty} \left\| \sum_{j=1}^l V^j(\cdot - x_n^j) + v_n^l \right\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} \\ &\leq \limsup_{n \rightarrow \infty} \left(\left\| \sum_{j=1}^l V^j(\cdot - x_n^j) \right\|_{L^{\frac{4s}{d}+2}} + \|v_n^l\|_{L^{\frac{4s}{d}+2}} \right)^{\frac{4s}{d}+2} \\ &\leq \limsup_{n \rightarrow \infty} \left\| \sum_{j=1}^{\infty} V^j(\cdot - x_n^j) \right\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2}. \end{aligned} \quad (3.5)$$

By the elementary inequality

$$\left| \sum_{j=1}^l |a_j|^{\frac{4s}{d}+2} - \sum_{j=1}^l |a_j|^{\frac{4s}{d}+2} \right| \leq C \sum_{j \neq k} |a_j| |a_k|^{\frac{4s}{d}+1},$$

we have

$$\begin{aligned} \int \left| \sum_{j=1}^l V^j(x - x_n^j) \right|^{\frac{4s}{d}+2} dx &\leq \sum_{j=1}^l \int |V^j(x - x_n^j)|^{\frac{4s}{d}+2} dx + C \sum_{j \neq k} \int |V^j(x - x_n^j)| |V^k(x - x_n^k)|^{\frac{4s}{d}+1} dx \\ &\leq \sum_{j=1}^l \int |V^j(x - x_n^j)|^{\frac{4s}{d}+2} dx + C \sum_{j \neq k} \int |V^j(x + x_n^k - x_n^j)| |V^k(x)|^{\frac{4s}{d}+1} dx. \end{aligned}$$

Using the pairwise orthogonality (3.1), the Hölder inequality implies that $V^j(\cdot + x_n^k - x_n^j) \rightarrow 0$ in H^s as $n \rightarrow \infty$ for any $j \neq k$. This leads to the mixed terms in the sum (3.5) vanish as $n \rightarrow \infty$. We thus get

$$m^{\frac{4s}{d}+2} \leq \sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2}.$$

We next use the sharp Gagliardo-Nirenberg inequality (1.4) to estimate

$$\sum_{j=1}^{\infty} \|V^j\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} \leq \frac{2s+d}{d} \|Q\|_{L^2}^{-\frac{4s}{d}} \sup_{j \geq 1} \|V^j\|_{L^2}^{\frac{4s}{d}} \sum_{j=1}^{\infty} \|V^j\|_{H^s}^2. \quad (3.6)$$

By (3.4), we infer that

$$\sum_{j=1}^{\infty} \|V^j\|_{H^s}^2 \leq \limsup_{n \rightarrow \infty} \|v_n\|_{H^s}^2 \leq M^2.$$

Therefore,

$$\sup_{j \geq 1} \|V^j\|_{L^2}^{\frac{4s}{d}} \geq \frac{d}{d+2s} \frac{m^{\frac{4s}{d}+2}}{M^2} \|Q\|_{L^2}^{\frac{4s}{d}}.$$

Since the series $\sum_{j \geq 1} \|V^j\|_{L^2}^2$ is convergent, the supremum above is attained. In particular, there exists j_0 such that

$$\|V^{j_0}\|_{L^2}^{\frac{4s}{d}} \geq \frac{d}{d+2s} \frac{m^{\frac{4s}{d}+2}}{M^2} \|Q\|_{L^2}^{\frac{4s}{d}}.$$

By a change of variables, we write

$$v_n(x + x_n^{j_0}) = V^{j_0}(x) + \sum_{\substack{1 \leq j \leq l \\ j \neq j_0}} V^j(x + x_n^{j_0} - x_n^j) + \tilde{v}_n^l(x),$$

where $\tilde{v}_n^l(x) := v_n^l(x + x_n^{j_0})$. The pairwise orthogonality of the family $(x_n^j)_{j \geq 1}$ implies

$$V^j(\cdot + x_n^{j_0} - x_n^j) \rightharpoonup 0 \text{ weakly in } H^s,$$

as $n \rightarrow \infty$ for every $j \neq j_0$. We thus get

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0} + \tilde{v}^l, \quad \text{as } n \rightarrow \infty, \quad (3.7)$$

where \tilde{v}^l is the weak limit of $(\tilde{v}_n^l)_{n \geq 1}$. On the other hand,

$$\|\tilde{v}^l\|_{L^{\frac{4s}{d}+2}} \leq \limsup_{n \rightarrow \infty} \|\tilde{v}_n^l\|_{L^{\frac{4s}{d}+2}} = \limsup_{n \rightarrow \infty} \|v_n^l\|_{L^{\frac{4s}{d}+2}} \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

By the uniqueness of the weak limit (3.7), we get $\tilde{v}^l = 0$ for every $l \geq j_0$. Therefore, we obtain

$$v_n(\cdot + x_n^{j_0}) \rightharpoonup V^{j_0}.$$

The sequence $(x_n^{j_0})_{n \geq 1}$ and the function V^{j_0} now fulfill the conditions of Theorem 1.2. The proof is complete. \square

4. BLOWUP CONCENTRATION

Theorem 4.1 (Blowup concentration). *Let*

$$\left\{ \begin{array}{l} d = 1, \quad s \in (\frac{1}{3}, 1) \setminus \{\frac{1}{2}\}, \quad \alpha = 4s, \quad u_0 \in H^s \text{ non-radial,} \\ d = 2, \quad s \in (\frac{1}{2}, 1), \quad \alpha = 2s, \quad u_0 \in H^s \text{ non-radial,} \\ d = 3, \quad s \in [\frac{3}{5}, \frac{3}{4}], \quad \alpha = \frac{4s}{3}, \quad u_0 \in H^s \text{ radial,} \\ d = 3, \quad s \in (\frac{3}{4}, 1), \quad \alpha = \frac{4s}{3}, \quad u_0 \in H^s \text{ non-radial,} \\ d \geq 4, \quad s \in [\frac{d}{2d-1}, 1), \quad \alpha = \frac{4s}{d}, \quad u_0 \in H^s \text{ radial.} \end{array} \right. \quad (4.1)$$

Assume that the corresponding solution u to (1.1) blows up at finite time $0 < T < \infty$. Let $a(t) > 0$ be such that

$$a(t) \|u(t)\|_{H^s}^{\frac{1}{s}} \rightarrow \infty, \quad (4.2)$$

as $t \uparrow T$. Then there exists $x(t) \in \mathbb{R}^d$ such that

$$\liminf_{t \uparrow T} \int_{|x-x(t)| \leq a(t)} |u(t, x)|^2 dx \geq \int |Q(x)|^2 dx, \quad (4.3)$$

where Q is the unique solution to (1.3).

Remark 4.2. • The condition (4.1) comes from the local theory (see Table 1).

- By the blowup rate given in Corollary 2.7, we have

$$\|u(t)\|_{\dot{H}^s} > \frac{C}{\sqrt{T-t}},$$

for $t \uparrow T$. Rewriting

$$\frac{1}{a(t)\|u(t)\|_{\dot{H}^s}^{\frac{1}{s}}} = \frac{2^s \sqrt{T-t}}{a(t)} \frac{1}{(\sqrt{T-t}\|u(t)\|_{\dot{H}^s})^{\frac{1}{s}}} < C \frac{2^s \sqrt{T-t}}{a(t)},$$

we see that any function $a(t) > 0$ satisfying $\frac{2^s \sqrt{T-t}}{a(t)} \rightarrow 0$ as $t \uparrow T$ fulfills the conditions of Theorem 4.1.

Proof of Theorem 4.1. Let $(t_n)_{n \geq 1}$ be a sequence such that $t_n \uparrow T$ and $g \in \mathcal{G}$. Set

$$\lambda_n := \left(\frac{\|Q\|_{\dot{H}^s}}{\|u(t_n)\|_{\dot{H}^s}} \right)^{\frac{1}{s}}, \quad v_n(x) := \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n x).$$

By the blowup alternative, we see that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, we have

$$\|v_n\|_{L^2} = \|u(t_n)\|_{L^2} = \|u_0\|_{L^2},$$

and

$$\|v_n\|_{\dot{H}^s} = \lambda_n^s \|u(t_n)\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s},$$

and

$$E(v_n) = \lambda_n^{2s} E(u(t_n)) = \lambda_n^{2s} E(u_0) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This implies in particular that

$$\|v_n\|_{L^{\frac{4s}{d}+2}} \rightarrow \frac{d+2s}{d} \|Q\|_{\dot{H}^s}^2, \quad \text{as } n \rightarrow \infty.$$

The sequence $(v_n)_{n \geq 1}$ satisfies the conditions of Theorem 1.2 with

$$m^{\frac{4s}{d}+2} = \frac{d+2s}{d} \|Q\|_{\dot{H}^s}^2, \quad M^2 = \|Q\|_{\dot{H}^s}^2.$$

Therefore, there exists a sequence $(x_n)_{n \geq 1}$ in \mathbb{R}^d such that up to a subsequence,

$$v_n(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V \text{ weakly in } H^s,$$

as $n \rightarrow \infty$ with $\|V\|_{L^2} \geq \|Q\|_{L^2}$. In particular,

$$v(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V \text{ weakly in } L^2.$$

This implies for every $R > 0$,

$$\liminf_{n \rightarrow \infty} \int_{|x| \leq R} \lambda_n^{\frac{d}{2}} |u(t_n, \lambda_n x + x_n)|^2 dx \geq \int_{|x| \leq R} |V(x)|^2 dx,$$

or

$$\liminf_{n \rightarrow \infty} \int_{|x-x_n| \leq R\lambda_n} |u(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V(x)|^2 dx.$$

Since

$$a(t_n)\|u(t_n)\|_{\dot{H}^s}^{\frac{1}{s}} = \frac{a(t_n)}{\lambda_n}\|Q\|_{\dot{H}^s}^{\frac{1}{s}},$$

the assumption (4.2) implies $\frac{a(t_n)}{\lambda_n} \rightarrow \infty$ as $n \rightarrow \infty$. We thus get

$$\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V(x)|^2 dx,$$

for every $R > 0$, which means that

$$\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t_n)} |u(t_n, x)|^2 dx \geq \int |V(x)|^2 dx \geq \int |Q(x)|^2 dx.$$

Since the sequence $(t_n)_{n \geq 1}$ is arbitrary, we infer that

$$\liminf_{t \uparrow T} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx \geq \int |Q(x)|^2 dx.$$

But for every $t \in (0, T)$, the function $y \mapsto \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx$ is continuous and goes to zero at infinity. As a result, we get

$$\sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx = \int_{|x-x(t)| \leq a(t)} |u(t, x)|^2 dx,$$

for some $x(t) \in \mathbb{R}^d$. This shows (4.3). The proof is complete. \square

5. LIMITING PROFILE WITH MINIMAL MASS

Let us start with the following characterization of solution with minimal mass.

Lemma 5.1. *Let $d \geq 1$ and $0 < \alpha < 1$. If $u \in H^s$ is such that $\|u\|_{L^2} = \|Q\|_{L^2}$ and $E(u) = 0$, then u is of the form*

$$u(x) = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda x + x_0),$$

for some $\theta \in \mathbb{R}, \lambda > 0$ and $x_0 \in \mathbb{R}^d$.

Proof. Since $E(u) = 0$, we have

$$\|u\|_{\dot{H}^s}^2 = \frac{d}{d+2s} \|u\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2}.$$

Thus

$$\frac{\|u\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2}}{\|u\|_{L^2}^{\frac{4s}{d}} \|u\|_{\dot{H}^s}^2} = \frac{d+2s}{d} \|u\|_{L^2}^{-\frac{4s}{d}} = \frac{d+2s}{d} \|Q\|_{L^2}^{-\frac{4s}{d}} = C_{\text{GN}},$$

where C_{GN} is the sharp constant in (1.4). By the characterization of the sharp constant to the Gagliardo-Nirenberg inequality (1.4) (see e.g. [10, Section 3]), we learn that u is of the form $u(x) = aQ(\lambda x + x_0)$ for some $a \in \mathbb{C}^*$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$. On the other hand, since $\|u\|_{L^2} = \|Q\|_{L^2}$, we have $|a| = \lambda^{\frac{d}{2}}$. This shows the result. \square

We now have the following limiting profile of blowup solutions with minimal mass.

Theorem 5.2 (Limiting profile with minimal mass). *Let d, s, α and u_0 be as in (4.1). Assume that the corresponding solution u to (1.1) blows up at finite time $0 < T < \infty$. If $\|u_0\|_{L^2} = \|Q\|_{L^2}$, then there exist $\theta(t) \in \mathbb{R}, \lambda(t) > 0$ and $x(t) \in \mathbb{R}^d$ such that*

$$e^{i\theta(t)} \lambda^{\frac{d}{2}}(t) u(t, \lambda(t) \cdot + x(t)) \rightarrow Q \text{ strongly in } H^s,$$

as $t \uparrow T$.

Proof. We will show that for any $(t_n)_{n \geq 1}$ satisfying $t_n \uparrow T$, there exist a subsequence still denoted by $(t_n)_{n \geq 1}$, sequences of $\theta_n \in \mathbb{R}, \lambda_n > 0$ and $x_n \in \mathbb{R}^d$ such that

$$e^{it\theta_n} \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightarrow Q \text{ strongly in } H^s \text{ as } n \rightarrow \infty. \quad (5.1)$$

Let $(t_n)_{n \geq 1}$ be a sequence such that $t_n \uparrow T$. Set

$$\lambda_n := \left(\frac{\|Q\|_{\dot{H}^s}}{\|u(t_n)\|_{\dot{H}^s}} \right)^{\frac{1}{s}}, \quad v_n(x) := \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n x).$$

By the blowup alternative, we see that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, we have

$$\|v_n\|_{L^2} = \|u(t_n)\|_{L^2} = \|u_0\|_{L^2} = \|Q\|_{L^2}, \quad (5.2)$$

and

$$\|v_n\|_{\dot{H}^s} = \lambda_n^s \|u(t_n)\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s}, \quad (5.3)$$

and

$$E(v_n) = \lambda_n^{2s} E(u(t_n)) = \lambda_n^{2s} E(u_0) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This yields in particular that

$$\|v_n\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} \rightarrow \frac{d+2s}{d} \|Q\|_{\dot{H}^s}^2, \quad \text{as } n \rightarrow \infty. \quad (5.4)$$

The sequence $(v_n)_{n \geq 1}$ satisfies the conditions of Theorem 1.2 with

$$m^{\frac{4s}{d}+2} = \frac{d+2s}{d} \|Q\|_{\dot{H}^s}^2, \quad M^2 = \|Q\|_{\dot{H}^s}^2.$$

Therefore, there exists a sequence $(x_n)_{n \geq 1}$ in \mathbb{R}^d such that up to a subsequence,

$$v_n(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V \text{ weakly in } H^s,$$

as $n \rightarrow \infty$ with $\|V\|_{L^2} \geq \|Q\|_{L^2}$. Since $v_n(\cdot + x_n) \rightharpoonup V$ weakly in H^s as $n \rightarrow \infty$, the semi-continuity of weak convergence and (5.2) imply

$$\|V\|_{L^2} \leq \liminf_{n \rightarrow \infty} \|v_n\|_{L^2} \leq \|Q\|_{L^2}.$$

This together with the fact $\|V\|_{L^2} \geq \|Q\|_{L^2}$ show that

$$\|V\|_{L^2} = \|Q\|_{L^2} = \lim_{n \rightarrow \infty} \|v_n\|_{L^2}. \quad (5.5)$$

Therefore

$$v_n(\cdot + x_n) \rightarrow V \text{ strongly in } L^2 \text{ as } n \rightarrow \infty.$$

On the other hand, the Gagliardo-Nirenberg inequality (1.4) shows that $v_n(\cdot + x_n) \rightarrow V$ strongly in $L^{\frac{4s}{d}+2}$ as $n \rightarrow \infty$. Indeed, by (5.3),

$$\begin{aligned} \|v_n(\cdot + x_n) - V\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} &\lesssim \|v_n(\cdot + x_n) - V\|_{L^2}^{\frac{4s}{d}} \|v_n(\cdot + x_n) - V\|_{\dot{H}^s}^2 \\ &\lesssim (\|Q\|_{\dot{H}^s} + \|V\|_{\dot{H}^s})^2 \|v_n(\cdot + x_n) - V\|_{L^2}^{\frac{4s}{d}} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Moreover, using (5.4) and (5.5), the sharp Gagliardo-Nirenberg inequality (1.4) yields

$$\|Q\|_{\dot{H}^s}^2 = \frac{d}{d+2s} \lim_{n \rightarrow \infty} \|v_n\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} = \frac{d}{d+2s} \|V\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} \leq \left(\frac{\|V\|_{L^2}}{\|Q\|_{L^2}} \right)^{\frac{4s}{d}} \|V\|_{\dot{H}^s}^2 = \|V\|_{\dot{H}^s}^2,$$

or $\|Q\|_{\dot{H}^s} \leq \|V\|_{\dot{H}^s}$. By the semi-continuity of weak convergence and (5.3),

$$\|V\|_{\dot{H}^s} \leq \liminf_{n \rightarrow \infty} \|v_n\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s}.$$

Therefore,

$$\|V\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s} = \lim_{n \rightarrow \infty} \|v_n\|_{\dot{H}^s}. \quad (5.6)$$

Combining (5.5), (5.6) and using the fact $v_n(\cdot + x_n) \rightharpoonup V$ weakly in H^s , we conclude that

$$v_n(\cdot + x_n) \rightarrow V \text{ strongly in } H^s \text{ as } n \rightarrow \infty.$$

In particular, we have

$$E(V) = \lim_{n \rightarrow \infty} E(v_n) = 0.$$

This shows that there exists $V \in H^s$ such that

$$\|V\|_{L^2} = \|Q\|_{L^2}, \quad E(V) = 0.$$

By Lemma 5.1, we have $V(x) = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda x + x_0)$ for some $\theta \in \mathbb{R}, \lambda > 0$ and $x_0 \in \mathbb{R}^d$. Thus

$$v_n(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightarrow V = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda \cdot + x_0) \text{ strongly in } H^s \text{ as } n \rightarrow \infty.$$

Redefining variables as

$$\bar{\lambda}_n := \lambda_n \lambda^{-1}, \quad \bar{x}_n := \lambda_n \lambda^{-1} x_0 + x_n,$$

we get

$$e^{-i\theta} \bar{\lambda}_n^{\frac{d}{2}} u(t_n, \bar{\lambda}_n \cdot + \bar{x}_n) \rightarrow Q \text{ strongly in } H^s \text{ as } n \rightarrow \infty.$$

This proves (5.1) and the proof is complete. \square

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