Stability of nonnegative isotropic curvature under continuous deformations of the metric

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Abstract

Using a method introduced by R. Bamler to study the behavior of scalar curvature under continuous deformations of Riemannian metrics, we prove that if a sequence of smooth Riemannian metrics g_i on a fixed compact manifold M has isotropic curvature bounded from below by a nonnegative function $\underline{u}$, and if g_i converge in C^0 norm to a smooth metric g, then g has isotropic curvature bounded from below by $\underline{u}$. The proof also works for various other bounds from below on the curvature, such has non-negative curvature operator.

A major trend in modern Riemannian geometry is to understand the geometry of a metric space (X, d) which is a limit of a sequence of smooth manifolds $(M_i, g_i)_{i\in\mathbb{N}}$ in terms of given geometric information on the elements of the sequence $(M_i, g_i)_{i\in\mathbb{N}}$. In a lot of cases $(M_i, g_i)_{i\in\mathbb{N}}$ is supposed to converge to (X, d) in the Gromov–Hausdorff (abbreviated as GH) sense. In this context one can show the following results:

- If $(M_i, g_i)_{i\in\mathbb{N}}$ GH converges to a smooth manifold (M, g) and each (M_i, g_i) has sectionnal curvature greater than K, then (M, g) has sectionnal curvature greater than K. This follows from the synthetic theory of lower sectionnal curvature bounds as exposed in [BBI01].
- If $(M_i^n, g_i)_{i\in\mathbb{N}}$ GH converges to a smooth manifold (M^n, g) and each (M_i^n, g_i) has Ricci curvature greater than K, then (M^n, g) has Ricci curvature greater than K. Note that collapsing is ruled out in this case. The result is a consequence of the theory of $CD(K, N)$ spaces (see [LV09] and [Stu05]).

*Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050), UPEM, UPEC, CNRS, F-94010, Créteil, France.
e-mail: thomas.richard@u-pec.fr

The case of lower bounds on the scalar curvature is much less clear. As the example of the product of a radius $1/i$ 2-sphere with a negative scalar curvature manifold shows, a result as above cannot be true under the sole assumption of Gromov-Hausdorff convergence, at least if collapsing is allowed. Example 6.1 in the preprint [BDS17] is actually a 3 dimensional singular limit of a sequence of 3-manifolds with positive scalar curvature which cannot be reasonably thought of as having positive scalar curvature.

In [Gro14], Gromov layed the first bricks of a possible synthetic theory of lower bounds on scalar curvature and proved :

Theorem 0.1 (Gromov [Gro14])**.** *Let M be a smooth manifold, $\underline{u} : M \to \mathbb{R}$ be a continuous function, and $(g_i)_{i\in\mathbb{N}}$ be a sequence of possibly non-complete C^2 Riemannian metrics. Assume that:*

- *For every $i \in \mathbb{N}$, $\mathrm{Scal}_{g_i}(x) \geq \underline{u}(x)$.*
- *g_i converges in C^0 norm to a C^2 Riemannian metric g.*

Then $\mathrm{Scal}_g(x) \geq \underline{u}(x)$.

Here and in the rest of the paper, by "g_i converges in C^0 norm to g" we will mean that $\sup_M |g_i - g|$ goes to zero as i goes to infinity where the pointwise norm of the 2-tensors $g_i - g$ is taken with respect to a fixed background Riemannian metric.

Gromov's proof uses arguments from the theory of minimal hypersurfaces. In [Bam16], Bamler gave an alternative proof of this result using Ricci flow. In this paper we will adapt Bamler's method to other type of curvature lower bounds and show the following theorem:

Theorem 0.2. *Let M be a compact smooth manifold and $\underline{u}$ be a lower semi-continuous nonnegative function on M. Let g_i be a sequence of complete smooth metrics with bounded curvature on M which converges in C^0 norm to a smooth metric g. Then:*

- *if for every $i \in \mathbb{N}$ the lowest eigenvalue of the curvature operator of g_i at any $x \in M$ is bigger than $\underline{u}(x)$, then the lowest eigenvalue of the curvature operator of g at any $x \in M$ is bigger than $\underline{u}(x)$.*
- *if for every $i \in \mathbb{N}$ the isotropic curvature of g_i at any $x \in M$ is bigger than $\underline{u}(x)$, then the isotropic curvature[1] of g at any $x \in M$ is bigger than $\underline{u}(x)$.*

Then $\mathrm{R}_g - \underline{u}\,\mathrm{I} \in \mathcal{C}$.

This will actually follow from the more general Theorem 0.4 below.

To state it, we need to recall some notation: let A and B be two symmetric tensors on an inner product space E, we define the algebraic curvature tensor $A \wedge B : \Lambda^2 E \to \Lambda^2 E$ by:

$$(A \wedge B)(x \wedge y) = \tfrac{1}{2}\left(Ax \wedge By + Bx \wedge Ay\right).$$

We will also denote by $\mathrm{I} : \Lambda^2 E \to \Lambda^2 E$ the identity operator.

[1] We say that (M, g) has isotropic curvature greater than k at $x \in M$ if, for any orthonormal 4-frame $(e_1, \ldots, e_4)$ of $T_x M$, $R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4) + R(e_2, e_3, e_2, e_3) + R(e_2, e_4, e_2, e_4) - 2R(e_1, e_2, e_3, e_4) \geq k$. See Appendix B for more details.

Definition 0.3. *A curvature cone $\mathcal{C} \subset S^2_B\Lambda^2\mathbb{R}^n$ is said to satisfy condition $(*)$ if:*

- *$\mathcal{C}$ is Ricci flow invariant.*
- *If $\mathrm{R} \in \mathcal{C}$ then $\mathrm{Ric}(\mathrm{R}) \wedge \mathrm{id} \subset \mathcal{C}$.*

The notion of Ricci flow invariant curvature cone is recalled in appendix A.
We mention here two examples of curvature cones which satisfy conditon $(*)$:

- The cone of curvature operators with positive isotropic curvature satisfies $(*)$. This follows from [Ngu10] or [BS09] and the two lemmas proven in the appendix.
- if $\mathcal{C}$ is Ricci flow invariant and $\{\mathrm{R} \geq 0\} \subset \mathcal{C} \subset \{\mathrm{Ric}(\mathrm{R}) \geq 0\}$, then it satisfies condition $(*)$: indeed the eigenvalues of $\mathrm{Ric}(\mathrm{R}) \wedge \mathrm{id}$ are $\frac{\lambda_i+\lambda_j}{2}$ where the λ_i's are the eigenvalues of $\mathrm{Ric}(\mathrm{R})$ which are nonnegative by assumption. This includes the cones of nonnegative curvature operators, 2-nonnegative curvature operators as well as the NNIC1 and NNIC2 cones.

Theorem 0.4. *Let $\mathcal{C}$ be a curvature cone which satisfies condition $(*)$. Let M be a compact smooth manifold and $\underline{u}$ be a lower semi-continuous nonnegative function on M. Let g_i be a sequence of complete smooth metrics with bounded curvature on M such that:*

- *As i goes to infinity, g_i converges in C^0 norm to a smooth metric g.*
- *For every i, the curvature operator of g_i satisfies $\mathrm{R}_{g_i} - \underline{u}\,\mathrm{I} \in \mathcal{C}$.*

Then $\mathrm{R}_g - \underline{u}\,\mathrm{I} \in \mathcal{C}$.

Picking for $\mathcal{C}$ the NNIC cone or the cone of nonnegative curvature operators, we recover Theorem 0.2

The proof of this result follows roughly Bamler's proof in [Bam16] with some differences:

- the evolution equation for the curvature operator along the Ricci flow is not as nice as the evolution equation for the scalar curvature, this is why we have not been able to handle the case of a lower bound $\underline{u}$ of arbitrary sign.
- we require compactness of the manifold M because we are not able to localize the argument as Bamler did. Our proof would work if we assumed completeness and bounded curvature of all the metrics involved, in order to be able to apply the maximum principle.
- we study the behavior of the heat flow as t goes to 0 using hessian estimates based on the maximum principle, whereas Bamler uses heat kernel estimates. This choice has been made to make the proof more self-contained.

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1. Bounding the curvature from below by the heat flow

Proposition 1.1. *Let $\mathcal{C}$ be a curvature cone which satisfy condition $(*)$. Let $(M, g(t))_{t\in[0,T)}$ be a solution to the Ricci flow and $u(t,\cdot)$ be a nonnegative solution to the heat equation such that:*

$$\mathrm{R}_{g(0)} - u(0,\cdot)\,\mathrm{I} \in \mathcal{C}$$

Then

$$\mathrm{R}_{g(t)} - u(t,\cdot)\,\mathrm{I} \in \mathcal{C}$$

for all $t \in [0,T)$

Proof. We will apply Hamilton's maximum principle (see [Ham86], Theorem 4.3) to $\mathrm{L}(t) = \mathrm{R}_{g(t)} - u(t,\cdot)\,\mathrm{I}$. It satisfies the evolution equation:

$$\begin{aligned}(\partial_t - \Delta)\,\mathrm{L} &= 2Q(\mathrm{R})\\ &= 2Q(\mathrm{L}) + 4u\,Q(\mathrm{L},\mathrm{I}) + 2u^2 Q(\mathrm{I})\\ &= 2Q(\mathrm{L}) + 4u\,\mathrm{Ric}(\mathrm{L}) \wedge \mathrm{id} + 2(n-1)u^2\,\mathrm{I}\end{aligned}$$

where $Q(\mathrm{R}) = \mathrm{R}^2 + \mathrm{R}^{\#}$ and we have used the notations and results from [BW08].

$(n-1)u^2\,\mathrm{I} \in \mathcal{C}$ since $\mathcal{C}$ is a curvature cone, $2u\,\mathrm{Ric}(\mathrm{L}) \wedge \mathrm{id} \in \mathcal{C}$ whenever $\mathrm{L} \in \mathcal{C}$ by condition $(*)$ and the fact that u is nonnegative, $Q(\mathrm{L}) \in T_{\mathrm{L}}\mathcal{C}$ whenever $\mathrm{L} \in \mathcal{C}$ since $\mathcal{C}$ is Ricci flow invariant. Thus we have that $(\partial_t - \Delta)\,\mathrm{L} \in T_{\mathrm{L}}\mathcal{C}$ whenever $\mathrm{L} \in \mathcal{C}$. Hamilton's maximum principe implies that $\mathrm{L}(t) \in \mathcal{C}$ if $\mathrm{L}(0) \in \mathcal{C}$. □

2. A variation on Koch and Lamm's Ricci flow of C^0 metrics.

We review here the theory of Ricci flow of C^0 metrics develloped by Koch and Lamm in [KL15] and [KL12]. The precise version of the theory we need here is formulated in terms of Ricci flow background rather than a static background metric, and is very similar to the theory developped by Deruelle and Lamm in [DL16] formulated with an expanding Ricci soliton as a background.

Let $g(t)_{t\in[0,T)}$ be a Ricci flow on a compact manifold M. We consider the following problem:

$$\begin{cases}\partial_t \bar{g} = -2\,\mathrm{Ric}_{\bar{g}} + \mathcal{L}_{W(\bar{g},g)}\bar{g}\\ \bar{g}(0) = \bar{g}_0\end{cases} \tag{RicDT}$$

where $W(\bar{g}, g)$ is the vector field given in local coordinates by: $W^k = g^{ij}\left(\bar{\Gamma}^k_{ij} - \Gamma^k_{ij}\right)$ where $\bar{\Gamma}^k_{ij}$ and Γ^k_{ij} are the Christoffel symbols of $\bar{g}(t)$ and $g(t)$ (respectively).

If we set $h = \bar{g} - g$, a computation shows that the equation above is equivalent to:

$$\begin{aligned}(\partial_t - \Delta_{L,g(t)})h =& \nabla_l\left(\left((g+h)^{lm} - g^{lm}\right)\nabla_m h_{ik}\right)\\ &+ \bar{g}^{-1} \circledast \bar{g}^{-1} \circledast \nabla h \circledast \nabla h\\ &+ \left((g+h)^{-1} - g^{-1}\right) \circledast R_{g(t)} \circledast h\\ =& \mathcal{F}[h]\end{aligned}$$

where ∇ denotes the covariant derivative with respect to $g(t)$, $R_{g(t)}$ is the curvature tensor of $g(t)$ and $\Delta_{L,g(t)}$ is the Lichnerowicz Laplacian with respect to $g(t)$ (see [CLN06], p. 109). If A and B are tensors, $A \circledast B$ denotes any tensor built by tracing $A \otimes B$.

Following [KL12], [KL15] and [DL16], let $X_{T'}$ be the space of time dependent symmetric 2-tensors h which satisfy:

$$\|h\|_X = \sup_{M\times[0,T')} |h| + \sup_{(x,R^2)\in M\times(0,T')} \left(R\|\bar{\nabla}h\|_{2,x,R} + \|\sqrt{t}\bar{\nabla}h\|^+_{n+4,x,R} \right) < +\infty$$

where:

$$\|h\|_{p,x,R} = \left(\frac{\displaystyle\int_0^{R^2} \int_{B_{\bar{g}(s)}(x,R)} |h|^p dv_{\bar{g}(s)} ds}{\displaystyle\int_0^{R^2} \mathrm{vol}_{\bar{g}(s)}\, B_{\bar{g}(s)}(x,R) ds} \right)^{1/p}$$

$$\|h\|^+_{p,x,R} = \left(\frac{\displaystyle\int_{\frac{1}{2}R^2}^{R^2} \int_{B_{\bar{g}(s)}(x,R)} |h|^p dv_{\bar{g}(s)} ds}{\displaystyle\int_{\frac{1}{2}R^2}^{R^2} \mathrm{vol}_{\bar{g}(s)}\, B_{\bar{g}(s)}(x,R) ds} \right)^{1/p}.$$

For a fixed initial condition h_0 small enough in C^0 norm, one can consider the map Φ, defined on $X_{T'}$ for T' small enough, which sends a small enough time dependent symmetric 2-tensor $k(t)$ to the solution $h(t)$ of the linear problem:

$$\begin{cases} (\partial_t - \Delta_{L,g(t)})h = \mathcal{F}[k] \\ h(0) = h_0. \end{cases}$$

Lemma 2.1. *Provided T' and the C^0 norm of h_0 are small enough, $\Phi : X_{T'} \to X_{T'}$ is a strict contraction from a small ball in $X_{T'}$ to itself.*

The proof of this result follows the same route as the proof of Theorem 4.3 in [KL12], with adjustments needed to handle the fact that the background metric is evolving by Ricci flow. These adjustments have been carried out in details in [DL16], though our situation requires much less delicate estimates since we are only interested in short time existence and uniqueness, whereas [DL16] study the long time behavior of the solutions.

Banach's fixed point theorem can then be applied to show that Φ has a unique fixed point and get the following existence and uniqueness theorem:

Theorem 2.2. *Let (M, g_0) be a compact smooth manifold with a smooth riemannian metric, let $(g(t))_{t\in[0,T)}$ be the smooth Ricci flow starting from g_0.*

Then there exists $\varepsilon > 0$ and $T' \in (0, T)$ such that if $\bar{g}_0$ is a metric with $|\bar{g}_0 - g_0| < \varepsilon$ then there exists a solution $(\bar{g}(t))_{t\in[0,T')}$ to the equation:

$$\partial_t \bar{g}(t) = -2\,\mathrm{Ric}_{\bar{g}(t)} + \mathcal{L}_{W(\bar{g}(t),g(t))}\bar{g}(t)$$

with $\bar{g}(0) = \bar{g}_0$.

Moreover for every $k > 0$*, there exist constants* $C_k > 0$ *such that:*

$$t^{k/2} \sup_M |\nabla^k(\bar{g}(t) - g(t))| \leq C_k \sup_M |\bar{g}_0 - g_0| \tag{1}$$

for any $t \in (0, T')$.

The solution $\bar{g}(t)$ *is unique among all solutions which satisfy the above estimate for* $k = 0$ *and* $k = 1$.

3. Hölder continuity of the heat flow in a Ricci flow background

In this section, we will show an estimate on solutions to the heat equation which will be used to control how fast a solution can deviate from its initial contion.

We start with the following well known lemma:

Lemma 3.1. *Let* $(M, g(t))$ *be a solution to the Ricci flow and* $u(t)$ *be a* C^2 *solution to the heat equation* $\partial_t u = \Delta_{g(t)} u$ *then :*

$$\sup_M |\nabla u(t)| \leq \sup_M |\nabla u(0)|.$$

Proof. Once one notices that Bochner's formule implies

$$(\partial_t - \Delta_{g(t)})|\nabla u(t)|^2 = -2|\nabla^2 u(t)|^2 \leq 0,$$

this is a straightforward consequence of the maximum principle. □

Proposition 3.2. *Let* $(M, g(t))$ *be Ricci flow such that* $\sup_M |\mathrm{R}_{g(t)}| \leq A/t$ *for some* $A > 0$*, let* $u(t)$ *be a* C^2 *solution to* $\partial_t u = \Delta_{g(t)} u$*. Then for every* $\alpha > 1/2$ *there exists* $T'(A, \alpha) \in (0, T)$ *such that for every* $t \in (0, T')$*:*

$$\sup_M |\nabla^2 u(t)| \leq \frac{\sup_M |\nabla u(0)|}{t^\alpha}.$$

Proof. We first compute, using the evolution equation for the Hessian given in [CLN06] Lemma 2.33:

$$\begin{aligned}(\partial_t - \Delta_{g(t)})|\nabla^2 u|^2 &= \mathrm{Ric}_{g(t)} * \nabla^2 u * \nabla^2 u + 2\left\langle (\Delta_L - \Delta)\nabla^2 u, \nabla^2 u \right\rangle - 2|\nabla^3 u|^2 \\ &= \mathrm{R}_{g(t)} * \nabla^2 u * \nabla^2 u - 2|\nabla^3 u|^2 \\ &\leq \frac{C_n A}{t}|\nabla^2 u|^2\end{aligned}$$

where C_n is a dimensional constant.

We now fix $\alpha > 1/2$ and set $\delta = 2\alpha - 1 > 0$ and $F(t) = t^{1+\delta}|\nabla^2 u|^2 + |\nabla u|^2$. F satisfies:

$$(\partial_t - \Delta_{g(t)})F \leq \left(C_n A t^\delta + (1+\delta)t^\delta - 2\right)|\nabla^2 u|^2.$$

The right hand side is negative for $t \leq \left(\frac{2}{C_n A+1+\delta}\right)^{1/\delta} = T'$. Thus the maximum principle implies that $\sup_M F(t) \leq \sup_M F(0) = \sup_M |\nabla u(0)|^2$ for $t \in (0, T')$.

Since $t^{1+\delta}|\nabla^2 u(t)|^2 \leq F(t)$, we have the required estimate. □

Proposition 3.3. *Let $\bar{g}(t)$ be a solution to equation* (RicDT) *given by Theorem 2.2, and $\bar{u}$ be a C^2 solution to:*

$$\partial_t \bar{u} = \Delta_{\bar{g}(t)} \bar{u} + \left\langle W, \bar{\nabla} \bar{u} \right\rangle .$$

Then for every $\beta \in (0, \frac{1}{2})$, there exist constants $C, T' > 0$ depending only on $\sup_M |\bar{g}_0 - g_0|$ and $\sup_M |\bar{\nabla} \bar{u}(0)|$ such that:

$$\sup_M |\bar{u}(t) - \bar{u}(0)| \leq C t^{\beta}$$

for $t \in [0, T')$.

Proof. Let $W = W(\bar{g}, g)$ be the vector field built from the solution $\bar{g}(t)$ to equation (RicDT) given by Theorem 2.2. Let us remark that $|\mathrm{R}_{\bar{g}(t)}| \leq A/t$ and $|W(\bar{g}(t), g(t))| \leq B/\sqrt{t}$ thanks to the estimate (1).

Let φ_t be the flow of the vector field $-W$. Then set $\tilde{g}(t) = \varphi_t^* \bar{g}(t)$ and $\tilde{u}(t) = \varphi_t^* \bar{u}(t)$. We have that:

$$\begin{cases} \partial_t \tilde{g} = -2 \operatorname{Ric}_{\tilde{g}} \\ \partial_t \tilde{u} = \Delta_{\tilde{g}} \tilde{u} \end{cases}$$

We will have that $|\mathrm{R}_{\tilde{g}(t)}| \leq A/t$ since the same estimate was true for $\bar{g}(t)$. Thus we can apply Propositions 3.1 and 3.2 to get that:

$$\sup_M |\tilde{\nabla} \tilde{u}| \leq C_1, \ \sup_M |\tilde{\nabla}^2 \tilde{u}| \leq \frac{C_2}{t^{\alpha}}$$

for any $\alpha > \frac{1}{2}$ and $t \leq T(\alpha, A)$. Those same estimates will thus hold for $\bar{u}$.

We can then write:

$$\begin{aligned} |\bar{u}(t) - \bar{u}(0)| &\leq \int_0^t |\Delta_{\bar{g}} \bar{u}| + |\left\langle W, \bar{\nabla} \bar{u} \right\rangle| dt \\ &\leq \int_0^t \sqrt{n} |\bar{\nabla}^2 \bar{u}| + |W| \, |\bar{\nabla} \bar{u}| dt \\ &\leq \int_0^t \sqrt{n} C_2 t^{-\alpha} + C_1 B t^{-1/2} \, dt \\ &\leq C t^{1-\alpha}. \end{aligned}$$

□

4. Proof of the main result

Proof of Theorem 0.4. In this section we prove Theorem 0.4.

For any lower semi continuous $\underline{u} : M \to \mathbb{R}$, one can find a sequence of v_k of smooth functions such that $v_k \leq \underline{u}$ and v_k pointwisely converges to $\underline{u}$. Thus we can assume that $\underline{u} : M \to \mathbb{R}$ is C^∞.

Let $g(t)$ be the Ricci flow of g and $g_i(t)$ be the solution to equation (RicDT) such that $g_i(0) = g_i$.

Thanks to Theorem 2.2, we have that $g_i(t)$ converges in $C^\infty_{loc}((0,T) \times M)$ to $g(t)$. Set $u_i(t)$ be the solution to:

$$\partial_t u_i = \Delta_{g_i(t)} u_i + \langle W(g(t), g_i(t)), \nabla u_i \rangle$$

with $u_i(0) = \underline{u}$.

Thanks to the maximum principle $u_i(t)$ is bounded uniformly in $i \in \mathbb{N}$. We also have uniform bounds on the Hessian and the gradient of each $u_i(t)$ thanks to the results of section 3. Thus, up to a subsequence, $u_i(t)$ converges locally uniformly on $(0,T) \times M$ to a solution $u(t)$ of the equation:

$$\partial_t u = \Delta_{g(t)} u.$$

Thanks to proposition 3.3, we have that for each i, $\sup_M |u_i(t) - \underline{u}| \leq Ct^{1/4}$. Hence $\sup_M |u(t) - \underline{u}| \leq Ct^{1/4}$ and $u(t)$ uniformly converges to $\underline{u}$ as t goes to 0.

Moreover, up to a pull back by a time dependent diffeomorphism, $g_i(t)$ and $u_i(t)$ satisfy the hypothesis of proposition 1.1. Hence we have that $\mathrm{R}_{g_i(t)} - u_i(t)\,\mathrm{I} \in \mathcal{C}$.

Since $g_i(t)$ converges in C^2 to $g(t)$ for every fixed $t > 0$, we have that :

$$\mathrm{R}_{g(t)} - u(t)\,\mathrm{I} \in \mathcal{C}.$$

We now let t go to 0 to get :

$$\mathrm{R}_g - \underline{u}\,\mathrm{I} \in \mathcal{C}.$$

This ends the proof of Theorem 0.4. □

A. Curvature cones and the Ricci flow

We gather here some definitions on Ricci flow invariant curvature cones for convenience of the reader. For a more detailed exposition of this topic see [Ric14].

Recall that the space of algebraic curvature operators, denoted by $S^2_B\Lambda^2\mathbb{R}^n$, is the space of symmetric endomorphisms $\mathrm{R} : \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ which satisfy the Bianchi identity:

$$\langle \mathrm{R}(x \wedge y), z \wedge t \rangle + \langle \mathrm{R}(y \wedge z), x \wedge t \rangle + \langle \mathrm{R}(z \wedge x), y \wedge t \rangle = 0.$$

The orthogonal group $O(n)$ acts on $S^2_B\Lambda^2\mathbb{R}^n$ by:

$$\langle g \cdot \mathrm{R}(x \wedge y), z \wedge t \rangle = \langle \mathrm{R}(gx \wedge gy), gz \wedge gt \rangle .$$

Definition A.1. *A curvature cone is closed convex cone $\mathcal{C} \subset S^2_B\Lambda^2\mathbb{R}^n$ which is $O(n)$ invariant and contains the identity operator $\mathrm{I} : \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ in its interior.*

Thanks to the $O(n)$ invariance of $\mathcal{R}$, it makes thanks to see $\mathcal{C}$ as a subset of $S^2_B\Lambda^2 T_xM$ for each point x in a Riemannian manifold (M, g), and thus it makes sense to say that the curvature operator R_g of a Riemannian manifold (M, g) belongs to $\mathcal{C}$.

Recall that, once Uhlenbeck's trick has been applied, the curvature operator $\mathrm{R}_{g(t)}$ of a Ricci flow $(M, g(t))$ satisfies

$$\partial_t \mathrm{R}_{g(t)} = \Delta_{g(t)}\mathrm{R}_{g(t)} + Q(\mathrm{R}_{g(t)})$$

where $Q(\cdot)$ can be seen as a quadratic vector field $Q : S^2_B\Lambda^2\mathbb{R}^n \to S^2_B\Lambda^2\mathbb{R}^n$. We will write $Q(\cdot, \cdot)$ for the associated bilinear map.

With Hamilton's tensor maximum principle in mind, we have the following definition:

Definition A.2. *A curvature cone $\mathcal{C}$ is said to be Ricci flow invariant if $Q(\mathrm{R}) \in T_{\mathrm{R}}\mathcal{C}$ whenever $\mathrm{R} \in \mathcal{C}$.*

Examples of Ricci flow invariant curvature cones include the cones of curvature operator which are nonnegative or 2-nonnegative as symmetric quadratic forms, the cone of NNIC curvature operators and the related NNIC1 and NNIC2 cones.

B. Isotropic curvature and Ricci curvature

If R is a curvature operator, we set $\mathrm{R}_{ijkl} = \langle \mathrm{R}(e_i \wedge e_j), e_k \wedge e_l\rangle$. Recall that R is said to have nonnegative isotropic curvature (in short NNIC) if, for any orthonormal 4-frame (e_1, e_2, e_3, e_4), we have:

$$\mathcal{IC}_{1234}(\mathrm{R}) = \mathrm{R}_{1313} + \mathrm{R}_{1414} + \mathrm{R}_{2323} + \mathrm{R}_{2424} - 2\mathrm{R}_{1234} > 0.$$

Recall that a symmetric tensor is said to be k-nonnegative if the sum of its k smallest eigenvalues is nonnegative.

These two lemmas show that that the cone of curvature with nonnegative isotropic curvature satisfy condition $(*)$.

Lemma B.1. *Let* R *be a NNIC curvature operator, then* Ric(R) *is* 4*-nonnegative.*

Proof. Let $(e_i)_{i=1,\dots,n}$ be any orthonormal basis of $\mathbb{R}^n$.

If R is PIC, we have:

$$\mathrm{R}_{1i1i} + \mathrm{R}_{1j1j} + \mathrm{R}_{2i2i} + \mathrm{R}_{2j2j} - 2\mathrm{R}_{12ij} \geq 0.$$

Summing this and the same expression obtained by exchanging e_i and e_j, we get:

$$\mathrm{R}_{1i1i} + \mathrm{R}_{1j1j} + \mathrm{R}_{2i2i} + \mathrm{R}_{2j2j} \geq 0.$$

We sum together the $n-3$ terms corresponding to letting i ranging from 3 to n, excluding j, we get:

$$\mathrm{R}_{11} + \mathrm{R}_{22} + (n-4)(\mathrm{R}_{1j1j} + \mathrm{R}_{2j2j}) - 2\mathrm{R}_{1212} \geq 0$$

where R_{ii} stands for $\mathrm{Ric}(\mathrm{R})(e_i, e_i)$.

We now sum over j ranging from 3 to n. This gives:

$$(n-3)(\mathrm{R}_{11} + \mathrm{R}_{22} - 2\mathrm{R}_{1212}) \geq 0.$$

Thus we have $\mathrm{R}_{11} + \mathrm{R}_{22} \geq 2\mathrm{R}_{1212}$. Hence:

$$\begin{aligned}\mathrm{R}_{11} + \mathrm{R}_{22} + \mathrm{R}_{33} + \mathrm{R}_{44} =& \tfrac{1}{2}(\mathrm{R}_{11} + \mathrm{R}_{33}) + \tfrac{1}{2}(\mathrm{R}_{11} + \mathrm{R}_{44}) \\ &+ \tfrac{1}{2}(\mathrm{R}_{22} + \mathrm{R}_{33}) + \tfrac{1}{2}(\mathrm{R}_{22} + \mathrm{R}_{44}) \\ \geq& \mathrm{R}_{1313} + \mathrm{R}_{1414} + \mathrm{R}_{2323} + \mathrm{R}_{2424} \\ \geq& 0.\end{aligned}$$

Since this inequality is true for any orthonormal 4-frame (e_1, e_2, e_3, e_4), we have that $\mathrm{Ric}(\mathrm{R})$ is 4-positive. □

Lemma B.2. *Let A be a 4-nonnegative symmetric endomorphism, then $A \wedge \mathrm{id}$ is NNIC.*

Proof. Recall that if (e_i, e_j, e_k, e_l) come from an orthonormal frame, we have that:

$$(A \wedge \mathrm{id})_{ijkl} = \frac{1}{2}\left(A_{ik}\delta_{jl} - A_{il}\delta_{jk} + \delta_{ik}A_{jl} - \delta_{il}A_{jk}\right)$$

which implies that $(A \wedge \mathrm{id})_{ijkl} = 0$ if i, j, k, l are all distinct and that $(A \wedge \mathrm{id})_{ijij} = \frac{1}{2}(A_{ii} + A_{jj})$ if $i \neq j$.

Let $(e_1, \ldots, e_4)$ be any orthonormal 4-frame, and A be a symmetric endomorphism. Then:

$$\mathcal{IC}_{1234}(A \wedge \mathrm{id}) = A_{11} + A_{22} + A_{33} + A_{44} \geq 0$$

since A is 4-positive. □

References

[Bam16] Richard H. Bamler. A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature. *Math. Res. Lett.*, 23(2):325–337, 2016.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[BDS17] J. Basilio, J. Dodziuk, and C. Sormani. Sewing Riemannian Manifolds with Positive Scalar Curvature. *ArXiv e-prints*, March 2017.

[BS09] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22(1):287–307, 2009.

[BW08] Christoph Böhm and Burkhard Wilking. Manifolds with positive curvature operators are space forms. *Ann. of Math. (2)*, 167(3):1079–1097, 2008.

[CLN06] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.

[DL16] A. Deruelle and T. Lamm. Weak stability of Ricci expanders with positive curvature operator. *ArXiv e-prints*, January 2016.

[Gro14] Misha Gromov. Dirac and Plateau billiards in domains with corners. *Cent. Eur. J. Math.*, 12(8):1109–1156, 2014.

[Ham86] Richard S. Hamilton. Four-manifolds with positive curvature operator. *J. Differential Geom.*, 24(2):153–179, 1986.

[KL12] Herbert Koch and Tobias Lamm. Geometric flows with rough initial data. *Asian J. Math.*, 16(2):209–235, 2012.

[KL15] Herbert Koch and Tobias Lamm. Parabolic equations with rough data. *Math. Bohem.*, 140(4):457–477, 2015.

[LV09] John Lott and Cédric Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[Ngu10] Huy T. Nguyen. Isotropic curvature and the Ricci flow. *Int. Math. Res. Not. IMRN*, (3):536–558, 2010.

[Ric14] Thomas Richard. Curvature cones and the ricci flow. *Séminaire de théorie spectrale et géométrie*, 31:197–220, 2012-2014.

[Stu05] Karl-Theodor Sturm. Convex functionals of probability measures and nonlinear diffusions on manifolds. *J. Math. Pures Appl. (9)*, 84(2):149–168, 2005.