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Analysis of a dynamic contact problem with nonmonotone friction and non clamped boundary conditions†

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Abstract. We consider a dynamic process of frictional contact between a non clamped viscoelastic body and a foundation. We assume that the normal contact response depends on the depth of penetration of the foundation by the considered body, and the dependence between these two quantities is governed by normal compliance conditions. On the other hand, the friction force is assumed to be a nonmonotone function of the slip rate where the friction threshold also depends on the depth of the penetration. Our aim in this paper is twofold. The first one is to prove the existence and the uniqueness of a weak solution for the contact problem under consideration. The second one is to provide the numerical analysis of the process involving its semi-discrete and fully discrete approximation as well as estimation of the error for both numerical schemes and the validation of such a result.

Keywords: non clamped conditions, dynamic process, viscoelastic material, normal compliance, nonmonotone friction law, finite element method, error estimate.

2010 Mathematics Subject Classification: 65M15, 65M60, 74M10, 74M15, 74S05, 74S20.

1. Introduction

Mechanical contact phenomena occur in many branches of engineering sciences and everyday life. They appear for example when two or more parts of a device touch each other and in many other situations as well. The researchers take into account many various aspects when creating models of contact phenomena. For instance, physical parameters of materials, dynamics of the processes, phenomena like adhesion, wear, heat transfer, electrical conduction and friction provide a variety of possibilities in this field. As a consequence we are given a broad spectrum of complicated mathematical problems that require various mathematical techniques. Regarding this fact, one can confirm that mathematical modelling of contact problems has become an important and rapidly developing branch of science in last years. For recent mathematical results devoted to contact mechanics we refer to W. Han & M. Sofonea (2002); M. Shillor *et al.* (2004); M. Sofonea & A. Matei (2012, 2009) and all the works referenced therein.

The theory of hemivariational inequalities, which allows to represent nonmonotone and nonsmooth contact problems, is a relatively new approach. Early comprehensive references in the area are (Z. Naniewicz & P. D. Panagiotopoulos, 1995; P.D. Panagiotopoulos, 1993, 1985, 1995). For a more recent work, we refer to (S. Migórski *et al.*, 2013) and the references therein. We refer the reader to (Y. Ayyad *et al.*, 2009; M. Barboteu *et al.*, 2013, 2014; M. Barboteu *et al.* , 2015, 2008; M. Barboteu *et al.*, 2008, 2002; M. Campo *et al.*, 2005, 2006; P. Hild & Y. Renard, 2007; T. Laursen, 2002; P. Wriggers, 2002) for more discussion about various numerical aspects of contact problems in mechanics.

In this paper we deal with a mathematical model of a dynamic contact between a non clamped viscoelastic body and a foundation. The foundation is assumed to be composed by a deformable basis covered by a thin deformable layer with a different rigidity. Namely, the penetration of the foundation is modelled by a monotone relation between the normal stress and the normal displacement. However, the use of several deformable materials allows to characterize

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the hardening phenomena of the foundation after reaching a certain penetration. On the other hand, the dependence of the tangential stress with respect to the tangential velocity (slip rate) is modelled by a nonmonotone slip rate dependent friction law (cf. (M. Barboteu *et al.*, 2015; S. Migórski *et al.*, 2013)), in which the friction bound also depends explicitly on the depth of the penetration.

The first result of our paper provides the existence and uniqueness of a weak solution to the considered problem. The proof of the solvability is based on a fixed point technique where, for the existence of a solution to an auxiliary intermediate problem, we use an abstract theorem from (K. Bartosz, submitted). In the second part of the paper, we deal with a numerical analysis of the problem. To this end, we consider two numerical schemes, semi-discrete and fully discrete one. In the first case, the Galerkin spatial approximation is used, while the time is kept continuous. This method is called Faedo-Galerkin approximation. In the second scheme, both space and time are discretized. For both schemes we provide abstract theorems concerning error estimates. Moreover, in the case when the spatial Galerkin approximation is based on a finite element technique involving first order polynomials, we conclude that the error depends linearly on the discretization parameters, provided the solution satisfies additional regularity conditions.

The results obtained by the first two authors in (M. Barboteu *et al.* , 2015) and (M. Barboteu *et al.*, 2015) motivate the present paper. Namely in (M. Barboteu *et al.* , 2015), we obtained analogous error estimates result but for much simpler bilateral contact condition. In (M. Barboteu *et al.*, 2015), a nonmonotone slip rate dependent friction law has been used as well. However, for the contact conditions, a normal compliance model was coupled with a unilateral constraint. In that case, only a finite penetration of the foundation is allowed. In fact these frictional contact conditions introduce some difficulty in the mathematical model. In particular the variational formulation of such problem has the form of a non local variational inequality. This kind of relation is called in literature "very weak formulation". In contrast to (M. Barboteu *et al.*, 2015) we provide the existence and additionally the uniqueness of a solution to the problem that, in the variational formulation, has the form of a local inclusion. Hence we deal with the weak, but not very weak solution. From this point of view our result is stronger than the one obtained in (M. Barboteu *et al.*, 2015). On the other hand, the error estimates for the model studied in (M. Barboteu *et al.*, 2015) remain an open problem while in this work, the numerical analysis of the problem is provided. Furthermore, we also underline that the contact problems studied in (M. Barboteu *et al.* , 2015) and (M. Barboteu *et al.*, 2015) are assumed to use clamped boundary conditions, that is, the displacement vanishes at a part of the boundary Γ_D that has a positive Lebesgue measure. It is well known that the assumption of clamped boundary conditions helps a lot in the variational analysis of the problem. Namely, it allows to use the Korn inequality and introduce the norm $\|v\|_V^2 = \int_\Omega \varepsilon(v) : \varepsilon(v)\,dx$ on the space V used in variational formulation. As a consequence, the viscosity and elasticity tensors are coercive or strongly monotone with respect to the norm $\|\cdot\|_V$. In contrast, in this work, we consider non clamped case and deal with the space $V = H^1(\Omega;\mathbb{R}^d)$ with the usual Sobolev norm. Such an assumption leads to work with the viscosity operator that is not coercive nor strongly monotone in space V, but satisfies a slightly weaker condition (see (3.13)(c)). On the other hand, the elasticity operator does not have to satisfy a coercivity condition. It introduces a significant difficulty to be handled in the present paper. Moreover, in (M. Barboteu *et al.* , 2015) a smallness condition for the constants of the problem is assumed (see (3.22) of (M. Barboteu *et al.* , 2015)). In our paper we use a more sophisticated method that allows to avoid such constraint and makes the result even stronger.

The rest of the paper is structured as follows. In Section 2, we introduce notation that will be used in the rest of the paper and provide preliminary material. In Section 3, we describe the mechanical problem of our interest and provide its mathematical description in both classical and variational forms. Moreover, we formulate and prove the main result, existence and uniqueness theorem for the problem in variational form. In Section 4 and 5, we deal with error estimates for a semi-discrete and fully discrete approximation, respectively. Finally, in Section 6, we provide a numerical validation of the optimal error estimate established in Section 5.

2. Notation and preliminaries

In this section we present the notation and some preliminary material which will be of use later on. We use the notation $\mathbb{N}$ and $\mathbb{R}$ for the set of natural and real numbers respectively. The symbol $\mathbb{R}_+$ will represent the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0,+\infty)$. Given a normed space $(X,\|\cdot\|_X)$, and two real numbers $a < b$, we denote by $C(a,b;X)$ the space of all continuous functions $f\colon [a,b] \to X$ equipped with the norm $\|f\|_{C(a,b;X)} = \max_{x\in[a,b]} \|f(x)\|_X$ for all $f \in C(a,b;X)$. For a time dependent function f, we use symbols $\dot{f}$ and $\ddot{f}$ for its first and second time derivative, respectively. For a real valued function f, we denote by f_+ its positive part defined by $f_+(x) = \max\{f(x),0\}$ for all x. Let $d \in \{2,3\}$. Then, we denote by $\mathbb{S}^d$ the space of symmetric $d \times d$ matrices. The inner product and norm on $\mathbb{R}^d$ and

$\mathbb{S}^d$ are defined by

$$u \cdot v = u_i v_i, \qquad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{\frac{1}{2}} \qquad \text{for all } u, v \in \mathbb{R}^d,$$
$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{\frac{1}{2}} \qquad \text{for all } \sigma, \tau \in \mathbb{S}^d.$$

Here and below the indices i and j run between 1 and d, and unless stated otherwise, the summation convention over repeated indices is used.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ. We use the notation $x = (x_i)$ for a typical point in $\bar{\Omega}$ and we denote by $\nu = (\nu_i)$ the outward unit normal at Γ. Also, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$. We use standard notation for the Lebesgue and Sobolev spaces associated with Ω and Γ and, moreover, we consider the spaces

$$H = L^2(\Omega; \mathbb{R}^d), \qquad Q = L^2(\Omega; \mathbb{S}^d),$$
$$V = \{ u = (u_i) \mid \varepsilon(u) \in Q \}, \qquad Q_1 = \{ \tau \in Q \mid \mathrm{Div}\tau \in H \}.$$

Here $\varepsilon \colon V \to Q$ and $\mathrm{Div} \colon Q_1 \to H$ represent the deformation and divergence operators given by

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad \mathrm{Div}\sigma = (\sigma_{ij,j}).$$

The spaces H, Q, V and Q_1 are real Hilbert spaces endowed with the inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \qquad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} \, dx,$$
$$(u, v)_V = (u, v)_H + (\varepsilon(u), \varepsilon(v))_Q, \qquad (\sigma, \tau)_{Q_1} = (\sigma, \tau)_Q + (\mathrm{Div}\sigma, \mathrm{Div}\tau)_H,$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_V$ and $\|\cdot\|_{Q_1}$, respectively. We denote by v_ν and v_τ the normal and the tangential component of v on Γ, respectively, given by $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$. We denote by σ_ν and σ_τ the normal and tangential traces of σ, i.e. $\sigma_\nu = (\sigma\nu) \cdot \nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

We recall Theorem 2.25 of (S. Migórski *et al.*, 2013) concerning Green formula.

THEOREM 2.1 Let Ω be an open bounded and connected set of $\mathbb{R}^d$ with Lipschitz boundary Γ. Then

$$\int_\Omega \sigma : \varepsilon(v) \, dx + \int_\Omega \mathrm{Div}\sigma \cdot v \, dx = \int_\Gamma \sigma\nu \cdot v \, d\Gamma \tag{2.1}$$

for all $v \in V$ and $\sigma \in C^1(\bar{\Omega}; \mathbb{S}^d)$.

Now we pass to the definition of the subdifferential in the sense of Clarke.

Definition 2.2 Let X be a Banach space and X^* its dual. The Clarke generalized directional derivative of a locally Lipschitz function $\varphi \colon X \to \mathbb{R}$ at the point $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

The Clarke subdifferential of φ at x is a subset of X^* given by

$$\partial\varphi(x) = \{ \zeta \in X^* \mid \varphi^0(x; v) \geqslant \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X \}.$$

We will need the following Gronwall inequalities proved in (W. Han & M. Sofonea, 2002).

LEMMA 2.1 Let $f, g \in C(a, b; \mathbb{R})$ and g be nondecreasing. Assume that

$$f(t) \leqslant g(t) + c \int_a^t f(s) ds \quad \text{for all } t \in [a, b],$$

where c is a positive constant. Then

$$f(t) \leqslant g(t) e^{c(t-a)} \quad \text{for all } t \in [a, b].$$

LEMMA 2.2 Let $T > 0$ be given. For a positive integer N we define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying

$$e_n \leqslant \tilde{c} g_n + \tilde{c} \sum_{j=1}^{n} k e_j, \text{ for all } n = 1, \ldots, N$$

for a positive constant $\tilde{c}$ independent of N or k. Then there exists a positive constant c, independent of N or k, such that

$$\max_{1 \leqslant n \leqslant N} e_n \leqslant c \max_{1 \leqslant n \leqslant N} g_n.$$

We introduce now the Ehrling lemma proved in (R. Temam, 1984).

LEMMA 2.3 Let X_0, X and X_1 be three Banach spaces such that

$$X_0 \subset X \subset X_1,$$

the injection of X into X_1 being continuous and the injection of X_0 into X is compact. Then, for each $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that

$$\|v\|_X \leqslant \varepsilon \|v\|_{X_0} + C(\varepsilon) \|v\|_{X_1} \text{ for all } v \in X_0.$$

At the end of this section, we recall the Young inequality that will be used several times in the paper.

$$ab \leqslant \frac{1}{2} a^2 + \frac{1}{2} b^2 \text{ for all } a, b > 0, \tag{2.2}$$

and, in general,

$$ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \text{ for all } a, b, \varepsilon > 0. \tag{2.3}$$

In the rest of the paper, we denote by C a generic positive constant that may differ from place to place.

3. Mechanical problem and variational formulation

In this section, we describe a mechanical contact problem and present its classical mathematical formulation. Next we introduce a list of assumptions on the data of the problem and pass to its variational formulation. The main result of this section is Theorem 3.1 that guarantees an existence and uniqueness of a weak solution to the problem under consideration.

3.1 *Classical formulation*

We consider a viscoelastic body that occupies in its reference configuration an open bounded region $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega = \Gamma$. The body may undergo a deformation under the influence of volume and boundary forces. The boundary Γ is divided into two disjoint parts Γ_N and Γ_C, namely $\Gamma = \overline{\Gamma_N} \cup \overline{\Gamma_C}$ and $\Gamma_N \cap \Gamma_C = \emptyset$. We assume that the external forces may act on the part Γ_N and the contact of the body with a foundation is possible on the part Γ_C. Our interest is to describe the behaviour of the body in a time interval $[0,T]$, where $T > 0$. We denote by $u(x,t)$ and $\sigma(x,t)$ the displacement and the stress at point $x \in \Omega$ at moment $t \in [0,T]$, respectively. Keeping notation introduced in Section 2 for vectors and tensors, we consider the following classical formulation of the dynamic contact problem.

Problem $\mathscr{P}$. *Find a displacement field $u\colon \Omega \times (0,T) \to \mathbb{R}^d$ and a stress field $\sigma\colon \Omega \times (0,T) \to \mathbb{S}^d$ such that*

$$\sigma = \mathscr{C}\varepsilon(\dot{u}) + \mathscr{E}\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\rho \ddot{u} - \operatorname{Div} \sigma = f_0 \quad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\sigma \nu = f_2 \quad \text{on } \Gamma_N \times (0,T), \tag{3.3}$$

$$-\sigma_\nu = p(u_\nu) + \frac{1}{r}(u_\nu - g)_+ \quad \text{on } \Gamma_C \times (0,T), \tag{3.4}$$

$$\begin{cases} \|\sigma_\tau\|_{\mathbb{R}^d} \leqslant \mu(\|\dot{u}_\tau\|_{\mathbb{R}^d})\,(p(u_\nu) + \frac{1}{r}(u_\nu - g)_+) \\ -\sigma_\tau = \mu(\|\dot{u}_\tau\|_{\mathbb{R}^d})\,(p(u_\nu) + \frac{1}{r}(u_\nu - g)_+) \frac{\dot{u}_\tau}{\|\dot{u}_\tau\|_{\mathbb{R}^d}} \text{ if } \dot{u}_\tau \neq 0 \end{cases} \quad \text{on } \Gamma_C \times (0,T), \tag{3.5}$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \quad \text{in } \Omega. \tag{3.6}$$

Now we shortly describe the physical meaning of relations (3.1)-(3.6). Equation (3.1) represents a viscoelastic constitutive law where $\mathscr{C}$ and $\mathscr{E}$ denote the viscosity and the elasticity tensor, respectively. Relation (3.2) is the equation of motion that governs the evolution of the mechanical state of the body. Here ρ is the mass density and f_0 is the density of applied volume forces. The traction boundary condition (3.3) states that the stress vector $\sigma\nu$ is given on part Γ_N of the boundary, and is equal to the boundary force of density f_2. Now we explain the contact condition (3.4) that expresses the dependence of the normal stress σ_ν on the normal displacement u_ν on the contact part of boundary Γ_C. Recall that it models the contact with a foundation which is made by a deformable basis covered by a thin layer made of elastic material, of thickness g. As far as the normal displacement does not reach the bound g, the contact is described with a normal compliance condition $-\sigma_\nu = p(u_\nu)$. Once the depth of the penetration exceeds thickness g, the additional contribution of normal reaction, $\frac{1}{r}(u_\nu - g)$, occurs due to the resistance of the deformable basis of the foundation. Nevertheless, the basis is not perfectly rigid, namely it allows to be penetrated. The given constant $1/r > 0$ represents the rigidity of the basis. Condition (3.5) represents the slip rate dependent friction law, where the value $\mu(\|\dot{u}_\tau\|_{\mathbb{R}^d})(p(u_\nu) + \frac{1}{r}(u_\nu - g)_+)$ plays the role of the friction bound and μ is not necessary monotone with respect to the slip rate $\|\dot{u}_\tau\|_{\mathbb{R}^d}$. Hence we may deal with a nonmonotone friction law. Finally, in (3.6) we impose the initial conditions.

In the study of Problem $\mathscr{P}$ we assume the following properties on the data.

$H(\mathscr{C})$: The viscosity operator $\mathscr{C}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) } \mathscr{C}(\cdot,\varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d; \\ \text{(b) } (\mathscr{C}(x,\varepsilon_1) - \mathscr{C}(x,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geqslant c_2 \|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d} \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ \qquad \text{a.e. } x \in \Omega \text{ with } c_2 > 0; \\ \text{(c) } \|\mathscr{C}(x,\varepsilon_1) - \mathscr{C}(x,\varepsilon_2)\|_{\mathbb{S}^d} \leqslant L_{\mathscr{C}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ \qquad \text{a.e. } x \in \Omega \text{ with } L_{\mathscr{C}} > 0; \\ \text{(d) } \mathscr{C}(x,0) = 0 \text{ a.e. } x \in \Omega. \end{cases}$$

$H(\mathscr{E})$: The elasticity operator $\mathscr{E}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a bounded, symmetric, nonnegatively definite fourth order tensor, i.e.

$$\begin{cases} \text{(a) } \mathscr{E}_{ijkl} \in L^\infty(\Omega),\ 1 \leqslant i,j,k,l \leqslant d; \\ \text{(b) } \mathscr{E}(x)\sigma \cdot \tau = \sigma \cdot \mathscr{E}(x)\tau \text{ for all } \sigma, \tau \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c) } \mathscr{E}(x)\tau \cdot \tau \geqslant 0 \text{ for all } \tau \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \end{cases}$$

$H(p)$: The normal compliance function $p\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) } p(\cdot,s) \text{ is measurable on } \Gamma_C \text{ for all } s \in \mathbb{R}; \\ \text{(b) } |p(x,s_1) - p(x,s_2)| \leqslant L_p |s_1 - s_2| \text{ for all } s_1, s_2 \in \mathbb{R}, \\ \qquad \text{a.e. } x \in \Gamma_C \text{ with } L_p > 0; \\ \text{(c) } p(x,s) \leqslant c_3 \text{ for all } s \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C \text{ with } c_3 > 0; \\ \text{(d) } p(x,s) = 0 \text{ for all } s \leqslant 0, \text{ a.e. } x \in \Gamma_C. \end{cases}$$

$H(\mu)$: The friction bound $\mu : [0,\infty) \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) } \mu \text{ is continuous;} \\ \text{(b) } |\mu(s)| \leqslant c_4 \text{ for all } s \geqslant 0, \text{ with } c_4 > 0; \\ \text{(c) } \mu(s_1) - \mu(s_2) \geqslant -\lambda(s_1 - s_2) \text{ for all } s_1 > s_2 \geqslant 0 \text{ with } \lambda > 0. \end{cases}$$

$H(g)$: The gap function satisfies $g \in L^2(\Gamma_C)$ and $g(x) \geqslant 0$ a.e. $x \in \Gamma_C$.

$H(f)$: The force and the traction densities satisfy

$$f_0 \in L^2(0,T;H), \quad f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d)).$$

Also, we assume that the initial values satisfy

H_0: $u_0 \in V$ and $u_1 \in H$.

Let us consider an auxiliary function $q\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$q(x,s) = p(x,s) + \frac{1}{r}(s - g(x))_+ \text{ for all } s \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C.$$

Using $H(p)$ and $H(g)$ we obtain

$H(q)$: The function q satisfies

$$\begin{cases} \text{(a) } q(\cdot,s) \text{ is measurable on } \Gamma_C \text{ for all } s \in \mathbb{R}; \\ \text{(b) } |q(x,s_1) - q(x,s_2)| \leqslant L_q |s_1 - s_2| \text{ for all } s_1, s_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C, \\ \quad \text{with } L_q = L_p + \frac{1}{r}; \\ \text{(c) } q(x,s) \leqslant c_3 + \frac{1}{r}|s| \text{ for all } s \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C; \\ \text{(d) } q(x,s) = 0 \text{ for all } s \leqslant 0, \text{ a.e. } x \in \Gamma_C. \end{cases} \tag{3.7}$$

Now we introduce the function $\tilde{p}$, defined by

$$\tilde{p}(x,s) = \min\{q(x,s), \tilde{c_3}\} \text{ for all } s \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C.$$

$H(\tilde{p})$: The function $\tilde{p}\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) } \tilde{p}(\cdot,s) \text{ is measurable on } \Gamma_C \text{ for all } s \in \mathbb{R}; \\ \text{(b) } |\tilde{p}(x,s_1) - \tilde{p}(x,s_2)| \leqslant \tilde{L_p} |s_1 - s_2| \text{ for all } s_1, s_2 \in \mathbb{R}, \\ \quad \text{a.e. } x \in \Gamma_C \text{ with } \tilde{L_p} = L_q; \\ \text{(c) } \tilde{p}(x,s) \leqslant \tilde{c_3} \text{ for all } s \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C \text{ with } \tilde{c_3} > 0; \\ \text{(d) } \tilde{p}(x,s) = 0 \text{ for all } s \leqslant 0, \text{ a.e. } x \in \Gamma_C. \end{cases}$$

Note that such a function was only introduced for mathematical reasons; from the practical point of view, if $\tilde{c_3}$ is large enough, $q(u_\nu) = \tilde{p}(u_\nu)$ for all u, solution of Problem $\mathscr{P}$.

3.2 *Variational formulation*

We turn now to the variational formulation of Problem $\mathscr{P}$.

Let V and H be the spaces defined in Section 2. We denote by $\langle \cdot,\cdot \rangle_{V^* \times V}$ and by $(\cdot,\cdot)_H$ the duality pairing between V and its dual V^* and the inner product in H, respectively. Identifying H with its dual, we have an evolution triple $V \subset H \subset V^*$ with dense, continuous and compact embeddings. We denote by $\gamma\colon V \to L^2(\Gamma_C;\mathbb{R}^d)$ the trace operator and for the element $v \in V$ we still denote by v its trace γv. By the Sobolev trace theorem there exists a positive constant c_0 depending only on the domain Ω and Γ_C such that

$$\|v\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant c_0 \|v\|_V \qquad \text{for all } v \in V. \tag{3.8}$$

In what follows we need the spaces $\mathscr{V} = L^2(0,T;V)$, $\mathscr{H} = L^2(0,T;H)$ and $\mathscr{W} = \{v \in \mathscr{V} \mid \dot{v} \in \mathscr{V}^*\}$ where the time derivative involved in the definition of $\mathscr{W}$ is understood in the sense of vector valued distributions. Equipped with the norm $\|v\|_{\mathscr{W}} = \left(\|v\|^2_{\mathscr{V}} + \|\dot{v}\|^2_{\mathscr{V}^*}\right)^{1/2}$ the space $\mathscr{W}$ becomes a separable Hilbert space. It is well known that the embeddings $\mathscr{W} \subset C(0,T;H)$ and $\{w \in \mathscr{V} \mid \dot{w} \in \mathscr{W}\} \subset C(0,T;V)$ are continuous. Now we provide the following lemma.

LEMMA 3.1 For all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|v\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant \varepsilon \|v\|_V + C(\varepsilon)\|v\|_H \text{ for all } v \in V. \tag{3.9}$$

Proof. We take $\delta \in (0, \frac{1}{2})$. Since the embedding $V \subset H^{1-\delta}(\Omega;\mathbb{R}^d)$ is compact and the embedding $H^{1-\delta}(\Omega;\mathbb{R}^d) \subset H$ is continuous, we can apply Lemma 2.3. Thus, for any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that for all $v \in V$

$$\|v\|_{H^{1-\delta}(\Omega;\mathbb{R}^d)} \leqslant \varepsilon \|v\|_V + C(\varepsilon)\|v\|_H. \tag{3.10}$$

Furthermore, we denote by $i\colon V \to H^{1-\delta}(\Omega;\mathbb{R}^d)$, the continuous embedding operator, by $\gamma_1\colon H^{1-\delta}(\Omega;\mathbb{R}^d) \to H^{\frac{1}{2}-\delta}(\Gamma;\mathbb{R}^d)$ the compact trace operator and by $j\colon H^{\frac{1}{2}-\delta}(\Gamma;\mathbb{R}^d) \to L^2(\Gamma_C;\mathbb{R}^d)$ the continuous embedding operator. Thus $\gamma = j \circ \gamma_1 \circ i$ is linear, continuous and compact. Using above notation, we have

$$\|v\|_{L^2(\Gamma_C;\mathbb{R}^d)} = \|\gamma v\|_{L^2(\Gamma_C;\mathbb{R}^d)} = \|(j \circ \gamma_1 \circ i)v\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant c\|iv\|_{H^{1-\delta}(\Omega;\mathbb{R}^d)} \text{ for all } v \in V$$

with $c > 0$. This together with (3.10) completes the proof. □

We introduce the function $j : \mathbb{R}^d \to \mathbb{R}$ defined by

$$j(z) = \int_0^{\|z\|_{\mathbb{R}^d}} \mu(s)\, ds \text{ for all} z \in \mathbb{R}^d.$$

The properties of the function j are summarized in the next lemma.

LEMMA 3.2 If the assumptions $H(\mu)(a)$–(b) hold, then the function j is locally Lipschitz, and

$$\|\boldsymbol{\xi}\|_{\mathbb{R}^d} \leqslant c_4 \text{ for all } \boldsymbol{\xi} \in \partial j(z),\ z \in \mathbb{R}^d.$$

Furthermore, if the assumption $H(\mu)(c)$ holds then

$$(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\cdot(z_1 - z_2) \geqslant -\lambda\|z_1 - z_2\|^2_{\mathbb{R}^d} \text{ for all } \boldsymbol{\xi}_i \in \partial j(u_i)\ u_i \in \mathbb{R}^d,\ i = 1,2.$$

Proof. First, we have to prove that j is locally Lipschitz. In order to do so, let $z \in \mathbb{R}^d$ and $r > 0$. For $z_1, z_2 \in B(z,r)$, we get

$$|j(z_1) - j(z_2)| \leqslant \max_{x \in B(z,r)} \mu(\|x\|_{\mathbb{R}^d})\|z_1 - z_2\|_{\mathbb{R}^d}.$$

With $H(\mu)(a)$ and Proposition 5.6.28(ii) in (Z. Denkowski *et al.* , 2003) we have the following characterization of the Clarke subdifferential ∂j of j.

$$\partial j(z) = \begin{cases} B(0,\mu(0)) & \text{for } z = 0, \\ \mu(\|z\|_{\mathbb{R}^d})\dfrac{z}{\|z\|_{\mathbb{R}^d}} & \text{for } z \neq 0. \end{cases} \tag{3.11}$$

Then, the other properties follow straightforwardly. □ Now we observe that due to (3.11), the contact condition (3.5) is equivalent to the following subdifferential inclusion

$$-\sigma_\tau \in \tilde{p}(u_\nu)\partial j(\dot{u}_\tau) \quad \text{on } \Gamma_C \times (0,T). \tag{3.12}$$

We introduce the operators $A\colon V \to V^*$, $B\colon V \to V^*$, and the function $f\colon (0,T) \to V^*$, defined by

$$\langle Au, v\rangle_{V^*\times V} = \int_\Omega \mathscr{C}\varepsilon(u) : \varepsilon(v)\, dx \quad \text{for all } u, v \in V,$$
$$\langle Bu, v\rangle_{V^*\times V} = \int_\Omega \mathscr{E}\varepsilon(u) : \varepsilon(v)\, dx \quad \text{for all } u, v \in V,$$
$$\langle f(t), v\rangle_{V^*\times V} = (f_0(t), v)_H + (f_2(t), v)_{L^2(\Gamma_N;\mathbb{R}^d)} \quad \text{for all } v \in V.$$

Using assumptions $H(\mathscr{C})$ and $H(\mathscr{E})$ we easily find that operators A and B satisfy

$$\begin{cases} \text{(a) } \|Au - Av\|_{V^*} \leqslant L_{\mathscr{C}}\|u - v\|_V \text{ for all } u, v \in V; \\ \text{(b) } A0 = 0; \\ \text{(c) } \langle Au - Av, u - v\rangle_{V^*\times V} \geqslant c_2\|u - v\|^2_V - c_2\|u - v\|^2_H \text{ for all } u, v \in V. \end{cases} \tag{3.13}$$

and

$$\begin{cases} \text{(a) } B \in \mathscr{L}(V;V^*) \text{ is symmetric;} \\ \text{(b) } \langle Bv, v\rangle_{V^*\times V} \leqslant L_{\mathscr{E}}\|v\|^2_V \text{ for all } v \in V, \text{ where } L_{\mathscr{E}} = \|B\|_{\mathscr{L}(V,V^*)}; \\ \text{(c) } \langle Bv, v\rangle_{V^*\times V} \geqslant 0 \text{ for all } v \in V. \end{cases} \tag{3.14}$$

We consider a multivalued mapping $S^2_{\partial j}\colon L^2(\Gamma_C;\mathbb{R}^d) \to 2^{L^2(\Gamma_C;\mathbb{R}^d)}$ defined by

$$\boldsymbol{\xi} \in S^2_{\partial j}(v) \;\Leftrightarrow\; \boldsymbol{\xi} \in L^2(\Gamma_C;\mathbb{R}^d) \text{ and } \boldsymbol{\xi}(x) \in \partial j(v(x)) \text{ for a.e. } x \in \Gamma_C,\ v \in L^2(\Gamma_C;\mathbb{R}^d).$$

It follows directly from $H(\mu)$ and Lemma 3.2 that $S^2_{\partial j}$ satisfies

$$\begin{cases} \text{(a) } \|\boldsymbol{\xi}\|_{L^\infty(\Gamma_C;\mathbb{R}^d)} \leqslant c_4 \text{ for all } \boldsymbol{\xi} \in S^2_{\partial j}(v),\ v \in L^2(\Gamma_C;\mathbb{R}^d); \\ \text{(b) } (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2, v_1 - v_2)_{L^2(\Gamma_C;\mathbb{R}^d)} \geqslant -\lambda \|v_1 - v_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)} \text{ for all } \boldsymbol{\xi}_i \in S^2_{\partial j}(v_i), \\ \quad v_i \in L^2(\Gamma_C;\mathbb{R}^d),\ i = 1,2. \end{cases} \tag{3.15}$$

Now we give the following variational formulation of Problem $\mathscr{P}$.

Problem $\mathscr{P}_V$ *Find a displacement field $u \in \mathscr{V}$ with $\dot{u} \in \mathscr{W}$ such that*

$$\langle \rho \ddot{u}(t) + A\dot{u}(t) + Bu(t) - f(t), v\rangle_{V^*\times V} + (\tilde{p}(u_\nu(t)), v_\nu)_{L^2(\Gamma_C;\mathbb{R})} \tag{3.16}$$

$$+(\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t), v_\tau)_{L^2(\Gamma_C;\mathbb{R}^d)} = 0 \;\; for\, all\;\; v \in V,\;\; a.e.\;\; t \in (0,T),$$

and

$$u(0) = u_0, \quad \dot{u}(0) = u_1, \tag{3.17}$$

where the corresponding friction density $\boldsymbol{\xi} \in L^2(0,T,L^2(\Gamma_C;\mathbb{R}^d))$ satisfies

$$\boldsymbol{\xi}(t) \in S^2_{\partial j}(\dot{u}_\tau(t)) \quad for\, a.e.\; t \in (0,T). \tag{3.18}$$

Note that formally we obtain (3.16) multiplying the equation of motion (3.1) by a test function $v \in V$, integrating over Ω, applying the Green formula (2.1), the definitions of operators A and B and function f and using (3.12).

We complete this section with an existence and uniqueness result for Problem $\mathscr{P}_V$.

THEOREM 3.1 Assume $H(\mathscr{C})$, $H(\mathscr{E})$, $H(\tilde{p})$, $H(\mu)$, $H(g)$, $H(f)$ and H_0. Then there exists a unique solution u of Problem $\mathscr{P}_V$.

Before the proof of Theorem 3.1 we consider an auxiliary problem. To this end we fix $\eta \in C(0,T;V)$ and define the function $f_\eta \in \mathscr{V}^*$ by the formula

$$\langle f_\eta(t), v\rangle_{V^*\times V} = \langle f(t), v\rangle_{V^*\times V} - (\tilde{p}(\eta_\nu(t)), v_\nu)_{L^2(\Gamma_C;\mathbb{R})} \text{ for all } v \in V,\ \text{a.e. } t \in (0,T).$$

It follows from the property $H(f)$, $H(\tilde{p})$(c) and (3.8), that function f_η is well defined.

We consider the following problem.

Problem $\mathscr{P}_{V_\eta}$ *Find $u_\eta \in \mathscr{V}$ with $\dot{u}_\eta \in \mathscr{W}$ such that*

$$\langle \rho \ddot{u}_\eta(t) + A\dot{u}_\eta(t) + Bu_\eta(t), v\rangle_{V^*\times V} + (\tilde{p}(\eta_\nu(t))\boldsymbol{\xi}_\eta(t), v_\tau)_{L^2(\Gamma_C;\mathbb{R}^d)} = \langle f_\eta(t), v\rangle_{V^*\times V} \tag{3.19}$$

$$for\, all\;\; v \in V,\;\; a.e.\;\; t \in (0,T),$$

and

$$u_\eta(0) = u_0, \quad \dot{u}_\eta(0) = u_1 \tag{3.20}$$

with

$$\boldsymbol{\xi}_\eta(t) \in S^2_{\partial j}(\dot{u}_{\eta\tau}(t)) \quad for\, a.e.\; t \in (0,T). \tag{3.21}$$

Our aim is to study the existence of solution of Problem $\mathscr{P}_{V_\eta}$. To this end we will consider a more general problem. Namely, we define the functional $J_\eta\colon (0,T)\times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ given by

$$J_\eta(t,u) = \int_{\Gamma_C} \tilde{p}(x, \eta_\nu(x,t))\, j(u(x))\, d\Gamma \text{ for all } u \in L^2(\Gamma_C;\mathbb{R}^d).$$

The next lemma deals with properties of the functional J.

LEMMA 3.3 If the assumptions $H(\tilde{p})$ and $H(\mu)$ hold, then functional J_η satisfies

(i) $J_\eta(\cdot,u)$ is measurable for all $u \in L^2(\Gamma_C;\mathbb{R}^d)$;

(ii) $J_\eta(t,\cdot)$ is locally Lipschitz for a.e. $t \in (0,T)$.

Proof. The properties (i) and (ii) follow from Theorem 3.47(ii) and (iii) of (S. Migórski *et al.*, 2013), respectively. □

Note that Lemma 3.3 (ii) ensures that under assumptions $H(\tilde{p})$ and $H(\mu)$, functional J_η is locally Lipschitz with respect to the second variable for a.e. $t \in (0,T)$. Hence, for a.e. $t \in (0,T)$ there exist Clarke subdifferential of J_η with respect to the second variable denoted by $\partial J_\eta(t,\cdot)$. Now we deal with its properties.

LEMMA 3.4 If the assumptions $H(\tilde{p})$ and $H(\mu)$ hold, then for all $u \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$, we have

$$\partial J_\eta(t,u) \subset \int_{\Gamma_C} \partial[\tilde{p}(x,\eta_v(x,t))\,j(u(x))]\,d\Gamma, \tag{3.22}$$

where $\partial[\tilde{p}(x,\eta_v(x,t))\,j(u(x))]$ denotes the Clarke subdifferential of the function $\tilde{p}(x,\eta_v(x,t))\,j(u(x))$ with respect to u.

Proof. Lemma 3.4 follows from Theorem 3.47 of (S. Migórski *et al.*, 2013). □

REMARK 3.1 The inclusion (3.22) is understood in the sense that for each $\theta \in \partial J_\eta(t,u)$ there exists $\zeta \in L^2(\Gamma_C;\mathbb{R}^d)$ such that

$$(\theta,v)_{L^2(\Gamma_C;\mathbb{R}^d)} = \int_{\Gamma_C} \zeta(x)\cdot v(x)\,d\Gamma \text{ for all } v \in L^2(\Gamma_C;\mathbb{R}^d),$$

and

$$\zeta(x) \in \partial[\tilde{p}(x,\eta_v(x,t))\,j(u(x))] \text{ for a.e. } x \in \Gamma_C.$$

Moreover, since the function $\tilde{p}(x,\eta_v(x,t))$ does not depend on u, we have $\partial[\tilde{p}(x,\eta_v(x,t))\,j(u(x))] = \tilde{p}(x,\eta_v(x,t))\partial j(u(x))$. Hence, the last inclusion is equivalent to

$$\zeta(x) = \tilde{p}(x,\eta_v(x,t))\xi(x) \text{ for a.e. } x \in \Gamma_C,$$

with $\xi(x) \in \partial j(u(x))$ for a.e. $x \in \Gamma_C$. Additionally, by Lemma 3.2, we have $\xi \in L^2(\Gamma_C;\mathbb{R}^d)$.

LEMMA 3.5 If the assumptions $H(\tilde{p})$ and $H(\mu)$ hold, then

(i) $\partial J_\eta(\cdot,u)$ is measurable for all $u \in L^2(\Gamma_C;\mathbb{R}^d)$;

(ii) $\partial J_\eta(t,u)$ is nonempty, convex and weakly* compact subset of $L^2(\Gamma_C;\mathbb{R}^d)$ for all $u \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$;

(iii) the mapping $\partial J_\eta(t,\cdot)$ is upper semicontinuous from the strong topology of $L^2(\Gamma_C;\mathbb{R}^d)$ into weak topology in $L^2(\Gamma_C;\mathbb{R}^d)$ for a.e. $t \in (0,T)$;

(iv) $\|\zeta\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant \tilde{c}_3 c_4 \mathrm{meas}(\Gamma_C)$ for all $\zeta \in \partial J_\eta(t,u)$, $u \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$.

Proof. The property (i) follows from Lemma 3.3 and Proposition 3.23 of (S. Migórski *et al.*, 2013). The properties (ii) and (iii) follow from Proposition 3.23(iv) and (vi) respectively of (S. Migórski *et al.*, 2013). Finally, (iv) follows from $H(\tilde{p})$(c), $H(\mu)$(b) and Lemma 3.4. □

Now, we introduce the operator $\gamma_\tau\colon V \to L^2(\Gamma_C;\mathbb{R}^d)$ defined by $\gamma_\tau u = (\gamma u)_\tau = u_\tau$ for all $u \in V$. In what follows, $\gamma_\tau^*\colon L^2(\Gamma_C;\mathbb{R}^d) \to V^*$ denotes its adjoint operator given by

$$\langle \gamma_\tau^*\xi, v\rangle_{V^*\times V} = (\xi,\gamma_\tau v)_{L^2(\Gamma_C;\mathbb{R}^d)} \text{ for all } \xi \in L^2(\Gamma_C;\mathbb{R}^d),\ v \in V. \tag{3.23}$$

Consider the following problem.

Problem $\mathscr{Q}_{V_\eta}$ *Find $u_\eta \in \mathscr{V}$ with $\dot{u}_\eta \in \mathscr{W}$ such that (3.20) holds and*

$$\rho\ddot{u}_\eta(t) + A\dot{u}_\eta(t) + Bu_\eta(t) + \gamma_\tau^*\partial J_\eta(t,\gamma_\tau\dot{u}(t)) \ni f_\eta(t) \text{ in } V^* \text{ for a.e. } t \in (0,T).$$

LEMMA 3.6 Assume $H(\mathscr{C})$, $H(\mathscr{E})$, $H(\tilde{p})$, $H(\mu)$, $H(g)$, $H(f)$ and H_0. Then there exists a solution of Problem $\mathscr{Q}_{V_\eta}$.

Proof. Having in mind properties of multivalued mapping $\partial J_\eta(\cdot,v)$ provided by Lemma 3.5, we are in a position to apply Theorem 24 of (K. Bartosz, submitted) concerning the solvability of a class of dynamic inclusions that covers problem $\mathscr{Q}_{V_\eta}$ as a special case. Hence, we deduce that Problem $\mathscr{Q}_{V_\eta}$ has a solution. □

LEMMA 3.7 Assume $H(\mathscr{C})$, $H(\mathscr{E})$, $H(\tilde{p})$, $H(\mu)$, $H(g)$, $H(f)$ and H_0. Then there exists a unique solution of Problem $\mathscr{P}_{V_\eta}$.

Proof. By Lemma 3.6, we know, that there exists a solution u_η of Problem $\mathscr{Q}_{V_\eta}$. Thus, in particular $u_\eta \in \mathscr{V}$, $\dot{u}_\eta \in \mathscr{W}$, u_η satisfies (3.20) and, for a.e. $t \in (0,T)$, there exists $\theta_\eta(t) \in L^2(\Gamma_C;\mathbb{R}^d)$ such that

$$\left\langle \rho\ddot{u}_\eta(t) + A\dot{u}_\eta(t) + Bu_\eta(t) + \gamma_\tau^*\theta_\eta(t), v\right\rangle_{V^*\times V} = \left\langle f_\eta(t), v\right\rangle_{V^*\times V} \quad \text{for all } v \in V \text{ a.e. } t \in (0,T); \tag{3.24}$$

$$\theta_\eta(t) \in \partial J_\eta(t, \gamma_\tau \dot{u}_\eta(t)) \text{ for a.e. } t \in (0,T). \tag{3.25}$$

Using (3.25), Lemma 3.4 and Remark 3.1, we claim, that for a.e. $t \in (0,T)$, there exists $\xi_\eta(t) \in L^2(\Gamma_C;\mathbb{R}^d)$ such that

$$(\theta_\eta(t), v)_{L^2(\Gamma_C;\mathbb{R}^d)} = \int_{\Gamma_C} \tilde{p}(x, \eta_v(x,t))\xi_\eta(t)(x)\cdot v(x)\, d\Gamma \text{ for all } v \in V, \tag{3.26}$$

and

$$\xi_\eta(x,t) \in \partial j(\gamma_\tau \dot{u}(x,t)) \text{ for a.e. } x \in \Gamma_C. \tag{3.27}$$

It follows from (3.23) and (3.26) that

$$\left\langle \gamma_\tau^*\theta_\eta(t), v\right\rangle_{V^*\times V} = (\tilde{p}(\eta_v(t))\xi_\eta(t), v_\tau)_{L^2(\Gamma_C;\mathbb{R}^d)} \text{ for all } v \in V. \tag{3.28}$$

Combining (3.24) with (3.28) we claim that u_η satisfies (3.19). Moreover, it follows from (3.27) that ξ_η satisfies (3.21). We conclude that u_η is a solution of Problem $\mathscr{P}_{V_\eta}$.

To show the uniqueness, suppose that u_η^1 and u_η^2 solve Problem $\mathscr{P}_{V_\eta}$. Thus, there exist $\boldsymbol{\xi}_\eta^1, \boldsymbol{\xi}_\eta^2 \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$ such that for $i = 1,2$, we have

$$\langle \rho\ddot{u}_\eta^i(t) + A\dot{u}_\eta^i(t) + Bu_\eta^i(t), v\rangle_{V^*\times V} + (\tilde{p}(\eta_v(t))\boldsymbol{\xi}_\eta^i(t), v_\tau)_{L^2(\Gamma_C;\mathbb{R}^d)} = \langle f_\eta, v\rangle_{V^*\times V} \quad \text{for all } v \in V, \text{ a.e. } t \in (0,T) \tag{3.29}$$

with

$$\boldsymbol{\xi}_\eta^i(t) \in S^2_{\partial j}(\dot{u}^i_{\eta\tau}(t)) \quad \text{for a.e. } t \in (0,T), \tag{3.30}$$

and

$$u_\eta^i(t) = u_0, \quad \dot{u}_\eta^i(t) = u_1. \tag{3.31}$$

We subtract equation (3.29) with $i = 2$ from the equation (3.29) with $i = 1$ and take $v = \dot{r}(t)$, where $r(t) = u_\eta^1(t) - u_\eta^2(t)$. We integrate the result over interval $(0,t)$ and, using (3.9), (3.13)-(3.15) and $H(\tilde{p})$(c), we obtain

$$\|\dot{r}(t)\|_H^2 + \int_0^t \|\dot{r}(s)\|_V^2\, ds \leqslant C\int_0^t \|\dot{r}(s)\|_H^2\, ds \text{ for all } t \in [0,T].$$

By Lemma 2.1, we conclude, that $\|\dot{r}(t)\|_H = 0$ for all $t \in [0,T]$. Thus, using (3.31), we have

$$\|r(t)\|_H \leqslant \|r(0)\|_H + \int_0^t \|\dot{r}(s)\|_H\, ds = 0,$$

and we get $u_\eta^1(t) = u_\eta^2(t)$, which completes the proof of uniqueness. □

Now, we define the operator $\Lambda : C(0,T;V) \to C(0,T;V)$ by $\Lambda\eta = u_\eta$, where u_η is the unique solution of Problem $\mathscr{P}_{V_\eta}$. We remind that $\{w \in \mathscr{V} \mid \dot{w} \in \mathscr{W}\} \subset C(0,T;V)$, and thus operator Λ is well defined.

LEMMA 3.8 Assume $H(\mathscr{C})$, $H(\mathscr{E})$, $H(\tilde{p})$, $H(\mu)$, $H(g)$, $H(f)$ and H_0. Then there exists a unique fixed point of Λ.

Proof. Let $\eta^1, \eta^2 \in C(0,T;V)$ and $u_{\eta^i} = \Lambda\eta^i$ for $i = 1,2$. By the definition of operator Λ, we know, that u_{η^i} is a solution of Problem $\mathscr{P}_{V_{\eta^i}}$, $i = 1,2$. Let ξ_{η^i} denote the function corresponding to u_{η^i} such that the pair $(u_{\eta^i}, \xi_{\eta^i})$ satisfy (3.19)-(3.21) with $\eta = \eta^i$, $i = 1,2$. For simplicity, we denote $u^i = u_{\eta^i}$ and $\xi^i = \xi_{\eta^i}$, $i = 1,2$. We subtract equation (3.19) with $i = 2$ from the equation (3.19) with $i = 1$, and take $v = \dot{u}^1(t) - \dot{u}^2(t)$ to obtain

$$\begin{aligned}
&\rho\left(\ddot{u}^1(t) - \ddot{u}^2(t), \dot{u}^1(t) - \dot{u}^2(t)\right)_H + \langle A\dot{u}^1(t) - A\dot{u}^2(t), \dot{u}^1(t) - \dot{u}^2(t)\rangle_{V^*\times V} \\
&+ \langle Bu^1(t) - Bu^2(t), \dot{u}^1(t) - \dot{u}^2(t)\rangle_{V^*\times V} \\
&+ \left(\tilde{p}(\eta_v^1(t))\boldsymbol{\xi}_\tau^1(t) - \tilde{p}(\eta_v^2(t))\boldsymbol{\xi}_\tau^2(t), \dot{u}_\tau^1(t) - \dot{u}_\tau^2(t)\right)_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&= \left(\tilde{p}(\eta_v^2(t)) - \tilde{p}(\eta_v^1(t)), \dot{u}_v^1(t) - \dot{u}_v^2(t)\right)_{L^2(\Gamma_C;\mathbb{R})}.
\end{aligned} \tag{3.32}$$

Now, using $H(\tilde{p})$(a)-(c), (3.9), (3.15) and (2.3), we estimate

$$\begin{aligned}
&\left(\tilde{p}(\eta_v^1(t))\boldsymbol{\xi}_\tau^1(t) - \tilde{p}(\eta_v^2(t))\boldsymbol{\xi}_\tau^2(t), \dot{u}_\tau^1(t) - \dot{u}_\tau^2(t)\right)_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&= \left(\tilde{p}(\eta_v^1(t))(\boldsymbol{\xi}_\tau^1(t) - \boldsymbol{\xi}_\tau^2(t)), \dot{u}_\tau^1(t) - \dot{u}_\tau^2(t)\right)_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&+ \left((\tilde{p}(\eta_v^1(t)) - \tilde{p}(\eta_v^2(t)))\boldsymbol{\xi}_\tau^2(t), \dot{u}_\tau^1(t) - \dot{u}_\tau^2(t)\right)_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&\geqslant -\tilde{c}_3\lambda\|\dot{u}^1(t) - \dot{u}^2(t)\|^2_{L^2(\Gamma_C;\mathbb{R}^d)} - c_4\tilde{L_p}\|\eta^1(t) - \eta^2(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)}\|\dot{u}^1(t) - \dot{u}^2(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&\geqslant -\varepsilon\tilde{c}_3\lambda\|\dot{u}^1(t) - \dot{u}^2(t)\|^2_V - C(\varepsilon)\tilde{c}_3\lambda\|\dot{u}^1(t) - \dot{u}^2(t)\|^2_H \\
&- \varepsilon\|\dot{u}^1(t) - \dot{u}^2(t)\|^2_V - \frac{c_4\tilde{L_p}^2c_0^2}{4\varepsilon}\|\eta^1(t) - \eta^2(t)\|^2_V
\end{aligned} \tag{3.33}$$

for $\varepsilon > 0$. Using $H(\tilde{p})$(b),(3.9) and (2.3), we obtain

$$\left(\tilde{p}(\eta_v^2(t)) - \tilde{p}(\eta_v^1(t)), \dot{u}_v^1(t) - \dot{u}_v^2(t)\right)_{L^2(\Gamma_C;\mathbb{R})} \leqslant \varepsilon\|\dot{u}^1(t) - \dot{u}^2(t)\|^2_V + \frac{\tilde{L_p}^2c_0^2}{4\varepsilon}\|\eta^1(t) - \eta^2(t)\|^2_V. \tag{3.34}$$

Integrate over t for $t \in [0,T]$, taking $\varepsilon > 0$ small enough and applying (3.13)(c), (3.14)(c) and (3.33)–(3.34) in (3.32), we derive

$$\begin{aligned}
\|\dot{u}^1(t) - \dot{u}^2(t)\|^2_H + \int_0^t \|\dot{u}^1(s) - \dot{u}^2(s)\|^2_V\,ds &\leqslant C\int_0^t \|\eta^1(s) - \eta^2(s)\|^2_V\,ds \\
&+ C\int_0^t \|\dot{u}^1(s) - \dot{u}^2(s)\|^2_H\,ds \text{ for all } t \in [0,T].
\end{aligned} \tag{3.35}$$

Applying Lemma 2.1, we find, that

$$\|\dot{u}^1(t) - \dot{u}^2(t)\|^2_H \leqslant C\int_0^t \|\eta^1(s) - \eta^2(s)\|^2_V\,ds \quad \text{for all } t \in [0,T].$$

Integrating and combining this with (3.35), we obtain

$$\int_0^t \|\dot{u}^1(s) - \dot{u}^2(s)\|^2_V\,ds \leqslant C\int_0^t \|\eta^1(s) - \eta^2(s)\|^2_V\,ds \tag{3.36}$$

Using (3.36), we get

$$\|\Lambda\eta^1(t) - \Lambda\eta^2(t)\|_V = \|u^1(t) - u^2(t)\|_V \leqslant \int_0^t \|\dot{u}^1(s) - \dot{u}^2(s)\|^2_V \leqslant C\int_0^t \|\eta^1(s) - \eta^2(s)\|^2_V\,ds.$$

Using Proposition 3.1 of (M. Sofonea & A. Matei, 2012), we obtain the thesis. □

Now, we pass to the proof of Theorem 3.1.

Proof. It is easy to observe that every function u is a solution of Problem $\mathscr{P}_V$ if and only if it is a fixed point of operator Λ. Thus, by Lemma 3.8, we obtain the existence of a unique solution of Problem $\mathscr{P}_V$. □

4. Spatially semi-discrete approximation

In this section we consider and analyse a spatially semi-discrete approximation of Problem $\mathscr{P}_V$. Let $V^h \subset V$ be a finite dimensional subspace of space V and $h > 0$ denote the spatial discretization parameter. Let $u_0^h, u_1^h \in V^h$ be suitable approximations of u_0 and u_1, characterized by

$$(u_0^h - u_0, v^h)_V = 0, \quad (u_1^h - u_1, v^h)_H = 0 \quad \text{for all } v^h \in V^h. \tag{4.1}$$

The semi-discrete approximation of Problem $\mathscr{P}_V$ is the following.

Problem $\mathscr{P}_V^h$ *Find a displacement field $u^h \in L^2(0,T;V^h)$ with $\dot{u}^h, \ddot{u}^h \in L^2(0,T;V^h)$ such that*

$$\begin{aligned} &\langle \rho \ddot{u}^h(t) + A\dot{u}^h(t) + Bu^h(t) - f(t), v^h\rangle_{V^*\times V} + (\tilde{p}(u_\nu^h(t)), v_\nu^h)_{L^2(\Gamma_C;\mathbb{R})} \\ &+(\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h, v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)} = 0 \quad \textit{for all } v^h \in V^h, \textit{ a.e. } t \in (0,T) \end{aligned} \tag{4.2}$$

and

$$u^h(0) = u_0^h, \quad \dot{u}^h(0) = u_1^h, \tag{4.3}$$

where the corresponding friction density $\boldsymbol{\xi}^h \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$ satisfies

$$\boldsymbol{\xi}^h(t) \in S_{\partial j}^2(\dot{u}_\tau^h(t)) \ \textit{ for a.e. } t \in (0,T). \tag{4.4}$$

In what follows, we provide a result on the error estimate between the solutions of Problem $\mathscr{P}_V$ and its semi-discrete approximation Problem $\mathscr{P}_V^h$.

THEOREM 4.1 Assume $H(\mathscr{C})$, $H(\mathscr{E})$, $H(\tilde{p})$, $H(\mu)$, $H(g)$, $H(f)$ and H_0. Let u and u^h be solutions of Problems $\mathscr{P}_V$ and $\mathscr{P}_V^h$, respectively. Then, we have

$$\begin{aligned} &\|\dot{u} - \dot{u}^h\|_{C(0,T;H)}^2 + \|\dot{u} - \dot{u}^h\|_{\mathscr{V}}^2 + \|u - u^h\|_{C(0,T;V)}^2 \\ &\leqslant C(\|u_0 - u_0^h\|_V^2 + \|u_1 - u_1^h\|_H^2 + \|u_1 - v^h(0)\|_H^2 \\ &+\|\dot{u} - v^h\|_{\mathscr{V}}^2 + \|\ddot{u} - \dot{v}^h\|_{\mathscr{V}^*}^2 + \|\dot{u}_\tau - v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))}) \end{aligned} \tag{4.5}$$

for all $v^h \in L^2(0,T;V^h)$ with $\dot{v}^h \in \mathscr{V}^*$.

Proof. Let $v^h \in L^2(0,T;V^h)$. For a.e. $t \in (0,T)$, we take the same element $v^h(t) \in V^h$ as a test function in (3.16) and (4.2). Next, we subtract (4.2) from (3.16) and get

$$\begin{aligned} &\langle \rho(\ddot{u}(t) - \ddot{u}^h(t)) + A(\dot{u}(t)) - A(\dot{u}^h(t)) + B(u(t) - u^h(t)), v^h(t)\rangle_{V^*\times V} \\ &+(\tilde{p}(u_\nu(t)) - \tilde{p}(u_\nu^h(t)), v_\nu^h(t))_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t) - \tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h(t), v_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)} = 0. \end{aligned} \tag{4.6}$$

We observe that (4.6) holds in particular for $v^h = \dot{u}^h(t)$. Hence, we obtain

$$\begin{aligned} &\langle \rho(\ddot{u}(t) - \ddot{u}^h(t)) + A(\dot{u}(t)) - A(\dot{u}^h(t)) + B(u(t) - u^h(t)), v^h(t)\rangle_{V^*\times V} \\ &+(\tilde{p}(u_\nu(t)) - \tilde{p}(u_\nu^h(t)), v_\nu^h(t))_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t) - \tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h(t), v_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)} \\ &= \langle \rho(\ddot{u}(t) - \ddot{u}^h(t)) + A(\dot{u}(t)) - A(\dot{u}^h(t)) + B(u(t) - u^h(t)), \dot{u}^h(t)\rangle_{V^*\times V} \\ &+(\tilde{p}(u_\nu(t)) - \tilde{p}(u_\nu^h(t)), \dot{u}_\nu^h(t))_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t) - \tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h(t), \dot{u}_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)}. \end{aligned} \tag{4.7}$$

Using (4.7), it is easy to get

$$\begin{aligned}
&\langle \rho(\ddot{u}(t)-\ddot{u}^h(t))+A(\dot{u}(t))-A(\dot{u}^h(t))+B(u(t)-u^h(t)),\dot{u}(t)-\dot{u}^h(t)\rangle_{V^*\times V} \\
&+(\tilde{p}(u_\nu(t))-\tilde{p}(u_\nu^h(t)),\dot{u}_\nu(t)-\dot{u}_\nu^h(t))_{L^2(\Gamma_C;\mathbb{R})} \\
&+(\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t)-\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h(t),\dot{u}_\tau(t)-\dot{u}_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&=\langle \rho(\ddot{u}(t)-\ddot{u}^h(t))+A(\dot{u}(t))-A(\dot{u}^h(t))+B(u(t)-u^h(t)),\dot{u}(t)-v^h(t)\rangle_{V^*\times V} \\
&+(\tilde{p}(u_\nu(t))-\tilde{p}(u_\nu^h(t)),\dot{u}_\nu(t)-v_\nu^h(t))_{L^2(\Gamma_C;\mathbb{R})} \\
&+(\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t)-\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h(t),\dot{u}_\tau(t)-v_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)}.
\end{aligned} \tag{4.8}$$

Therefore, after an elementary manipulation, we derive

$$\begin{aligned}
&\langle \rho(\ddot{u}(t)-\ddot{u}^h(t))+A(\dot{u}(t)-\dot{u}^h(t))+B(u(t)-u^h(t)),\dot{u}(t)-\dot{u}^h(t)\rangle_{V^*\times V} \\
&+(\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}(t)-\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h(t),\dot{u}_\tau(t)-\dot{u}_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&=\langle \rho(\ddot{u}(t)-\ddot{u}^h(t))+A(\dot{u}(t)-\dot{u}^h(t))+B(u(t)-u^h(t)),\dot{u}(t)-v^h(t)\rangle_{V^*\times V} \\
&+(\tilde{p}(u_\nu(t))-\tilde{p}(u_\nu^h(t)),(\dot{u}_\nu^h(t)-\dot{u}_\nu(t))+(\dot{u}_\nu(t)-v_\nu^h(t)))_{L^2(\Gamma_C;\mathbb{R})} \\
&+(\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t)-\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}(t),\dot{u}_\tau(t)-v_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&+(\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}(t)-\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}^h(t),\dot{u}_\tau(t)-v_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&+(\tilde{p}(u_\nu^h(t))\boldsymbol{\xi}(t)-\tilde{p}(u_\nu(t))\boldsymbol{\xi}(t),\dot{u}_\tau(t)-\dot{u}_\tau^h(t))_{L^2(\Gamma_C;\mathbb{R}^d)}.
\end{aligned} \tag{4.9}$$

In what follows we integrate (4.9) over $[0,t]$ where $t\in[0,T]$. We can easily show that

$$\begin{aligned}
&\int_0^t \langle \rho(\ddot{u}(s)-\ddot{u}^h(s)),\dot{u}(s)-\dot{u}^h(s)\rangle_{V^*\times V}\,ds=\frac{\rho}{2}\int_0^t \frac{d}{ds}\|\dot{u}(s)-\dot{u}^h(s)\|_H^2\,ds \\
&=\frac{\rho}{2}\|\dot{u}(t)-\dot{u}^h(t)\|_H^2-\frac{\rho}{2}\|u_1-u_1^h\|_H^2.
\end{aligned} \tag{4.10}$$

Applying (3.13)(c), we get

$$\begin{aligned}
&\int_0^t \langle A\dot{u}(s)-A\dot{u}^h(s),\dot{u}(s)-\dot{u}^h(s)\rangle_{V^*\times V}\,ds \\
&\geqslant c_2\int_0^t \|\dot{u}(s)-\dot{u}^h(s)\|_V^2\,ds-c_2\int_0^t \|\dot{u}(s)-\dot{u}^h(s)\|_H^2\,ds.
\end{aligned} \tag{4.11}$$

It follows from (3.14)(a) that

$$\begin{aligned}
&\int_0^t \langle Bu(s)-Bu^h(s),\dot{u}(s)-\dot{u}^h(s)\rangle_{V^*\times V}\,ds \\
&=\frac{1}{2}\int_0^t \frac{d}{ds}\langle Bu(s)-Bu^h(s),u(s)-u^h(s)\rangle_{V^*\times V}\,ds \\
&=\frac{1}{2}\langle Bu(t)-Bu^h(t),u(t)-u^h(t)\rangle_{V^*\times V}-\frac{1}{2}\langle Bu(0)-Bu^h(0),u(0)-u^h(0)\rangle_{V^*\times V} \\
&\geqslant -\frac{1}{2}L_{\mathscr{E}}\|u_0-u_0^h\|_V^2.
\end{aligned} \tag{4.12}$$

From (3.9), $H(\tilde{p})$(c), (3.18), (4.4) and (3.15)(b), we have

$$\begin{aligned}
&\int_0^t (\tilde{p}(u_\nu^h(s))\boldsymbol{\xi}(s) - \tilde{p}(u_\nu^h(s))\boldsymbol{\xi}^h(s), \dot{u}_\tau(s) - \dot{u}_\tau^h(s))_{L^2(\Gamma_C;\mathbb{R}^d)}\, ds \\
&\geqslant -\lambda \tilde{c_3} \int_0^t \|\dot{u}_\tau(s) - \dot{u}_\tau^h(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}\, ds \\
&\geqslant -\lambda \tilde{c_3} \left(\varepsilon \int_0^t \|\dot{u}(s)\, ds - \dot{u}^h(s)\|_V^2\, ds + C(\varepsilon) \int_0^t \|\dot{u}(s) - \dot{u}^h(s)\|_H^2\, ds \right).
\end{aligned} \tag{4.13}$$

In what follows, we apply the integration by parts formula and use (2.2) and (2.3). Hence, we obtain

$$\begin{aligned}
&\int_0^t \langle \ddot{u}(s) - \ddot{u}^h(s), \dot{u}(s) - v^h(s)\rangle_{V^*\times V} ds = (\dot{u}(t) - \dot{u}^h(t), \dot{u}(t) - v^h(t))_H \\
&-(\dot{u}(0) - \dot{u}^h(0), \dot{u}(0) - v^h(0))_H - \int_0^t \langle \dot{u}(s) - \dot{u}^h(s), \ddot{u}(s) - \dot{v}^h(s)\rangle_{V^*\times V} ds \\
&\leqslant \|\dot{u}(t) - \dot{u}^h(t)\|_H \|\dot{u}(t) - v^h(t)\|_H + \|u_1 - u_1^h\|_H \|u_1 - v^h(0)\|_H \\
&+ \int_0^t |\langle \dot{u}(s) - \dot{u}^h(s), \ddot{u}(s) - \dot{v}^h(s)\rangle_{V^*\times V}| ds \\
&\leqslant \frac{1}{4}\|\dot{u}(t) - \dot{u}^h(t)\|_H^2 + \|\dot{u}(t) - v^h(t)\|_H^2 + \frac{1}{2}\|u_1 - u_1^h\|_H^2 + \frac{1}{2}\|u_1 - v^h(0)\|_H^2 \\
&+\varepsilon \int_0^t \|\dot{u}(s) - \dot{u}^h(s)\|_V^2\, ds + \frac{1}{4\varepsilon}\|\ddot{u} - \dot{v}^h\|_{\mathscr{V}^*}^2
\end{aligned} \tag{4.14}$$

with $\varepsilon > 0$. Applying Lipschitz continuity of operators A and B (see (3.13)(a) and (3.14)(b), respectively), and using (2.2) and (2.3), we get

$$\begin{aligned}
\int_0^t \langle A\dot{u}(s) - A\dot{u}^h(s), \dot{u}(s) - v^h(s)\rangle_{V^*\times V} ds &\leqslant \int_0^t L_{\mathscr{C}} \|\dot{u}(s) - \dot{u}^h(s)\|_V \|\dot{u}(s) - v^h(s)\|_V\, ds \\
&\leqslant \varepsilon \int_0^t \|\dot{u}(s) - \dot{u}^h(s)\|_V^2\, ds + \frac{L_{\mathscr{C}}^2}{4\varepsilon}\|\dot{u} - v^h\|_{\mathscr{V}}^2
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\int_0^t \langle Bu(s) - Bu^h(s), \dot{u}(s) - v^h(s)\rangle_{V^*\times V} ds &\leqslant \int_0^t L_{\mathscr{E}} \|u(s) - u^h(s)\|_V \|\dot{u}(s) - v^h(s)\|_V\, ds \\
&\leqslant \frac{1}{2}\int_0^t \|u(s) - u^h(s)\|_V^2\, ds + \frac{L_{\mathscr{E}}^2}{2}\|\dot{u} - v^h\|_{\mathscr{V}}^2.
\end{aligned} \tag{4.16}$$

Using (3.8), and $H(\tilde{p})$(b), we have

$$\begin{aligned}
&\int_0^t (\tilde{p}(u_\nu(s)) - \tilde{p}(u_\nu^h(s)), \dot{u}_\nu^h(s) - \dot{u}_\nu(s))_{L^2(\Gamma_C;\mathbb{R})} ds \\
&\leqslant \int_0^t c_0^2 \tilde{L_p} \|u(s) - u^h(s)\|_V \|\dot{u}^h(s) - \dot{u}(s)\|_V ds \\
&\leqslant \frac{c_0^4 \tilde{L_p}^2}{4\varepsilon} \int_0^t \|u(s) - u^h(s)\|_V^2 ds + \varepsilon \int_0^t \|\dot{u}^h(s) - \dot{u}(s)\|_V^2
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\int_0^t (\tilde{p}(u_\nu(s)) - \tilde{p}(u_\nu^h(s)), \dot{u}_\nu(s) - v_\nu^h(s))_{L^2(\Gamma_C;\mathbb{R})} ds \\
&\leqslant \int_0^t c_0^2 \tilde{L_p} \|u(s) - u^h(s)\|_V \|\dot{u}(s) - v(s)\|_V ds \\
&\leqslant \frac{c_0^4 \tilde{L_p}^2}{2} \int_0^t \|u(s) - u^h(s)\|_V^2 ds + \frac{1}{2}\|\dot{u} - v^h\|_{\mathscr{V}}^2.
\end{aligned} \tag{4.18}$$

On the other hand, using again (3.8), $H(\tilde{p})$(a)-(c), (3.15)(a), (2.2) and (2.3), we get

$$\begin{aligned}
&\int_0^t (\tilde{p}(u_\nu(s))\boldsymbol{\xi}(s) - \tilde{p}(u_\nu^h(s))\boldsymbol{\xi}(s), \dot{u}_\tau(s) - v_\tau^h(s))_{L^2(\Gamma_C;\mathbb{R}^d)} ds \\
&\leqslant \int_0^t c_0^2 \tilde{L_p} c_4 \|u(s) - u^h(s)\|_V \|\dot{u}(s) - v^h(s)\|_V ds \\
&\leqslant \frac{c_0^4 \tilde{L_p}^2 c_4^2}{2} \int_0^t \|u(s) - u^h(s)\|_V^2 ds + \frac{1}{2}\|\dot{u} - v^h\|_{\mathscr{V}}^2,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&\int_0^t (\tilde{p}(u_\nu^h(s))\boldsymbol{\xi}(s) - \tilde{p}(u_\nu^h(s))\boldsymbol{\xi}^h(s), \dot{u}_\tau(s) - v_\tau^h(s))_{L^2(\Gamma_C;\mathbb{R}^d)} ds \\
&\leqslant 2\tilde{c_3} c_4 \int_0^t \|\dot{u}_\tau(s) - v_\tau^h(s)\|_{L^2(\Gamma_C;\mathbb{R}^d)} ds \leqslant 2\tilde{c_3} c_4 \sqrt{T} \|\dot{u}_\tau - v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))}
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\int_0^t (\tilde{p}(u_\nu^h(s))\boldsymbol{\xi}(s) - \tilde{p}(u_\nu(s))\boldsymbol{\xi}(s), \dot{u}_\tau(s) - \dot{u}_\tau^h(s))_{L^2(\Gamma_C;\mathbb{R}^d)} ds \\
&\leqslant \int_0^t c_0^2 \tilde{L_p} c_4 \|u(s) - u^h(s)\|_V \|\dot{u}(s) - \dot{u}^h(s)\|_V ds \\
&\leqslant \frac{c_0^4 \tilde{L_p}^2 c_4^2}{4\varepsilon} \int_0^t \|u(s) - u^h(s)\|_V^2 ds + \varepsilon \int_0^t \|\dot{u}(s) - \dot{u}^h(s)\|_{\mathscr{V}}^2 ds.
\end{aligned} \tag{4.21}$$

We integrate (4.9) and combine (4.10)–(4.21) to obtain

$$\begin{aligned}
&\frac{\rho}{4}\|\dot{u}(t) - \dot{u}^h(t)\|_H^2 + (c_2 - (\lambda \tilde{c_3} + 3 + \rho)\varepsilon) \int_0^t \|\dot{u}(s) - \dot{u}^h(s)\|_V^2 ds \\
&\leqslant \frac{1}{2} L_{\mathscr{E}} \|u_0 - u_0^h\|_V^2 + \frac{\rho}{2}\|u_1 - u_1^h\|_H^2 + \rho \|\dot{u}(t) - v^h(t)\|_H^2 + \frac{\rho}{2}\|u_1 - u_1^h\|_H^2 \\
&+ \frac{\rho}{2}\|u_1 - v(0)^h\|_H^2 + \frac{\rho}{4\varepsilon}\|\ddot{u} - \dot{v}^h\|_{\mathscr{V}^*}^2 + \left(\frac{L_{\mathscr{C}}^2}{4\varepsilon} + \frac{L_{\mathscr{E}}^2}{2} + 1\right)\|\dot{u} - v^h\|_{\mathscr{V}}^2 \\
&+ \left(\frac{1}{2} + c_0^4\left(\tilde{L_p}^2 + \tilde{L_p}^2 c_4^2\right)\left(\frac{1}{2} + \frac{1}{4\varepsilon}\right)\right) \int_0^t \|u(s) - u^h(s)\|_V^2 ds \\
&+ 2\tilde{c_3} c_4 \sqrt{T} \|\dot{u}_\tau - v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} + (c_2 + \lambda \tilde{c_3} C(\varepsilon)) \int_0^t \|\dot{u}(s) - \dot{u}^h(s)\|_H^2 ds.
\end{aligned} \tag{4.22}$$

Next, we recall that

$$\|u(s) - u^h(s)\|_V^2 \leqslant 2\|u_0 - u_0^h\|_V^2 + 2T \int_0^s \|\dot{u}(z) - \dot{u}^h(z)\|_V^2 dz.$$

Integrating this over $[0,t]$, we get

$$\int_0^t \|u(s) - u^h(s)\|_V^2 dt \leqslant 2T\|u_0 - u_0^h\|_V^2 + 2T \int_0^t \left(\int_0^s \|\dot{u}(z) - \dot{u}^h(z)\|_V^2 dz\right) ds. \tag{4.23}$$

Let us introduce the following function

$$y(t) = \|\dot{u}(t) - \dot{u}^h(t)\|_H^2 + \int_0^t \|\dot{u}(s) - \dot{u}^h(s)\|_V^2 ds \tag{4.24}$$

and the notation

$$\begin{aligned}
\alpha =& \|u_0 - u_0^h\|_V^2 + \|u_1 - u_1^h\|_H^2 + \|u_1 - v^h(0)\|_H^2 + \max_{t\in[0,T]} \|\dot{u}(t) - v^h(t)\|_H^2 \\
&+ \|\dot{u} - v^h\|_{\mathscr{V}}^2 + \|\ddot{u} - \dot{v}^h\|_{\mathscr{V}^*}^2 + \|\dot{u}_\tau - v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))}.
\end{aligned} \tag{4.25}$$

We fix $\varepsilon = \frac{1}{2}c_2(\lambda \tilde{c_3} + 3 + \rho)^{-1}$. Then $c_2 - (\lambda \tilde{c_3} + 3 + \rho)\varepsilon > 0$ and, combining (4.22) with (4.23), we get

$$y(t) \leqslant C\alpha + C \int_0^t y(s) ds. \tag{4.26}$$

From Lemma 2.1, we deduce

$$y(t) \leqslant C\alpha e^{CT}. \tag{4.27}$$

Since $t \in [0,T]$ is arbitrary, it follows that

$$\|\dot{u} - \dot{u}^h\|^2_{C(0,T;H)} + \|\dot{u} - \dot{u}^h\|^2_{\mathscr{V}} \leqslant C\alpha. \tag{4.28}$$

Hence

$$\|u - u^h\|^2_{C(0,T;V)} \leqslant 2\|u_0 - u_0^h\|^2_V + 2T\|\dot{u} - \dot{u}^h\|^2_{\mathscr{V}} \leqslant C\alpha. \tag{4.29}$$

We recall that the embedding $\{v \in \mathscr{V} \mid \dot{v} \in \mathscr{V}^*\} \subset C(0,T;H)$ is continuous, i.e.

$$\max_{t\in[0,T]} \|\dot{u}(t) - v^h(t)\|^2_H = \|\dot{u} - v^h\|_{C(0,T;H)} \leqslant C(\|\dot{u} - v^h\|^2_{\mathscr{V}} + \|\ddot{u} - \dot{v}^h\|^2_{\mathscr{V}^*}). \tag{4.30}$$

Combining (4.25), (4.27) and (4.30), we get the thesis. □ Theorem 4.1 is valid for any finite dimensional subspace V^h of V. In applications, V^h is usually taken to be a finite element space. As a particular example, assume Ω is a polygonal/polyhedral domain and $\{\mathscr{T}^h\}$ is a regular family of finite element triangulations of $\overline{\Omega}$ into triangles ($d = 2$) or tetrahedrons ($d = 3$). For an element $T \in \mathscr{T}^h$, denote by $P_1(T)$ the space of polynomials of a total degree less than or equal to one in T. Then we can use the linear element space of continuous piecewise affine functions

$$V^h = \{v^h \in [C(\overline{\Omega})]^d \mid v^h|_T \in [P_1(T)]^d \text{ for all } T \in \mathscr{T}^h\}. \tag{4.31}$$

Then, by considering the previous finite element space V^h and some additional regularity conditions, we obtain the following corollary which provides the optimal order error estimates.

COROLLARY 4.1 Keep the assumptions stated in Theorem 4.1. Assume Ω is a polygonal/polyhedral domain, and $\{V^h\}$ is the family of linear element spaces defined by (4.31), corresponding to a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. Let u and u^h be solutions of Problems P_V and P_V^h, respectively. Assume $u_0 \in H^2(\Omega;\mathbb{R}^d)$, $u_1 \in H^1(\Omega;\mathbb{R}^d)$, and take $u_0^h, u_1^h \in V^h$ to be projections of u_0 and u_1 defined by (4.1). Under the following regularity condition

$$\dot{u} \in L^2(0,T;H^2(\Omega;\mathbb{R}^d)),\ \ddot{u} \in L^2(0,T;H^2(\Omega;\mathbb{R}^d)),\ \dot{u}_\tau \in L^2(0,T;H^2(\Gamma_C;\mathbb{R}^d)),$$

we have the optimal order error estimate

$$\|u - u^h\|_{C(0,T;V)} + \|\dot{u} - \dot{u}^h\|_{C(0,T;H)} + \|\dot{u} - \dot{u}^h\|_{\mathscr{V}} \leqslant Ch. \tag{4.32}$$

Proof. Note that under the stated regularity assumptions, for a.e. $t \in [0,T]$, $\dot{u}(t)$, $\ddot{u}(t)$ are continuous on $\overline{\Omega}$, and $\dot{u}_\tau(t)$ is continuous on Γ_C. Let $v^h(t) = \Pi^h\dot{u}(t) \in V^h$ be the finite element interpolant of $\dot{u}(t)$, for a.e. $t \in [0,T]$. Note that $v_\tau^h(t) = (\Pi^h\dot{u}(t))_\tau$ is the continuous piecewise linear interpolant of $\dot{u}_\tau(t)$ on Γ_C. Moreover, $\dot{v}^h(t)$ is the continuous piecewise linear interpolant of $\ddot{u}(t)$. Then, by the standard finite element interpolation error estimates ((K. Atkinson & W. Han, 2009; S.C. Brenner & L.R. Scott, 2008; P.G. Ciarlet, 1978)), we have the following approximation properties

$$\begin{aligned}
&\|\dot{u}(t) - v^h(t)\|_V \leqslant ch\|\dot{u}(t)\|_{H^2(\Omega;\mathbb{R}^d)},\\
&\|\ddot{u}(t) - \dot{v}^h(t)\|_{V^*} \leqslant ch\|\ddot{u}(t)\|_{H^2(\Omega;\mathbb{R}^d)},\\
&\|\dot{u}_\tau(t) - v_\tau^h(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant ch^2\|\dot{u}_\tau\|_{H^2(\Gamma_C;\mathbb{R}^d)}
\end{aligned}$$

and

$$\begin{aligned}
&\|u_0 - u_0^h\|_V \leqslant ch\,\|u_0\|_{H^2(\Omega;\mathbb{R}^d)},\\
&\|u_1 - u_1^h\|_H \leqslant ch\,\|u_1\|_{H^1(\Omega;\mathbb{R}^d)}.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\|\dot{u} - v^h\|_{\mathscr{V}} \leqslant ch\|\dot{u}\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))},\\
&\|\ddot{u} - \dot{v}^h\|_{\mathscr{V}^*} \leqslant ch\|\ddot{u}\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))},\\
&\|\dot{u}_\tau - v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} \leqslant ch^2\|\dot{u}_\tau\|_{L^2(0,T;H^2(\Gamma_C;\mathbb{R}^d))}.
\end{aligned}$$

Then the error bound (4.32) is a consequence of (4.5). □

5. Fully discrete error estimates

We introduce a fully discrete approximation of Problem $\mathscr{P}_V$ in order to bound the error of the fully discrete solutions. To this end, we consider again a finite dimensional subspace V^h of space V, where $h > 0$ denote the spatial discretization parameter. Moreover, on the time interval $[0,T]$, we consider a positive integer N and we define the time step size $k = T/N$ and the time nodal points $t_n = nk,\ 0 \leqslant n \leqslant N$.

For any time dependent function h we use notation $h_n = h(t_n)$ for $n = 1,...,N$ and for any sequence $\{x_n\}_{n=0}^N$ we denote $\delta x_n = (x_n - x_{n-1})/k$ for $n = 1,...,N$.

The fully discrete approximation of Problem $\mathscr{P}_V$ is the following.

Problem $\mathscr{P}_V^{hk}$ *Find a displacement field* $\{u_n^{hk}\}_{n=0}^N \subset V^h$ *such that*

$$\langle \rho\,\delta w_n^{hk} + Aw_n^{hk} + Bu_n^{hk} - f_n, v^h\rangle_{V^*\times V} + (\tilde{p}(u_{n\nu}^{hk}), v_\nu^h)_{L^2(\Gamma_C;\mathbb{R})} \tag{5.1}$$
$$+(\tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)} = 0 \quad \textit{for all } v^h \in V^h,\ n = 1,...,N,$$

and

$$u_0^{hk} = u_0^h, \tag{5.2}$$

where the sequence $\{w_n^{hk}\}_{n=0}^N \subset V^h$ *is defined by*

$$w_0^{hk} = u_1^h, \tag{5.3}$$

and

$$w_n^{hk} = \delta u_n^{hk},\ \text{for all } n = 1,...,N, \tag{5.4}$$

and the corresponding friction density $\{\boldsymbol{\xi}_n^{hk}\}_{n=0}^N \subset L^2(\Gamma_C;\mathbb{R}^d)$ *satisfies*

$$\boldsymbol{\xi}_n^{hk} \in S_{\partial j}^2(w_{n\tau}^{hk})\ \ \textit{a.e. on } \Gamma_C,\ n = 1,...,N. \tag{5.5}$$

Note that (5.4) is equivalent to

$$u_n^{hk} = u_0^{hk} + \sum_{j=1}^n k w_j^{hk} \text{ for all } n = 1,...,N. \tag{5.6}$$

In what follows, we present the theorem concerning error estimate between the solution of Problem $\mathscr{P}_V$ and the solution of its fully discrete approximation, Problem $\mathscr{P}_V^{hk}$.

THEOREM 5.1 Assume $H(\mathscr{C})$, $H(\mathscr{E})$, $H(\tilde{p})$, $H(\mu)$, $H(g)$, $H(f)$ and H_0. Let u and u^{hk} be solutions of Problems $\mathscr{P}_V$ and $\mathscr{P}_V^{hk}$, respectively. Moreover, suppose that the solution u has the following regularity

$$u \in C^2(0,T;H) \cap C^1(0,T;V), \quad \dot{u}_\nu \in C(0,T;L^2(\Gamma_C;\mathbb{R})), \quad \dot{u}_\tau \in C(0,T;L^2(\Gamma_C;\mathbb{R}^d)). \tag{5.7}$$

Then

$$\max_{1\leqslant n\leqslant N} \|u_n - u_n^{hk}\|_V^2 + \max_{1\leqslant n\leqslant N} \|w_n - w_n^{hk}\|_H^2 + \sum_{j=1}^N k\|w_j - w_j^{hk}\|_V^2 \tag{5.8}$$
$$\leqslant C\Big[\sum_{j=1}^N k(\|\dot{w}_j - \delta w_j\|_H^2 + \|w_j - v_j^h\|_V^2 + \|w_{j\tau} - v_{j\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)})$$
$$+\|u_0 - u_0^h\|_V^2 + \|w_0 - w_0^{hk}\|_H^2 + \max_{1\leqslant n\leqslant N} \|w_n - v_n^h\|_H^2 + \|w_1 - v_1^h\|_H^2$$
$$+k^2\|u\|_{H^2(0,T;V)}^2 + \sum_{j=1}^{N-1} \frac{1}{k}\|w_j - v_j^h - (w_{j+1} - v_{j+1}^h)\|_H^2\Big]$$

for all $\{v_j^h\}_{j=1}^N \subset V^h$.

Proof. We take $v = v^h$ in (3.16) and we combine it with (5.1) in order to get

$$\langle \rho(\dot{w}_n - \delta w_n^{hk}) + A(w_n - w_n^{hk}) + B(u_n - u_n^{hk}), v^h \rangle_{V^* \times V} \tag{5.9}$$
$$+ (\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), v_\nu^h)_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_{n\nu})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)} = 0.$$

Note that (5.9) holds, in particular, for $v^h = w_n^{hk}$. Hence, we have

$$\langle \rho(\dot{w}_n - \delta w_n^{hk}) + A(w_n - w_n^{hk}) + B(u_n - u_n^{hk}), v^h \rangle_{V^* \times V} \tag{5.10}$$
$$+ (\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), v_\nu^h)_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_{n\nu})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)}$$
$$= \langle \rho(\dot{w}_n - \delta w_n^{hk}) + A(w_n - w_n^{hk}) + B(u_n - u_n^{hk}), w_n^{hk} \rangle_{V^* \times V}$$
$$+ (\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), w_{n\nu}^{hk})_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_{n\nu})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, w_{n\tau}^{hk})_{L^2(\Gamma_C;\mathbb{R}^d)}.$$

After a reformulation of (5.10) under the form $(., w_n - w_n^{hk}) = (., w_n - v^h)$, we have

$$\langle \rho(\dot{w}_n - \delta w_n^{hk}) + A(w_n - w_n^{hk}) + B(u_n - u_n^{hk}), w_n - w_n^{hk} \rangle_{V^* \times V} \tag{5.11}$$
$$+ (\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), w_{n\nu} - w_{n\nu}^{hk})_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_{n\nu})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, w_{n\tau} - w_{n\tau}^{hk})_{L^2(\Gamma_C;\mathbb{R}^d)}$$
$$= \langle \rho(\dot{w}_n - \delta w_n^{hk}) + A(w_n - w_n^{hk}) + B(u_n - u_n^{hk}), w_n - v^h \rangle_{V^* \times V}$$
$$+ (\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), w_{n\nu} - v_\nu^h)_{L^2(\Gamma_C;\mathbb{R})} + (\tilde{p}(u_{n\nu})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, w_{n\tau} - v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)}.$$

Therefore, after an elementary manipulation, we infer

$$(\rho(\delta w_n - \delta w_n^{hk}), w_n - w_n^{hk})_H + \langle A(w_n - w_n^{hk}), w_n - w_n^{hk} \rangle_{V^* \times V} \tag{5.12}$$
$$+ (\tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, w_{n\tau} - w_{n\tau}^{hk})_{L^2(\Gamma_C;\mathbb{R}^d)} =$$
$$(\rho(\dot{w}_n - \delta w_n), w_n^{hk} - w_n)_H + (\rho(\delta w_n - \delta w_n^{hk}), w_n - v^h)_H$$
$$+ (\rho(\dot{w}_n - \delta w_n), w_n - v^h)_H + \langle A(w_n - w_n^{hk}), w_n - v^h \rangle_{V^* \times V}$$
$$+ \langle B(u_n - u_n^{hk}), (w_n^{hk} - w_n) + (w_n - v^h) \rangle_{V^* \times V}$$
$$+ (\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), (w_{n\nu}^{hk} - w_{n\nu}) + (w_{n\nu} - v_\nu^h))_{L^2(\Gamma_C;\mathbb{R})}$$
$$+ (\tilde{p}(u_{n\nu})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n, w_{n\tau} - v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)}$$
$$+ (\tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, w_{n\tau} - v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)}$$
$$+ (\tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu})\boldsymbol{\xi}_n, w_{n\tau} - w_{n\tau}^{hk})_{L^2(\Gamma_C;\mathbb{R}^d)}.$$

Applying the identity $2(a-b,a)_H = \|a-b\|_H^2 + \|a\|_H^2 - \|b\|_H^2$ with $a = w_n - w_n^{hk}$ and $b = w_{n-1} - w_{n-1}^{hk}$, we obtain

$$\frac{\rho}{2k}(\|w_n - w_n^{hk}\|_H^2 - \|w_{n-1} - w_{n-1}^{hk}\|_H^2) \leqslant \rho((\delta w_n - \delta w_n^{hk}), w_n - w_n^{hk})_H. \tag{5.13}$$

From (3.13)(c), we get

$$c_2\|w_n - w_n^{hk}\|_V^2 - c_2\|w_n - w_n^{hk}\|_H^2 \leqslant \langle A(w_n - w_n^{hk}), w_n - w_n^{hk} \rangle_{V^* \times V}. \tag{5.14}$$

Using (3.9), $H(\tilde{p})$(c) and (3.15)(b), we have

$$(\tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, w_{n\tau} - w_{n\tau}^{hk})_{L^2(\Gamma_C;\mathbb{R}^d)} \geqslant -\lambda\tilde{c}_3\|w_{n\tau} - w_{n\tau}^{hk}\|_{L^2(\Gamma_C;\mathbb{R}^d)}^2 \tag{5.15}$$
$$\geqslant -\lambda\tilde{c}_3(\varepsilon\|w_n - w_n^{hk}\|_V^2 + C(\varepsilon)\|w_n - w_n^{hk}\|_H^2).$$

By the definitions of spaces H and V, we easily have $\|v\|_H \leqslant \|v\|_V$ for all $v \in V$. Hence, using (2.3) and (2.2), we can estimate

$$(\rho(\dot{w}_n - \delta w_n), w_n^{hk} - w_n)_H \leqslant \rho\|\dot{w}_n - \delta w_n\|_H \|w_n^{hk} - w_n\|_V$$
$$\leqslant \varepsilon\|w_n^{hk} - w_n\|_V^2 + \frac{\rho^2}{4\varepsilon}\|\dot{w}_n - \delta w_n\|_H^2, \tag{5.16}$$

and

$$(\rho(\dot{w}_n - \delta w_n), w_n - v^h)_H \leqslant \rho\|\dot{w}_n - \delta w_n\|_H \|w_n - v^h\|_V$$
$$\leqslant \frac{1}{2}\|w_n - v^h\|_V^2 + \frac{\rho^2}{2}\|\dot{w}_n - \delta w_n\|_H^2. \tag{5.17}$$

Applying Lipschitz continuity of operators A and B (see (3.13)(a) and (3.14)(b), respectively), and applying using (2.3) and (2.2), we get

$$\langle A(w_n - w_n^{hk}), w_n - v^h\rangle_{V^*\times V} \leqslant L_{\mathscr{C}}\|w_n - w_n^{hk}\|_V \|w_n - v^h\|_V$$
$$\leqslant \varepsilon\|w_n - w_n^{hk}\|_V^2 + \frac{L_{\mathscr{C}}^2}{4\varepsilon}\|w_n - v^h\|_V^2, \tag{5.18}$$

$$\langle B(u_n - u_n^{hk}), w_n^{hk} - w_n\rangle_{V^*\times V} \leqslant L_{\mathscr{E}}\|u_n - u_n^{hk}\|_V \|w_n^{hk} - w_n\|_V$$
$$\leqslant \varepsilon\|w_n^{hk} - w_n\|_V^2 + \frac{L_{\mathscr{E}}^2}{4\varepsilon}\|u_n - u_n^{hk}\|_V^2 \tag{5.19}$$

and

$$\langle B(u_n - u_n^{hk}), w_n - v^h\rangle_{V^*\times V} \leqslant L_{\mathscr{E}}\|u_n - u_n^{hk}\|_V \|w_n - v^h\|_V$$
$$\leqslant \frac{L_{\mathscr{E}}^2}{2}\|u_n - u_n^{hk}\|_V^2 + \frac{1}{2}\|w_n - v^h\|_V^2. \tag{5.20}$$

Using (3.8), $H(\tilde{p})$(b), (2.3) and (2.2), we have

$$(\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), w_{n\nu}^{hk} - w_{n\nu})_{L^2(\Gamma_C;\mathbb{R})} \leqslant c_0^2\tilde{L_p}\|u_n - u_n^{hk}\|_V \|w_n - w_n^{hk}\|_V$$
$$\leqslant \varepsilon\|w_n - w_n^{hk}\|_V^2 + \frac{\tilde{L_p}^2 c_0^4}{4\varepsilon}\|u_n - u_n^{hk}\|_V^2 \tag{5.21}$$

and

$$(\tilde{p}(u_{n\nu}) - \tilde{p}(u_{n\nu}^{hk}), w_{n\nu} - v_\nu^h)_{L^2(\Gamma_C;\mathbb{R})} \leqslant c_0^2\tilde{L_p}\|u_n - u_n^{hk}\|_V \|w_n - v^h\|_V$$
$$\leqslant \frac{\tilde{L_p}^2 c_0^4}{2}\|u_n - u_n^{hk}\|^2 + \frac{1}{2}\|w_n - v^h\|_V^2. \tag{5.22}$$

On the other hand, from (3.8), $H(\tilde{p})$(b), (3.15)(a), (2.3) and (2.2), we obtain

$$(\tilde{p}(u_{n\nu})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n, w_{n\tau} - v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant c_0^2\tilde{L_p}c_4\|u_n - u_n^{hk}\|_V \|w_n - v^h\|_V$$
$$\leqslant \frac{c_0^4 c_4^2 \tilde{L_p}^2}{2}\|u_n - u_n^{hk}\|_V^2 + \frac{1}{2}\|w_n - v^h\|_V^2, \tag{5.23}$$

$$(\tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n^{hk}, w_{n\tau} - v_\tau^h)_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant 2\tilde{c}_3 c_4\|w_{n\tau} - v_\tau^h\|_{L^2(\Gamma_C;\mathbb{R}^d)} \tag{5.24}$$

and

$$(\tilde{p}(u_{n\nu}^{hk})\boldsymbol{\xi}_n - \tilde{p}(u_{n\nu})\boldsymbol{\xi}_n, w_{n\tau} - w_{n\tau}^{hk})_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant c_0^2 \tilde{L_p} c_4 \|u_n - u_n^{hk}\|_V \|w_n - w_n^{hk}\|_V$$

$$\leqslant \frac{c_0^4 c_4^2 \tilde{L_p}^2}{4\varepsilon} \|u_n - u_n^{hk}\|_V^2 + \varepsilon \|w_n - w_n^{hk}\|_V^2. \tag{5.25}$$

Thus, from (5.13)–(5.25), we get

$$\frac{\rho}{2k}(\|w_n - w_n^{hk}\|_H^2 - \|w_{n-1} - w_{n-1}^{hk}\|_H^2) + (c_2 - (\lambda \tilde{c_3} + 5)\,\varepsilon)\|w_n - w_n^{hk}\|_V^2 \tag{5.26}$$

$$\leqslant (\rho(\delta w_n - \delta w_n^{hk}), w_n - v^h)_H + (c_2 + \lambda \tilde{c_3} C(\varepsilon))\|w_n - w_n^{hk}\|_H^2$$

$$+ \left(\frac{1}{2} + \frac{1}{4\varepsilon}\right)\left(L_{\mathscr{E}}^2 + c_0^4(1 + c_4^2)\tilde{L_p}^2\right)\|u_n - u_n^{hk}\|_V^2 + \rho^2\left(\frac{1}{2} + \frac{1}{4\varepsilon}\right)\|\dot{w}_n - \delta w_n\|_H^2$$

$$+ \left(2 + \frac{L_{\mathscr{C}}^2}{4\varepsilon}\right)\|w_n - v^h\|_V^2 + 2\tilde{c_3} c_4 \|w_{n\tau} - v_\tau^h\|_{L^2(\Gamma_C;\mathbb{R}^d)}.$$

We take $\varepsilon = \frac{1}{2} c_2 (\lambda \tilde{c_3} + 5)^{-1}$. Then

$$c_2 - (\lambda \tilde{c_3} + 5)\varepsilon > 0. \tag{5.27}$$

Now, we replace n by j and take $v^h = v_j^h \in V^h$ in (5.26). We sum the above inequalities over j from 1 to n, use (5.27), and after a minor reformulation, we to obtain

$$\|w_n - w_n^{hk}\|_H^2 + \sum_{j=1}^n k\|w_j - w_j^{hk}\|_V^2 \leqslant C \sum_{j=1}^n k((\delta w_j - \delta w_j^{hk}), w_j - v_j^h)_H \tag{5.28}$$

$$+ C \sum_{j=1}^n k\left(\|\dot{w}_j - \delta w_j\|_H^2 + \|u_j - u_j^{hk}\|_V^2 + \|w_j - v_j^h\|_V^2 + \|w_{j\tau} - v_{j\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right)$$

$$+ C(\varepsilon) \sum_{j=1}^n k\|w_j - w_j^{hk}\|_H^2 + \|w_0 - w_0^{hk}\|_H^2.$$

In what follows, we deal with the first term of the right hand side of (5.28). We have

$$\sum_{j=1}^n k\left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h\right)_H = \sum_{j=1}^n \left((w_j - w_j^{hk}) - (w_{j-1} - w_{j-1}^{hk}), w_j - v_j^h\right)_H$$

$$= (w_0 - w_0^{hk}, w_1 - v_1^h)_H + \sum_{j=1}^{n-1}\left(w_j - w_j^{hk}, (w_j - v_j^h) - (w_{j+1} - v_{j+1}^h)\right)_H$$

$$+ (w_n - w_n^{hk}, w_n - v_n^h)_H \leqslant \frac{1}{2}\|w_0 - w_0^{hk}\|_H^2 + \frac{1}{2}\|w_1 - v_1^h\|_H^2 + \frac{1}{2}\|w_n - w_n^{hk}\|_H^2$$

$$+ \frac{1}{2}\|w_n - v_n^h\|_H^2 + \sum_{j=1}^{n-1}\|w_j - w_j^{hk}\|_H \|w_j - v_j^h - (w_{j+1} - v_{j+1}^h)\|_H$$

$$\leqslant \frac{1}{2}(\|w_0 - w_0^{hk}\|_H^2 + \|w_1 - v_1^h\|_H^2 + \|w_n - w_n^{hk}\|_H^2 + \|w_n - v_n^h\|_H^2)$$

$$+ \sum_{j=1}^{n-1}\frac{k}{4}\|w_j - w_j^{hk}\|_H^2 + \sum_{j=1}^{n-1}\frac{1}{k}\|w_j - v_j^h - (w_{j+1} - v_{j+1}^h)\|_H^2. \tag{5.29}$$

We recall the following classical inequality

$$\|u_j - u_j^{hk}\|_V \leqslant \|u_0 - u_0^h\|_V + \sum_{p=1}^j k\|w_p - w_p^{hk}\|_V + I_j, \tag{5.30}$$

where I_j is defined by the following relation

$$I_j = \| \int_0^{t_j} w(s)ds - \sum_{p=1}^{j} kw_p \|_V \leqslant k\|u\|_{H^2(0,T;V)}.$$

Thus

$$\|u_j - u_j^{hk}\|_V^2 \leqslant 3\Big(\|u_0 - u_0^h\|_V^2 + \Big(\sum_{p=1}^{j} k\|w_p - w_p^{hk}\|_V \Big)^2 + k^2\|u\|_{H^2(0,T;V)}^2 \Big)$$

$$\leqslant C\Big(\|u_0 - u_0^h\|_V^2 + j\sum_{p=1}^{j} k^2\|w_p - w_p^{hk}\|_V^2 + k^2\|u\|_{H^2(0,T;V)}^2 \Big). \tag{5.31}$$

Therefore

$$\sum_{j=1}^{n} k\|u_j - u_j^{hk}\|_V^2 \leqslant CT\Big(\|u_0 - u_0^h\|_V^2 + \sum_{j=1}^{n} k^2 \sum_{p=1}^{j} \|w_p - w_p^{hk}\|_V^2 + k^2\|u\|_{H^2(0,T;V)}^2 \Big). \tag{5.32}$$

Now, we introduce the notation

$$e_n = \|w_n - w_n^{hk}\|_H^2 + \sum_{j=1}^{n} k\|w_j - w_j^{hk}\|_V^2, \tag{5.33}$$

and

$$g_n = \sum_{j=1}^{n} k(\|\dot{w}_j - \delta w_j\|_H^2 + \|w_j - v_j^h\|_V^2 + \|w_{j\tau} - v_{j\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)}) \tag{5.34}$$

$$+\|u_0 - u_0^h\|_V^2 + \|w_0 - w_0^{hk}\|_H^2 + \|w_1 - v_1^h\|_H^2 + \|w_n - v_n^h\|_H^2$$

$$+k^2\|u\|_{H^2(0,T;V)}^2 + \sum_{j=1}^{n-1} \frac{1}{k}\|w_j - v_j^h - (w_{j+1} - v_{j+1}^h)\|_H^2.$$

Then, we can rewrite (5.28) as follows

$$e_n \leqslant Cg_n + C\sum_{j=1}^{n} ke_j, \ n = 1, ..., N. \tag{5.35}$$

We are now in a position to apply Lemma 2.2 for (5.35) and obtain

$$\max_{1\leqslant n\leqslant N} (\|w_n - w_n^{hk}\|_H^2 + \sum_{j=1}^{n} k\|w_j - w_j^{hk}\|_V^2) \tag{5.36}$$

$$\leqslant C\Big[\sum_{j=1}^{N} k(\|\dot{w}_j - \delta w_j\|_H^2 + \|w_j - v_j^h\|_V^2 + \|w_{j\tau} - v_{j\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)})$$

$$+\|u_0 - u_0^h\|_V^2 + \|w_0 - w_0^{hk}\|_H^2 + \max_{1\leqslant n\leqslant N} \|w_n - v_n^h\|_H^2 + \|w_1 - v_1^h\|_H^2$$

$$+k^2\|u\|_{H^2(0,T;V)}^2 + \sum_{j=1}^{N-1} \frac{1}{k}\|w_j - v_j^h - (w_{j+1} - v_{j+1}^h)\|_H^2 \Big].$$

Combining this with (5.31), we obtain the thesis. As for the Section 4, by considering additional regularity assumptions, we deduce the following optimal error estimate result. □

COROLLARY 5.1 Keep the assumptions stated in Theorem 4.1. Assume Ω is a polygonal/polyhedral domain, and $\{V^h\}$ is the family of linear element spaces defined by (4.31), corresponding to a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. Let u and $\{u_n^{hk}\}_{n=1}^N$ be solutions of Problems P_V and P_V^{hk}, respectively.

Assume $u_0 \in H^2(\Omega;\mathbb{R}^d)$, $u_1 \in H^1(\Omega;\mathbb{R}^d)$, and take $u_0^h, u_1^h \in V^h$ to be projections of u_0 and u_1, defined by (4.1). Under the following regularity condition

$$u \in C^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap H^3(0,T;H), \quad \dot{u}_\tau \in C(0,T;H^2(\Gamma_C;\mathbb{R}^d)),$$

we have the optimal order error estimate

$$\max_{1\leqslant n\leqslant N} \|u_n - u_n^{hk}\|_V + \max_{1\leqslant n\leqslant N} \|w_n - w_n^{hk}\|_H \leqslant C(h+k) \tag{5.37}$$

with $C > 0$ independent of h and k.

Proof. Let $v_j^h \in V^h$ be the finite element interpolant of u_j, $t \in [0,T]$ and $1 \leqslant j \leqslant N$. Then from ((W. Han & M. Sofonea, 2002)) we have

$$k\sum_{j=1}^{N} \|\dot{w}_j - \delta w_j\|_H^2 \leqslant ck^2 \|w\|^2_{H^2(0,T;H)},$$

$$\frac{1}{k}\sum_{j=1}^{N-1} \|(w_j - v_j^h) - (w_{j+1} - v_{j+1}^h)\|_H^2 \leqslant ch^2 \|u\|^2_{H^2(0,T;V)}.$$

Then, similarly to the proof of Corollary 4.1, we obtain (5.37) from the estimate (5.8). □

6. Numerical validation of the fully discrete error estimate

The aim of this section is to provide a numerical evidence of the convergence of the discrete scheme established in Section 5. The numerical solution is based on a iterative procedure which leads to a sequence of convex programming problems already used in (M. Barboteu *et al.*, 2013, 2014; M. Barboteu *et al.* , 2015; M. Barboteu *et al.*, 2015). For each “convexification" iteration, the value of the friction coefficient $\mu(\|w_\tau\|_{\mathbb{R}^d})$ is fixed to a given value depending on the tangential velocity solution w_τ found in the previous iteration. Then, the resulting nonsmooth convex iterative problems are solved by classical numerical methods. As a consequence, the frictional contact conditions can be treated by using a numerical approach based on the combination of a penalized method for the normal compliance terms with the augmented Lagrangian method (cf (P. Alart & A. Curnier, 1991; T. Laursen, 2002; P. Wriggers, 2002)) for the friction conditions. To this end, we consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the frictional contact element used for the geometrical discretization of the interface Γ_3. In the case presented below, the discretization is based on “node-to-rigid" contact element, which is composed by one node of Γ_3 and one Lagrange multiplier node. For more details on the discretization step and Computational Contact Mechanics, we refer to (P. Alart & A. Curnier, 1991; H.B. Khenous *et al.*, 2006,?; T. Laursen, 2002; P. Wriggers, 2002).

Numerical example. We consider the physical setting depicted in Figure 1. There, $\Omega = (0,L)\times(0,L) \subset \mathbb{R}^2$ with $L > 0$ and

$$\Gamma_N = [0,L]\times\{L\},\ \Gamma_C = [0,L]\times\{0\}.$$

The domain Ω represents the cross section of a three-dimensional linearly viscoelastic body subjected to the action of body forces during the dynamic process in such a way that the plane stress hypothesis is assumed. The body, located at a height of h, can impact with friction an obstacle on the part $\Gamma_C = [0,L]\times\{0\}$ of the boundary when it reaches the ground. Note that we used the viscosity in (3.1) for mathematical reasons, so far, but from the practical point of view, one may take the viscosity as small as one wishes. Here, since it is not our main interest, we choose to neglect it.

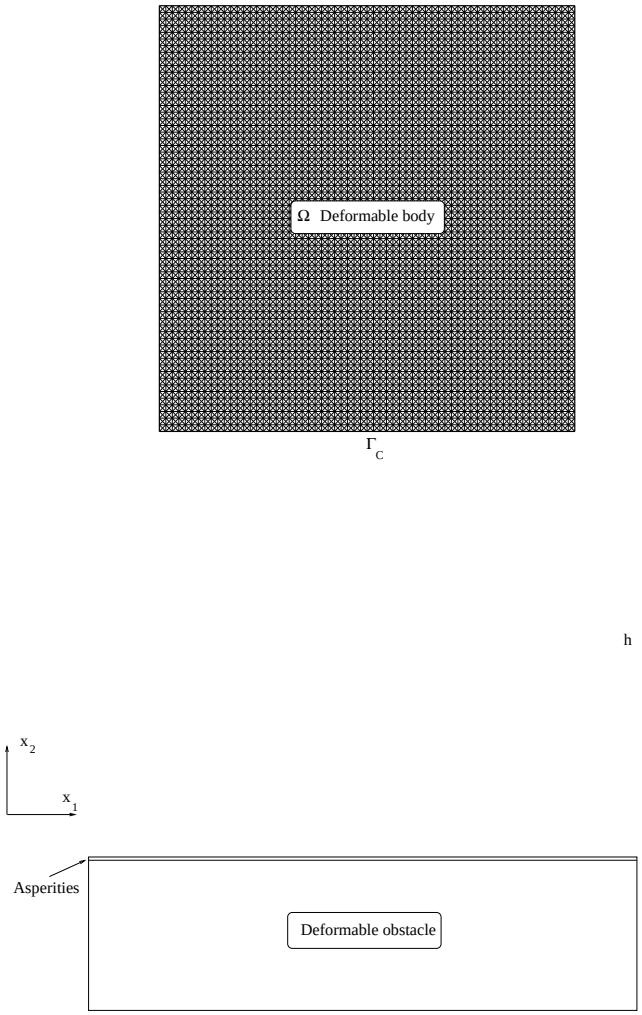

FIG. 1. Reference configuration of the two-dimensional body.

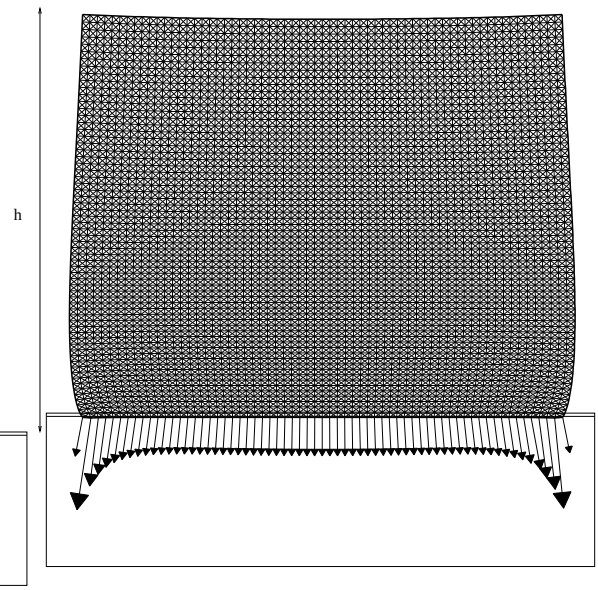

FIG. 2. Deformed mesh with the associated frictional contact forces on Γ_c

Therefore, the material response is governed by an elastic linear constitutive law defined by the elasticity tensor $\mathscr{E}$ given by

$$(\mathscr{E}\tau)_{\alpha\beta} = \frac{E\kappa}{(1+\kappa)(1-2\kappa)}(\tau_{11}+\tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \quad 1 \leqslant \alpha,\beta \leqslant 2,\ \forall \tau \in \mathbb{S}^2.$$

Here, E and κ are Young's modulus and Poisson's ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker delta.

The normal compliance function p is defined by

$$p(r) = c_v r_+.$$

For the coefficient of friction we choose a function $\mu : \mathbb{R}^d \to \mathbb{R}_+$ of the form

$$\mu(\|\dot{u}_\tau\|) = (a-b)\cdot e^{-\alpha\|\dot{u}_\tau\|} + b, \tag{6.1}$$

with $a, b, \alpha > 0$, $a \geqslant b$. Such a slip weakening phenomenon appears in the study of geophysical problems, see (C.H. Scholz, 1990) for details. Indeed, in this case the coefficient of friction decreases with the slip from the value a to the limit value b. And, for this reason, the corresponding friction law can be characterized as being nonmonotone. For the computation we use the following data:

$$\begin{aligned}
&L = 1\ m, \quad \rho = 1\ kg/m^3, \quad T = 2\ s,\\
&u_0 = (0,0)\ m, \quad u_1 = (0,-0.4)\ m/s, \quad g = 5.01\times 10^{-3}\ m, \quad h = 1\ m,\\
&E = 1\ Gpa, \quad \kappa = 0.3, \quad f_0 = (0,-0.2)\ GPa, \quad f_2 = (0,0)\ GPa.m,\\
&c_v = 1\ GPa, \quad \tfrac{1}{r} = 5\ GPa, \quad a = 1, \quad b = 0.2, \quad \alpha = 2000.
\end{aligned}$$

Error estimate. In order to check the convergence of the discrete scheme and to illustrate the optimal error estimate obtained in Section 5, we report in Figure 3 numerical solution errors in the energy norm defined by

$$\|u\|_E := \langle Bu, u\rangle_{V^*\times V}^{1/2}$$

which is equivalent to the norm $\|u\|_V$. Since the true solution u is not available, we use instead, the numerical solution corresponding to a fine discretization of Ω as the "reference" solution u_{ref} in computing the solution errors. Here, the numerical solution with $h = 1/256$ is taken to be the "reference" solution u_{ref}. This fine discretization corresponds to a problem with 132612 degrees of freedom, 131329 elements and was computed in 120573 CPU time (expressed in seconds) on a IBM computer equipped with Intel Dual core processors (Model 5148, 2.33 GHz). We observe in Figure 3 that the curve of the numerical error estimate is asymptotically linear, which is consistent with the theoretically predicted optimal linear convergence of the numerical solution established in Section 5.

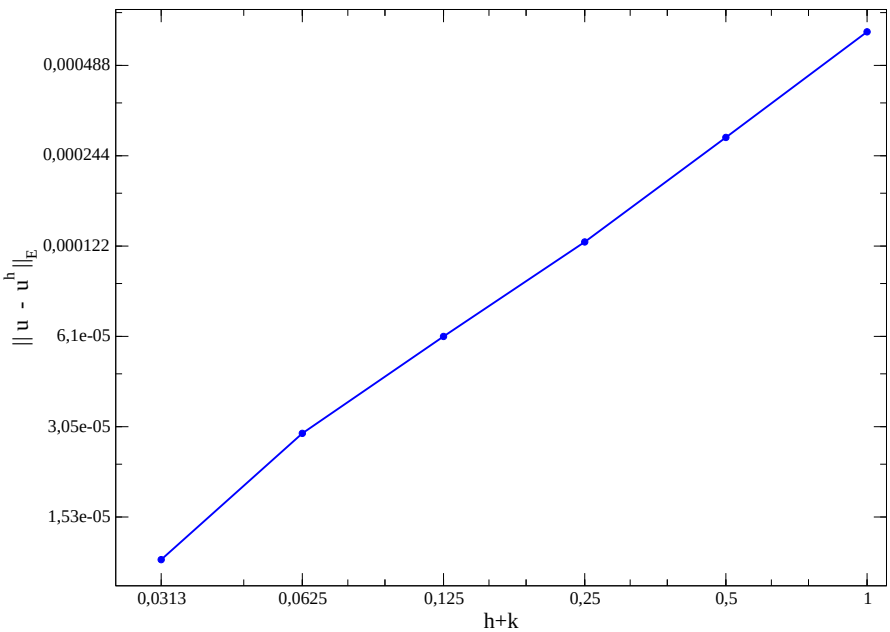

FIG. 3. Numerical errors.

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