Analysis of a dynamic frictional contact problem for hyperviscoelastic material with non-convex energy density

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Abstract
Using the time approximation method we obtain the existence of a weak solution for the dynamic contact problem with damping and a non-convex stored elastic energy function. On the contact boundary we assume the normal compliance law and the generalization of the Coulomb friction law which allows for non-monotone dependence of the friction force on the tangential velocity. The existence result is accompanied by two numerical examples, one of them showing lack of uniqueness for the numerical solution.

Keywords
Dynamics, hyperelasticity, viscosity, normal compliance, non-monotone friction, weak formulation, time approximation, existence result, non-uniqueness, numerical simulations

1. Introduction

The mathematical analysis of dynamic frictional contact problems within the framework of hyperelasticity today still remains a subject of research which is incomplete and difficult to apprehend. Indeed, this challenging field to work on involves several difficulties such as the non-convexity of the stored energy of the material and the non-monotonicity of the frictional contact laws. Therefore, the establishment of the existence of variational weak solutions represents a non-trivial and very interesting issue for such nonlinear elastodynamic problems. In the context of convex energy materials, there exist numerous references dealing with the variational solvability of dynamic contact problems. Without being exhaustive, we can cite the works [1–17] as well as all the references they include.

In this work, we study the system of partial differential equations which models the evolution of the displacement of a body made from hyperviscoelastic material in frictional contact with a foundation. Our results are twofold: we obtain the result on the existence of a weak solution and we present the examples of the numerical simulations. The existence result does not need either the assumption that the elastic stored energy function is convex, or that its derivative is Lipschitz. The argument is based on the time approximation method and the

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so-called propagation of regularity argument established in [18] and later developed in [19]. The novelty of the model with respect to [18, 19] is that we assume *contact boundary conditions*: the normal compliance condition and the multi-valued friction condition. This last condition is described by a multi-valued and non-monotone relation between the friction force density and normal velocity (see [17, 20]), which leads to the partial differential inclusion rather than the equation. Still, it is possible to use the arguments of [18, 19] to get the solution existence. Theoretical results are complemented by numerical examples, one of them demonstrating that, at least on a numerical level, we observe a lack of solution uniqueness for the analyzed problem.

The structure of the article is as follows: the problem is formulated in Section 2, where the assumptions are also presented and discussed, and the main result (Theorem 2.3) is stated. Sections 3 and 4 are devoted to the proof of Theorem 2.3. Finally, in Section 5 two numerical examples are presented and discussed.

2. Problem statement, assumptions, and main result

Formulation of the problem. Let $\Omega \subset \mathbb{R}^d$, where $d \in \{2, 3\}$, be a reference domain occupied by a nonlinearly viscoelastic body and let $(0, T)$ be the time interval of interest. The boundary of Ω is sufficiently smooth (Lipschitz) and divided into three relatively open and mutually disjoint parts $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_C} \cup \overline{\Gamma_N}$. Each of these parts is measurable with respect to $(d-1)$-dimensional Hausdorff measure $\mathcal{H}_{d-1}$, and can possibly be an empty set. By $\mathbb{M}^{d\times d}$ we denote the space of the matrices $d \times d$. The symbol $|\cdot|$ denotes, depending on the context, the norm in $\mathbb{R}^d$ or $\mathbb{M}^{d\times d}$. We are looking for the displacement function $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$ and the stress tensor $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{M}^{d\times d}$ such that

$$\rho\ddot{\boldsymbol{u}} = \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$
$$\boldsymbol{\sigma} = \mathcal{G}(\nabla\boldsymbol{u}) + \mathcal{A}(\nabla\dot{\boldsymbol{u}}) \quad \text{in} \quad \Omega \times (0, T), \tag{2}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \tag{3}$$
$$\dot{\boldsymbol{u}}(0) = \boldsymbol{v}(0) = \boldsymbol{u}_1 \quad \text{in} \quad \Omega, \tag{4}$$
$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_D \times (0, T), \tag{5}$$
$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_1 \quad \text{on} \quad \Gamma_N \times (0, T), \tag{6}$$
$$-\sigma_\nu = p_\nu(u_\nu) \quad \text{on} \quad \Gamma_C \times (0, T), \tag{7}$$
$$-\boldsymbol{\sigma}_\tau \in |\sigma_\nu|\partial j(\dot{\boldsymbol{u}}_\tau) \quad \text{on} \quad \Gamma_C \times (0, T). \tag{8}$$

In the above formulation $\operatorname{Div}\boldsymbol{\sigma} = \boldsymbol{\sigma}_{ij,j}$ and $\nabla\boldsymbol{u} = \boldsymbol{u}_{i,j}$, where we use the summation convention over the repeated indexes and the index after a comma denotes the derivative taken with respect to the corresponding variable. The vector $\boldsymbol{\nu}$ is the unit outer normal vector on $\partial\Omega$ and $\boldsymbol{\tau}$ is its associated tangent vector. We use the decompositions of displacements and stresses (and in general, all vector functions) on the boundary into their normal and tangent components

$$u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}, \quad \sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{\nu}, \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}.$$

The function $\boldsymbol{f}_0 : \Omega \times (0, T) \to \mathbb{R}^d$ is the density of mass forces, and $\boldsymbol{f}_1 : \Gamma_N \times (0, T) \to \mathbb{R}^d$ is the density of boundary tractions. The number $\rho > 0$ denotes the mass density of the material and is assumed to be a constant in the rest of the paper, for the sake of simplicity. The function $\mathcal{G} : \mathbb{M}^{d\times d} \to \mathbb{M}^{d\times d}$ is the elasticity operator, which can be nonlinear, and $\mathcal{A} : \mathbb{M}^{d\times d} \to \mathbb{M}^{d\times d}$ is the linear viscosity operator. The normal compliance function $p_\nu : \mathbb{R} \to \mathbb{R}$ is assumed to attain only non-negative values and hence (8) can be equivalently rewritten as

$$-\boldsymbol{\sigma}_\tau \in p_\nu(u_\nu)\partial j(\dot{\boldsymbol{u}}_\tau) \quad \text{on} \quad \Gamma_C \times (0, T).$$

The friction multifunction $\partial j : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is the Clarke subdifferential of a certain locally Lipschitz potential $j : \mathbb{R}^d \to \mathbb{R}$. Multifunctions of this type are commonly used to describe friction (see [17, 21]). As j is defined on the finite-dimensional space $\mathbb{R}^d$, its Clarke subdifferential is given by

$$\partial j(\boldsymbol{s}) = \overline{\operatorname{conv}}\left\{\lim_{n\to\infty}\nabla j(\boldsymbol{s}_n) \;:\; \boldsymbol{s}_n \notin S \cup N,\ \boldsymbol{s}_n \to \boldsymbol{s},\ \text{sequence}\,\{\nabla j(\boldsymbol{s}_n)\}\ \text{is convergent}\right\},$$

where S is the Lebesgue null set of non-differentiability points of j and N is any Lebesgue null set. For the general definition of the Clarke subdifferential, when the domain of j can be an infinite-dimensional Banach space, as well as for its properties, the reader is referred to [22, 23].

Assumptions on the problem data. We take $p \geq 2$. By p' we denote its conjugate exponent, such that $1/p + 1/p' = 1$. We also define the exponent q such that

$$q \in \begin{cases} (1,\infty) & \text{if} \quad d = 2, \\ (1,\infty) & \text{if} \quad d = 3 \text{ and } p \geq 3, \\ (1, 2p/(3-p)) & \text{if} \quad d = 3 \text{ and } p \in [2,3). \end{cases}$$

We will denote the exponent conjugate of q by q'.

Lemma 2.1. *The choice of p and q guarantees that the trace operator $\gamma : W^{1,p}(\Omega)^d \to L^q(\partial\Omega)^d$ is compact and the trace operator $\gamma_\delta : W^{1-\delta,p}(\Omega)^d \to L^q(\partial\Omega)^d$ is continuous for a small constant $\delta > 0$.*

Proof. By [24, Theorem 1.4.3.2, p. 26] the embedding $W^{1,p}(\Omega)^d \subset W^{1-\delta,p}(\Omega)^d$ is compact for a small constant $\delta > 0$. Let us first consider the situation $p < d$. This holds only if $d = 3$ and $p \in [2,3)$. If $q \in (1, 2p/(3-p))$, we can find small $\epsilon, \delta > 0$ such that $q = 2p/(3 - p + p(\delta + \epsilon))$. Using the Sobolev embedding theorem we obtain $W^{1-\delta,p}(\Omega)^d \subset W^{s,q}(\Omega)^d$, where $s = \epsilon + 1/q$ (see [24, Theorem 1.4.4.1, p. 27]). Now the trace theorem for fractional Sobolev spaces (see [24, Theorem 1.5.1.2, p. 37]) implies the continuity of the trace operator $\gamma_s : W^{s,q}(\Omega)^d \to L^q(\partial\Omega)^d$. We can compose the two embeddings (one of them being compact) and the trace operator to obtain the assertion for the case $p < d$. If $p = d$, then [24, Theorem 1.4.4.1, p. 27] implies the continuity of embedding $W^{1-\delta,p}(\Omega)^d \subset W^{s,q}(\Omega)^d$ for $q = (d-1)/(\delta + \epsilon)$ and $s = \epsilon + 1/q$, and [24, Theorem 1.5.1.2, p. 37] implies the continuity of the trace operator $\gamma_s : W^{s,q}(\Omega)^d \to L^q(\partial\Omega)^d$. Finally, if $p > d$, [24, Theorem 1.4.4.1, p. 27] implies that we can always choose small $\delta > 0$ such that the embedding $W^{1-\delta,p}(\Omega)^d \subset C(\overline{\Omega})^d$ is continuous, and, clearly, the trace $\gamma_C : C(\overline{\Omega})^d \to L^q(\partial\Omega)^d$ is continuous for any $q \in (1,\infty)$. Hence, the assertion for the case $p > d$ also holds. □

We make the following assumptions.

(H1) There exists $\Phi \in C^1(\mathbb{M}^{d\times d})$ such that $\mathcal{G}(\boldsymbol{M}) = \frac{\partial \Phi(\boldsymbol{M})}{\partial \boldsymbol{M}}$ for every matrix $\boldsymbol{M} \in \mathbb{M}^{d\times d}$.

(H2) It holds that $|\mathcal{G}(\boldsymbol{M})| \leq C_{\mathcal{G}}(1 + |\boldsymbol{M}|^{p-1})$ for every $\boldsymbol{M} \in \mathbb{M}^{d\times d}$ with a constant $C_{\mathcal{G}} \geq 0$. Moreover, $\alpha|\boldsymbol{M}|^p - \beta \leq \Phi(\boldsymbol{M}) \leq C_\Phi(|\boldsymbol{M}|^p + 1)$ for every $\boldsymbol{M} \in \mathbb{M}^{d\times d}$ with constants $\alpha, \beta, C_\Phi \geq 0$.

(H3) $(\mathcal{G}(\boldsymbol{M}) - \mathcal{G}(\overline{\boldsymbol{M}})) : (\boldsymbol{M} - \overline{\boldsymbol{M}}) \geq -K|\boldsymbol{M} - \overline{\boldsymbol{M}}|^2$ for every $\boldsymbol{M}, \overline{\boldsymbol{M}} \in \mathbb{M}^{d\times d}$ with a constant $K \geq 0$.

(H4) $\mathcal{A} : \mathbb{M}^{d\times d} \to \mathbb{M}^{d\times d}$ is linear and bounded with $\mathcal{A}(\boldsymbol{M}) : \boldsymbol{M} \geq \lambda|\boldsymbol{M}|^2$ for every $\boldsymbol{M} \in \mathbb{M}^{d\times d}$ with a constant $\lambda > 0$.

(H5) It holds that $p_v \in C(\mathbb{R}; [0,\infty))$ and p_v is monotone increasing with $p_v(s) = 0$ for every $s \leq 0$. We assume that $|p_v(s)| \leq C_p(1 + |s|^{q-1})$ for every $s > 0$ with a constant $C_p \geq 0$.

(H6) The function $j : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz. Moreover, $\max_{\boldsymbol{\zeta} \in \partial j(\boldsymbol{s})} |\boldsymbol{\zeta}| \leq C_j$ for every $\boldsymbol{s} \in \mathbb{R}^d$ with a constant $C_j > 0$ and $\boldsymbol{\zeta} \cdot \boldsymbol{s} \geq 0$ for every $\boldsymbol{s} \in \mathbb{R}^d$ and $\boldsymbol{\zeta} \in \partial j(\boldsymbol{s})$.

(H7) $\boldsymbol{f}_0 \in L^2(0,T; L^{p_1}(\Omega))$ and $\boldsymbol{f}_1 \in L^2(0,T; L^{p_2}(\Gamma_N))$, where

$$p_1 \begin{cases} \in (1,\infty) & \text{if} \quad d = 2, \\ = \frac{6}{5} & \text{if} \quad d = 3, \end{cases} \quad \text{and} \quad p_2 \begin{cases} \in (1,\infty) & \text{if} \quad d = 2, \\ = \frac{4}{3} & \text{if} \quad d = 3. \end{cases}$$

Assumptions (H1) to (H3) on the relation between the elastic stress and displacement gradient are the same as in Friesecke and Dolzmann's paper [18]. Assumption (H1) means that the relation between the elastic part of the stress tensor and the displacement gradient is given through the derivative of the stored energy function Φ. This function is not required to be convex, indeed, in such a case $\mathcal{G}$ would be monotone, and, clearly, (H3) is more general. In fact one can easily prove that (H3) is equivalent to the statement that $\boldsymbol{M} \to \mathcal{G}(\boldsymbol{M}) + K\boldsymbol{M}$ is monotone, which in turn is equivalent to the fact that the functional $\boldsymbol{M} \to \Phi(\boldsymbol{M}) + \frac{K}{2}|\boldsymbol{M}|^2$ is convex, that is, Φ is a quadratic perturbation of a convex functional. If $\Phi \in C^2(\mathbb{M}^{d\times d})$ the assumption (H3) is more general than the Andrews–Ball condition [25],

$$(\mathcal{G}(\boldsymbol{M}) - \mathcal{G}(\overline{\boldsymbol{M}})) : (\boldsymbol{M} - \overline{\boldsymbol{M}}) \geq 0 \quad \text{for all} \quad |\boldsymbol{M}|, |\overline{\boldsymbol{M}}| \geq R \quad \text{and some} \quad R > 0;$$

see [18, Lemma 1.1]. Also note that we do not require a global Lipschitz condition on $\mathcal{G}$, for example, the law given by $\Phi(\boldsymbol{M}) = |\boldsymbol{M}|^p$ does not lead to globally Lipschitz $\mathcal{G}$, and (H1) to (H3) still hold.

The coercivity in (H4) can be replaced with the more general assumption

$$\int_\Omega \mathcal{A}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{u}\, d\boldsymbol{x} \geq \lambda \|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)^{d\times d}} - \kappa \|\boldsymbol{u}\|^2_{L^2(\Omega)^d} \quad \text{for every} \quad \boldsymbol{u} \in H^1(\Omega)^d,$$

which by the second Korn inequality (see e.g. [26]) would allow us to consider the situation where $\mathcal{A}$ is a function of the symmetric part of $\nabla \boldsymbol{u}$, that is, $\mathcal{A}(\nabla \boldsymbol{u}) = \overline{\mathcal{A}}\left(\frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\intercal)\right)$. The argument in the proof of existence in Sections 3 and 4 is already quite technical and involved, so to make the exposition simpler we deal with $\mathcal{A}(\boldsymbol{M}) : \boldsymbol{M} \geq \lambda |\boldsymbol{M}|^2$. The interested reader is asked to modify the estimates below to account for this more general situation.

Note that since the elasticity term $\mathcal{G}$ is not assumed to be monotone we need coercivity of $\mathcal{A}$ to obtain the solvability of the discretized problem (see Lemma 3.1), a priori estimates for the solutions of the discretized problem (see Theorem 3.2), and propagation of the regularity property needed to pass to the limit in the term with $\mathcal{G}$ (see Lemma 4.6). We leave open the question of whether the existence result presented here holds if $\mathcal{A}$ is non-coercive (for example $\mathcal{A} = 0$), but the method presented here does not work if we want to keep non-monotone $\mathcal{G}$ without coercivity of $\mathcal{A}$.

We consider the normal compliance condition (7) which allows for the infinite penetration of the foundation. We leave open the question of whether the proposed method can be generalized to the case of non-penetration condition

$$\sigma_\nu + p_\nu(u_\nu) \leq 0, \quad u_\nu \leq g, \quad (\sigma_\nu + p_\nu(u_\nu))(u_\nu - g) = 0 \quad \text{on} \quad \Gamma_C \times (0, T).$$

We expect that, using the formulation based on variational inequalities, the presented existence result should also hold for the non-penetration case, in the framework of the article by Barboteu et al. [1].

Typically, linear growth is needed in the boundary conditions on the contact boundary; see [17]. We make use of Lemma 2.1, whence due to the Sobolev embedding on the boundary we can consider the growth condition with power $q - 1$. The condition $\boldsymbol{\zeta} \cdot \boldsymbol{s} \geq 0$ in (H6) means that the friction force is directed opposite to velocity, which is natural. The constant C_j represents the maximum static friction coefficient. The bound by C_j in the friction term signifies that the kinetic friction cannot exceed the maximum static friction. The multifunction ∂j can be non-monotone and does not have to satisfy the one-sided Lipschitz relation like in (H3). This lack of monotonicity signifies that kinetic friction is less than static friction and rapid drops of friction with the increase of velocity are allowed.

Also note that, in contrast to results such as [27, 28] we make no assumptions on the smallness of the constants present in the model. The reason is that we prove only existence and not uniqueness of the solution. Arguments based on the Banach fixed point theorem used in [27, 28] typically need the strengthened assumptions involving smallness of the constants present in the model, and they give stronger results, namely solution uniqueness. In the case of our model, due to the type of nonlinearity $\mathcal{G}$ we do not expect solution uniqueness even if we assume smallness of constants present in the boundary terms. Moreover, we have numerical evidence that the solution is expected to be non-unique; see Section 5.

Weak formulation and the main theorem. Proceeding in a standard way we get the following weak form of the above problem, where $\boldsymbol{\xi} : \Omega \to \mathbb{R}^d$ is a test function which satisfies the same Dirichlet boundary conditions as the sought solution $\boldsymbol{u}$:

$$\begin{aligned} &\int_\Omega \rho \ddot{\boldsymbol{u}} \cdot \boldsymbol{\xi}\, d\boldsymbol{x} + \int_\Omega \mathcal{A}(\nabla \dot{\boldsymbol{u}}) : \nabla \boldsymbol{\xi}\, d\boldsymbol{x} + \int_\Omega \mathcal{G}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{\xi}\, d\boldsymbol{x} + \int_{\Gamma_C} p_\nu(u_\nu)\xi_\nu\, dS + \int_{\Gamma_C} p_\nu(u_\nu)\boldsymbol{\eta} \cdot \boldsymbol{\xi}_\tau\, dS \\ &\quad = \int_{\Gamma_N} \boldsymbol{f}_1 \cdot \boldsymbol{\xi}\, dS + \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{\xi}\, d\boldsymbol{x}, \end{aligned} \tag{9}$$

with $\boldsymbol{\eta}(\boldsymbol{x}, t) \in \partial j(\dot{\boldsymbol{u}}_\tau(\boldsymbol{x}, t))$, and the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$. To give more meaning to the weak form we need several Sobolev-type spaces. We denote $H = L^2(\Omega)^d$ and define $V = \{\boldsymbol{v} \in W^{1,p}(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_{\mathbf{D}}\}$, and we use the standard notation for the Sobolev–Bochner spaces. Duality pairing between V' and V is denoted by $\langle \cdot, \cdot \rangle$ and the norm of V is denoted by $\| \cdot \|$. For spaces other than V the corresponding norms are denoted by appropriate subscripts. We also define $W = \{\boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_{\mathbf{D}}\}$. Clearly $V \subset W \subset H \subset W' \subset V'$ with all embeddings being continuous. We are now in a position to introduce the definition of the weak solution for the considered problem.

Definition 2.2. The function $\boldsymbol{u} \in L^\infty(0,T;V)$ with $\dot{\boldsymbol{u}} \in L^\infty(0,T;H) \cap L^2(0,T;W)$ and $\ddot{\boldsymbol{u}} \in L^2(0,T;V')$ is a weak solution for problem (1)–(8) if there exists $\boldsymbol{\eta} \in L^\infty(0,T;L^\infty(\Omega)^d)$ with $\boldsymbol{\eta}(\boldsymbol{x},t) \in \partial j(\dot{\boldsymbol{u}}_\tau(\boldsymbol{x},t))$ a.e. in $\Gamma_C \times (0,T)$ such that for any $\boldsymbol{\xi} \in L^2(0,T;V)$ it holds that

$$\begin{aligned}\int_0^{\mathrm{T}} \langle \rho \ddot{\boldsymbol{u}}(t), \boldsymbol{\xi}(t)\rangle \, dt + \int_0^{\mathrm{T}} \int_\Omega \mathcal{A}(\nabla \dot{\boldsymbol{u}}(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt + \int_0^{\mathrm{T}} \int_\Omega \mathcal{G}(\nabla \boldsymbol{u}(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt \\ + \int_0^{\mathrm{T}} \int_{\Gamma_C} p_\nu(u_\nu(t)) \xi_\nu(t) \, dS \, dt + \int_0^{\mathrm{T}} \int_{\Gamma_C} p_\nu(u_\nu(t)) \boldsymbol{\eta}(t) \cdot \boldsymbol{\xi}_\tau(t) \, dS \, dt \\ = \int_0^{\mathrm{T}} \int_{\Gamma_N} \boldsymbol{f}_1(t) \cdot \boldsymbol{\xi}(t) \, dS \, dt + \int_0^{\mathrm{T}} \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt, \end{aligned} \tag{10}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1. \tag{11}$$

We formulate the main theoretical result of our article.

Theorem 2.3. *If assumptions (H1) to (H7) hold and $\boldsymbol{u}_0 \in V$, $\boldsymbol{u}_1 \in H$ then the problem* (1)–(8) *has a weak solution.*

The proof of the theorem relies on the time approximation method and is given in Sections 3 and 4. Throughout Sections 3 and 4 we always assume that $\boldsymbol{u}_0 \in V$, $\boldsymbol{u}_1 \in H$ and that the assumptions (H1) to (H7) hold. By C we will denote a generic positive constant which may change from line to line.

3. Time-discretized problem: Existence and a priori estimates

The regularity of $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$ given in (H7) implies that there exists $\boldsymbol{f} \in L^2(0,T;W')$ such that

$$\int_0^{\mathrm{T}} \langle \boldsymbol{f}(t), \boldsymbol{\xi}(t)\rangle_{W' \times W} \, dt = \int_0^{\mathrm{T}} \int_{\Gamma_N} \boldsymbol{f}_1(t) \cdot \boldsymbol{\xi}(t) \, dS \, dt + \int_0^{\mathrm{T}} \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt$$

for every $\boldsymbol{\xi} \in L^2(0,T;W)$.

Let us introduce some material below concerning the time discretization. We consider a uniform discretization of the time interval of interest $(0,T)$ characterized by the discrete time instants $t_j = jh_N$ for $j \in \{0,\dots,N\}$ where $h_N = T/N$ is the time step and N the number of time steps. For simplicity we will write h in place of h_N and $h \to 0$ in place of $N \to \infty$. In order to establish the time approximation of the problem governed by the equation (10) and the initial conditions (11), we consider the discrete displacement $\boldsymbol{u}_h^j$ and the discrete velocity $\boldsymbol{v}_h^j$ which are the approximations of the displacement $\boldsymbol{u}(t)$ and the velocity $\dot{\boldsymbol{u}}(t)$ respectively at the time $t = t_j$ for $j \in \{0,\dots,N\}$. Then, the discretizations of the velocity $\dot{\boldsymbol{u}}(t)$ and the acceleration $\ddot{\boldsymbol{u}}(t)$ are given by an implicit backward Euler scheme as follows:

$$\dot{\boldsymbol{u}}(t_j) \approx \boldsymbol{v}_h^j = \frac{\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}}{h} \quad \text{and} \quad \ddot{\boldsymbol{u}}(t_j) \approx \frac{\boldsymbol{v}_h^j - \boldsymbol{v}_h^{j-1}}{h} = \frac{\boldsymbol{u}_h^j - 2\boldsymbol{u}_h^{j-1} + \boldsymbol{u}_h^{j-2}}{h^2} \quad \text{for} \quad j \in \{1,\dots,N\}. \tag{12}$$

Moreover, we approximate $\boldsymbol{u}_1 \in H$ by a sequence $\boldsymbol{u}_{1h} \in V$ such that $\boldsymbol{u}_{1h} \to \boldsymbol{u}_1$ strongly in H as $h \to 0$. We can always choose the sequence $\boldsymbol{u}_{1h}$ such that for a constant $C > 0$ we have

$$\|\nabla \boldsymbol{u}_{1h}\|_{L^2(\Omega)^{d\times d}} \le \frac{C}{\sqrt{h}}. \tag{13}$$

Indeed, if $\|\nabla \boldsymbol{u}_{1h}\|_{L^2(\Omega)^{d\times d}}$ does not go to infinity as $h \to 0$, the assertion is clear, and if it goes to infinity, we can always define a new sequence

$$\hat{\boldsymbol{u}}_{1h} = \boldsymbol{u}_{1\hat{h}} \quad \text{where} \quad \hat{h} = \min\left\{\bar{h} \ : \ \|\nabla \boldsymbol{u}_{1\bar{h}}\|_{L^2(\Omega)^{d\times d}} \le \frac{C}{\sqrt{h}}\right\}.$$

The bound will be used in Lemma 4.1 to obtain the estimate on the piecewise linear interpolant of the velocity from the bound on the piecewise constant one. We also define $\boldsymbol{u}_h^0 = \boldsymbol{u}_0$, $\boldsymbol{v}_h^0 = \boldsymbol{u}_{1h}$ and $\boldsymbol{u}_h^{-1} = \boldsymbol{u}_0 - h\boldsymbol{u}_{1h}$. Denoting the interval $I_h^j = (t_{j-1}, t_j]$ we can define the zeroth-order quasi-interpolant of $\boldsymbol{f}$ as

$$\boldsymbol{f}_h(t) = \boldsymbol{f}_h^j \quad \text{for} \quad t \in I_h^j \quad \text{with} \quad \boldsymbol{f}_h^j = \frac{1}{h}\int_{I_h^j} \boldsymbol{f}(t)\,dt \quad \text{for} \quad j \in \{1,\dots,N\}.$$

Using [29, Lemma 3.3] it follows that

$$\|\boldsymbol{f}_h\|_{L^2(0,T;W')} \le \|\boldsymbol{f}\|_{L^2(0,T;W')} \quad \text{and} \quad \lim_{h\to 0}\|\boldsymbol{f}_h - \boldsymbol{f}\|_{L^2(0,T;W')} = 0. \tag{14}$$

Therefore, we are in position to formulate the time-discretized problem as follows.

Problem ($\mathbf{P}_h$). Find $\{\boldsymbol{u}_h^j\}_{j=1,\dots,N} \subset V$ and $\{\boldsymbol{\eta}_h^j\}_{j=1,\dots,N} \subset L^\infty(\Gamma_C)^d$ such that for $j \in \{1,\dots,N\}$ we have

$$\begin{aligned}\int_\Omega \rho\frac{\boldsymbol{u}_h^j - 2\boldsymbol{u}_h^{j-1} + \boldsymbol{u}_h^{j-2}}{h^2}\cdot\boldsymbol{\xi}\,d\boldsymbol{x} + \int_\Omega \mathcal{A}\left(\frac{\nabla\boldsymbol{u}_h^j - \nabla\boldsymbol{u}_h^{j-1}}{h}\right) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x} + \int_\Omega \mathcal{G}(\nabla\boldsymbol{u}_h^j) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x}\\ + \int_{\Gamma_C} p_\nu(u_{h\nu}^j)\xi_\nu\,dS + \int_{\Gamma_C} p_\nu(u_{h\nu}^j)\boldsymbol{\eta}_h^j\cdot\boldsymbol{\xi}_\tau\,dS = \langle \boldsymbol{f}_h^j, \boldsymbol{\xi}\rangle_{W'\times W} \quad \text{for every} \quad \boldsymbol{\xi} \in V,\end{aligned} \tag{15}$$

with

$$\boldsymbol{\eta}_h^j \in S_{\partial j}^\infty\left(\frac{\boldsymbol{u}_{h\tau}^j - \boldsymbol{u}_{h\tau}^{j-1}}{h}\right), \tag{16}$$

that is, $\boldsymbol{\eta}_h^j$ is the L^∞ selection of ∂j at $(\boldsymbol{u}_{h\tau}^j - \boldsymbol{u}_{h\tau}^{j-1})/h$ meaning that $\boldsymbol{\eta}_h^j(\boldsymbol{x}) \in \partial j\left((\boldsymbol{u}_{h\tau}^j(\boldsymbol{x}) - \boldsymbol{u}_{h\tau}^{j-1}(\boldsymbol{x}))/h\right)$ for a.e. $\boldsymbol{x} \in \Gamma_C$ and $\boldsymbol{\eta}_h^j \in L^\infty(\Gamma_C)^d$.

Lemma 3.1. *Let $h_0 = 2\lambda/K$. Time-discretized problem (P_h) has a solution for $h \in (0, h_0]$.*

Proof. We proceed by induction. As $\boldsymbol{u}_h^{-1}$ and $\boldsymbol{u}_h^0$ are known, it suffices to prove that if $\boldsymbol{u}_h^{j-1} \in V$ and $\boldsymbol{u}_h^{j-2} \in V$ are given, there exist $\boldsymbol{u}_h^j \in V$ and $\boldsymbol{\eta}_h^j \in L^\infty(\Gamma_C)^d$ satisfying (15) and (16). We rewrite (15) as

$$\begin{aligned}&\frac{\rho}{h^2}\int_\Omega \boldsymbol{u}_h^j\cdot\boldsymbol{\xi}\,d\boldsymbol{x} + \frac{1}{h}\int_\Omega \mathcal{A}(\nabla\boldsymbol{u}_h^j) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x} + \int_\Omega \mathcal{G}(\nabla\boldsymbol{u}_h^j) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x}\\ &\qquad + \int_{\Gamma_C} p_\nu(u_{h\nu}^j)\xi_\nu\,dS + \int_{\Gamma_C} p_\nu(u_{h\nu}^j)\boldsymbol{\eta}_h^j\cdot\boldsymbol{\xi}_\tau\,dS\\ &= \frac{\rho}{h^2}\int_\Omega (2\boldsymbol{u}_h^{j-1} - \boldsymbol{u}_h^{j-2})\cdot\boldsymbol{\xi}\,d\boldsymbol{x} + \frac{1}{h}\int_\Omega \mathcal{A}(\nabla\boldsymbol{u}_h^{j-1}) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x} + \langle \boldsymbol{f}_h^j, \boldsymbol{\xi}\rangle_{W'\times W} \quad \text{for every} \quad \boldsymbol{\xi} \in V.\end{aligned} \tag{17}$$

First, we prove that the right-hand side of the above equation defines a linear and continuous functional of the variable $\boldsymbol{\xi} \in V$. The assertion follows from the obvious estimate which uses (H4):

$$\begin{aligned}&\left|\frac{\rho}{h^2}\int_\Omega (2\boldsymbol{u}_h^{j-1} - \boldsymbol{u}_h^{j-2})\cdot\boldsymbol{\xi}\,d\boldsymbol{x} + \frac{1}{h}\int_\Omega \mathcal{A}(\nabla\boldsymbol{u}_h^{j-1}) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x} + \langle \boldsymbol{f}_h^j, \boldsymbol{\xi}\rangle_{W'\times W}\right|\\ &\quad \le \frac{\rho}{h^2}\int_\Omega (2|\boldsymbol{u}_h^{j-1}| + |\boldsymbol{u}_h^{j-2}|)|\boldsymbol{\xi}|\,d\boldsymbol{x} + \frac{1}{h}\int_\Omega |\mathcal{A}(\nabla\boldsymbol{u}_h^{j-1})||\nabla\boldsymbol{\xi}|\,d\boldsymbol{x} + \|\boldsymbol{f}_h^j\|_{W'}\|\boldsymbol{\xi}\|_W\\ &\quad \le \frac{C}{h^2}(\|\boldsymbol{u}_h^{j-1}\|_H + \|\boldsymbol{u}_h^{j-2}\|_H)\|\boldsymbol{\xi}\|_H + \frac{C}{h}\|\nabla\boldsymbol{u}_h^{j-1}\|_{L^2(\Omega)^{d\times d}}\|\nabla\boldsymbol{\xi}\|_{L^2(\Omega)^{d\times d}} + C\|\boldsymbol{f}_h^j\|_{W'}\|\boldsymbol{\xi}\| \le C\|\boldsymbol{\xi}\|.\end{aligned}$$

To deal with the left-hand side of (17) we introduce the three operators $H_1, H_2 : V \to V'$ and $H_3 : V \to 2^{V'}$ as follows:

$$\langle H_1 \boldsymbol{w}, \boldsymbol{\xi} \rangle = \frac{1}{h} \int_\Omega \mathcal{A}(\nabla \boldsymbol{w}) : \nabla \boldsymbol{\xi} \, d\boldsymbol{x} + \int_\Omega \mathcal{G}(\nabla \boldsymbol{w}) : \nabla \boldsymbol{\xi} \, d\boldsymbol{x},$$

$$\langle H_2 \boldsymbol{w}, \boldsymbol{\xi} \rangle = \frac{\rho}{h^2} \int_\Omega \boldsymbol{w} \cdot \boldsymbol{\xi} \, d\boldsymbol{x},$$

$$\boldsymbol{\zeta} \in H_3(\boldsymbol{w}) \Leftrightarrow \langle \boldsymbol{\zeta}, \boldsymbol{\xi} \rangle = \int_{\Gamma_C} p_\nu(w_\nu) \xi_\nu \, dS + \int_{\Gamma_C} p_\nu(w_\nu) \boldsymbol{\eta} \cdot \boldsymbol{\xi}_\tau \, dS \quad \text{with} \quad \boldsymbol{\eta} \in S^\infty_{\partial j} \left(\frac{\boldsymbol{w}_\tau - \boldsymbol{u}^{j-1}_{h\tau}}{h} \right).$$

We will prove that the multi-valued operator $H_1 + H_2 + H_3 : V \to 2^{V'}$ is surjective. To this end we will use Theorem A.6 in Appendix 1. We will first show that H_1, H_2, and H_3 are well-defined, pseudomonotone and bounded.

H_1 is well defined and bounded. By (H2) and (H4) for any $\boldsymbol{w}, \boldsymbol{\xi} \in V$

$$\begin{aligned} |\langle H_1 \boldsymbol{w}, \boldsymbol{\xi} \rangle| &\le \frac{1}{h} \int_\Omega |\mathcal{A}(\nabla \boldsymbol{w})| |\nabla \boldsymbol{\xi}| \, d\boldsymbol{x} + \int_\Omega |\mathcal{G}(\nabla \boldsymbol{w})| |\nabla \boldsymbol{\xi}| \, d\boldsymbol{x} \\ &\le \frac{C}{h} \int_\Omega |\nabla \boldsymbol{w}| |\nabla \boldsymbol{\xi}| \, d\boldsymbol{x} + C \int_\Omega (1 + |\nabla \boldsymbol{w}|^{p-1}) |\nabla \boldsymbol{\xi}| \, d\boldsymbol{x} \le \frac{C}{h} \|\boldsymbol{w}\| \|\boldsymbol{\xi}\| + C \|\boldsymbol{\xi}\| + C \|\boldsymbol{w}\|^{p-1} \|\boldsymbol{\xi}\|. \end{aligned}$$

H_1 is monotone. Taking $\boldsymbol{w}_1, \boldsymbol{w}_2 \in V$ and using (H3) and (H4), we obtain

$$\begin{aligned} &\langle H_1 \boldsymbol{w}_1 - H_1 \boldsymbol{w}_2, \boldsymbol{w}_1 - \boldsymbol{w}_2 \rangle \\ &\quad = \int_\Omega (\mathcal{G}(\nabla \boldsymbol{w}_1) - \mathcal{G}(\nabla \boldsymbol{w}_2)) : (\nabla \boldsymbol{w}_1 - \nabla \boldsymbol{w}_2) \, d\boldsymbol{x} + \frac{1}{h} \int_\Omega \mathcal{A}(\nabla \boldsymbol{w}_1 - \nabla \boldsymbol{w}_2) : (\nabla \boldsymbol{w}_1 - \nabla \boldsymbol{w}_2) \, d\boldsymbol{x} \\ &\quad \ge -K \|\nabla \boldsymbol{w}_1 - \nabla \boldsymbol{w}_2\|^2_{L^2(\Omega)^{d\times d}} + \frac{\lambda}{h} \|\nabla \boldsymbol{w}_1 - \nabla \boldsymbol{w}_2\|^2_{L^2(\Omega)^{d\times d}} \ge 0, \end{aligned}$$

provided $h \le \frac{\lambda}{K}$.

H_1 is continuous. Assume that $\boldsymbol{u}_n \to \boldsymbol{u}$ in V. We must prove that

$$\sup_{\|\boldsymbol{\xi}\|=1} |\langle H_1 \boldsymbol{u}_n - H_1 \boldsymbol{u}, \boldsymbol{\xi} \rangle| \to 0 \quad \text{as} \quad n \to \infty.$$

Take $\boldsymbol{\xi} \in V$ with $\|\boldsymbol{\xi}\| = 1$ and estimate

$$|\langle H_1 \boldsymbol{u}_n - H_1 \boldsymbol{u}, \boldsymbol{\xi} \rangle| \le \|\mathcal{G}(\nabla \boldsymbol{u}_n) - \mathcal{G}(\nabla \boldsymbol{u})\|_{L^{p'}(\Omega)^{d\times d}} + \frac{C}{h} \|\nabla \boldsymbol{u}_n - \nabla \boldsymbol{u}\|_{L^2(\Omega)^{d\times d}}.$$

Clearly the second term converges to zero. To see that the first term also converges to zero it is sufficient to use the fact that, for a subsequence, $\nabla \boldsymbol{u}_n(\boldsymbol{x}) \to \nabla \boldsymbol{u}(\boldsymbol{x})$ a.e. in Ω and $|\nabla \boldsymbol{u}_n(\boldsymbol{x})| \le h(\boldsymbol{x})$ with $h \in L^p(\Omega)$, the continuity of $\mathcal{G}$ (see (H1)), the growth condition on $\mathcal{G}$ (see (H2)), and the Lebesgue dominated convergence theorem.

We can immediately see that H_2 is monotone, linear, and bounded. Hence, pseudomonotonicity of H_1 and H_2 follows from Lemma A.5.

H_3 is pseudomonotone. The proof is similar to the one in [30]. We will use Lemma A.3. We first obtain the boundedness of H_3. For $\boldsymbol{w}, \boldsymbol{\xi} \in V$ and $\boldsymbol{\zeta} \in H_3(\boldsymbol{w})$ we get, using (H5) and (H6),

$$\begin{aligned} |\langle \boldsymbol{\zeta}, \boldsymbol{\xi} \rangle| &\le \int_{\Gamma_C} |p_\nu(w_\nu)| |\xi_\nu| \, dS + C_j \int_{\Gamma_C} |p_\nu(w_\nu)| \, |\boldsymbol{\xi}_\tau| \, dS \le C \int_{\Gamma_C} (1 + |w_\nu|^{q-1}) |\boldsymbol{\xi}| \, dS \\ &\le C \left(\|\boldsymbol{\xi}\|_{L^1(\Gamma_C)^d} + \|\boldsymbol{\xi}\|_{L^q(\Gamma_C)^d} \|\boldsymbol{w}\|^{q-1}_{L^q(\Gamma_C)^d} \right) \le C(1 + \|\boldsymbol{w}\|^{q-1}) \|\boldsymbol{\xi}\|. \end{aligned}$$

Hence $\|\boldsymbol{\zeta}\|_{V'} \leq C(1 + \|\boldsymbol{w}\|^{q-1})$ and the boundedness is proved. We show that H_3 has non-empty and convex values. For $\boldsymbol{w} \in V$ denote $\boldsymbol{v} = (\boldsymbol{w}_\tau - \boldsymbol{u}_{h\tau}^{j-1})/h$. We have $\boldsymbol{v}_\tau \in L^q(\Gamma_C)^d$ and $v_\nu \in L^q(\Gamma_C)$. Let $\{\boldsymbol{v}_n\}_{n=1}^\infty$ be a sequence of simple (i.e. piecewise constant and measurable) functions such that $\boldsymbol{v}_n(\boldsymbol{x}) \to \boldsymbol{v}_\tau(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Gamma_C$. Let $\boldsymbol{\eta}_n : \Gamma_C \to \mathbb{R}^d$ be such that $\boldsymbol{\eta}_n(\boldsymbol{x}) \in \partial j(\boldsymbol{v}_n(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_C$. As $|\boldsymbol{\eta}_n(\boldsymbol{x})| \leq C_j$ almost everywhere, for a subsequence still denoted by the same index we have

$$\boldsymbol{\eta}_n \to \boldsymbol{\eta} \quad \text{weakly} - * \text{ in} \quad L^\infty(\Gamma_C)^d.$$

Since ∂j is known to be an upper-semicontinuous multifunction (see [23, Proposition 5.6.10]) by the convergence theorem of Aubin and Cellina (see [31, Theorem 7.2.2]) it follows that $\boldsymbol{\eta}(\boldsymbol{x}) \in \partial j(\boldsymbol{v}(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_C$. Hence $\boldsymbol{\eta} \in S^\infty_{\partial j}(\boldsymbol{v})$. Consider the mapping

$$V \ni \boldsymbol{\xi} \to \left(\int_{\Gamma_C} p_\nu(w_\nu)\xi_\nu \, dS + \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta} \cdot \boldsymbol{\xi}_\tau \, dS \right) \in \mathbb{R}.$$

It is straightforward to check that it is linear and continuous and hence $H_3(\boldsymbol{w})$ is non-empty. The fact that $H_3(\boldsymbol{w})$ is convex follows immediately from the fact that $\partial j(\boldsymbol{s})$ is always a convex set. We will show that if $\boldsymbol{w}_n \to \boldsymbol{w}$ weakly in V and $\boldsymbol{\zeta}_n \to \boldsymbol{\zeta}$ weakly in V' are the sequences such that $\boldsymbol{\zeta}_n \in H_3(\boldsymbol{w}_n)$ then $\boldsymbol{\zeta} \in H_3(\boldsymbol{w})$ and $\langle \boldsymbol{\zeta}_n, \boldsymbol{w}_n \rangle \to \langle \boldsymbol{\zeta}, \boldsymbol{w} \rangle$. It will follow that H_3 is generalized pseudomonotone and that $H_3(\boldsymbol{w})$ is a closed set (since, in particular, we can take $\boldsymbol{w}_n := \boldsymbol{w}$ and $\boldsymbol{\zeta}_n \to \boldsymbol{\zeta}$ strongly in V'). For any $\boldsymbol{\xi} \in V$

$$\langle \boldsymbol{\zeta}_n, \boldsymbol{\xi} \rangle = \int_{\Gamma_C} p_\nu(w_{n\nu})\xi_\nu \, dS + \int_{\Gamma_C} p_\nu(w_{n\nu})\boldsymbol{\eta}_n \cdot \boldsymbol{\xi}_\tau \, dS \quad \text{with} \quad \boldsymbol{\eta}_n \in S^\infty_{\partial j}\left(\frac{\boldsymbol{w}_{n\tau} - \boldsymbol{u}_{h\tau}^{j-1}}{h} \right).$$

Compactness of the trace $\gamma : V \to L^q(\Gamma_C)^d$ implies that

$$\begin{aligned} w_{n\nu} &\to w_\nu \quad \text{strongly in} \quad L^q(\Gamma_C), \\ \boldsymbol{w}_{n\tau} &\to \boldsymbol{w}_\tau \quad \text{strongly in} \quad L^q(\Gamma_C)^d. \end{aligned}$$

Hence, for a subsequence, $w_{n\nu}(\boldsymbol{x}) \to w_\nu(\boldsymbol{x})$ and $\boldsymbol{w}_{n\tau}(\boldsymbol{x}) \to \boldsymbol{w}_\tau(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Gamma_C$ with $|w_{n\nu}(\boldsymbol{x})| \leq h(\boldsymbol{x})$ and $|\boldsymbol{w}_{n\tau}(\boldsymbol{x})| \leq h(\boldsymbol{x})$ with $h(\boldsymbol{x}) \in L^q(\Gamma_C)$. We can estimate

$$\begin{aligned} |p_\nu(w_{n\nu}(\boldsymbol{x}))\xi_\nu(\boldsymbol{x})| &\leq C(1 + |w_{n\nu}(\boldsymbol{x})|^{q-1})|\xi_\nu(\boldsymbol{x})| \leq C(|\xi_\nu(\boldsymbol{x})| + |\xi_\nu(\boldsymbol{x})|^q + |h(\boldsymbol{x})|^q), \\ |p_\nu(w_{n\nu}(\boldsymbol{x}))w_{n\nu}(\boldsymbol{x})| &\leq C(1 + |w_{n\nu}(\boldsymbol{x})|^{q-1})|w_{n\nu}(\boldsymbol{x})| \leq C(|h(\boldsymbol{x})| + |h(\boldsymbol{x})|^q). \end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \int_{\Gamma_C} p_\nu(w_{n\nu})\xi_\nu \, dS &= \int_{\Gamma_C} p_\nu(w_\nu)\xi_\nu \, dS, \\ \lim_{n\to\infty} \int_{\Gamma_C} p_\nu(w_{n\nu})w_{n\nu} \, dS &= \int_{\Gamma_C} p_\nu(w_\nu)w_\nu \, dS. \end{aligned}$$

Now,

$$\int_{\Gamma_C} p_\nu(w_{n\nu})\boldsymbol{\eta}_n \cdot \boldsymbol{\xi}_\tau \, dS = \int_{\Gamma_C} (p_\nu(w_{n\nu}) - p_\nu(w_\nu))\boldsymbol{\eta}_n \cdot \boldsymbol{\xi}_\tau \, dS + \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta}_n \cdot \boldsymbol{\xi}_\tau \, dS.$$

We use the Lebesgue dominated convergence theorem to deduce that

$$\lim_{n\to\infty} \int_{\Gamma_C} (p_\nu(w_{n\nu}) - p_\nu(w_\nu))\boldsymbol{\eta}_n \cdot \boldsymbol{\xi}_\tau \, dS = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\Gamma_C} (p_\nu(w_{n\nu}) - p_\nu(w_\nu))\boldsymbol{\eta}_n \cdot \boldsymbol{w}_{n\tau} \, dS = 0.$$

In a manner similar to the proof that $H_3(\boldsymbol{w})$ is non-empty, we have, for a subsequence,

$$\boldsymbol{\eta}_n \to \boldsymbol{\eta} \quad \text{weakly} - * \text{ in} \quad L^\infty(\Gamma_C)^d,$$

with $\boldsymbol{\eta} \in S^{\infty}_{\partial j}((\boldsymbol{w}_\tau - \boldsymbol{u}_{h\tau}^{j-1})/h)$. Since $p_\nu(w_\nu)\boldsymbol{\xi}_\tau \in L^1(\Gamma_C)^d$ and $p_\nu(w_\nu)\boldsymbol{w}_{n\tau} \to p_\nu(w_\nu)\boldsymbol{w}_\tau$ strongly in $L^1(\Gamma_C)^d$, it follows that, for a subsequence,

$$\int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta}_n \cdot \boldsymbol{\xi}_\tau \, dS \to \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta} \cdot \boldsymbol{\xi}_\tau \, dS,$$
$$\int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta}_n \cdot \boldsymbol{w}_{n\tau} \, dS \to \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta} \cdot \boldsymbol{w}_\tau \, dS.$$

Hence

$$\begin{aligned}\langle \boldsymbol{\zeta}, \boldsymbol{\xi} \rangle &= \lim_{n\to\infty} \langle \boldsymbol{\zeta}_n, \boldsymbol{\xi} \rangle = \lim_{n\to\infty} \left(\int_{\Gamma_C} p_\nu(w_{n\nu})\xi_\nu \, dS + \int_{\Gamma_C} p_\nu(w_{n\nu})\boldsymbol{\eta}_n \cdot \boldsymbol{\xi}_\tau \, dS \right) \\ &= \int_{\Gamma_C} p_\nu(w_\nu)\xi_\nu \, dS + \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta} \cdot \boldsymbol{\xi}_\tau \, dS,\end{aligned}$$

where the convergence must hold for the whole sequence. It follows that $\boldsymbol{\zeta} \in H_3(\boldsymbol{w})$. Moreover,

$$\begin{aligned}\lim_{n\to\infty} \langle \boldsymbol{\zeta}_n, \boldsymbol{w}_n \rangle &= \lim_{n\to\infty} \left(\int_{\Gamma_C} p_\nu(w_{n\nu})\boldsymbol{w}_{n\nu} \, dS + \int_{\Gamma_C} p_\nu(w_{n\nu})\boldsymbol{\eta}_n \cdot \boldsymbol{w}_{n\tau} \, dS \right) \\ &= \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{w}_\nu \, dS + \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta} \cdot \boldsymbol{w}_\tau \, dS = \langle \boldsymbol{\zeta}, \boldsymbol{w} \rangle,\end{aligned}$$

and, again, the convergence holds for the whole sequence. The proof of generalized pseudomonotonicity and of pseudomonotonicity of H_3 is complete.

To use Theorem A.6 we must estimate from below the quantity $\langle \boldsymbol{\zeta} - L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1} \rangle$ for $\boldsymbol{\zeta} \in (H_1 + H_2 + H_3)(\boldsymbol{w})$ and $L \in V'$. We have

$$\begin{aligned}\langle \boldsymbol{\zeta} - L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1} \rangle &= \frac{\rho}{h^2} \int_\Omega \boldsymbol{w} \cdot (\boldsymbol{w} - \boldsymbol{u}_h^{j-1}) \, d\boldsymbol{x} + \frac{1}{h} \int_\Omega \mathcal{A}(\nabla \boldsymbol{w}) : (\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}) \, d\boldsymbol{x} \\ &\quad + \int_\Omega \mathcal{G}(\nabla \boldsymbol{w}) : (\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}) \, d\boldsymbol{x} + \int_{\Gamma_C} p_\nu(w_\nu)(w_\nu - u_{h\nu}^{j-1}) \, dS \\ &\quad + \int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta} \cdot (\boldsymbol{w}_\tau - \boldsymbol{u}_{h\tau}^{j-1}) \, dS - \langle L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1} \rangle,\end{aligned}$$

with $\boldsymbol{\eta} \in S^{\infty}_{\partial j}((\boldsymbol{w}_\tau - \boldsymbol{u}_{h\tau}^{j-1})/h)$. We estimate from below five integrals in the above formula. We have

$$\begin{aligned}\frac{\rho}{h^2} \int_\Omega \boldsymbol{w} \cdot (\boldsymbol{w} - \boldsymbol{u}_h^{j-1}) \, d\boldsymbol{x} &\geq \frac{\rho}{h^2} \left(\|\boldsymbol{w}\|_H^2 - \|\boldsymbol{w}\|_H \|\boldsymbol{u}_h^{j-1}\|_H \right) \\ &\geq \frac{\rho}{2h^2} \left(\|\boldsymbol{w}\|_H^2 - \|\boldsymbol{u}_h^{j-1}\|_H^2 \right).\end{aligned}$$

Moreover, by the coercivity and boundedness of $\mathcal{A}$ (see (H4)),

$$\begin{aligned}\frac{1}{h} \int_\Omega \mathcal{A}(\nabla \boldsymbol{w}) : (\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}) \, d\boldsymbol{x} &= \frac{1}{h} \int_\Omega \left(\mathcal{A}(\nabla \boldsymbol{w}) - \mathcal{A}(\nabla \boldsymbol{u}_h^{j-1}) + \mathcal{A}(\nabla \boldsymbol{u}_h^{j-1}) \right) : (\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}) \, d\boldsymbol{x} \\ &\geq \frac{\lambda}{h} \|\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}\|_{L^2(\Omega)^{d\times d}}^2 + \frac{1}{h} \int_\Omega \mathcal{A}(\nabla \boldsymbol{u}_h^{j-1}) : (\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}) \, d\boldsymbol{x} \\ &\geq \frac{\lambda}{h} \|\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}\|_{L^2(\Omega)^{d\times d}}^2 - \frac{\epsilon}{h} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)^{d\times d}}^2 - \frac{C(\epsilon)}{h} \|\nabla \boldsymbol{u}_h^{j-1}\|_{L^2(\Omega)^{d\times d}}^2,\end{aligned}$$

where $\epsilon > 0$ is arbitrary and $C(\epsilon)$ depends on ϵ and $\|\mathcal{A}\|_{\mathcal{L}(\mathbb{M}^{d\times d};\mathbb{M}^{d\times d})}$. Now note that by (H3)

$$(\mathcal{G}(\boldsymbol{M}_1) + K\boldsymbol{M}_1 - \mathcal{G}(\boldsymbol{M}_2) - K\boldsymbol{M}_2) : (\boldsymbol{M}_1 - \boldsymbol{M}_2) \geq 0 \quad \text{for} \quad \boldsymbol{M}_1, \boldsymbol{M}_2 \in \mathbb{M}^{d\times d}.$$

Since

$$\mathcal{G}(\boldsymbol{M}) + K\boldsymbol{M} = \frac{\partial}{\partial M}\left(\Phi(M) + \frac{K}{2}|\boldsymbol{M}|^2\right),$$

it follows that the function

$$\boldsymbol{M} \to \Phi(M) + \frac{K}{2}|\boldsymbol{M}|^2$$

is convex, and

$$\Phi(M_1) + \frac{K}{2}|\boldsymbol{M}_1|^2 - \Phi(M_2) - \frac{K}{2}|\boldsymbol{M}_2|^2 \leq (\mathcal{G}(\boldsymbol{M}_1) + K\boldsymbol{M}_1) : (\boldsymbol{M}_1 - \boldsymbol{M}_2),$$

and we deduce that

$$\Phi(M_1) - \Phi(M_2) - \frac{K}{2}|\boldsymbol{M}_1 - \boldsymbol{M}_2|^2 \leq \mathcal{G}(\boldsymbol{M}_1) : (\boldsymbol{M}_1 - \boldsymbol{M}_2) \quad \text{for} \quad \boldsymbol{M}_1, \boldsymbol{M}_2 \in \mathbb{M}^{d\times d}. \tag{18}$$

It follows that

$$\int_\Omega \mathcal{G}(\nabla \boldsymbol{w}) : (\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1})\, d\boldsymbol{x} \geq \int_\Omega \Phi(\nabla \boldsymbol{w})\, d\boldsymbol{x} - \int_\Omega \Phi(\nabla \boldsymbol{u}_h^{j-1})\, d\boldsymbol{x} - \frac{K}{2}\|\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}\|^2_{L^2(\Omega)^{d\times d}}.$$

We can use (H2) to note that

$$\int_\Omega \Phi(\nabla \boldsymbol{w})\, d\boldsymbol{x} - \int_\Omega \Phi(\nabla \boldsymbol{u}_h^{j-1})\, d\boldsymbol{x} \geq \alpha\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}} - C_\Phi\|\nabla \boldsymbol{u}_h^{j-1}\|^p_{L^p(\Omega)^{d\times d}} - (\beta + C_\Phi)|\Omega|.$$

By (H5) and (H6) we have

$$\int_{\Gamma_C} p_\nu(w_\nu)\boldsymbol{\eta} \cdot (\boldsymbol{w}_\tau - \boldsymbol{u}_{h\tau}^{j-1})\, dS \geq 0.$$

Assumption (H5) implies that, defining $P(s) = \int_0^s p_\nu(r)\, dr$, it holds that

$$p_\nu(s)(s-t) \geq P(s) - P(t) \geq -P(t) = -\int_0^t p(r)\, dr \geq -\left|\int_0^t C_p(1 + |r|^{q-1})\, dr\right| = -C_p|t| - C_p\frac{|t|^q}{q},$$

for $s, t \in \mathbb{R}$, and hence

$$\int_{\Gamma_C} p_\nu(w_\nu)(w_\nu - u_{h\nu}^{j-1})\, dS \geq -C\left(1 + \|u_{h\nu}^{j-1}\|^q_{L^q(\Gamma_C)}\right) \geq -C\left(1 + \|\boldsymbol{u}_h^{j-1}\|^q\right).$$

We summarize all estimates

$$\begin{aligned}\langle \boldsymbol{\zeta} - L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1}\rangle \geq\ & \frac{\rho}{2h^2}\left(\|\boldsymbol{w}\|^2_H - \|\boldsymbol{u}_h^{j-1}\|^2_H\right) - \frac{\epsilon}{h}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)^{d\times d}} - \frac{C(\epsilon)}{h}\|\nabla \boldsymbol{u}_h^{j-1}\|^2_{L^2(\Omega)^{d\times d}} \\ & + \alpha\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}} - C_\Phi\|\nabla \boldsymbol{u}_h^{j-1}\|^p_{L^p(\Omega)^{d\times d}} - (\beta + C_\Phi)|\Omega| + \left(\frac{\lambda}{h} - \frac{K}{2}\right)\|\nabla \boldsymbol{w} - \nabla \boldsymbol{u}_h^{j-1}\|^2_{L^2(\Omega)^{d\times d}} \\ & - C\left(1 + \|\boldsymbol{u}_h^{j-1}\|^q\right) - \|L\|_{V'}\|\boldsymbol{w}\| + \langle L, \boldsymbol{u}_h^{j-1}\rangle.\end{aligned}$$

As $h_0 = 2\lambda/K$, treating all terms dependent on $\boldsymbol{u}_h^{j-1}$ as constants, which may depend on h, we obtain

$$\begin{aligned}&\langle \boldsymbol{\zeta} - L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1}\rangle \\ &\quad \geq \alpha\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}} + \frac{\rho}{2h^2}\|\boldsymbol{w}\|^2_H - \frac{\epsilon}{h}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)^{d\times d}} - \frac{C(\epsilon)}{h}\|\nabla \boldsymbol{u}_h^{j-1}\|^2_{L^2(\Omega)^{d\times d}} - C\|\boldsymbol{w}\| - C.\end{aligned}$$

Since

$$\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)^{d\times d}} \le 1 + \|\nabla \boldsymbol{w}\|^p_{L^2(\Omega)^{d\times d}} \le 1 + C\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}},$$

we can choose ϵ small enough that the term $\epsilon/h\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)^{d\times d}}$ is absorbed in $\alpha\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}}$, whence

$$\langle \boldsymbol{\zeta} - L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1}\rangle \ge \frac{\alpha}{2}\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}} + \frac{\rho}{2h^2}\|\boldsymbol{w}\|^2_H - C\|\boldsymbol{w}\| - C. \tag{19}$$

We will use the following Gagliardo–Nirenberg interpolation inequality valid for $\boldsymbol{w} \in V$:

$$\|\boldsymbol{w}\|_{L^p(\Omega)^d} \le C_1\|\nabla \boldsymbol{w}\|^{\alpha}_{L^p(\Omega)^{d\times d}}\|\boldsymbol{w}\|^{1-\alpha}_H + C_2\|\boldsymbol{w}\|_H \quad \text{with} \quad \alpha = \frac{d(p-2)}{d(p-2)+2p}.$$

Observe that the quantity $|||\boldsymbol{w}||| = \|\boldsymbol{w}\|_H + \|\nabla \boldsymbol{w}\|_{L^p(\Omega)^{d\times d}}$ is the norm on V equivalent to $\|\boldsymbol{w}\|$. Indeed,

$$|||\boldsymbol{w}||| = \|\boldsymbol{w}\|_H + \|\nabla \boldsymbol{w}\|_{L^p(\Omega)^{d\times d}} \le C\|\boldsymbol{w}\|_{L^p(\Omega)^d} + \|\nabla \boldsymbol{w}\|_{L^p(\Omega)^{d\times d}} \le C\|\boldsymbol{w}\|,$$

and

$$\begin{aligned}\|\boldsymbol{w}\| &= \|\boldsymbol{w}\|_{L^p(\Omega)^d} + \|\nabla \boldsymbol{w}\|_{L^p(\Omega)^{d\times d}}\\ &\le C_1\|\nabla \boldsymbol{w}\|^{\alpha}_{L^p(\Omega)^{d\times d}}\|\boldsymbol{w}\|^{1-\alpha}_H + C_2\|\boldsymbol{w}\|_H + \|\nabla \boldsymbol{w}\|_{L^p(\Omega)^{d\times d}}\\ &\le C_1|||\boldsymbol{w}|||^{\alpha}|||\boldsymbol{w}|||^{1-\alpha} + C_2|||\boldsymbol{w}||| + |||\boldsymbol{w}||| \le C|||w|||.\end{aligned}$$

Using the estimates $\|\boldsymbol{w}\|_H \le \rho/(4h^2)\|\boldsymbol{w}\|^2_H + C$ and $\|\nabla \boldsymbol{w}\|_{L^p(\Omega)^{d\times d}} \le \alpha/4\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}} + C$, the bound (19) yields

$$\begin{aligned}\langle \boldsymbol{\zeta} - L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1}\rangle &\ge \frac{\alpha}{2}\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}} + \frac{\rho}{2h^2}\|\boldsymbol{w}\|^2_H - C(\|\nabla \boldsymbol{w}\|_{L^p(\Omega)^{d\times d}} + \|\boldsymbol{w}\|_H) - C\\ &\ge \frac{\alpha}{4}\|\nabla \boldsymbol{w}\|^p_{L^p(\Omega)^{d\times d}} + \frac{\rho}{4h^2}\|\boldsymbol{w}\|^2_H - C.\end{aligned}$$

If we choose R_0 large enough that

$$\frac{\rho}{4h^2}\left(\frac{R_0}{2}\right)^2 > C \quad \text{and} \quad \frac{\alpha}{4}\left(\frac{R_0}{2}\right)^p > C,$$

then

$$|||\boldsymbol{w}||| \ge R_0 \Rightarrow \langle \boldsymbol{\zeta} - L, \boldsymbol{w} - \boldsymbol{u}_h^{j-1}\rangle > 0.$$

If necessary, we increase R_0 such that $R_0 \ge \left|\left|\left|\boldsymbol{u}_h^{j-1}\right|\right|\right|$, and the surjectivity holds by Theorem A.6. □

We pass to the a priori estimates which hold for the solutions of the time-discretized problems.

Theorem 3.2. *Let $0 < h_0 < \lambda/(2K)$. For all $h \in (0, h_0]$ and for all $j \in \{1, \dots, N\}$ the following estimates hold:*

$$\|\boldsymbol{v}_h^j\|_H \le C, \qquad \|\nabla \boldsymbol{u}_h^j\|_{L^p(\Omega)^{d\times d}} \le C,$$

$$\sum_{j=1}^N \|\boldsymbol{v}_h^j - \boldsymbol{v}_h^{j-1}\|^2_H \le C, \qquad h\sum_{j=1}^N \|\nabla \boldsymbol{v}_h^j\|^2_{L^2(\Omega)^{d\times d}} \le C,$$

$$\|\boldsymbol{u}_h^j\|_H \le C.$$

Proof. For simplicity we will drop the index h in the estimates. Taking the test function $\boldsymbol{\xi} = \boldsymbol{u}^j - \boldsymbol{u}^{j-1}$ in (15), we obtain, for $j \in \{1, \dots, N\}$,

$$
\begin{aligned}
0 = &\underbrace{\int_\Omega \rho \frac{\boldsymbol{u}^j - 2\boldsymbol{u}^{j-1} + \boldsymbol{u}^{j-2}}{h^2} \cdot (\boldsymbol{u}^j - \boldsymbol{u}^{j-1})\, d\boldsymbol{x}}_{I_1} + \underbrace{\int_\Omega \mathcal{A}\left(\frac{\nabla \boldsymbol{u}^j - \nabla \boldsymbol{u}^{j-1}}{h}\right) : (\nabla \boldsymbol{u}^j - \nabla \boldsymbol{u}^{j-1})\, d\boldsymbol{x}}_{I_2} \\
&+ \underbrace{\int_\Omega \mathcal{G}(\nabla \boldsymbol{u}^j) : (\nabla \boldsymbol{u}^j - \nabla \boldsymbol{u}^{j-1})\, d\boldsymbol{x}}_{I_3} + \underbrace{\int_{\Gamma_C} p_\nu(u_\nu^j)(u_\nu^j - u_\nu^{j-1})\, dS}_{I_4} \\
&+ \underbrace{\int_{\Gamma_C} p_\nu(u_\nu^j)\boldsymbol{\eta}^j \cdot (\boldsymbol{u}_\tau^j - \boldsymbol{u}_\tau^{j-1})\, dS}_{I_5} - \underbrace{\langle \boldsymbol{f}^j, \boldsymbol{u}^j - \boldsymbol{u}^{j-1} \rangle_{W' \times W}}_{I_6}.
\end{aligned}
$$

We deal with all terms separately.

$$
I_1 = \int_\Omega \rho(\boldsymbol{v}^j - \boldsymbol{v}^{j-1}) \cdot \boldsymbol{v}^j\, d\boldsymbol{x} = \frac{\rho}{2}\|\boldsymbol{v}^j\|_H^2 - \frac{\rho}{2}\|\boldsymbol{v}^{j-1}\|_H^2 + \frac{\rho}{2}\|\boldsymbol{v}^j - \boldsymbol{v}^{j-1}\|_H^2.
$$

The term I_2 is estimated straightforwardly from (H4):

$$
I_2 \geq \frac{\lambda}{h}\|\nabla \boldsymbol{u}^j - \nabla \boldsymbol{u}^{j-1}\|_{L^2(\Omega)^{d\times d}}^2.
$$

We estimate I_3 using (18)

$$
I_3 \geq \int_\Omega \Phi(\nabla \boldsymbol{u}^j)\, d\boldsymbol{x} - \int_\Omega \Phi(\nabla \boldsymbol{u}_h^{j-1})\, d\boldsymbol{x} - \frac{K}{2}\|\nabla \boldsymbol{u}^j - \nabla \boldsymbol{u}_h^{j-1}\|_{L^2(\Omega)^{d\times d}}^2.
$$

To estimate I_5 observe that $P(s) = \int_0^s p_\nu(r)\, dr$ is a convex function with $P'(s) = p_\nu(s)$ and it holds that

$$
p_\nu(s)(s - t) \geq P(s) - P(t) \quad \text{for} \quad s, t \in \mathbb{R},
$$

and hence

$$
I_4 \geq \int_{\Gamma_C} P(u_\nu^j)\, dS - \int_{\Gamma_C} P(u_\nu^{j-1})\, dS.
$$

From (H5) and (H6) it follows that

$$
I_5 \geq 0.
$$

We estimate I_6:

$$
I_6 \leq \|\boldsymbol{f}_j\|_{W'}\|\boldsymbol{u}^j - \boldsymbol{u}^{j-1}\|_W \leq \frac{\epsilon}{h}\left(\|\boldsymbol{u}^j - \boldsymbol{u}^{j-1}\|_H^2 + \|\nabla \boldsymbol{u}^j - \nabla \boldsymbol{u}^{j-1}\|_{L^2(\Omega)^{d\times d}}^2\right) + C(\epsilon)h\|\boldsymbol{f}_j\|_{W'}^2.
$$

Combining all the above estimates we obtain

$$
\begin{aligned}
&\frac{\rho}{2}\|\boldsymbol{v}^j\|_{L^2(\Omega)^d}^2 - \frac{\rho}{2}\|\boldsymbol{v}^{j-1}\|_{L^2(\Omega)^d}^2 + \frac{\rho}{2}\|\boldsymbol{v}^j - \boldsymbol{v}^{j-1}\|_{L^2(\Omega)^d}^2 \\
&\quad + \int_\Omega \Phi(\nabla \boldsymbol{u}^j)\, d\boldsymbol{x} - \int_\Omega \Phi(\nabla \boldsymbol{u}_h^{j-1})\, d\boldsymbol{x} + \left(\frac{\lambda}{h} - \frac{K}{2} - \frac{\epsilon}{h}\right)\|\nabla \boldsymbol{u}^j - \nabla \boldsymbol{u}_h^{j-1}\|_{L^2(\Omega)^{d\times d}}^2 \\
&\quad + \int_{\Gamma_C} P(u_\nu^j)\, dS - \int_{\Gamma_C} P(u_\nu^{j-1})\, dS \leq \frac{\epsilon}{h}\|\boldsymbol{u}^j - \boldsymbol{u}^{j-1}\|_{L^2(\Omega)^d}^2 + C(\epsilon)h\|\boldsymbol{f}_j\|_{W'}^2.
\end{aligned}
$$

We choose $\epsilon = \lambda/2$. Using the fact that $h_0 < \lambda/(2K)$ it follows that

$$\frac{\rho}{2}\|\boldsymbol{v}^j\|_H^2 - \frac{\rho}{2}\|\boldsymbol{v}^{j-1}\|_H^2 + \frac{\rho}{2}\|\boldsymbol{v}^j - \boldsymbol{v}^{j-1}\|_H^2 + \int_\Omega \Phi(\nabla \boldsymbol{u}^j)\,d\boldsymbol{x} - \int_\Omega \Phi(\nabla \boldsymbol{u}_h^{j-1})\,d\boldsymbol{x}$$
$$+ \frac{\lambda}{4}h\|\nabla \boldsymbol{v}^j\|_{L^2(\Omega)^{d\times d}}^2 + \int_{\Gamma_C} P(u_v^j)\,dS - \int_{\Gamma_C} P(u_v^{j-1})\,dS \le \frac{\lambda}{2}h\|\boldsymbol{v}^j\|_H^2 + Ch\|\boldsymbol{f}_j\|_{W'}^2.$$

Adding the above inequalities for $j = 1$ to l, where $l \in \{1, \dots, N\}$, yields

$$\frac{\rho}{2}\|\boldsymbol{v}^l\|_H^2 + \frac{\rho}{2}\sum_{j=1}^{l}\|\boldsymbol{v}^j - \boldsymbol{v}^{j-1}\|_H^2 + \int_\Omega \Phi(\nabla \boldsymbol{u}^l)\,d\boldsymbol{x} + \frac{\lambda}{4}h\sum_{j=1}^{l}\|\nabla \boldsymbol{v}^j\|_{L^2(\Omega)^{d\times d}}^2 + \int_{\Gamma_C} P(u_v^l)\,dS$$
$$\le \frac{\rho}{2}\|\boldsymbol{u}_{1h}\|_H^2 + \int_\Omega \Phi(\nabla \boldsymbol{u}_0)\,d\boldsymbol{x} + \int_{\Gamma_C} P(u_{0v})\,dS + \frac{\lambda}{2}h\sum_{j=1}^{l}\|\boldsymbol{v}^j\|_H^2 + Ch\sum_{j=1}^{l}\|\boldsymbol{f}_j\|_{W'}^2.$$

Noting that by (H5) values of P are non-negative, that the norm $\|\boldsymbol{u}_{1h}\|_H$ is bounded due to the strong convergence $\boldsymbol{u}_{1h} \to \boldsymbol{u}_1$ in H, and that, by (14),

$$h\sum_{j=1}^{l}\|\boldsymbol{f}_j\|_{W'}^2 \le h\sum_{j=1}^{N}\|\boldsymbol{f}_j\|_{W'}^2 = \|\boldsymbol{f}_h\|_{L^2(0,T;W')}^2 \le \|\boldsymbol{f}\|_{L^2(0,T;W')}^2,$$

using (H2), we get the estimate

$$\rho\|\boldsymbol{v}^l\|_H^2 + \rho\sum_{j=1}^{l}\|\boldsymbol{v}^j - \boldsymbol{v}^{j-1}\|_H^2 + 2\alpha\|\nabla \boldsymbol{u}^l\|_{L^p(\Omega)^{d\times d}}^p + \frac{\lambda}{2}h\sum_{j=1}^{l}\|\nabla \boldsymbol{v}^j\|_{L^2(\Omega)^{d\times d}}^2 \le C + \lambda h\sum_{j=1}^{l}\|\boldsymbol{v}^j\|_H^2.$$

We are in a position to use the discrete Gronwall lemma (see [32, Theorem 1.46, p. 26] or [33, Theorem 6.1, p. 339]), which immediately yields the first four estimates of the theorem. To obtain the estimate for $\|\boldsymbol{u}^j\|_H$ note that

$$\sum_{j=1}^{l}\boldsymbol{v}^j = \frac{\boldsymbol{u}^l - \boldsymbol{u}_0}{h}.$$

Hence $\|\boldsymbol{u}^l\|_H \le h\sum_{j=1}^{l}\|\boldsymbol{v}^j\|_H + \|\boldsymbol{u}_0\|_H \le hlC + \|\boldsymbol{u}_0\|_H \le CT + \|\boldsymbol{u}_0\|_H$, and the proof is complete. □

Remark 3.3. The constant h_0 in Lemma 3.1 is equal to $2\lambda/K$ while in Theorem 3.2 it is less than $\lambda/(2K)$. It follows that we can take h_0 from Theorem 3.2 and Lemma 3.1 still holds.

4. Convergence of the discretization scheme

We define the piecewise constant and piecewise linear interpolants built on the solutions of the time-discrete problem both for the displacement $\boldsymbol{u}$, the velocity $\boldsymbol{v}$, and for the selection of the multi-valued term $\boldsymbol{\eta}$:

$$\boldsymbol{u}_h(t) = \boldsymbol{u}^j + (\boldsymbol{u}^j - \boldsymbol{u}^{j-1})\frac{t - t_j}{h} \quad \text{for} \quad t \in [t_{j-1}, t_j], \quad \overline{\boldsymbol{u}}_h(t) = \begin{cases} \boldsymbol{u}^j & \text{for } t \in (t_{j-1}, t_j], \\ \boldsymbol{u}^0 & \text{for } t = 0, \end{cases}$$

$$\boldsymbol{v}_h(t) = \boldsymbol{v}^j + (\boldsymbol{v}^j - \boldsymbol{v}^{j-1})\frac{t - t_j}{h} \quad \text{for} \quad t \in [t_{j-1}, t_j], \quad \overline{\boldsymbol{v}}_h(t) = \begin{cases} \boldsymbol{v}^j & \text{for } t \in (t_{j-1}, t_j], \\ \boldsymbol{v}^0 & \text{for } t = 0, \end{cases}$$

$$\overline{\boldsymbol{\eta}}_h(t) = \boldsymbol{\eta}^j \quad \text{for} \quad t \in (t_{j-1}, t_j].$$

Clearly $\dot{\boldsymbol{u}}_h(t) = \overline{\boldsymbol{v}}_h(t)$ for a.e. $t \in (0, T)$. From Theorem 3.2 we immediately obtain the following estimates.

Lemma 4.1. *There exist $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0]$ we have the bounds*

$$\|\overline{\boldsymbol{u}}_h\|_{L^\infty(0,T;V)} \le C, \qquad \|\boldsymbol{u}_h\|_{L^\infty(0,T;V)} \le C, \tag{20}$$
$$\|\overline{\boldsymbol{v}}_h\|_{L^\infty(0,T;H)} \le C, \qquad \|\boldsymbol{v}_h\|_{L^\infty(0,T;H)} \le C, \tag{21}$$
$$\|\nabla\overline{\boldsymbol{v}}_h\|_{L^2(0,T;L^2(\Omega)^{d\times d})} \le C, \qquad \|\nabla\boldsymbol{v}_h\|_{L^2(0,T;L^2(\Omega)^{d\times d})} \le C. \tag{22}$$

Proof. In Theorem 3.2 we have obtained the bound

$$\|\boldsymbol{u}_h^j\|_H + \|\nabla\boldsymbol{u}_h^j\|_{L^p(\Omega)^{d\times d}} \le C \quad \text{for} \quad j \in \{1,\dots,N\}.$$

As $\boldsymbol{u}_h^0 = \boldsymbol{u}_0 \in V$ the bounds (20) follow from the equivalence of norms $|||\cdot|||$ and $\|\cdot\|$ on V obtained in the proof of Lemma 3.1. Bounds (21) are immediate noting that $\|\boldsymbol{v}_h^0\|_H$ is bounded as $\boldsymbol{v}_h^0 = \boldsymbol{u}_{1h}$ converges strongly in H to $\boldsymbol{u}_1$. The bound on $\|\nabla\overline{\boldsymbol{v}}_h\|_{L^2(0,T;L^2(\Omega)^{d\times d})}$ is immediate. To get the bound of $\|\nabla\boldsymbol{v}_h\|_{L^2(0,T;L^2(\Omega)^{d\times d})}$ note that for $t \in (t_{j-1}, t_j)$ we have

$$\begin{aligned}\|\nabla\boldsymbol{v}_h(t)\|_{L^2(\Omega)^{d\times d}} &\le \|\nabla\boldsymbol{v}_h^j\|_{L^2(\Omega)^{d\times d}}\frac{t-t_{j-1}}{h} + \|\nabla\boldsymbol{v}_h^{j-1}\|_{L^2(\Omega)^{d\times d}}\frac{t_j-t}{h}\\ &\le \|\nabla\boldsymbol{v}_h^j\|_{L^2(\Omega)^{d\times d}} + \|\nabla\boldsymbol{v}_h^{j-1}\|_{L^2(\Omega)^{d\times d}},\end{aligned}$$

and hence

$$\begin{aligned}\|\nabla\boldsymbol{v}_h\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})} &\le \sum_{j=1}^N \int_{I_h^j} 2\|\nabla\boldsymbol{v}_h^j\|^2_{L^2(\Omega)^{d\times d}} + 2\|\nabla\boldsymbol{v}_h^{j-1}\|^2_{L^2(\Omega)^{d\times d}}\,dt\\ &\le 4h\sum_{j=1}^N \|\nabla\boldsymbol{v}_h^j\|^2_{L^2(\Omega)^{d\times d}} + 2h\|\nabla\boldsymbol{v}_h^0\|^2_{L^2(\Omega)^{d\times d}}.\end{aligned}$$

The assertion follows from Theorem 3.2 and (13) as $\boldsymbol{v}_h^0 = \boldsymbol{u}_{1h}$. □

Directly from the estimates in Lemma 4.1 we get the following result.

Lemma 4.2. *For a subsequence of $h \to 0$ we have*

$$\overline{\boldsymbol{u}}_h \to \overline{\boldsymbol{u}} \text{ weakly-* in } L^\infty(0,T;V), \tag{23}$$
$$\boldsymbol{u}_h \to \boldsymbol{u} \text{ weakly-* in } L^\infty(0,T;V), \text{weakly-* in } W^{1,\infty}(0,T;H), \text{weakly in } H^1(0,T;W), \tag{24}$$
$$\overline{\boldsymbol{v}}_h \to \overline{\boldsymbol{v}} \text{ weakly-* in } L^\infty(0,T;H), \text{weakly in } L^2(0,T;W), \tag{25}$$
$$\boldsymbol{v}_h \to \boldsymbol{v} \text{ weakly-* in } L^\infty(0,T;H), \text{weakly in } L^2(0,T;W). \tag{26}$$

Define the piecewise constant interpolant for the acceleration by

$$\overline{\boldsymbol{w}}_h(t) = \boldsymbol{w}_h^j \text{ for } t \in (t_{j-1}, t_j] \text{ and } j \in \{1,\dots,N\} \text{ where } \boldsymbol{w}_h^j = \frac{\boldsymbol{v}_h^j - \boldsymbol{v}_h^{j-1}}{h}.$$

Clearly

$$\boldsymbol{w}_h^j = \frac{\boldsymbol{u}_h^j - 2\boldsymbol{u}_h^{j-1} + \boldsymbol{u}_h^{j-2}}{h^2} \text{ for } j \in \{1,\dots,N\},$$

and

$$\dot{\boldsymbol{v}}_h(t) = \overline{\boldsymbol{w}}_h(t) \quad \text{for a.e.} \quad t \in (0,T). \tag{27}$$

In the following lemma we derive some further estimates.

Lemma 4.3. *There exist $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0]$ we have the bounds*

$$\|\mathcal{G}(\nabla \overline{\boldsymbol{u}}_h)\|_{L^\infty(0,T;L^{p'}(\Omega)^{d\times d})} \le C, \tag{28}$$
$$\|\overline{\boldsymbol{w}}_h\|_{L^2(0,T;V')} \le C. \tag{29}$$

Moreover, for a subsequence of $h \to 0$ we have the convergences

$$\mathcal{G}(\nabla \overline{\boldsymbol{u}}_h) \to \overline{\mathcal{G}} \quad \textit{weakly-}^* \textit{ in} \quad L^\infty(0,T;L^{p'}(\Omega)^{d\times d}), \tag{30}$$
$$\overline{\boldsymbol{w}}_h \to \overline{\boldsymbol{w}} \quad \textit{weakly in} \quad L^2(0,T;V'), \tag{31}$$
$$\boldsymbol{v}_h \to \boldsymbol{v} \quad \textit{weakly in} \quad H^1(0,T;V'). \tag{32}$$

Proof. Assumption (H2) implies that

$$|\mathcal{G}(\nabla \boldsymbol{u}_h^j(\boldsymbol{x}))|^{p'} \le C(1 + |\nabla \boldsymbol{u}_h^j(\boldsymbol{x})|^p),$$

and hence

$$\|\mathcal{G}(\nabla \boldsymbol{u}_h^j)\|^{p'}_{L^{p'}(\Omega)^{d\times d}} \le C\left(1 + \|\nabla \boldsymbol{u}_h^j\|^p_{L^p(\Omega)^{d\times d}}\right),$$

which immediately gives (28) and (30) by Theorem 3.2. We derive the estimate for $\overline{\boldsymbol{w}}_h$. Let $\boldsymbol{\xi} \in V$. By (15)

$$\rho\langle \boldsymbol{w}_h^j, \boldsymbol{\xi}\rangle = \rho(\boldsymbol{w}_h^j, \boldsymbol{\xi}) = \underbrace{-\int_\Omega \mathcal{A}(\nabla \boldsymbol{v}_h^j) : \nabla \boldsymbol{\xi}\, d\boldsymbol{x}}_{I_1} \underbrace{-\int_\Omega \mathcal{G}(\nabla \boldsymbol{u}_h^j) : \nabla \boldsymbol{\xi}\, d\boldsymbol{x}}_{I_2}$$
$$\underbrace{-\int_{\Gamma_C} p_\nu(u_{h\nu}^j)\xi_\nu\, dS}_{I_3} \underbrace{-\int_{\Gamma_C} p_\nu(u_{h\nu}^j)\boldsymbol{\eta}_h^j \cdot \boldsymbol{\xi}_\tau\, dS}_{I_4} + \underbrace{\langle \boldsymbol{f}_h^j, \boldsymbol{\xi}\rangle_{W'\times W}}_{I_5}.$$

Using the bounds of Theorem 3.2 and estimates (H2), (H4), (H5), and (H6), we estimate all terms:

$$|I_1| \le C\int_\Omega |\nabla \boldsymbol{v}_h^j||\nabla \boldsymbol{\xi}|\, d\boldsymbol{x} \le C\|\nabla \boldsymbol{v}_h^j\|_{L^2(\Omega)^{d\times d}}\|\nabla \boldsymbol{\xi}\|_{L^2(\Omega)^{d\times d}} \le C\|\nabla \boldsymbol{v}_h^j\|_{L^2(\Omega)^{d\times d}}\|\boldsymbol{\xi}\|,$$

$$|I_2| \le C\int_\Omega (1 + |\nabla \boldsymbol{u}_h^j|^{p-1})|\nabla \boldsymbol{\xi}|\, d\boldsymbol{x} \le C\|\nabla \boldsymbol{\xi}\|_{L^p(\Omega)^{d\times d}}(1 + \|\nabla \boldsymbol{u}_h^j\|^{p-1}_{L^p(\Omega)^{d\times d}}) \le C\|\boldsymbol{\xi}\|.$$

To estimate $|I_3|$ we will need the equivalence between norms $|||\cdot|||$ and $\|\cdot\|$ on V:

$$|I_3| \le \int_{\Gamma_C} |p_\nu(u_{h\nu}^j)||\xi_\nu|\, dS \le C_p\int_{\Gamma_C} (1 + |u_{h\nu}^j|^{q-1})|\xi_\nu|\, dS \le C\|\xi_\nu\|_{L^q(\Gamma_C)}(1 + \|u_{h\nu}^j\|^{q-1}_{L^q(\Gamma_C)})$$
$$\le C\|\boldsymbol{\xi}\|_{L^q(\Gamma_C)^d}(1 + \|\boldsymbol{u}_h^j\|^{q-1}_{L^q(\Gamma_C)^d}) \le C\|\boldsymbol{\xi}\|(1 + \|\boldsymbol{u}_h^j\|^{q-1}) \le C\|\boldsymbol{\xi}\|.$$

The estimate of $|I_4|$ is similar to that of $|I_3|$:

$$|I_3| \le \int_{\Gamma_C} |p_\nu(u_{h\nu}^j)||\boldsymbol{\eta}_h^j||\boldsymbol{\xi}_\tau|\, dS \le C\int_{\Gamma_C} (1 + |u_{h\nu}^j|^{q-1})|\boldsymbol{\xi}_\tau|\, dS \le C\|\boldsymbol{\xi}_\tau\|_{L^q(\Gamma_C)^d}(1 + \|u_{h\nu}^j\|^{q-1}_{L^q(\Gamma_C)})$$
$$\le C\|\boldsymbol{\xi}\|_{L^q(\Gamma_C)^d}(1 + \|\boldsymbol{u}_h^j\|^{q-1}_{L^q(\Gamma_C)^d}) \le C\|\boldsymbol{\xi}\|(1 + \|\boldsymbol{u}_h^j\|^{q-1}) \le C\|\boldsymbol{\xi}\|.$$

Finally,

$$|I_5| \le \|\boldsymbol{f}_h^j\|_{W'}\|\boldsymbol{\xi}\|_W \le C\|\boldsymbol{f}_h^j\|_{W'}\|\boldsymbol{\xi}\|.$$

It follows that

$$\|\boldsymbol{w}_h^j\|_{V'} \le C(1 + \|\nabla \boldsymbol{v}_h^j\|_{L^2(\Omega)^{d\times d}} + \|\boldsymbol{f}_h^j\|_{W'}) \quad \text{for} \quad t \in (t_{j-1}, t_j),$$

and

$$\|\overline{\boldsymbol{w}}_h\|^2_{L^2(0,T;V')} = h\sum_{j=1}^{N}\|\boldsymbol{w}_h^j\|^2_{V'} \le C\left(1 + h\sum_{j=1}^{N}\|\nabla \boldsymbol{v}_h^j\|^2_{L^2(\Omega)^{d\times d}} + h\sum_{j=1}^{n}\|\boldsymbol{f}_h^j\|^2_{W'}\right).$$

Using (14) and Theorem 3.2 we obtain the assertions (29) and (31). The convergence (32) follows from (31) as well as (26) and (27). □

We get the following result.

Lemma 4.4. *It holds that*

$$\overline{\boldsymbol{u}} = \boldsymbol{u}, \quad \overline{\boldsymbol{v}} = \boldsymbol{v}, \quad \boldsymbol{v} = \dot{\boldsymbol{u}}, \quad \overline{\boldsymbol{w}} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{u}},$$

and we have the regularity

$$\boldsymbol{u} \in L^\infty(0,T;V), \quad \dot{\boldsymbol{u}} \in L^\infty(0,T;H)\cap L^2(0,T;W), \quad \ddot{\boldsymbol{u}} \in L^2(0,T;V').$$

Proof. We need to demonstrate that $\overline{\boldsymbol{u}} = \boldsymbol{u}$ and that $\overline{\boldsymbol{v}} = \boldsymbol{v}$. To this end we calculate

$$\|\overline{\boldsymbol{u}}_h(t) - \boldsymbol{u}_h(t)\|_H = \frac{t-t_j}{h}\|\boldsymbol{u}_h^j - u_h^{j-1}\|_H \quad \text{for} \quad t \in (t_{j-1}, t_j).$$

Hence

$$\|\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h\|^2_{L^2(0,T;H)} = \frac{h}{3}\sum_{j=1}^{N}\|\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}\|_H^2 = \frac{h^3}{3}\sum_{j=1}^{N}\|\boldsymbol{v}_h^j\|_H^2 \le Ch^2,$$

where we used the estimate of Theorem 3.2. So

$$\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h \to 0 \quad \text{strongly in} \quad L^2(0,T;H),$$

but (23) and (24) imply that

$$\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h \to \overline{\boldsymbol{u}} - \boldsymbol{u} \quad \text{weakly in} \quad L^2(0,T;H),$$

whence $\overline{\boldsymbol{u}} = \boldsymbol{u}$. Similarly, by the estimate of Theorem 3.2,

$$\|\overline{\boldsymbol{v}}_h - \boldsymbol{v}_h\|^2_{L^2(0,T;H)} = \frac{h}{3}\sum_{j=1}^{N}\|\boldsymbol{v}_h^j - \boldsymbol{v}_h^{j-1}\|_H^2 \le Ch,$$

whence

$$\overline{\boldsymbol{v}}_h - \boldsymbol{v}_h \to 0 \quad \text{strongly in} \quad L^2(0,T;H),$$

and by (25) and (26)

$$\overline{\boldsymbol{v}}_h - \boldsymbol{v}_h \to \overline{\boldsymbol{v}} - \boldsymbol{v} \quad \text{weakly in} \quad L^2(0,T;H),$$

whence the assertion that $\overline{\boldsymbol{v}} = \boldsymbol{v}$ follows easily. □

We can pass to the limit with $h \to 0$ in (15) which gives us the following result.

Theorem 4.5. *We have $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$. Moreover, for any $\boldsymbol{\xi} \in L^2(0,T;V)$,*

$$\begin{aligned}\int_0^{\mathrm{T}}\langle\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{\xi}(t)\rangle\,dt + \int_0^{\mathrm{T}}\int_\Omega \mathcal{A}(\nabla\dot{\boldsymbol{u}}(t)) : \nabla\boldsymbol{\xi}(t)\,d\boldsymbol{x}\,dt + \int_0^{\mathrm{T}}\int_\Omega \overline{\mathcal{G}}(t) : \nabla\boldsymbol{\xi}(t)\,d\boldsymbol{x}\,dt \\ + \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(u_\nu(t))\xi_\nu(t)\,dS\,dt + \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(u_\nu(t))\boldsymbol{\eta}(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt = \int_0^{\mathrm{T}}\langle\boldsymbol{f}(t),\boldsymbol{\xi}(t)\rangle_{W'\times W}\,dt,\end{aligned} \tag{33}$$

where $\boldsymbol{\eta} \in L^\infty(0,T;L^\infty(\Omega)^d)$ is such that $\boldsymbol{\eta}(\boldsymbol{x},t) \in \partial j(\dot{\boldsymbol{u}}_\tau(\boldsymbol{x},t))$ a.e. in $\Gamma_C \times (0,T)$.

Proof. Note that for $t \in (0,T)$ and $\boldsymbol{\zeta} \in V$

$$\langle \boldsymbol{\zeta}, \dot{\boldsymbol{u}}(t) - \boldsymbol{v}_h(t)\rangle = \langle \boldsymbol{\zeta}, \dot{\boldsymbol{u}}(0) - \boldsymbol{v}_h(0)\rangle + \int_0^t \langle \boldsymbol{\zeta}, \ddot{\boldsymbol{u}}(s) - \overline{\boldsymbol{w}}_h(s)\rangle \, ds.$$

Integrating the last inequality with respect to t we get

$$\int_0^{\mathrm{T}} \langle \boldsymbol{\zeta}, \dot{\boldsymbol{u}}(t) - \boldsymbol{v}_h(t)\rangle \, dt = T\langle \boldsymbol{\zeta}, \dot{\boldsymbol{u}}(0) - \boldsymbol{v}_h(0)\rangle + \int_0^{\mathrm{T}} \langle \boldsymbol{\zeta}, \ddot{\boldsymbol{u}}(t) - \overline{\boldsymbol{w}}_h(t)(T-t)\rangle \, dt,$$

whence, from Lemma 4.4,

$$T\langle \boldsymbol{\zeta}, \dot{\boldsymbol{u}}(0) - \boldsymbol{u}_{1h}\rangle = \int_0^{\mathrm{T}} \langle \boldsymbol{\zeta}, \boldsymbol{v}(t) - \boldsymbol{v}_h(t)\rangle \, dt - \int_0^{\mathrm{T}} \langle \boldsymbol{\zeta}(T-t), \overline{\boldsymbol{w}}(t) - \overline{\boldsymbol{w}}_h(t)\rangle \, dt.$$

Passing with h to zero it follows that $\boldsymbol{u}_{1h} \to \dot{\boldsymbol{u}}(0)$ weakly in V', but since we know that $\boldsymbol{u}_{1h} \to \boldsymbol{u}_1$ strongly in H it follows that $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$. Moreover from (24) it follows that $\boldsymbol{u}_h(0) \to \boldsymbol{u}(0)$ and, as $\boldsymbol{u}_h(0) = \boldsymbol{u}_0$, it follows that $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Taking $\boldsymbol{\xi} \in L^2(0,T;V)$ we rewrite (15) as the following equation valid for a.e. $t \in (0,T)$:

$$\begin{aligned}\int_\Omega \rho \overline{\boldsymbol{w}}_h(t) \cdot \boldsymbol{\xi}(t) \, d\boldsymbol{x} + \int_\Omega \mathcal{A}(\nabla \overline{\boldsymbol{v}}_h(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} + \int_\Omega \mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \\ + \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t)) \xi_\nu(t) \, dS + \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t)) \overline{\boldsymbol{\eta}}_h(t) \cdot \boldsymbol{\xi}_\tau(t) \, dS = \langle \boldsymbol{f}_h(t), \boldsymbol{\xi}(t)\rangle_{W' \times W},\end{aligned} \tag{34}$$

with

$$\overline{\boldsymbol{\eta}}_h(t) = \boldsymbol{\eta}_h^j \in S_{\partial j}^\infty(\boldsymbol{v}_{h\tau}^j) \quad \text{for} \quad t \in (t_{j-1}, t_j].$$

After integration from 0 to T in time we get

$$\begin{aligned}\int_0^{\mathrm{T}} \int_\Omega \rho \overline{\boldsymbol{w}}_h(t) \cdot \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt + \int_0^{\mathrm{T}} \int_\Omega \mathcal{A}(\nabla \overline{\boldsymbol{v}}_h(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt + \int_0^{\mathrm{T}} \int_\Omega \mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt \\ + \int_0^{\mathrm{T}} \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t)) \xi_\nu(t) \, dS \, dt + \int_0^{\mathrm{T}} \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t)) \overline{\boldsymbol{\eta}}_h(t) \cdot \boldsymbol{\xi}_\tau(t) \, dS \, dt = \int_0^{\mathrm{T}} \langle \boldsymbol{f}_h(t), \boldsymbol{\xi}(t)\rangle_{W' \times W} \, dt.\end{aligned} \tag{35}$$

We must pass to the limit with h to zero in the six terms in the above equation. First of all (31) together with Lemma 4.4 implies that

$$\int_0^{\mathrm{T}} \int_\Omega \rho \overline{\boldsymbol{w}}_h(t) \cdot \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt \to \int_0^{\mathrm{T}} \langle \rho \ddot{\boldsymbol{u}}(t), \boldsymbol{\xi}(t)\rangle \, dt.$$

Next, as the Nemytskii mapping $\mathcal{A} : L^2(0,T : L^2(\Omega)^{d\times d}) \to L^2(0,T : L^2(\Omega)^{d\times d})$ is linear and continuous, and hence also weakly sequentially continuous, the convergence (26) implies that

$$\int_0^{\mathrm{T}} \int_\Omega \mathcal{A}(\nabla \overline{\boldsymbol{v}}_h(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt \to \int_0^{\mathrm{T}} \int_\Omega \mathcal{A}(\nabla \dot{\boldsymbol{u}}(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt.$$

Next, the convergence (30) implies that

$$\int_0^{\mathrm{T}} \int_\Omega \mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(t)) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt \to \int_0^{\mathrm{T}} \int_\Omega \overline{\mathcal{G}}(t) : \nabla \boldsymbol{\xi}(t) \, d\boldsymbol{x} \, dt.$$

Furthermore (14) implies that

$$\int_0^{\mathrm{T}} \langle \boldsymbol{f}_h(t), \boldsymbol{\xi}(t)\rangle_{W' \times W} \, dt \to \int_0^{\mathrm{T}} \langle \boldsymbol{f}(t), \boldsymbol{\xi}(t)\rangle_{W' \times W} \, dt.$$

To complete the proof of the theorem we must pass to the limit in the two boundary integrals. To this end we first derive the bound on $\overline{\boldsymbol{u}}_h$ in $BV^2(0,T;H)$, and on $\overline{\boldsymbol{v}}_h$ in $BV^2(0,T;V')$; see Definition A.7.

There exist the natural numbers $0 = m_0 < m_1 < \ldots < m_M = N$ such that

$$\|\overline{\boldsymbol{u}}_h\|^2_{BV^2(0,T;H)} = \sum_{k=1}^{M} \|\boldsymbol{u}_h^{m_k} - \boldsymbol{u}_h^{m_{k-1}}\|^2_H.$$

Next,

$$\|\boldsymbol{u}_h^{m_k} - \boldsymbol{u}_h^{m_{k-1}}\|^2_H = \left\| \sum_{j=m_{k-1}+1}^{m_k} (\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}) \right\|^2_H \le \left(\sum_{j=m_{k-1}+1}^{m_k} \left\| \boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1} \right\|_H \right)^2,$$

$$\le (m_k - m_{k-1}) \sum_{j=m_{k-1}+1}^{m_k} \left\| \boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1} \right\|^2_H \le (m_k - m_{k-1}) \sum_{j=1}^{N} \left\| \boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1} \right\|^2_H.$$

Hence, using Theorem 3.2,

$$\|\overline{\boldsymbol{u}}_h\|^2_{BV^2(0,T;H)} \le N \sum_{j=1}^{N} \left\| \boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1} \right\|^2_H = Th \sum_{j=1}^{N} \left\| \boldsymbol{v}_h^j \right\|^2_H \le CT^2. \tag{36}$$

In a similar way we obtain, using (29),

$$\|\overline{\boldsymbol{v}}_h\|^2_{BV^2(0,T;V')} \le N \sum_{j=1}^{N} \left\| \boldsymbol{v}_h^j - \boldsymbol{v}_h^{j-1} \right\|^2_{V'} = Th \sum_{j=1}^{N} \left\| \boldsymbol{w}_h^j \right\|^2_{V'} \le CT. \tag{37}$$

Lemma 4.1 implies that $\overline{\boldsymbol{u}}_h$ is bounded in $L^q(0,T;V)$. Since it is also bounded in $BV^2(0,T;H)$ by (36), Theorem A.8 implies that $\overline{\boldsymbol{u}}_h$ is relatively compact in $L^q(0,T;W^{1-\delta,p}(\Omega)^d)$ for any small $\delta > 0$. Since the trace operator is linear and continuous from $W^{1-\delta,p}(\Omega)^d$ to $L^q(\Gamma_C)^d$ it follows that

$$\overline{\boldsymbol{u}}_h \to \boldsymbol{u} \quad \text{strongly in} \quad L^q(0,T;L^q(\Gamma_C)^d). \tag{38}$$

Now, by Lemma 4.1, $\overline{\boldsymbol{v}}_h$ is bounded in $L^2(0,T;W)$. Hence, the compactness of embedding $W \subset H^{1-\delta}(\Omega)^d$, the continuity of the trace operator from $H^{1-\delta}(\Omega)^d$ to $L^2(\Gamma_C)^d$, and Theorem A.8 imply that

$$\overline{\boldsymbol{v}}_h \to \dot{\boldsymbol{u}} \quad \text{strongly in} \quad L^2(0,T;L^2(\Gamma_C)^d). \tag{39}$$

For a subsequence, denoted by the same index,

$$\overline{\boldsymbol{u}}_h(\boldsymbol{x},t) \to \overline{\boldsymbol{u}}(\boldsymbol{x},t) \quad \text{for a.e.} \quad (\boldsymbol{x},t) \in \Gamma_C \times (0,T) \quad \text{and} \quad |\overline{\boldsymbol{u}}_h(\boldsymbol{x},t)| \le h_1(\boldsymbol{x},t), \tag{40}$$

where $h_1 \in L^q(\Gamma_C \times (0,T))$. Assume for a moment that $\boldsymbol{\xi} \in L^{\max\{2,q\}}(0,T;V)$ (if $q > 2$ we will later use the density argument to show that in fact one can take $\boldsymbol{\xi} \in L^2(0,T;V)$). Then $\xi_\nu \in L^q(\Gamma_C \times (0,T))$. Continuity of p_ν implies that

$$p_\nu(\overline{u}_{h\nu})\xi_\nu \to p_\nu(\overline{u}_\nu)\xi_\nu \quad \text{for a.e.} \quad (\boldsymbol{x},t) \in \Gamma_C \times (0,T).$$

Moreover

$$|p_\nu(\overline{u}_{h\nu})\xi_\nu| \le C(1 + |\overline{u}_{h\nu}|^{q-1})|\xi_\nu| \le c(|\xi_\nu| + |\xi_\nu|^q + |h_1|^q),$$

and the right-hand side is a function in $L^1(\Gamma_C \times (0,T))$. We can use the Lebesgue dominated convergence theorem to deduce that

$$\int_0^{\mathrm{T}} \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_\nu(t)\, dS\, dt \to \int_0^{\mathrm{T}} \int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\xi_\nu(t)\, dS\, dt. \tag{41}$$

Observe that

$$\|p_\nu(\overline{u}_{h\nu})\|^2_{L^2(0,T;L^{q'}(\Gamma_C))} = \int_0^{\mathrm{T}} \|p_\nu(\overline{u}_{h\nu}(t))\|^2_{L^{q'}(\Gamma_C)}\,dt \le \int_0^{\mathrm{T}} \left(\int_{\Gamma_C} |p_\nu(\overline{u}_{h\nu}(t))|^{q'}\,dS\right)^{\frac{2}{q'}}\,dt$$
$$\le \int_0^{\mathrm{T}} \left(\int_{\Gamma_C} C(1+|\overline{u}_{h\nu}(t)|^{q-1})^{q'}\,dS\right)^{\frac{2}{q'}}\,dt \le C\left(1+\int_0^{\mathrm{T}}\left(\int_{\Gamma_C}|\overline{u}_{h\nu}(t)|^q\,dS\right)^{\frac{2}{q'}}\,dt\right)$$
$$\le C\left(1+\int_0^{\mathrm{T}}\|\overline{u}_{h\nu}(t)\|^{2(q-1)}_{L^q(\Gamma_C)}\,dt\right) \le C\left(1+\int_0^{\mathrm{T}}\|\overline{\boldsymbol{u}}_h(t)\|^{2(q-1)}\,dt\right) \le C(1+T\|\overline{\boldsymbol{u}}_h\|^{2(q-1)}_{L^\infty(0,T;V)}) \le C.$$

If $q > 2$ then we can approximate $\boldsymbol{\xi} \in L^2(0,T;V)$ by a strongly convergent sequence $\{\boldsymbol{\xi}_n\} \subset L^q(0,T;V)$. We obtain

$$\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_\nu(t)\,dS\,dt = \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_{n\nu}(t)\,dS\,dt + \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))(\xi_\nu(t)-\xi_{n\nu}(t))dS\,dt.$$

For the first integral on the right-hand side we have

$$\lim_{h\to 0}\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_{n\nu}(t)\,dS\,dt = \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\xi_{n\nu}(t)\,dS\,dt,$$

and the second integral is bounded in the following way:

$$\left|\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))(\xi_\nu(t)-\xi_{n\nu}(t))\,dS\,dt\right| \le \|p_\nu(\overline{u}_{h\nu})\|_{L^2(0,T;L^{q'}(\Gamma_C))}\|\xi_\nu-\xi_{n\nu}\|_{L^2(0,T;L^q(\Gamma_C))}$$
$$\le C\|\boldsymbol{\xi}-\boldsymbol{\xi}_n\|_{L^2(0,T;V)}.$$

In a similar way

$$\left|\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))(\xi_\nu(t)-\xi_{n\nu}(t))\,dS\,dt\right| \le C\|\boldsymbol{\xi}-\boldsymbol{\xi}_n\|_{L^2(0,T;V)}.$$

Hence

$$\left|\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_\nu(t)\,dS\,dt - \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\xi_\nu(t)\,dS\,dt\right|$$
$$\le \left|\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_{n\nu}(t)\,dS\,dt - \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\xi_{n\nu}(t)\,dS\,dt\right| + C\|\boldsymbol{\xi}-\boldsymbol{\xi}_n\|_{L^2(0,T;V)},$$

meaning that

$$\limsup_{h\to 0}\left|\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_\nu(t)\,dS\,dt - \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\xi_\nu(t)\,dS\,dt\right| \le C\|\boldsymbol{\xi}-\boldsymbol{\xi}_n\|_{L^2(0,T;V)},$$

for any $n \in \mathbb{N}$, and we obtain (41) for $\boldsymbol{\xi} \in L^2(0,T;V)$. To pass to the limit in the term with the multi-valued expression note that as $\overline{\boldsymbol{\eta}}_h$ is bounded in $L^\infty(0,T;L^\infty(\Gamma_C))^d$, for a subsequence we have

$$\overline{\boldsymbol{\eta}}_h \to \boldsymbol{\eta} \quad \text{weakly-}^* \text{ in} \quad L^\infty(0,T;L^\infty(\Gamma_C)^d). \tag{42}$$

We can write

$$\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\overline{\boldsymbol{\eta}}_h(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt$$
$$= \int_0^{\mathrm{T}}\int_{\Gamma_C}(p_\nu(\overline{u}_{h\nu}(t))-p_\nu(\overline{u}_\nu(t)))\overline{\boldsymbol{\eta}}_h(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt + \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\overline{\boldsymbol{\eta}}_h(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt.$$

It is straightforward to verify that $p_\nu(\overline{u}_\nu)\boldsymbol{\xi}_\tau \in L^1(\Gamma_C \times (0,T))$, and hence

$$\lim_{h\to 0}\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\overline{\boldsymbol{\eta}}_h(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt = \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\boldsymbol{\eta}(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt.$$

To pass to the limit in the first integral note that

$$\left|\int_0^{\mathrm{T}}\int_{\Gamma_C}(p_\nu(\overline{u}_{h\nu}(t)) - p_\nu(\overline{u}_\nu(t)))\overline{\boldsymbol{\eta}}_h(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt\right| \le C\int_0^{\mathrm{T}}\int_{\Gamma_C}|p_\nu(\overline{u}_{h\nu}(t)) - p_\nu(\overline{u}_\nu(t))||\boldsymbol{\xi}_\tau(t)|\,dS\,dt.$$

The proof that the last integral converges to zero as $h \to 0$ exactly follows the lines of the proof of (41). We have proved that

$$\lim_{h\to 0}\int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\overline{\boldsymbol{\eta}}_h(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt = \int_0^{\mathrm{T}}\int_{\Gamma_C} p_\nu(\overline{u}_\nu(t))\boldsymbol{\eta}(t)\cdot\boldsymbol{\xi}_\tau(t)\,dS\,dt.$$

We need to prove that $\boldsymbol{\eta}(\boldsymbol{x},t) \in \partial j(\dot{\boldsymbol{u}}_\tau(\boldsymbol{x},t))$ a.e. in $\Gamma_C \times (0,T)$. Note that $\overline{\boldsymbol{\eta}}_h(\boldsymbol{x},t) \in \partial j(\overline{\boldsymbol{v}}_\tau(\boldsymbol{x},t))$ a.e. in $\Gamma_C \times (0,T)$. Indeed, the inclusion $\boldsymbol{\eta}_h^j(\boldsymbol{x}) \in \partial j(\boldsymbol{v}_{h\tau}^j(\boldsymbol{x}))$ may not hold on a null set in Γ_C, so, for $t \in (t_{j-1},t_j)$ the inclusion $\overline{\boldsymbol{\eta}}_h(\boldsymbol{x},t) \in \partial j(\overline{\boldsymbol{v}}_{h\tau}(\boldsymbol{x},t))$ does not hold on a null set in $\Gamma_C \times (t_{j-1},t_j)$. Hence, the inclusion $\overline{\boldsymbol{\eta}}_h(\boldsymbol{x},t) \in \partial j(\overline{\boldsymbol{v}}_{h\tau}(\boldsymbol{x},t))$ may not hold on a null set in $\Gamma_C \times (0,T)$. The convergence (42) together with (39), by the convergence theorem of Aubin and Cellina (see [31, Theorem 7.2.2]) immediately implies the required assertion. □

It only remains to verify that $\overline{\mathcal{G}}(t) = \mathcal{G}(\nabla \boldsymbol{u}(t))$ for a.e. $t \in (0,T)$. For this step we first need to obtain the auxiliary lemma on the so-called *propagation of regularity*. The proof of this lemma mostly follows the lines of the proof of [18, Proposition 3.1].

Lemma 4.6. *Both sequences $\nabla \boldsymbol{u}_h$ and $\nabla \overline{\boldsymbol{u}}_h$ converge to $\nabla \boldsymbol{u}$ strongly in $L^2(0,T;L^2(\Omega)^{d\times d})$. More precisely,*

$$\lim_{h\to 0}\int_0^{\mathrm{T}} \|\nabla \boldsymbol{u}(t) - \nabla \overline{\boldsymbol{u}}_h(t)\|^2_{L^2(\Omega)^{d\times d}}\,dt = 0, \tag{43}$$

$$\lim_{h\to 0}\int_0^{\mathrm{T}} \|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}_h(t)\|^2_{L^2(\Omega)^{d\times d}}\,dt = 0. \tag{44}$$

Proof. We know that both sequences $\{\overline{\boldsymbol{v}}_h\}$ and $\{\boldsymbol{v}_h\}$ are bounded in $L^2(0,T;W)$; see Lemma 4.1. Moreover, $\{\overline{\boldsymbol{v}}_h\}$ is bounded in $BV^2(0,T;V')$ (see estimate (37)), and $\{\dot{\boldsymbol{v}}_h\}$ is bounded in $L^2(0,T;V')$ (see Lemma 4.3), remembering that $\dot{\boldsymbol{v}}_h = \overline{\boldsymbol{w}}_h$. By the Aubin–Lions compactness lemma the sequence $\{\boldsymbol{v}_h\}$ is relatively compact in $L^2(0,T;H)$ and by Theorem A.8 so is the sequence $\{\overline{\boldsymbol{v}}_h\}$. Convergences (25) and (26) together with Lemma 4.4 imply that, for a subsequence still indexed by h,

$$\overline{\boldsymbol{v}}_h \to \boldsymbol{v} = \dot{\boldsymbol{u}} \quad \text{strongly in} \quad L^2(0,T;H), \tag{45}$$

$$\boldsymbol{v}_h \to \boldsymbol{v} = \dot{\boldsymbol{u}} \quad \text{strongly in} \quad L^2(0,T;H). \tag{46}$$

In a similar way estimates (20), (21), and (36) imply that

$$\overline{\boldsymbol{u}}_h \to \boldsymbol{u} \quad \text{strongly in} \quad L^2(0,T;H),$$

$$\boldsymbol{u}_h \to \boldsymbol{u} \quad \text{strongly in} \quad L^2(0,T;H).$$

Since $\boldsymbol{u}_h \to \boldsymbol{u}$ strongly in $H^1(0,T;H)$ we deduce that

$$\boldsymbol{u}_h \to \boldsymbol{u} \quad \text{strongly in} \quad C([0,T];H). \tag{47}$$

First we show that it is sufficient to demonstrate (44), and the convergence (43) follows. Indeed,

$$\|\nabla \overline{\boldsymbol{u}}_h(t) - \nabla \boldsymbol{u}(t)\|_{L^2(\Omega)^{d\times d}} \le \|\nabla \boldsymbol{u}_h(t) - \nabla \boldsymbol{u}(t)\|_{L^2(\Omega)^{d\times d}} + \frac{t-t_j}{h}\|\nabla \boldsymbol{u}_h^j - \nabla \boldsymbol{u}_h^{j-1}\|_{L^2(\Omega)^{d\times d}},$$

for $t \in (t_{j-1}, t_j)$. Hence

$$\int_{I_h^j} \|\nabla \overline{\boldsymbol{u}}_h(t) - \nabla \boldsymbol{u}(t)\|^2_{L^2(\Omega)^{d\times d}}\, dt \le 2\int_{I_h^j} \|\nabla \boldsymbol{u}_h(t) - \nabla \boldsymbol{u}(t)\|^2_{L^2(\Omega)^{d\times d}}\, dt + \frac{2h}{3}\|\nabla \boldsymbol{u}_h^j - \nabla \boldsymbol{u}_h^{j-1}\|^2_{L^2(\Omega)^{d\times d}}.$$

It follows that

$$\begin{aligned}
\|\nabla \overline{\boldsymbol{u}}_h - \nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})} &\le 2\|\nabla \boldsymbol{u}_h - \nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})} + \frac{2h}{3}\sum_{j=1}^{N} \|\nabla \boldsymbol{u}_h^j - \nabla \boldsymbol{u}_h^{j-1}\|^2_{L^2(\Omega)^{d\times d}} \\
&\le 2\|\nabla \boldsymbol{u}_h - \nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})} + \frac{2h^3}{3}\sum_{j=1}^{N} \|\nabla \boldsymbol{v}_h^j\|^2_{L^2(\Omega)^{d\times d}} \\
&\le 2\|\nabla \boldsymbol{u}_h - \nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})} + Ch^2,
\end{aligned}$$

where, in the last estimate, we have used Theorem 3.2. We pass to the proof of (44). Take $\boldsymbol{\xi} \in L^2(0, T+h; V)$ and extend $\overline{\boldsymbol{v}}_h$ to $(-h, T)$ by taking $\overline{\boldsymbol{v}}(t) = \boldsymbol{v}_h^0 = \boldsymbol{u}_{1h}$ on the interval $(-h, 0)$. Clearly

$$\frac{\boldsymbol{u}_h^j - 2\boldsymbol{u}_h^{j-1} + \boldsymbol{u}_h^{j-2}}{h^2} = \frac{\boldsymbol{v}_h^j - \boldsymbol{v}_h^{j-1}}{h} = \frac{\overline{\boldsymbol{v}}_h(t) - \overline{\boldsymbol{v}}_h(t-h)}{h} \quad \text{for} \quad t \in (t_{j-1}, t_j), j \in \{1, \dots, N\}.$$

Hence, we can rewrite (15) in the following way, where $s \in (0, T)$:

$$\begin{aligned}
&\int_0^s \int_\Omega \rho \frac{\overline{\boldsymbol{v}}_h(t) - \overline{\boldsymbol{v}}_h(t-h)}{h} \cdot \boldsymbol{\xi}(t)\, d\boldsymbol{x}\, dt + \int_0^s \int_\Omega \mathcal{A}(\nabla \overline{\boldsymbol{v}}_h(t)) : \nabla \boldsymbol{\xi}(t)\, d\boldsymbol{x}\, dt \\
&\quad + \int_0^s \int_\Omega \mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(t)) : \nabla \boldsymbol{\xi}(t)\, d\boldsymbol{x}\, dt + \int_0^s \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_\nu(t)\, dS\, dt \\
&\quad + \int_0^s \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\overline{\boldsymbol{\eta}}_h(t) \cdot \boldsymbol{\xi}_\tau(t)\, dS\, dt = \int_0^s \langle \boldsymbol{f}_h(t), \boldsymbol{\xi}(t)\rangle_{W'\times W}\, dt.
\end{aligned}$$

We rearrange the first term

$$\begin{aligned}
&\int_0^s \int_\Omega \rho \frac{\overline{\boldsymbol{v}}_h(t) - \overline{\boldsymbol{v}}_h(t-h)}{h} \cdot \boldsymbol{\xi}(t)\, d\boldsymbol{x}\, dt \\
&\quad = \int_0^s \int_\Omega \rho \overline{\boldsymbol{v}}_h(t) \cdot \frac{\boldsymbol{\xi}(t)}{h}\, d\boldsymbol{x}\, dt - \int_{-h}^{s-h} \int_\Omega \rho \overline{\boldsymbol{v}}_h(t) \cdot \frac{\boldsymbol{\xi}(t+h)}{h}\, d\boldsymbol{x}\, dt \\
&\quad = \int_0^s \int_\Omega \rho \overline{\boldsymbol{v}}_h(t) \cdot \frac{\boldsymbol{\xi}(t) - \boldsymbol{\xi}(t+h)}{h}\, d\boldsymbol{x}\, dt - \int_{-h}^{0} \int_\Omega \rho \boldsymbol{v}_0^h \cdot \frac{\boldsymbol{\xi}(t+h)}{h}\, d\boldsymbol{x}\, dt \\
&\qquad + \int_{s-h}^{s} \int_\Omega \rho \overline{\boldsymbol{v}}_h(t) \cdot \frac{\boldsymbol{\xi}(t+h)}{h}\, d\boldsymbol{x}\, dt.
\end{aligned}$$

We have the following weak form of the discretized problem valid for any $\boldsymbol{\xi} \in L^2(0, s+h; V)$:

$$\begin{aligned}
&\int_0^s \int_\Omega \rho \overline{\boldsymbol{v}}_h(t) \cdot \frac{\boldsymbol{\xi}(t) - \boldsymbol{\xi}(t+h)}{h}\, d\boldsymbol{x}\, dt - \int_{-h}^{0} \int_\Omega \rho \boldsymbol{v}_0^h \cdot \frac{\boldsymbol{\xi}(t+h)}{h}\, d\boldsymbol{x}\, dt + \int_{s-h}^{s} \int_\Omega \rho \overline{\boldsymbol{v}}_h(t) \cdot \frac{\boldsymbol{\xi}(t+h)}{h}\, d\boldsymbol{x}\, dt \qquad (48) \\
&\quad + \int_0^s \int_\Omega \mathcal{A}(\nabla \overline{\boldsymbol{v}}_h(t)) : \nabla \boldsymbol{\xi}(t)\, d\boldsymbol{x}\, dt + \int_0^s \int_\Omega \mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(t)) : \nabla \boldsymbol{\xi}(t)\, d\boldsymbol{x}\, dt \\
&\quad + \int_0^s \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\xi_\nu(t)\, dS\, dt + \int_0^s \int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\overline{\boldsymbol{\eta}}_h(t) \cdot \boldsymbol{\xi}_\tau(t)\, dS\, dt = \int_0^s \langle \boldsymbol{f}_h(t), \boldsymbol{\xi}(t)\rangle_{W'\times W}\, dt.
\end{aligned}$$

We take $\boldsymbol{\xi} = \overline{\boldsymbol{u}}_h - \boldsymbol{u}$ in (48) (we extend $\boldsymbol{f}$, $\boldsymbol{u}$, $\boldsymbol{u}_h$, and $\overline{\boldsymbol{u}}$ to $(0, 2T)$; in fact we can continue the recursive solving of the discretized scheme to the interval $(0, 2T)$ and all estimates and convergence we have obtained on the interval

$(0,T)$ will remain valid for $(0,2T)$) and subtract from the resultant equation (33) tested with $\chi_{(0,s)}(\boldsymbol{u}_h-\boldsymbol{u})$ ($\chi_{(0,s)}$ is the characteristic function of $(0,s)$). The term with $\ddot{\boldsymbol{u}}$ is integrated by parts as follows:

$$\begin{aligned}
&\int_0^s \langle \rho\ddot{\boldsymbol{u}}(t), \boldsymbol{u}_h(t)-\boldsymbol{u}(t)\rangle\, dt\\
&\quad= -\int_0^s\int_\Omega \rho\dot{\boldsymbol{u}}(t)\cdot(\dot{\boldsymbol{u}}_h(t)-\dot{\boldsymbol{u}}(t))\,d\boldsymbol{x}\,dt + \int_\Omega \rho\dot{\boldsymbol{u}}(s)\cdot(\boldsymbol{u}_h(s)-\boldsymbol{u}(s))\,d\boldsymbol{x} - \int_\Omega \rho\dot{\boldsymbol{u}}(0)\cdot(\boldsymbol{u}_h(0)-\boldsymbol{u}(0))\,d\boldsymbol{x}\\
&\quad= -\int_0^s\int_\Omega \rho\dot{\boldsymbol{u}}(t)\cdot(\dot{\boldsymbol{u}}_h(t)-\dot{\boldsymbol{u}}(t))\,d\boldsymbol{x}\,dt + \int_\Omega \rho\dot{\boldsymbol{u}}(s)\cdot(\boldsymbol{u}_h(s)-\boldsymbol{u}(s))\,d\boldsymbol{x}.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\underbrace{\int_0^s\int_\Omega \rho\bar{\boldsymbol{v}}_h(t)\frac{\overline{\boldsymbol{u}}_h(t)-\boldsymbol{u}(t)-(\overline{\boldsymbol{u}}_h(t+h)-\boldsymbol{u}(t+h))}{h} + \rho\dot{\boldsymbol{u}}(t)\cdot(\dot{\boldsymbol{u}}_h(t)-\dot{\boldsymbol{u}}(t))\,d\boldsymbol{x}\,dt}_{T_1}\\
&\quad\underbrace{-\int_{-h}^0\int_\Omega \rho\boldsymbol{v}_0^h\frac{\overline{\boldsymbol{u}}_h(t+h)-\boldsymbol{u}(t+h)}{h}\,d\boldsymbol{x}\,dt}_{T_2}\\
&\quad\underbrace{+\int_{s-h}^s\int_\Omega \rho\bar{\boldsymbol{v}}_h(t)\frac{\overline{\boldsymbol{u}}_h(t+h)-\boldsymbol{u}(t+h)}{h} - \rho\dot{\boldsymbol{u}}(s)\cdot\frac{\boldsymbol{u}_h(s)-\boldsymbol{u}(s)}{h}\,d\boldsymbol{x}\,dt}_{T_3}\\
&\quad\underbrace{+\int_0^s\int_\Omega \mathcal{A}(\nabla\bar{\boldsymbol{v}}_h(t)):(\nabla\overline{\boldsymbol{u}}_h(t)-\nabla\boldsymbol{u}(t)) - \mathcal{A}(\nabla\dot{\boldsymbol{u}}(t)):(\nabla\boldsymbol{u}_h(t)-\nabla\boldsymbol{u}(t))\,d\boldsymbol{x}\,dt}_{T_4}\\
&\quad\underbrace{+\int_0^s\int_\Omega \mathcal{G}(\nabla\overline{\boldsymbol{u}}_h(t)):(\nabla\overline{\boldsymbol{u}}_h(t)-\nabla\boldsymbol{u}(t)) - \overline{\mathcal{G}}(t):(\nabla\boldsymbol{u}_h(t)-\nabla\boldsymbol{u}(t))\,d\boldsymbol{x}\,dt}_{T_5}\\
&\quad\underbrace{+\int_0^s\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))(\overline{u}_{h\nu}(t)-u_\nu(t)) - p_\nu(u_\nu(t))(u_{h\nu}(t)-u_\nu(t))\,dS\,dt}_{T_6}\\
&\quad\underbrace{+\int_0^s\int_{\Gamma_C} p_\nu(\overline{u}_{h\nu}(t))\overline{\boldsymbol{\eta}}_h(t)\cdot(\overline{\boldsymbol{u}}_{h\tau}(t)-\boldsymbol{u}_\tau(t)) - p_\nu(u_\nu(t))\boldsymbol{\eta}(t)\cdot(\boldsymbol{u}_{h\tau}(t)-\boldsymbol{u}_\tau(t))\,dS\,dt}_{T_7}\\
&\underbrace{=\int_0^s \langle \boldsymbol{f}_h(t),\overline{\boldsymbol{u}}_h(t)-\boldsymbol{u}(t)\rangle_{W'\times W} - \langle \boldsymbol{f}(t),\boldsymbol{u}_h(t)-\boldsymbol{u}(t)\rangle_{W'\times W}\,dt}_{T_8}.
\end{aligned}$$

We estimate terms $T_1,\dots,T_8$. We will use the notation $\eta(h)$ for terms tending to zero as $h\to 0$ uniformly with respect to $s\in(0,T)$. Terms $T_1,\dots,T_5$ are estimated exactly as in [18, Proposition 3.1]. The estimate for $|T_1|$ is as follows:

$$\begin{aligned}
|T_1| &\le \rho\|\bar{\boldsymbol{v}}_h\|_{L^2(0,T;H)}\left(\|\bar{\boldsymbol{v}}_h-\dot{\boldsymbol{u}}\|_{L^2(h,T+h;H)} + \left\|\dot{\boldsymbol{u}}-\frac{\boldsymbol{u}-\boldsymbol{u}(\cdot-h)}{h}\right\|_{L^2(h,T+h;H)}\right)\\
&\quad+\rho\|\dot{\boldsymbol{u}}\|_{L^2(0,T;H)}\|\bar{\boldsymbol{v}}_h-\dot{\boldsymbol{u}}\|_{L^2(0,T;H)} = \eta(h).
\end{aligned}$$

To estimate $|T_2|$ and $|T_3|$ we need

$$\|\overline{\boldsymbol{u}}_h-\boldsymbol{u}_h\|_{L^\infty(0,T;H)} = \sup_{j\in\{1,\dots,N\}}\|\boldsymbol{u}_j^h-\boldsymbol{u}_j^{h-1}\|_H = h\sup_{j\in\{1,\dots,N\}}\|\boldsymbol{v}_j^h\|_H = \eta(h). \tag{49}$$

Using the above bound and (47),

$$|T_2| \le \rho\|\boldsymbol{v}_0^h\|_H\|\overline{\boldsymbol{u}}_h - \boldsymbol{u}\|_{L^\infty(0,h;H)} = \eta(h) + \rho\|\boldsymbol{v}_0^h\|_H\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^\infty(0,T;H)} = \eta(h).$$

We pass to the estimate of $|T_3|$, where we use (47) and (49) in the space $L^\infty(0,2T;H)$ and the bound (21):

$$\begin{aligned}|T_3| &\le \rho\|\overline{\boldsymbol{v}}_h\|_{L^\infty(-h,T;H)}\|\overline{\boldsymbol{u}}_h - \boldsymbol{u}\|_{L^\infty(0,T+h;H)} + \rho\|\dot{\boldsymbol{u}}\|_{L^\infty(0,T;H)}\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^\infty(0,T;H)}\\ &\le \rho\|\overline{\boldsymbol{v}}_h\|_{L^\infty(-h,T;H)}\|\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h\|_{L^\infty(0,2T;H)}\\ &\qquad + \rho\left(\|\dot{\boldsymbol{u}}\|_{L^\infty(0,T;H)} + \|\overline{\boldsymbol{v}}_h\|_{L^\infty(-h,T;H)}\right)\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^\infty(0,2T;H)} = \eta(h).\end{aligned}$$

We pass to estimates of T_4, T_5, and T_6:

$$\begin{aligned}T_4 &= \int_0^s\int_\Omega \mathcal{A}(\nabla\overline{\boldsymbol{v}}_h(t)) : (\nabla\overline{\boldsymbol{u}}_h(t) - \nabla\boldsymbol{u}(t)) - \mathcal{A}(\nabla\dot{\boldsymbol{u}}(t)) : (\nabla\boldsymbol{u}_h(t) - \nabla\boldsymbol{u}(t))\,d\boldsymbol{x}\,dt\\ &= \int_0^s\int_\Omega \mathcal{A}(\nabla\overline{\boldsymbol{v}}_h(t) - \nabla\dot{\boldsymbol{u}}(t)) : (\nabla\boldsymbol{u}_h(t) - \nabla\boldsymbol{u}(t)) + \mathcal{A}(\nabla\overline{\boldsymbol{v}}_h(t)) : (\nabla\overline{\boldsymbol{u}}_h(t) - \nabla\boldsymbol{u}_h(t))\,d\boldsymbol{x}\,dt\\ &= \frac{1}{2}\int_\Omega \mathcal{A}(\nabla\boldsymbol{u}_h(s) - \nabla\boldsymbol{u}(s)) : (\nabla\boldsymbol{u}_h(s) - \nabla\boldsymbol{u}(s))\,d\boldsymbol{x} + \sum_{j=1}^N \int_{(0,s)\cap I_h^j}(t_j - t)\int_\Omega \mathcal{A}(\nabla\boldsymbol{v}_h^j) : \nabla\boldsymbol{v}_h^j\,d\boldsymbol{x}\,dt\\ &\ge \frac{\lambda}{2}\|\nabla\boldsymbol{u}_h(s) - \nabla\boldsymbol{u}(s)\|^2_{L^2(\Omega)^{d\times d}}.\end{aligned}$$

$$\begin{aligned}T_5 &= \int_0^s\int_\Omega (\mathcal{G}(\nabla\overline{\boldsymbol{u}}_h(t)) - \mathcal{G}(\nabla\boldsymbol{u}(t))) : (\nabla\overline{\boldsymbol{u}}_h(t) - \nabla\boldsymbol{u}(t)) + \mathcal{G}(\nabla\boldsymbol{u}(t))) : (\nabla\overline{\boldsymbol{u}}_h(t) - \nabla\boldsymbol{u}(t))\\ &\qquad - \overline{\mathcal{G}}(t) : (\nabla\boldsymbol{u}_h(t) - \nabla\boldsymbol{u}(t))\,d\boldsymbol{x}\,dt\\ &\ge -K\int_0^s\int_\Omega |\nabla\overline{\boldsymbol{u}}_h(t) - \nabla\boldsymbol{u}(t)|^2\,d\boldsymbol{x}\,dt - \eta(h) \ge -2K\int_0^s \|\nabla\boldsymbol{u}_h(t) - \nabla\boldsymbol{u}(t)\|^2_{L^2(\Omega)^{d\times d}}\,dt - \eta(h).\end{aligned}$$

$$\begin{aligned}|T_6| &\le \int_0^{\mathrm{T}}\int_{\Gamma_C} |p_\nu(\overline{u}_{h\nu}(t))||\overline{u}_{h\nu}(t) - u_\nu(t)| + |p_\nu(u_\nu(t))||u_{h\nu}(t) - u_\nu(t)|\,dS\,dt\\ &\le C\int_0^{\mathrm{T}}\int_{\Gamma_C} \left(1 + |\overline{u}_{h\nu}(t)|^{q-1}\right)|\overline{u}_{h\nu}(t) - u_\nu(t)| + \left(1 + |u_\nu(t)|^{q-1}\right)|u_{h\nu}(t) - u_\nu(t)|\,dS\,dt\\ &\le C\int_0^{\mathrm{T}} \left(1 + \|\overline{u}_{h\nu}(t)\|^{q-1}_{L^q(\Gamma_C)}\right)\|\overline{u}_{h\nu}(t) - u_\nu(t)\|_{L^q(\Gamma_C)} + \left(1 + \|u_\nu(t)\|^{q-1}_{L^q(\Gamma_C)}\right)\|u_{h\nu}(t) - u_\nu(t)\|_{L^q(\Gamma_C)}\,dt.\end{aligned}$$

The bound (20) implies that $\overline{u}_{h\nu}$ is bounded in $L^\infty(0,T;L^q(\Gamma_C))$. Moreover, as $u \in L^\infty(0,T;V)$ we have $u_\nu \in L^\infty(0,T;L^q(\Gamma_C))$. Hence

$$|T_6| \le C\int_0^{\mathrm{T}} \|\overline{u}_{h\nu}(t) - u_\nu(t)\|_{L^q(\Gamma_C)} + \|u_{h\nu}(t) - u_\nu(t)\|_{L^q(\Gamma_C)}\,dt.$$

Convergences (24) and (25) imply, by the Aubin–Lions lemma, that

$$\boldsymbol{u}_h \to \boldsymbol{u} \quad \text{strongly in} \quad L^q(0,T;W^{1-\delta,p}(\Omega)^d).$$

Hence, similarly to in (38), we have

$$\boldsymbol{u}_h \to \boldsymbol{u} \quad \text{strongly in} \quad L^q(0,T;L^q(\Gamma_C)^d).$$

This fact together with (38) implies that

$$|T_6| \le \eta(h).$$

The estimate on $|T_7|$ is derived in the same way as the estimate for $|T_6|$, so we do not present its detailed derivation. We only note that we use the fact that $|\boldsymbol{\eta}(\boldsymbol{x},t)| \le C_j$ and $|\overline{\boldsymbol{\eta}}_h(\boldsymbol{x},t)| \le C_j$ for a.e. $(\boldsymbol{x},t) \in \Omega \times (0,T)$:

$$|T_7| \le \eta(h).$$

The term T_8 can be rewritten as follows:

$$T_8 = \int_0^s \langle \boldsymbol{f}_h(t) - \boldsymbol{f}(t), \overline{\boldsymbol{u}}_h(t) - \boldsymbol{u}(t)\rangle_{W'\times W} + \langle \boldsymbol{f}(t), \overline{\boldsymbol{u}}_h(t) - \boldsymbol{u}_h(t)\rangle_{W'\times W}\, dt,$$

whence we have the following estimate which uses (14) and the bounds of Lemma 4.1:

$$\begin{aligned}
|T_8| &\le C\|\boldsymbol{f}_h - \boldsymbol{f}\|_{L^2(0,T;W')}(\|\overline{\boldsymbol{u}}_h\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T;V)}) + \int_0^{\mathrm{T}} \|\boldsymbol{f}(t)\|_{W'}\|\overline{\boldsymbol{u}}_h(t) - \boldsymbol{u}_h(t)\|_W\, dt \\
&\le \eta(h) + \sum_{j=1}^{N} \int_{I_h^j} \|\boldsymbol{f}(t)\|_{W'} \frac{t_j - t}{h} \|\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}\|_W\, dt \le \eta(h) + h\int_0^{\mathrm{T}} \|\boldsymbol{f}(t)\|_{W'}\|\overline{\boldsymbol{v}}_h(t)\|_W\, dt \\
&\le \eta(h) + h\|\boldsymbol{f}\|_{L^2(0,T;W')}\|\overline{\boldsymbol{v}}_h\|_{L^2(0,T;W)} = \eta(h).
\end{aligned}$$

Summarizing all the bounds we obtain the estimate

$$\frac{\lambda}{2}\|\nabla \boldsymbol{u}_h(s) - \nabla \boldsymbol{u}(s)\|^2_{L^2(\Omega)^{d\times d}} \le 2K\int_0^s \|\nabla \boldsymbol{u}_h(t) - \nabla \boldsymbol{u}(t)\|^2_{L^2(\Omega)^{d\times d}}\, dt + \eta(h)$$

valid for all $s \in (0,T)$, whence the assertion of the lemma follows easily by the Gronwall inequality. □

Remark 4.7. The above lemma can be generalized to the situation where the initial displacement $\boldsymbol{u}_0$ is approximated by a sequence $\boldsymbol{u}_{0h} \in V$. Then, in accordance with [18], we would need $\boldsymbol{u}_{0h} \to \boldsymbol{u}_0$ weakly in V and $\boldsymbol{u}_{0h} \to \boldsymbol{u}_0$ strongly in W.

In the last step of the proof we show that $\overline{\mathcal{G}} = \mathcal{G}(\nabla \boldsymbol{u})$.

Theorem 4.8. *We have $\overline{\mathcal{G}}(t) = \mathcal{G}(\nabla \boldsymbol{u}(t))$ for a.e. $t \in (0,T)$, and hence the function $\boldsymbol{u}$ is the weak solution given by Definition 2.2.*

Proof. Lemma 4.6 implies that, for a subsequence

$$\nabla \overline{\boldsymbol{u}}_h(\boldsymbol{x},t) \to \nabla \boldsymbol{u}(\boldsymbol{x},t) \quad \text{for a.e.} \quad (\boldsymbol{x},t) \in \Omega \times (0,T),$$

and by the continuity of $\mathcal{G}$,

$$\mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(\boldsymbol{x},t)) \to \mathcal{G}(\nabla \boldsymbol{u}(\boldsymbol{x},t)) \quad \text{for a.e.} \quad (\boldsymbol{x},t) \in \Omega \times (0,T).$$

Fix $\epsilon > 0$. By the Egorov theorem there exists the set $A_\epsilon \subset \Omega \times (0,T)$ with $\mu_{d+1}((\Omega \times (0,T)) \setminus A_\epsilon) < \epsilon$ such that

$$\lim_{h\to 0} \sup_{(\boldsymbol{x},t)\in A_\epsilon} |\mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(\boldsymbol{x},t)) - \mathcal{G}(\nabla \boldsymbol{u}(\boldsymbol{x},t))| = 0. \tag{50}$$

Convergence (30) implies that

$$\mathcal{G}(\nabla \overline{\boldsymbol{u}}_h) \to \overline{\mathcal{G}} \quad \text{weakly in} \quad L^{q'}(A_\epsilon)^{d\times d}. \tag{51}$$

Take $\boldsymbol{\eta} \in L^q(A_\epsilon)^{d\times d}$:

$$\begin{aligned}
&\int_{A_\epsilon} \big(\overline{\mathcal{G}}(\boldsymbol{x},t) - \mathcal{G}(\nabla \boldsymbol{u}(\boldsymbol{x},t))\big) : \boldsymbol{\eta}\, d\boldsymbol{x}\, dt \\
&= \int_{A_\epsilon} \big(\overline{\mathcal{G}}(\boldsymbol{x},t) - \mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(\boldsymbol{x},t))\big) : \boldsymbol{\eta}\, d\boldsymbol{x}\, dt + \int_{A_\epsilon} (\mathcal{G}(\nabla \overline{\boldsymbol{u}}_h(\boldsymbol{x},t)) - \mathcal{G}(\nabla \boldsymbol{u}(\boldsymbol{x},t))) : \boldsymbol{\eta}\, d\boldsymbol{x}\, dt.
\end{aligned}$$

Both integrals converge to zero as $h \to 0$, the first one from (51), and the second one by (50). Since $\epsilon > 0$ was arbitrary, the assertion follows readily. □

5. Numerical simulations

The aim of this section is to provide some numerical simulations in order to characterize the mechanical behaviour of the dynamic hyperviscoelastic contact problem with normal compliance conditions and non-monotone friction analyzed in the previous sections.

First we formulate the fully discrete version of the problem. By k we will denote a constant signifying the space discretization parameter. We choose the sequence of finite-dimensional spaces approximating the space V from inside, that is, $\overline{\bigcup_{k=1}^{\infty} V_k}^{V} = V$. We also need the discrete spaces $Y_{k\nu}$ and $Y_{k\tau}$ of functions defined on the contact boundary and the projection operators $\Pi_{k\nu} : V_k \to Y_{k\nu}$ and $\Pi_{k\tau} : V_k \to Y_{k\tau}$. The spaces $Y_{k\nu}$ and $Y_{k\tau}$ contain the functions which are piecewise constant on the appropriately discretized contact boundary, and the use of the Haslinger projections $\Pi_{k\nu}$ and $\Pi_{k\tau}$ allows us to discretize the boundary contact condition in a way which is convenient from the point of view of numerical computations, because it allows us to choose only one selection of the multifunction on the whole element of the contact boundary; see [34] for details. Then, with the previous notation, the fully discrete problem is formulated as follows.

Problem ($\mathbf{P}_{hk}$). Find $\{\boldsymbol{u}_{hk}^j\}_{j=1,\dots,N} \subset V_k$ and $\{\boldsymbol{\eta}_{hk}^j\}_{j=1,\dots,N} \subset Y_{k\tau}$ such that for $j \in \{1,\dots,N\}$ we have

$$\int_\Omega \rho \frac{\boldsymbol{u}_{hk}^j - 2\boldsymbol{u}_{hk}^{j-1} + \boldsymbol{u}_{hk}^{j-2}}{h^2} \cdot \boldsymbol{\xi}\, d\boldsymbol{x} + \int_\Omega \mathcal{A}\left(\frac{\nabla \boldsymbol{u}_{hk}^j - \nabla \boldsymbol{u}_{hk}^{j-1}}{h}\right) : \nabla \boldsymbol{\xi}\, d\boldsymbol{x} + \int_\Omega \mathcal{G}(\nabla \boldsymbol{u}_{hk}^j) : \nabla \boldsymbol{\xi}\, d\boldsymbol{x}$$
$$+ \int_{\Gamma_C} p_\nu(\Pi_{k\nu}\boldsymbol{u}_{hk}^j)\Pi_{k\nu}\boldsymbol{\xi}\, dS + \int_{\Gamma_C} p_\nu(\Pi_{k\nu}\boldsymbol{u}_{hk}^j)\boldsymbol{\eta}_{hk}^j \cdot \Pi_{k\tau}\boldsymbol{\xi}\, dS = \langle \boldsymbol{f}_h^j, \boldsymbol{\xi}\rangle_{W'\times W} \quad \text{for every} \quad \boldsymbol{\xi} \in V_k,$$

with

$$\boldsymbol{\eta}_{hk}^j(\boldsymbol{x}) \in \partial j\left(\Pi_{k\tau}\left(\frac{\boldsymbol{u}_{hk}^j - \boldsymbol{u}_{hk}^{j-1}}{h}\right)(\boldsymbol{x})\right).$$

In the examples presented below V_k are piecewise linear finite element spaces and $Y_{k\nu}$ and $Y_{k\tau}$ are spaces of piecewise constant functions on the contact boundary. The above formulation presents only one possibility of the time stepping scheme. In numerical simulations we use another scheme, given in [35].

Based on a particular example of non-convex hyperviscoelastic energy density $\Phi(\mathbf{F})$ presented in Section 5.1, we carry out numerical simulations of two academic but non-trivial hyperelastodynamic contact problems with friction: the impact of a ring against a foundation (Section 5.3) and the buckling of a slender stem that comes into contact with an upper and a lower obstacle (Section 5.4). Section 5.3 is devoted to giving a quantitative and qualitative validation of the specific hyperviscoelastic model compared to a well-known Ogden-type hyperelastic model. Finally, in Section 5.4, the chosen example enables us to highlight the question of possible non-uniqueness of the numerical solution of Problem (P_{hk}). Note that Section 5.2 gives a brief overview of the main traits of the numerical methods used to solve Problem (P_{hk}).

5.1. Specific constitutive laws: Hyperviscoelastic energy density and frictional contact conditions

Here, we present the specific constitutive laws used to model the dynamic hyperviscoelastic contact problem with normal compliance conditions and non-monotone friction P_V introduced in Section 2.

Specific non-convex hyperviscoelastic energy density $\Phi(\mathbf{F})$. The stress–strain behaviour of hyperelastic materials is highly nonlinear and a simple modulus of elasticity is no longer sufficient. The constitutive behaviour of hyperelastic material is characterized by the stress tensor $\boldsymbol{\sigma}$ which derives from an internal hyperelastic energy density $\Phi(\mathbf{F})$, that is, $\boldsymbol{\sigma} = \partial_{\mathbf{F}}\Phi(\mathbf{F})$. Here $\mathbf{F}$ is the deformation gradient defined by $\mathbf{F} = \mathbf{I} + \nabla \boldsymbol{u}$ and $\partial_{\mathbf{F}}$ represents the differential with respect to the variable $\mathbf{F}$; see [36] for details. Due to the frame material requirement, the internal hyperelastic energy density $\Phi(\mathbf{F})$ is expressed in terms of the three strain invariants $I_1 = \mathrm{tr}(\mathbf{C})$, $I_2 = \mathrm{tr}(\mathrm{cof}(\mathbf{C}))$ and $I_3 = \det(\mathbf{C})$ of the right Cauchy–Green strain tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$.

The hyperviscoelastic energy density considered in this work is based on the Yeoh model that is used for foam-like or rubber-like materials [37, 38]. In 1993, Yeoh [39] proposed a phenomenological model in the form of a third-order polynomial based only on the first invariant I_1. It can be used for the characterization of carbon black filled rubber and can capture upturn of stress–strain curves. In this work, we consider a variant of Yeoh's model with nearly incompressible behaviour by considering a penalization term depending on the third invariant I_3 [37, 40]. This model has good fit over a large strain range and can simulate various modes of deformation with

limited data. This model does not include any dependency on I_2. The sensitivity of the strain energy function to variation in I_2 is generally much smaller than its sensitivity to variation in I_1. It appears that eliminating the terms containing I_2 from the strain energy function improves the ability of the models to predict the behaviour of complex deformation states when limited test data is available. This variant of Yeoh's model is also called the reduced polynomial model for nearly incompressible rubber and it can take the following form:

$$\Phi(\mathbf{F}) = \sum_{i=1}^{N} C_i \left(\frac{||\mathbf{F}||^2}{(\det(\mathbf{F}))^{2/3}} - 3 \right)^i + D(\det(\mathbf{F}) - 1)^2, \tag{52}$$

where C_i are positive constants and D is a positive penalization coefficient. The given value N (equal to 1, 2 or 3) is related to the number of terms in the strain energy function that we want to consider. Values of $N > 2$ are rarely used due to the difficulty of fitting such a large number of material properties to experimental data. Note that we have $||\mathbf{F}|| = \sqrt{I_1}$ and $\det(\mathbf{F}) = \sqrt{I_3}$. This density enables us to satisfy both the mathematical hypothesis and the mechanical requirements (material frame indifference, infinite amount of energy necessary to expand a body infinitely, and near-incompressibility). More precisely, for the case of incompressible strains ($I_3 = 1$), it can be shown that the Andrews–Ball-type condition (monotonicity condition of the stress tensor for large strains; see [18, 19, 41]) is verified. In the case of nearly incompressible strains, the monotonicity condition at infinity can be proved for $N = 1$ and for the assumption where $\det(\mathbf{F})$ is very near 1. Furthermore, the hyperelastic model is extended to viscosity with the addition of a damping term as proposed in [18]:

$$\Phi_\omega(\dot{\mathbf{F}}) = \frac{\omega}{4} \left(\mathrm{tr}(\dot{\mathbf{F}}^2) + \mathrm{tr}((\dot{\mathbf{F}}^{\mathrm{T}})^2) \right), \tag{53}$$

where ω is the damping coefficient.

Therefore, we consider a stress tensor $\boldsymbol{\sigma}$ characterized by a viscoelastic Kevin–Voigt law which permits us to fulfil the fundamental requirement of infinitesimal frame indifference,

$$\boldsymbol{\sigma}(\mathbf{F}, \dot{\mathbf{F}}) = \frac{\partial \Phi(\mathbf{F})}{\partial \mathbf{F}} + \frac{\partial \Phi_\omega(\dot{\mathbf{F}})}{\partial \dot{\mathbf{F}}}. \tag{54}$$

Thereby, the stress tensor takes the following form:

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{F}, \dot{\mathbf{F}}) = & \sum_{i=1}^{N} i \, \frac{C_i}{(\det(\mathbf{F}))^{2/3}} \left(\frac{||\mathbf{F}||^2}{(\det(\mathbf{F}))^{2/3}} - 3 \right)^{i-1} \mathbf{F} \\ & - \frac{2}{3} \sum_{i=1}^{N} i \, C_i \frac{||\mathbf{F}||^2}{(\det(\mathbf{F}))^{5/3}} \left(\frac{||\mathbf{F}||^2}{(\det(\mathbf{F}))^{2/3}} - 3 \right)^{i-1} \mathrm{cof}(\mathbf{F}) \\ & + 2D(\det(\mathbf{F}) - 1)\mathrm{cof}(\mathbf{F}) + \frac{\omega}{2} \left(\dot{\mathbf{F}} + \dot{\mathbf{F}}^{\mathrm{T}} \right). \end{aligned} \tag{55}$$

As mentioned above, by considering a high deformation range of nearly incompressible nature, it can be seen that the behaviour of stress tensor $\boldsymbol{\sigma}$ is of monotonic type when the strains are large.

Normal compliance conditions and non-monotone friction law. The normal compliance conditions (7) are characterized by a compliance function p_ν which leads to the following behaviour: when there is separation ($u_\nu < 0$), the normal contact reaction of the body on the foundation vanishes ($\sigma_\nu = 0$) and when there is penetration ($0 \leq u_\nu$), the normal contact reaction is uniquely determined. For the numerical simulations, we consider a compliance function p_ν that takes the form

$$p_\nu(r) = c_\nu (r_+)^2 \tag{56}$$

where $(r_+) = \max\{0, r\}$ is a non-negative prescribed function which vanishes for negative arguments and c_ν is a positive constant related to the stiffness of the foundation. To summarize, the normal contact reaction $-\sigma_\nu$ is expressed as

$$-\sigma_\nu = c_\nu ((u_\nu)_+)^2. \tag{57}$$

The friction is modelled by a non-monotone law (8) in which the friction bound is characterized by a subdifferential of a non-convex superpotential j which depends on the tangential velocity $\dot{\boldsymbol{u}}_\tau$. In the following, for the numerical simulations, we consider a function $j : \mathbb{R}^d \to \mathbb{R}$ defined by

$$j(\boldsymbol{s}) = \int_0^{|\boldsymbol{s}|} \mu(t)\, dt, \tag{58}$$

where μ corresponds to the friction coefficient μ which depends on the variable $\boldsymbol{s}$. Then, for this particular choice the condition (8) leads to the following subdifferential inclusion:

$$-\boldsymbol{\sigma}_\tau \in \sigma_\nu \mu(|\dot{\boldsymbol{u}}_\tau|)\partial|\dot{\boldsymbol{u}}_\tau| \text{ on } \Gamma_C. \tag{59}$$

In this case, the friction bound $\sigma_\nu \mu(|\dot{\boldsymbol{u}}_\tau|)$ depends both on the depth of the penetration u_ν and on the tangential velocity $|\dot{\boldsymbol{u}}_\tau|$:

$$\begin{array}{ll} |\boldsymbol{\sigma}_\tau| \le \sigma_\nu \mu(|\dot{\boldsymbol{u}}_\tau|) & \text{if} \quad \dot{\boldsymbol{u}}_\tau = \boldsymbol{0}, \\ -\boldsymbol{\sigma}_\tau = \sigma_\nu \mu(|\dot{\boldsymbol{u}}_\tau|)\frac{\dot{\boldsymbol{u}}_\tau}{|\dot{\boldsymbol{u}}_\tau|} & \text{if} \quad \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0}, \end{array} \quad \text{on } \Gamma_C. \tag{60}$$

The strict inequality in (60) holds in the stick zone and the equality holds in the slip zone. This physical model of slip-dependent friction was introduced in [42] for the geophysical context of earthquake modelling and it was also studied in [9, 20, 43]. For the simulations, the following friction coefficient $\mu : \mathbb{R}^d \to \mathbb{R}$ is considered:

$$\mu(|\dot{\boldsymbol{u}}_\tau|) = (a - b) \cdot e^{-\alpha|\dot{\boldsymbol{u}}_\tau|} + b, \tag{61}$$

with $a, b, \alpha > 0$, $a \ge b$.

5.2. Numerical treatment

Since the numerical methods for solving hyperelastic problems with non-monotone contact and friction conditions have been presented in detail in [1, 35, 44] and in order to keep the paper of reasonable length, a succinct presentation of the main traits of the numerical treatment of Problem (P_{hk}) is given below.

In order to solve a nonlinear elastodynamic problem, we have to use time integration schemes with long-term time integration accuracy and stability. Indeed, when nonlinear dynamic problems are considered, the standard implicit schemes (θ-method, Newmark schemes, midpoint or Hilber-Hughes-Taylor methods) lose their unconditional stability. Therefore, we have to use implicit schemes with energy conservation properties (or energy consistent properties in the case of viscosity or friction phenomena) such as those described in [45–49]. Then, in this work we consider an energy-consistent scheme to solve the problem (P_{hk}) based on recent energy-controlling time integration methods for nonlinear elastodynamics that are described in detail in [35]. This numerical scheme is based on a specific form of the discretization of the normal compliance condition and the procedure of equivalent mass matrix [50] in order to conserve energy during the impact.

Furthermore, since non-convex potentials are considered to model both the hyperelastic law and the friction law, classical convex programming methods cannot be used to solve problem (P_{hk}). A numerical technique described in [1, 44] to solve this kind of non-convex problem is to use a "convexification" iterative procedure which leads to a sequence of convex programming problems. Then, the resulting non-smooth convex iterative problems are solved by classical numerical methods that can be found in [48, 51]. For instance, we use a combination of the penalized method and the augmented Lagrangian method to treat the frictional contact conditions. The resulting nonlinear system is solved by a semi-smooth Newton method. Details on the discretization step and computational contact mechanics, including algorithms similar to that used here, can be found in [25, 48, 50–52].

5.3. Impact of a ring on a foundation

The interest of this example is to provide validation for the hyperviscoelastic frictional contact model considered in Section 5.1 compared to results obtained by using the well-known Ogden hyperelastic model. This non-trivial example, introduced by Laursen in [48], concerns an academic frictional impact problem that is the impact with friction of a hyperviscoelastic ring against a foundation.

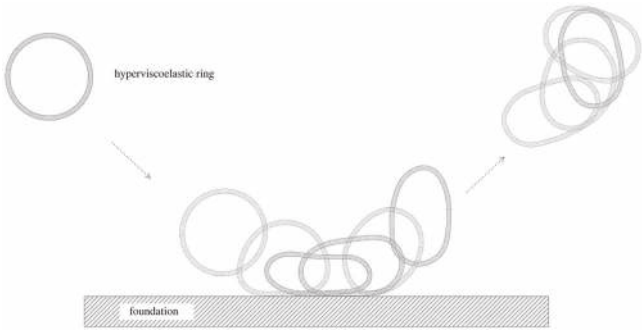

Figure 1. Sequence of the deformed hyperviscoelastic ring before, during and after impact.

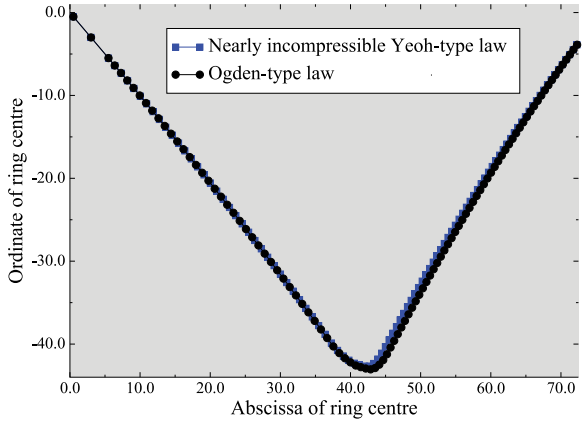

Figure 2. Trajectories of the ring centre for the two hyperelastic models during the time process ($t \in [0, T]$ with $T = 5$ s).

For each hyperelastic model, we use the viscous damping term given in (53). Details on the physical setting of the problem are given below:

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ 81 \leq (x_1 - 100)^2 + (x_2 - 100)^2 \leq 100\right\},$$
$$\Gamma_D = \varnothing, \qquad \Gamma_N = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ (x_1 - 100)^2 + (x_2 - 100)^2 = 81\right\},$$
$$\Gamma_C = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ (x_1 - 100)^2 + (x_2 - 100)^2 = 100\right\}.$$

The domain Ω represents the cross-section of a three-dimensional deformable body under the plane stress hypothesis. The ring is thrown with an initial velocity at a 45° angle toward a foundation as depicted in Figure 1. The foundation is given by $\left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 \leq 0\right\}$. For the discretization, we use 1664 elastic nodes and 128 Lagrange multiplier nodes. For the numerical experiments, the data are

$$\rho = 1000 \text{ kg/m}^3, \quad T = 10 \text{ s}, \quad k = \tfrac{1}{300},$$
$$\mathbf{u}_0 = (0, 0) \text{ m}, \quad \mathbf{u}_1 = (10, -10) \text{ m/s}, \quad \boldsymbol{f}_0 = (0, 0) \text{ N/m}^2, \quad \boldsymbol{f}_1 = (0, 0) \text{ N/m},$$
$$C_1 = 0.5 \text{ MPa}, \quad C_2 = 0.5 \times 10^{-2} \text{ MPa}, \quad C_3 = 0.5 \times 10^{-4} \text{ MPa}, \quad D = 100 \text{ MPa}, \quad \omega = 10^{-4},$$
$$c_v = 1000, \quad a = 1.5, \quad b = 0.5, \quad \alpha = 100.$$

The numerical results presented in Figure 2 and in Table 1 provide a qualitative and quantitative comparison between the Ogden-type model and the nearly incompressible Yeoh-type model. The compressible material response, considered for the Ogden constitutive law (see [53]) is characterized by the following energy density:

$$\Phi(\mathbf{F}) = c_1(I_1 - 3) + c_2(I_2 - 3) + d(I_3 - 1) - (c_1 + 2c_2 + d) \ln I_3,$$

with $c_1 = 0.5$ MPa, $c_2 = 0.5 \times 10^{-2}$ MPa and $d = 0.35$ MPa. In Figure 2 the trajectories of the ring centre are plotted for the whole time process ($t \in [0, T]$ with $T = 5$ s). We note that the trajectories are quite similar before, during and after the impact of the ring on the foundation.

Table 1. Relative errors between the numerical solutions obtained with the Ogden-type model and the nearly-incompressible Yeoh-type model.

Time instants (configurations)	t = 0.1 s (before impact)	t = 2.3 s (during impact)	t = 0.4 s (after impact)
$\frac{\|\boldsymbol{u}^{hk}_{\text{Yeoh}} - \boldsymbol{u}^{hk}_{\text{Ogden}}\|}{\|\boldsymbol{u}^{hk}_{\text{Ogden}}\|}$	1.1443%	1.0831%	1.3725%

In Table 1, the relative error estimates between the numerical solutions computed with the Ogden-type model and those computed with the nearly incompressible Yeoh-type model are calculated at three time instants ($t = 1$ s, $t = 2.3$ s and $t = 4$ s) corresponding to three configurations of deformation of the ring (before, during and after impact). Note that these results provide good numerical evidence for the small difference of the between the numerical solutions obtained with the two hyperelastic models. Indeed, the relative error estimates are of the order of 1%.

5.4. Buckling of a stem and contact with upper and lower obstacles

The interest of this second example is to highlight the non-uniqueness of the solution of the hyperviscoleastic problem with contact and friction conditions, due to the sensitivity of the solution with respect to the geometric setting. To this end, we consider numerical simulations based on the buckling of a slender hyperelastic stem with upper and lower contact obstacles which put in evidence the fact that for a very small perturbation of the mesh we may possibly obtain two different solutions for the same level of loading.

The physical setting used for this numerical example is depicted in Figure 3. The deformable body is a rectangle, $\Omega = (0, 20) \times (0, 1) \subset \mathbb{R}^2$, and its boundary Γ is split as follows:

$$\begin{aligned}
&\Gamma_D = (\{0\} \times [0, 1]) \cup (\{20\} \times [0, 1]),\\
&\Gamma_N = \varnothing,\\
&\Gamma_C = ([0, 20] \times \{0\}) \cup ([0, 20] \times \{1\}).
\end{aligned}$$

As for the first numerical example, the domain Ω represents the cross-section of a three-dimensional deformable beam subjected to the action of displacement compressions on the right part of Γ_D in such a way that a plane stress hypothesis is assumed. On the part $\{0\} \times [0, 1]$ of Γ_D the body is clamped and, therefore, the displacement field vanishes there. Imposed displacement compressions act on the part $\{20\} \times [0, 1]$ of the boundary Γ_D and the part Γ_N is traction-free. No vertical body forces are assumed to act on the hyperviscoelastic beam. The beam may come into frictional contact with two obstacles on the parts $[0, 20] \times \{0\}$ and $[0, 20] \times \{1\}$ of the boundary Γ_C. The lower and upper obstacles are given respectively by the sets $\left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 \leq -2\right\}$ and $\left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 \geq 3\right\}$. For the discretization, we use 3096 elastic nodes and 256 Lagrange multiplier nodes. For the numerical experiments, the data are

$$\begin{aligned}
&\rho = 1000 \text{ kg/m}^3, \quad T = 0.8 \text{ s}, \quad k = \tfrac{1}{40},\\
&\boldsymbol{f}_0 = (0, 0) \text{ N/m}^2, \quad \mathbf{u}_D = (-3, 0) \text{ m}, \quad \text{on } \{20\} \times [0, 1],\\
&C_1 = 0.2 \text{ MPa}, \quad C_2 = 0.5 \times 10^{-3} \text{ MPa}, \quad C_3 = 0.5 \times 10^{-4} \text{ MPa}, \quad D = 100 \text{ MPa}, \quad \omega = 10^{-4},\\
&c_v = 1000, \quad a = 0.2, \quad b = 0.1, \quad \alpha = 100.
\end{aligned}$$

In order to show the sensitivity of numerical solutions with respect to the geometric setup and the possible multiplicity of numerical solutions, we consider a procedure of random perturbations of the mesh. More precisely, a very small relative perturbation (1.10^{-4}) is introduced randomly on two points of the mesh: the points $(20, 0)$ and $(20, 1)$ are considered arbitrary. Then we realized several series of numerical tests with these meshes perturbed randomly. Consequently, we obtained two types of buckling configuration of the deformed stem, represented in Figures 4 and 5.

These simulation results depicted in Figures 4 and 5 put in evidence the fact that we obtain two different solutions for the same level of loading. This phenomenon highlights the non-uniqueness of the solution of the

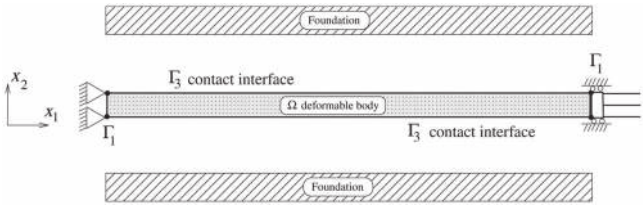

Figure 3. Two buckling configurations of the deformed hyperviscoelastic stem.

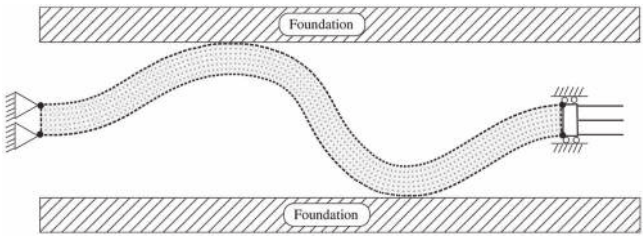

Figure 4. First buckling configuration of the deformed hyperviscoelastic stem.

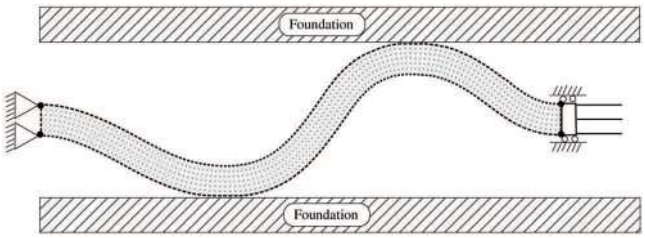

Figure 5. Second buckling configuration of the deformed hyperviscoelastic stem.

hyperviscoelastic problem (P_{hk}), due to the sensitivity with respect to the geometric setting of the numerical example considered in this section.

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Appendix 1

We recall some notions from the abstract theory of multi-valued operators in reflexive Banach spaces which are used in the article.

Definition A.1. A multi-valued operator $B : X \to 2^{X'}$ where X is a reflexive Banach space is called multi-valued pseudomonotone if

(i) $B(u)$ is closed, bounded, convex, and non-empty for all $u \in X$;
(ii) B is upper semicontinuous from each finite-dimensional subspace of X to X' endowed with weak topology;
(iii) if $\{u_n\}$ is a sequence in X and $\{u_n^*\}$ is a sequence in X' such that $u_n \to u$ weakly in X, $u_n^* \in B(u_n)$ and $\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \leq 0$, then for each $v \in X$ there exists $u^*(v) \in B(u)$ such that $\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle \geq \langle u^*(v), u - v\rangle$.

Definition A.2. A multi-valued operator $B : X \to 2^{X'}$ where X is a reflexive Banach space is called generalized pseudomonotone if for any sequences $\{u_n\}$ in X and $\{u_n^*\}$ in X' such that $u_n \to u$ weakly in X, $u_n^* \to u^*$ weakly in X' and $u_n^* \in B(u_n)$ if we have $\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \leq 0$, then $u^* \in B(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

Lemma A.3. See Proposition 2.2 in [21]. *If X is a reflexive Banach space and $B : X \to 2^{X'}$ is a generalized pseudomonotone and bounded operator such that for each $u \in X$ the set $B(u)$ is non-empty, convex, and closed, then B is multi-valued pseudomonotone.*

Lemma A.4. See Proposition 2.4 in [21]. *A sum of two multi-valued pseudomonotone operators on a reflexive Banach space X is multi-valued pseudomonotone.*

We can associate with a single-valued operator $B : V \to V'$ a multi-valued operator such that its value at each point is a singleton.

Lemma A.5. See Proposition 32.7 in [54] and Proposition 2.3 in [21]. *If $B : V \to V'$ is monotone and continuous then it is multi-valued pseudomonotone.*

We recall a theorem on the surjectivity of pseudomonotone operators.

Theorem A.6. See Theorem 2.2 in [55]. *Let $B : X \to 2^{X'}$ be a multi-valued pseudomonotone bounded operator and $L \in X'$. Assume that there exists $u_0 \in X$ and $R \geq \|u_0\|_X$ such that*

$$\langle \eta - L, u - u_0\rangle > 0$$

for all $u \in X$ with $\|u\|_X = R$ and all $\eta \in B(u)$. Then the inclusion

$$L \in B(u)$$

has a solution.

Definition A.7. Let $I = [a, b]$ be the finite time interval and let $u : I \to X$ be the function from I to a Banach space X. We define the q-variation seminorm as

$$\|u\|^q_{BV^q(I;X)} = \sup \left\{ \sum_{i=1}^{k} \|u(t_k) - u(t_{k-1})\|^q_X \ : \ k \in \mathbb{N}, a = t_0 < t_1 < \ldots < t_k = b \right\}.$$

By $BV^q(I; X)$ we denote the set of all functions $u : I \to X$ such that their q-variation seminorm is finite.

For Banach spaces X, Y such that $X \subset Y$ with a continuous embedding, define

$$M^{p,q}(I; X, Y) = L^p(I; X) \cap BV^q(I; Y).$$

We have the following theorem; see [56, Theorem 1].

Theorem A.8. *Let $1 \leq p, q < \infty$. Moreover, let $X_1 \subset X_2 \subset X_3$ be Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact, and the embedding $X_2 \subset X_3$ is continuous. If a set $\mathcal{S} \subset M^{p,q}(I; X_1, X_3)$ is bounded then it is relatively compact in $L^p(I; X_2)$.*