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A non-intrusive reduced basis approach for parametrized heat transfer problems

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Abstract

Computation Fluid Dynamics (CFD) simulation has become a routine design tool for i) predicting accurately the thermal performances of electronics set ups and devices such as cooling system and ii) optimizing configurations. Although CFD simulations using discretization methods such as finite volume or finite element can be performed at different scales, from component/board levels to larger system, these classical discretization techniques can prove to be too costly and time consuming, especially in the case of optimization purposes where similar systems, with different design parameters have to be solved sequentially. The design parameters can be of geometric nature or related to the boundary conditions. This motivates our interest on model reduction and particularly on reduced basis methods. As is well documented in the literature, the offline/online implementation of the standard RB method (a Galerkin approach within the reduced basis space) requires to modify the original CFD calculation code, which for a commercial one may be problematic even impossible. For this reason, we have proposed in a previous paper, with an application to a simple scalar convection diffusion problem, an alternative non-intrusive reduced basis approach (NIRB) based on a two-grid finite element discretization. Here also the process is two stages: *offline*, the construction of the reduced basis is performed on a fine mesh; *online* a new configuration is simulated using a coarse mesh. While such a coarse solution, can be computed quickly enough to be used in a rapid decision process, it is generally not accurate enough for practical use. In order to retrieve accuracy, we first project every such coarse solution into the reduced space, and then further improve them via a rectification technique. The purpose of this paper is to generalize the approach to a CFD configuration.

Keywords : Non-intrusive method; Reduced basis method; Parametric studies; Heat transfer; CFD.

1. Introduction

During the past fifty years electronic devices and systems kept becoming smaller and smaller, this growing need for miniaturization led to an increasing high heat production. To avoid any possible failure or malfunction of electronics devices and ensure their reliability, it is essential to maintain the temperature of the electronic components below an acceptable upper limit. Cooling of electronic systems is consequently essential in controlling the component temperature and avoiding any hot spot. Designing cooling systems for miniaturized electronics devices presents difficult challenges to mechanical engineers and analysts. Meanwhile, computational modeling is gaining popularity, particularly Computation Fluid Dynamics (CFD) modeling which has become a routine design tool for predicting accurately thermal

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performance of electronics cooling system. Although CFD modeling can be used at different scales, from component/board levels to larger system, classical discretization techniques such as finite volume or finite element methods can prove to be limited by memory space and long calculation times, which can be problematic. Inexpensive and accurate computational tools to predict the fluid flow and heat transfer can be very useful, specially when thermal analysis is done at the end of the design process where time constraints are greatest, hence our focus on model reduction and particularly to reduced basis methods (see [9, 14, 17, 19]).

Reduced basis method exploits the parametric structure of the governing PDEs to construct rapidly, convergent and computationally efficient approximations. Previous work on the reduced basis method in numerical fluid dynamics has been carried out by [12, 16, 20] and more particularly for the Navier-Stokes equations [7, 18, 23, 27, 28] which requires treatment of non-linearities and non-affine parametric dependence. More recent works with turbulent flows can be found in [2, 26]. Let σ be a set of parameters associated to our physical system, these methods rely on the fact that when the parameters vary, the manifold of solutions is often of small (Kolmogorov) dimension. In this instance, there exists a set of N particular values of σ taken in \mathcal{D} (the parameter space) from which one can build a basis. This basis, called reduced basis, is made of the solutions $u(\sigma_1), \dots, u(\sigma_N)$ and can approach any solution $u(\sigma)$, $\sigma \in \mathcal{D}$. Thus, when the σ_i are well chosen⁵, the size of the reduced basis is much smaller compared to the number of degrees of freedom of the problem discretized by a classical method (finite element, finite volume, or other). The standard reduced basis method is a Galerkin approach within the reduced basis approximation space, thus the reduced basis approximation of the “truth” solution is obtained by the resolution of a small dimensional linear system. One of the keys of this technique is the decomposition of the computational work into *offline/online* stages. However, the decomposition of the matrices into *offline/online* pieces requires modifying the calculation code, leading to an intrusive procedure. Examples of the standard reduced basis method applied to heat transfer problems can be found in [8, 21, 24, 25]. In some situation — with a commercial CFD code for example — it is not possible to perform all the *offline* computations required to have a inexpensive and fast *online* stage. For this reason, we proposed to use an less intrusive reduced basis method, introduced in [3, 4], where coarse triangulations are used to compute coarse approximation during the *online* stage. Recently, other non-intrusive Reduced Order Methods (ROM) for fluid dynamics have been proposed [5, 29], those ROM are based on proper orthogonal decomposition and radial basis function (RBF) to compute the coefficients of the reduced model. In our method, in order to retrieve the same accuracy as the high fidelity model, we first project every coarse solution into the reduced space and then further improve them via a rectification technique.

The aim of this paper is to provide tests to validate and generalize our method for heat transfer problems. In Section 2, we provide a brief introduction to reduced basis methods and the methodology of the non-intrusive reduced basis method. In Section 3, we give a brief description of simple models of cooling devices; we formulate the physical system, the governing equations and boundary conditions. In Section 4, we discuss our numerical experimentations and present the results and conclusions.

2. Methodology

Let us consider the nonlinear parametrized PDEs describing our physical system, over a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 . In these governing equations, θ represent the temperature, u_i the components of the velocity vector field \mathbf{u} in the x_i -directions, p the pressure and σ is a set of n_p

⁵A classical and efficient approach to choose the σ_i is the greedy method [1, 11].

parameters related to physical properties or boundary conditions. We denote by $\mathcal{D} \subset \mathbb{R}^{n_p}$ the set of parameters.

We now introduce the variational formulation of our parametrized PDEs :

$$\left\{ \begin{array}{l} \text{for a given } \sigma \in \mathcal{D}, \text{ find } (\theta, \mathbf{u}, p) \equiv (\theta(\sigma), \mathbf{u}(\sigma), p(\sigma)) \in X \times Y \times Q \text{ such that} \\ \mathcal{F}(\theta, \mathbf{u}, p; T, \mathbf{v}, q; \sigma) = (0, \mathbf{0}, 0)^t, \quad \forall (T, \mathbf{v}, q) \in X \times Y \times Q, \end{array} \right. \quad (1)$$

where \mathcal{F} is a functional with the appropriate properties, X, Y and Q are appropriate functional spaces. Let $\{\mathcal{T}_h\}_h$ be a family of regular triangulation of Ω , we denote by X_h, Y_h and Q_h finite element approximation subspaces of respectively X, Y and Q over \mathcal{T}_h . The discrete velocity space Y_h and the discrete pressure space Q_h are chosen in order to satisfy the inf-sup condition [6, 13, 22].

The finite element discretization of (1) is as follows :

$$\left\{ \begin{array}{l} \text{for a given } \sigma \in \mathcal{D}, \text{ find } (\theta_h, \mathbf{u}_h, p_h) \equiv (\theta_h(\sigma), \mathbf{u}_h(\sigma), p_h(\sigma)) \in X_h \times Y_h \times Q_h \text{ such that} \\ \mathcal{F}(\theta_h, \mathbf{u}_h, p_h; T, \mathbf{v}, q; \sigma) = (0, \mathbf{0}, 0)^t, \quad \forall (T, \mathbf{v}, q) \in X_h \times Y_h \times Q_h, \end{array} \right. \quad (2)$$

For significantly fine meshes \mathcal{T}_h and adequately chosen discretization spaces X_h, Y_h and Q_h , the conforming finite element solution $(\theta_h, \mathbf{u}_h, p_h)$, solution to (2), is accurate enough to be considered as a good approximation of the exact solution, named “truth solutions”. However, because of the high dimension of the associated discretization spaces, solving the finite element problem (2) with different values of σ can prove to be too costly and time consuming.

The idea of reduced basis methods is to provide an economical and accurate approximation to the “truth” approximation $(\theta_h, \mathbf{u}_h, p_h)$ by using approximation spaces made up of few suitable samples of solutions to the parametrized PDEs. This relies on the fact that when the parameters vary, the set of solutions is often of small Kolmogorov width, thus implying that the manifold of all solutions can be approximated within a (small) finite space of well-chosen solutions to the parametrized problem (2). In that case, there exists a set of parameters $S_N = (\sigma_1, \sigma_2, \dots, \sigma_N) \in \mathcal{D}^N$ such that for any $\sigma \in \mathcal{D}$, the truth solution can be approximated by a linear combination of the particular solutions associated to $\sigma_i \in S_N$. To distinguish each physical component (temperature, velocity and pressure), we add superscripts θ, \mathbf{u} or p in our notation, for example we denote by $(\sigma_1^\theta, \sigma_2^\theta, \dots, \sigma_{N^\theta}^\theta)$ the set of parameters — with size N^θ — used to generate the reduced basis approximation space associated to the temperature component of the solution. For each physical component, we introduced the associated reduced basis approximation spaces as

$$\begin{aligned} X_h^N &= \text{span}\{\theta_h(\sigma_i^\theta), 1 \leq i \leq N^\theta\} = \text{span}\{\phi_i^\theta, 1 \leq i \leq N^\theta\}, \\ Y_h^N &= \text{span}\{\mathbf{u}_h(\sigma_i^\mathbf{u}), 1 \leq i \leq N^\mathbf{u}\} = \text{span}\{\phi_i^\mathbf{u}, 1 \leq i \leq N^\mathbf{u}\}, \\ Q_h^N &= \text{span}\{p_h(\sigma_i^p), 1 \leq i \leq N^p\} = \text{span}\{\phi_i^p, 1 \leq i \leq N^p\}, \end{aligned}$$

where the $(\phi_i^\theta)_i \in X_h, (\phi_i^\mathbf{u})_i \in Y_h, (\phi_i^p)_i \in Q_h$ are H^1 -orthonormal basis sets (obtained respectively from $(\theta_h(\sigma_i^\theta))_i, (\mathbf{u}_h(\sigma_i^\mathbf{u}))_i$ and $(p_h(\sigma_i^p))_i$ by a orthogonalization process), called reduced basis functions. The standard reduced basis method consists in a Galerkin approach within the low dimensional spaces X_h^N, Y_h^N and Q_h^N .

$$\left\{ \begin{array}{l} \text{for a given } \sigma \in \mathcal{D}, \text{ find } (\theta_h^N, \mathbf{u}_h^N, p_h^N) \in X_h^N \times Y_h^N \times Q_h^N \text{ such that} \\ \mathcal{F}(\theta_h^N, \mathbf{u}_h^N, p_h^N; T, \mathbf{v}, q; \sigma) = (0, \mathbf{0}, 0)^t, \quad \forall (T, \mathbf{v}, q) \in X_h^N \times Y_h^N \times Q_h^N. \end{array} \right. \quad (3)$$

Note that, in order to ensure stable, good approximation of the pressure, supremizer-enrichment of the velocity reduced basis space may be performed, see for example [6, 13, 22]⁶.

One of the key ingredients of the method is the decomposition of the computational work into *offline* and *online* stages. During the *offline* stage the reduced basis functions are computed — providing the above defined spaces X_h^N, Y_h^N, Q_h^N — as well as all parameter-independent quantities. This is an expensive stage that is done only once, whereas parameter-dependent quantities are computed during the *online* stage together with the resolution to (3). Considering that the dimension of the reduced basis space is quite smaller compared to the finite element one, solving the reduced problem (3) is much less expensive than the “truth” finite element problem (2).

The construction of the associated discrete system to be solved is thus classically the corner stone of the global process. Indeed, since the construction of the reduced discrete system associated to the variational form $\mathcal{F}(\theta_h^N, \mathbf{u}_h^N, p_h^N; T, v, q; \sigma)$ has to be done for each new value of σ , to perform efficiently the online stage, one has to be able to isolate the parametric contributions. Thus, all σ -independent quantities, high dimensional matrices and vectors used in the construction of the reduced discrete system are computed only once and saved during the offline stage. This part of the offline stage requires entering and modifying the simulation code used to compute the truth finite element approximations, leading to an intrusive method, which can be problematic. When the simulation code is locked and used as a black box — which is often the case in the industrial framework — the parametric decomposition is not possible, which prevent us from building each new reduced discrete system quickly for a new value of σ . This take away the benefit of the reduced basis method, thus to overcome it, we propose to use an alternative method, less intrusive, where coarse triangulations are used to compute finite element solutions during the *online* stage. These “coarse” approximations are projected into the reduced basis approximation spaces X_h^N, Y_h^N, Q_h^N , and then improve via a rectification technique introduced in [3, 4]. All these improvements after the coarse approximation is computed can be done in a different, more versatile code than the one used to do the simulation.

Let $\{\mathcal{T}_H\}_H$ be a family of “coarse” regular triangulation of Ω , such that $H \gg h$, we denote by X_H, Y_H and Q_H the coarse finite element spaces associated to this mesh, and $(\theta_H, \mathbf{u}_H, p_H)$ the “coarse” finite element approximation. Since the computation of the “coarse” finite element approximation, for $H \gg h$ is less expensive than the “truth” approximation, using the simulation code — during the online stage — with the mesh size H (chosen adequately) to construct a reduced solution is still cheap enough.

In order to understand our non-intrusive reduced basis method, let us indicate that the idea of standard reduced basis methods (3) is to compute an inexpensive and accurate approximation of the projection of the solution $(\theta_h(\sigma), \mathbf{u}_h(\sigma), p_h(\sigma))$ on $X_h^N \times Y_h^N \times Q_h^N$, by evaluating the coefficients appearing in the decomposition of $\theta_h^N(\sigma)$, $\mathbf{u}_h^N(\sigma)$ and $p_h^N(\sigma)$ in the basis $(\phi_i^\theta)_i$, $(\phi_i^{\mathbf{u}})_i$ and $(\phi_i^p)_i$.

The best linear combination – measured in H^1 -norm – of the reduced basis functions is provided by the H^1 -orthogonal projection from X_h into the space X_h^N for the temperature and the H^1 -orthogonal projection operator from Y_h into the space Y_h^N , for the velocity.

In our method, an alternative linear combination of the reduced basis functions has been chosen, using the optimal coefficients $\{\beta_j^{\theta, h}(\sigma)\}_{1 \leq j \leq N_\theta}$ and $\{\beta_j^{\mathbf{u}, h}(\sigma)\}_{1 \leq j \leq N_{\mathbf{u}}}$ involved in the L^2 -orthogonal projection

⁶However, in our context, we are not much interested here in the precise value of the pressure that is (only) a Lagrange multiplier to the divergence free condition), thus using the fact that the reduced velocity space is composed of divergence free functions the pressure does not even appear in the reduced set of equations (3), if no supremizer for the pressure space is added to X_h^N .

of $\theta_h(\sigma)$ on X_h^N and $\mathbf{u}_h(\sigma)$ on Y_h^N . As shown in [3, 4], these coefficients, defined as follows

$$\beta_i^{\theta,h}(\sigma) = \left(\theta_h(\sigma), \phi_i^\theta \right)_{L^2}, \quad \text{and} \quad \beta_j^{\mathbf{u},h}(\sigma) = \left(\mathbf{u}_h(\sigma), \phi_j^\mathbf{u} \right)_{L^2} \quad \text{with } 1 \leq i \leq N^\theta \text{ and } 1 \leq j \leq N^\mathbf{u}, \quad (4)$$

are still good enough but require the knowledge of the fine solution intervening respectively in the decomposition of $\theta_h(\sigma)$ and $\mathbf{u}_h(\sigma)$. The non-intrusive reduced basis method aims at computing a cheap, yet accurate enough approximation of these coefficients by using

$$\beta_i^{\theta,H}(\sigma) = \left(\theta_H(\sigma), \phi_i^\theta \right)_{L^2}, \quad \text{and} \quad \beta_j^{\mathbf{u},H}(\sigma) = \left(\mathbf{u}_H(\sigma), \phi_j^\mathbf{u} \right)_{L^2} \quad \text{with } 1 \leq i \leq N^\theta \text{ and } 1 \leq j \leq N^\mathbf{u}, \quad (5)$$

as substitutes. While ‘‘coarse’’ finite element approximations $(\theta_H(\sigma), \mathbf{u}_H(\sigma))$ can be computed quickly enough to be used in model reduction techniques, they may not be accurate enough for practical use. We have proven in [3, 4] on a simpler example that this first NIRB approximation provides some improvement in the accuracy with respect to the coarse solution. To further improve the accuracy of this technique we have also proposed to perform a simple ‘‘rectification’’ that allows to ensure that for the set of parameters $S_{N^\theta} \equiv \{\sigma_i^\theta\}_{i, 1 \leq i \leq N^\theta}$, (and resp. $S_{N^\mathbf{u}} \equiv \{\sigma_i^\mathbf{u}\}_{i, 1 \leq i \leq N^\mathbf{u}}$) used in the construction of the reduced basis, the method returns exactly $\theta_h(\sigma_i^\theta)$ (and resp. $\mathbf{u}_h(\sigma_i^\mathbf{u})$). Indeed, contrarily to $\theta_h(\sigma)$ and $\mathbf{u}_h(\sigma)$ that we don’t want to compute for a large number of values of σ , the set $\{\theta_h(\sigma_i^\theta)\}_{i, 1 \leq i \leq N^\theta}$ and $\{\mathbf{u}_h(\sigma_i^\mathbf{u})\}_{i, 1 \leq i \leq N^\mathbf{u}}$ have actually already been computed to build the reduced basis spaces. It is thus desirable that the non-intrusive reduced basis approach provides these truth solutions. To do so, we identify the matrices $R^{\theta,N} \in \mathbb{R}^{N^\theta \times N^\theta}$ and $R^{\mathbf{u},N} \in \mathbb{R}^{N^\mathbf{u} \times N^\mathbf{u}}$ such that :

$$\sum_{j=1}^{N^\theta} R_{ij}^{\theta,N} \beta_j^{\theta,H}(\sigma_k) = \beta_i^{\theta,h}(\sigma_k), \quad \forall 1 \leq i, k \leq N^\theta,$$

$$\sum_{j=1}^{N^\mathbf{u}} R_{ij}^{\mathbf{u},N} \beta_j^{\mathbf{u},H}(\sigma_k) = \beta_i^{\mathbf{u},h}(\sigma_k), \quad \forall 1 \leq i, k \leq N^\mathbf{u}.$$

Thus, the matrix $R^{\theta,N}$ is equal to

$$R^{\theta,N} = \left(\begin{array}{c|ccc} \beta_1^{\theta,h}(\sigma_1^\theta) & \cdots & \beta_1^{\theta,h}(\sigma_{N^\theta}^\theta) \\ \vdots & \vdots & \vdots \\ \beta_{N^\theta}^{\theta,h}(\sigma_1^\theta) & \cdots & \beta_{N^\theta}^{\theta,h}(\sigma_{N^\theta}^\theta) \end{array} \right) \times \left(\begin{array}{c|ccc} \beta_1^{\theta,H}(\sigma_1^\theta) & \cdots & \beta_1^{\theta,H}(\sigma_{N^\theta}^\theta) \\ \vdots & \vdots & \vdots \\ \beta_{N^\theta}^{\theta,H}(\sigma_1^\theta) & \cdots & \beta_{N^\theta}^{\theta,H}(\sigma_{N^\theta}^\theta) \end{array} \right)^{-1}$$

and the matrix $R^{\mathbf{u},N}$ is equal to

$$R^{\mathbf{u},N} = \left(\begin{array}{c|ccc} \beta_1^{\mathbf{u},h}(\sigma_1^\mathbf{u}) & \cdots & \beta_1^{\mathbf{u},h}(\sigma_{N^\mathbf{u}}^\mathbf{u}) \\ \vdots & \vdots & \vdots \\ \beta_{N^\mathbf{u}}^{\mathbf{u},h}(\sigma_1^\mathbf{u}) & \cdots & \beta_{N^\mathbf{u}}^{\mathbf{u},h}(\sigma_{N^\mathbf{u}}^\mathbf{u}) \end{array} \right) \times \left(\begin{array}{c|ccc} \beta_1^{\mathbf{u},H}(\sigma_1^\mathbf{u}) & \cdots & \beta_1^{\mathbf{u},H}(\sigma_{N^\mathbf{u}}^\mathbf{u}) \\ \vdots & \vdots & \vdots \\ \beta_{N^\mathbf{u}}^{\mathbf{u},H}(\sigma_1^\mathbf{u}) & \cdots & \beta_{N^\mathbf{u}}^{\mathbf{u},H}(\sigma_{N^\mathbf{u}}^\mathbf{u}) \end{array} \right)^{-1}.$$

So, for each new value of σ , we will replace the $\beta_i^{\theta,h}(\sigma)$ and $\beta_i^{\mathbf{u},h}(\sigma)$ coefficients by respectively

$$\beta_i^{\theta_H,h}(\sigma) = \sum_{j=1}^{N^\theta} R_{ij}^{\theta,N} \beta_j^{\theta,H}(\sigma)$$

and

$$\beta_i^{\mathbf{u}_{H,h}}(\sigma) = \sum_{j=1}^{N_{\mathbf{u}}} R_{ij}^{\mathbf{u},N} \beta_j^{\mathbf{u},H}(\sigma),$$

to eventually build an improved reduced basis approximation

$$\left\{ \begin{array}{l} \theta_{H,h}^N = \sum_{i,j=1}^{N_{\theta}} R_{ij}^{\theta,N} \beta_j^{\theta,H}(\sigma) \phi_i^{\theta,N} \\ \mathbf{u}_{H,h}^N = \sum_{i,j=1}^{N_{\mathbf{u}}} R_{ij}^{\mathbf{u},N} \beta_j^{\mathbf{u},H}(\sigma) \phi_i^{\mathbf{u},N}. \end{array} \right. \quad (6)$$

3. Application to heat transfer problems

One of the most prominent industrial applications of heat transfer problems is the cooling and thermal control of electronic devices and circuitry. Cooling systems are generally divided into two categories: passive (rely on the thermo-dynamics of conduction and convection to complete the heat transfer process) and active (require an external powered device as fans or pumps). In these paper we choose to study the heat transfer due to natural convection in a heated cavity and heat transfer inside a simple cooling system for electronics components.

3.1. The governing equations

The fluid is supposed incompressible and driven by external forces – the gravity acting on the mass. The governing equations are the Navier-Stokes equations, in the Cartesian coordinates they are given by :

- Continuity equation :

$$\nabla \cdot \mathbf{u} = 0 \quad (7)$$

- Momentum equation :

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p = \rho \mathbf{g} \quad (8)$$

- Thermal energy equation :

$$\rho C_p \left(\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta \right) - k \Delta \theta = 0 \quad (9)$$

| | | | |
|----------|--|-------|--|
| ρ | density | C_p | heat capacity |
| k | thermal conductivity | μ | dynamic viscosity |
| κ | thermal diffusivity ($= \frac{k}{\rho C_p}$) | ν | kinematic viscosity ($= \frac{\mu}{\rho}$) |

Table 1: Nomenclature of fluid's properties

Considering a reference state in which the density ρ_{ref} and the pressure p_{ref} are so that $\nabla p_{\text{ref}} = \rho_{\text{ref}} \mathbf{g}$ and writing $p = p_{\text{ref}} + \tilde{p}$ and $\rho = \rho_{\text{ref}} + \tilde{\rho}$, equation (8) becomes

$$\left(1 + \frac{\tilde{\rho}}{\rho_{\text{ref}}} \right) \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \frac{1}{\rho_{\text{ref}}} \nabla \tilde{p} - \nu_{\text{ref}} \Delta \mathbf{u} = \frac{\tilde{\rho}}{\rho_{\text{ref}}} \mathbf{g}, \quad \text{with } \nu_{\text{ref}} = \frac{\mu}{\rho_{\text{ref}}},$$

and equation (9) becomes

$$\left(1 + \frac{\tilde{\rho}}{\rho_{\text{ref}}}\right) \left(\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta\right) - \kappa_{\text{ref}} \Delta \theta = 0, \quad \text{with } \kappa_{\text{ref}} = \frac{k}{\rho_{\text{ref}} C_p},$$

We place ourselves within the *Boussinesq approximation*⁷ for steady state, with

$$\tilde{\rho} = \rho_{\text{ref}} \beta (\theta - \theta_{\text{ref}}),$$

where β is the volumetric coefficient of the thermal expansion and θ_{ref} some reference temperature for which $\rho_{\text{ref}} = \rho$, here we took $\theta_{\text{ref}} = 295$ Kelvin's degrees. By applying this approximation, the momentum equation (8) becomes :

$$\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho_{\text{ref}}} \nabla \tilde{p} - \nu_{\text{ref}} \Delta \mathbf{u} = \beta (\theta - \theta_{\text{ref}}) \mathbf{g}, \quad (10)$$

and the thermal energy equation (9) becomes

$$-\kappa_{\text{ref}} \Delta \theta + \mathbf{u} \cdot \nabla \theta = 0. \quad (11)$$

3.2. Application 1 : Natural convection in a 2D cavity

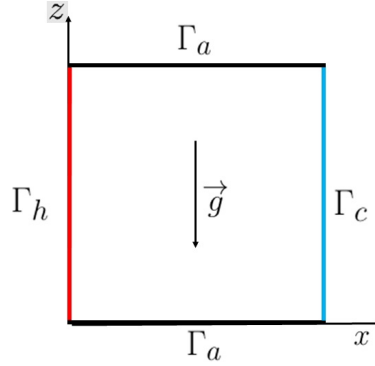


Figure 1: Geometry for a differentially heated square cavity

In this first example we investigate the case of a flow enclosed in a two-dimensional (x, z) differentially heated square cavity (see Figure 1). The left wall is maintained at the hot temperature T_h , the right one at the cold temperature $T_c = T_{\text{ref}}$ whereas the walls at top and bottom are adiabatic.

The governing equations (7),(11) and (10) are made dimensionless by introducing $\Delta \theta = T_h - T_c$ (the temperature difference), L (the distance between region of high temperature and region of low temperature), the Grashof number $Gr = \frac{g\beta\Delta\theta L^2}{\nu^2}$ (the ratio between the buoyant forces and viscous forces acting on the fluid) and the Prandtl number $Pr = \frac{\nu}{\kappa}$ (the ratio between the kinematic viscosity and the thermal diffusivity). The following dimensionless quantities are introduced:

$$X = \frac{x}{L}, \quad Z = \frac{z}{L}, \quad \mathbf{U} = \frac{\mathbf{u}}{\sqrt{g\beta\Delta\theta L}}, \quad P = \frac{\tilde{p}}{g\beta\Delta\theta L\rho_{\text{ref}}}, \quad \Theta = \frac{\theta - T_c}{T_h - T_c}$$

⁷Boussinesq approximation states that the thermo-physical properties of the fluid (ρ , μ , C_p and k) are assumed to be constant and independent of temperature except in the gravity term. Hence the variation of the density $\frac{\tilde{\rho}}{\rho_{\text{ref}}}$ due to changes in the temperature can be neglected except in the gravity term.

The dimensionless velocity $\mathbf{U} = (u_x, u_z)$, the dimensionless temperature θ , the pressure p of the fluid satisfy the following equations :

$$\left\{ \begin{array}{ll} -\frac{1}{\sqrt{Gr}} \Delta \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P = (0, 1)^T \Theta & \text{in } \Omega, \\ \nabla \cdot \mathbf{U} = 0 & \text{in } \Omega, \\ \mathbf{U} = 0 & \text{on } \partial\Omega, \\ \\ -\frac{1}{Pr\sqrt{Gr}} \Delta \Theta + \mathbf{U} \cdot \nabla \Theta = 0 & \text{in } \Omega, \\ \Theta = 0 & \text{on } \Gamma_c, \\ \Theta = 1 & \text{on } \Gamma_h, \\ \frac{\partial \Theta}{\partial n} = 0 & \text{on } \Gamma_a. \end{array} \right. \quad (12)$$

The varying parameters of this example are the Grashof number $Gr \in [10^3, 10^6]$ and the Prandtl number $Pr \in [0.5, 1]$.

3.3. Application 2 : Cooling system of electronic devices

In this second example we are interested by the variation of the temperature in a simplified system that is inspired by a suggested cooling system for electronic device (a small box with two electronic components).

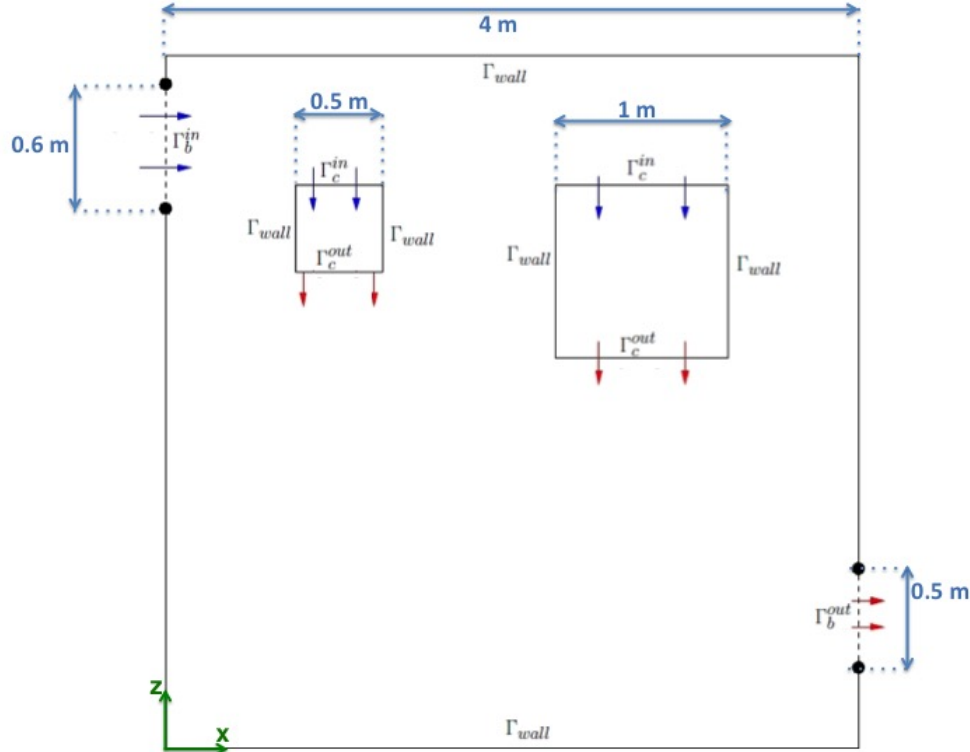


Figure 2: Geometry of the simplified cooling system for electronic devices

The model is a two dimensional square where the cooling air enters the box at the Γ_b^{in} boundary and flows around the two electronic components. Those two components produce heat energy at the boundary Γ_c^{out} , and due to a convection phenomenon the heat energy is carried out by the fluid flow through the box to finally exits from the boundary Γ_b^{out} (see Figure 2).

On the inflow section Γ_b^{in} a Poiseuille's velocity profile $\mathbf{v}_{in} = v_b \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$ is imposed, where v_b is the velocity magnitude and $f(x)$ a parabolic function. On the outflow section Γ_b^{out} , homogenous Neumann condition is imposed. For the flow passing through the electronic components, we have chosen to model the interaction by non homogenous Dirichlet condition v_c imposed to the second component of the velocity on the inflow Γ_c^{in} and similarly on the outflow sections Γ_c^{out} .

We impose a temperature of θ_b on the inflow section Γ_b^{in} and a temperature of θ_c on the outflow section Γ_c^{out} , whereas the walls Γ_{wall} , the outflow section Γ_b^{out} and the inflow section Γ_c^{in} are all adiabatic.

The velocity $\mathbf{u} = (u_x, u_z)$, the temperature θ , the pressure p of the fluid satisfy the following equations :

$$\left\{ \begin{array}{ll} \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho_{ref}} \nabla \tilde{p} - \nu_{ref} \Delta \mathbf{u} = \beta(\theta - \theta_{ref}) \mathbf{g}, \\ \mathbf{u} \cdot \nabla \theta - \kappa_{ref} \Delta \theta = 0, \\ \mathbf{u} = \mathbf{v}_{in} & \text{on } \Gamma_b^{in} \\ \frac{\partial \mathbf{u}}{\partial n} = 0 & \text{on } \Gamma_b^{out} \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_{wall} \\ u_z = -v_c, u_x = 0 & \text{on } \Gamma_c^{out} \\ u_z = -v_c, \frac{\partial u_x}{\partial n} = 0 & \text{on } \Gamma_c^{in} \\ \theta = \theta_b & \text{on } \Gamma_b^{in}, \\ \theta = \theta_c & \text{on } \Gamma_c^{out}, \\ \frac{\partial \theta}{\partial n} = 0 & \text{on } \partial\Omega \setminus \{\Gamma_c^{out} \cup \Gamma_b^{in}\}. \end{array} \right. \quad (13)$$

The varying parameters are the velocities $v_b \in [0.5; 2]$, $v_c \in [0.1; 0.4]$ (in mm/s), the imposed temperatures $\theta_b \in [288; 292]$ and $\theta_c \in [295; 315]$ (in Kelvin). For simplicity we will also denoted by $\sigma = (v_b, \theta_b, v_c, \theta_c)$ the set of parameters.

4. Numerical results

The blackbox software to compute the pressure, velocity and temperature ‘‘truth’’ and ‘‘coarse’’ approximations is the finite element code ‘‘FreeFem++’’[10]. The \mathbb{P}_2 - \mathbb{P}_1 Taylor-Hood finite element has been used to build the velocity and pressure approximation spaces and the \mathbb{P}_2 finite element has been used to build the temperature approximation space over various coarse and fine meshes : \mathcal{T}_H and \mathcal{T}_h .

4.1. Application 1 : Natural convection in a 2D cavity

In this example, we consider heat transfer due to natural convection inside a heated cavity as introduced in section 3.2. To start with, we shall focus here on the approximation of the only temperature θ , hence, in the remainder subsection, we have simplified the notation by removing the superscript θ on N^θ .

In order to build the reduced basis space X_h^N , we have computed a “training” sample made of a series of $n_{train} = 93$ fine finite element approximations of (12) for a Prandtl number varying between 0.5 and 1 and a Grashoff number varying between 10^3 and 10^6 (see Figure 4).

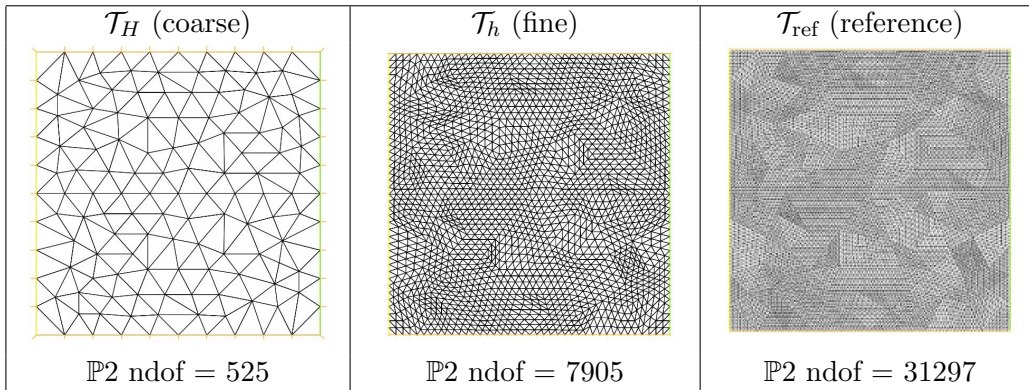


Figure 3: Embedded meshes used to build finite element approximation spaces

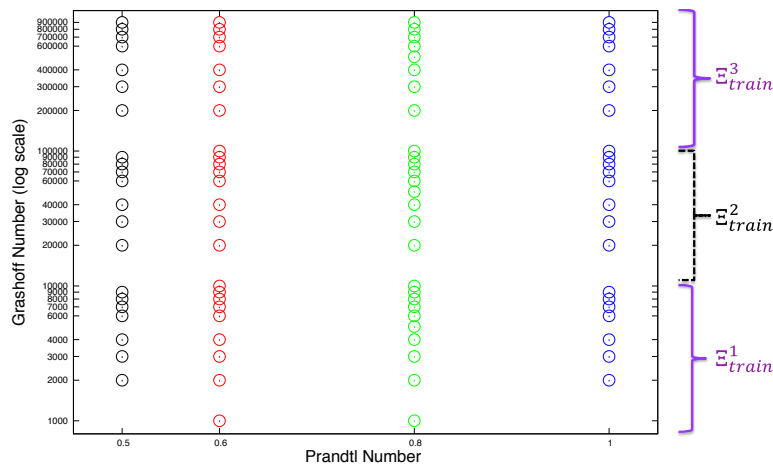


Figure 4: Parameters distribution in $\Xi_{train} = [0.5 : 1] \times [10^3 : 10^6]$

Let us denote by, Ξ_{train}^F , the “ full training” set of points, in \mathcal{D} , associated to our complete “training” sample of finite element approximations. In order to better take into account the disparity between the solutions of lower and higher Rayleigh number ($Ra = Gr \times Pr$), we have decomposed this initial “training” parameter set into three parts: $\Xi_{train}^F = \Xi_{train}^1 \cup \Xi_{train}^2 \cup \Xi_{train}^3$ with $\Xi_{train}^1 = [0.5 : 1] \times [10^3 : 10^4]$, $\Xi_{train}^2 = [0.5 : 1] \times [10^4 : 10^5]$ and $\Xi_{train}^3 = [0.5 : 1] \times [10^5 : 10^6]$ (see figure 4).

In order to obtain an optimal set of parameters $\{\sigma_1, \dots, \sigma_{N_{max}}\}$ from a given training sample $\Xi_{train} \subset \Xi_{train}^F$, we resort to a greedy sampling procedure given in Algorithm 1.

Algorithm 1 Greedy's algorithm used to choose $\{\sigma_1, \dots, \sigma_{N_{max}}\}$

Given $\Xi_{train} = (\sigma_1, \dots, \sigma_{n_t}) \in \mathcal{D}^{n_t}$, $n_t \gg 1$ with $n_t \leq n_{train}$ and $\Xi_{train} \subset \Xi_{train}^F$
 Choose randomly σ_1 , $\rightarrow S_1 = \{\sigma_1\}$ and $X_h^1 = \text{span}\{\theta_h(\sigma_1)\}$

Set $\xi_1 = \frac{\theta_h(\sigma_1)}{\|\theta_h(\sigma_1)\|_{L^2}}$

for $n = 2$ to N_{max} **do**

$$\sigma_n = \arg \max_{\sigma \in \Xi_{train}} \frac{\|\theta_h(\sigma) - \sum_{i=1}^{n-1} (\theta_h(\sigma), \xi_i)_{L^2} \xi_i\|_{L^2}}{\|\theta_h(\sigma)\|_{L^2}}$$

$$S_n = S_{n-1} \cup \sigma_n \text{ and } X_h^k = X_h^{n-1} + \text{span}\{\theta_h(\sigma_n)\}$$

$$\text{Compute } \tilde{\xi}_n = \theta_h(\sigma_n) - \sum_{i=1}^{n-1} \xi_i (\theta_h(\sigma_n), \xi_i)_{L^2} \text{ and set } \xi_n = \frac{\tilde{\xi}_n}{\|\tilde{\xi}_n\|_{L^2}}$$

end for

Besides, in order to determine the reduced basis's optimal size we propose to look at the behavior of the average error of the rectification process for all the $\sigma \in \Xi_{train} \setminus S_N$ as N increase. In order to do so, we computed temporary rectification matrices T_N associated to the functional ξ_N . We identify the matrices $T_N \in \mathbb{R}^{N \times N}$ such that :

$$\sum_{j=1}^N T_{ij}^N (\theta_H(\sigma_k), \xi_j)_{L^2} = (\theta_h(\sigma_k), \xi_i)_{L^2}, \quad \forall 1 \leq i, k \leq N \leq N_{max} \text{ and } \sigma_k \in S_N.$$

Thus, the matrix T_N is equal to

$$T_N = \begin{pmatrix} \gamma_1^{\theta, h}(\sigma_1) & \cdots & \gamma_1^{\theta, h}(\sigma_N) \\ \vdots & \vdots & \vdots \\ \gamma_N^{\theta, h}(\sigma_1) & \cdots & \gamma_N^{\theta, h}(\sigma_N) \end{pmatrix} \times \begin{pmatrix} \gamma_1^{\theta, H}(\sigma_1) & \cdots & \gamma_1^{\theta, H}(\sigma_N) \\ \vdots & \vdots & \vdots \\ \gamma_N^{\theta, H}(\sigma_1) & \cdots & \gamma_N^{\theta, H}(\sigma_N) \end{pmatrix}^{-1}$$

with $\gamma_i^{\theta, h}(\sigma) = (\theta_h(\sigma), \xi_i)_{L^2}$ and $\gamma_i^{\theta, H}(\sigma) = (\theta_H(\sigma), \xi_i)_{L^2}$ for $1 \leq i \leq N$.

In the remainder of this subsection we define by P_N the projection operator such that $P_N \theta^\delta(\sigma) =$

$$\sum_{i=1}^N \xi_i (\theta_\delta(\sigma), \xi_i)_{L^2} \text{ with } \delta = H \text{ or } h \text{ and } R_N P_N \text{ the rectification operator such that } R_N P_N \theta^H(\sigma) =$$

$$\sum_{i,j=1}^N T_{i,j} (\theta_H(\sigma), \xi_j)_{L^2} \xi_i.$$

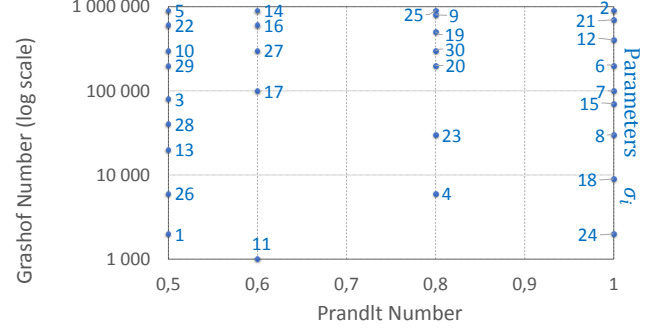
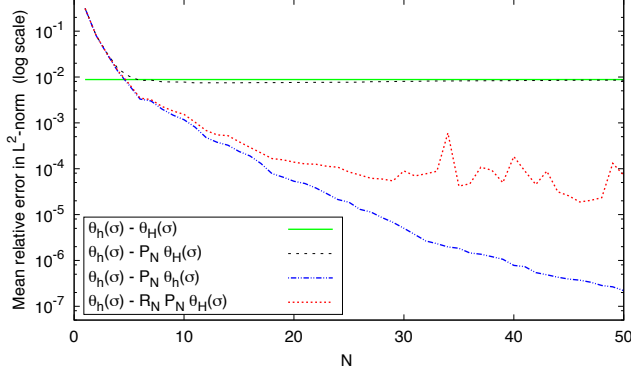


Figure 5: Average errors (left) and parameters distribution (right) during the reduced basis construction stage with $\Xi_{train} = \Xi_{train}^F$

Reminding that, as proven in [3, 4, 15], we have

$$\|\theta(\sigma) - R_N P_N \theta_H(\sigma)\|_{H^1} \leq c_1(N)(h + H^2) + \varepsilon(N)$$

The deterioration of the rectified solution $R_N P_N \theta_H(\sigma)$ observed when N is large in figures 6 and 7 below, confirms that constant $c_1(N)$ is growing with N .

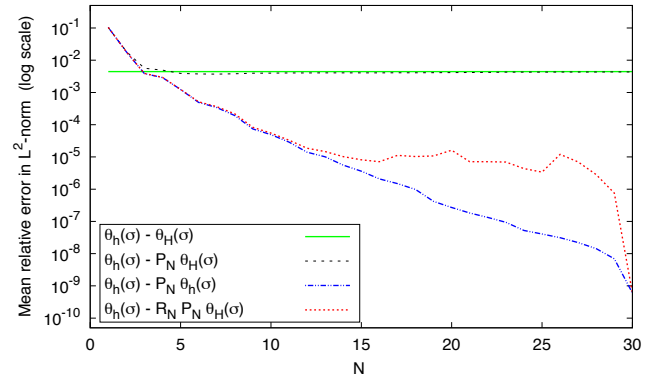
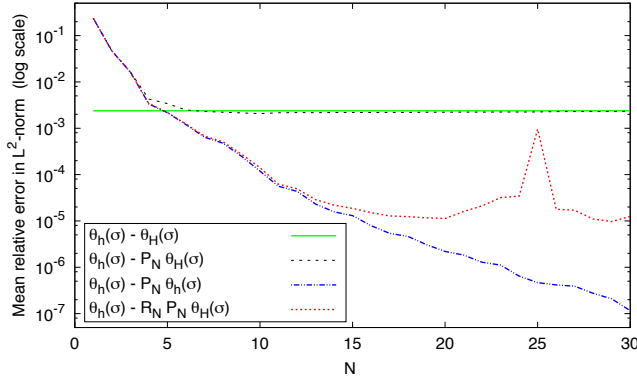


Figure 6: Average errors during the reduced basis construction stage with $\Xi_{train} = \Xi_{train}^1 \cup \Xi_{train}^2$ (left) and with $\Xi_{train} = \Xi_{train}^2$ (right)

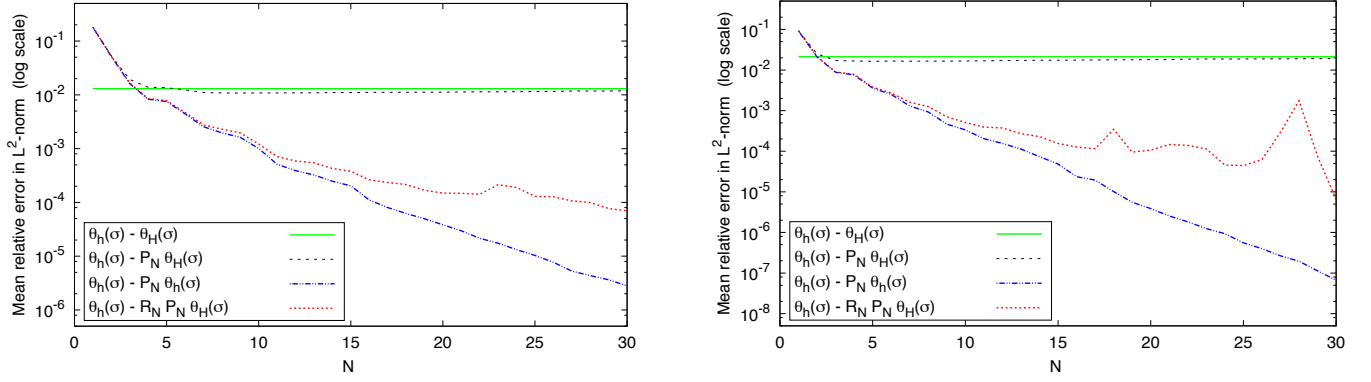


Figure 7: Mean errors during the reduced basis construction stage with $\Xi_{train} = \Xi_{train}^F \setminus \Xi_{train}^1$ (left) and $\Xi_{train} = \Xi_{train}^3$ (right)

To get around this, we have decided to stop the enrichment of the reduced basis space when the rectification error increases too rapidly (see table 2).

| Parameter space Ξ_{train} | N_{max} |
|---|-----------|
| Ξ_{train}^F | 30 |
| $\Xi_{train}^1 \cup \Xi_{train}^2$ | 20 |
| Ξ_{train}^2 | 15 |
| $\Xi_{train}^F \setminus \Xi_{train}^1$ | 20 |
| Ξ_{train}^3 | 20 |

Table 2: Value of N_{max}^θ for a variety of Ξ_{train}

Once we have determined, for each parameter space Ξ_{train} , the set of parameters $\{\sigma_1, \dots, \sigma_{N_{max}}\}$ that will be used to generate X_N^h , we solve, for several values of $N \leq N_{max}$, the following eigenvalue problem : find $(\lambda_k, \Phi_k) \in \mathbb{R}^N \times \mathbb{R}^N$, $1 \leq k \leq N$ such that

$$K^N \Phi_k = \lambda_k M^N \Phi_k \quad \text{with } M_{i,j}^N = \int_{\Omega} \xi_i \xi_j \text{ and } K_{i,j}^N = \int_{\Omega} \nabla \xi_i \nabla \xi_j.$$

Which will provide N basis functions of X_h^N orthogonalized in L^2 and H^1 norm, defined as

$$\phi_k^{\theta,N} = \frac{1}{\sqrt{\lambda_k}} \sum_{i=1}^N (\Phi_k)_i \xi_i.$$

To validate the reduced basis functions, we have looked at the convergence rate of the projection operator Π_N for some particular solutions $\theta_h(\sigma)$, with $\sigma \in \mathcal{D} \setminus \Xi_{train}$ depending on the parameters space Ξ_{train} (figures 8 and 9).

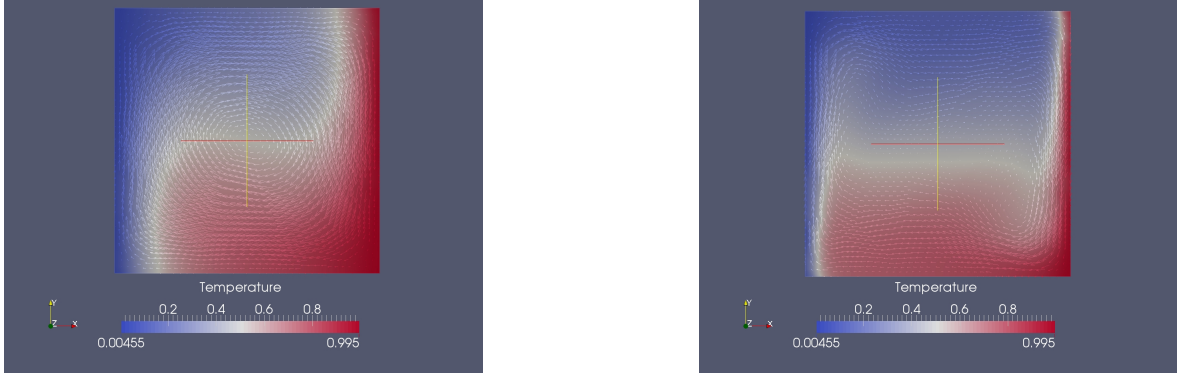


Figure 8: Temperature field $\theta_h(\sigma)$ (on the left with $\sigma = (0.75; 25000)$ and on the right with $\sigma = (0.7; 750000)$)

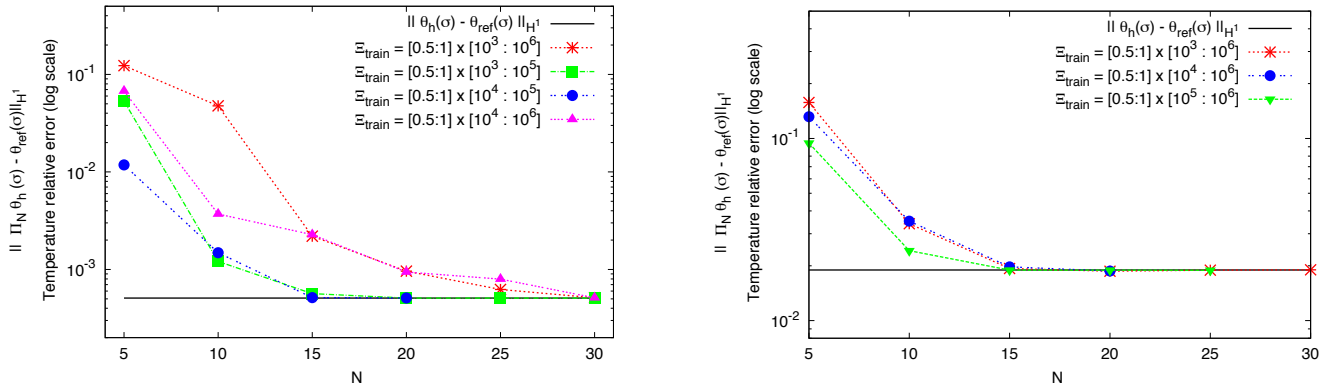


Figure 9: Convergence rate of the reduced basis's projection measured in H^1 -norm (on the left with $\sigma = (0.75; 25000)$ and on the right with $\sigma = (0.7; 750000)$)

In the figure 10 we can see the convergence rate of the improved reduced basis approximation $\theta_{H,h}^N$ as function of N for some particular value of $\sigma \in \mathcal{D} \setminus \Xi_{train}$ depending on the parameters space Ξ_{train} .

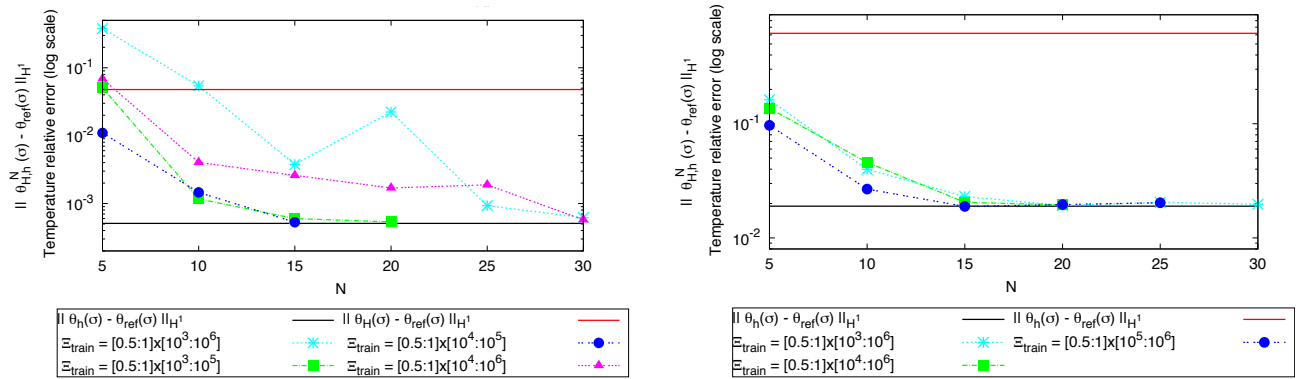


Figure 10: Convergence rate of the improved reduced basis approximation $\theta_{H,h}^N$ as function of N measured in H^1 -norm during the online stage (on the left with $\sigma = (0.75; 25000)$ and on the right with $\sigma = (0.7; 750000)$)

As we expected, we were able to reach the same accuracy as the fine finite element solution and to

reduced significantly the computational times (see table 3).

| F.E.M. Fine sol. | NIRB method (on line stage) | | | |
|---------------------|--------------------------------|---------------|-------|--------|
| | FEM coarse sol. | Rectification | | |
| | | N=15 | N=20 | N=30 |
| 7min28 | 26 sec | 5 sec | 9 sec | 19 sec |

Table 3: Average execution's times

4.2. Application 2 : Cooling system of electronic devices

In this second example, we are interested in the evaluation of the velocity and the temperature inside a simplified cooling system for electronics devices.

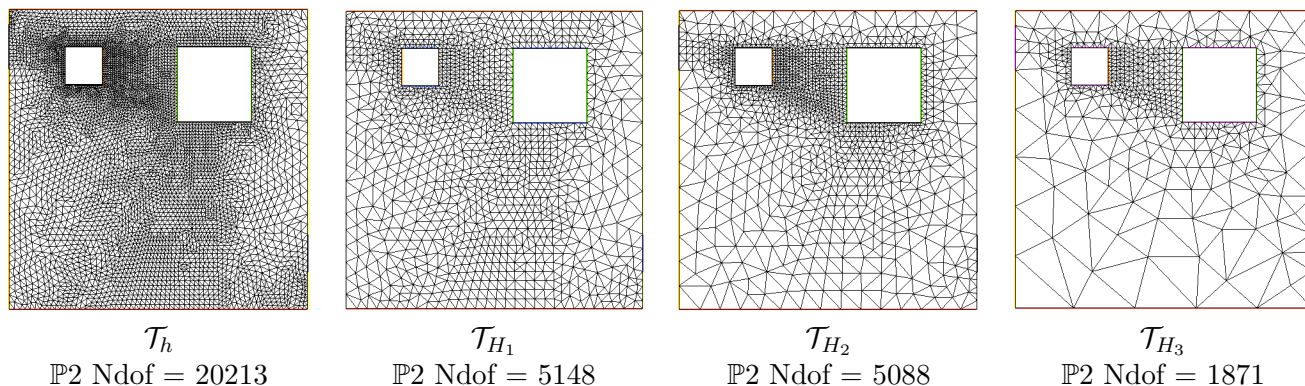


Figure 11: From the left to the right : fine mesh (\mathcal{T}_h), embedded coarse mesh (\mathcal{T}_{H_1}), non embedded coarse mesh (\mathcal{T}_{H_2}), non embedded very coarse mesh (\mathcal{T}_{H_3}).

In order to build the reduced spaces X_h^N and Y_h^N we have used a fine finite element mesh with 20,213 degrees of freedom (mesh \mathcal{T}_h in Figure 11). In order to test the NIRB on various coarse meshes, we have computed a sample of coarse and fine finite element approximations of problem (13), using embedded or non embedded meshes (see figure 11). The associated parameters space — denoted by Ξ^{train} — is of size 120, and composed of parameters within the following values:

| θ_b | v_b | v_c | θ_c |
|------------|-------|-------|------------|
| 288 | 0.5 | 0.1 | 295 |
| 292 | 1 | 0.2 | 300 |
| | 2 | 0.3 | 305 |
| | | 0.4 | 310 |
| | | | 315 |

To verify the non-intrusive reduced basis method, we have compared the fine finite element solution for $\sigma = (\theta_b; v_b; v_c; \theta_c) = (290; 1.5; 0.25; 312)$ (see figure 12) with different reduced solutions (see Figure 13).

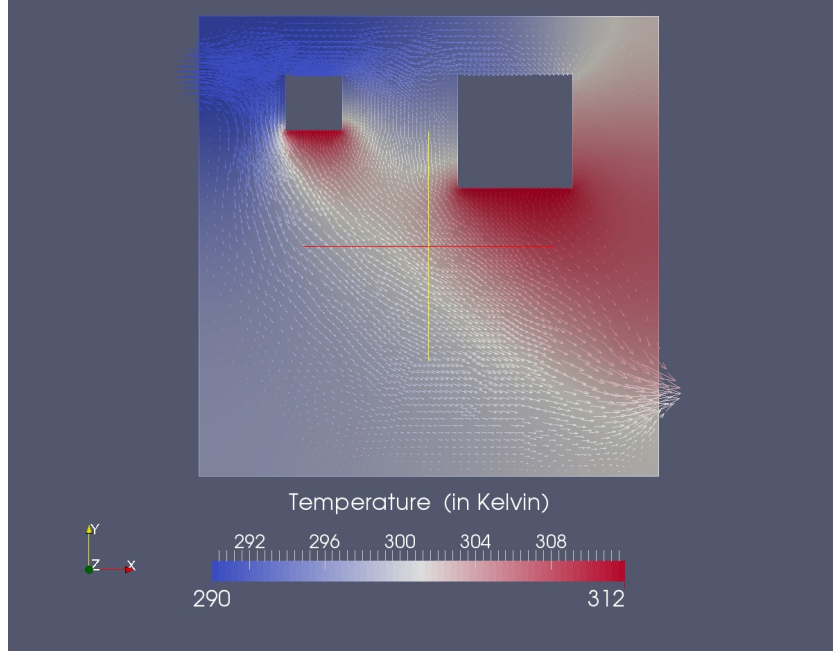


Figure 12: Temperature field for $\sigma = (\theta_b; v_b; v_c; \theta_c) = (290; 1.5; 0.25; 312)$

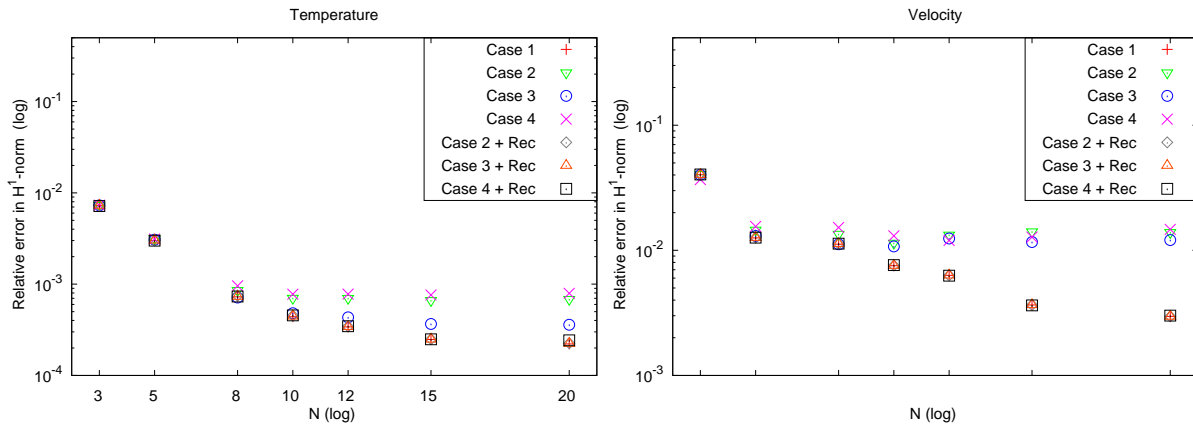


Figure 13: Relative error between the fine FEM solution and the various reduced solutions measured in H^1 -norm (temperature on the left and velocity on the right).

- **Case 1:**

In this example, we want to see the error due only to the reduced basis size N . To do so, as in the previous section, we have computed the L^2 projection of the fine finite element solution on the reduced basis space.

For a given N , those solutions are the best approximation that we can expect in the reduced basis space.

- **Case 2, 3 and 4:**

In those examples, we wanted to see how the choice of the coarse mesh \mathcal{T}_H affect the NIRB method

when there is no rectification.

We have used embedded coarse mesh ($\mathcal{T}_H = \mathcal{T}_{H_1}$) and not embedded coarse meshes ($\mathcal{T}_H = \mathcal{T}_{H_2}$ or \mathcal{T}_{H_3}) to compute the reduced approximation in respectively the case 2, 3 and 4.

We notice that as N goes larger the error between the different reduced solutions and the FEM solution goes smaller to finally reach a threshold. This is due to the fact that the coarse finite element's error become more significant than the reduced basis size's error.

Remark : *From a practical viewpoint, we want to use a finite element software able to compute the coefficients $\gamma_j^{H_i}(\sigma)$ easily and in a way that it is not too time consuming. Indeed, in order to do compute those coefficients we resort to numerical integration. However, in the case when the meshes are not embedded a supplementary difficulty arise from the fact that the reduced basis functions and the coarse solutions do not belong to a same finite element space. We want to avoid an extra interpolation error due to the interpolation of the coarse solution to the reference mesh. It is made by the use of a numerical quadrature formula over each triangle K_h based Gauss points, here we choose a \mathbb{P}_2 -unisolvant formula (with 7 points). For example $\int_{K_h} \theta_{H_i}(\sigma) \xi_h^j$ will be replaced*

by $\sum_{p=1}^7 \omega_p \times \theta_{H_i}(x_p; \sigma) \times \xi_h^j(x_p)$, where ω_p are the weight of the quadrature point. Then to compute $\theta_{H_i}(x_p; \sigma)$, we just have to find in which triangle of \mathcal{T}_{H_i} belongs the point x_p , to avoid any interpolation of $\theta_{H_i}(\sigma)$ on K_h . In the software that we have used for our numerical experiments, this step was done automatically.

- **Case 2 + Rec, 3 + Rec and 4 + Rec :** In those examples we wanted to see the influence of the rectification on the reduced solution. We observe that with this rectification we are able to reach the same accuracy as if we have projected the reference finite element solution on the reduced basis space, even with the coarsest mesh. In particular, when using coarse mesh \mathcal{T}_{H_3} , we are able to divide the computational times by 10 without any loss of accuracy (see figures 14 and 15).

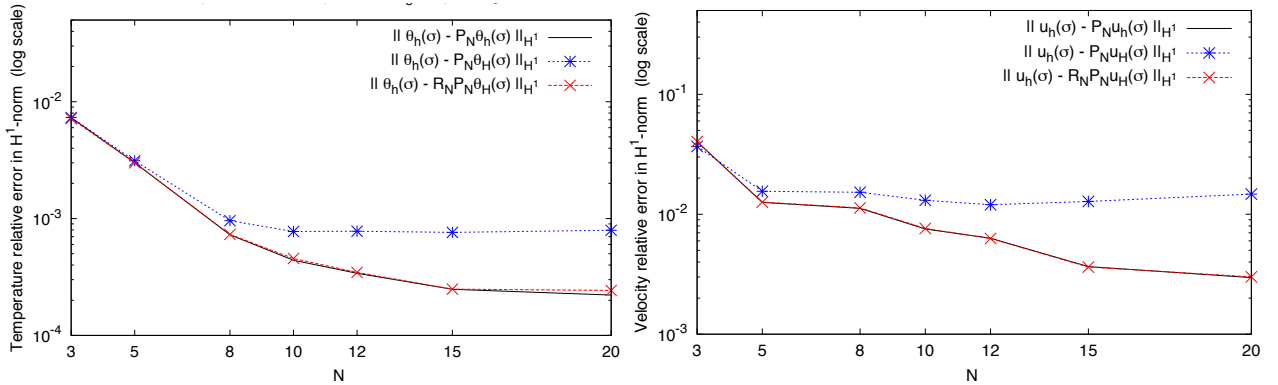


Figure 14: Relative error between the fine FEM solution the various reduced solutions measured in H^1 -norm with $T_H = T_{H_3}$ (temperature on the left and velocity on the right).

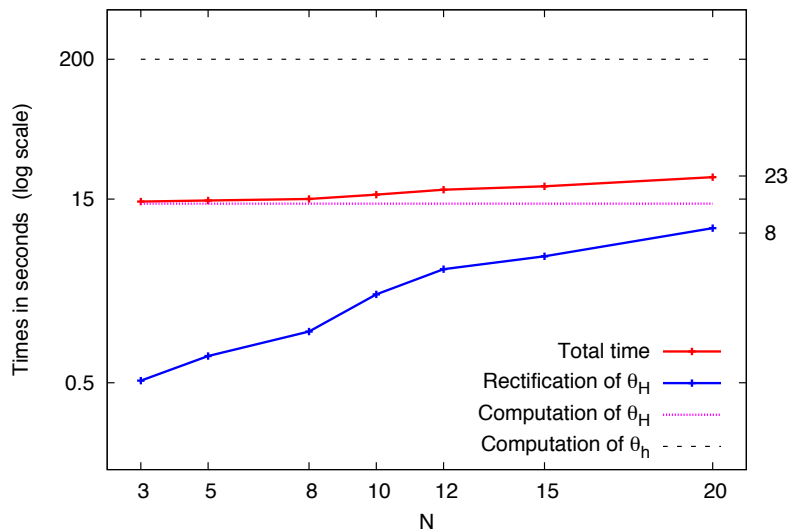


Figure 15: Comparison of CPU Times between the fine FEM and the NIRB method

4.3. Conclusion

The present paper illustrates the capability of non-intrusive reduced basis method to return computationally efficient and accurate thermal flux for simple model of cooling systems. Coarse triangulations have been used to compute finite element solutions during the online stage. In order to regain accuracy, we first project these solutions into the RB space, and then improve them via a rectification technique. Our numerical results prove that the two ingredients i) projection ii) rectification, are indissociable in order to get the expected accuracy. With this approach, we are still able to significantly reduce our computational cost without modifying the CFD code, and it is thus user-friendly. The only thing that we need to get out the plain CFD finite element software are: i) the meshes (fine and coarse) and ii) the nodal values of each fields we want to approximate, in order to export these data in a simple finite element code (as e.g. Matlab, FEEL++, FreeFem++) where simple L^2 and H^1 scalar products are performed. Our simulations also illustrate that the coarse and fine meshes do not need to be embedded.

Acknowledgment

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