Some trends and tools for the study of time-delay systems

Jean-Pierre Richard

Some Trends and Tools for the Study of Time Delay Systems

Jean-Pierre RICHARD
Ecole Centrale de Lille, LAIL - URA CNRS D1440,
B.P. 48, 59651 Villeneuve d'Ascq Cedex-France
e-mail: jprichard@ec-lille.fr

ABSTRACT

This survey presents some of the recent developments in time-delay systems theory. The general situation of this kind of model in the engineering sciences is recalled and some classes of models are presented. Then, stability and control properties are discussed. Of course, due to the large number of references involved in this field, this overview cannot pretend to exhaustiveness but, the author hopes, contributes to clarify some of the differences between ordinary and hereditary systems.

1. Introduction.

Time-delays are known to be natural components of the dynamic processes in physics, mechanics, biology, ecology, physiology, economics, epidemiology, population dynamics, chemistry, aeronautics, aerospace, to name a few. Even if the process itself does not include delay phenomena, the actuators or sensors that are involved in its automatic control usually introduce such time lags. This explains the great number of works devoted to such class of systems. During the last decades, the field of equations with delays, also called hereditary equations or functional differential equations (*FDEs*) has been making significant breakthroughs in its practice, which are no longer only a specialist's field: the reader can see for instance the -English writing- monographs by Bellman and Cooke [5] (frequency domain approach, integer functions), Krasovskii [66] (extension of the direct Lyapunov's method), Halanay [48] (extension of the Popov's theory), Lakshmikantham and Leela [68] (differential inequalities, comparison approach), El'sgol'ts and Norkin [31] (stability, metric spaces), Driver [28], Hale [49] (topological stability methods), MacDonald [77] (biological applications), Burton [14] (direct Lyapunov's method, periodic solutions), Kolmanovskii and Nosov [62] (comprehensive introduction to stability, with examples), Gorecki *et al.* [44] (characteristic function, infinite-dimensional tools), Stépán [109] (characteristic function, robotics), Hale and Verduyn Lunel [50] (completed from [49]), Gopalsamy [39] (stability and oscillations based on ecology examples), Kolmanovskii and Myshkis [61] (deterministic and stochastic FDEs with a lot of concrete examples), Diekmann *et al.* [27] (operator theory approach), Kolmanovskii and Shaikhet [65] (optimal control, self-adjusting systems), Malek-Zavarei and Jamshidi [78], and some very recent collective works: Dugard and Verriest [29], Richard and Kolmanovskii [101], Loiseau and Rabah [76]. Other contributions, in French, can be found in [87] and 7 Ph.D. defended from 1994 to 1997 (see references). Last but not least, many contributions were done by other eastern scientists, which books are in Russian. Today, some international workshops and a lot of specialized sessions are regularly organized on this topic.

This interest is probably motivated by two points: on the one hand, the fundamental aspects are quite exciting for scientists, because specific properties of delay systems are often surprising. On the other hand, the applications are of real economic interest: together with the increasing expectations of dynamic performances, engineers need the models to behave more closely to the real processes, and the number of FDE-models used in the sciences and in applied areas has been growing up tremendously: applications range from physiology and enzyme kinetics to whaling control and foodwebs, from neural networks to laser optics, from studies of engines to the theory of business cycles, from transportation and communication systems to chemical and metallurgical processing, from traffic and power control to water resources systems, from flight mechanics to robot-manipulators, from flexible structures to mechanics of viscoelasticity, from idle speed to air-fuel ratio control problems, from telerobotic systems and earth-controlled satellite devices to bio-thermo-chemical processes...

One could think that for "small" values of the delays, the simplest approach would consist in neglecting or replacing them by rational approximations: unfortunately, ignoring effects which are adequately represented by FDEs is not a general alternative, since it can lead to potentially disastrous consequences in terms of stability and control design. The real nature of FDEs is to be infinite-dimension, and finite-dimension approximations cannot be used as soon as the model has to be accurate enough, the behaviors to be faster, or the control to be robust.

Moreover, several studies showed that voluntary introduction of delays in feedback laws can also benefit the control (for instance, stabilization [1][41] and deadbeat control [120] of ODEs, or finite-spectrum assignment of FDEs [46][120][9]).

The huge variety of applications gives new breath to some older parts of FDEs theory[1] and generates many new ones. Along with the traditional classes, new types of FDEs are being introduced and widely used in mathematical modelling, for example, stochastic FDEs, equations with im-

[1] The study of hereditary equations began during the 18th century with Bernoulli, Euler, Lagrange,Condorcet, then sporadically followed till the begining of 20th with Volterra,... to mention some of the most famous names. But, in the 1930's, the growing number of technical control problems showed the need of some global, mathematic statement of the question (in particular, for the initial value), which was provided by the paper of Myshkis in 1949 [85] who defined the notion and classification of FDEs.

pulses, hybrid and large scale FDEs, distributed (partial) FDEs, systems with fractional dimension, equations with state dependent time lags, n-D systems, and so on... Techniques to investigate modern problems of FDEs theory include many parts of real and complex analysis, functional analysis, operator theory, dynamic systems, theory of stochastic processes, theory of semi-groups, theory of systems over rings, topological methods,...

It would not be possible to present here -or even, to be aware of!- all researches results and trends in such a huge field: we shall consider the reduced area of some interesting control questions related to modelling, stabilization and controllability. Lastly, some references concerning control are recalled.

2. Models for delay systems

Functional differential equations, notion of state

A classical hypothesis in the modelling of physical processes is to assume, in the autonomous case, that the future behavior of the deterministic system can be summed up in its only, present *state*. This leads to "Ordinary" Differential Equations (ODEs), described by a n-vector $x(t)$ moving in Euclidean space R^n:

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), t, u(t)\right), \qquad t \geq t_0; \qquad (1) \\ x(t_0) &= x_0 \in R^n, \end{aligned}$$

(as throughout this paper, we don't consider the case of implicit systems). Dots indicate the time-derivatives, $u(t)$ denotes the input (control or disturbances).

However, in numerous cases (see many examples in [61]), some "aftereffect" cannot be neglected in the modelling, which means one has to take into account an irreducible influence of the past: it is clear that, for instance, the simple delay equation

$$\dot{x}(t) = -x(t-h), \qquad (2)$$

has several solutions (for $h = \pi/2$: $\sin t$, $\cos t$,...) that get the same value at an infinite number of instants. Then, the *state* cannot anymore be a vector $x(t)$ defined at a discrete value of time t: in Functional Differential Equations (FDEs), it is a function x_t corresponding to the past time-interval $[t-h,\, t]$, where h is a positive, irreducible-to-zero constant (Shimanov's notation, 1960).

This *argument deviation*, *i.e.* the "time-delay" h, may be bounded or infinite: it represents the maximal value of all the (possibly time-varying) delay phenomena in the process, a "memory time horizon". Note that the equations as (2) are also called *differential-difference* equations, since both kinds of operators are involved in. Two classes of hereditary models are considered: the retarded systems and the neutral ones (the mathematical class of *advanced* systems, $h < 0$, is not in this study for obvious reasons of causality).

Retarded systems with input $u(t)$ can generally be described by FDEs as

$$\begin{aligned} \dot{x}(t) &= f\left(x_t, t, u_t\right), \qquad t \geq t_0, \qquad (3) \\ x_t(\theta) &= x(t+\theta), \qquad -h \leq \theta \leq 0, \\ u_t(\theta) &= u(t+\theta), \qquad -h \leq \theta \leq 0, \\ x(\theta) &= \varphi(\theta), \qquad t_0 - h \leq \theta \leq t_0. \end{aligned}$$

The nature of physics is known to be nonlinear, and such equations arise very often in the literature (see e.g. [61]). The vector $x(t)$ will be called, here, the *solution at time* t (it is also called "instantaneous state" [62]). Note that the functional notation x_t needs the initial condition φ for equation (3) to be prescribed on the interval $[-h, 0]$. In fact, it is more natural to consider as state-space the set of continuous functions mapping the interval $[-h, 0]$ into R^n, denoted $\mathcal{C}$ throughout this paper. Under some conditions on f [49], for a given, continuous function $u(t)$, $t \in R$, and any given function y of $\mathcal{C}$, there exists a unique solution of (3) such that $x(t) = z(t)$ for $t \in [-h, 0]$ (here, z may have the meaning of an initial condition). This result can easily be proved using a step-by-step method. For instance, let us consider (2), the initial function $\varphi(\theta) = 1$ for any θ in $[-h, 0]$. Then, equation (2) on the first time interval $[0, h]$ gives $\dot{x}(t) = 1$ or $x(t) = 1 - t$. Expression of $x(t)$ can then be obtained on $[h, 2h]$ by using the same scheme of proof, and so on. The resulting solution is a succession of polynomial functions of t with increasing degree on each $[kh, (k+1)h]$.

Classification as a *neutral* system implies that in the modelling procedure, the same, highest derivation order concerns some component of $x(t)$ at both time t and past time(s) $t' < t$, which implies an increased mathematical complexity. The following system is a neutral type one:

$$\dot{x}(t) = f\left(x_t, t, \dot{x}_t, u_t\right), \qquad (4)$$

with the usual, particular case

$$Fx_t = f\left(x_t, t, u_t\right), \qquad (5)$$

where $F : \mathcal{C} \to R^n$ is some regular (to avoid implicit systems) operator with deviating argument on time, for instance,

$$Fx_t = x(t) - Dx(t-\omega),$$

with D a constant matrix. Such models arise from the approximation of hyperbolic, distributed parameters equations with mixed initial and derivative boundary condition, as wave propagation in processes including steam pipes or lossless transmission lines, for instance [62][50][61][84], but can also be encountered in robotics (manipulators in contact with rigid environment [92]). In this case, due to the contained difference-equation involving $\dot{x}(t)$, the trajectory may "replicate", all along the time, any time-discontinuity of the initial condition $\varphi(t)$, even if f and D present many smoothness properties (while retarded systems will "smooth" it: the differentiability degree of the solution increases with time).

Nature is nonlinear, we said, but linear models are very useful: they are a bit more easy to deal with and constitute a good basis for investigating many properties of delay systems. The time-invariant model is

$$\begin{aligned} \dot{x}(t) &= \sum_{k=1}^{q} D_k\, \dot{x}(t-\omega_k) \qquad (6) \\ &\quad + \sum_{i=0}^{k} \left(A_i x(t-h_i) + B_i u(t-h_i)\right) \\ &\quad + \sum_{j=1}^{r} \int_{t-\tau_j}^{t} \left(C_j(\theta)\, x(\theta) + G_j(\theta)\, u(\theta)\right) d\theta, \end{aligned}$$

where $h_0 = 0$, the matrix A_0 (constant) represents *instantaneous* feedback gains; A_i, $i > 0$ (constant), represent *discrete-delay* phenomena; the last integrals correspond to *distributed-delay* effects, which influence is weighted by the C_j over the time intervals $[t-\tau_j,\, t]$; D_i are the *neutral* part, and B_i, $G_j(s)$ are input matrices. Here,

$h = \max_{i,j,k}\{h_i, \tau_j, \omega_k\}$. Many physical systems can also be approximated by such models (see for instance [34] and references herein) with, mainly, only one neutral delay ($q = 1$).

Note that, in (6), $C_j = -C_k$ for some (j,k) permits to consider "discrete-and-distributed" effects as $\int_{t-\tau_j}^{t-\tau_k} C_j x(\theta)d\theta$, and also that some additional approximation (here again, without any guarantee) may allow to replace the distributed effects by a sum of discrete ones, by considering that

$$\int_{t-\tau}^{t} C(\theta)\, x(\theta)d\theta \approx \frac{\tau}{d}\sum_{i=1}^{d} \alpha_i C(\frac{i\tau}{d})x(t-\frac{i\tau}{d}),$$

with constant, tuning coefficients $\alpha_i \in R$.

Due to this simplification, many results deal with the particular case of discrete-delay systems defined by

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-h_i) + \sum_{i=1}^{k} B_i u(t-h_i). \quad (7)$$

The special class of systems with *commensurate delays* is particularly investigated, which delays $h_i = i\delta$ are all integer multiples of a same positive, constant, basic delay δ (then, $h = k\delta$). In the following, we shall see that several important results are now available for the design of linear, commensurate models.

Solution of linear, discrete-delay systems

There exist several numerical methods (see [18]) for the construction of solutions for FDEs, that are mainly using step-by-step approaches, *i.e.* iterative resolution over time intervals $[jh,(j+1)h]$ by means of classical ODEs procedures (Euler, Runge-Kutta,...), with continuity at instants jh. Concerning the general solution of (3), we know [62][50] that some Lipschitz properties on f ensure the existence of a unique solution for given φ and $u_{t\in R}$. Of course, as in the ODEs case, the explicit solution is not known, but we can illustrate here the question of linear systems with commensurate delays, described by:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{k} (A_i x(t-i\delta) + B_i u(t-i\delta)) \quad (8)\\ y(t) &= \sum_{i=0}^{k} C_i x(t-i\delta), \quad x(\theta) = \varphi(\theta)\ (-k\delta \le \theta \le 0), \end{aligned}$$

which solution [115][78] from $t_0 = 0$ is

$$\begin{aligned} x(t;t_0,\varphi,u) &= F(t)\varphi(0) + \sum_{i=0}^{k}\int_{-i\delta}^{0} F(t\text{-}\theta\text{-}i\delta)A_i\varphi(\theta)d\theta \\ &\quad + \int_0^t F(t-\theta)\left[\sum_{i=0}^{k} B_i u(\theta-i\delta)\right]d\theta. \end{aligned}$$

Here, $F(t)$ is the *fundamental matrix*, solution of $\dot{F}(t) = \sum_{i=0}^{k}(A_i F(t-i\delta))$, $F(0) = I$, $F(t<0) = 0$. Of course, the main difficulty is to calculate $F(t)$. It can be done, by means of infinite-series development, with the *Kirillova-Churakova operators*,

$$\begin{aligned} P_{q+1}(j) &= \sum_{i=0}^{k} A_i P_q(j-i), \quad (9)\\ P_0(0) &= I,\ P_q(j) = 0 \text{ for } i \text{ or } j < 0. \end{aligned}$$

Then, for any integer $\lambda > 0$,

$$F(t) = \sum_{j=0}^{\lambda}\sum_{q=j}^{\infty}\frac{1}{q!}P_q(j)(t-j\delta)^q \text{ for } t \in [0,\lambda\delta].$$

Operators in infinite dimension

It is possible to imbed the delay systems in the larger class of infinite-dimensional systems. This approach, based on some abstract state-space formulation in terms of operators, may benefit of the appropriate definitions of controllability/stabilizability, observability/detectability,... that were defined in this very general framework. Among the rich literature concerning these models, just mention here [25][80][24][6][57]. For simplicity, following [105], we reduce this presentation to linear, single-delay systems, say

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_0 u(t), \quad (10)\\ y(t) &= C_0 x(t). \end{aligned}$$

We note $\mathcal{L}_2 = \mathcal{L}_2([-h,0];R^n)$ the set of functions $[-h,0] \to R^n$ with integrable square (the solutions of delay systems in infinite dimension can be reduced [80] to initial functions φ belonging to $\mathcal{L}_2$). The behavior is represented by the variable $\overline{x}$, $\overline{x}(t) = [x(t), x_t] = [x^0(t), x^1(t)]$, belonging to the Hilbert space $\mathcal{M}_2 = \mathcal{M}_2([-h,0];R^n) = R^n \times \mathcal{L}_2$. Then, system (10) can be described by

$$\begin{aligned} \dot{\overline{x}}(t) &= \widetilde{A}\overline{x}(t) + \widetilde{B}u(t),\\ y(t) &= \widetilde{C}\overline{x}(t), \quad (11) \end{aligned}$$

where the operator $\widetilde{A}$ is unbounded, closed, dense in space $\mathcal{M}_2$, and is defined by

$$\overline{x}(t) \to \widetilde{A}\,\overline{x}(t) = [A_0 x^0(t) + A_1 x^1(t)(-h),\ \frac{dx^1(t)(\theta)}{d\theta}],$$

while the operators $\widetilde{B}$, $R^m \to \mathcal{M}_2$, and $\widetilde{C}$, $\mathcal{M}_2 \to R^p$, are bounded, defined by

$$\begin{aligned} u(t) &\to \widetilde{B}u(t) = [B_0 u(t),\ 0],\\ \overline{x}(t) &\to \widetilde{C}\overline{x}(t) = y(t). \end{aligned}$$

This representation allows one to use the properties of semi-groups and, from the general theory of infinite-dimensional, differential equations, the solution is uniquely defined by

$$\overline{x}(t;\overline{x}_0,u) = S(t)\overline{x}_0 + \int_0^t S(t-\theta)\widetilde{B}u(\theta)d\theta \quad (12)$$

where the family $\{S(t), t \ge 0\}$ is the continuous semigroup of operators spanned by $\widetilde{A}$ [49], verifying

$$\begin{aligned} S(0) &= I, \qquad S(t+s) = S(t)S(s),\\ \frac{d}{dt}(S(t)\overline{x}_0) &= \widetilde{A}S(t)\overline{x}_0 \qquad t,s \ge 0. \end{aligned}$$

This approach is then a direct extension of the exponential solutions for unbounded operators (if $h = 0$, then $\widetilde{A}$ is bounded and $S(t) = e^{\widetilde{A}t}$). We considered here a simple class of discrete delays: however, infinite dimension allows to consider, with an unmodified notation, distributed delays as well. This may be the main advantage of these general models, but also presents some drawbacks: for instance, all control laws that are obtainable from such a formalism are designed under an undiscernible, distributed form, $u(t) = \int_{-h}^{0} F(\theta)x(t+\theta)d\theta$, that does not allow any *a priori* preference of a (often more easy to implement) discrete feedback.

Geometric approach: systems over polynomial rings

For nonlinear systems in particular, FDEs appear as a very convenient tool, with a good compromise between generality and simplicity. For linear ones (and only in this case), the classical geometric approach (in the sense of Wonham) for linear ODEs has been generalized, up to

a certain point, to systems with coefficients over a ring [110]. Then, the basic idea is to translate these results to the context of delay systems [116][54][83][71][16][17]: instead of defining vectors and matrices over the field of real numbers R, leading to the vector space R^n, one uses $\mathbf{R}=R[\nabla]^2$, the commutative ring of polynomials in the delay operator ∇ with, for commensurate-delays system (8), $\nabla(f)(t)=f(t-\delta)$. Then, (8) becomes

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}(\nabla)x(t)+\mathbf{B}(\nabla)u(t), \\ y(t) &= \mathbf{C}(\nabla)x(t), \qquad (13) \\ \mathbf{A}(\nabla) &\in R^{n\times n}[\nabla], \mathbf{B}(\nabla)\in R^{n\times m}[\nabla], \mathbf{C}(\nabla)\in R^{p\times n}[\nabla] \end{aligned}$$

where x belongs to the free state-module $\mathbf{R}^n=R[\nabla]^n$, u and y to he associated input and output free modules (a *module* is the equivalent of vector-space over a ring). Obviously, the absence of inverse on $R[\nabla]$ corresponds to the impossibility to realize the advance operator ∇^{-1}. The main advantage of such modelling is its apparent finite dimension: in addition, since $R[\nabla]$ is a principal ideal domain[3], many results concerning the Smith form and invariant polynomials can be used in the realization theory. The solutions are directly determined by inversion of the Laplace transform:

$$\begin{aligned} y(s) &= C_s(sI-A_s)^{-1}B_s u(s)+C_s(sI-A_s)^{-1}\varphi(s), \\ M_s &\triangleq \mathbf{M}(e^{-\delta s}) \text{ for } M=A,B, \text{or } C. \qquad (14) \end{aligned}$$

Several results are obtained in modelling [15], stabilization [46], controllability [105] and observability [99] indexes, decoupling control [106] and disturbance rejection [16][17]. It is to be noted that any polynomial feedback $u(t)=-\mathbf{F}(\nabla)x(t)+v(t)$, $\mathbf{F}\in R[\nabla]^{m\times n}$ ensures the resulting system to stay in the class (13).

In what concerns the systems with non-commensurate (but constant) delays, it is also possible to use the same approach by considering the ring of polynomials in several delay operators ∇_1, ∇_2,..., *i.e.* , $R[\nabla_1,\nabla_2,...]$. For distributed delays, some convolution operator [54] or a ring of distributions [123] are to be introduced, with an additional complexity (indeed, it seems preferable to extend these polynomial models to rational, realizable ones, as described in the next section).

However, such a *polynomial* class of control laws appears to be limited for several advanced controllers which concrete realizations need either *rational* fractions (precompensators by state or output feedback [96][97], neutral and 2-D systems [30][124][98]) or *distributed* delays (finite spectrum assignment [120], as we shall see). These lacks will be filled up by the generalized, rational and algebraic types of models, to be presented in the two following subsections.

Systems over rational rings

In [96][97][98] the above-mentioned realizability of concrete controllers was emphasized, by working with matrices $\mathbf{A},\mathbf{B},\mathbf{C}$ defined over the larger subring $\mathcal{R}_u(\nabla)$ of the irreducible, rational fractions in ∇, which denominator has a non-zero constant term:

$$\mathcal{R}_u(\nabla)=\{n(\nabla)=p(\nabla)/q(\nabla)\in R(\nabla),\ q(z=0)\neq 0\}.$$

$\mathcal{R}_u(\nabla)$ coincides with the ring of proper fractions in ∇^{-1}[30][124], but also with *non-anticipative* operators[4]. For example, $\frac{1}{1+\nabla}$ (*i.e.*, $y(t)=-y(t-\delta)+u(t)$) belongs to $\mathcal{R}_u(\nabla)$, while $\frac{1}{\nabla}$ (*i.e.*, $y(t)=u(t+\delta)$) does not. This is expressed in the following properties [97]:

Definition. *The transfer matrix $M(s,\nabla)\in R(s,\nabla)^{p\times m}$ is said to be causal if it has a realization over $\mathcal{R}_u(\nabla)$, i.e. if there exist matrices $\mathbf{A}(\nabla),\mathbf{B}(\nabla),\mathbf{C}(\nabla),\mathbf{D}(\nabla)$ defined over $\mathcal{R}_u(\nabla)$ such that*

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}(\nabla)x(t)+\mathbf{B}(\nabla)u(t), \\ y(t) &= \mathbf{C}(\nabla)x(t)+\mathbf{D}(\nabla)u(t), \qquad (15) \\ M(s,\nabla) &= \mathbf{C}(\nabla)(sI-\mathbf{A}(\nabla))^{-1}\mathbf{B}(\nabla)+\mathbf{D}(\nabla). \end{aligned}$$

Theorem. *The transfer function $n(s,\nabla)=p(s,\nabla)/q(s,\nabla)\in R(s,\nabla)$, with $p(s,\nabla)=p_0(\nabla)+...+s^r p_r(\nabla)$ and $q(s,\nabla)=q_0(\nabla)+...+s^k q_k(\nabla)$ is causal iff it belongs to the ring $\mathcal{R}_c\subset R(s,\nabla)$ defined by:*

1) *$n(s,\nabla)$ is s-proper (i.e., $r\leq k$)*
2) *$q_k(\nabla)\in R_u(\nabla)$.*

The advantage of models over $\mathcal{R}_u(\nabla)$ is that any dynamic feedback law also defined over $\mathcal{R}_u(\nabla)$ makes the resulting systems remain in the same class (whereas $R[\nabla]$-polynomial systems (13) change of class and become $\mathcal{R}_u(\nabla)$-rational).

Algebraic formalism: Laplace transform models for distributed delays

As in the previous case, the Laplace transform with algebraic formalism has to be restricted to linear models with commensurate delays. It is well known that the discrete-delay effect, denoted $\nabla(f)(t)=f(t-\delta)$ in the previous section, leads to the operator $e^{-s\delta}$ in Laplace transform. Then, algebraic formalism is near to the previous one $\mathcal{R}_u(\nabla)$, but explicitely considers this relation between the derivative s and delay $e^{-s\delta}$ operators. In 1985, Kamen, Khargonekar and Tannenbaum [56] introduced the set $\mathcal{G}$ of the realizable, distributed-delays transfers which Laplace transforms can be expressed as rational functions of s and $e^{-s\delta}$, in short, $\mathcal{G}=\{\mathcal{L}(\text{distributed delay})\in R(s,e^{-s\delta})\}$[5]. Bretheé and Loiseau [9][10][76][101] recently characterized this set $\mathcal{G}$ in a complete way and defined an other set, the ring of the so-called *pseudo-polynomials* (because they are analytic functions), $\mathcal{E}=R[e^{-s\delta}]\cup\mathcal{G}$, which is isomorphic to the quasi-polynomials ring $R[s,e^{-s\delta}]$. For instance, $F(s)$ is the Laplace transform[6] of the distributed transfer

$$\begin{aligned} u &\rightarrow y, \qquad y(t)=\int_{h_1}^{h_2} f(\theta)u(t-\theta)d\theta, \\ \frac{Y(s)}{U(s)} &= F(s)=\int_{h_1}^{h_2} f(\theta)e^{-s\theta}d\theta \qquad (16) \end{aligned}$$

(or, the zero-holder operator $\frac{1-e^{-sh}}{s}$ is obtained with $h_1=$

[2] We shall use the classical notations $R[\nabla]$ for the ring of polynomials in ∇ with coefficients in R, and $R(\nabla)$ for the ring of rational fractions in ∇ with coefficients in R.

[3] See [15][96]. An *ideal* $\mathcal{I}$ is an additive sub-group of a commutative, integral ring $\mathcal{R}$, which is invariant by product with elements of $\mathcal{R}$. It is principal if it is generated by a single element ($\mathcal{I}=a\mathcal{R}$). $\mathcal{R}$ is a principal ideal domain (*PID*) if any ideal of $\mathcal{R}$ is principal. In this case, the submodule $\mathcal{V}$ of $\mathcal{R}^n$ is $\mathcal{R}$-closed if there is a submodule $\mathcal{W}$ such that $\mathcal{V}\oplus\mathcal{W}=\mathcal{R}^n$. The closure $\overline{\mathcal{V}}$ of a submodule $\mathcal{V}$ of $\mathcal{R}^n$ is defined by $\overline{\mathcal{V}}=\{x\in\mathcal{R}^n, \exists a\in\mathcal{R}, a\neq 0, ax\in\mathcal{V}\}$. Note that $\overline{\mathcal{V}}\supset\mathcal{V}$ while $\dim\overline{\mathcal{V}}=\dim\mathcal{V}$.

[4] *Causality* is equivalent *properness* defined in [30][124], which transfers were formulated with $z=\nabla^{-1}$.

[5] Initially, the introduced set Θ was defined by Laplace transforms of the distributed-delays with Laplace transform in the set $R(s)[e^{-sh}]$, but in this case there is an isomorphism with $\mathcal{G}$ [9].

[6] in fact, finite Laplace transform is used in this case [79].

$0, h_2 = h$ and the kernel $f(\theta) = 1$ over $[0, h]$, and $f(\theta) = 0$ elsewhere). The main result is that $\mathcal{E}$ is a domain of Bézout [10], which interest for finite-spectrum assignment will be emphasized in the section "Control". Note that analogous conclusion was simultaneously obtained [38] on the basis of systems over $R[s, e^{-s\delta}, e^{s\delta}]$.

2-D models and neutral systems

It was remarked [124][73][74] that 2-D models can be used for the modelling and control of delay systems. The Roesser models (1975) are describing a two-operators system:

$$\begin{aligned} sX &= A_0X + A_2Z + B_0U, \\ \omega Z &= A_3X + DZ + B_3U, \\ Y &= C_1X + C_2Z. \end{aligned} \tag{17}$$

Here, (s, ω) respectively correspond to derivation and h-advance operators. If one considers, for instance, $A_2 = I$, $A_3 = A_1 + DA_0$, $B_3 = B_1 + DB_0$, then (17) corresponds to the neutral system:

$$\dot{x}\,(t) - D\,\dot{x}\,(t\text{-}h) = A_0x(t) + A_1x(t\text{-}h) + B_0u(t) + B_1u(t\text{-}h),$$

which is a special case of (33). Such Roesser models, in turn, allow to use previous realization results [30] for stabilization [124], and some matrix factorizations for model matching [73][74]. Equivalence with the question of realization over $\mathcal{R}_u(\nabla)$ was shown in [96][98].

Rational and finite-horizon approximations

The most common approach for control of time delay systems has been the approximation by some rational, then finite-dimension, approximations, generally based on the truncation of some infinite series. Such estimations are generally inappropriate for time-varying delays. It can be achieved by methods as the well known Padé approximations [12][69], Hankel operator methods for infinite dimensional systems [55][37], Laguerre-Fourrier series [94][70][32] or spline approximations [3]. A case study is given in [37] on the basis of L_∞ error. However, two specific, linked problems arise with that kind of simplification: together with the problem of choosing the truncation order (hence, the dimension of the approximation), it is very difficult to prove the stability of a closed loop on the basis of such reduced model [104][45].

The subsystems description [93][115] is an other way to achieve a finite-dimensional model: in the case of a system with commensurate delays, for instance (8), the approximation is made by considering a *finite time horizon*. For any variable $z(t)$, one denotes $z_i(\theta) \triangleq z(\theta + i\delta)$ and $Z_i(\theta) \triangleq [z_0(\theta), z_1(\theta), ..., z_i(\theta)]^T$. $X_i(\theta)$ is the variable of subsystem (S_i), available on the time-interval $[0, \delta]$, with increasing size $i + 1$. The behavior of (8) on the time interval $[0, (i + 1)\delta]$ is described by:

$$\begin{aligned} \dot{X}_i\,(\theta) &= \underline{A}_iX_i(\theta) + \underline{B}_iU_i(\theta) + \underline{R}_i\Phi_i(\theta), \\ Y_i(\theta) &= \underline{C}_iX_i(\theta), \quad \text{for } \theta \in [0, \delta], \\ \underline{M}_i &= \begin{pmatrix} M_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ M_i & \cdots & M_0 \end{pmatrix}, \quad \begin{matrix} M = A, B, \text{ or } C \\ M_i = 0 \text{ if } i > k \end{matrix} \\ \underline{R}_i &= \begin{pmatrix} A_1 & \cdots & A_{i+1} \\ \vdots & . & \vdots \\ A_{i+1} & \cdots & A_{2i+1} \end{pmatrix}, \Phi_i(\theta) = \begin{pmatrix} \varphi(\theta\text{-}\delta) \\ \varphi(\theta\text{-}2\delta) \\ \vdots \end{pmatrix} \end{aligned}$$

This technique is linked in its principle to the previously mentioned, step-by-step procedure. It was mainly used in [93][115] for studying controllability properties. However, it appears to be limited because of the size of involved matrices, larger and larger as time increases, with continuity problems in the junction of trajectories from model (S_i) to (S_{i+1}).

3. Stability of delay systems

Delays are reputed to destabilize the control loops. Indeed, system (2) showed that, for constant initial function $\varphi(\theta) \equiv 1$, the delay h is a source of oscillations in the time response (replacing it by 0 makes oscillations disappear). But, on the contrary, the following example

$$\ddot{y}\,(t) + \omega_0^2 y(t) - ky(t - h) = 0, \tag{18}$$

shows that delay can also have some stabilizing effect: if, in (18), h is zero, system is oscillating or unstable. However it was remarked [1] that some values of $h > 0$ and $k > 0$ make the system converge to zero.

Obviously, the ability to analyze the stability of a process is a basic need for the validation of any closed-loop controller. For instance, let us mention the following, related aspects:

Asymptotic stability: do the solutions converge toward the operating point, for sufficiently small initial perturbations? This is the basic, local, asymptotic stability property.

Robustness with regard to the parameters: what are the admissible bound-values of the (constant or varying) parameters that assure the convergence? For delay systems particularly, the question is to know the maximal values of the delays (and, sometimes the minimal one) that keep the stability property. If this bound is infinite, the process exhibits the strong property of independent-of-delay stability (*i.o.d. stability*). But the assumptions for i.o.d. stability, too, may be very strong in practice and it may be preferable to look for delay-dependent conditions (*d.d. stability*), as soon as the user has information about the possible ranges of the delay variations.

Stability domains with regard to the variables: what set of initial conditions will make the state surely converge towards the equilibrium? This question may be meaningless in linear conditions of behavior, but becomes crucial for wide-range, nonlinear models: in this last case, answering is necessary for providing the admissible changes of operating points, or for determining whether bounded additive perturbations on the state may destabilize the closed loop system.

Guaranteed, exponential decreasing rate: what is the exponential rate of convergence, this means, the velocity of the final controlled process? This point aims to compare the behaviour with a first order, ordinary system: it is related to $\alpha-$stability (see below the definition).

Positive invariance: how to be sure that a trajectory will not go out of a predetermined domain of constraints? Such constraints may be introduced on the state (for physical security reasons), or on the control variables (for energy-limiting considerations).

In this section, some stability analysis methods are given with illustrative examples. Starting with some mathematical background on the stability of FDEs, we then propose a classification of the corresponding meth-

ods: the first part applies to linear models (foundations, frequency-domain and root-locus methods, matrix-based methods, complex plane methods, time-varying aspects); a second part deals with time-domain approaches, that are applicable to both linear and nonlinear models (first Lyapunov method, Lyapunov-Krasovkii functionals, Lyapunov-Razumikhin functions, comparison methods). The reader can find some more complete presentation in the previously mentioned books as [62][29] or also in [19]. As for ODEs, the stability property is classically defined for system (3) in free motion ($u \equiv 0$), that is supposed to have an unique solution, with an equilibrium solution at zero:

$$\begin{aligned} \dot{x}(t) &= f(x_t, t), \qquad t \geq t_0, \\ x(\theta) &= \varphi(\theta), \qquad t_0 - h \leq \theta \leq t_0, \\ f(0,t) &= 0, \qquad \forall t. \end{aligned} \tag{19}$$

The solution is denoted $x(t; t_0, \varphi)$ or, briefly, $x(t)$. Mainly, the concepts are the same than for ODEs, but replacing the norm of initial values by some uniform norm of function $\|\varphi\| \triangleq \max_{-h \leq \theta \leq 0} |\varphi(\theta)|$, $|x|$ denoting a norm of vector x.

Definition. *The zero solution of system (19) is:*

1) *Stable if for any $\varepsilon > 0$ and any t_0 there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x(t; t_0, \varphi)| < \varepsilon$ for all $\varphi \in \mathcal{C}$ verifying $\|\varphi\| < \delta$ and for all $t \geq t_0$.*

2) *Asymptotically stable if it is stable and if, for any solution $x(t)$ of the stability problem 1), we have $\lim_{t \to \infty} |x(t)| = 0$.*

3) *Uniformly, asymptotically stable if 1) hold with $\delta = \delta(\varepsilon)$ and 2) holds with uniform limit ($t \to \infty$).*

4) *$\alpha-$stable if 3) holds with the more constraining condition $\lim_{t \to \infty} |e^{\alpha t} x(t)| = 0$, with $\alpha > 0$. The constant α is called a guaranteed (exponential) decay rate of (19).*

5) *Uniformly, asymptotically stable independently of the delays (shortly, i.o.d. stable) if it is uniformly, asymptotically stable for all positive values of the upper bound h.*

In engineering practice, the parameters of a model are known with a finite precision only; then, the model can be considered as the sum of two terms:

$$\dot{x}(t) = f(x_t, t) + \Delta f(x_t, t)$$

where the first part f represents the nominal model, and the second part Δf represents the uncertainties on the model. All we know about this second term is that it belongs to a certain functional family $\mathcal{D} \subset \mathcal{C}$. Generally, the nominal model is linear and uncertainties are described by their bounds: for instance, in case of *unstructured uncertainties*, $\mathcal{D}$ is the set of continuous functions such that $\|\Delta f(x_t, t)\| \leq \delta \|x_t\|$ (norms), and for *structured uncertainties*, $|\underline{F} x_t| \leq |\Delta f(x_t, t)| \leq |\overline{F} x_t|$ (absolute values) with linear, delayed mappings $\underline{F}$ and $\overline{F}$. This yields the following definition.

Definition. *The zero solution of system (19) is robustly (asymptotically) stable with regard to set $\mathcal{D}$ if it is (asymptotically) stable for any $\Delta f \in \mathcal{D}$.*

Basic stability property in the linear case

In what concerns the stability of the linear equations (6), the necessary and sufficient condition (N.S.C.) is also a straightforward generalization of ODE's theory, based on the research of some particular, nontrivial, exponential solution $x(t) = e^{st}$.

Theorem. *The zero equilibrium of retarded system (6), with $C_j(s) = C_j$ constant and $D_k = 0$, is asymptotically stable iff all the characteristic roots (s) of the characteristic function (20), $p(s) = 0$, have negative real parts,*

$$p(s) = \det\left(sI_n - A_0 - \sum_{i=1}^{m} A_i e^{-sh_i} + \sum_{j=1}^{r} C_j \frac{1 - e^{s\tau_j}}{s}\right). \tag{20}$$

Note that in this case there can only be a finite number of unstable roots, which is no more true in the neutral case: if $D_k \neq 0$, the term $-s \sum_{i=1}^{k} D_k e^{-s\omega_k}$ is to be added inside of the determinant. In the complex plane, there may be infinite branches of roots tending to the imaginary axis: conditions based on the sign of the real parts must then be considered with a great care [61]. But, assuming the stability of the difference equation $x(t) + \sum_{i=1}^{k} D_k x(t - \omega_k) = 0$ (cf. (33)) makes them hold [31].

Then, checking such conditions is much harder than in the ODE's case: $p(s) = q(s, e^{-s})$ is not a polynomial in s and there is no equivalent to the Routh-Hurwitz test. Hand-calculating the characteristic roots of the very simple, scalar example

$$\dot{x}(t) = -ax(t) - bx(t-h), \quad x \in R, \tag{21}$$

($s + a + be^{-hs} = 0$) illustrates how difficult it can be to carry a direct analysis of the transcendental equation (20) for systems with certain dimension, or for designing some tuning parameters. We shall see in the following that there are many stability criteria, but none of them gives necessary and sufficient conditions which are simple and practical at the same time[7]. These methods are presented in a synthetic way in [19][23][87][29].

As a general observation concerning delay-dependent and i.o.d. criteria, it is worth noticing that, when applied to the prototypical system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad x \in R^n, \tag{22}$$

the first class needs the matrix $A_0 + A_1$ to be Hurwitz, while the second, of course more constraining, demands this condition for A_0. Lastly remark that the consideration of multiple delays is accompanied by a huge increase of computational complexity (see an evaluation in terms of *NP*-hardness in [113]).

Linear systems: frequency-domain, root-locus

Then, the stability analysis of a linear, time-invariant system with delays is grounded on its characteristic equation (20). Of course, as the characteristic function $p(s)$ depends on the delay, a system may be stable for some set $\{\omega_k, h_i, \tau_j\}$ and unstable for an other set. The extensions of Routh-Hurwitz criterion proposed by Pontryagin (1942) or Chebotarev (1949) [62] are seldom applicable in practice. Besides, the rational approximations (as Padé's ones) are not very relevant, since the study has to be carried up to an undetermined order. Fortunately, there are some interesting methods that allow the analysis of the characteristic equation in a necessary and sufficient way: in addition to the Tsypkin i.o.d. stability criterion[8], let us mention here the Pontryagin method (for commensurate

[7] However, for (21), explicit N.S.C. are known: (21) is asymptotically stable for any value h iff $a + b > 0$ and $a \geq |b|$; it is asymptotically stable for any value of h less than h^* iff $b > |a|$ and $h^* = (b^2 - a^2)^{-1/2} \arccos(-a/b)$.

[8] Restricted to single-delay, open-loop-stable transfert functions

delays), the D-partition approach (dividing the space of the parameters into several regions, which boundaries correspond to critical stability), the methods by τ-partition (for commensurate delays, dividing the study on intervals of delays) as, in particular, the interesting method of Walton and Marshall (1987, commensurate delays, polynomial analysis) or the similar, pseudo-delay approaches (Rekasius 1980, Hertz, Jury and Zeheb 1984, 1987) and methods by Kamen (1980-1983, commensurate delays, i.o.d. conditions, methods based on 2-variable polynomials in (s,z), $z=e^{-\delta s}$).

The general drawback of these necessary and sufficient conditions, restricted to constant delays, is to be uneasy when several parameters are to be tuned. Lastly mention the Chebotarev method, which theory needs to check an infinite number of determinants but, conversely, can be used as a necessary condition of stability. Description and examples are given in [62][19][23][87][29].

Linear systems: methods in the complex plane

Classical stability conditions such as Nyquist or Mykhailov-Leonhard criteria are easily generalizable to systems with delays. Indeed, the argument principle, central core of these criteria, is still applicable since the number of the unstable roots in the complex plane is finite. The induced methods [62][19][87][29] generally apply, in a necessary and sufficient way, to constant but non necessarily commensurate delays. They yield computational difficulties when many combinations are to be checked, with complex parameters.

Linear systems: matrix-based methods

Several results are expressed in terms of sufficient (but non necessary) conditions, involving the matrix measures and norms[9]. Compared with previous frequency approaches, this lack of necessity is compensated by the relative easiness of the implementation. We shall see in a next section, devoted to the comparison approach, that some of these approaches may also hold for nonlinear models. Among the various methods, we just recall here the very representative Mori, Fukuma and Kuwahara criterion (1981), for single-delay systems, which further gave rise to generalized formulations (for instance, Tokumaru *et al.* 1975, Brierley *et al.* 1982, Hmamed 1986, Mori and Kokame 1989, Dambrine and Richard 1993, Kolmanovskii 1995, Goubet *et al.* 1997). These other statements can be found, with examples, in [19][87][29].

Theorem. *The system (22) is i.o.d. stable if* $\mu(A_0)+\|A_1\|<0$. *Moreover, its solution verifies* $\|x(t;0,\varphi)\| \leq \|\varphi\| e^{-\sigma t}$ $(t \geqslant 0)$, *where* σ *is the real solution of equation* $1+\frac{\sigma}{\mu(A_0)}+\frac{\|A_1\|}{\mu(A_0)}e^{\sigma h}=0$.

Other results for commensurate-delays systems, by Chen (1994) and Su (1995), involve generalized eigenvalues[10] and matrix pencils techniques [90]. They need to check matrices of increased order, obtained by sums and products of Kronecker. Delay-dependent or i.o.d. criteria can be obtained (see [29]). The main difficulty is here the high-dimensional computations of large-scale pencils (the dimension is multiplying with the number of delays).

Linear, time varying systems

Except some matrix-based methods, the previous results do not apply anymore if the delay is time-varying. The following example has been shown [52] to be unstable[11]:

$$\begin{aligned} \dot{x}(t) &= -ax(t)-bx(t-h(t)) \\ h(t) &= t-kT,\ \forall t\in\,]kT,(k+1)T] \quad (\text{then } h(t)\leq T) \\ T &= 1, \qquad a=3.5, \qquad b=4, \end{aligned}$$

while for any constant value of the delay $h(t)\equiv h\leq 1$, its characteristic roots have negative real parts. Inversely, for $a=-1$, $b=1.5$, it is asymptotically stable, while linear time-invariant conditions don't hold (see also [19]). Then, even if the works which take into account time-varying delays are rather fewer, they are of practical interest when designing the control of a process which delays variations are actually non negligeable. The next methods (presented for nonlinear systems) have then to be developed and used for this time-varying case, as for the classical ODEs. For instance, the simple system (2) was shown to be asymptotically stable for time-varying $h(t)$ if $h(t)\leq h<1$ (note that this condition is only sufficient). The special case of linear, periodical-time varying delay systems (*i.e.* (7) in free motion with periodic A_i, h_i) received particular attention, with the generalization of the monodromy operators and *characteristic multipliers* encountered in the Floquet-Lyapunov theory (Stokes 1962, Halanay 1966, see [49][62][19]): but, here again, delays imply an increasing complexity.

Nonlinear systems: the first Lyapunov method

The first Lyapunov method [31] still holds for the system

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{k} A_i x(t-h_i)+q(t,x_t) \qquad (23) \\ q(t,x_t) &= q(t,x(t),x(t-\tau_1(t)),\ldots x(t-\tau_k(t)) \\ h_0 &= 0,\ h_i=\text{constant},\ \ \tau_j(t)\in[0,\tau_i]\ \text{continuous}, \end{aligned}$$

with a function q such that, for any u_i, $\|u_i\|\leq\varepsilon \Rightarrow \|q(t,u_0,\ldots,u_k)\|\leq\beta_\varepsilon(\|u_0\|+\ldots+\|u_k\|)$, with constant β_ε uniformly decreasing to 0 as $\varepsilon\to 0$. The "tangent", linearized system is, as usual, defined by

$$\dot{x}(t)=\sum_{i=0}^{k} A_i x(t-h_i). \qquad (24)$$

Theorem. *If system (24) is asymptotically stable, then the zero solution of (23) is, too. If (24) has at least one characteristic root with positive real part, then the zero solution of (23) is unstable.*

This result can be followed by some small-delays approximation theorem (obtained by continuity of the characteristic roots with regard to delays h_i). Here, "small" is to be understood as "sufficiently small".

Theorem. *If* $\sum_{i=0}^{k} A_i$ *is a Hurwitz matrix, then the zero solution of (24) is asymptotically stable for small values of the delays* h_i. *If this matrix is unstable, then the zero solution of (24) is unstable for small values of the delays*

$\frac{e^{-hs}P(s)}{Q(s)+e^{-hs}P(s)}$, the Tsypkin N.S.C. demands polynomials P (degree $n-1$) and Q (stable, degree n) to satisfy $|P(j\omega)|>|Q(j\omega)|$ for all $\omega\in R$.

[9] The measure (or logarithmic norm) $\mu(A)$ of a matrix A, associated to a norm, is $\mu(A)=\lim_{\varepsilon\to 0}\frac{\|I+\varepsilon A\|-1}{\varepsilon}$; matrix norm is $\|A\|=\sup_{x\in R^n}\frac{\|Ax\|}{\|x\|}$. Measure may be negative, norm must be nonnegative.

[10] A generalized eigenvalue of matrices A and B is a complex number λ such that $det(A-\lambda B)=0$ (the number of finite generalized eigenvalue is at most equal to the rank of B).

[11] This kind of time-varying delay corresponds to T-periodic sampling.

h_i. *If* 0 *is a single eigenvalue of this matrix, the other having negative real parts, then the zero solution of (24) is stable for small values of the delays* h_i.

Nonlinear/linear systems: time-domain methods
The next subsections present three approaches based on time-domain, FDEs representations. The direct method of Lyapunov has been extended to FDEs in two different ways: the first one, due to Krasovskii (1963), uses a *functional* generalization of the notion of Lyapunov function; the other one (Razumikhin, 1956) keeps the classical approach of Lyapunov *functions* but applies it to a certain type of solutions. For further details, the reader can refer to [62][87][29]. A third approach is based on *comparison* systems, and will be lastly presented (see details in [19][40][29]).

These very general time-domain approaches apply to both linear and nonlinear systems:

- *in the linear case*, they contribute to many results on *robust stability*, whereas previous necessary and sufficient conditions are quite limited (see a survey of the induced, sufficient conditions in the first chapter of [29]);

- *in the nonlinear case*, they simply appear as the only way to the stability analysis.

Functional approach of Lyapunov-Krasovskii
This section gives a short overview of the first class of the above-mentioned, time-domain methods. In order to extend the Lyapunov's direct method to FDEs, Krasovskii (1963) proposed to consider functionals instead of classical Lyapunov functions. This generalization permits in particular to obtain some converse theorems. It is based on the following, classical result:

Theorem. *System (19) is asymptotically stable if there exists a continuous functional* $V(t,\varphi) : R \times C \longrightarrow R^+$, *which is positive-definite, decrescent, admitting an infinitesimal upper limit, and whose full derivative* $\dot{V}(t,x_t)$ *along the motions of (19) is negative definite over a neighborhood of the origin.*

Among the particular choices of the functional V, several authors proposed stability conditions for linear systems (22) with the following, "generalized quadratic form"

$$V(x_t) = x(t)^T P x(t) + \int_{-h}^{0} x(t+\theta)^T S x(t+\theta) d\theta.$$

This functional, applied to the linear systems, leads to sufficient conditions in the form of Riccati equations, as follows (see for instance [87]).

Theorem. *System (22) is i.o.d. stable if there exist positive-definite, symmetric matrices P, S, R verifying the following, auxiliary Riccati equation*

$$A_0^T P + P A_0 + P A_1 S^{-1} A_1^T P + S + R = 0. \qquad (25)$$

Other i.o.d. conditions [107][29] were formulated in terms of Riccati equations. More complex functionals lead to delay-dependent conditions, available for discrete-single [91], discrete-multiple [29][64] and distributed [63] delays.

Moreover, many such Riccati-type results where translated in terms of linear matrix·inequalities (LMIs, see [8]) [29][91][88][87]. For instance, the condition (25) can be equivalently checked by means of LMIs, as

$$\begin{pmatrix} A_0^T P + P A_0 + S & P A_1 \\ A_1^T P & -S \end{pmatrix} < 0.$$

Note that the major part of the delay-dependent conditions were obtained by using some other formulation of the initial, generally linear system. For instance, the following system,

$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t-h_i). \qquad (26)$$

can be written under the three following forms [64],

$$\dot{x}(t) = Ax(t) - \sum_{i,j=1}^{m} A_{ij} \int_{t-h_{ij}}^{t-h_j} x(s) ds, \qquad (27)$$

$$\dot{x}(t) = Ax(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} \dot{x}(s) ds, \qquad (28)$$

$$\frac{d}{dt}\left[x(t) + \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} x(s) ds \right] = Ax(t), \qquad (29)$$

with notation

$$A = \sum_{i=1}^{m} A_i,\ A_{ij} = A_i A_j,\ h_{ij} = h_i + h_j,\ h = \sum_{i=1}^{m} h_i.$$

Each formulation can be studied by specific Lyapunov-Krasovskii functionals, leading to the three different Riccati equations [64],

$$A^T P + PA + mRh + P \sum_{i,j=1}^{m} h_i A_{ij} R^{-1} B_{ij}^T P = -Q, \qquad (30)$$

$$A^T P + PA + \sum_{i=1}^{m} (h_i P A_i R^{-1} B_i^T P + mh A_i^T R A_i) = -Q, \qquad (31)$$

$$-Q = A^T P + PA + \sum_{i=1}^{m} R_i h_i + \sum_{i,j=1}^{m} A^T P A_i R_i^{-1} A_i^T P A h_i. \qquad (32)$$

Then, the system (26) is asymptotically stable if for some symmetric positive matrices R_i and Q there exists a positive solution, P, of one of the equations (30), (31), (32). Note that, due to the neutral-like nature of the third equation (29), an additional condition must be assumed with (32): $x(t) + \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} x(s) ds = 0$ is asymptotically stable (see the section "neutral systems").

However, in the general, nonlinear case, finding a suitable functional V can be compared... *to an art!* [13][60] This question, already encountered with ODEs models, is enforced for FDEs. A formal procedure to construct Lyapunov functionals V for concrete equations with delay was proposed by Kolmanovskii [60][58]. Basic features of this procedure are as follows: represent the right-hand side of the equation as a sum of two terms, first of which has the form of an instantaneous negative feedback; construct a Lyapunov function v for the auxiliary ordinary differential equation corresponding to the first term; obtain functional V from by change of the arguments of v. Note that various steps of the procedure can be implemented non-uniquely.

The Lyapunov-Razumikhin approach
Because of the complexity of the construction of a Lyapunov functional for nonlinear models, Razumikhin (1956) proposed another generalization of Lyapunov second method, keeping the idea of Lyapunov *functions* $V(x(t))$ (and not functional $V(x_t)$). The great difference is that the derivative of the chosen Lyapunov function has to be negative only for special solutions of the system (very roughly speaking, the idea is to check the sign of

the derivative $\dot{V}$ only when the state function may go out of a set $V(x) = \text{constant}$).

Theorem. *Let $u(\rho)$, $v(\rho)$, $w(\rho)$ and $p(\rho)$ $(R^+ \to R^+)$ be continuous, nondecreasing functions, positive for $\rho > 0$, $u(0) = v(0) = 0$ and $p(\rho) > \rho$ for $\rho > 0$. If there is a continuous function $V : R \times R^n \to R$ such that $u(\|x\|) \leq V(t,x) \leq v(\|x\|)$ for any (x,t), and $\dot{V}(t,x(t)) \leq -w(\| x(t) \|)$ for states x_t verifying $\{\forall\theta \in [-h,0], V(t+\theta, x(t+\theta)) < p(V(t,x(t)))\}$, then the zero solution of (19) is uniformly asymptotically stable.*

A practical corollary was given in [117], changing the last condition into: $\dot{V}(t,x(t)) \leq -w(\| x(t) \|)$ *for states x_t verifying $\forall\theta \in [-h,0]$, $\| x(t+\theta) \| < \eta \| x(t) \|$ for an $\eta > 1$.*

The LaSalle principle

The invariance principle of LaSalle (1960) is a well known extension of the Lyapunov functions theory, that allows to study of asymptotic behavior of ODEs solutions (in particular, the boundedness properties). It involves the notion of positive invariance of sets, that can be easily generalized to FDEs. Then, the LaSalle invariance principle was extended to retarded, time-invariant systems, by using either the Krasovskii functional procedure (Hale, 1965 [49]) or the Razumikhin functions one [47]. We don't give here the statements (see for instance [19]).

The comparison approach

The direct stability analysis of a complex system often remains too cumbersome or even can be impossible to perform. An alternative, indirect way is to proceed *via* a simpler system, called *comparison system*. This notion was originally defined for ODEs [81] and then, extended to FDEs [68]. Firstly, we present a wide definition of the idea.

Definition. *A system $(\mathcal{A})$ is said to be a comparison system of a system $(\mathcal{B})$ with regard to the property $\mathcal{P}$ (for example, stability of its zero solution), if the verification of property $\mathcal{P}$ for system $(\mathcal{A})$ implies the same property for system $(\mathcal{B})$.*

For instance, the first-order approximation of a nonlinear ordinary differential equation may be viewed as a comparison system with regard to the local, uniform asymptotic stability. However, most of comparison systems rely on differential inequalities [68][58] and vector-Lyapunov functions[12] [20][21][19][42], which tools constitute the framework of the approach. Major part of the referenced results use a Razumikhin approach in their proof. The next definition is a continuation of the previous one.

Definition. *Let $V : R^n \to R_+^k$ (with $k \leq n$) be a continuous, positive function such that $V(x) = 0 \Leftrightarrow x = 0$. Assume that, along the solutions of (19), the right-hand time-derivative (Dini derivative) of $y(t) = V(x(t))$ satisfies the functional differential inequality $D^+y(t) \leq \mathcal{F}(t, y_t)$. Then system $\dot{z}(t) = \mathcal{F}(t, z_t)$ is an overvaluing system of (19) with respect to the function V if the inequality $V(x(t)) \leq z(t)$ holds for any $t \geq t_0$ as soon as it holds for initial times $t \in [t_0 - h, t_0]$.*

Using the assumptions made on V, it is simple to prove that an overvaluing system is also a comparison system with regard to stability or asymptotic stability. Conditions on functional $\mathcal{F}$ to be an overvaluing system are called comparison principles (as, in ODEs case, the so-called Ważewski conditions): some of them, very general, are recalled in [19][29]. A particular but interesting comparison principle can be mentioned here for illustration, providing an exponential convergence rate γ (this lemma was proved in [114] for a single-delay inequality, and under this two-delay form in [42]).

Lemma. *Let C, D_1 and D_2 be $n \times n$ matrices with real entries and let $x(t)$ be a solution of the differential inequality $(t \geq 0)$,*

$$\begin{aligned} \dot{x}(t) &\leq g(x_t), \\ g(x_t) &= -Cx(t) + D_1 \sup_{0\leq\theta\leq h_1} x(t-\theta) + D_2 \sup_{0\leq\theta\leq h_2} x(t-\theta). \end{aligned}$$

Assume that $D_1 \geq 0$, $D_2 \geq 0$, that the off-diagonal entries of C are non positive, and that $(-C + D_1 + D_2)$ is the opposite of an M-matrix[13]. Then the solution $x(t)$ of this inequality is overvalued by the asymptotically stable *solution $z(t)$ of the differential equation $\dot{z}(t) = g(z_t)$, $t \geq 0$, with initial condition $0 \leq x(\theta) \leq z(\theta)$ for $h \leq \theta \leq 0$ $(h = -\max\{h_1, h_2\})$.*

If in addition $(-C + D_1 + D_2)$ is irreducible, then there is a constant $\gamma > 0$ and a constant vector $k_\gamma > 0$ such that $x(t) \leq k_\gamma e^{-\gamma t}$ for $t \geq 0$. Here, γ and k are obtained in the following way: γ is the positive real solution of the equation $\lambda_m(A_\gamma) = -\gamma$, where $A_\gamma = -C + D_1 e^{\gamma h_1} + D_2 e^{\gamma h_2}$. k_γ is a positive, importance eigenvector of A_γ associated with the importance eigenvalue $\lambda_m(A_\gamma)$.

Vector-norms (each entry of V is a scalar norm of a subvector x_i of x) constitute a particular case of the general (but hard to solve) vector-Lyapunov function(al)s: they lead to systematic determination of comparison systems in many cases of FDEs [19][40][29][111]. Applying this tool on a nonlinear system with a single delay h, a systematic construction of matrices C, D_1, and possibly D_2 (with $h_1 = h$, $h_2 = 2h$) is given in [42][29], leading to the following, simple conditions ($|.|$ denotes here the entry-to-entry absolute value of vectors or matrices, M^* denotes the matrix obtained from M by replacing all its off-diagonal entries by their absolute values).

Theorem. *The zero equilibrium of the uncertain system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-h(t)) \\ &\quad + f(x(t),t) + g(x(t-h(t),t) \\ |f(x,t)| &\leq F|x|, \quad |g(x,t)| \leq G|x|, \\ h(t) &\leq h, \quad B = B' + B'' \end{aligned}$$

is asymptotically stable if the matrix $M = (A + B')^* + |B''| + F + G + h[|B'A| + |B'B| + |B'|(F+G)]$ is Hurwitz.

This result was accompanied with the determination of positively invariant sets, and convergence rate. Note that it does not need A to be Hurwitz, but $A+B'$. Such a result is very closed to the matrix-based methods seen in the linear case[14]. Recently, results using the same comparison

[12] Vector-Lyapunov functions were simultaneously introduced for ODEs in [4] and [81].

[13] a matrix A is the opposite of an M-matrix if all its off-diagonal elements are non-negative and if A is Hurwitz. This latter condition may be easily tested by verifying that all its successive, principal minors are negative. Such matrix A has a real eigenvalue $\lambda_m(A)$ which is greater than the real parts of all others. $\lambda_m(A)$ is called the importance eigenvalue of A. If A is irreductible (*i.e.* if it is not similar to a bloc-triangular matrix), then there is an associated, importance eigenvector u_m of A verifying $u_m > 0$.

[14] For unstructured perturbations $\|f(x,t)\| \leq \alpha\|x\|$, $\|g(x,t)\| \leq \beta\|x\|$, the condition can be stated in terms of measures as $\mu(A + B') + \|B''\| + \alpha + \beta + h(\|B'A\| + \|B'B\| + \|B'\|(\alpha+\beta)) < 0$.

approach for discrete-plus-distributed delay systems have also been obtained [111] on the basis of transformations such as (27).

This shows that some comparison results are directly workable: even if the main question with such procedure may be its non-uniqueness (dependence with regard to the chosen state basis and to the decomposition $B = B' + B''$), it provides information about both qualitative and quantitative aspects that were presented at the beginning of this section devoted to stability [29]. In our opinion, this simplicity, compared with the wideness of the admissible models and possible applications, constitutes the main point of the comparison approach.

The case of neutral systems

The above mentioned functional approach is also fruitful for neutral systems [108][62][50][61]. In this case, the procedure can be generalized to a bit more complex one, involving functionals $V(Fx_t)$ with notation referring to (5) and the stability of the operator F also has to be checked: F is stable if the zero solution of the equation $Fx_t = 0$ is uniformly, asymptotically stable. For instance, considering the (usual) case $Fx_t = x(t) - Dx(t-h)$, with constant matrix D, a necessary stability condition for the linear, neutral system

$$\dot{x}(t) - D\dot{x}(t-h) = \sum_{i=0}^{k} A_i x(t-h_i) \qquad (33)$$

to be stable is that D has eigenvalues inside the unit circle (or, equivalently, is Schur-Cohn stable).

Theorem. *Consider the equation*

$$\frac{d}{dt} Fx_t = f(x_t), \qquad (34)$$

with $f : \mathcal{C} \to R^n$ taking bounded sets of $\mathcal{C}$ into bounded sets of R^n. Suppose F is stable, and that $u(\rho)$, $v(\rho)$, $w(\rho)$ are continuous and nondecreasing functions, cancelling at $\rho = 0$ and positive elsewhere. If there is a continuous function $V : R \times \mathcal{C} \to R$ such that $u(\|F\varphi\|) \leq V(t,\varphi) \leq v(\|\varphi\|)$ and, along the motions, $\dot{V}(t, x_t) \leq -w(\| x(t) \|)$, then the zero solution of (34) is uniformly asymptotically stable.

On this basis, Riccati equations can be constructed [108][29] similarly to the retarded case. In 1979, Kolmanovskii and Nosov [62] also defined the principles of *f-stability* and *degenerate functionals* ($V(x_t)$ is said to be degenerate because it may cancel even if the function x_t is not identically zero) for stability study of nonlinear, neutral equations. Results based on the comparison approach were given by Tchangani *et al.* [29][112], together with estimates of the stability domains and asymptotic-behaviors bounds.

Stabilization

Many studies are devoted to stabilization of time-delay systems. The previous stability criteria are of course directly involved in such control study, but some of them are more useful regarding to the kind of stabilization problem.

Concerning the linear, time-invariant models, the methods are related to the controllability properties, with a great interest in the *finite-spectrum assignment* problem. Since the stability tests are to be made on the characteristic equation (by previously presented N.S.C.), they are much simpler in the particular case of finite-spectrum assignment, since the aim is then to obtain a polynomial equation (hence, with finite number of roots). This aspect will be presented in the following section "Control".

Concerning *robust stabilization* of linear models with constant or nonlinear, time-varying parametric uncertainties (see for instance [29]), the methods are mainly based on the time-domain Krasovskii approach or on the comparison approach; both allow one to deal with time-varying delays, whereas the frequency-domain and complex-plane methods generally need the delays to be constant.

The problem of stabilization with *input-disturbances* can be treated by means of H_∞ norms: this involves time-domain approaches, mainly the Krasovskii generalized quadratic functionals, leading to Riccati equations or LMIs (see for instance [120][87][88] and references herein).

It is to be mentioned that the main part of these two last categories (robustness-type results) are dealing with systems with memoryless input (*i.e.* no delay on the control), which imposes a real restriction: the delay phenomenon is often induced by the actuators or sensors. A possible solution to this problem consists in introducing an integrator in the control: the simple system $\dot{x}(t) = u(t-h)$ is then transformed by ($u(t) = x_2(t)$) into:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-h), \\ \dot{x}_2(t) &= v(t). \end{aligned}$$

Lastly, *constrained stabilizing control* is mainly grounded on the positive-invariance property, which is a little bit more difficult to handle in functional spaces. Several results have been obtained in the case of linear delay systems [89][43][51] and nonlinear ones [22][29].

4. Structural properties

Controllability and observability of delay systems have been studied through different modelling approaches. Two large classes of properties can be distinguished:

- the *functional* ones aim to reach a function $\phi \in \mathcal{C}$ at time t, this means to make the behavior reach some predetermined function $x_t \in \mathcal{C}$; they mainly correspond to the infinite-dimensional models; among them, the *spectral* properties only concern the eigenvalues, thus, problems of stabilization or observation.

- the *point-wise* ones consider the problem of reaching the point $x \in R^n$ (solution at a given time); they can be studied through all the above-mentioned classes of models.

Many authors contributed to this study: surveys can be found in [105][67][57]. Correspondences between different properties in an unifying framework (in the module theory) were given by Fliess and Mounier [84][33][76]. In this paper, we shall mainly deal with the notions related to controllability.

Functional controllability properties

Infinite-dimensional models as (11) received several controllability definitions [25][80], that are of *functional* type. The following ones refer to system (11) with solution (12), and are also called "*approximate controllability*":

Definition. *The state $\overline{x}_0$ is $\mathcal{M}_2$-controllable at time t to $\overline{x}_1 \in \mathcal{M}_2([-h,0]; R^n)$ if there is a sequence of controls $\{u_i\}$ defined in $\mathcal{L}_2([0,t]; R^m)$ such that $\overline{x}(t; \overline{x}_0, u_i)$ converges to $\overline{x}_1$ (in the sense of the norm over $\mathcal{M}_2$). The system (11) is $\mathcal{M}_2$-controllable at time t if all states $\overline{x}_0$ are $\mathcal{M}_2$-controllable at time t to any $\overline{x}_1 \in \mathcal{M}_2([-h,0]; R^n)$.*

$\mathcal{M}_2$-controllability was characterized by N.S.C. [25] but the conditions are not so easy to check. This notion corresponds to a restricted notion (*approximate*) if ones compares it with the *$\mathcal{M}_2$-strict controllability at time t* defined in [25]: it only involves the *limit* trajectories that can be obtained by sequences $\{u_i\}$, because the domain of definition of operator $\widetilde{A}$ is strictly included in $\mathcal{M}_2$ (only its adherence is equal to $\mathcal{M}_2$), while the strict notion needs a unique, concrete control law u to exist. Other forms of controllability were defined [80] for $\overline{x}_0 = 0$.

For systems with delayed control but without delay on the state variables, the notion of *absolute controllability* was defined (together with simple N.S.C.) in [93].

Definition. *The linear system with commensurate delays (8) with the restriction $\forall k \geq 1$, $A_i = 0$ is absolutely controllable if, for any initial condition $\{x_0,\ u(t)_{t\in[-k\delta,\ 0]}\}$, there is a time $t_1 > 0$ and a bounded control law $u(t)$ such that $x(t_1) = 0$ with $u(t) = 0$ for all $t \in [t_1 - k\delta,\ t_1]$.*

Theorem. *The system (8) with $A_i = 0\ \forall k \geq 1$, is absolutely controllable iff* $\operatorname{rank}\left[E, A_0E, ..., A_0^{n-1}E\right] = n$, *with* $E = \sum_{i=0}^{k} e^{-i\delta A_0} B_i$.

Absolute controllability is actually a functional property, since it implies the ability of maintaining $x(t)$ at zero on a time interval $[t_1, t_1 + k\delta]$. However, the main problem is contained in its very demanding definition, needing $u(t) = 0$ for all $t \in [t_1 - k\delta,\ t_1]$: such an "ending free-motion" is too constraining in general.

An other property was defined by Weiss: the R^n-functional controllability [121][93] (see also in [105]), in which there is not this zero-input constraint. The definition for a single-delay system without delay on the input is as follows.

Definition. *The linear system (10) is (ψ, R^n)-controllable (with regard to some function $\psi \in \mathcal{C}$) if, for any initial condition $\varphi \in \mathcal{C}$, there is a finite time $t_1 > 0$ and a control law $u(t) \in \mathcal{L}_2\left([0, t_1 + h], R^m\right)$ such that $x(t; \varphi, u) = \psi(t - t_1 - h)$ for all $t \in [t_1,\ t_1 + h]$.*

This property can be checked by generalizing the notion grammian [121] as in equation (35).

Theorem. *The linear system (10) is $(0, R^n)$-controllable (i.e. with regard to $\psi = 0$) if*

1) *there is a finite time $t_1 > 0$ such that*

$$\operatorname{rank}\left(\int_0^{t_1} F(t_1 - \theta) B_0 B_0^T F(t_1 - \theta)^T d\theta\right) = n, \qquad (35)$$

with $F(t)$ solution of $\dot{F}(t) = A_0F(t) + A_1F(t-h)$, $F(0) = I$, $F(t_1) = 0$,

2) *the equation $A_0x(t-h) + B_0u(t) = 0$, $t \in [t_1, t_1 + h]$ has a solution $u(t) \in \mathcal{L}_2\left([t_1, t_1 + h], R^m\right)$.*

Condition 1) ensures R^n-point-wise controllability at time t_1, whereas condition 2) allows to maintain the solution at the origin after t_1. Condition 1) can be replaced by simpler point-wise controllability conditions, which are recalled in the next subsection.

Spectral properties

The following spectral properties, as we shall see, constitute very interesting bases for effective control of linear systems. Spectral controllability can be seen as a functional controllability property, but it only applies to the problem of controlling the *spectrum* of the linear system (8) with model over ring (13),

$$\sigma(A) = \left\{s \in C,\quad \det\left(sI - \mathbf{A}(e^{-\delta s})\right)\right\}, \qquad (36)$$

in such a way it belongs to some region of the complex, left half-plane. Of course, spectral properties concern the problem of stabilization (functional controllability to zero), but they also have been related to behavioral properties [103][102]. We shall not consider here the infinite-dimension models, however the spectral properties can also be tested in this framework (see [57]).

Definitions. *The system (8) or (13) is spectrally controllable if, for any $s \in C$,*

$$rank\left[sI - \mathbf{A}(e^{-\delta s}),\ \mathbf{B}(e^{-\delta s})\right] = n. \qquad (37)$$

It is spectrally observable if, for any $s \in C$,

$$rank\left[sI - \mathbf{A}(e^{-\delta s})^T,\ \mathbf{C}^T(e^{-\delta s})\right] = n. \qquad (38)$$

It is stabilizable if there exists a causal control law which makes it asymptotically stable. It is detectable if there exists a causal, asymptotic observer of the solution $x(t) \in R^n$.

Theorem. *The system (8) is stabilizable iff (37) holds for any $s \in C$, $\operatorname{Re}(s) \geq 0$. It is detectable iff (38) holds for any $s \in C$, $\operatorname{Re}(s) \geq 0$.*

This result was proved in a constructive way in [9], which work also studied the question of the realization:

Theorem. *Any causal transfer matrix on $R(s, e^{-\delta s})$ has a realization that is stabilizable and spectrally observable. It also has a realization that is detectable and spectrally controllable.*

It was also remarked that in what concerns delay systems, the notion of *minimal realization* (in the sense of spectral controllability and spectral observability) does not always exist (the transfer $\frac{1+e^{-2s}}{s+e^{-s}\pi/2}$ was taken as example).

Point-wise controllability properties

Many other works have been devoted to point-wise structural properties: a main one is the so-called *euclidean-space controllability*, or *R^n-controllability*, that means, let us recall it, defined for trajectories considered in the vector-space R^n. These works probably started with Kirillova, Churakova and Gabasov (1967), Buckalo (1968), Weiss (1970), Zmood (1974) (see [105]). All approaches use the same definition, but lead to different conditions (sometimes equivalent). We recall here the definition for linear systems with single delay (note it corresponds to the notion of *reachability*), but it can easily be extended to multiple delay systems with input delays, when the delays are commensurate.

Definition. *The linear system (8) is R^n-controllable at time t_1 if, for any initial condition $\varphi \in \mathcal{C}$ and $x_1 \in R^n$, there is a time $t_1 > 0$ and a control law $u(t) \in \mathcal{L}_2\left([0, t_1], R^m\right)$ such that $x(t_1; \varphi, u) = x_1$.*

It is R^n-controllable if there exists such a time t_1.

It is strongly R^n-controllable if it is R^n-controllable at any time $t_1 > 0$.

If the above R^n-controllability property is restricted to $x_1 = 0$, then the system is R^n-controllable to the origin.

Then, the evolution of the trajectory after t_1 is not constrained by these definitions (the trajectory may not stay at x_1, contrarily to the non-delayed case and to functional

controllability). Two other differences with ODEs are to be remarked: the time t_1, in general cannot be smaller than the delay δ (except in the rare case of *strong* controllability) and the R^n-controllability is not equivalent to the R^n-controllability to the origin (the difference corresponds to the so-called *point-wise completeness*, which additional property makes the two definitions equivalent. Completeness can be checked by matrix-type N.S.C. due to Zwerkin (1971).

Many criteria give R^n-controllability conditions (for instance, eqn. (35) is a condition due to Weiss). The basic one uses the Kirillova-Churakova operators (9):

Theorem. *The single-delay system (10) is R^n-controllable to the origin if* [15]

$$n = \text{rank}[P_0(0)B_0, P_1(0)B_0, P_1(1)B_0, P_2(0)B_0, P_2(1)B_0, P_2(2)B_0, ..., P_{n-1}(n-1)B_0].$$

Controllability over rings

The following, point-wise notions are of algebraic type and are detailled in [67][105][96]: roughly speaking, *strong controllability* implies the existence of a non-anticipative feedback control based on the past values of the solution, *i.e.* $x(t)$, $x(t-\delta)$, $x(t-2\delta)$,... that one can say to be of "polynomial type". *Weak controllability* just needs a "rational" feedback to exist, and the resulting control law may be anticipative (thus, non realizable). A link between this form of controllability and the subsystems description [93][115] is given in [96].

Definition. *System over ring (13) is controllable over the ring $R[\nabla]$ or "strongly controllable", if there exists a control law of polynomial type $u(t) = f(x, \nabla x, \nabla^2 x, ...)$ allowing to reach, any element of the module $R^n[\nabla]$ from any initial state $x_0 \in R^n[\nabla]$. It is controllable over the field $R(\nabla)$ or "weakly controllable", if there exists a control law of rational type $u(t) = f(x, \nabla x, \nabla^2 x, ..., \nabla^{-1} x, \nabla^{-2} x,)$ allowing to reach any element of the module $R^n[\nabla]$ from any initial state $x_0 \in R^n[\nabla]$.*

The following theorem (see a more complete version in [67]) uses the notations

$$\langle \mathbf{A}/\mathbf{B} \rangle = [\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, ..., \mathbf{A}^{n-1}\mathbf{B}],$$

and $\langle \mathbf{A}/\operatorname{Im}\mathbf{B} \rangle$ for the controllability submodule associated to the pair $(\mathbf{A}, \mathbf{B})$, *i.e.*

$$\langle \mathbf{A}/\operatorname{Im}\mathbf{B} \rangle = \operatorname{Im}\mathbf{B} + \mathbf{A}^2 \operatorname{Im}\mathbf{B} + ... + \mathbf{A}^{n-1} \operatorname{Im}\mathbf{B}.$$

Theorem. *The following, equivalent conditions are necessary and sufficient for system (13) to be strongly controllable –i.e. over $R[\nabla]$:*

1) $\langle \mathbf{A}(\nabla)/\operatorname{Im}\mathbf{B}(\nabla) \rangle = R^n[\nabla]$;

2) *the Smith form of* $\langle \mathbf{A}(\nabla)/\operatorname{Im}\mathbf{B}(\nabla) \rangle$ *is* $[I_{n\times n} \mid 0]$;

3) $\text{rank}[sI - \mathbf{A}(z) \mid \mathbf{B}(z)] = n$ *for all s and z in C.*

Theorem. *The following, equivalent conditions are necessary and sufficient for system (13) to be weakly controllable –i.e. over $R(\nabla)$:*

1) $\text{rank}\langle \mathbf{A}(\nabla)/\mathbf{B}(\nabla) \rangle = n$;

2) *all the diagonal elements of the Smith form of* $\langle \mathbf{A}(\nabla)/\operatorname{Im}\mathbf{B}(\nabla) \rangle$ *are nonzero*;

3) $\text{rank}[sI - \mathbf{A}(z) \mid \mathbf{B}(z)] = n$ *for all s and at least one z in C.*

[15] *This condition is also necessary if the system is point-wise complete.*

In the first statement, condition 2) means that $\langle \mathbf{A}(\nabla)/\mathbf{B}(\nabla) \rangle$ is $\mathcal{R}_u(\nabla)$-closed (see footnote 3). In the second statement, it is not.

In the same framework of models over rings, the notion of *controllability indices* has been extended to delay systems by Sename, Picard and Lafay [105][106][96], giving interesting informations about the smallest time t_1 that is needed for given state variables (controllability submodules) to reach the expected value x_1 in R^n. This question is clearly illustrated by the very simple example [105] $\dot{x}_1(t) = u_1(t)$, $\dot{x}_2(t) = u_2(t-h)$ where different, minimal delay times are needed for the control of x_1 and x_2. Then, each of these indices are represented by a class and an order in this class: class reflects the minimal delay, whereas order corresponds to the classical notion for systems without delay (*i.e.*, the lengthes of controllability chains). Controllability submodules are also associated (see details in [96]).

Relations between the controllabilities

The survey paper [67] makes appear the following implications (and other additional ones, using the notion of torsion submodules):

Theorem. *In the case of a linear system with commensurate delays (8), the following implications hold:*

1) *Strong controllability, over $R[\nabla]$ $\Rightarrow$ Absolute controllability $\Rightarrow$ Weak controllability, over $R(\nabla)$ $\Rightarrow$ R^n-controllability.*

2) *Approximate controllability $\Rightarrow$ Spectral controllability $\Rightarrow$ Weak controllability, over $R(\nabla)$.*

Remark that it makes conclude that strong controllability is a very demanding property: in fact, it means that the system can be controlled as if it was not including any delay.

Remarks on the observability

The previous notions of controllability (strong, weak, spectral,...) can be transposed to observability (see [96][99] and references herein). Indexes and classes can be used for determining the minimum time needed by an observer to construct the point $x(t)$. General solution can be obtained for retarded systems by means of realizations over $\mathcal{R}_u(\nabla)$ but, in what concerns neutral systems, this problem of reconstruction is still open [96].

5. Control

Since the Smith "posicast control" (1957) and predictor (1959), control of delay systems has been widely considered. A great part of the practices was based on approximation methods, which are not necessarily convenient when significant uncertainties -including delays variations- are involved in the process. We have previously mentioned some approaches in the subsection "Stabilization" but, of course, the quasi-totality of the control methods received attempts of generalization. This section will provide a glance over some present trends.

Spectrum assignment

In the 70s, some papers emphasized the interest of using distributed-delays controllers for discrete-delays plants [7][83][54][80][79] (see also [118][120][9]): such operators, placed in the feedback loop, allow a reduction of the spectrum $\sigma(A)$ (36) to a finite set. Contrarily to the problem (initiated by Osipov in 1965, see [79]) of shifting an ar-

bitrary but finite number of eigenvalues, finite-spectrum assignment does not require the preliminary knowledge of the spectrum $\sigma(A)$; moreover, stability of the closed loop is easy to check, since the characteristic function (20) becomes a polynomial. The following simple, scalar example [9] can illustrate the idea:

$$\dot{y}(t) = y(t) + u(t-1), \qquad \frac{Y(s)}{U(s)} = \frac{e^{-s}}{s-1},$$

$$u(t) = -2\int_0^1 e^{\theta} u(t-\theta)d\theta - 2ey(t) + v(t).$$

Here, from (16), the control $u(t)$ achieves a finite-spectrum assignment at $s = -1$. The following result was proved (necessity [79], sufficiency [119]):

Theorem. *The system is n-assignable iff it is spectrally controllable.*

Several algorithms followed, proposing calculation of the corresponding feedback in the general case. A complete algebraic formalism was recently proposed [9][10][75], based on the set of pseudo-polynomials $\mathcal{E}$ (see before in section "Algebraic formalism"). The sketch of solution is as follows: if the expected finite spectrum is defined by the polynomial equation $\phi(s) = 0$, $\phi \in R[s]$, if the process is described by $Y(s)/U(s) = p(s)/q(s)$, $p, q \in R[s, e^{-s}]$, and if the control is to be calculated as $U(s) = p_c(s)/q_c(s)$, $p_c, q_c \in \mathcal{E}$, then the problem has a solution if $qq_c + pp_c = \phi$ has a solution. This last condition holds because $\mathcal{E}$ is a domain of Bézout. On these bases, the robustness aspects now remain to be studied.

Some other control aspects

Many control problems can be studied by means of models over rings, then for the synthesis of discrete-delayed feedback laws: *disturbance decoupling* [16] and *block-decoupling* [17], *model matching* [98], *pre-compensators design* [97]. Some overview and results can be found in [96].

Concerning *optimal control*, many results and references can be found in [65] (also considering stochastic FDEs). [36][26] considered some approaches by approximation of infinite-dimensional Riccati equations, and additional results and references on LQG control are given in [95].

Self-adjusting control with reference model (with identification problem) is considered in [65]. *H_∞-robustness* results and references can be found in [120][88]. *Feedback linearization* of delay systems was considered in [35], and *vibrational control* in [72][29]. *Constrained control* (with invariant or saturated control) was considered in [22][89][29][43][51][2], and *deadbeat control* of ODEs by means of delays can be found in [120]. Lastly mention some trends in CRONE *control* (french abbreviation of "robust control with non-integer order of derivation") [53].

Of course, if control can be studied with some success in the case of unperturbed, linear, time-invariant models, it is clear that, in more complex cases, the domain remains widely opened.

6. Conclusion

This overview of three aspects of delay systems - modelling, stability and controllability- makes appear four points of view:

1- the *functional* point of view: FDEs and infinite-dimensional models, Lyapunov-Krasovskii functionals for stability, functional and spectral controllability properties;

2- the *point-wise* one: models over rings, Lyapunov-Razumikhin functions for stability, point-wise R^n-controllability properties;

3- the *approximative* one: mainly based on classical, finite-dimension simplifications, followed by usual criteria for ordinary differential systems.

Only the two first approaches do take into consideration the specific characters of delay systems. Roughly speaking, the first class is the only one that allows to consider nonlinear behaviors.

Acknowledgments

In 1993, several teams (mainly in France, but also in Europe, Russia and US) initiated a working group cooperation on delay systems, which movement was encouraged by the french CNRS (*Centre National de la Recherche Scientifique*). The author would like to acknowledge all the colleagues of this group and specially Doctor M. Dambrine, Professors V. B. Kolmanovskii and J.-F. Lafay for fruitful discussions. An other particular emphasis is given to the contributions of Doctors S. I. Niculescu, O. Sename and D. Bethé: their Ph.D. manuscripts highly contributed to the present paper.

References

[1] Abdallah G., Dorato P., Benitez-Read J., Byrne R., *Delayed positive feedback can stabilize oscillatory systems*, Proc. American Control Con., p. 3106-3107, 1993.

[2] Assan J., Lafay J.-F., Perdon A.-M., *On $(A + BF)$-invariance for systems over a Principal Ideal Domain*, Proc. 36th IEEE Conf. Decision and Control, San Diego, CA, 1997.

[3] Banks H. T. and Kappel F., *Spline approximations for functional differential equations*, J. Differential Equations, No. 34, p. 496-522, 1979.

[4] Bellman R., *Vector Lyapunov Functions*, J. SIAM Control, Ser. A, No. 1, p. 33-34, 1962.

[5] Bellman R. and Cooke K. L., *Differential difference equations*, Academic Press, New York, 1963.

[6] Bensoussan A., Da Prato G., Delfour M. C., Mitter S. K., *Representation and control of infinite dimensional systems*, Systems & Control: foundation and applications, Vol. 1&2, Birkhäuser, Boston, 1993.

[7] Bhat K. P. M. and Koivo H. N., *Modal characterization of controllability and observability in time-delay systems*, IEEE Trans. Aut. Control, Vol. 21, No. 2, p. 292-293, 1976.

[8] Boyd S., El Ghaoui L., Feran E., Balakrishnan V., *Linear matrix inequalities in system and control theory*, SIAM studies in Applied Math., 15, 1994.

[9] Bethé D., *Contribution à l'étude de la stabilisation des systèmes linéaires à retards*, Ph.D. *(in French)*, IRCyN, Université de Nantes, Ecole Centrale de Nantes, December 1997.

[10] Bethé D. and Loiseau J.-J., *A result that could bear fruit for the control of delay-differential systems*, Proc. 4th IEEE Mediterranean Symposium

on New Directions in Control and Automation, MSCA'96, Maleme, Krete, Greece, June 10–13 1996.

[11] BRETHÉ D. AND LOISEAU J.-J., *Stabilization of time-delay systems*, Proc. IFAC Conf. System Structure and Control, Nantes, France. p. 115-128, 1996.

[12] BULTHEEL A. AND VAN BAREL M., *Padé techniques for model reduction in linear system theory: a survey*, Journal of Computational Applied Mathematics, 14, p. 401-438.

[13] BURTON T. A., *The Volterra integral and differential equations*, Academic Press, New York, 1983.

[14] BURTON T. A., *Stability and periodic solutions of ordinary and functional differential equations*, Academic Press, Orlando, Vol. 178, 1985.

[15] CONTE G. AND PERDON A.-M., *Systems over a principal ideal domain: a polynomial approach*, SIAM J. Control and Optimization, Vol. 20, No. 1, p. 112-124, 1982.

[16] CONTE G. AND PERDON A.-M., *The disturbance decoupling problem for systems over a ring*, SIAM J. Control and Optimization, Vol. 33, No. 3, 1995.

[17] CONTE G. AND PERDON A.-M., *Noninteracting control problems for delay-differential systems via systems over rings*, Proc. of the CNRS Conf.. "Analyse et commande des systèmes avec retards", Nantes, France, p. 101-114, 1996 [76].

[18] CRYER C. W., *Numerical methods for functional differential equations*, in *Delay and functional differential equations and their applications*, Academic Press, New York, 1972.

[19] DAMBRINE M., *Contribution à l'étude de la stabilité des systèmes à retards*, Ph.D. *(in French)*, Laboratoire d'Automatique et d'Informatique Industrielle de Lille, Ecole Centrale de Lille, October 1994.

[20] DAMBRINE M. AND RICHARD J.-P., *Stability analysis of time-delay systems,* Dynamic Syst. & Applications, No. 2, p. 405-414, 1993.

[21] DAMBRINE M. AND RICHARD J.-P., *Stability and stability domains analysis for nonlinear, differential-difference equations,* Dynamic Syst. & Applications, No. 3, p. 369-378, 1994.

[22] DAMBRINE M., RICHARD J.-P. AND BORNE P., *Feedback control of time-delay systems with bounded control and state*, J. Mathematical Problems in Engineering, first issue, 1, p. 77-87, 1995.

[23] DAMBRINE M., NICULESCU S. I., GOUBET-BARTHOLOMÉÜS A., DUGARD L., DION J.-M., RICHARD J.-P., *Stabilité et stabilisation des systèmes à retards: un tour d'horizon,* (in French) Proc. of the CNRS Conf. "Analyse et commande des systèmes avec retards", Nantes, France, p. 79-100, 1996.

[24] DELFOUR M. AND KARRAKCHOU., *State-space theory of linear time invariant systems with delays in state, control and observation variables*, parts I&II, J. Math. Anal & Appl., Vol. 125, p. 361-452, 1987.

[25] DELFOUR M. AND MITTER S., *Controllability, observability and optimal feedback control of affine, hereditary, differential systems*, SIAM J. Contr. Optim., No. 10, p. 298-328, 1972.

[26] DE SANTIS A., GERMANI A., JETTO L., *Approximation of the algebraic Riccati equation in the Hilbert space of Hilbert-Schmidt operators*, SIAM J. Contr. Optim., No. 4, p. 847-874, 1993

[27] DIEKMANN O., VON GILS S. A., VERDUYN LUNEL S. M., WALTHER H.-O., *Delay equations, functional, complex and nonlinear analysis*, Applied Math. Sciences, 110, Springer Verlag, 1995.

[28] DRIVER R. D., *Ordinary and delay differential equations*, Applied Math. Sciences, Springer, 1977.

[29] DUGARD L. AND VERRIEST E. I. (EDS), *Stability and control of time-delay systems*, Lecture Notes in Control and Information Sciences No 228, Springer Verlag, 337 pages, 1997.

[30] EISING R., *Realization and stabilization of 2-D systems*, IEEE Trans. Aut. Control, Vol. 23, p. 793-799, 1978.

[31] EL'SGOL'TS L. E. AND NORKIN S. B., *Introduction to the theory and application of differential equations with deviating arguments*, Academic Press, New York, 1973.

[32] FERNANDES J. M. AND FERREIRA A. R., *The generalized Laguerre formula for approximating time delays*, Proc. IEEE-IMACS Conf. on Computational Engineering in Syst. Applications, Lille, France, Vol. 1, p. 405-409, 1996.

[33] FLIESS M. AND MOUNIER H., *Interpretation and comparison of various types of delay system controllabilities*, Proc.IFAC Conf. System Structure and Control, Nantes, France. p. 330-335, 1995.

[34] FLIESS M. MOUNIER H., ROUCHON P., RUDOLPH J., *Examples of linear systems with delays,* Proc. of the CNRS Conf. "Analyse et commande des systèmes avec retards", Nantes, France, p. 1-18, 1996.

[35] GERMANI A., MANES C., PEPE P., *Linearization of input-output mapping for nonlinear delay systems via static state feedback*, Proc. IEEE-IMACS Conf. on Computational Engineering in Syst. Applications, Lille, France, Vol. 1, p. 599-602, 1996.

[36] GIBSON J. S., *Linear quadratic optimal control of hereditary differential systems: infinite-dimensional Riccati equations and numerical approximation*, SIAM J. Contr. Optim., No. 31, p. 95-139, 1983.

[37] GLADER C., HÖGNAS G., MÄKILÄ P. M., TOIVONEN H. T., *Approximation of delay systems: a case study*, Int. J. Control, Vol. 53, No. 2, p. 369-390, 1991.

[38] GLÜSING-LÜERSSEN H., *A behavioral approach to delay-differential systems*, SIAM J. Contr. Optim., Vol. 35, No. 2, p. 480-499, 1997.

[39] GOPALSAMY K., *Stability and oscillations in delay differential equations of population dynamics*, Mathematics and Applications Vol. 74, Kluwer Acad. Publ., 1992.

[40] GOUBET-BARTHOLOMÉÜS A., *Sur la stabilité et la stabilisation des systèmes retardés: Critères dépendant des retards*, Ph.D. *(in French)*, Laboratoire d'Automatique et d'Informatique Industrielle de Lille, Ecole Centrale de Lille, December 1996.

[41] GOUBET-BARTHOLOMÉÜS A., DAMBRINE M., RICHARD J.-P., *An extension of stability criteria for linear and nonlinear time delay systems*, Proc. IFAC Conf. System Structure and Control, Nantes, France. p. 278–283, 1996.

[42] GOUBET -BARTHOLOMÉÜS A., DAMBRINE M., RICHARD J.-P., *Stability of perturbed systems with time-varying delay*, Systems and Control Letters, No.31, p. 155-163, 1997.

[43] GOUBET -BARTHOLOMÉÜS A., DAMBRINE M., RICHARD J.-P., *Bounded domains and constrained control of linear time-delays systems*, Special issue J. Européen des Systèmes Automatisés, No. 6, 1997.

[44] GORECKI H., FUKSA S., GRABOWSKI P., KORYTOWSKI A., *Analysis and synthesis of time delay systems*, John Wiley & Sons, 1989.

[45] GU G., KHARGONEKAR P. P., LEE E. B., MISRA P., *Finite-dimensional approximations of unstable infinite-dimensional systems*, SIAM J. Contr. Optim., 30, p. 704-716, 1992.

[46] HABETS L., *Algebraic and computational aspects of time-delay systems*, Ph. D., Eindhoven Univ. of Technology, The Netherland, 1994.

[47] HADDOCK J. R. AND TERJÉKI J., *Lyapunov-Razumikhin functions and an invariance principle for functional differential equations*, J. of Differential Equations, Vol. 48, p. 95-122, 1983.

[48] HALANAY A., *Differential equations: stability, oscillations, time lags*, Academic Press, New York, 1966.

[49] HALE J. K., *Theory of functional differential equations*, Springer-Verlag, New York, 1977.

[50] HALE J. K. AND VERDUYN LUNEL S. M., *Introduction to functional differential equations*, Applied Math. Sciences, 99, Springer Verlag, N.Y., 1991.

[51] HENNET J.-C. AND TARBOURIECH S., *Stability conditions of constrained delay systems via positive invariance*, to appear in Int. J. of Robust and Nonlinear Control, 1997.

[52] HIRAI K. AND SATOH Y., *Stability of a system with variable time-delay*, IEEE Trans. Aut. Control, Vol. 25, No. 3, p. 552-554, 1980.

[53] HOTZEL R. AND FLIESS M., *On linear systems with a fractional derivation: introductory theory and examples*, Special issue of "Mathematics and Computers in Simulation", Elsevier Sc. Publ., January 1998.

[54] KAMEN E., *On an operator theory of linear systems with pure and distributed delays*, Proc. IEEE Conf. on Decision and Control, 1976. *An operator theory of linear functional differential equations*, J. of Differential Equations, Vol. 27, p. 274-297, 1978.

[55] KAMEN E. W., KHARGONEKAR P. P., TANNENBAUM A., *Stabilization of time-delay systems using finite-dimensional compensators*, IEEE Trans. Aut. Control, Vol. 30, No.1, p. 75-78, 1985.

[56] KAMEN E. W., KHARGONEKAR P. P., TANNENBAUM A., *Proper stable Bézout factorization and feedback control of linear time-delay systems*, Int. J. Control, 43, p. 837-857, 1986.

[57] KARRAKCHOU J. AND RABAH R., *Quelques éléments sur la controlabilité des systèmes en dimension infinie, (in French)* Proc. of the CNRS Conf. "Analyse et commande des systèmes avec retards", Nantes, France, p. 61-78, 1996.

[58] KOLMANOVSKII V. B., *Stability of some nonlinear functional differential equations*, J. Nonlinear Differential Equations, No 2, p. 185-198, 1995.

[59] KOLMANOVSKII V. B., *Application of differential inequalities for stability of some functional differential equations*, J. Nonlinear Analysis, Theory, Methods & Applic., Vol. 25, No. 9-10, p. 1017-1028, 1995.

[60] KOLMANOVSKII V. B., *The stability of certain retarded systems with variable coefficients*, J. Appl. Maths. Mechs., Vol. 59, No. 1, p. 65-74, 1995.

[61] KOLMANOVSKII V. B. AND MYSHKIS A., *Applied theory of functional differential equations*, Mathematics and Applications Vol. 85, Kluwer Acad. Publ., 1992.

[62] KOLMANOVSKII V. B. AND NOSOV V. R., *Stability of functional differential equations*, Academic Press, London, 1986.

[63] KOLMANOVSKII V. B. AND RICHARD J.-P., *Stability of some systems with distributed delay*, JESA, European Journal of Automatic Control, No. 6, 1997.

[64] KOLMANOVSKII V. B. AND RICHARD J.-P., *Stability of systems with pure, discrete multi-delays*, to appear in Proc. IFAC Syst., Struct. & Control, Nantes, France, 8-10 July 1998.

[65] KOLMANOVSKII V. B. AND SHAIKHET L. E., *Control of systems with after effect*, American Mathematical Society, RI, Vol. 157, 1996, 359 pages.

[66] KRASOVSKII N. N., *Stability of motion* (translation by J. Brenner), Stanford Univ. Press, 1963.

[67] LAFAY J.-F., FLIESS M., MOUNIER H., SENAME O., *Sur la commandabilité des systèmes linéaires à retards,* (in French) Proc. of the CNRS Conf. "Analyse et commande des systèmes avec retards",Nantes, France, p. 19-42, 1996.

[68] LAKSHMIKANTHAM V. AND LEELA S., *Differential and integral inequalities*, Vol. II, Academic Press, New York, 1969.

[69] LAM J., *Convergence of a class of Padé's approximations for delay systems*, Int. J. Control, 52, p. 989-1008, 1990.

[70] LAM J., *Analysis of Laguerre formula for approximating delay systems*, IEEE Trans. Aut. Control, Vol. 39, No. 7, p. 1517-1521, 1994.

[71] LEE E. B. AND OLBROT A., *Observability and related structural results for linear hereditary systems*, Int. J. Control, 34, p. 1061-1078, 1981.

[72] LEHMAN B., BENTSMAN J., VERDUYN LUNEL S. VERRIEST E. I., *Vibrational control of nonlinear time-lag systems with bounded delay: averaging theory, stabilizability and transient behavior*, IEEE Trans. Aut. Control, No. 5, p. 898-912, 1994.

[73] LOISEAU J.-J. AND BRETHÉ D., *The use of 2-D systems theory for the control of time-delay systems*, Special issue J. Européen des Systèmes Automatisés, No. 6, 1997.

[74] LOISEAU J.-J. AND BRETHÉ D., *2-D exact model matching with stability, the structural approach*, Bulletin of the Polish Acad. of Sc. - Technical Sciences, Vol. 45, No. 2, p. 309-317, 1997.

[75] LOISEAU J.-J. AND BRETHÉ D., *An effective algorithm for finite spectrum assignment of single-input systems with delays*, Special issue of "Mathematics & Computers in Simulation", Elsevier Sc. Publ., January 1998.

[76] LOISEAU J.-J. AND RABAH R. (EDS), *Analysis and Control of Time-Delay Systems*, Special issue J. Européen des Systèmes Automatisés, No. 6, 1997.

[77] MACDONALD, *Time lags in biological models*, Lecture Notes in Biomathematics, 27, Springer Verlag, 1978.

[78] MALEK-ZAVAREI M. AND JAMSHIDI M., *Time-delay systems*, North-Holland. 1996.

[79] MANITIUS A. AND OLBROT A. W., *Finite spectrum assignment problem for systems with delays*, IEEE Trans. Aut. Control, Vol. 24, No. 4, p. 541-553, 1979.

[80] MANITIUS A. AND TRIGGIANI R., *Function space controllability of linear retarded systems: a derivation from abstract operators conditions*, SIAM J. Contr. Opt., Vol. 16, No. 4, p. 599-645, 1978.

[81] MATROSOV V. M., *On the theory of stability of motion*, Prikl. Mat. Meh., Vol. 26, No. 6, p. 992-1002, 1962, *Comparison principle and vector Lyapunov functions*, Diff. Uranh., 4, p. 1374-1386, 1968.

[82] MOOG C. H., CASTRO R., VELASCO M., *The disturbance decoupling problem for nonlinear systems with multiple time-delays: static feedback solutions*, Proc. IEEE-IMACS Conf. on Computational Engineering in Syst. Applications, Lille, France, Vol. 1 p. 596-598, 1996.

[83] MORSE A. S., *Ring models for delay differential systems*, Automatica, 12, p. 529-531, 1976.

[84] MOUNIER H., *Propriétés structurelles des systèmes linéaires à retards: aspects théoriques et pratiques*, Ph.D. *(in French)*, Laboratoire de Signaux et Systèmes, Université Paris-Sud, October 1995.

[85] MYSHKIS A. D., *General theory of differential equations with delay*, Uspehi Mat. Naut (N.S.), Vol. 4, No. 33; p. 99-141 *(in Russian)*, 1949, English transl. in Transl. AMS, No. 55, p. 1-62, 1951.

[86] NICULESCU S. I., *Sur la stabilité et la stabilisation des systèmes à états retardés*, Ph.D. *(in French)*, Laboratoire d'Automatique de Grenoble, INPG, February 1996.

[87] NICULESCU S. I., *Systèmes à retard. Aspects qualitatifs sur la stabilité et la stabilisation, (in French)* Diderot Multimedia, Paris, series "Nouveaux Essais", 1997.

[88] NICULESCU S. I., $\mathcal{H}_\infty$ *memoryless control with an* α*-stability constraint for time delays systems: an LMI approach*, to appear in IEEE Trans. Aut. Control.

[89] NICULESCU S. I., DION J.-M. AND L. DUGARD S. I., *Robust stabilization for uncertain time-delay systems containing saturating actuators*, IEEE Trans. Aut. Control, Vol. 41, No. 5, p. 742-747, 1996.

[90] NICULESCU S. I. AND IONESCU V., *On delay-independent stability criteria: a matrix pencil approach*, to appear in IMA Journal Math. Contr. Information.

[91] NICULESCU S.I., TROFINO-NETO A., DION J.M., DUGARD L., *Delay-dependent stability of linear systems with delayed state: an LMI approach*, Proc. 34th. IEEE Conf. on Decision and Control, New Orleans, p. 1495-1505, 1995.

[92] NICULESCU S. I. AND BROGLIATO B., *On force measurements time-delays in control of constrained manipulators*, Proc. IFAC Syst., Struct. & Control, Nantes, France, p. 266-271, 1995.

[93] OLBROT A. W., *On controllability of linear systems with time delay in control*, IEEE Trans. Aut. Control, p. 664-666, 1972.

[94] PARTINGTON J. R., *Approximation of delay systems by Fourrier-Laguerre series*, Automatica, 29, p. 569-572, 1991.

[95] PEPE P., *Il controllo LQG dei sistemi con ritardo*, Ph.D. *(in Italian)*, Dept. of Electrical Eng., Univ. L'Aquila, Italy, 1996.

[96] PICARD P., *Sur l'observabilité et la commande des systèmes linéaires à retards modélisés sur un anneau*, Ph.D. *(in French)*, IRCyN, Université de Nantes, Ecole Centrale de Nantes, October 1996.

[97] PICARD P., LAFAY J.-F., KUČERA V., *Feedback realization of nonsingular precompensators for linear systems with delays*, IEEE Trans. Aut. Control, Vol? 42, No. 6, 1997.

[98] PICARD P., LAFAY J.-F., KUČERA V., *Model matching for linear systems with delays and 2-D systems*, Automatica, Vol. 34, No. 2, 1998.

[99] PICARD P., SENAME O., LAFAY J.-F., *Observers and observability indices for linear systems with delays*, Proc. IEEE-IMACS Conf. on Computational Engineering in Syst. Applications, Lille, France, Vol. 1, p. 81-86, 1996.

[100] RICHARD J.-P., GOUBET-BARTHOLOMÉÜS A., TCHANGANI P. A., M. DAMBRINE, *Nonlinear delay systems: tools for quantitative approach to stabilization*, Chap. 10 (218-240) of "Stability and control of time-delay systems", Lecture Notes in Control and Information Sciences No 228, Springer Verlag, 1997.

[101] RICHARD J.-P. AND KOLMANOVSKII V. (EDS), *Delay Systems*, Special issue of "Mathematics and Computers in Simulation", Elsevier Sc. Publ., January 1998.

[102] ROCHA P., *Behavioral interpretation of weak and strong controllability for delay-differential systems*, Proc. IFAC Syst., Struct. & Control, Nantes, France, p. 506-510, 1995.

[103] ROCHA P. AND WILLEMS J., *Controllability for delay-differential systems*, Proc. 33rd IEEE Conf. Decision and Control, Lake Buena Vista, Florida, p. 2894-2897, 1994.

[104] SALAMON D., *Structure and stability of finite-dimensional approximations for functional differential equations*, SIAM J. Contr. Opt., 23, p. 928-951, 1985.

[105] SENAME O., *Sur la commandabilité et le découplage des systèmes linéaires à retards*, Ph.D. *(in French)*, Laboratoire d'Automatique de Nantes, Ecole Centrale de Nantes, October 1994.

[106] SENAME O., RABAH R., LAFAY J.-F., *Decoupling without prediction of linear systems with delays: a structural approach*, Syst. Contr. Lett., vol. 25, pp 387-395, 1995.

[107] SHEN J.C., CHEN B.S., KUNG F.C., *Memoryless stabilization of uncertain dynamic delay systems: Riccati equation approach*, IEEE Trans. Aut. Control, 36, p. 638-640, 1991.

[108] SLEMROD M. AND INFANTE E. F., *Asymptotic stability criteria for linear systems of differential equations of neutral type and their discrete analogues*, J. Math. Anal. Appl., 38, 1972, 399-415.

[109] STÉPÁN G., *Retarded dynamical systems; stability and characteristic functions*, Research Notes in Math. Series, 210, John Wiley & Sons, 1989.

[110] SONTAG E., *Linear systems over rings: a survey*, Ricerche di Automatica 7, 1976, and *Linear systems over rings: a (partial) update survey*, Proc. IFAC'81, Kyoto, 1981.

[111] TCHANGANI A. P., DAMBRINE M., RICHARD, J.-P., KOLMANOVSKII V. B., *Stability of linear differential equations with distributed delay*, to appear in J. Nonlinear Analysis: Theory, Methods and Applications, 1998.

[112] TCHANGANI A. P, DAMBRINE M., RICHARD J.-P., *Stability, attraction domains and ultimate boundedness for nonlinear neutral systems*, special issue of Mathematics and Computers in Simulation, Elsevier Sc. Publ., January 1997.

[113] TOKER O. AND OZBAY H., *Complexity issues in robust stability of linear delay-differential systems*, Math.,Control, Signals, Syst., 9, p. 386-400, 1996.

[114] TOKUMARU, H., ADACHI, N., AMEMIYA, T., *Macroscopic stability of interconnected systems*, 6-th IFAC Congress, ID44.4, 1975

[115] TSOI A. C., *Recent advances in the algebraic system theory of delay differential equations*, Chapter 5 of the book *Recent theoretical developments in control*, p. 67-127, Gregson M. J. Ed., Academic Press, 1978.

[116] TZAFESTAS S. G. AND PARASKEVOPOULOS P. N., *On the decoupling of multivariable systems with time delays*, Int. J. Control, Vol. 17, p. 405-415, 1973.

[117] XU B. AND LIU Y., *An improved Razumikhin-type theorem and its applications*, IEEE Trans. Aut. Control, Vol. 39, No. 4, p. 839-841, 1994.

[118] WATANABE K., *Finite spectrum assignment and observer for multivariable systems with commensurate delays*, IEEE Trans. Aut. Control, Vol. 31, No. 6, p. 543-550, 1986.

[119] WATANABE K., ITO M., KANEKO M., *Finite spectrum assignment problem for systems with multiple commensurate delays in state variables*, Int. J. Control, Vol. 38, No. 5, p. 506-508, 1983.

[120] WATANABE K., NOBUYAMA E., KOJIMA K., *Recent advances in control of time-delay systems A tutorial review*, Proc. 35th Conf. Decision & Control, Kobe, Japan, p. 2083-2089, 1996

[121] WEISS L., *On the controllability of delay-differential equations*, SIAM J. Cont. Optim., Vol. 5, No. 4, p. 575-587, 1967.

[122] WEISS L., *An algebraic criterion for controllability of linear systems with time delay*, IEEE Trans. Aut. Control, Vol. 15, No. 4, p. 443-444, 1970.

[123] YAMAMOTO Y., *Reachability of a class of infinite-dimensional linear systems: an external approach with applications to neutral systems*, SIAM J. Cont. Optim., 27, p. 217-234, 1989.

[124] ZAK S. H., LEE E. B., LU W.-S., *Realizations of 2-D filters and time delay systems*, IEEE Trans. Circuits Syst., Vol. CAS-33, p. 1241-1244, 1986.

Jean-Pierre Richard *is Professor of Automatic Control at the Ecole Centrale de Lille (France). Born in Montpellier (France) in 1956, he has obtained by 1979 the Engineer Diploma of the "Institut Industriel du Nord" (French "Grande École") and, the same year, he graduated from the University of Lille to the "Diplôme d'Études Approfondies" in Electronics. In 1981, he obtained the Ph.D. in Automatic Control, and the DSc in Physical Sciences in 1984, both in the University of Lille. After starting his teaching activities in 1979, he joined in 1981 the École Centrale de Lille as an Assistant Professor. Now, he is Full Professor since 1989. He has also been a visiting Professor in Casablanca (Morocco) and is the Director of GRAISyHM (French abbreviation for Research Group in Integrated Automatic Control and Man-Machine Systems), grouping about 250 persons from 12 French Laboratories.*

Pr. Richard researches are devoted to the analysis and control of complex, nonlinear systems, and in particular to the linked stability questions. His current interests in this field concern ordinary differential equations and functional equations with delays. He has successfully directed 12 Ph.D. and published over 110 communications, including 6 books and 40 papers in journals or collective volumes.

He is a Senior Member of the IEEE, Member of the Russian Academy of Nonlinear Sciences, Member of the IFNA and of the New York Academy of Sciences, and has belonged to the IPC of 7 international conferences (IEEE, IFAC, IMACS). He has organized about 20 invited sessions in his research field and is a reviewer for 7 international journals. He recently was the guest editor of a special issue of the IMACS journal "Mathematics and Computers in Simulation", devoted to delay systems.