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► **To cite this version:**

Hélène Barucq, Henri Calandra, Aurélien Citrain, Julien Diaz, Christian Gout. Spectral Element Method and Discontinuous Galerkin approximation for elasto-acoustic problems. MATHIAS – TOTAL Symposium on Mathematics, Oct 2017, Paris, France. hal-01690670

HAL Id: hal-01690670

<https://hal.science/hal-01690670v1>

Submitted on 23 Jan 2018

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On the coupling of Spectral Element Method with Discontinuous Galerkin approximation for elasto-acoustic problems.

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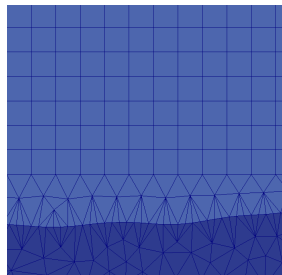
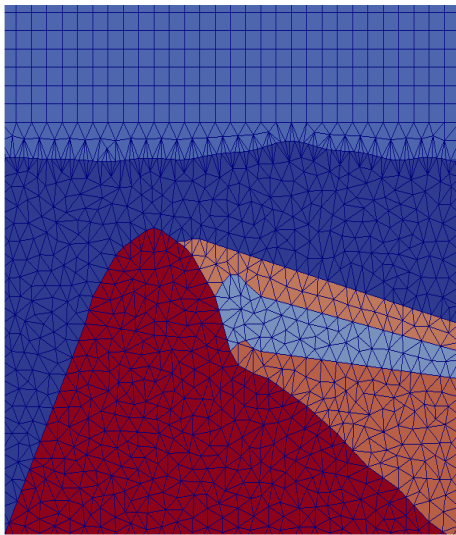
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MATHIAS 2017 October 25-27



Why using hybrid meshes?



- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water)
- Allows the coupling of numerical methods in order to reduce the computational cost

$x \in \Omega \subset \mathbb{R}^d$, $t \in [0, T]$, $T > 0$:

$$\begin{cases} \rho(x) \frac{\partial v}{\partial t}(x, t) &= \nabla \cdot \underline{\underline{\sigma}}(x, t) \\ \frac{\partial \underline{\underline{\sigma}}}{\partial t}(x, t) &= \underline{\underline{C}}(x) \underline{\underline{\epsilon}}(v(x, t)) \end{cases}$$

With :

- $\rho(x)$ the density
- $C(x)$ the elasticity tensor
- $\epsilon(x, t)$ the deformation tensor
- $v(x, t)$, the wavespeed
- $\underline{\underline{\sigma}}(x, t)$ the strain tensor

Software written in **Fortran 90** for wave propagation simulation in the **time domain**

Features

Simulation:

- on various types of meshes (**unstructured triangle, structured quadrangle, hybrid**)
 - on **heterogeneous media (acoustic, elastic and elasto-acoustic)**
-
- **Discontinuous Galerkin (DG)** on **quadrangle, triangle and hybrid mesh**
 - **Spectral Element Method (SEM)** only on **quadrangle mesh**
 - with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**
 - with **p-adaptivity, multi-order** computation...

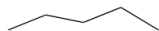
- 1 Numerical Methods
- 2 Comparison DG/SEM on structured quadrangle mesh
- 3 DG/SEM coupling

- 1 Numerical Methods
 - Discontinuous Galerkin Method (DG)
 - Spectral Element Method (SEM)
 - Advantages of each method

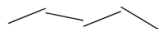
Use discontinuous functions :



mesh

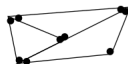


continuous

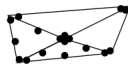


discontinuous

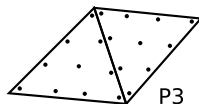
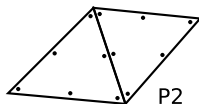
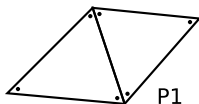
h adaptivity :



p adaptivity :

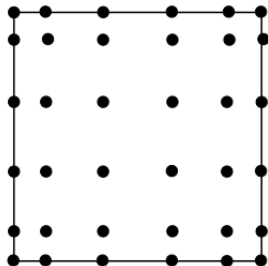


Degrees of freedom necessary on each cell :



General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on L^2 -norm)



- $\int f(x) dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j)$
- $\varphi_i(\xi_j) = \delta_{ij}$

Main change with DG

- DG discontinuous, SEM continuous
- Need to define local to global numbering
- Global matrices needed for SEM
- Basis functions computed differently

DG

- Element per element computation (*hp*-adaptivity)
- Time discretization quasi explicit (block diagonal mass matrix)
- Simple to parallelize

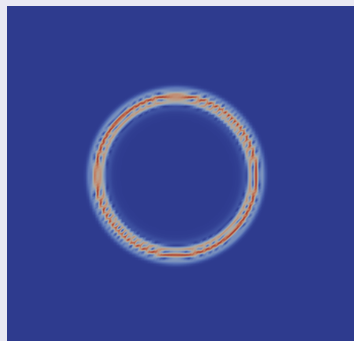
SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method
- Reduces the computational cost when you use structured meshes in comparison with DG
- Simplifies the mass and stiff matrices (mass matrix diagonal)

2 Comparison DG/SEM on structured quadrangle mesh

- Description of the test cases
- Comparative tables

Physical parameters



P wavespeed	1000 m.s^{-1}
Density	1 kg.m^{-3}

Second order **Ricker Source** in P wave
($f_{peak} = 10\text{Hz}$)

General context

- **Acoustic homogeneous** medium
- Four different meshes : **10000 cells**, **22500 cells**, **90000 cells**, **250000 cells**
- CFL computed using **power iteration** method
- **Leap-Frog** time scheme
- **Four threads** parallel execution with **OpenMP**

- Comparison between numerical solution and analytical solution obtained using the software Garbmore

Quadrangle mesh 10000 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.99e-3	2.5e-2	61.96	500
SEM	4.9e-3	1.3e-1	0.73	204
SEM(DG CFL)	1.99e-3	4.8e-2	1.48	502

Quadrangle mesh 22500 elements:

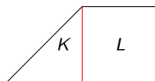
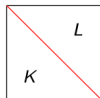
	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	252.20	750
SEM	3.26e-3	7e-2	2.42	306
SEM(DG CFL)	1.33e-3	1.2e-2	4.70	751

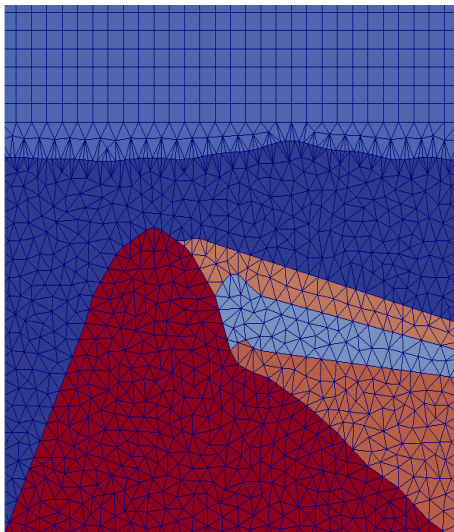
SEM fifty time much faster on a mesh with 22500 cells than DG

- 3 DG/SEM coupling
 - Hybrid meshes structures
 - Variational formulation
 - Space discretization

- Need to couple P_k and Q_k structures.
- Need to extend or split some of the structures (e.g. neighbour indexes)
- Necessity to define new face matrices

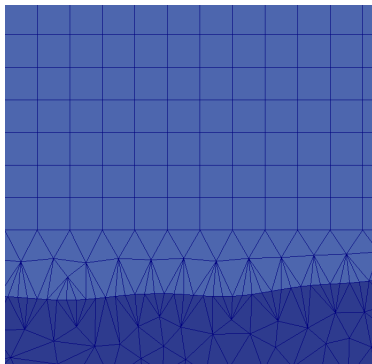
$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L$$





Global context

- Domain in two parts : $\Omega_{h,1}$ (**structured quadrangle + SEM**), $\Omega_{h,2}$ (**unstructured triangle + DG**)
- w_1, w_2 the tests-function and ξ_1, ξ_2 the tests-tensors



SEM variational formulation :

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out}} (\sigma_1 n_1) \cdot w_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out}} (C\xi_1 n_1) \cdot v_1 \end{cases}$$

DG variational formulation :

$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [[w_2]] \cdot n_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out}} (C\xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} [[C\xi_2]] \cdot n_2 \end{cases}$$

SEM variationnal formulation :

$$\left\{ \begin{array}{l} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{DG/SEM}} (\sigma_1 n_1) \cdot w_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C \xi_1)) \cdot v_1 + \int_{\Gamma_{out}} (C \xi_1 n_1) \cdot v_1 + \int_{\Gamma_{DG/SEM}} (C \xi_1 n_1) \cdot v_1 \end{array} \right.$$

DG variationnal formulation :

$$\left\{ \begin{array}{l} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [[w_2]] \cdot n_2 + \int_{\Gamma_{DG/SEM}} (\sigma_2 n_2) \cdot w_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C \xi_2)) \cdot v_2 + \int_{\Gamma_{out}} (C \xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} [[C \xi_2]] \cdot n_2 + \int_{\Gamma_{DG/SEM}} (C \xi_2 n_2) \cdot v_2 \end{array} \right.$$

Computation steps

- Simplify the coupling terms and separates the two parts + put $\sigma \cdot n = 0$

$$\left\{ \begin{array}{l} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \frac{1}{2} \int_{\Gamma_{DG/SEM}} (\sigma_1 + \sigma_2) n_1 \cdot w_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \frac{1}{2} \int_{\Gamma_{DG/SEM}} (C\xi_1 n_1) \cdot (v_1 + v_2) \end{array} \right.$$

$$\left\{ \begin{array}{l} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [[w_2]] \cdot n_2 - \frac{1}{2} \int_{\Gamma_{DG/SEM}} (\sigma_1 + \sigma_2) n_1 \cdot w_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} [[C\xi_2]] \cdot n_2 - \frac{1}{2} \int_{\Gamma_{DG/SEM}} (C\xi_2 n_1) \cdot (v_1 + v_2) \end{array} \right.$$

- φ_i : SEM basis functions
- ψ_j : DG basis functions

$$\begin{cases} M_{\mathbf{v}_1} \partial_t \mathbf{v}_{h,1} + R_{\sigma_1} \sigma_{h,1} + R_{\sigma_2}^{2,1} \sigma_{h,2} = 0 \\ M_{\sigma_1} \partial_t \sigma_{h,1} + R_{\mathbf{v}_1} \mathbf{v}_{h,1} + R_{\mathbf{v}_2}^{2,1} \mathbf{v}_{h,2} = 0 \end{cases}$$

- $M_{ij} = \int_{\Omega} \varphi_i \varphi_j \approx \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \sum_{k=1}^{(r+1)^d} \omega_k \varphi_i(\xi_k) \varphi_j(\xi_k) = \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \omega_i \delta_{i,j}$ the mass matrix
- $R_{p_{ij}} = \int_{\Omega} \varphi_i \frac{\partial \varphi_j}{\partial p}$ stiffness matrix

Matrix of DG/SEM coupling :

$$R_{\sigma_2, ij}^{2,1} = \int_{\partial\Omega_1 \cap \partial\Omega_2} \psi_i \varphi_j$$

$$\begin{cases} \rho M_{v_2} \partial_t \mathbf{v}_{h,2} + R_{\sigma_2} \boldsymbol{\sigma}_{h,2} - R_{\sigma_1}^{1,2} \boldsymbol{\sigma}_{h,1} = 0 \\ M_{\sigma_2} \partial_t \boldsymbol{\sigma}_{h,2} + R_{v_2} \mathbf{v}_{h,2} - R_{v_1}^{1,2} \mathbf{v}_{h,1} = 0 \end{cases}$$

- $M_{ij}^K = \int_K \psi_i^K \psi_j^K$ mass matrix,
- $R_{p_{ij}}^K = \int_K \psi_i^K \frac{\partial \psi_j^K}{\partial p}$ stiffness matrix,
- $R_{p_{ij}}^{K,L} = \int_{\partial K \cap \partial L} \psi_i^K \psi_j^L n_K \cdot e_p$ the mass-face matrices

Two new matrices which come from the DG/SEM hybridation $R_{\star}^{1,2}$. Block composed :

$$R_{v_1}^{1,2} = R_{\sigma_1}^{1,2} = -\frac{1}{2} \int_{\partial \Omega_2 \cap \partial K_1} \psi_j^{K_2} \varphi_i \quad \forall i, j = 1..N_m \quad (1)$$

Conclusion

- 1 As expected, SEM is more efficient on structured quadrangle mesh than DG
- 2 Show the utility on the use of hybrid meshes and method coupling (reduce computational cost,...)
- 3 Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability

Perspectives

- Implement DG/SEM coupling on the code (2D)
- Develop h-adaptivity for the structured part
- Develop DG/SEM coupling in 3D

Thank you for your attention !

Questions?