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## THE TRANSPORT EQUATION AND ZERO QUADRATIC VARIATION PROCESSES

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**ABSTRACT.** We analyze the transport equation driven by a zero quadratic variation process. Using the stochastic calculus via regularization and the Malliavin calculus techniques, we prove the existence, uniqueness and absolute continuity of the law of the solution. As an example, we discuss the case when the noise is a Hermite process.

**1. Introduction.** Transport phenomena arise in many research fields; geosciences, physics, biology, even in social sciences, for naming just a few. The linear transport equation,

$$\partial_t u(t, x) + b(t, x) \cdot \nabla u(t, x) = 0, \quad (1)$$

emerges as a model for the concentration (density) of a pollutant in a flow, and may be considered as a particular case of the convection/advection equation when the flow under consideration is incompressible (i.e. has zero divergence).

To point out some applications of this equation, we refer the reader to the works of Lions ([12, 13]) for a detailed exposition of its use in fluid dynamics, the work of Dafermos [3] for its relation to conservation laws, and the work of Perthame [22] to understand the relevance of equations like (1) in biology. Transport equations also appear in models for ocean salinity, see [4].

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In this paper we analyze a stochastic transport equation. The following one-dimensional Cauchy problem is considered: given an initial-data  $u_0$ , find  $u(t, x; \omega) \in \mathbb{R}$ , satisfying

$$\begin{cases} \partial_t u(t, x; \omega) + \partial_x u(t, x; \omega) \left( b(t, x) + \frac{dZ_t}{dt}(\omega) \right) = 0, \\ u(t_0, x) = u_0(x), \end{cases} \quad (2)$$

with  $(t, x) \in U_T = [0, T] \times \mathbb{R}$ ,  $\omega \in \Omega$ ,  $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  a given vector field, and the noise  $(Z_t)_{t \geq 0}$  is a stochastic process with zero quadratic variation. Problem (2) may be understood as a model for the concentration (density) of a pollutant in a flow where the velocity field has a random perturbation.

The stochastic transport equation driven by the standard Brownian motion was first addressed in Kunita's books (see [10], [11]). More recently it has been studied by several authors; in [1] the linear additive case is considered, existence and uniqueness of weak  $L^p$ -solutions and a representation for the general solution were shown. The non-blow-up problem is addressed for the multiplicative case with Stratonovich form in [5]. In [7] the authors have shown that the introduction of a multiplicative noise in the PDE allows some improvements in the traditional hypothesis needed to prove that the problem is well-posed, this is extended later to a non-linear case in [20]. A new uniqueness result is obtained in [14] by means of Wiener-chaos decomposition, and working on the associated Kolmogorov equation. The extension of the model to the fractional Brownian noise has been done in [21], where the existence of density of the solution and Gaussian estimates of the density were proven.

Our purpose is to solve the equation (2) and to analyze the properties of its solution in the case when the noise is a more general stochastic process, possibly non-Gaussian. We will focus on the situation when the noise  $Z$  in (2) is a stochastic process with zero quadratic variation, this is well defined in the next section of the paper.

The reason why we chose such a noise is that the stochastic integration theory in the sense of Russo-Vallois (see [24], [25]) can be applied to it. In fact, the stochastic integral in (2) will be understood as a symmetric integral in the Russo-Vallois sense with respect to the noise  $Z$ . Besides, in most of the papers cited in the previous paragraph, the Itô-Wentzell formula plays a crucial role in the characterization of the solution, consider a zero quadratic variation process is as far as one can go in order to prove that characterization, avoiding the presence of second order terms (see Proposition 9 in [8]). Among the zero covariance processes lies the fractional Brownian motion (for  $H \geq 1/2$ ), a self-similar process that find some of their applications in various kind of phenomena, going from hydrology and surface modelling to network traffic analysis and mathematical finance, to name a few.

In this paper, first, the existence and uniqueness of the solution to (2) is proved, by using the so-called method of characteristics that comes from the works of Kunita ([9], [10]). Then it is show that the solution can be expressed as the initial data  $u_0$  applied to the inverse flow associated to the transport equation. Later, using the techniques of Malliavin calculus on this representation, it is possible to prove the existence of the density of this solution. This will be done by showing that the Malliavin derivative of the solution is strictly positive. To this end, the explicit expression of the Malliavin derivative must be calculated and controlled.

The outline of this paper is as follows: In Section 2 we present the basics definitions of the integration theory in the sense of Russo-Vallois, stochastic calculus via

regularization, and the Malliavin techniques. The existence and uniqueness of weak solution to problem (2) is proved in Section 3. Section 4 provides the demonstration of the existence of the density of the solution. Finally in Section 5 we illustrate the results considering a Hermite process as the driving noise.

**2. Preliminaries.** In this section, we recall the notions from the stochastic calculus via regularization and from Malliavin calculus that we will use in what follows. More details can be found in [25] or [17].

**2.1. Stochastic calculus via regularization.** Throughout this paper  $T$  will be a fixed positive real number. We recall the definition of the symmetric integral  $d^\circ X$  that will appear in (2) and in the definition of the solution in formula (5).

Assume  $(X_t)_{t \geq 0}$  is a continuous process and  $(Y_t)_{t \geq 0}$  is a process with paths in  $L^1_{loc}(\mathbb{R}^+)$ , i.e. for any  $b > 0$ ,  $\int_0^b |Y_t| dt < \infty$  a.s. The generalized stochastic integrals (forward, backward and symmetric) are defined through a regularization procedure see [24], [25]. Here we recall only the definition of the symmetric integral (actually, since we are dealing with zero quadratic variation processes, the three integrals will coincide). Let  $I^0(\varepsilon, Y, dX)$  be the  $\varepsilon$ -symmetric integral

$$I^0(\varepsilon, Y, dX)(t) = \int_0^t Y_s \frac{(X_{s+\varepsilon} - X_{s-\varepsilon})}{2\varepsilon} ds, \quad t \geq 0.$$

The symmetric integral  $\int_0^t Y_s d^\circ X_s$  is defined as

$$\int_0^t Y_s d^\circ X_s := \lim_{\varepsilon \rightarrow 0} I^0(\varepsilon, Y, dX)(t), \quad (3)$$

for every  $t \in [0, T]$ , provided the limit exists in the *ucp* sense (uniformly on compacts in probability).

In a similar way, the covariation or generalized bracket,  $[X, Y]_t$  of two stochastic processes  $X$  and  $Y$  is defined as the limit *ucp* when  $\varepsilon$  goes to zero of

$$[X, Y]_{\varepsilon, t} = \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) ds, \quad t \geq 0.$$

Note that  $[X, Y]$  coincide with the classical bracket when  $X$  and  $Y$  are semi-martingales.

A process  $X$ , such that  $[X, X]$  exists, is called finite quadratic variation processes. If  $[X, X] \equiv 0$  we say that  $X$  is a zero quadratic variation process. Our integrand in (2) will be such a zero quadratic variation process.

**2.2. Malliavin derivative.** We also present the elements from the Malliavin calculus that will be used in the paper. We refer to [17] for a more complete exposition. Consider  $\mathcal{H}$  a real separable Hilbert space and  $(B(\varphi), \varphi \in \mathcal{H})$  an isonormal Gaussian process on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , that is, a centered Gaussian family of random variables such that  $\mathbb{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$ .

We denote by  $D$  the Malliavin derivative operator that acts on smooth functions of the form  $F = g(B(\varphi_1), \dots, B(\varphi_n))$  ( $g$  is a smooth function with compact support and  $\varphi_i \in \mathcal{H}, i = 1, \dots, n$ )

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_n)) \varphi_i.$$

It can be checked that the operator  $D$  is closable from  $\mathcal{S}$  (the space of smooth functionals as above) into  $L^2(\Omega; \mathcal{H})$  and it can be extended to the space  $\mathbb{D}^{1,p}$  which is the closure of  $\mathcal{S}$  with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}F^p + \mathbb{E}\|DF\|_{\mathcal{H}}^p.$$

We denote by  $\mathbb{D}^{k,\infty} := \bigcap_{p \geq 1} \mathbb{D}^{k,p}$  for every  $k \geq 1$ . In this paper,  $\mathcal{H}$  will be the standard Hilbert space  $L^2([0, T])$ .

We will make use of the chain rule for the Malliavin derivative (see Proposition 1.2.4 in [17]). That is, if  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable with bounded derivative and  $F \in \mathbb{D}^{1,2}$ , then  $\varphi(F) \in \mathbb{D}^{1,2}$  and

$$D\varphi(F) = \varphi'(F)DF. \quad (4)$$

An important role of the Malliavin calculus is that it provides criteria for the existence of the density of a random variable. Here we will use the following result: if  $F$  is a random variable in  $\mathbb{D}^{1,2}$  such that  $\|DF\|_{\mathcal{H}} > 0$  almost surely, then  $F$  admits a density with respect to the Lebesgue measure (see e.g. Theorem 2.1.3 in [17]).

**3. Existence and uniqueness.** This section presents the results concerning the existence of the weak solution for the stochastic transport equation (2) driven by a zero quadratic variation process.

Let  $(\Omega, \mathcal{F}, P)$  be a fixed probability space and  $(W_t)_{t \in [0, T]}$  a standard Wiener process on it. Consider a continuous process  $(Z_t)_{t \geq 0}$ , adapted to the filtration generated by  $W$ , and such that

$$[Z, Z]_t = 0, \quad \forall t \geq 0.$$

The quadratic variation  $[Z, Z]$  is understood in the sense of stochastic calculus via regularization, as introduced in Section 2.1.

We consider the one-dimensional Cauchy problem (2) driven by the process  $Z$ , with a given initial-data  $u_0$ .

We recall the notion of weak solution to (2) (see [7]).

**Definition 3.1.** A stochastic process  $u \in L^\infty(\Omega \times [0, T] \times \mathbb{R})$  is called a weak  $L^p$ -solution of the Cauchy problem (2), if for any  $\varphi \in C_c^\infty(\mathbb{R})$ ,  $\int_{\mathbb{R}} u(t, x)\varphi(x)dx$  is an adapted real valued process which has a continuous modification, finite covariation, and for all  $t \in [0, T]$ ,  $\mathbb{P}$ -almost surely

$$\begin{aligned} \int_{\mathbb{R}} u(t, x)\varphi(x)dx &= \int_{\mathbb{R}} u_0(x)\varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) b(s, x)\partial_x\varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x)b'(s, x)\varphi(x) dx ds + \int_0^t \int_{\mathbb{R}} u(s, x)\partial_x\varphi(x) dx d^\circ Z_s, \end{aligned}$$

where  $b'(s, x)$  denotes the derivative of  $b(s, x)$  with respect to the spatial variable  $x$ , and the integral  $d^\circ Z$  is a symmetric integral defined via regularization (see (3)).

Following the arguments presented in [21], the existence and uniqueness of the weak solution to (2) follows immediately.

**Proposition 1.** *Assume that  $b \in L^\infty((0, T); C_b^1(\mathbb{R}))$ . Then there exists a  $C^1(\mathbb{R})$  stochastic flow of diffeomorphism  $(X_{s,t}, 0 \leq s \leq t \leq T)$ , that satisfies*

$$X_{s,t}(x) = x + \int_s^t b(u, X_{s,u}(x))du + Z_t - Z_s \quad (5)$$

for every  $x \in \mathbb{R}^d$ . Moreover, given  $u_0 \in L^\infty(\mathbb{R})$ , the stochastic process

$$u(t, x) := u_0(X_t^{-1}(x)), \quad t \in [0, T], x \in \mathbb{R} \quad (6)$$

is the unique weak  $L^\infty$ -solution of the Cauchy problem 2, where  $X_t := X_{0,t}$  for every  $t \in [0, T]$ .

*Proof.* The demonstration follows closely the lines of the proof of Theorem 1 in [21], which is based on the Itô formula in the Russo-Vallois sense for functions depending on  $\omega$ . Although in [21] the noise is a fractional Brownian motion with Hurst parameter  $H > \frac{1}{2}$ , the only property of the fBm needed in the demonstration is the fact that, for  $H > \frac{1}{2}$ , is a zero quadratic variation process. Therefore, all the steps in the proof of Theorem 1 in [21] remain valid when the noise is a general zero quadratic variation process.  $\square$

From (6), it is clear that the properties of the solution to the transport equation (2), in particular the existence of its density, will depend on the initial condition  $u_0$  and on the properties of the inverse flow  $X_t^{-1}$ . Since later in the paper we will assume on  $u_0$  as much regularity as needed, we focus on the analysis of the inverse flow. Let's start by describing its dynamic.

**Lemma 3.2.** *Assume  $b \in L^\infty((0, T); C_b^1(\mathbb{R})) \cap C((0, T) \times \mathbb{R})$ . Then the inverse flow satisfies the backward stochastic equation*

$$Y_{s,t}(x) = x - \int_s^t b(r, Y_{r,t}(x)) dr - (Z_t - Z_s) \quad (7)$$

for every  $0 \leq s \leq t \leq T$  and for every  $x \in \mathbb{R}$ .

Moreover,  $Y$  is the unique process that satisfies (7) with  $Y_{s,s}(x) = x$ .

*Proof.* Analogously to the proof of Lemma 2 in [21].  $\square$

**Remark 1.** If we set  $R_{t,x}(u) = Y_{t-u,t}(x)$  for  $u \in [0, t]$ , and  $x \in \mathbb{R}$ , then we have

$$R_{t,x}(u) = x - \int_0^u b(t-a, R_{t,x}(a)) da - (Z_t - Z_{t-u}). \quad (8)$$

This can be easily seen from (7) by making the change of variable  $a = t - r$ . If we set

$$B_t(a, x) = -b(t-a, x)$$

for  $a \leq t$ ,  $x \in \mathbb{R}$ , and

$$Z_{u,t} = -(Z_t - Z_{t-u})$$

for  $t \in [0, T]$  and  $u \leq t$ , then (8) becomes

$$R_{t,x}(u) = x + \int_0^u B_t(a, R_{t,x}(a)) da + Z_{u,t}. \quad (9)$$

We will actually use the above equation in order to obtain the properties of the inverse flow.

**4. The Malliavin derivative and the density of the solution.** We will show that the solution to (2) is Malliavin differentiable and, using the techniques of the Malliavin calculus, that it admits a density with respect to the Lebesgue measure. From the representation (6), it is enough to focus on the Malliavin derivative of the inverse flow  $X_t^{-1}$  whose dynamic is governed by (7) or (9).

**4.1. Malliavin differentiability of the inverse flow.** Throughout this section we assume that

$$b \in L^\infty((0, T); C_b^1(\mathbb{R})) \cap C((0, T) \times \mathbb{R}). \quad (10)$$

The noise  $Z$  is a zero quadratic variation process, adapted to the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  such that  $Z_t \in \mathbb{D}^{1,2}$  for every  $t \in [0, T]$ . We suppose also that

$$\sup_{t \in [0, T]} \mathbb{E} |Z_t|^2 < \infty \text{ and } \sup_{t \in [0, T]} \mathbb{E} \|DZ_t\|_{L^2([0, T])}^2 < \infty. \quad (11)$$

**Lemma 4.1.** *Under hypothesis (10) and (11), the equation (9) has an unique solution.*

*Proof.* Using classic Picard iterations is clear that there exists an unique solution to equation (9). We refer the reader to the proofs of Lemma 5 in [6] or Lemma 2.2.1 in [17] for similar results.  $\square$

Now, the goal is to show the Malliavin differentiability of the solution to (9).

**Proposition 2.** *Assume (10) and (11). Then, for  $x \in \mathbb{R}, t \in [0, T], u \leq t$ , the random variable  $R_{t,x}(u)$  given by (9) belongs to  $\mathbb{D}^{1,2}$ . Moreover, the Malliavin derivative of  $R_{t,x}(u)$  satisfies*

$$D_\alpha R_{t,x}(u) = \int_0^u B'_t(s, R_{t,x}(s)) D_\alpha R_{t,x}(s) ds + D_\alpha Z_{u,t} \quad (12)$$

for every  $\alpha < t$ .

*Proof.* Fix  $t \in [0, T], x \in \mathbb{R}$ . Define the usual iterations

$$R_{t,x}^{(0)}(u) = x \text{ for every } t \in [0, T]$$

and for  $n \geq 0$ ,

$$R_{t,x}^{(n+1)}(u) = x + \int_0^u B_t(s, R_{t,x}^{(n)}(s)) ds + Z_{u,t}$$

with  $0 \leq u \leq t$ .

It is clear that  $R_{t,x}^{(0)}(u)$  is  $\mathcal{F}_t$ -measurable and belongs to  $\mathbb{D}^{1,2}$ . By a trivial induction argument and using Proposition 1.2.4 in [17], we have that  $R_{t,x}^{(n)}(u)$  is  $\mathcal{F}_t$ -measurable and  $R_{t,x}^{(n)}$  belongs to  $\mathbb{D}^{1,2}$  for every  $n \geq 0$ . This implies that

$$D_\alpha R_{t,x}^{(n)}(u) = 0 \text{ if } \alpha > t.$$

Our proof follows the following standard arguments: first, we notice the  $L^2$  convergence of  $R_{t,x}^{(n)}(u)$  to  $R_{t,x}(u)$ . Secondly, we prove an uniform bound on the sequence of Malliavin derivatives of  $R_{t,x}^{(n)}(u)$  and then we conclude the Malliavin differentiability of  $R_{t,x}(u)$  along with the expression of its Malliavin derivative.

Concerning the first step, we only remind that (see e.g. the proof of Lemma 5 in [6]), for every  $u \in [0, T]$

$$\sup_{t \geq u} \mathbb{E} \left| R_{t,x}^{(n)}(u) - R_{t,x}(u) \right|^2 \xrightarrow{n \rightarrow \infty} 0. \quad (13)$$

Let us use the notation

$$m_T := \sup_{t \in [0, T]} \sup_{u \in [0, t]} \mathbb{E} \|DZ_{u,t}\|_{L^2([0, T])}^2 \quad (14)$$

which is a finite positive constant due to (11). We now show that

$$\sup_{n \geq 1} \sup_{t \geq u} \mathbb{E} \|DR_{t,x}^{(n)}(u)\|_{L^2([0,T])}^2 < \infty. \quad (15)$$

For  $u \in [0, T]$  fixed, denote by

$$g_u^n = \sup_{t \geq u} \mathbb{E} \|DR_{t,x}^{(n)}(u)\|_{L^2([0,T])}^2 = \sup_{t \geq u} \mathbb{E} \int_0^T (D_\alpha R_{t,x}^{(n)}(u))^2 d\alpha.$$

Then, using the fact that  $b'$  is bounded

$$\begin{aligned} g_u^{(n+1)} &= \sup_{t \geq u} \mathbb{E} \left( \int_0^T \int_0^u B'_t(a, R_{t,x}^{(n)}(a)) D_\alpha R_{t,x}^{(n)}(a) da + D_\alpha Z_{u,t} \right)^2 d\alpha \\ &\leq C(T) \sup_{t \geq u} \left( \mathbb{E} \int_0^T d\alpha \int_0^u da (D_\alpha R_{t,x}^{(n)}(a))^2 + m_T \right) \\ &\leq C(T) \sup_{t \geq u} \left( \int_0^u da \sup_{t \geq a} \mathbb{E} \int_0^T d\alpha (D_\alpha R_{t,x}^{(n)}(a))^2 + m_T \right) \\ &= C(T) \left( \int_0^u g_a^n da + m_T \right), \end{aligned}$$

because  $a \leq u \leq t$ . By taking the supremum over  $n \geq 1$ , and then applying Gronwall lemma, we get

$$\sup_{n \geq 1} g_u^n \leq C_1(T) e^{C_2(T)} < \infty.$$

So, (15) holds. This, together with (13) and Lemma 1.2.3 in [17] implies that

$$R_{t,x}(u) \in \mathbb{D}^{1,2} \text{ for every } t \in [0, T],$$

and the sequence of derivatives  $(DR_{t,x}^{(n)}(u))_{n \geq 0}$  is convergent in  $L^2(\Omega \times [0, T])$  to  $DR_{t,x}(u)$ , hence (12) holds true.  $\square$

**4.2. The density of the solution.** Let  $Y_{s,t}(x)$  be the inverse flow given by (7), with  $0 \leq s \leq t \leq T$  and  $x \in \mathbb{R}$ . Recall the notation  $R_{t,x}(u) = Y_{t-u,t}(x)$  if  $u < t$ , and the hypothesis on the noise  $Z$ .

From (9) and Proposition 2, it follows that  $Y_{s,t}(x)$  is Malliavin differentiable. Our next step is to find the expression of its Malliavin derivative.

**Proposition 3.** *Assume that  $b$  satisfies (10),  $Z$  satisfies (11) and let  $Y$  be given by (7). Then for every  $s \leq t, \alpha \leq t$  and  $x \in \mathbb{R}$ , we have*

$$\begin{aligned} D_\alpha Y_{s,t}(x) &= 1_{(0,T)}(\alpha) e^{-\int_s^t b'(u, Y_{u,t}(x)) du} \int_s^t b'(u, Y_{u,t}(x)) D_\alpha(Z_{u,t}) e^{\int_u^t b'(r, Y_{r,t}(x)) dr} du \\ &\quad + D_\alpha Z_{s,t}. \end{aligned}$$

*Proof.* First we notice that, if  $\alpha > t$ ,  $D_\alpha Y_{s,t}(x) = 0$ , because  $Y_{s,t}(x)$  is  $\mathcal{F}_t$ -measurable. Now, let's observe that

$$D_\alpha R_{t,x}(s) = \int_0^s B'_t(u, R_{t,x}(u)) D_\alpha R_{t,x}(u) du + D_\alpha Z_{s,t},$$



where we recall the notation  $Z_{s,t} = -(Z_t - Z_{t-s})$  for  $0 \leq s \leq t \leq T$ . Considering the previous relation as an integral equation of functions depending on the variable  $s$ , regularizing  $Z$  by standard mollifiers, applying the method of variations of parameters and then integration by parts, we obtain

$$\begin{aligned} D_\alpha R_{t,x}(s) &= D_\alpha Z_{s,t} \\ &+ e^{\int_0^s B'_t(u, R_{t,x}(u)) du} \int_0^s B'_t(u, R_{t,x}(u)) D_\alpha(Z_{u,t}) e^{-\int_0^u B'_t(r, R_{t,x}(r)) dr} du \end{aligned}$$

Writing this in terms of the inverse flow  $Y$ ,

$$\begin{aligned} D_\alpha Y_{t-s,t}(x) &= D_\alpha Z_{s,t} + e^{-\int_0^s b'(t-u, Y_{t-u,x}(u)) du} \cdot \\ &\int_0^s b'(t-u, Y_{t-u,x}(u)) D_\alpha(Z_{u,t}) e^{\int_0^u b'(t-r, Y_{t-r,t}(r)) dr} du. \end{aligned}$$

Using the changes of variables  $u' = t - u$ ,  $r' = t - r$  and the notation  $s' = t - s$  we obtain the desired result.  $\square$

Let us prove that the random variable  $Y_{s,t}$  admits a density, for every  $0 \leq s \leq t \leq T$ .

**Proposition 4.** *Fix  $0 \leq s \leq t \leq T$ . Assume (10) and (11). In addition we will suppose that for every  $0 < s < t \leq T$*

$$\|DZ_t\|_{L^2([t-s, T])}^2 = \int_{t-s}^T (D_\alpha Z_t)^2 d\alpha > 0, \text{ almost surely.} \quad (16)$$

*Then the law of  $Y_{s,t}(x)$  is absolutely continuous with respect to the Lebesgue measure.*

*Proof.* By Theorem 2.1.3 in [17], we need to prove that

$$\|DY_{s,t}\|_{L^2([0, T])}^2 = \int_0^T (D_\alpha Y_{s,t})^2 d\alpha > 0 \text{ almost surely.}$$

We will use the inequality

$$\int_0^T (D_\alpha Y_{s,t})^2 d\alpha \geq \int_{t-s}^T (D_\alpha Y_{s,t})^2 d\alpha$$

and the fact that, for  $\alpha \in (t-s, T)$  and  $u \in (s, t)$ , we have

$$D_\alpha Z_{s,t} = -D_\alpha Z_t \text{ and } D_\alpha Z_{u,t} = -D_\alpha Z_t$$

since  $D_\alpha Z_{t-s} = D_\alpha Z_{t-u} = 0$ . Therefore

$$\begin{aligned} &\int_0^T (D_\alpha Y_{s,t})^2 d\alpha \geq \int_{t-s}^T (D_\alpha Y_{s,t})^2 d\alpha \\ &= \int_{t-s}^T \left( -e^{-\int_s^t b'(u, Y_{u,t}(x)) du} \int_s^t b'(u, Y_{u,t}(x)) D_\alpha Z_t e^{\int_u^t b'(r, Y_{r,t}(x)) dr} du - D_\alpha Z_t \right)^2 d\alpha \\ &= \int_{t-s}^T (D_\alpha Z_t)^2 \left( e^{-\int_s^t b'(u, Y_{u,t}(x)) du} \int_s^t b'(u, Y_{u,t}(x)) e^{\int_u^t b'(r, Y_{r,t}(x)) dr} du + 1 \right)^2 d\alpha \end{aligned}$$

We claim that

$$e^{-\int_s^t b'(u, Y_{u,t}(x)) du} \int_s^t b'(u, Y_{u,t}(x)) e^{\int_u^t b'(r, Y_{r,t}(x)) dr} du + 1 > C > 0. \quad (17)$$

Indeed, considering the boundedness of  $b'$ , and the inequalities

$$e^{-(t-s)\|b'\|_\infty} \leq e^{\pm \int_s^t b'(a, Y_{a,t}(x)) da} \leq e^{(t-s)\|b'\|_\infty}, \quad (18)$$

we may note that

$$\begin{aligned} & e^{-\int_s^t b'(u, Y_{u,t}(x)) du} \int_s^t b'(u, Y_{u,t}(x)) e^{\int_u^t b'(r, Y_{r,t}(x)) dr} du \\ & \geq \inf_{u \in (s,t); x \in \mathbb{R}} b'(u, x) \cdot (t-s) \cdot e^{-2(t-s)\|b'\|_\infty} \\ & \geq -\|b'\|_\infty \cdot (t-s) \cdot e^{-2(t-s)\|b'\|_\infty} \\ & = f(\|b'\|_\infty \cdot (t-s)), \end{aligned}$$

with  $f(x) = -x \exp(-2x)$ . The function  $f$  attains its minimum at  $x = 1/2$ , with  $f(1/2) = -1/2 \exp(-1) > -1$ , this prove our claim. The conclusion follows from condition (16).  $\square$

Condition (16) ensures the existence of the density of the noise  $Z_t$  for each  $t$ . This property is then transferred to the solution.

Let us conclude the existence of the density of the solution to the transport equation.

**Theorem 4.2.** *Let  $u(t, x)$  be the solution to the transport equation (2). Assume that  $u_0 \in C^1(\mathbb{R})$  such that there exists  $C > 0$  with  $(u_0'(x))^2 \geq C$  for every  $x \in \mathbb{R}$ . Then, for every  $t \in [0, T]$  and for every  $x \in \mathbb{R}$ , the random variable  $u(t, x)$  is Malliavin differentiable. Moreover  $u(t, x)$  admits a density with respect to the Lebesgue measure.*

*Proof.* By formula (6),  $u(t, x) = u_0(Y_{0,t}(x))$  and then we get the Malliavin differentiability of  $u(t, x)$  from Proposition 2. The chain rule for the Malliavin derivative (4) implies

$$D_\alpha u(t, x) = u_0'(Y_{0,t}(x)) D_\alpha Y_{0,t}(x)$$

and then, from the above result and the condition imposed on the initial value  $u_0$

$$\int_0^T (D_\alpha u(t, x))^2 d\alpha > 0$$

almost surely for every  $t \in [0, T], x \in \mathbb{R}$ . This implies that the random variables  $u(t, x)$  admits a density.  $\square$

**5. An example: The Hermite process.** In this section we will give an example of a class of stochastic processes that satisfies the conditions required for the noise  $Z$  in (2). Recall that we assumed that the noise  $Z$  is an adapted square integrable process, with zero quadratic variation in the Russo-Vallois sense, Malliavin differentiable, and satisfies (11) and (16).

The class of processes we consider is those of Hermite processes. The Hermite process of order  $q \geq 1$  lives in the Wiener chaos of order  $q$ , and it is defined as a multiple stochastic integral with respect to the standard Brownian motion. Its representation is related to the Wiener integral representation of the fractional Brownian motion. We recall that the fractional Brownian process  $(B_t^H)_{t \in [0,1]}$  with Hurst parameter  $H \in (0, 1)$  can be written as

$$B_t^H = \int_0^t K^H(t, s) dW_s, \quad t \in [0, 1] \quad (19)$$

where  $(W_t, t \in [0, T])$  is a standard Wiener process, and if  $H > \frac{1}{2}$ , the kernel  $K^H(t, s)$  has the expression

$$K^H(t, s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du$$

where  $t > s$  and  $c_H = \left( \frac{H(2H-1)}{\beta(2-2H, H-1/2)} \right)^{1/2}$  and  $\beta(\cdot, \cdot)$  is the Beta function. For  $t > s$ , the kernel's derivative is

$$\frac{\partial K^H}{\partial t}(t, s) = c_H \left( \frac{s}{t} \right)^{1/2-H} (t-s)^{H-3/2}.$$

We denote by  $(Z_t^{(q,H)})_{t \in [0, T]}$  the Hermite process with *self-similarity parameter*  $H \in (1/2, 1)$ . For  $t \in [0, T]$  it is given by

$$Z_t^{(q,H)} = d(H) \int_0^t \dots \int_0^t dW_{y_1} \dots dW_{y_q} \cdot \left( \int_{y_1 \vee \dots \vee y_q}^t \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du \right), \quad (20)$$

where  $K^{H'}$  is the usual kernel of the fractional Brownian motion that appears in (19) and

$$H' = 1 + \frac{H-1}{q} \iff (2H'-2)q = 2H-2.$$

The covariance of  $Z^{(q,H)}$  is identical to that of fBm, namely

$$\mathbb{E} \left[ Z_s^{(q,H)} Z_t^{(q,H)} \right] = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}). \quad (21)$$

The constant  $d(H)$  is chosen to have variance equal to 1.

The Hermite process  $Z^{(q,H)}$  is  $H$ -self-similar and it has stationary increments, the mean square of the increment is given by

$$\mathbb{E} \left[ \left| Z_t^{(q,H)} - Z_s^{(q,H)} \right|^2 \right] = |t-s|^{2H}; \quad (22)$$

as a consequence, using the self-similarity and the stationarity of the increments of  $Z^H$ , it follows from Kolmogorov's continuity criterion (see theorem 2.2.3 in [19]) that  $Z^{(q,H)}$  has Hölder-continuous paths of any exponent  $\delta < H$ . For  $q = 1$ ,  $Z^{(1,H)}$  is standard fBm with Hurst parameter  $H$ , while for  $q \geq 2$  the Hermite process is not Gaussian. In the case  $q = 2$  this stochastic process is known as the Rosenblatt process.

We will use the notation  $Z^H := Z^{(q,H)}$ . Also, denote by  $L^H$  the kernel of the Hermite process

$$L_t^H(y_1, \dots, y_q) = 1_{(y_1 \vee \dots \vee y_q \leq t)} \int_{y_1 \vee \dots \vee y_q}^t \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du. \quad (23)$$

We can write  $Z_t^H = I_q(L^H)$  with  $I_q$  being the multiple integral of order  $q$  with respect to the Wiener process  $W$ . We refer the reader to the manuscript [28] and references there in for a deeper discussion on Hermite processes and other self-similar processes.

It is immediate to see that  $Z^H$  has zero quadratic variation as defined in Section 2.

**Lemma 5.1.**  $Z^H$  is a zero quadratic variation process.

*Proof.* Indeed, from (22),

$$\frac{1}{\varepsilon} \mathbb{E} \int_0^t (Z_{s+\varepsilon}^H - Z_s^H)^2 ds = t\varepsilon^{2H-1} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

□

We will show that the process  $Z^H$  satisfies the assumptions imposed throughout the paper for the noise  $Z$  appearing in (2). Since it is defined as a multiple integral, the random variable  $Z_t^H$  is clearly Malliavin differentiable for every  $t \in [0, T]$ .

**Lemma 5.2.** The Hermite process satisfies (11) and (16).

*Proof.* Clearly, using (21)

$$\sup_{t \in [0, T]} \mathbb{E} |Z_t^H|^2 = T^{2H} < \infty.$$

Also, with  $L^H$  given by (23),

$$D_\alpha Z_t^H = q I_{q-1}(L_t^H(\cdot, \alpha)),$$

the properties of multiple Wiener-Itô integrals gives

$$\mathbb{E} \int_0^T (D_\alpha Z_t^H)^2 d\alpha = q \mathbb{E} [I_q(L_t^H)]^2 = q \mathbb{E} (Z_t^H)^2 = qt^{2H}.$$

This implies

$$\sup_{t \in [0, T]} \mathbb{E} \int_0^T (D_\alpha Z_t^H)^2 d\alpha < \infty.$$

The condition (16) is satisfied because  $Z_t^H$  is a multiple integral of order  $q$ . A classical result by Shikogawa [27] (see also [15], Corollary 5.2) says that, if  $q \geq 1$  is an integer, and  $f$  a of  $L^2(\mathbb{R}^q)$  with  $\|f\|_{L^2(\mathbb{R}^q)} \neq 0$ , then the  $q$ -multiple Wiener-Itô integral of  $f$  has a density and satisfies (16). □

**Remark 2.** In the case  $q = 1$ , that is,  $Z^H$  is a fractional Brownian motion, a deeper analysis of the density of the solution to (2) can be done. In particular, it is possible to prove Gaussian bounds for the density, see [21].

## REFERENCES

- [1] P. Catuogno and C. Oliveira,  $L^p$  solutions of the stochastic transport equation. *Random Operators and Stochastic Equations*, **21** (2013), 125–134.
- [2] P. L. Chow, *Stochastic Partial Differential Equations*, 2<sup>nd</sup> edition. Advances in Applied Mathematics. CRC Press, Boca Raton, FL, 2015.
- [3] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 3<sup>rd</sup> edition, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), 325. Springer-Verlag, 2010.
- [4] J. Duan, H. Gao and B. Schmalfuss. Stochastic dynamics of a coupled atmosphere-ocean model, *Stochastics and Dynamics*, **2** (2002), 357–380.
- [5] F. Fedrizzi and F. Flandoli. Noise prevents singularities in linear transport equations, *Journal of Functional Analysis*, **264** (2013), 1329–1354.
- [6] M. Ferrante and C. Rovira, Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter  $H > \frac{1}{2}$ , *Bernoulli* **12** (2006), 85–100.
- [7] F. Flandoli, M. Gubinelli and E. Priola, Well-posedness of the transport equation by stochastic perturbation, *Invent. Math.*, **180** (2010), 1–53.
- [8] F. Flandoli and F. Russo. Generalized integration and stochastic ODEs, *Annals of Probability*, **30** (2002), 270–292.

- [9] H. Kunita Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms, in *Ecole d'été de probabilités de Saint-Flour XII - 1982*, Lecture Notes in Mathematics (1984) 143–303, Springer, Berlin.
- [10] H. Kunita, First order stochastic partial differential equations, in *Proceedings of the Taniguchi International Symposium on Stochastic Analysis*, North-Holland Mathematical Library, (1984) 249–269.
- [11] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, 1990.
- [12] P. L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I: Incompressible Models*. Oxford Lecture Series in Mathematics and its applications, 3 (1996), Oxford University Press.
- [13] P. L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. II: Compressible Models*. Oxford Lecture Series in Mathematics and its applications, 10 (1998), Oxford University Press.
- [14] M. Maurelli. Wiener chaos and uniqueness for stochastic transport equation, *Comptes Rendus Mathématique*, **349** (2011), 669–672.
- [15] I. Nourdin. *Selected Aspects of Fractional Brownian Motion*. (2013), Bocconi-Springer.
- [16] I. Nourdin and F. Viens. Density formula and concentration inequalities with Malliavin calculus, *Electronic Journal of Probability*, **14** (2009), 2287–2309.
- [17] D. Nualart. *Malliavin Calculus and Related Topics*. 2<sup>nd</sup> edition, Springer New York 2006.
- [18] D. Nualart and L. Quer-Sardanyons, Gaussian density estimates for solutions to quasi-linear stochastic partial differential equations, *Stoch. Process. Appl.*, **119** (2009), 3914–3938.
- [19] B. Oksendal, *Stochastic Differential Equations*. Springer-Verlag 2003.
- [20] C. Olivera . Well-posedness of first order semilinear PDE's by stochastic perturbation, *Non-linear Anal.*, **96** (2014), 211–215.
- [21] C. Olivera and C. A. Tudor. The density of the solution to the transport equation with fractional noise, *Journal of Mathematical Analysis and Applications*, **431** (2015), 57–72.
- [22] B. Perthame. *Transport Equations in Biology*. Series 'Frontiers in Mathematics', Birkhauser, 2007.
- [23] V. Pipiras and M. Taqqu, Integration questions related to the fractional Brownian motion, *Probability Theory and Related Fields*, **118** (2001), 251–281.
- [24] F. Russo and P. Vallois, Forward, backward and symmetric stochastic integration, *Probab. Theory Rel. Fileds*, **97** (1993), 403–421.
- [25] F. Russo and P. Vallois. Elements of stochastic calculus via regularization, in *Séminaire de Probabilités XL*, Lecture Notes in Mathematics, **1899** (2007), 147–186.
- [26] M. Sanz-Solé. *Malliavin Calculus. With Applications to Stochastic Partial Differential Equations*. Fundamental Sciences, EPFL Press, Lausanne (2005).
- [27] I. Shigekawa. Derivatives of Wiener functionals and absolute continuity of induced measures. *J. Math. Kyoto Univ.* **20** (1980), 263–289.
- [28] C. A. Tudor. *Analysis of Variations for Self-similar Processes*. Springer 2013.

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