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# A general decomposition theory for the 1-2-3 Conjecture and locally irregular decompositions<sup>☆</sup>

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## Abstract

How can one distinguish the adjacent vertices of a graph through an edge-weighting? In the last decades, this question has been attracting increasing attention, which resulted in the active field of distinguishing labellings.

One of its most popular problems is the one where neighbours must be distinguishable via their incident sums of weights. An edge-weighting verifying this is said neighbour-sum-distinguishing. The popularity of this notion arises from two reasons. A first one is that designing a neighbour-sum-distinguishing edge-weighting showed up to be equivalent to turning a simple graph into a locally irregular (*i.e.*, without neighbours with the same degree) multigraph by adding parallel edges, which is motivated by the concept of irregularity in graphs. Another source of popularity is probably the influence of the famous 1-2-3 Conjecture, which claims that such weightings with weights in  $\{1, 2, 3\}$  exist for graphs with no isolated edge.

The 1-2-3 Conjecture has recently been investigated from a decompositional angle, via so-called locally irregular decompositions, which are edge-partitions into locally irregular subgraphs. Through several recent studies, it was shown that this concept is quite related to the 1-2-3 Conjecture. However, the full connexion between all those concepts was not clear.

In this work, we propose an approach that generalizes all concepts above, involving coloured weights and sums. As a consequence, we get another interpretation of several existing results related to the 1-2-3 Conjecture. We also come up with new related conjectures, to which we give some support.

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## 1. Introduction

The current work is mainly related to the well-known **1-2-3 Conjecture**, which is defined accordingly to the upcoming notions. Let  $G$  be a graph, and let  $\omega$  be an edge-weighting (assigning weights among  $\{1, \dots, k\}$ ) of  $G$ . From  $\omega$ , one can design the vertex-colouring  $\sigma$  of  $G$  where each vertex  $v$  gets assigned, as its colour  $\sigma(v)$ , the sum of weights (called its *weighted degree*) assigned to its incident edges. That is, for every vertex  $v$  of  $G$  we have

$$\sigma(v) := \sum_{u \in N(v)} \omega(uv),$$

where  $N(v)$  denotes the set of neighbours of  $v$ . In case  $\sigma$  is actually a proper vertex-colouring of  $G$ , *i.e.*, we have  $\sigma(u) \neq \sigma(v)$  for every two adjacent vertices  $u$  and  $v$ , then we call  $\omega$  *neighbour-sum-distinguishing*.

For any two graphs  $G$  and  $H$ , we say that  $G$  *has no isolated  $H$*  if no connected component of  $G$  is isomorphic to  $H$ . Note that all graphs with no isolated edge admit neighbour-sum-distinguishing edge-weightings (consider *e.g.* an inductive argument). Graphs with no such connected components are thus called *nice*, with respect to this edge-weighting notion. The 1-2-3 Conjecture, posed in 2004 by Karoński, Łuczak and Thomason [9], asks whether, for every nice graph, we can design neighbour-sum-distinguishing 3-edge-weightings, *i.e.*, using weights 1, 2, 3 only. More precisely, if we denote by  $\chi_{\Sigma}(G)$  the least  $k$  such that a nice graph  $G$  admits a neighbour-sum-distinguishing  $k$ -edge-weighting, then it is believed that  $\chi_{\Sigma}(G) \leq 3$  should always hold.

**1-2-3 Conjecture** (Karoński, Łuczak, Thomason [9]). *For every nice graph  $G$ , we have  $\chi_{\Sigma}(G) \leq 3$ .*

Despite many active investigations in the last decade, the 1-2-3 Conjecture is still wide open to date. These investigations have been mainly focused on 1) proving the 1-2-3 Conjecture for new classes of nice graphs, 2) proving general constant upper bounds on  $\chi_{\Sigma}$ , and 3) studying side aspects of the 1-2-3 Conjecture. As the literature on this topic is vast, a brief summary of some of these investigations is deferred to the next section.

The current work is also related to **locally irregular decompositions**, which were considered as a decompositional approach towards understanding some aspects behind the 1-2-3 Conjecture. In general, by a *decomposition* of a graph, we mean an edge-colouring where each colour class yields a graph with particular properties. A *locally irregular graph* is a graph in which every two adjacent vertices have distinct degrees. By a *locally irregular decomposition* of a graph, we thus mean a decomposition into locally irregular

graphs. Sticking to the edge-colouring point of view, we will also sometimes instead speak of a *locally irregular edge-colouring*.

Locally irregular decompositions relate to the 1-2-3 Conjecture through, notably, the following arguments. In a sense, the graphs  $G$  that are the “most convenient” for the 1-2-3 Conjecture are those which verify  $\chi_\Sigma(G) = 1$ . Those graphs are precisely the locally irregular ones. Also, in particular contexts, locally irregular decompositions can be turned into neighbour-sum-distinguishing edge-weightings. Perhaps the best illustration of that claim is the fact that, in any regular graph, a neighbour-sum-distinguishing 2-edge-weighting yields a decomposition into two locally irregular graphs, and vice versa.

Similarly as for neighbour-sum-distinguishing edge-weightings, there exist graphs which do not admit any locally irregular decomposition; but, this time, the class of *exceptional graphs* is much wider (consider *e.g.* any path of odd length). An exceptional graph (with respect to locally irregular decompositions) is also called an *exception*, for short. Conversely, a graph that is not exceptional is said *decomposable*. For a decomposable graph  $G$ , we denote by  $\chi'_{\text{irr}}(G)$  the least  $k$  such that  $G$  admits a locally irregular  $k$ -edge-colouring. Similarly as in the case of the 1-2-3 Conjecture, it is believed that every decomposable graph should decompose into at most three locally irregular graphs, as conjectured by some of the authors of the present paper:

**Conjecture 1.1** (Baudon, Bensmail, Przybyło, Woźniak [3]). *For every decomposable graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 3$ .*

Conjecture 1.1 was first verified for a few graph classes. Also, general constant upper bounds on  $\chi'_{\text{irr}}$  were recently exhibited. For the sake of keeping the introduction short, we again refer the reader to the next section for a survey of some of these results.

In this work, we aim at introducing a general decompositional theory enclosing neighbour-sum-distinguishing edge-weightings and locally irregular decompositions. This theory is based on the following observations. A locally irregular  $\ell$ -edge-colouring of a graph  $G$  is, put differently, a decomposition of  $G$  into graphs  $G_1, \dots, G_\ell$  verifying  $\chi_\Sigma(G_1), \dots, \chi_\Sigma(G_\ell) = 1$ . The other way around, a neighbour-sum-distinguishing  $k$ -edge-weighting of  $G$  can be seen as a 1-edge-colouring where the only colour class induces a graph, that is precisely  $G$ , whose value of  $\chi_\Sigma$  is  $k$ .

These observations lead us to combine the notions of neighbour-sum-distinguishing edge-weightings and locally irregular edge-colourings, in the following way. Let  $\ell, k \geq 1$  be two integers, and  $G$  be a graph. To each edge of  $G$ , we assign, via a colouring  $\omega$ , a pair  $(\alpha, \beta)$ , where  $\alpha \in \{1, \dots, \ell\}$  and  $\beta \in \{1, \dots, k\}$ , which can be regarded as a *coloured weight* (with value  $\beta$  and colour  $\alpha$ ). Now, for every vertex  $v$  of  $G$ , and every colour  $\alpha \in \{1, \dots, \ell\}$ , one can compute the *weighted  $\alpha$ -degree*  $\sigma_\alpha(v)$ , being the sum of weights

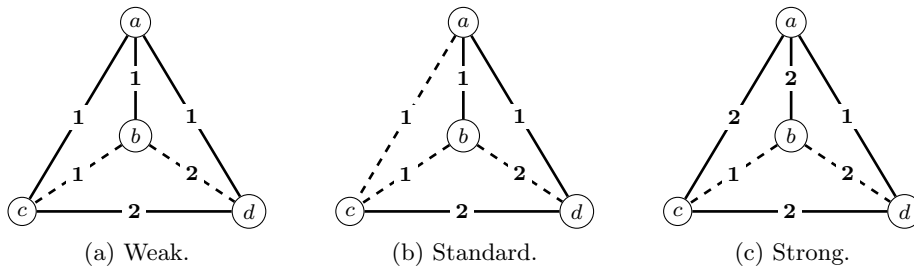


Figure 1: Three  $(2, 2)$ -colourings of  $K_4$ .

with colour  $\alpha$  incident to  $v$ . So, to every vertex  $v$  is associated a palette  $(\sigma_1(v), \dots, \sigma_\ell(v))$  of  $\ell$  coloured weighted degrees.

When working on variants of the 1-2-3 Conjecture, the intent is to design edge-weightings  $\omega$  that allow to distinguish the adjacent vertices, accordingly to some distinction condition. When dealing with the notions introduced in the last paragraph, there are many ways for asking for distinction, as several coloured sums are available; in this work, we will focus on the following three distinction variants, which sound the most natural to us:

- *Weak distinction*: two adjacent vertices  $u$  and  $v$  of  $G$  are considered distinguished if there is an  $\alpha \in \{1, \dots, \ell\}$  such that  $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ .
- *Standard distinction*: two adjacent vertices  $u$  and  $v$  of  $G$  are considered distinguished if, assuming  $\omega(uv) = (\alpha, \beta)$ , we have  $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ .
- *Strong distinction*: two adjacent vertices  $u$  and  $v$  of  $G$  are considered distinguished if, for every  $\alpha \in \{1, \dots, \ell\}$ , we have  $\sigma_\alpha(u) = \sigma_\alpha(v) = 0$ , or  $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ .

Assuming  $\omega$  verifies one of the weak, standard or strong distinction condition for every pair of adjacent vertices, we say that  $\omega$  is a *weak*, *standard* or *strong*  $(\ell, k)$ -edge-colouring, and that  $G$  is *weakly*, *standardly* or *strongly*  $(\ell, k)$ -coloured. We also say that  $G$  is *weakly*, *standardly* or *strongly*  $(\ell, k)$ -colourable, if there are  $\ell', k' \geq 1$  with  $\ell' \leq \ell$  and  $k' \leq k$  such that  $G$  can be weakly, standardly or strongly  $(\ell', k')$ -coloured, respectively.

We provide, in Figure 1, an illustration of these concepts on  $K_4$ , the complete graph on four vertices, where the two colours are represented by solid and dashed edges. By the “incident solid sum” of a vertex, we here mean the sum of weights assigned to its incident solid edges. It can be checked that, in Figure 1.(a), the depicted  $(2, 2)$ -colouring is a weak colouring. It is however not a standard  $(2, 2)$ -colouring as vertices  $c$  and  $d$  are joined by a solid edge but their incident solid sums equal 3. The colouring in Figure 1.(b) is a standard  $(2, 2)$ -colouring which is not a strong colouring,

in particular because vertices  $a$  and  $c$  both have incident solid sum 2. The colouring in Figure 1.(c) is a strong  $(2, 2)$ -colouring.

This paper is organized as follows. As already mentioned, the notions of weak, standard and strong  $(\ell, k)$ -colourings can be employed to generalize neighbour-sum-distinguishing edge-weightings and locally irregular edge-colourings. In Section 2, we explore these connexions. In particular, we recall known results and translate them in our new terminology.

Playing with the parameters  $\ell$  and  $k$  and the distinction conditions, we also come up with new problems, some of which we believe are of independent interest. In particular, we wonder whether almost all graphs can be weakly, standardly, or even strongly  $(2, 2)$ -coloured. If true, this would imply side decomposition results related to the 1-2-3 Conjecture. The strong, standard and weak versions of that question are formally introduced in Section 3. They are then studied in Sections 4, 5 and 6, respectively.

## 2. Previous results and connexions to $(\ell, k)$ -colourings

As a warm up, we start, in Section 2.1, by making first observations and remarks on weak, standard and strong colourings. We then survey, in Section 2.2, some of the results from literature that are directly connected to these notions. More precisely, we explain which notions in the literature are encompassed by weak, standard and strong colourings, and, by rephrasing known results under that new terminology, we exhibit first results.

### 2.1. Early observations

First of all, we note that, according to the definitions, every result holding for some version of  $(\ell, k)$ -colourings also holds for the weaker versions. This is why, throughout Sections 4 to 6, we start by considering strong colourings, then standard colourings, and, finally, weak colourings.

**Observation 2.1.** *A strong  $(\ell, k)$ -colouring is also a standard  $(\ell, k)$ -colouring. Analogously, a standard  $(\ell, k)$ -colouring is also a weak  $(\ell, k)$ -colouring.*

In general, though, it can be observed that the converse direction is not true, *i.e.*, that a given  $(\ell, k)$ -colouring does not necessarily fulfil stronger distinction conditions. A good illustration for that is the fact that  $K_3$  can be weakly  $(2, 2)$ -coloured but not standardly  $(2, 2)$ -coloured. There are situations, though, where all distinction conditions behave similarly. We state a few of them below.

First of all, we recall that, for some values of  $\ell$  and  $k$ , some versions of  $(\ell, k)$ -colourings are equivalent to other kinds of distinguishing colourings and weightings. Most of these observations are straightforward, and thus do not need a formal proof. In particular, it can easily be checked that some of these results do not hold for stronger or weaker versions of our colouring variants.

**Observation 2.2.** *Weak, standard and strong  $(1, k)$ -colourings and neighbour-sum-distinguishing  $k$ -edge-weightings are equivalent notions.*

**Observation 2.3.** *Standard  $(k, 1)$ -colourings and locally irregular  $k$ -edge-colourings are equivalent notions.*

Let  $G$  be a graph, and  $\omega$  be an edge-weighting of  $G$ . For each vertex  $v$  of  $G$ , one can compute its *multiset*  $\mu(v)$  of incident weights induced by  $\omega$ . We say that  $\omega$  is *neighbour-multiset-distinguishing* if no two adjacent vertices of  $G$  get the same multiset of incident weights. Note that having  $\sigma(u) \neq \sigma(v)$  for an edge  $uv$  of  $G$  implies that  $\mu(u) \neq \mu(v)$  (but the converse is not necessarily true). For this reason, neighbour-multiset-distinguishing edge-weightings have been studied as a weaker form of neighbour-sum-distinguishing edge-weightings.

The point for mentioning neighbour-multiset-distinguishing edge-weightings is that they relate to our notion of weak colourings.

**Observation 2.4.** *Weak  $(k, 1)$ -colourings and neighbour-multiset-distinguishing  $k$ -edge-weightings are equivalent notions.*

In Observation 2.2, we noticed that, for  $(1, k)$ -colourings, all three distinction conditions are equivalent. In the following result, we point out another context where the three colouring variants coincide.

**Observation 2.5.** *In regular graphs, weak, standard and strong  $(2, 1)$ -colourings are equivalent notions.*

## 2.2. Previous results

In this section, we restate, in our terminology, several results from the literature on distinguishing weightings and colourings to derive the existence of particular  $(1, k)$ - or  $(\ell, 1)$ -colourings. In other words, we here point out how our colouring concepts encapsulate existing distinguishing weightings and colourings.

This section is not intended to be a full survey on variants of the 1-2-3 Conjecture. Hence, we voluntarily focus on those existing results that are closely related to our investigations; for more details, please refer to the survey [12] by Seamone.

### 2.2.1. Neighbour-sum-distinguishing edge-weightings

Recall that, according to Observation 2.2, being strongly  $(1, k)$ -colourable is equivalent to being neighbour-sum-distinguishing  $k$ -edge-weightable. Thus, all general constant upper bounds on  $\chi_\Sigma$  yield results on strong colourability (hence on the weaker variants as well, recall Observation 2.1).

In the context of neighbour-sum-distinguishing edge-weightings, the leading conjecture is the 1-2-3 Conjecture. If true, that conjecture would imply that every nice graph is strongly  $(1, 3)$ -colourable. Recall that nice graphs are exactly those graphs with no isolated edges.

**Conjecture 2.6.** *Every nice graph is strongly  $(1, 3)$ -colourable.*

To date, the best result towards the 1-2-3 Conjecture was given by Kalkowski, Karoński and Pfender [8], who proved that  $\chi_\Sigma(G) \leq 5$  holds for every nice graph  $G$ . As said above, this result can be stated as follows, using our terminology.

**Theorem 2.7.** *Every nice graph is strongly  $(1, 5)$ -colourable.*

The 1-2-3 Conjecture was shown to hold for several common classes of nice graphs, such as complete graphs and 3-colourable graphs. There exist graphs  $G$  verifying  $\chi_\Sigma(G) = 3$ , such as complete graphs of order at least 3. One natural question is thus whether such graphs are easy to characterize. Dudek and Wajc settled the question in the negative [6], by showing that determining the exact value of  $\chi_\Sigma(G)$  is an NP-complete problem. Later on, Ahadi, Dehghan and Sadeghi [2] proved that this remains true when restricted to regular (cubic) graphs. This result is of prime interest, as all distinguishing weighing and colouring notions considered in this paper tend to be equivalent when 1) only two weights or colours are considered, and 2) the graph is regular (recall Observation 2.5). This result, by itself, directly establishes the general hardness of weak, standard and strong colourings.

It took some time to settle this complexity question for bipartite graphs. In a first work [7], Chang, Lu, Wu and Yu provided several sufficient conditions for a nice bipartite graph  $G$  to verify  $\chi_\Sigma(G) \leq 2$ . In particular, they showed that being connected and having one of the two partite sets of even cardinality is a sufficient condition, and, from this result, they easily proved that nice trees always admit neighbour-sum-distinguishing 2-edge-weightings. Later on, the full characterization of connected bipartite graphs  $G$  with  $\chi_\Sigma(G) = 3$  was given by Thomassen, Wu and Zhang [13], who proved that they are exactly the *odd multicacti*. These graphs can be constructed as follows. Start from  $m \geq 1$  cycles  $C_1, \dots, C_m$  whose lengths are at least 6 and congruent to 2 modulo 4, and colour the edges of the  $C_i$ 's using colours red and green alternately. Then, an odd multicactus is any connected graph obtained from the  $C_i$ 's via repeated applications of the following operation: pick two connected components  $G_1$  and  $G_2$ , and identify a green edge of  $G_1$  with a green edge of  $G_2$ . Said differently, an odd multicactus is obtained by identifying edges of particular cycles in a tree-like fashion. In particular, every cycle with length congruent to 2 modulo 4 is an odd multicactus.

**Theorem 2.8** (Thomassen, Wu, Zhang [13]). *A connected bipartite graph  $G$  verifies  $\chi_\Sigma(G) = 3$  if and only if  $G$  is an odd multicactus.*

### 2.2.2. Locally irregular edge-colourings

By Observation 2.3, we get that locally irregular  $k$ -edge-colourings are precisely standard  $(k, 1)$ -colourings. We thus survey some of the research



on locally irregular edge-colourings, as they transfer to standard  $(k, 1)$ -colourings.

As mentioned in Section 1, not all graphs decompose into locally irregular graphs, so one has to deal with so-called exceptions. In their first work on this topic [3], Baudon, Bensmail, Przybyło and Woźniak completely characterized all connected exceptions. Namely, connected exceptions include 1) odd-length paths, 2) odd-length cycles, and 3) the family  $\mathfrak{T}$  defined recursively as follows:

- The triangle  $K_3$  belongs to  $\mathfrak{T}$ .
- Every other graph in  $\mathfrak{T}$  can be constructed by 1) taking an auxiliary graph  $F$  being either a path of even length or a path of odd length with a triangle glued to one of its ends, then 2) choosing a graph  $G \in \mathfrak{T}$  containing a triangle with at least one vertex, say  $v$ , of degree 2 in  $G$ , and finally 3) identifying  $v$  with a vertex of degree 1 of  $F$ .

Note that all connected exceptions have maximum degree at most 3.

Thus, a graph is decomposable if and only if it has no exception as a connected component. Once the set of exceptions was characterized, Baudon, Bensmail, Przybyło and Woźniak conjectured that every decomposable graph  $G$  should decompose into at most three locally irregular graphs, *i.e.*,  $\chi'_{\text{irr}}(G) \leq 3$ . Due to Observation 2.3, this conjecture can be restated as follows:

**Conjecture 2.9.** *Every decomposable graph is standardly  $(3, 1)$ -colourable.*

The first constant upper bound on  $\chi'_{\text{irr}}$  is due to Bensmail, Merker and Thomassen [5], who proved that we have  $\chi'_{\text{irr}}(G) \leq 328$  for every decomposable graph  $G$ . This bound was recently improved down to 220 by Lužar, Przybyło and Soták [10]. We can thus state the following:

**Theorem 2.10.** *Every decomposable graph is standardly  $(220, 1)$ -colourable.*

Baudon, Bensmail, Przybyło and Woźniak verified Conjecture 2.9 for several decomposable graph classes [3], including complete graphs, some bipartite graphs, some Cartesian products, and regular graphs with degree at least  $10^7$ . Later on, Przybyło [11] verified the conjecture for graphs with minimum degree at least  $10^{10}$ . The complexity aspects were considered by Baudon, Bensmail and Sopena [4], who proved that, for a given graph  $G$ , deciding whether  $\chi'_{\text{irr}}(G) = 2$  is NP-complete in general, while determining  $\chi'_{\text{irr}}(G)$  can be done in polynomial time when  $G$  is a tree.

### 2.2.3. Neighbour-multiset-distinguishing edge-weightings

As mentioned in the previous section, all neighbour-sum-distinguishing edge-weightings are neighbour-multiset-distinguishing, but the converse is

not always true. The connexion between these two notions was first considered by Karoński, Łuczak and Thomason in the paper introducing the 1-2-3 Conjecture [9]. The first formal study of neighbour-multiset-distinguishing edge-weightings may be attributed to Addario-Berry, Aldred, Dalal and Reed, who, later on, gave improved results towards a “multiset version” of the 1-2-3 Conjecture [1]. In our terminology, this conjecture reads as follows:

**Conjecture 2.11.** *Every nice graph is weakly  $(3, 1)$ -colourable.*

So far, the best result towards Conjecture 2.11 is hence due to Addario-Berry, Aldred, Dalal and Reed, who proved that all nice graphs admit neighbour-multiset-distinguishing 4-edge-weightings [1].

**Theorem 2.12.** *Every nice graph is weakly  $(4, 1)$ -colourable.*

All graph classes verifying the 1-2-3 Conjecture also verify Conjecture 2.11. Additionally, the latter conjecture was also verified for graphs with minimum degree at least 1000, see [1].

### 3. New problems

As seen in Section 2, some of the  $(1, k)$ -colouring and  $(\ell, 1)$ -colouring variants correspond to distinguishing weighting and colouring notions already considered in the literature. In particular, for such values of  $\ell$  and  $k$ , there is still some gap between the corresponding conjectures and the best results we know to date. One way to get some sort of side progress, could be to prove the existence of  $(\ell, k)$ -colourings (for some distinction condition) where  $\ell + k$  or  $\max\{\ell, k\}$  is as small as possible.

In particular, the main problem we consider in the rest of this paper, which corresponds to minimizing  $\max\{\ell, k\}$ , and to which we could not find any obvious counterexample, reads as follows. By a *nicer graph*, we mean a graph with no isolated edges and triangles.

**Conjecture 3.1.** *Every nicer graph is strongly  $(2, 2)$ -colourable.*

The main reason for suspecting that  $K_2$  and  $K_3$  might be the only connected graphs admitting no strong  $(2, 2)$ -colourings is that they are the only connected exceptional graphs (recall the exact characterization in Subsection 2.2.2) admitting no neighbour-sum-distinguishing 2-edge-weightings.

**Observation 3.2.** *Every connected exception different from  $K_2$  and  $K_3$  verifies Conjecture 3.1.*

*Proof.* Let  $G$  be a connected exception different from  $K_2$  and  $K_3$ . We consider several cases corresponding to the three families of connected exceptions given by the definition:

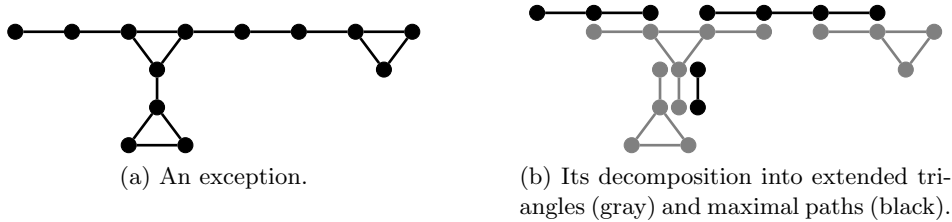


Figure 2: Decomposing an exception as described in the proof of Observation 3.2.

- If  $G$  is an odd-length path, then  $G$  is a connected bipartite graph different from an odd multicactus, thus verifies  $\chi_\Sigma(G) \leq 2$  according to Theorem 2.8, and hence admits strong  $(1, 2)$ -colourings.
- If  $G$  is an odd-length cycle with length at least 5, then  $G$  can be decomposed into two paths  $P_r, P_b$  with length at least 2. In particular, the end-vertices of  $P_r$  (and similarly  $P_b$ ) are not adjacent in  $G$ , and we have  $\chi_\Sigma(P_r), \chi_\Sigma(P_b) \leq 2$ . By considering a strong  $(1, 2)$ -colouring of  $P_r$  (with red weights) and a strong  $(1, 2)$ -colouring of  $P_b$  (with blue weights), we eventually get a strong  $(2, 2)$ -colouring of  $G$ .
- Finally assume that  $G \in \mathfrak{T} \setminus \{K_3\}$ . By contracting the triangles (there is at least one such) of  $G$  to vertices, we obtain a tree  $R(G)$  with maximum degree 3, whose some nodes (*triangle nodes*) correspond to triangles of  $G$ , while some nodes (*normal nodes*) correspond to real vertices. Furthermore, by definition, any path of  $R(G)$  joining two triangle nodes has odd length, and any path joining a triangle node and a pendant normal node has even length.

We can consider  $G$  as a collection of triangles with at most three pendant edges attached (*extended triangles*), and paths with one or two ends attached to a triangle (*maximal paths*) (see Figure 2 for an example). The pendant edges attached to the extended triangles, as well as the end-edges incident to triangles of the maximal paths, are called *attachment edges*. According to these definitions,  $G$  can be constructed from extended triangles and maximal paths by glueing their attachment edges. In particular, every attachment edge belongs to one extended triangle and one maximal path.

Necessarily  $R(G)$  has a degree-1 node  $r$ , being either a triangle node (pendant triangle in  $G$ ) or a normal node (pendant vertex in  $G$ ). Consider the (virtual) orientation of the edges of  $R(G)$  from  $r$  towards the leaves. We construct a strong  $(2, 2)$ -colouring (assigning weights coloured red and blue) iteratively, by extending a colouring along extended triangles and maximal paths following the ordering given by the orientation of the attachment edges. Since  $R(G)$  is a tree, note

that once an attachment edge is coloured, this provides a pre-colouring of the next extended triangle or maximal path to be coloured.

We start constructing the colouring from  $r$ . In  $G$ , node  $r$  corresponds either to an end-vertex of a maximal path  $P$  (normal node), or to a triangle  $T$  (triangle node). In the first case, let  $P := v_1 \dots v_{2k}$ ; then we just assign red weights  $1, 2, 2, 1, 1, \dots$  along  $P$ . In the second case, let  $T := v_1 v_2 v_3 v_1$ , and let  $v'_1$  denote, without loss of generality, the neighbour of  $v_1$  outside  $T$ ; we here assign red weight 1 to  $v_3 v_2$  and red weight 2 to  $v_2 v_1$ , and blue weight 1 to  $v_3 v_1$  and blue weight 2 to  $v_1 v'_1$ . In any case, it can be checked that the colouring is correct so far.

We now proceed to the general case, *i.e.*, we consider a maximal path  $P$  or extended triangle  $T$  whose one attachment edge is coloured, and we extend the colouring to all its other attachment edges in  $G$ . Consider first a maximal path  $P := v_1 \dots v_k$  whose attachment edge  $v_1 v_2$  was assigned, say, a red weight. We here extend the colouring to all edges of  $P$  by assigning red weights (with value 1 or 2) to its edges  $v_2 v_3, \dots, v_{k-1} v_k$  successively. Note that this can be done correctly, as, when a red weight is being assigned to an edge  $v_i v_{i+1}$ , we just have to make sure that the red sum of  $v_i$  avoids the red sum of  $v_{i-1}$ , which is possible since we have two red weights to play with.

We are left with the case where the colouring must be extended to an extended triangle  $T := v_1 v_2 v_3 v_1$  whose one attachment edge, say  $v_1 v'_1$ , was previously assigned, say, a red weight. We here consider cases depending on the number of additional attachment edges:

- If  $v_1 v'_1$  is the only attachment edge of  $T$ , then we assign a red weight to  $v_1 v_2$  so that the red sum of  $v_1$  does not get equal to the red sum of  $v'_1$ . We then assign blue weights  $1, 2$  or  $2, 1$  to  $v_1 v_3$  and  $v_3 v_2$  in such a way that the blue sum of  $v_1$  does not get equal to the blue sum of  $v'_1$ .
- Assume  $v_2 v'_2$  is the only other attachment edge of  $T$ . We here assign a red weight to  $v_1 v_3$  in such a way that the red sum of  $v_1$  does not get equal to the red sum of  $v'_1$ . We then assign blue weights  $1, 2, 1$  or  $2, 1, 1$  to  $v_2 v_1, v_2 v_3$  and  $v_2 v'_2$  in such a way that the blue sum of  $v_1$  does not get equal to the blue sum of  $v'_1$ .
- Lastly, assume  $v_2 v'_2$  and  $v_3 v'_3$  are attachment edges. First, we assign blue weight 1 to  $v_1 v_2$  and blue weight 2 to  $v_1 v_3$ . We now assign red weight 1 to  $v'_2 v_2$ , red weight  $\alpha$  to  $v_2 v_3$  and red weight 2 to  $v_3 v'_3$ , where  $\alpha$  is the red weight of  $v'_1 v_1$ .

In any of these cases, it can be checked that the colouring extension is correct. So this covers all cases of the proof.  $\square$

The rest of this paper is dedicated to providing evidences towards Conjecture 3.1. We do it gradually, by first considering, in Section 4, Conjecture 3.1 in its literal form. We then consider its standard version (in Section 5), before finally considering its weak version (in Section 6).

#### 4. Strong $(\ell, k)$ -colouring

In this section, we consider Conjecture 3.1 in its literal form, namely:

**Strong Conjecture.** *Every nicer graph is strongly  $(2, 2)$ -colourable.*

We verify the Strong Conjecture for nice complete graphs and bipartite graphs. Recall that every result on strong  $(2, 2)$ -colourings directly transfers to standard and weak  $(2, 2)$ -colourings.

We start off with complete graphs. For every  $n \geq 1$ , we denote by  $K_n$  the complete graph with order  $n$ .

**Theorem 4.1.** *For every  $n \geq 4$ , the graph  $K_n$  is strongly  $(2, 2)$ -colourable.*

*Proof.* We prove the claim by induction on  $n$ . To ease the proof, we prove a stronger statement, namely that every complete graph  $K_n$  admits a strong  $(2, 2)$ -colouring with red and blue weights such that either there is no vertex incident to red edges only, or there is no vertex incident to blue edges only.

As a base step, consider  $K = K_4$ . Note that  $K$  can be decomposed into two paths  $P_r$  and  $P_b$  of length 3. To get a strong  $(2, 2)$ -colouring, we proceed as follows. Consider first the edges of  $P_r$  from one end to the other, and assign them red weights 1, 2, 2, respectively. Similarly, then consider the edges of  $P_b$  from one end to the other, and assign them blue weights 1, 2, 2, respectively. Since  $P_r$  and  $P_b$  span all vertices of  $K$ , each vertex gets a non-zero red sum and a non-zero blue sum. This, by itself, guarantees that the additional requirement is fulfilled (*i.e.*, there is no monochromatic vertex). Now, due to how the red weights were assigned, it can easily be seen that the obtained red sums are 1, 2, 3, 4; hence no two vertices get the same red sums. As this is also the case for the blue sums, we have thus constructed a strong  $(2, 2)$ -colouring of  $K$ .

We now prove the general case. Let  $K = K_n$  (where  $n \geq 5$ ), and remove one vertex  $v$  from  $K$ . We end up with a graph isomorphic to  $K_{n-1}$ , which, by the induction hypothesis, admits a strong  $(2, 2)$ -colouring with colours red and blue. Furthermore, we may, without loss of generality, assume that, by this colouring, there is no vertex incident to red edges only. We extend this colouring to  $K$ , *i.e.*, to the edges incident to  $v$ , by assigning red weight 2 to all those edges. As a result, all red sums of the vertices of  $V(K) \setminus \{v\}$  rise by 2, and since every two of them were different, they still are after the extension. Now, note that the red sum of  $v$  is precisely  $2(n-1)$ , which is strictly greater than all the other red sums since all vertices of  $V(K) \setminus \{v\}$

are incident to blue edges. Furthermore, the blue sums of the vertices of  $V(K) \setminus \{v\}$  have not been altered, while  $v$  has blue sum 0 – so no two non-zero blue sums are the same. We thus get a strong  $(2, 2)$ -colouring of  $K$ , and it can be noted that no vertex is incident to blue edges only, as additionally required.  $\square$

We now prove the Strong Conjecture for bipartite graphs. Recall that a connected bipartite graph  $G$  verifies  $\chi_\Sigma(G) = 3$  if and only if it is an odd multicactus (Theorem 2.8).

**Theorem 4.2.** *Every nice bipartite graph  $G$  is strongly  $(2, 2)$ -colourable.*

*Proof.* We can assume that  $G$  is connected. If  $G$  is not an odd multicactus, then  $\chi_\Sigma(G) \leq 2$ , and, equivalently,  $G$  is strongly  $(1, 2)$ -colourable. So let us now assume that  $G$  is an odd multicactus. By construction, note that  $G$  necessarily has a degree-2 vertex  $v$ . Furthermore,  $G$  is 2-connected, so the graph  $G' := G - \{v\}$  is connected. Also,  $G'$  is not an odd multicactus (to be convinced of this, note that it has degree-1 vertices and that one of its partite sets is of even cardinality). So  $G'$  is strongly  $(1, 2)$ -colourable.

Consider thus a strong  $(1, 2)$ -colouring of  $G'$  assigning red weights. We extend this colouring to a strong  $(2, 2)$ -colouring of  $G$ , *i.e.*, to the edges  $u_1v$  and  $vu_2$  incident to  $v$ , by just assigning blue weights 1 and 2 to  $u_1v$  and  $vu_2$ , respectively. As no new edge was assigned a red weight, the adjacent red sums are still different in  $G$ . Furthermore, the only three non-zero blue sums are all different, as they are equal to 1, 2 and 3.  $\square$

In the rest of this section, we confirm that odd multicacti are a peculiar class of nice bipartite graphs for the distinguishing colouring notions we consider, in the following sense.

**Theorem 4.3.** *The connected nice bipartite graphs that cannot be strongly  $(1, 1)$ -,  $(1, 2)$ - or  $(2, 1)$ -coloured are exactly the odd multicacti.*

The proof of Theorem 4.3 relies on the following result on locally irregular decompositions of odd multicacti, which we believe is of independent interest, as there is still no known characterization of bipartite graphs  $G$  verifying  $\chi'_{\text{irr}}(G) \leq 2$ .

**Lemma 4.4.** *For every odd multicactus  $G$ , we have  $\chi'_{\text{irr}}(G) = 3$ .*

*Proof.* Let  $G$  be an odd multicactus. As such (recall the description in Subsection 2.2.1),  $G$  has edges coloured red and green “alternatively”. To avoid any confusion with the colours, in the rest of the proof we refer to the green edges of  $G$  as its *attachment edges*, while we refer to the red edges as its *support edges*.

Since  $G$  is an odd multicactus, by construction there has to be an attachment edge  $uv$  such that  $u$  and  $v$  are joined by several disjoint non-trivial

paths  $P_1, \dots, P_k$  of length congruent to 1 modulo 4, whose removal does not disconnect the graph. In some sense, the  $P_i$ 's are leaves in the tree representation of the construction of  $G$ . It is easy to see that, in a locally irregular 2-edge-colouring of  $G$ , necessarily every two subsequent support edges of the  $P_i$ 's must have different colours. Since the  $P_i$ 's have length congruent to 1 modulo 4, this means that, from the point of view of  $uv$ , colouring the  $P_i$ 's is similar to colouring  $k$  parallel edges joining  $uv$ . Said differently, if the multigraph  $G'$ , obtained by replacing the  $P_i$ 's by  $k$  parallel (attachment) edges joining  $u$  and  $v$ , admits no locally irregular 2-edge-colouring, so neither does  $G$ . This operation, consisting in contracting non-trivial paths joining a "leaf" attachment edge, is called a **contraction** below.

By repeatedly applying contractions (note that the argument above works even if the non-trivial paths have parallel attachment edges), we get a series of multigraphs  $G = G_0, G_1, \dots, G_m = G'$  such that 1) if  $G_{i+1}$  admits no locally irregular 2-edge-colourings, then so does not  $G_i$ , and 2)  $G'$  consists of two vertices joined by several parallel (attachment) edges. Now, it should be clear that  $G'$  admits no locally irregular 2-edge-colourings, which gives our conclusion for  $G$ .  $\square$

We can now prove Theorem 4.3:

*Proof of Theorem 4.3.* Let  $G$  be a connected nice bipartite graph. If  $G$  is not an odd multicactus, then  $\chi_\Sigma(G) \leq 2$  (Theorem 2.8), and hence  $G$  is strongly  $(1, 2)$ -colourable. So we may assume that  $G$  is an odd multicactus, and thus that  $G$  is not strongly  $(1, 2)$ -colourable. In that case, according to Lemma 4.4,  $G$  admits no locally irregular 2-edge-colourings, hence no strong  $(2, 1)$ -colourings.  $\square$

## 5. Standard $(\ell, k)$ -colouring

We here consider the standard weakening of Conjecture 3.1:

**Standard Conjecture.** *Every nicer graph is standardly  $(2, 2)$ -colourable.*

Note that a standard  $(\ell, k)$ -colouring is nothing but a decomposition into  $\ell$  graphs admitting neighbour-sum-distinguishing  $k$ -edge-weightings. From that perspective, it could be interesting to wonder whether graphs, in general, decompose into a constant number of graphs verifying the 1-2-3 Conjecture. We believe this is an interesting aspect to consider, as not many graphs are known to verify the 1-2-3 Conjecture.

Towards the Standard Conjecture, we thus also raise the following related conjecture, which is, in a sense, a weakening of the 1-2-3 Conjecture:

**Conjecture 5.1.** *Every nice graph decomposes into two graphs verifying the 1-2-3 Conjecture.*

In this section, towards the Standard Conjecture, we first improve Theorem 2.10 by showing that all nice graphs admit standard  $(40, 3)$ -colourings. We then prove the Standard Conjecture 5.1 for nicer 2-degenerate graphs and subcubic graphs, before proving Conjecture 5.1 for nice 9-colourable graphs.

### 5.1. Standard $(40, 3)$ -colourability

The proof of the following result follows the lines of one in [5], where Bensmail, Merker and Thomassen proved that decomposable graphs can be decomposed into at most 328 locally irregular graphs.

**Theorem 5.2.** *Every decomposable graph  $G$  is standardly  $(40, 3)$ -colourable.*

*Proof.* In  $G$ , we can find a locally irregular subgraph  $H_1$  such that  $G - E(H_1)$  has all of its connected components being of even size ([5], Lemma 2.1). If  $G$  already had even size, then  $H_1$  is empty. Still calling  $G$  the remaining graph, we can decompose  $G$  into a graph  $H_2$  with minimum degree at least  $10^{10}$  and a  $(2 \cdot 10^{10} + 2)$ -degenerate graph  $H_3$  whose all connected components are of even size ([5], Lemma 4.5). On the one hand, according to a result of Przybyło [11], we can decompose  $H_2$  into three (possibly empty) locally irregular graphs  $H_{2,1}, H_{2,2}, H_{2,3}$ . On the other hand,  $H_3$  can be decomposed into 36 bipartite graphs  $H_{3,1}, \dots, H_{3,36}$  whose all connected components are of even size ([5], Theorem 4.3).

Recall that every locally irregular graph  $H$  verifies  $\chi_\Sigma(H) = 1$ . Furthermore, all nice bipartite graphs verify the 1-2-3 Conjecture. From these arguments, using a set of 40 coloured weights 1, 2, 3 to independently weight the edges of each of the  $H_i$ 's and the  $H_{i,j}$ 's, we eventually get a standard  $(40, 3)$ -colouring of  $G$ .  $\square$

Since all connected nice exceptional graphs are 3-colourable, they verify the 1-2-3 Conjecture (see [12]), and are thus standardly  $(1, 3)$ -colourable. Together with Theorem 5.2, this yields the following:

**Theorem 5.3.** *Every nice graph  $G$  is standardly  $(40, 3)$ -colourable.*

### 5.2. The Standard Conjecture for 2-degenerate graphs and subcubic graphs

Recall that a graph is *2-degenerate* if every of its subgraphs has a vertex with degree at most 2. A *subcubic graph* is a graph  $G$  with maximum degree at most 3. If all vertices of  $G$  have degree precisely 3, then we call  $G$  *cubic*. Furthermore, if  $G$  is connected and not cubic, *i.e.*,  $G$  has vertices with degree 1 or 2, then we say that  $G$  is *strictly subcubic*.

We first prove the Standard Conjecture for 2-degenerate graphs (with a few exceptions). More precisely, we prove:

**Theorem 5.4.** *Every nicer 2-degenerate graph  $G$  is standardly  $(2, 2)$ -colourable.*



Our proof of Theorem 5.4 relies on the following lemma, which is proved later in this section.

**Lemma 5.5.** *Every nicer 2-degenerate graph  $G$  decomposes into two nice forests.*

*Proof of Theorem 5.4.* According to Lemma 5.5, we can decompose  $G$  into two forests  $F_r$  and  $F_b$  none of which has an isolated edge. Since every nice tree  $T$  verifies  $\chi_\Sigma(T) \leq 2$  (i.e., admits standard  $(1, 2)$ -colourings), each of  $F_r$  and  $F_b$ , independently, admits a standard  $(1, 2)$ -colouring; let  $\omega_r$  and  $\omega_b$  be any such for  $F_r$  and  $F_b$ , respectively. To get a standard  $(2, 2)$ -colouring of  $G$ , we consider all weights assigned by  $\omega_r$  and  $\omega_b$ , and colour red those weights originating from  $\omega_r$ , while we colour blue those weights originating from  $\omega_b$ .  $\square$

We are left with proving Lemma 5.5.

*Proof of Lemma 5.5.* Throughout the proof, which is by induction on  $|V(G)| + |E(G)|$ , we assume that  $G$  is connected. As a base case, it can be checked that the claim is true whenever  $|V(G)| \leq 4$ . In particular, under all conditions,  $G$  is either 1) a nice tree (in which case the claim holds trivially), 2) a triangle with a pendant vertex attached (which decomposes into two paths of length 2), 3) two triangles glued along an edge (which decomposes into a path of length 2 and a star with three leaves), or 4) a cycle of length 4 (which decomposes into two paths of length 2).

Let us thus proceed to the proof of the general case (in particular,  $|V(G)| \geq 5$ ). First assume that  $G$  has a degree-1 vertex  $v$ . Denote by  $u$  the neighbour of  $v$  in  $G$ , and let  $G' := G - \{v\}$ . Since  $|V(G)| \geq 5$ , note that  $G'$  cannot be  $K_2$  or  $K_3$ . So, by the induction hypothesis,  $G'$  decomposes into a red nice forest and a blue nice forest. Assuming  $u$  belongs to the red forest, we extend that decomposition to  $G$  by adding  $vu$  to the red forest.

Thus, we may assume that  $G$  has a degree-2 vertex  $v$ , with neighbours  $u_1, u_2$ . We distinguish two cases:

- First case:  $v$  is a cut-vertex. Let  $H_1$  and  $H_2$  be the two connected components of  $G - \{v\}$ , where  $u_i$  belongs to  $H_i$  for  $i = 1, 2$ , and set  $G_1 := H_1 + \{u_1v\}$  and  $G_2 := H_2 + \{u_2v\}$ . Since  $G$  has no degree-1 vertex, note that none of  $G_1$  and  $G_2$  is isomorphic to  $K_2$ . Also,  $v$  has degree 1 in both  $G_1$  and  $G_2$ , so none of  $G_1$  and  $G_2$  is isomorphic to  $K_3$ . By the induction hypothesis,  $G_1$  and  $G_2$  decompose into two nice forests. Note that these two decompositions, when combined in  $G$ , altogether form a decomposition of  $G$  into two nice forests.
- Second case:  $v$  is not a cut-vertex. Thus none of  $vu_1$  and  $vu_2$  is a cut-edge. Thus,  $G' := G - \{vu_1\}$  is not isomorphic to  $K_2$  or  $K_3$ , and,

by the induction hypothesis,  $G'$  decomposes into two nice forests, say red and blue. Assume  $vu_2$  belongs to the red forest. If  $u_1$  belongs to the blue forest, then we obtain a decomposition of  $G$  by adding  $vu_1$  to the blue forest. So assume  $u_1$  belongs to the red forest only. If  $u_1$  and  $u_2$  belong to different trees of the red forest, then we can directly add  $vu_1$  to the red forest.

Thus, lastly suppose that  $u_1$  and  $u_2$  belong to the same tree of the red forest. Note that when moving  $vu_2$  from the red to the blue forest, and adding  $vu_1$  to the blue forest, then the obtained blue subgraph remains a forest, and cannot have any tree isomorphic to  $K_2$ . The only problem, here, is that the red forest might now include a tree isomorphic to  $K_2$ . Since  $u_1$  and  $u_2$  belonged to the same tree of the red forest, this means that  $vu_2u_1$ , a path of length 2, was exactly a tree of the red forest. In that situation,  $u_2$  is a cut-vertex of  $G$ , and  $u_1$  also has degree 2 - its neighbours are  $v$  and  $u_2$ . Said differently,  $vu_1u_2v$  is a pendant triangle of  $G$  attached at  $u_2$ .

Now, since  $G$  is not  $K_3$ , then  $u_2$  belongs to the blue forest in the decomposition of  $G'$ . To obtain the desired decomposition of  $G$ , we can here just add  $vu_2$  to the blue forest (which indeed remains a forest), and add  $vu_1$  and  $u_1u_2$  to the red forest (to which we add a path of length 2).

This concludes the proof. □

We now extend the previous results to nicer subcubic graphs.

**Lemma 5.6.** *Every nicer subcubic graph  $G$  decomposes into two nice forests.*

*Proof.* Throughout the proof, which is by induction on  $|V(G)| + |E(G)|$ , it is assumed that  $G$  is connected. As the claim is true whenever  $|V(G)| \leq 4$  ( $G$  is either strictly subcubic and the result follows from Lemma 5.5, or isomorphic to  $K_4$ , which decomposes into two paths of length 3), we proceed to the proof of the general case.

We now consider the general case  $|V(G)| \geq 5$ . If  $G$  is strictly subcubic, then  $G$  is 2-degenerate, in which case the result follows from Lemma 5.5. So let us assume that  $G$  is cubic. Let  $v$  be a (degree-3) vertex of  $G$ , with neighbours  $u_1, u_2, u_3$ . Note that if all edges among the  $u_i$ 's exist, then  $G$  is  $K_4$  while  $|V(G)| \geq 5$ , a contradiction. Hence, assume without loss of generality that  $u_1u_2$  is not an edge of  $G$ . Consider the graph  $G' := G - \{v\} + \{u_1u_2\}$ . Note that, although  $G'$  might consist of up to two connected components, none of them is isomorphic to  $K_2$  or  $K_3$  as  $G$  is cubic. So all connected components are subcubic, and they decompose into two nice forests, say red and blue.

Consider the decomposition of  $G'$ . Suppose that  $u_1u_2$  belongs to the red forest. We consider the same decomposition in  $G$ , except that, since  $G$  does

not contain the edge  $u_1u_2$ , we replace it, in the red forest, by the two edges  $u_1v$  and  $vu_2$ . Note that, in  $G$ , the red subgraph remains a nice forest. It thus remains to add  $vu_3$  to either the red or blue forest. If  $u_3$  belongs to the blue forest, then we are done when adding  $vu_3$  to the blue forest. So assume that the two edges, different from  $vu_3$ , incident to  $u_3$  belong to the red forest. If  $v$  and  $u_3$  belong to different trees of the red forest, then we can freely add  $vu_3$  to the red forest. So lastly suppose that we are not in that case.

All of  $u_1, u_2, u_3$  belong to the same tree, say  $T$ , of the red forest. In  $T$ , let us assume that  $u_3$  is closer to  $u_2$  than it is closer to  $u_1$ . In other words, in  $T$ , the only path from  $u_3$  to  $u_1$  passes through  $u_2$ . Let us remove  $vu_2$  from  $T$ . In the red forest,  $T$  is disconnected into two trees  $T'$  and  $T''$ , where  $T'$  contains  $u_2$  and  $u_3$ , while  $T''$  contains  $v$  and  $u_1$ . Note that  $T'$  is not isomorphic to  $K_2$ , since  $u_3$  remains of degree 2 in that tree. If  $T''$  also has this property, then we get a desired decomposition of  $G$  when adding  $vu_2$  and  $vu_3$  to the blue forest (recall that  $u_3$  originally did not belong to the blue forest). So we may assume that  $T''$  is actually isomorphic to  $K_2$ , which means that  $u_1$  had degree 1 in  $T$ . In this situation, we obtain the desired decomposition of  $G$  by adding  $vu_1$  and  $vu_3$  to the blue forest.  $\square$

A similar proof as that used to prove Theorem 5.4, but using Lemma 5.6 instead of Lemma 5.5, now yields the following.

**Theorem 5.7.** *Every nicer subcubic graph is standardly  $(2, 2)$ -colourable.*

### 5.3. Conjecture 5.1 for 9-colourable graphs

To prove Conjecture 5.1 for all nicer 9-colourable graphs, we essentially prove that 9-colourable graphs, in general, decompose into two nice 3-colourable graphs. With such a result in hand, we can then use the fact that nice 3-colourable graphs verify the 1-2-3 Conjecture.

**Lemma 5.8.** *Assume that a nicer graph  $G$  can be 2-edge-coloured with red and blue so that the induced red subgraph  $G_R$  and blue subgraph  $G_B$  satisfy  $\chi(G_R) = r$  and  $\chi(G_B) = b$  with  $r, b \geq 2$ . Then  $G$  can be 2-edge-coloured in such a way that  $\chi(G_R) \leq r$ ,  $\chi(G_B) \leq b$ , and  $G_R$  and  $G_B$  are nice.*

*Proof.* The edges of  $G$  will be coloured or recoloured during the proof. Changing the colour of an edge actually means that that edge is added to one of  $G_R$  and  $G_B$ , and, conversely, removed from the second subgraph.

Let us start by raising a few comments on how edge additions and removals affect the parameters and structure of the subgraph we are interested in:

- Adding an edge to a graph can, in general, increase its chromatic number; however, the addition of a pendant edge, or, more generally,

of pendant paths does not increase the chromatic number (unless when the graph is edgeless). The addition of an edge such that at least one of its ends was not isolated in the graph does not increase the number of isolated edges.

- Removing edges from a graph can, in general, reduce the chromatic number. It can also produce new isolated edges; but this can only happen when the removed edge lies on a path and is incident with a pendant edge of this path.

The proof is by induction on  $|V(G)|$ . As it can easily be seen that the statement is true when  $|V(G)| \leq 4$ , we may consider the general case  $|V(G)| \geq 5$ . We can assume that  $G$  is connected. To show that  $G$  can be 2-edge-coloured as claimed, we consider three cases:

**Case 1.**  $G$  has a pendant edge, *i.e.*,  $\delta(G) = 1$ .

Let  $uv$  be a pendant edge of  $G$  with  $d(u) = 1$ . Any 2-edge-colouring of  $G$  with  $\chi(G_R) = r \geq 2$  and  $\chi(G_B) = b \geq 2$  induces a 2-edge-colouring of the graph  $G' := G - u$  with  $\chi(G'_R) = r \geq 2$  and  $\chi(G'_B) = b \geq 2$ , or  $G'$  becomes monochromatic. In the first case, since  $G'$  has at least four vertices, we may assume that the graphs  $G'_R$  and  $G'_B$  have no isolated edges. At least one colour, say red, is present at vertex  $v$ . Then we colour the edge  $uv$  red. In the second case, *i.e.*, when  $G'$  is monochromatic, say red, we colour the edge  $uv$  red. In both cases we do not increase the chromatic number of  $G'_R$ .

**Case 2.**  $\delta(G) = 2$ .

Let  $u$  be a vertex with  $d(u) = 2$ . Denote by  $v, w$  its neighbours and let  $G' := G - u$ . As above, any 2-edge-colouring of  $G$  with  $\chi(G_R) = r \geq 2$  and  $\chi(G_B) = b \geq 2$  induces a 2-edge-colouring of the graph  $G'$  with  $\chi(G'_R) = r \geq 2$  and  $\chi(G'_B) = b \geq 2$ , or  $G'$  becomes monochromatic. Suppose first that  $G'$  is connected. Then the lemma holds for  $G'$  and we may suppose that the graphs  $G'_R$  and  $G'_B$  have no isolated edges, or  $G'$  is monochromatic, say red, with  $\chi(G'_R) = r' \geq 2$ . If we are able to find in  $G'$  two edges of different colours incident with  $v$  (say red) and  $w$  (say blue), respectively, then we colour the edge  $uv$  red and the edge  $uw$  blue. Note that this adds pendant edges to  $G'_R$  and  $G'_B$ , so we do not increase the chromatic number of these graphs and we do not create isolated edges. If the vertices  $v, w$  are incident with edges of one colour only, say red, or  $G'$  is monochromatic, say red, then we colour both edges  $uv, uw$  blue. Again, we do not increase the chromatic number of the graphs  $G'_R$  and  $G'_B$  and we do not create isolated edges.

Consider now the case where  $G - u$  has two connected components  $G_1$  and  $G_2$ . If neither  $G_1$  nor  $G_2$  is isomorphic to  $K_3$ , then we apply induction hypothesis to  $G_1$  and  $G_2$  and proceed as above, that is:

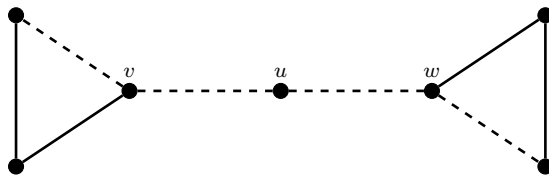


Figure 3: A decomposition into two nice bipartite graphs mentioned in the proof of Lemma 5.8. Solid edges stand for red edges. Dashed edges stand for blue edges.

- if we are able to find in  $G_1$  and  $G_2$  two edges of different colours incident with  $v$  (say red) and  $w$  (say blue), respectively, then we colour the edge  $uv$  red and the edge  $uw$  blue; and
- if the vertices  $v, w$  are incident with edges of one colour only, say red, then we colour both edges  $uv, uw$  blue.

Suppose now that only one of these connected components, say  $G_1$ , is isomorphic to  $K_3$  and denote its vertices by  $v, v_1, v_2$ . Now, we apply the induction hypothesis to  $G_2$  and if we are able to find an edge coloured, say, blue, incident with  $w$ , then we colour the edge  $uw$  blue. Next, we colour red the edges  $uv$  and  $vv_1$  and we colour blue the edges  $uv_2$  and  $v_2v_1$ .

If both connected components  $G_1$  and  $G_2$  are isomorphic to  $K_3$ , then one possible 2-edge-colouring without isolated edges is given in Figure 3.

**Case 3.**  $\delta(G) \geq 3$ .

We start from a 2-edge-colouring of  $G$  with  $\chi(G_R) = r \geq 2$  and  $\chi(G_B) = b \geq 2$  which minimizes the number of isolated edges in  $G_R$  and  $G_B$ . We will show that if the number of these is still positive, then we can get rid of any given such isolated edge, without creating a new one, and thus get a contradiction.

Let us suppose that  $uv$  is an isolated edge of  $G_B$ . Since  $d(u) \geq 3$  and  $d(v) \geq 3$ , neither  $u$  nor  $v$  is isolated in  $G_R$ . If the vertices  $u, v$  belong to two different connected components of  $G_R$ , then we recolour the edge  $uv$  red. Such an operation cannot increase the chromatic number of  $G_R$ .

Hence, the vertices  $u, v$  belong to one connected component of  $G_R$ . Then there is a *red path*  $P$  (containing only red edges) joining  $u$  and  $v$  in  $G_R$ . Denote this path by  $uw_1 \dots w_l v$ , where  $l \geq 1$ . Since the vertices  $u$  and  $v$  are of degree at least 3 in  $G$  and of degree 1 in the blue graph, they are of degree at least 2 in the red graph. Denote by  $u_1, \dots, u_p$  the neighbours of  $u$  in  $G_R$  and by  $v_1, \dots, v_q$  the neighbours of  $v$  in  $G_R$ , different from  $w_1, w_l$ , respectively. We have  $p, q \geq 1$ .

If  $p \geq 2$ , then we recolour  $uw_1$ , the first edge of the red path  $P$ , blue. From the point of view of the blue graph, we add to a connected component of  $G_B$  a pendant path  $w_1 uv$  of length 2. Since  $b \geq 2$ , this operation does not increase the chromatic number of  $G_B$ . From the point of view of the red graph, we delete an edge  $w_1 u$  but we do not create a new isolated edge.

Indeed, none of the red edges incident with  $u$  becomes isolated, because there remain  $p \geq 2$  of them. On the other hand, none of the edges incident with  $w_1$  becomes isolated because they are incident with the edge  $w_1w_2$  which lies on the path joining  $w_1$  with  $v$  and is not isolated even in the case  $w_2 = v$  because of the edge  $vv_1$ .

If  $p = 1$ , and by symmetry  $q = 1$ , then we can proceed as above except when the red degree of  $u_1$  is 1 *i.e.*,  $uu_1$  is a pendant edge in the red graph. Then we recolour this edge blue. Since  $b \geq 2$  this operation does not increase the chromatic number of  $G_B$ . Again, from the point of view of the blue graph, we add to a connected component of  $G_B$  a pendant path  $u_1uv$  of length 2, and from the point of view of the red graph, we delete a pendant edge. Both operations preserve the chromatic numbers of the red and blue graphs.  $\square$

We now prove the second key lemma of this section.

**Lemma 5.9.** *Every 9-colourable graph  $G$  decomposes into an  $r$ -colourable graph  $G_R$  and a  $b$ -colourable graph  $G_B$  with  $r, b \leq 3$ .*

*Proof.* Let  $G$  be a graph with  $\chi(G) \leq 9$ . It is easy to see that it is sufficient to consider the case where  $G$  is a complete graph of order  $n \leq 9$ . So let  $G = K_n$  and let  $x_1, \dots, x_n$  denote the vertices of  $G$ . We 2-edge-colour  $G$  with colours red and blue, yielding two subgraphs  $G_R$  and  $G_B$ , respectively, as follows. An edge  $x_i x_j$  is coloured red if and only if  $i = j \pmod{3}$ . Otherwise, *i.e.*, when  $i \neq j \pmod{3}$ , the edge  $x_i x_j$  is coloured blue.

Clearly,  $\chi(G_R) = 3$  for  $n \geq 3$ . Furthermore, since  $n \leq 9$ , there are at most three numbers congruent to 0 (or to 1, or to 2) modulo 3. Thus,  $G_B$  contains either isolated edges or triangles, so  $\chi(G_B) \leq 3$ .  $\square$

We are now ready to prove that nice 9-colourable graphs verify Conjecture 5.1.

**Theorem 5.10.** *Every nice 9-colourable graph  $G$  is standardly  $(2, 3)$ -colourable.*

*Proof.* If  $G$  is 3-colourable, then we have  $\chi_\Sigma(G) \leq 3$  (see [9]), or, in other words,  $G$  is standardly  $(1, 3)$ -colourable. Now assume that  $G$  is at least 4-chromatic. By Lemma 5.9, it can be decomposed into two 3-colourable graphs: an  $r$ -colourable graph  $G_R$  and a  $b$ -colourable graph  $G_B$  with  $r, b \leq 3$ . Since  $G$  is at least 4-chromatic we have also  $r, b \geq 2$ .

We distinguish two cases:

- If  $G$  has no isolated triangles, then, by Lemma 5.8, it can be decomposed into two nice graphs  $G_R$  and  $G_B$  with  $\chi(G_R) \leq r$  and  $\chi(G_B) \leq b$  with  $r, b \geq 3$ . So, both of them verify the 1-2-3 Conjecture, and thus admit standardly  $(1, 3)$ -colourings. Combining standardly  $(1, 3)$ -colourings of  $G_R$  and  $G_B$ , we get a standard  $(2, 3)$ -colouring of  $G$ .

- If  $G$  has isolated triangles, then we remove those from  $G$ , apply the previous point to get a  $(2, 3)$ -colouring of what is left, and then give the same color to all the edges of the isolated triangles. Finally, by weighting the edges of each isolated triangle with weights  $1, 2, 3$  (in any colour), we can extend the  $(2, 3)$ -colouring to  $G$ .  $\square$

We note that the approach above can be generalized to show that, in general, any nice graph  $G$  decomposes into a certain number, function of  $\chi(G)$ , of graphs fulfilling the 1-2-3 Conjecture.

## 6. Weak $(\ell, k)$ -colouring

We finally consider the weaker form of Conjecture 3.1:

**Weak Conjecture.** *Every nice graph is weakly  $(2, 2)$ -colourable.*

Towards the Weak Conjecture, we here first prove that all nice graphs are weakly  $(3, 2)$ - and  $(2, 4)$ -colourable. We then prove that graphs with minimum degree at least 59 are weakly  $(2, 3)$ -colourable.

### 6.1. Weak $(3, 2)$ - and $(2, 4)$ -colourability

We first prove that every nice graph is weakly  $(3, 2)$ -colourable.

**Theorem 6.1.** *Every nice graph  $G$  is weakly  $(3, 2)$ -colourable.*

Our proof of Theorem 6.1, relies on the following lemma proved by Addario-Berry, Aldred, Dalal and Reed in [1].

**Lemma 6.2.** *Every connected graph  $G$  with  $\chi(G) > 3$  has a tripartition  $V_0 \cup V_1 \cup V_2$  such that, for every vertex  $v \in V_i$ , we have:*

1.  $d_{V_{i+1 \pmod{3}}}(v) \geq d_{V_i}(v)$ , and
2.  $d_{V_{i+1 \pmod{3}}}(v) \geq 1$ .

We now proceed to the proof of Theorem 6.1.

*Proof of Theorem 6.1.* If  $G$  is 3-colourable, then  $G$  admits a neighbour-multiset-distinguishing 3-edge-weighting (we actually even have  $\chi_{\Sigma}(G) \leq 3$ ), which implies that  $G$  is weakly  $(3, 1)$ -colourable. So assume that  $\chi(G) > 3$ , and that  $G$  is connected. According to Lemma 6.2, there exists a tripartition  $V_0 \cup V_1 \cup V_2$  of  $V(G)$  where each vertex has at least as many neighbours (at least one) in the next part than it has in its own part. We construct a weak  $(3, 2)$ -colouring, assigning coloured weights red, blue and green, of  $G$  in the following way. We do this in two steps, first assigning colours to the edges, and then describing what weights should be assigned to them.

For two sets  $V_i, V_j$ , we denote by  $e(V_i, V_j)$  the edges of  $G$  with one end in  $V_i$  and one end in  $V_j$ . Also, by  $e(V_i)$  we denote the set  $e(V_i, V_i)$ . We colour the edges as follows:

- every edge in  $e(V_0) \cup e(V_0, V_1)$  is coloured red;
- every edge in  $e(V_1) \cup e(V_1, V_2)$  is coloured blue;
- every edge in  $e(V_2) \cup e(V_2, V_0)$  is coloured green.

Note that every edge of  $G$  is coloured. We now weight the edges in the following manner. For the sake of the exposition, we demonstrate the procedure for  $V_0$  and colour red, but it has to be performed in a similar way for  $V_1$  and  $V_2$ , using colour blue and green, respectively. We assign weight 1 to all edges in  $e(V_0)$ , and similarly for all edges in  $e(V_0, V_1)$ .

We now process all vertices of  $V_0$  following an arbitrary ordering. Let us assume we are now considering vertex  $v$ . Our goal is to make sure that the red sum of  $v$  is different from that of its neighbours in  $V_0$ . For this, let us denote by  $u_1, \dots, u_k$  the previously-treated neighbours of  $v$  in  $V_0$ . To alter the red sum of  $v$ , we are allowed to consider any neighbour  $u$  in  $V_1$ , and switch the (red) weight of  $vu$  from 1 to 2. Since  $v$  has  $d_{V_0}(v)$  neighbours in  $V_0$ , with  $d_{V_1}(v) \geq d_{V_0}(v) \geq k$ , by performing weight switches, we can make the red sum of  $v$  increase by any  $x \in \{0, \dots, d_{V_1}(v)\}$ . So there are up to  $k$  forbidden values as the red sum of  $v$ , while we have at least  $k + 1$  distinct ways to modify it. Hence, there is a way to switch 1's to 2's on the red edges going from  $v$  to  $V_1$ , so that no red sum conflict involving  $v$  and its neighbours in  $V_0$  remains.

Once this process has been applied for all vertices in  $V_0, V_1, V_2$ , we claim that we have obtained a weak  $(3, 2)$ -colouring of  $G$ . First, due to the process we have run, for every two adjacent vertices in  $V_0$  (resp.  $V_1, V_2$ ), we have distinct red (resp. blue, green) sums. Now, for every edge  $uv$  where  $u \in V_0$  and  $v \in V_1$ , we have that  $v$  has non-zero blue sum (because  $d_{V_2}(v) \geq 1$ ), while  $u$  has zero blue sum. So  $u$  and  $v$  are distinguished via their blue sums. Similarly, it can be noted that any two adjacent vertices in consecutive  $V_i$ 's are distinguished via a non-zero coloured sum and a zero coloured sum.  $\square$

We now prove that every nice graph can be weakly  $(2, 4)$ -coloured. This time, the proof is inspired by that in [8].

**Theorem 6.3.** *Every nice graph  $G$  is weakly  $(2, 4)$ -colourable.*

*Proof.* Assume that  $G$  is connected. Choose as  $v_n$  a vertex with degree at least 2 in  $G$ . Applying a depth-first search algorithm from  $v_n$ , we deduce an ordering  $v_1, \dots, v_n$  of the vertices of  $G$  where each  $v_i$  (but  $v_n$ ) has a *forward neighbour*, i.e., there is a  $j$  with  $j > i$  such that  $v_i v_j$  is an edge. Conversely, we call  $v_i$  a *backward neighbour* of  $v_j$ . An edge  $v_i v_j$  with  $i < j$  is a *forward edge* for  $v_i$  and a *backward edge* for  $v_j$ . By our choice of  $v_n$ , it has at least two backward neighbours.

We produce a weak  $(2, 4)$ -colouring (with red and blue weights) of  $G$  where, for every edge  $v_i v_j$ , vertices  $v_i$  and  $v_j$  are distinguished by their



red sums. We start with all edges of  $G$  being assigned red weight 2. We will then process all  $v_i$ 's in order. At each Step  $i$  with  $i < n$ , we will 1) define a set  $\Phi(v_i) := \{\phi(v_i), \phi(v_i) + 2\}$ , with  $\phi(v_i) \in \{0, 1\} \pmod{4}$ , of two possible red sums for  $v_i$ , so that, for every edge  $v_j v_i$  with  $j < i$ , we have  $\Phi(v_j) \neq \Phi(v_i)$  (so that  $\Phi(v_j) \cap \Phi(v_i)$  is empty), and 2) modify (either in value or colour) some weights incident to  $v_i$  so that its current red sum lies in  $\Phi(v_i)$ . More precisely, from the moment where  $\Phi(v_i)$  is defined, the red sum of  $v_i$  will always belong to that set. Eventually, at Step  $n$ , we will modify some weights incident to  $v_n$  so that no red sum conflict remains.

There are no backward constraints at Step 1, so we can just set  $\phi(v_1) = 2d(v_1)$ . Note that all required conditions are met so far. We now consider the general case, *i.e.*, Step  $i$  for some  $1 < i < n$ . For every  $j < i$ , the set  $\Phi(v_j)$  was defined, and the current red sum of  $v_j$  belongs to that set (Condition 1). Assume further that all forward edges incident to  $v_i$  currently have red weight 2 (Condition 2). Lastly (Condition 3), suppose that, for every backward edge  $v_j v_i$ , either:

- $v_j v_i$  has red weight 2 and  $v_i$  has red sum in  $\Phi(v_j)$  (Condition 3.a);

or, when  $v_i$  is the first (*i.e.*, with the lowest index) forward neighbour of  $v_j$ :

- $v_j v_i$  has red weight 1 and  $v_j$  has red sum  $\phi(v_j)$  (Condition 3.b); or
- $v_j v_i$  has red weight 3 and  $v_j$  has red sum  $\phi(v_j) + 2$  (Condition 3.c).

For each edge  $v_i v_j$  incident to  $v_i$ , either backward or forward, we will allow a *valid change* of its coloured weight, complying with the above Conditions:

- We allow to not alter any of the colour and weight of  $v_j v_i$ .
- If  $v_j v_i$  is a backward edge, we allow to change its colour or weight, as long as  $v_j$  complies with Condition 1:
  - if  $v_j v_i$  verifies Condition 3.a, we allow to change its red weight from 2 to 4 (if the red sum of  $v_j$  is  $\phi(v_j)$ ) or to any blue weight (if the red sum of  $v_j$  is  $\phi(v_j) + 2$ );
  - if  $v_j v_i$  verifies Condition 3.b, we allow to change its red weight from 1 to 3;
  - if  $v_j v_i$  verifies Condition 3.c, we allow to change its red weight from 3 to 1.
- If  $v_i v_j$  is the first forward edge of  $v_i$ , we allow to change its red weight to 1 or 3, as long as  $v_i$  eventually complies with Condition 3.

Assuming  $v_i$  has  $k \geq 1$  backward neighbours, by making valid changes backwards, we can make the red sum of  $v_i$  to reach a range of  $k + 1$  values (all having the same parity), in such a way that all backward neighbours comply with Condition 1. Furthermore, note that the valid change forwards makes the parity of the red sum of  $v_i$  change.

Hence, we can make the red sum of  $v_i$  reach any value within some interval  $\mathfrak{C} := \{\alpha, \dots, \alpha + 2k + 2\}$  by performing valid moves. We have to choose this value in such a way that the set  $\Phi(v_i)$  can correctly be defined from it. Since we need  $\Phi(v_i) \neq \Phi(v_j)$  for every backward neighbour  $v_j$ , the backward neighbours forbid at most  $2k$  red sums from  $\mathfrak{C}$ . Now, if a valid move forwards is performed, then, in case it is a decrement, we require the red sum of  $v_i$  to be  $\phi(v_i)$ , while we require it to be  $\phi(v_i) + 2$  otherwise. This condition might forbid the values  $\alpha$  and  $\alpha + 2k + 2$  of  $\mathfrak{C}$  as the red sum of  $v_i$ . So, in total, there is one value in  $\mathfrak{C}$  which is subject to no constraint; so we make the corresponding valid moves so that the red sum of  $v_i$  reaches this value, and define  $\Phi(v_i)$  accordingly. All conditions are met for the next Steps to be performed then.

We do not need to define a set  $\Phi(v_n)$ : at Step  $n$ , we just need to perform valid changes backwards so that the red sum of  $v_n$  gets different from that of its  $k \geq 2$  backward neighbours (though the notion of valid change was not defined for  $v_n$ , note that for every backward edge  $v_n v_j$  there is a way to increase or reduce the red sum of  $v_j$  by 2 without violating any of the Conditions). Start by performing all valid changes backwards that are decrements. By our choice of the  $\phi(v_i)$ 's, all backward red sums have value in  $\{0, 1\}$  modulo 4. If the red sum of  $v_n$  has thereby reached a value in  $\{2, 3\}$  modulo 4, then we are done. So we may assume that the red sum of  $v_n$  lies in  $\{0, 1\}$  modulo 4, and that at least one backward neighbour has the same red sum.

- If there is at least one backward neighbour  $v_i$  whose red sum is different to that of  $v_n$ , then we make a valid change on the edge  $v_n v_i$ ; this clearly yields a correct weighting.
- Now, if all backward neighbours have the same red sum as  $v_n$ , then we perform two valid changes backwards so that no red sum conflict remains.

We eventually get a weak  $(2, 4)$ -colouring of  $G$ . □

## 6.2. Graphs with $\delta \geq 59$ are weakly $(2, 3)$ -colourable

Before proceeding to the proof of the main result of this section, we first need to introduce two observations.

**Observation 6.4.** *Every graph  $G$  decomposes into two subgraphs  $G_1$  and  $G_2$  such that:*

- for every vertex  $v$  of  $G$ , we have  $d_{G_1}(v) \in \left\{ \left\lfloor \frac{d_G(v)}{2} \right\rfloor, \left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right\}$ , and
- for every even-degree vertex  $v$  of  $G$  except possibly one, we have  $d_{G_1}(v) = \frac{d_G(v)}{2}$ .

*Proof.* If the subset  $U \subseteq V$  of the vertices of odd degree in  $G$  is non-empty, add a new vertex  $u$  and join it by a single edge with every vertex in  $U$ ; denote the obtained graph by  $G'$  (if  $U = \emptyset$ , set  $G' = G$ ). As the degrees of all vertices in  $G'$  are even, there exists an Eulerian tour in it. We then traverse all edges of  $G'$  once along this Eulerian tour, starting at  $u$  if it exists, and colour them alternately red and blue. Then the red edges in  $G$  induce its subgraph  $G_1$  consistent with our requirements.  $\square$

**Observation 6.5.** *Every graph  $G$  has an orientation  $D$  such that, for every vertex  $v$ , we have  $d_D^+(v) \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ .*

*Proof.* Analogously as in the proof of Observation 6.4, if the subset  $U \subseteq V$  of the vertices of odd degree in  $G$  is non-empty, then add a new vertex  $u$  and join it by a single edge with every vertex in  $U$ ; denote the obtained graph by  $G'$  (if  $U = \emptyset$ , set  $G' = G$ ). As the degrees of all vertices in  $G'$  are even, there exists an Eulerian tour in it. By traversing it once we obtain an orientation of  $G'$  with equal in- and out-degrees for all vertices. This yields the desired orientation  $D$  of  $G$ .  $\square$

We now prove the main result:

**Theorem 6.6.** *Every graph  $G$  with  $\delta(G) \geq 59$  is weakly  $(2, 3)$ -colourable.*

*Proof.* We may suppose that  $G$  is connected. Let  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$  be the subgraphs of  $G$  obtained by applying Observation 6.4, where  $x$  is a vertex of even degree in  $G$  for which  $d_{G_1}(x) \neq \frac{d_G(x)}{2}$  if it exists (let  $x$  be any fixed vertex of  $G$  otherwise). We produce a weak  $(2, 3)$ -colouring of  $G$  by colouring and weighting  $G_1$  and  $G_2$  separately.

We colour red all edges of  $G_1$ , and blue all edges of  $G_2$ . Initially we weight all the edges with 2. Denote by  $\omega_1, \omega_2$  the temporary weightings of  $G_1, G_2$ . In what follows,  $\omega_1$  and  $\omega_2$  will be subject to changes, but, for the sake of the proof, we still call them  $\omega_1$  and  $\omega_2$ . At every step of our construction, the colour of any vertex  $v$ , denoted  $c(v)$ , will be understood as the pair  $(\sigma_{\omega_1}(v), \sigma_{\omega_2}(v))$ , where  $\sigma_{\omega_i}(v) := \sum_{u \in N_{G_i}(v)} \omega_i(uv)$  for  $i = 1, 2$ .

Let  $D_1$  and  $D_2$  be auxiliary orientations of  $G_1$  and  $G_2$ , respectively, consistent with Observation 6.5. It is straightforward to verify that for every  $v \in V \setminus \{x\}$  we then must have:

$$(d_{D_1}^+(v) + 1)(d_{D_2}^+(v) + 1) > 4d_G(v). \quad (1)$$

Let further  $O_i(v) := \{uv \in E_i : u \in N_{D_i}^+(v)\}$  denote the set of edges incident with a given vertex  $v$  which correspond to arcs out-going from  $v$  in  $D_i$ . Hence  $|O_i(v)| = d_{D_i}^+(v)$  for  $i = 1, 2$ .

Let us define the following family of pairwise disjoint four-element sets of pairs of integers

$$\mathfrak{B} = \{(2p, 2q), (2p, 2q + 1), (2p + 1, 2q), (2p + 1, 2q + 1)\} : p, q \in \mathbb{Z}.$$

Fix an arbitrary ordering  $v_1, \dots, v_n$  over the vertices in  $V$  with  $v_1 = x$ . We will analyse the  $v_i$ 's one after another consistently with this ordering. At each Step  $j$  we will choose some set  $B_j \in \mathfrak{B}$ , different from all such sets already fixed for the neighbours of  $v_j$  in  $G$ , and we will modify weights of the edges in  $O_1(v_j) \cup O_2(v_j)$  so that  $c(v_k) \in B_k$  for every  $k \leq j$ . Note that if we are able to achieve this using only weights 1, 2, 3 on the edges, after Step  $n$  we will then obtain a desired weighting of  $G_1$  and  $G_2$ .

Step 1 is trivial, so assume we analyse Step  $j$  for some  $j \in \{2, \dots, n\}$ , and thus far all our requirements have been fulfilled. Note that we cannot choose at most  $d_G(v_j)$  sets from  $\mathfrak{B}$  for  $v_j$  (these already assigned to neighbours of  $v_j$  in  $G$ ). We however may always modify the weight of every  $v_k v_j \in O_i(v_j)$  by 1 so that  $c(v_k) \in B_k$  (if  $v_k$  has already  $B_k$  assigned),  $i = 1, 2$ . This way we may obtain at least  $|O_1(v_j)| + 1$  distinct values of the first coordinate of  $c(v_j)$  and at least  $|O_2(v_j)| + 1$  distinct values for the second one. This altogether yields a list of available pairs of cardinality at least

$$(|O_1(v_j)| + 1)(|O_2(v_j)| + 1) \geq (d_{D_1}^+(v) + 1)(d_{D_2}^+(v) + 1) > 4d_G(v)$$

for  $c(v_j)$  due to Inequality (1) above. At least one of these pairs must thus not belong to any  $B_k$  for  $v_k \in N_G(v_j)$ . We choose any such pair and fix as  $B_j$  the set in  $\mathfrak{B}$  which includes this pair, performing at the same time modifications of weights (to 1 or 3) of (some of) the edges in  $O_1(v_j) \cup O_2(v_j)$  so that  $c(v_j) \in B_j$  afterwards. Note that by our choice of the sets  $O_i(v)$ , the weight of each edge might be modified only once, and hence belongs to  $\{1, 2, 3\}$ .

After Step  $n$ , we thus obtain desired weightings of  $G_1$  and  $G_2$ .  $\square$

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