HAL open science

Polyspectra of Ordered Signals

Bernard Picinbono

▶ To cite this version:

Bernard Picinbono. Polyspectra of Ordered Signals. IEEE Transactions on Information Theory, 1999, 45, pp.2239 - 2252. hal-01687877

HAL Id: hal-01687877
https://hal.science/hal-01687877

Submitted on 18 Jan 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Polyspectra of Ordered Signals

Bernard Picinbono, *Fellow, IEEE*

The pdf copy of the final published text can be obtained from the author at the folowing address : bernard.picinbono@lss.supelec.fr

Reference of this text : IEEE Trans. Inf. Theory, 45, Num. 7, pp. 2239-2252, 1999.

Abstract

Polyspectra are related to Fourier transforms of moment or cumulant functions of any order of random signals. They play an important role in many problems of signal analysis and processing. However, there are only a few statistical models giving explicitly the expression of polyspectra. Ordered signals are signals for which the explicit expression of the moment functions requires that the time instants appearing in these moments are put in an increasing order. There are many examples of such signals, the best known being the random telegraph signal constructed from a Poisson process. Some of these examples are presented and analyzed. The origin of the ordering structure is related with the point that real time is an oriented variable making a difference between past and future. This especially appears in Markov processes. The calculation of polyspectra is difficult because ordering is not adapted to Fourier analysis. By an appropriate grouping of various terms, the explicit expression of spectral moment functions is obtained. It shows in particular that many ordered signals present a normal density on the normal manifolds of the frequency domain and another contribution on the stationary manifold that is explicitly calculated.

Manuscript received

B. P. is with the Laboratoire des Signaux et Systèmes (L2S), a joint laboratory of the C.N.R.S. and the École Supérieure d'Électricité, Plateau de Moulon, 3 rue Joliot–Curie 91192, Gif sur Yvette, France. The L2S is associated with the University of Paris–Sud, France. E-mail: bernard.picinbono@lss.supelec.fr, .

The analysis of the structure of this expression allows us to discuss some relationships with normal distribution, central limit theorem, and time reversibility.

Index Terms

Central limit theorem, higher order statistics, normal distribution, signal analysis, spectral representation.

I. INTRODUCTION

POLYSPECTRA play an especially important role in the methods of signal analysis and processing using statistics of an order higher than two (HOS). From the structure of polyspectra it is possible to deduce various properties of signals that do not appear when using the power spectrum only. For example, many quite different signals can have the same correlation function, or the same power spectrum, but they can be distinguished by using HOS. Furthermore, there are various methods of signal processing using polyspectra in order to solve problems that cannot be solved by only using secondorder statistics. There are many papers dealing with properties and applications of bi- or trispectrum and it is not possible here to give a significant list of these papers. Restricting our attention to signal analysis, it is worth pointing out the papers where the general structure or symmetry of polyspectra are studied [1][4]. However, the mathematical expression of polyspectra is in general difficult to write explicitly. This expression is well known in the case of strictly white signals, of normal signals, of Poisson processes (see [5, Chs. 7 and 8]), and for linear processes or some nonlinear processes characterized by a Volterra expansion (see [6, p. 39]). Many papers deal with polyspectra of nonnormal signals [7][9]. However, without any additional specification, only general structure can be used in this case, because nonnormality does not introduce any precise definition of the statistics of a signal. Therefore, if we require explicit analytic expressions of polyspectra, it is necessary to introduce statistical models of signals that can represent physical phenomena and have a structure leading to possible explicit calculations. This is one of the main purposes of this paper. Many signals are related to physical phenomena or obtained by systems operating in real time. Therefore, the variable appearing in the signal is not only a point on a straight line, but an oriented variable because time is going in one sense only. This is the basis of the questions related with causality. This also appears in HOS in which we study mean values of a signal at time instants $t_1, t_2, ..., t_n$. From a mathematical point of view these instants are arbitrary, and this is especially true for calculating the polyspectra. Indeed, this calculation requires a Fourier transformation in which the variables are independent variables going from $-\infty$ to $+\infty$. However, from a physical point view these instants must sometimes be put in an increasing order. This is especially the case for signals in which the future is constructed from the present and the past. Therefore, for such signals, here called ordered signals and very common in practical problems, calculation of polyspectra requires specific methods analyzed here. The paper is organized as follows. In Section II, a short review of known concepts concerning moments and cumulants and their Fourier transforms is presented. Because of its importance in the analysis that follows, a specific emphasis is put on the stationary and normal manifolds in the frequency domain. There are surfaces characterizing the fact that a signal is stationary or normal, respectively. As relation to normality is an important issue, these manifolds will play an important role is the discussion that follows. Section III introduces the concept of ordered signals. These signals are characterized by the fact that the explicit expression of their moments or cumulants requires an ordered sequence of time instants. The simplest example of such signals is the famous random telegraph signal (RTS). Because of its simplicity it will be analyzed in detail. However, there are many other examples and some of them are presented and discussed. These signals frequently appear in physical applications because the ordering property is quite natural. For example, it appears with Markov processes for which past and future are independent, given the present. Then there is in the definition an ordering structure making a difference between past, present, and future. The principle of the calculation of the Fourier transforms of some ordered signals is presented. The difficulty comes from the fact that the explicit expression of the moment is known only for one permutation. Therefore, there exists different values of a moment of order and distinct domains for the calculation of the Fourier transform, which leads to a great deal of algebraic manipulations. Furthermore, the ordering property makes a strong difference between past and future, and, therefore, introduces in the time domain many unit step functions. As a consequence, the calculation of the Fourier transform leading to the polyspectra requires some specific and unusual properties of distributions. These calculations are used to obtain some polyspectra of ordered signals and to discuss their properties. One of the most important points to analyze is the value of the density on the normal manifolds and it is shown that for many ordered signals this density is the same as the one of a normal process (normal density). The last sections are devoted to the study of some consequences concerning central limit theorem and time reversibility. By central limit theorem we mean the fact that a nonnormal signal can become approximately normal after its filtering in a narrowband filter. On the other hand, reversibility means the fact that the statistical properties can remain invariant by time reversing. It is shown that these properties are strongly related to the value of the polyspectra on normal manifolds, which justifies the interest in analyzing their structures.

II. Review of Known Results

Let $x(t)$ be a real random signal with zero mean value. The variable refers to the time and most of the following results are valid in continuous and discrete time. The nth-order moment (or moment function) of $x(t)$ is defined by

$$m_n(\mathbf{t}) = E[x(t_1)x(t_2)...x(t_n)]. \tag{1}$$

If the stationarity assumption is introduced, this moment is a function of only $n-1$ variables. For $n=2$ we obtain the correlation function written as $\gamma t_1 - t_2)$. When $x(t)$ is normal, or Gaussian, the moment (1) is zero for n odd and for even values of n takes the form

$$m_{2k}^G(\mathbf{t}) = \sum_{\mathcal{P}} \prod \gamma(t_i - t_j), \tag{2}$$

where the sum with respect to the normal (or Gaussian) partitions $\mathcal{P}$ contains $(2k-1)!!$ terms generalizing in an obvious way the expression

$$m_4^G(\mathbf{t}) = \gamma(t_1 - t_2)\gamma(t_3 - t_4) + \gamma(t_1 - t_3)\gamma(t_2 - t_4) + \gamma(t_1 - t_4)\gamma(t_2 - t_3), \tag{3}$$

valid for k = 2. The letter G refers to the Gaussian structure.

Instead of using the moments defined by (1), it is sometimes more interesting to use the cumulants functions c_n introduced first in [10]. It is not appropriate to present here a discussion concerning respective advantages and disadvantages of moments and cumulants. Note simply that the cumulants of a normal signal are zero for $n > 2$. Furthermore, there is the same information in moments or cumulants and the expression allowing the calculation of cumulants in terms of moments is given in [6, p. 19].

Polyspectra are related to the Fourier transforms (FT) of moments or cumulants. Let $M_n(\{f_i\}) = m(\mathbf{f}) =$ be the FT of (1) called here spectral moment function. If $x(t)$ is stationary this function can be written (see [5, p. 238]) as

$$M_n(\{f_i\}) = \Gamma_{n-1}(f_1, f_2, ..., f_{n-1})\delta(\sum_{i=1}^{n} f_i), \tag{4}$$

where $\delta(.)$ is the Dirac distribution. This means that $M_n(\{f_i\}) = 0$, except on the manifold of the space $f_1 \times f_2 \times ... \times f_n$ defined by $\sum_{i=1}^{n} f_i = 0$ called stationary manifold. The function $\Gamma_{n-1}(.)$ is called the polyspectrum of order $n-1$. The polyspectrum of order 1 is the classical power spectrum $\Gamma(f)$ of $x(t)$, FT of its covariance function. For practical applications, bi- and trispectra are the most often used polyspectra. The same procedure can be applied to the cumulants, which introduces the cumulant spectral functions and polyspectra.

For the discussion that follows it is especially interesting to consider the normal case. By Fourier transformation of (2) we obtain

$$M_{2k}^{G}(\{f_i\}) = \sum_{\mathcal{P}} \prod \gamma(f_i)\delta(f_i + f_j), \tag{5}$$

where the sum is calculated with respect to the normal partitions $\mathcal{P}$. For $2k = 4$ this gives

$$M_{2k}^{G}(\{f_i\}) = \Gamma(f_1)\Gamma(f_3)\delta(f_1 + f_2)\delta(f_3 + f_4) + \Gamma(f_1)\Gamma(f_2)$$
$$\delta(f_1 + f_3)\delta(f_2 + f_4) + \Gamma(f_1)\Gamma(f_2)\delta(f_1 + f_1)\delta(f_2 + f_3). \tag{6}$$

This introduces the concept of normal manifolds and of normal density (see [5, p. 278]). It results from (5) that $M_{2k}^{G}(\{f_i\})$ is zero outside the manifolds defined by equations such that

$$f_1 + f_2 = 0, \;\; f_3 + f_4 = 0, \;\; f_{2k-1} + f_{2k} = 0. \tag{7}$$

There are $(2k-1)!!$ such manifolds defined by the $(2k-1)!!$ permutations of the s giving distinct equations of this form. These manifolds are called in the following normal manifolds and are clearly submanifolds of the stationary manifold defined after (4). Furthermore, the density on these normal manifolds is not arbitrary, but appears as a product of spectral densities of the signal and this structure is called a normal density. These properties can clearly be expressed in terms of polyspectra by applying (4) to (5). For example, the moment trispectrum of a normal signal is

$$\Gamma_3^{G}(f_1, f_2, f_3) = \Gamma(f_1)\Gamma(f_3)\delta(f_1 + f_2) + \Gamma(f_1)\Gamma(f_2)\delta(f_1 + f_3)$$
$$+\Gamma(f_1)\Gamma(f_2)\delta(f_2 + f_3). \tag{8}$$

The first normal manifold for the trispectrum is then defined by $f_1 + f_2 = 0$. In the space $f_1 \times f_2 \times f_3$ it is the plane defined by two straight lines: the axis defined by $f_1 = f_2 = 0$ and the bisectrix of the axes defined by $f_f 1 = -f_2$, $f_3 = 0$ On this plane the normal density is . The other two normal manifolds are defined similarly.

Conversely, if the function $M_n(\{f_i\})$ defined by (4) has the normal structure characterized by (5) for any value of , the moment (1) has the form (2). As the sequence of all the moments defines the probability distribution, a sequence of normal moments means that the signal is normal. As a result, a nonnormal process with a zero mean value has a spectral moment function that does not satisfy at least one of the following properties: 1) $M_{2k+1}(\{f_i\}) = 0$, 2) $M_{2k}(\{f_i\}) = 0$ outside the normal manifolds, and 3) the density on the normal manifolds is not normal. For example, spherically invariant processes (see [5, p. 299]) have spectral moment functions (4) equal to zero for $n = 2k + 1$ and outside the normal manifolds for $n = 2k$, but the density on the normal manifolds is not normal. In other words, the spectral moment functions are the same as those of normal signals, except the value of the density on the normal manifolds.

The structure of the density on the normal manifolds is important for various questions as ergodicity [11], central limit theorem for signals, and time reversibility analyzed in the following for ordered signals.

Let us consider the cumulant polyspectra noted $\Gamma_c(\{f_i\})$. They are defined exactly by the same procedure in which the moment function appearing in (1) is replaced by the cumulant function. These polyspectra are especially interesting in two cases. The first appears if the signal is normal. Indeed, the only nonzero polyspectrum of a normal signal is the power spectrum. This is an interesting characteristics or normality. The second case corresponds to the discrete-time strictly white noise. This means that $x[k]$ is a sequence of independent and identically distributed (i.i.d.) random variables with cumulants c_n. It results from the basic properties of cumulants that the cumulant polyspectrum appearing in (4) is now $\Gamma_{c,n-1} = c_n$. The same expression for the moments instead of cumulants is, of course, much more complex to write explicitly. However, for the following discussion it is interesting to study more carefully the relation between moment and cumulant polyspectra of discrete-time white noise. Let us first consider the case of the trispectrum. The moment trispectrum of white noise is

$$\Gamma_3(f_1, f_2, f_3) =$$

$$c_2^2[\delta(f_1 + f_2) + \delta(f_1 + f_3) + \delta(f_2 + f_3)] + c_4, \tag{9}$$

while the cumulant trispectrum is simply c_4. This shows that, contrary to the cumulant trispectrum, the moment trispectrum is not bounded in the space $f_1 \times f_2 \times f_3 \times$ because of the presence of the delta distributions appearing in this equation. This is often considered as the most important disadvantage of moment trispectrum with respect to cumulant trispectrum, and this appears not only for the trispectrum but for any polyspectrum. Furthermore, (9) shows that the spectral moment function $M_n(\{f_i\})$ appearing in (4) is distributed on the normal manifolds with a normal density and also on the stationary manifold with the density c_4. These points, shown for the trispectrum, are general for polyspectra of any order and lead to a geometrical interpretation of the relation between moments and cumulants. For this, consider the spectral moment function $M_6(\{f_i\})$ of white noise. This function is zero outside the stationary manifold but is also distributed on some of its submanifolds. There are three kinds of such manifolds. The first ones are the normal manifolds defined by equations such that $f_1+f_2 = 0$, $f_3+f_4 = 0$, $f_5+f_6 = 0$ and all their distinct permutations. The density on these manifolds is m_2^3, where m_n is the th-order moment of the i.i.d. random variables defining the white noise. The second submanifolds are on the type $f_1 + f_2 + f_3 = 0$, $f_4 + f_5 + f_6 = 0$ and the density is now m_3^2. Finally, there are the manifolds $f_1 + f_2 + f_3 + f_4 = 0$, $f_5 + f_6 = 0$, and all their distinct permutations, introducing the density $m_2 m_4$. In order to obtain a bounded spectral cumulant it suffices to subtract from the spectral moment all these terms and making this operation gives the relation giving moments in terms of cumulants. The same discussion can be presented for the continuoustime white noise by using known results concerning general properties of processes with independent increments.

The fact that the cumulant polyspectra are bounded is not specific to white noise and is true for many other signals. This is especially the case for linear processes [6], which means signals obtained at the output of a linear filter driven by a white noise. The cumulant function $c_n(t_1, t_2, ..., t_n)$ in the continuous-time case is given by

$$c_n(t_1, t_2, ..., t_n) = \\ c_n \int h(t_1 - \theta)h(t_2 - \theta)...h(t_n - \theta)d\theta \tag{10}$$

where c_n characterizes the driving white noise and $h(t)$ is the impulse response of the filter. The cumulant polyspectra are, therefore,

$$\Gamma_{c,n-1}(f_1, f_2, ..., f_{n-1}) \\ = c_n H(f_1)H(f_2)...H(f_{n-1})H(-f_1 - f_2 - ... - f_{n-1}), \tag{11}$$

where $H(f)$ is the frequency response of the filter. When this filter is real we can use the Hermitian symmetry $H(-f) = H^*(f)$ to simplify this expression. It shows obviously that if $H(f)$ is bounded the cumulant polyspectra are also bounded. However, it must be noted that this property is not valid for every signal and there are many counterexamples. The simplest is certainly that of spherically invariant processes introduced above. One can show, for example (see [5, p. 280]), that their cumulant trispectrum can be written as $k\Gamma_3^G(f_1, f_2, f_2)$, where $k = 0$ in the normal case. It is obvious from (8) that this spectrum is not bounded in the space $f_1 \times f_2 \times f_3$.

Furthermore, note that (10) introduces a necessary condition in the time domain that must satisfy a signal to be a linear process. By integration of (10) with respect to some time instants we obtain easily if $c_m \neq 0$

$$\int\int ... \int c_n(t_1, t_2, ..., t_m, t_{m+1}, t_{m+2}, ..., t_n)dt_{m+1}dt_{m+2}...dt_n \\ = \frac{c_n[H(0)]^{n-m}}{c_m} c_m(t_1, t_2, ..., t_m). \tag{12}$$

In this expression, it is useful to note that $H(0)$ is related to the power spectrum by the relation $\Gamma(0) = c_2|H(0)|^2$.

Linear processes are the most used in practice and (11) shows that their polyspectra possess a factorization property. Note, however, that a factorization such as (11) does not imply that the signal is a linear process. For this, (11) must be satisfied for any n and, furthermore, the coefficients appearing in this factorization must satisfy various conditions ensuring that a set of numbers c_n is a set of cumulants. These conditions appear in the famous moment problem of probability theory.

There is a point important to note concerning the HOS of random signals. Any second-order signal can be considered as generated by a linear filter driven by white noise. This only means that the signal and its linear model have the same correlation function. However, these two signals have no reason to be identical. This is no longer true for HOS and, for example, if (12) is not satisfied, there is no linear filter and white noise giving an output with the same statistics.

Note finally that when no a priori knowledge of a signal is introduced the calculation of the cumulant functions requires the use of the very complex expression giving these functions in terms of moments function and this is the reason why most of the calculations that follow are realized with moments, and cumulants are only sometimes introduced.

III. Ordered Signals

A. *Introduction and Definition*

The function $m_n(\{t_i\})$ defined by (1) is symmetric with respect to the variables t_i. This means that it is invariant under the n! permutations of these time instants. Let us call θ_i the time instants deduced from the t_is by the permutation ensuring that $\theta_i \leq \theta_{i+1}$. We shall say that the θ_is constitute the ordered permutation of the t_is. It is clear that $m_n(\{t_i\})$ is known as soon as $m_n(\{\theta_i\})$ is known.

There are signals for which $m_n(\{t_i\})$ is explicitly defined for any value of the s. This is, for example, the case of the normal signals, according to (3). However, this is not always the case.

Ordered signals are signals for which only $m_n(\{\theta_i\})$ is known, which means that the explicit expression of $m_n(\{t_i\})$ is known only for the ordered permutation of the s. Various examples of such signals will be presented hereafter. It is worth pointing out that for $n = 2$ this ordering property is described by the absolute value. For example, the correlation function $\exp[-(t_1 - t_2)^2]$ is defined for any set of t_is. On the other $\exp[-|t_1 - t_2|]$ hand, the correlation function takes two distinct explicit expressions according to the sign of $t_1 - t_2$.

It is well known that the symmetry property of $m_n(\{t_i\})$ has its counterpart in the Fourier domain. The obvious consequence is that the spectral moment function $M_n(\{f_i\})$ appearing in (4) is symmetric with respect to the f_is. This implies that this property is also valid for the polyspectrum $\Gamma(\{f_i\})$ appearing in (4). But, because of the delta term, the polyspectrum is invariant when replacing any frequency by $f_n = -(f_1 + f_2 + ...f_{n-1})$. This was used in many papers discussing the minimum domain of definition of the polyspectrum. On the other hand, the ordered property has no direct consequence on polyspectra because the structure of such polyspectra is a consequence of their calculation, which is the purpose of this paper.

There is a large class of signals for which the explicit expression of the moment function (1) requires that the distinct times instant t_i be classed in a given order. This order defines a specific permutation $(\{\theta_i\})$ of the time instants $(\{t_i\})$and the expression of $m_n(\{t_i\})$ is given in terms of the θ_is. This property is the origin of the expression of ordered signals.

It is worth pointing out that this is not in contradiction with the symmetry property of the moments meaning that $m_n(\{t_i\})$ is invariant under any permutation of the t_is. Indeed, any permutation of a given set of n instants t_i does not change the unique set of instants θ_i obtained by an ordering of the t_is.

We shall now present the simplest examples of ordered signals and afterwards discuss the origin of the ordering property.

B. Random Telegraph Signal (RTS)

It is a signal which only takes the values ± 1 of with the same probabilities, the changes of signs arising at the time instants t_i of a stationary Poisson point process of density λ. Its correlattion function is

$$m_2(t_1; t_2) = \gamma(t_1 - t_2) = \exp(-2\lambda|t_1 - t_2|)$$

The moments (1) are zero for odd values of n and for even values one obtains (see [5, p. 334])

$$m_{2k}(\{t_i\}) = \gamma(\theta_2 - \theta_1)\gamma(\theta_4 - \theta_3)...\gamma(\theta_{2k} - \theta_{2k-1}), \tag{13}$$

where the time instants θ_i are obtained by the unique permutation of the $2k$ distinct t_is such that $\theta_i < \theta_{i+1}$. This is the simplest example of ordered signal discussed hereafter. It is interesting to compare this expression with (3) valid for $k = 2$ and normal signals. In this case, there are three terms due to the three normal partitions, while (13) exhibits only one term due to the only one ordered permutation. Note especially that for this signal it is obviously much easier to work with moments than with cumulants. Indeed, using the classical expression giving cumulants in terms of moments leads to a very complicated mathematical expression. However, we see immediately that if the moments are simple to write explicitly the calculation of polyspectra remains complicated because of the ordered structure that is not appropriate for a simple Fourier transformation.

C. Random Jump Signals

In the studies concerning abrupt changes of the state of systems, one can describe the situation by the following signal. The starting point is once again a Poisson process of density λ. At each time instant p_i

of the process we choose a random variable V_i with a zero mean value. We assume that all these random variables are i.i.d. and also independent of the Poisson process. The signal $x(t)$ is defined as equal to V_i for $p_i \leq t < p_{i+1}$. This is, therefore, a constant signal with random jumps at the points of the Poisson process. Hence, the statistics of the signal are completely described by the moments m_n of the random variables V_i and by the density of the Poisson process. As before, it is assumed that $m_1 = 0$. Let us indicate the principle of the calculations of the moments functions.

Let θ_1 and θ_2 be two distinct arbitrary and ordered $(\theta_1 < \theta_2)$ time instants. If there is at least one point p_i of the process in the interval $[\theta_1, \theta_2]$, $x(\theta_1)$ and $x(\theta_2)$ are independent and $E[x(\theta_1)x(\theta_2)] = 0$. If there is no point in this interval, an event with probability $\exp[-\lambda(\theta_2 - \theta_1)]$, $E[x(\theta_1)x(\theta_2)] = m_2$ Thus for any t_1 and t_2, the correlation function of $x(t)$ is

$$m_2(t_1, t_2) = \gamma(t_1 - t_2) = m_2 \exp[-\lambda(|t_2 - t_1|)]. \tag{14}$$

Let us now calculate the third-order moment defined by (1). The reasoning is the same. Let θ_is be the time instants obtained by the ordered permutation of the t_is. Because of the assumption of zero mean value $E[x(t_1)x(t_2)x(t_3)] = 0$ as soon as there is at least one point p_i between θ_1 and θ_3. This yields

$$m_3(t_1, t_2, t_3) = m_3 \exp[-\lambda(|\theta_3 - \theta_1|)]. \tag{15}$$

and the interesting point is that this expression does not depend on the intermediary time θ_2.

Consider now the fourth-order moment (1). The θ_is define still the ordered permutation of the distinct s. Let X be the random variable

$$X = x(\theta_1)x(\theta_2)x(\theta_3)x(\theta_4).$$

Because of the assumption of independence and zero mean value, $E(X) \neq 0$ only if there is no point of the Poisson process in the intervals $[\theta_1, \theta_2]$ and $[\theta_3, \theta_4]$. Moreover, if there is also no point in $[\theta_1, \theta_2]$, then $E(X) = m_4$, and if there is at least one point in this interval, we have $E(X) = m_2^2$. Combining these results we deduce that

$$m_4((\{t_i\}) = m_2^2[1 - \exp\{-\lambda(\theta_3 - \theta_2)\}][\exp\{-\lambda(\theta_2 - \theta_1 + \theta_4 - \theta_3)\}]$$
$$+m_4 \exp\{-\lambda(\theta_4 - \theta_1)\}, \tag{16}$$

This can be expressed in the form

$$m_4((\{t_i\}) = \gamma(\theta_2 - \theta_1)\gamma(\theta_4 - \theta_3) + (m_4 - m_2^2) \exp\{-\lambda(\theta_4 - \theta_1)\}, \tag{17}$$

It is worth pointing out that $(m_4 - m_2^2)$ is the variance of the random variable V_i^2 and if $(m_4 = m_2^2)$, then V_i^2 is no longer random. This implies that the random variables can only take the values $\pm\sqrt{m_2}$, and, as the mean is zero, these two values have the same probability. In this case, the trajectories of the signal $x(t)$ are similar to those of the RTS, with the big difference that the changes of the sign are random instead of deterministic. That is the reason we obtain the term λ instead of 2λ in the correlation function (14). Furthermore, (17) is a sum of two terms. The first one is similar to (13) valid for the RTS while the latter has the same structure as (15).

The same principles can be used for the calculations of the moments (1) for any value of . However, the expressions become more and more tedious to write explicitly.

It is worth noting that the RTS and the random jump signal have the same exponential correlation function, even though they are quite different signals. The RTS only takes two values while the random jump signal takes the possible values of the random variable V which is arbitrary, provided that its moments do exist. First this means that all the methods using only the correlation function will give the same result. This is especially the case in the linear prediction, and therefore the prediction innovation is the same. As a consequence they have the same linear representation. Second, this shows the interest of HOS, because these two signals can be distinguished by third-order moments, or by their trispectrum.

Finally, let us show that the random-jump signal cannot be a linear process. Indeed, applying (12) and noting that for $n = 2$ and $n = 3$ moments and cumulants are equal yields

$$\int m_2(t_1, t_2, t_3) dt_3 = \alpha\gamma(t_1 - t_2), \tag{18}$$

where α is a constant and $\gamma(\tau)$ is the covariance function. This is a necessary condition in terms of moments to obtain a linear process. By choosing $t_2 = 0$ and $t_1 > 0$ we obtain that the integral appearing in (18) is proportional to $(t_1 + 2/\lambda)\exp() - \lambda t_1$, which means that (18) is not satisfied.

This model can be generalized by replacing the random variables V_i by random signals $v_i(t)$ with the same assumptions of independence. Assuming that all these signals have the same statistical properties and reasoning the same way as previously yields the covariane function

$$\gamma(t_1 - t_2) = \exp[-\lambda(|t_2 - t_1|)]c(t_2 - t_1), \tag{19}$$

where $c(t_2 - t_1)$ is the correlation function common to all the signals of the model. Similarly, the third-order moment takes the form

$$m_3(t_1, t_2, t_3) = \mu_3(t_1, t_2, t_3)\exp[-\lambda(\theta_3 - \theta_1)], \tag{20}$$

where $\mu_3(t_1, t_2, t_3)$ is the third-order moment function (1) common to all the signals $v_i(t)$ of the model. Finally, the fourth-order moment is

$$m_4((\{t_i\}) = \gamma(\theta_2 - \theta_1)\gamma(\theta_4 - \theta_3)$$

$$+[\mu_4((\{t_i\}) - c(\theta_2 - \theta_1)c(\theta_4 - \theta_3] \exp[-\lambda(\theta_4 - \theta_1)], \tag{21}$$

where $\mu_4(.)$ is the moment (1) of the signals $v_i(t)$. All these examples of fourth-order moments exhibit two common properties: the first is that the explicit expression of $m_4((\{t_i\})$ requires an ordering of the time instants ; the latter is that, even though the expressions are different, the correlation function $\gamma(.)$ of the signal appears in the form $\gamma(\theta_2 - \theta_1)\gamma(\theta_4 - \theta_3$.

D. Origin of the Ordering Property and Extensions

The RTS is one of the simplest example of continuous-time Markov process and the ordering property appears for many other Markov processes. Furthermore, the point common to all the signals presented before is that they are obtained from a Poisson process, which means a process with independent increments. It is precisely this property that introduces the ordering procedure for the calculations of higher order moments. Indeed, the time instants must be taken in an increasing order to use the independence property of successive increments. As a result it is tempting to verify whether or not other ordered signals can be obtained from other processes with independent increments. In the continuous-time cases, the degree of freedom is not very large because the only other kind of such process is the Brownian motion, introducing independent increments with normal distribution.

In this perspective there is a well-known signal that can be considered. It is the phase-noise signal introduced in the studies of frequency or phase stability of oscillators and also in the problems of coherence of laser light [12]. It is a complex signal defined by

$$z(t) = \exp\left[j\omega_0 t + \int_0^t dw(\theta) + \Phi\right], \tag{22}$$

where $w(t)$ is a Brownian motion of with diffusion constant c and Φ a random phase uniformly distributed between 0 and $2^p i$. This assumption ensures the stationarity of the signal and also the circularity property [13]. As a consequence, the only nonzero second-order moment is the correlation function $\gamma(\tau) = E[z(t)z^*(t-\tau)]$ which is

$$\gamma(\tau) = \exp[j\omega_0\tau + E\{\exp[j \int_{t-\tau}^t dw(\theta)]\} \tag{23}$$

The integral appearing in this equation is a normal random variable with variance $c\tau$, which implies that the correlation function of the signal is $\exp(j\omega_0\tau)\exp(-c|\tau|)$ By using the same definition of the θ_is as previously and also again the property of independent increments we find that

$$E[z(\theta_2)z^*(\theta_1)z(\theta_4)z(\theta_2)] = \gamma(\theta_2 - \theta_1)\gamma(\theta_4 - \theta_3), \tag{24}$$

which is similar to (13) and introduces an ordering structure. Because of the circularity property all the other fourth-order moments are zero. However, there is a strong difference with the case of the RTS.

Indeed, when permuting the time instants in (26) we must also permute the position of the complex conjugate, which introduces a strong difference with the real case and various changes in the calculations that follow. That is why we shall restrict the following analysis to real signals.

On the other hand, many of the previous results can be extended to the case of compound Poisson processes. These processes appear in many physical situations, and especially in statistical optics (see [5, p. 345]). A compound Poisson process is a Poisson process with a random density. This means that in all the previous calculations of moments it is necessary to take an expectation with respect to the positive random variable λ. It is even possible to assume that λ is replaced by a positive random process $\lambda(t)$, but this case is not analyzed in this paper.

Finally, note that the transposition to the discrete-time does not introduce any difficulty. Any sequence of i.i.d. random variables defines a random-walk process which has independent increments. From this process various ordered signals can easily be defined. For the calculations of polyspectra the difference is that integrals are replaced by series and the frequency domain is now limited to a finite interval, say $[-1/2, +1/2]$.

IV. Cobclusion

The study of very common signals, such as the RTS, introduces the property of time ordering. This property means that the calculation of fourth-order moments, and more generally moments of an order higher than two, requires a permutation such that the time instants are put in an increasing order. This ordering property is a consequence of the fact that time is oriented and appears in many examples of signals representing physical phenomena in real time. The HOS properties of many ordered signals are very simple to express in the time domain and various examples of higher order moment functions have been presented. On the other hand, the ordering property is not at all adapted to Fourier transformation leading to polyspectra. Indeed, in this transformation time is only a variable of integration and orientation of time does not play any role.

We presented a general method in order to calculate the polyspectra of ordered signals. The principle of this method is to decompose the domain of integration with respect to time in an appropriate way taking into account the ordering property of the signal. Furthermore, the grouping of various terms coming from this decomposition is facilitated by using some nonstandard properties of distributions. This method was especially used for some specific signals such as the RTS or various signals deduced from Poisson processes. A closed form of their polyspectra has been obtained and there are very few nonlinear models for which this is possible.

One of the most important points to study is the structure of the polyspectra on the normal manifold. If the polyspectra on these manifolds have a normal density, there are simple relationships between cumulant and moment polyspectra. Furthermore this property allows one to determine whether or not a nonnormal signal can become normal after narrowband filtering, which is a spectral approach of the central limit theorem.

Even if the ordered signals presented here are not normal, the calculation of polyspectra shows that they present a normal density on the normal manifolds and another contribution on the stationary manifold that was explicitly calculated.

REFERENCES

[1] D. Brillinger, An introduction to polyspectra, *Ann. Math. Stat.,* vol. 36, pp. 1351-1374, 1965.

[2] V. Chandran and S. Elgar, A general procedure for the derivation of principal domains of higher-order spectra, *IEEE Trans. Signal Processing,* vol. 42, pp. 229-233, 1994.

[3] M. V. Hinich and H. Messer, On the principal domain of the discrete bispectrum, *IEEE Trans. Signal Processing,* vol. 43, pp. 2130-2134, 1995.

[4] J. W. Dalle Molle and M. V. Hinich, Trispectral analysis of stationary random times series, *J. Acoust. Soc. Amer.,* vol. 97, pp. 2963-2978, 1995.

[5] B. Picinbono, *Random Signals and Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[6] D. Brillinger, *Time Series Analysis, Data Analysis and Theory..* New York: Holt, Rinehart and Winston, 1975.

[7] R. F. Dwyer, Fourth-order spectra of Gaussian amplitude-modulated sinusoids, *J. Acoust. Soc. Amer.,* vol. 90, pp. 918-926, 1991.

[8] P. Bondon, M. Benidir, and B. Picinbono, Polyspectrum modeling using linear or quadratic filters, *IEEE Trans. Signal Processing,* vol. 41, pp. 692-702, 1993.

[9] S. Nam and E. Powers, Application of higher order spectral analysis to cubically nonlinear system identification, *IEEE Trans. Signal Processing,* vol. 42, pp. 1746-1765, 1994.

[10] V. Leonov and A. Shiryaev, On a method of calculation of semiinvariants, *Theory Probab. Applic.,* vol. 5, pp. 460464, 1960.

[11] B. Picinbono, Ergodicity and fourth-order spectral moments, *IEEE Trans. Inform. Theory,* vol. 43, pp. 1273-1276, 1997.

[12] B. Picinbono and E. Boileau, Coherence properties of optical fieldsI: Higher order coherence functions and phase fluctuations, *J. Opt. Soc. Amer.,* vol. 58, pp. 784-789, 1968.

[13] B. Picinbono, On circularity, *IEEE Trans. Signal Processing,* vol. 42, pp. 3473-3482, 1994.