HAL open science

Noncrossing partitions, Bruhat order and the cluster complex

Philippe Biane, Matthieu Josuat-Vergès

To cite this version:

Philippe Biane, Matthieu Josuat-Vergès. Noncrossing partitions, Bruhat order and the cluster complex. Annales de l'Institut Fourier, 2019, 69 (5), pp.2241-2289. 10.5802/aif.3294 . hal-01687706

HAL Id: hal-01687706

https://hal.science/hal-01687706

Submitted on 18 Jan 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NONCROSSING PARTITIONS, BRUHAT ORDER AND THE CLUSTER COMPLEX

PHILIPPE BIANE AND MATTHIEU JOSUAT-VERGÈS

Abstract. We introduce two order relations on finite Coxeter groups which refine the absolute and the Bruhat order, and establish some of their main properties. In particular we study the restriction of these orders to noncrossing partitions and show that the intervals for these orders can be enumerated in terms of the cluster complex. The properties of our orders permit to revisit several results in Coxeter combinatorics, such as the Chapoton triangles and how they are related, the enumeration of reflections with full support, the bijections between clusters and noncrossing partitions.

1. Introduction

Let W be a Coxeter group with S a simple system of generators. There exists several natural order relations on W, namely the left or right weak order, the Bruhat order and the absolute order (this last order is associated to the length function with respect to the generating set of all reflections, see below). Two elements $v, w \in W$ such that vw^{-1} is a reflection are always comparable with respect to both the absolute and the Bruhat order. In this paper we introduce two order relations on W, which we denote by $\sqsubset$ and $\ll$, which encode this situation, namely for any pair v, w as above, such that $v < w$ (here $<$ denotes the absolute order), we define $v \sqsubset w$ if $v <_B w$ (here $<_B$ is the Bruhat order) and $v \ll w$ if $w <_B v$, then extend $\sqsubset$ and $\ll$ to order relations on W by transitivity. We believe that these two orders are important tools for understanding noncrossing partitions, clusters, and their interrelations. Bessis [7, Section 6.4] suggested to study how the different orders on W are related. The present paper can be considered as a first step in this direction.

Some versions of the orders $\sqsubset$ and $\ll$ were considered before this work. An order called $\ll$ on the set of classical noncrossing partitions was introduced independently by Belinschi and Nica [6, 24] in the context of noncommutative probabilities, and by Senato and Petrullo [26] in order to study Kerov polynomials. The notion of noncrossing partition can be defined in terms of the geometry of the symmetric group [8] and it has been extended to general Coxeter systems: a set of noncrossing partitions can be associated to some Coxeter element c in W as $NC(W, c) = \{w \in W \mid w \leqslant c\}$ (see [7, 11]). The

Key words and phrases. Noncrossing partitions, Bruhat order, cluster complex.

classical case corresponds to c being the cycle $(1, \ldots, n)$ in the symmetric group S_n. In this context, the order $\sqsubset$ on $NC(W, c)$ was introduced by the second author in [21], with a different definition, in order to give a refined enumeration of maximal chains in $NC(W, c)$.

After defining the two order relations $\sqsubset$ and $\ll$, we will consider the restriction of these orders to the set of noncrossing partitions and give a more direct characterization of the pairs $v, w \in NC(W, c)$ with $v \sqsubset w$ or $v \ll w$. We will also introduce interval partitions for arbitrary finite Coxeter groups, which generalize the classical interval partitions. These partitions play an important role in this study. Then we will consider intervals for the two orders. These turn out to be closely related to the cluster complex of Fomin and Zelevinsky [19]. Originated from the theory of cluster algebras, the cluster complex is a simplicial complex with vertex set the almost positive roots of W (see Section 6 for details), associated to a standard Coxeter element in a finite Coxeter group. Connections between noncrossing partitions and the cluster complex were first observed via an identity called the $F = M$ theorem, conjectured by Chapoton [14]. We will see that the introduction of the two orders $\sqsubset$ and $\ll$ sheds new light on these relations. In particular an explicit bijection between the facets of the cluster complex and the noncrossing partitions associated with the same Coxeter element was given by Reading [28], using the notion of c-sortable elements. Another bijection was given in [12] in the case of bipartite Coxeter elements. We will recast this last bijection in terms of the orders $\sqsubset$ and $\ll$, which will allow us to extend it to arbitrary standard Coxeter elements, using the definition of the cluster complex by Reading. We will also give a bijection between the intervals for $\ll$ and faces of the positive cluster complex. We also show that intervals of height k for $\sqsubset$ are equienumerated with the faces of the cluster complex of size $n-k$ and give a bijective proof in the case where c is a bipartite Coxeter element. Finally the orders $\ll$ and $\sqsubset$ will allow us to revisit the Chapoton triangles, to give new proofs of their properties and to refine them.

This paper is organized as follows. In Section 2 we recall basic facts about finite reflections groups, root systems and Coxeter elements. In Section 3 we define the noncrossing partitions and recall their main properties. We also prove Proposition 3.5, showing a relation between between the Bruhat order and the Kreweras complement on $NC(W, c)$, which plays a crucial role in this work. Section 4 is the central part of this paper, in it we introduce the two order relations $\sqsubset$ and $\ll$, which are the main subject of this paper, we characterize these relations and obtain some of their basic properties. We give a few examples in low rank in Section 5. In Section 6 we study the cluster complex associated to a standard Coxeter element, as defined by Reading [28]. This simplicial complex is defined with the help of a binary relation for which we give a new characterization. We also recall facts about nonnesting partitions and Chapoton triangles. In Section 7 we give some enumerative properties of the intervals of the two order relations which exhibit several connections with the cluster complex. In the following section,

using the orders $\sqsubset$ and $\ll$, we generalize the bijection of [4] between noncrossing partitions and maximal faces of the cluster complex to encompass arbitrary standard Coxeter elements, then we use this to give a bijection between intervals for the order $\ll$ and faces of the positive cluster complex. The final section is devoted to some properties of the Chapoton triangles.

2. Finite Coxeter or real reflection groups

We fix notations, recall some basic facts about real reflection groups and refer to [10, 20] for general information about these.

2.1. **Finite reflection groups, roots, reflections and inversions.**

2.1.1. Let V be a finite dimensional euclidian space and $W \subset O(V)$ be a finite real reflection group. Fix a fundamental chamber C, its unit inward normals $\alpha_1, \dots, \alpha_n$ are the simple positive roots and the associated set of fundamental reflections $S = \{s_1, \dots, s_n\}$ generates W, then (W, S) is a Coxeter system. We denote by ℓ the length function associated to S. The set of all reflections in W is denoted by T, it is a conjugacy class in W. Each reflection $t \in T$ has a fixed hyperplane and the unit normal of this fixed hyperplane, which has positive inner product with C, is a positive root normal to this hyperplane which we denote $r(t)$. The set of positive roots Π is thus in bijection with T. The chamber C is the dual cone of the positive span of positive roots.

The set of negative roots is $-\Pi$ and the set of all roots is $\Pi \cup (-\Pi)$. The *Bruhat order*, denoted by $\leqslant_B$, is defined as the transitive closure of the covering relations $v \lessdot_B w$ if $vw^{-1} \in T$ and $\ell(v) < \ell(w)$. A *left inversion* (respectively, *right inversion*) of $w \in W$ is a $t \in T$ such that $\ell(tw) < \ell(w)$ (respectively, $\ell(wt) < \ell(w)$). The set of left (respectively, right) inversions is denoted by $\mathrm{Inv}_L(w)$ (respectively, $\mathrm{Inv}_R(w)$). Note that a bijection $\mathrm{Inv}_L(w) \to \mathrm{Inv}_R(w)$ is given by $t \mapsto w^{-1}tw$.

Proposition 2.1 ([10], Proposition 4.4.6)**.** *For $w \in W$ and $t \in T$, we have:*

$$w\big(r(t)\big) = \begin{cases} -r(wtw^{-1}) & \text{if } t \text{ is a right inversion of } w, \\ r(wtw^{-1}) & \text{otherwise.} \end{cases} \tag{1}$$

In particular, t is a right inversion of w if and only if $w(r(t)) \in (-\Pi)$ and a left inversion if and only if $w^{-1}(r(t)) \in (-\Pi)$.

Corollary 2.2. *If $t_1, t_2 \in T$ commute then $t_1 \in \mathrm{Inv}_R(w) \Leftrightarrow t_1 \in \mathrm{Inv}_R(wt_2)$ i.e. $w \leqslant_B wt_1 \Leftrightarrow wt_2 \leqslant_B wt_2t_1$.*

If a simple reflection $s \in S$ is a right (repectively, left) inversion of w, it is called a right (repectively, left) *descent* of w.

If $w = s_{i_1}s_{i_2}\dots s_{i_r}$ is a reduced expression for w then the r left inversions of w are the reflections of the form

$$(s_{i_1}s_{i_2}\dots s_{i_{l-1}})s_{i_l}(s_{i_1}s_{i_2}\dots s_{i_{l-1}})^{-1}, \qquad l = 1, \dots, r, \tag{2}$$

and of course there is a similar formula for right inversions.

The *support* of $w \in W$, denoted by $\mathrm{supp}(w)$, is the subset of S containing the simple reflections appearing in some reduced expression of w. It does not depend on the chosen reduced expression since any two of them are related by a sequence of braid moves (see [10]). Equivalently, $\mathrm{supp}(w)$ is the smallest J such that $w \in W_J$, see Section 2.2. We say that w has *full support*, or that w is *full*, if $\mathrm{supp}(w) = S$. Using the explicit formula in (2), we have:

$$\mathrm{supp}(t) \subset \mathrm{supp}(w)$$

if t is a right or left inversion of w.

Remark 2.3. A root $r \in \Pi$ is in Δ iff it cannot be written as a sum $\sum_{r\in\Pi} c_r r$ where the coefficients c_v are $\geqslant 0$ and at least two of them are nonzero. This characterizes the set of simple roots in Π. Also, the simple roots form the unique set of n positive roots having the property that the scalar product of any pair is nonpositive, see [20].

2.1.2. *Coxeter elements.* A *standard Coxeter element* in (W, S) is a product $s_{i_1} \cdots s_{i_n}$ of all the simple reflections in some order. It is known that all standard Coxeter elements are conjugate in W, but in general they do not form a full conjugacy class. An element which is conjugated to some standard Coxeter element is called a Coxeter element. In this paper we will mostly consider standard Coxeter elements.

Lemma 2.4. *If c is a standard Coxeter element and $s \in S$ a right or left descent of c, then scs is also a standard Coxeter element. All standard Coxeter elements are connected to each other via such transformations.*

See [20, Section 3.16] for a proof.

Let $S = S_+ \cup S_-$ be a partition such that all s_i in S_+ commute and all s_i in S_- commute (such a partition always exists), then the standard Coxeter element $c = c_+c_-$ where $c_\pm = \prod_{s\in S_\pm} s$ is called a *bipartite Coxeter element*.

2.1.3. *Absolute length and the absolute order.* The *absolute length* is the length function associated to the generating set T:

$$\ell_T(w) = \min\{k \geqslant 0 \,|\, w \text{ can be expressed as a product of } k \text{ reflections}\}. \tag{3}$$

This quantity has a geometric interpretation (see [11, Proposition 2.2]):

$$\ell_T(w) = n - \dim(\mathrm{Fix}(w)) \tag{4}$$

where $\mathrm{Fix}(w) = \ker(w - I)$.

We call a factorization $v = v_1 \dots v_k$ in W *minimal* if

$$\ell_T(v) = \ell_T(v_1) + \ldots + \ell_T(v_k). \tag{5}$$

The following elementary lemma is well known, cf. [7].

Lemma 2.5. *Let $v = v_1 \dots v_k$ be a minimal factorization, then for any subsequence $i_1 \dots i_l$ with $1 \leqslant i_1 < \ldots < i_l \leqslant k$ the factorization $v_{i_1} \dots v_{i_l}$ is minimal, moreover $v_{i_1} \dots v_{i_l} \leqslant v$.*

Proof. If $i_1 = 1, i_2 = 2, \dots, i_l = l$ the statement is a simple consequence of the triangle inequality for ℓ_T. In the general case observe that for any $i < k$ the factorization $v = v_1 \dots v_{i-1}\hat{v}_i\hat{v}_{i+1}v_{i+2}\dots v_k$ with $\hat{v}_i = v_{i+1}$ and $\hat{v}_{i+1} = v_{i+1}^{-1}v_iv_{i+1}$ is again a minimal. Using this observation we can move successively $v_{i_1}, v_{i_2}, \dots, v_{i_l}$ to the beginning and reduce to the first case. □

Remark 2.6. The preceding lemma implies that, contrary to the case of reduced decomposition into simple reflections, in the case of a minimal factorization into reflections every subword of a reduced word is reduced. In order to avoid confusion, in this paper we will reserve the expressions "reduced word" and "subword" to the case of factorizations into simple reflections, related to the Bruhat order and reserve the expression "minimal factorization" to the case of factorizations satisfying (5).

The absolute length (3) allows to define an order relation on W, the *absolute order* denoted here by $\leqslant$:

$$v \leqslant w \quad \text{if} \quad \ell_T(w) = \ell_T(v^{-1}w) + \ell_T(v).$$

In particular a cover relation for this order, denoted by $v \lessdot w$, holds if and only if $vw^{-1} \in T$ and $\ell_T(v) = \ell_T(w) - 1$. The following properties of the absolute order are immediate or follow directly from Lemma 2.5.

Proposition 2.7. *The absolute order is invariant under conjugation and inversion, namely for all $u, v, w \in W$ one has*

$$v \leqslant w \Longleftrightarrow v^{-1} \leqslant w^{-1} \Longleftrightarrow uvu^{-1} \leqslant uwu^{-1}. \tag{6}$$

Let $u, v, w \in W$ be such that $u \leqslant v \leqslant w$ then $u^{-1}v \leqslant u^{-1}w$ and $u \leqslant uv^{-1}w \leqslant w$.

2.2. **Parabolic subgroups.**

2.2.1. Let $J \subset S$, then the *standard parabolic subgroup* W_J is the subgroup generated by J. If $s \in S$ we will also use the notation $W_{\langle s \rangle}$ for the parabolic subgroup associated with $S \backslash \{s\}$. A *parabolic subgroup* is any subgroup conjugate to some W_J.

The parabolic subgroups have the form

$$P = \{w \in W \ : \ w(x) = x \text{ for all } x \in E\}$$

for some subspace $E \subset V$, moreover if P is a parabolic subgroup and

$$\mathrm{Fix}(P) = \{x \in V \ : \ w(x) = x \text{ for all } w \in P\}$$

then P is itself a reflection group in $O(\mathrm{Fix}(P)^{\perp})$. In particular, to each $w \in W$ we can associate a parabolic subgroup $\Gamma(w)$:

$$\Gamma(w) = \{v \in W \ : \ \mathrm{Fix}(w) \subset \mathrm{Fix}(v)\}. \tag{7}$$

2.2.2. *Simple generators and roots.* Let $P \subset W$ be a parabolic subgroup. Then P is itself a reflection group, with reflection set $T \cap P$, and its roots form a subset of those of W. A natural set of positive roots for P is $\Pi(P) = \Pi \cap P$. Accordingly, there is a unique set of simple roots $\Delta(P)$ (see Remark 2.3 above) and a set of simple reflections $S(P) = r^{-1}(\Delta(P))$. We have $\Delta(P) \subset \Pi$. Note that $\Delta(P) \subset \Delta$ does not hold in general, this only happens for standard parabolic subgroups.

Since there are several definitions of these sets $\Delta(P)$ or $S(P)$ given in the literature, in order to apply results from various references, it is in order to check that they are all equivalent.

Remark 2.8. A set of positive roots is $\Delta(P)$ for some parabolic subgroup P if and only if they are such that the scalar product of any pair is nonpositive.

Proposition 2.9. *The simple reflections of P are the reflections $t \in T \cap P$ satisfying* $\mathrm{Inv}_R(t) \cap P = \{t\}$.

Proof. A reflection $t \in P \cap T$ is simple as an element of P, if and only if it has only one inversion as an element of P. Since $\mathrm{Inv}_R(t) \cap P$ is the right inversion set of t as an element of P, the result follows. □

Proposition 2.10. *The set $\Delta(P) \subset \Pi(P)$ is the unique simple system of P such that the fundamental chamber of P contains that of W.*

Proof. Since $\Pi(P) \subset \Pi$, the positive span of $\Pi(P)$ is included in that of Π. Taking the dual cone reverses inclusion, so the fundamental chamber of P contains that of W. □

2.2.3. *The Bruhat graph.* The Bruhat graph on the vertex set W is defined by putting an oriented arrow $w \to v$ if $vw^{-1} \in T$ and $\ell(v) < \ell(w)$. The unoriented underlying graph is the Cayley graph of W with the reflections as generating set.

Proposition 2.11 ([17])**.** *Let (W, S) be a Coxeter system and P a parabolic subgroup, then the restriction of the Bruhat graph to P is the Bruhat graph of P for its canonical generators.*

In other words, if $\leqslant_{B_P}$ is the Bruhat order on P and if $v, w \in P$ and vw^{-1} is a reflection then

$$v \leqslant_B w \Longleftrightarrow v \leqslant_{B_P} w.$$

This implies in particular that for $v, w \in P$ one has

$$v \leqslant_{B_P} w \Longrightarrow v \leqslant_B w.$$

The converse implication does not hold in general, see (10) below.

3. Noncrossing partitions

We refer to [1] for the general facts on this subject.

3.1. **Definition of noncrossing partitions.** Let c be a standard Coxeter element, then the set of noncrossing partitions associated to c, denoted by $NC(W,c)$, is the set of all $w \in W$ such that $w \leqslant c$. In the case where W is the symmetric group S_{n+1} with the Coxeter generators $s_i = (i, i+1)$ and $c = s_1 \dots s_n$ is the cycle $(1, 2, 3, \dots, n, n+1)$ one can associate to any $w \in NC(W,c)$ the partition given by its cycle decomposition. This coincides with the classical notion of noncrossing partition as defined by Kreweras, see [22], [8]. In the sequel we denote by NC_n the set of classical noncrossing partitions.

Endowed with the order $\leqslant$ the set $NC(W,c)$ is a lattice [11]. Since all Coxeter elements are conjugate, the isomorphism class of this lattice structure does not depend on c. The map $w \mapsto \Gamma(w)$ which associates a parabolic subgroup to a noncrossing partition is injective, moreover one has

$$v \leqslant w \Longleftrightarrow \Gamma(v) \subset \Gamma(w) \tag{8}$$

(see for example [1, Section 5.1.3]). In particular, the codimension of the fixed subspace, which is also the absolute length (see (4)) gives a rank function on $NC(W,c)$, which is a rank function for the lattice structure.

Parabolic subgroups can be considered as generalized set partitions so that the map Γ gives a way to consider noncrossing partitions as particular set partitions. In the classical case of noncrossing partitions of $[1,n]$ for all $i < j < k < l$, putting $t_1 = (i,k)$, $t_2 = (j,l)$ neither product t_1t_2 or t_2t_1 belongs to NC_n and this property characterizes the crossing of two transpositions with disjoint supports. This leads to the following definition of a property which will play an important role later.

Definition 3.1. (Bessis [7, Definition 2.1.1]). Two reflections t_1, t_2 are called *c-noncrossing* if either $t_1t_2 \leqslant c$ or $t_2t_1 \leqslant c$.

Remark 3.2.

i) Bessis uses the notation $\|_c$ to denote this relation, however this conflicts with the use of the same notation by Reading [28] to denote another relation. We will use Reading's notation later on (see Definition 6.3) so we will just say that two reflections are c-noncrossing when needed, without using a specific notation for this relation.

ii) Observe that if two reflections t_1, t_2 do not commute then one cannot have both $t_1t_2 \leqslant c$ and $t_2t_1 \leqslant c$ (this follows from the fact that $\Gamma(t_1t_2) = \Gamma(t_2t_1) = \langle t_1, t_2 \rangle$ and the injectivity of $w \mapsto \Gamma(w)$ on $NC(W,c)$).

One can define noncrossing partitions in the same way when c is a (general) Coxeter element, i.e. is conjugated to some standard Coxeter element. Then any noncrossing partition $w \leqslant c$ is a Coxeter element of a parabolic subgroup, see [7, Lemma 1.4.3]. However our results crucially depend on properties of standard Coxeter elements, therefore in the following we will only consider such Coxeter elements.

Proposition 3.3. *Let $w \in NC(W,c)$, then there exists an indexing $\mathfrak{s}_1, \dots, \mathfrak{s}_k$ of the simple generators of $\Gamma(w)$ such that*

$$w = \mathfrak{s}_1 \cdots \mathfrak{s}_k.$$

Proof. This follows from results of Reading [28]. More precisely, Theorem 6.1 from [28] shows that w is the product, in some order, of the so-called cover reflections of a Coxeter-sortable element. By Lemma 3.1 of the same reference these cover reflections are the simple generators of a parabolic subgroup.

Alternatively, this follows from the results of Brady and Watt [12]. In the case of a bipartite Coxeter element, [12, Proposition 5.1] gives a way to compute a valid $\mathfrak{s}_k$, reducing the problem to finding $\mathfrak{s}_1, \dots, \mathfrak{s}_{k-1}$, which can be done inductively. Using Lemma 2.4, it remains only to see how the result is transferred from c to scs where s is a left descent of c. Let $w = \mathfrak{s}_1 \cdots \mathfrak{s}_k$ be the factorization of $w \in NC(W,c)$ as a product of simple generators. If $s \notin \{\mathfrak{s}_1, \dots, \mathfrak{s}_k\}$, we have $sws = (s\mathfrak{s}_1 s) \cdots (s\mathfrak{s}_k s) \leqslant scs$ and the factors form a simple system, otherwise we can assume $s = \mathfrak{s}_1$ and we have $sws = \mathfrak{s}_2 \cdots \mathfrak{s}_k \mathfrak{s}_1 \leqslant scs$. □

In other words, each $w \in NC(W,c)$ is a standard Coxeter element of its own parabolic subgroup $\Gamma(w)$, considered as a reflection group. Observe that this property actually depends only on w and not on the standard Coxeter element c such that $w \leqslant c$.

From *ii*) of Remark 3.2 we deduce

Proposition 3.4. *Let $c = t_1 \dots t_n = t'_1 \dots t'_n$ be two minimal factorizations of c into reflections, where $t'_1, \dots, t'_n$ is a permutation of $t_1, \dots, t_n$, then one can pass from one factorization to the other by a succession of transpositions of neighbourhing commuting reflections $t_i t_j \to t_j t_i$.*

Proof. By induction on n, the length of c. If $t_1 = t'_1$ then use the induction hypothesis for the Coxeter element $t_1 c \in \Gamma(t_1 c)$. If $t_1 = t'_j$ with $j > 1$ then $t_1 t'_i \leqslant c$ and $t'_i t_1 \leqslant c$ for all $i < j$ therefore t_1 commutes with all t'_i with $i < j$ and we can move it to the left to reduce to the preceding case. □

3.2. **Generalized Catalan and Narayana numbers.** The number of elements of $NC(W,c)$ is the generalized Catalan number $\mathrm{Cat}(W)$. If (W,S) is an irreducible system one has

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1}$$

where the e_i are the exponents of W and h is the Coxeter number, i.e. the order of c as a group element. Analogously the number of elements of $NC(W,c)$ of rank k is the generalized Narayana number $\mathrm{Nar}_k(W)$, see e.g. [1].

Let $NCF(W,c) \subset NC(W,c)$ denote the set of noncrossing partitions with full support. Its cardinality is the *positive Catalan number* $\mathrm{Cat}^+(W)$, given

by

$$\mathrm{Cat}^+(W) = \prod_{i=1}^{n} \frac{h + e_i - 1}{e_i + 1}.$$

The numbers $\mathrm{Cat}(W)$ and $\mathrm{Cat}^+(W)$ are related by inclusion-exclusion.

The *positive Narayana number* $\mathrm{Nar}_k^+(W)$ is the number of noncrossing partitions of rank k with full support. These numbers are related with Narayana numbers by inclusion-exclusion.

Define the *Fuß-Catalan numbers* $\mathrm{Cat}^{(k)}(W)$ as

$$\mathrm{Cat}^{(k)}(W) = \prod_{i=1}^{n} \frac{kh + e_i + 1}{e_i + 1}. \tag{9}$$

This is the number of chains $w_1 \leqslant \cdots \leqslant w_k$ in $NC(W, q)$, see [14]. Note that this is a polynomial in k. It can be seen as a rescaling of the polynomial

$$\sum_{w \in W} t^{\ell_T(w)} = \prod_{i=1}^{n} (1 + te_i).$$

3.3. **The Kreweras complement.** For any $w \in NC(W, c)$ one defines its Kreweras complement as $K(w) = w^{-1}c$. The map K is bijective, and an anti-automorphism of the lattice structure on $NC(W, c)$. The map K is not involutive, rather $K^2(w) = c^{-1}wc$ is an automorphism of $NC(W, c), \leqslant)$. The inverse anti-automorphism of K is $K^{-1}(w) = cw^{-1}$. We sometimes denote K_c to mark the dependence on c.

The Kreweras complement has a remarkable compatibility with respect to the Bruhat order, which will play a crucial role in this paper.

Proposition 3.5.

i) Let $v, w \in NC(W, c)$ with $v \lessdot w$, then we have:

$$v >_B w \Longrightarrow K(v) >_B K(w).$$

ii) The converse result holds under a supplementary hypothesis: let $v, w \in NC(W, c)$ with $v \lessdot w$, and suppose v has full support in $NC(W, c)$, then we have:

$$K(v) >_B K(w) \Longrightarrow v >_B w.$$

Before the proof we need a lemma.

Lemma 3.6. *Let $w \in NC(W, c)$ and let t be a reflection such that $tw < w$ and $tc <_B c$, then $tw <_B w$. The same holds with right multiplication: if $wt < w$ and $ct <_B c$, then $wt <_B w$.*

Proof. Since $tc <_B c$ it follows that tc is a subword of length $n-1$ of a reduced expression for c therefore $tc \in W_{\langle s \rangle}$ for some $s \in S$ and $tc < c$, moreover s is in the support of t since $c \notin W_{\langle s \rangle}$. As $t < w \leqslant c$ by Proposition 2.7 one has $tw \leqslant tc$ therefore $tw \in W_{\langle s \rangle}$. One has $t \notin W_{\langle s \rangle}$ and $t = (tw)w^{-1}$ with $tw \in W_{\langle s \rangle}$, therefore $w^{-1} \notin W_{\langle s \rangle}$ and s belongs to the support of w. It follows

that w cannot be obtained as a reduced subword of tw thus $tw <_B w$. The case of right multiplication is analogous. □

Proof of Proposition 3.5. We first prove $i)$. Suppose that $v = tw$ for some reflection, that $v < w$ and that $w <_B v$, then $tc >_B c$ by Lemma 3.6. One has $K(v) = K(w)c^{-1}tc$ and $r(c^{-1}tc) = c^{-1}(r(t))$ by Proposition 2.1 therefore $K(w)(r(c^{-1}tc)) = w^{-1}(r(t)) \in \Pi$, again by Proposition 2.1 since $tw >_B w$. It follows that $c^{-1}tc$ is not a right inversion for $K(w)$ and $K(v) >_B K(w)$.

Now we prove $ii)$. Let $v = tw < w \leqslant c$ for a reflection t. If $tc <_B c$ then $tc \in W_{\langle s\rangle}$ for some $s \in S$ and $tc < c$ therefore $v < tc \in W_{\langle s\rangle}$. We deduce from this that, if v has full support, then $tc >_B c$. If $K(w) <_B K(v)$ then, again by Proposition 2.1 one has $r(c^{-1}tc) = c^{-1}(r(t))$ and $w^{-1}(r(t)) = K(w)(r(c^{-1}tc)) \in \Pi$ therefore t is not a left inversion for w and $tw >_B w$. □

3.4. **An involutive automorphism for bipartite Coxeter elements.** Let $c = c_+c_-$ be a bipartite Coxeter element.

Proposition 3.7. *The map L defined by*

$$L(w) = c_+wc_-$$

is an involutive anti-automorphism of the poset $NC(W, c)$.

Proof. The map L is the composition of the maps $w \to w^{-1}$ which is an isomorphism from $NC(W, c)$ to $NC(W, c^{-1})$, the isomorphism $w \mapsto c_+wc_+$ from $NC(W, c^{-1})$ to $NC(W, c)$ and the Kreweras map K which is an anti-automorphism of $NC(W, c)$, therefore L is an anti-automorphism. The fact that it is an involution follows from $c_\pm^2 = e$. □

We call the map L the *bipartite complement* on $NC(W, c)$.

Like the Kreweras complement, the bipartite complement is compatible with the Bruhat order.

Proposition 3.8.

i) Let $v, w \in NC(W, c)$ with $v \lessdot w$, then we have:

$$v >_B w \Longrightarrow L(v) >_B L(w).$$

ii) Let $v, w \in NC(W, c)$ with $v \lessdot w$, and suppose that v has full support, then we have:

$$L(v) >_B L(w) \Longrightarrow v >_B w.$$

Proof. We first prove $i)$. Suppose that $v = tw$ for some reflection t, that $v < w$ and $w <_B v$, then $tc >_B c$ by Lemma 3.6. It follows, applying repeatedly Proposition 2.1, that $t \notin S_+$ and $r(c_+tc_+) = c_+(r(t))$. Applying the same reasoning to $v = w(w^{-1}tw)$ gives that $c(w^{-1}tw) >_B c$ and $w^{-1}tw \notin S_-$ therefore, since $w^{-1}(r(t)) = r(w^{-1}tw)$, one has $r(c_w^{-1} - twc_-) = c_-w^{-1}(r(t))$. One has $L(v) = c_+tc_+L(w)$ and $L(w)^{-1}(r(c_+tc_+) = r(c_w^{-1} - twc_-)$ and c_+tc_+ is not a left inversion of $L(w)$, thus $L(v) >_B L(w)$.

We now prove $ii)$. Let $v = tw < w$ for some reflection t and suppose that v has full support. Arguing as in to the proof of $ii)$ in Proposition 3.5 we

get $tc >_B c$ moreover let $t' = v^{-1}tv$ then $v = wt'$ and, similarly, $ct' >_B c$. In particular, t is not a left inversion of c_+ and t' is not a right inversion of $c-$. Suppose now that $Lv >_B Lw$ i.e. $c_+vc_- >_B c_+tvc_-$ then c_+tc_+ is a left inversion of c_+vc_- and, since $c_+(r(t))$ is positive we see that $c_-v^{-1}(r(t))$ is a negative root. Since t' is not a right inversion of c_- we see that $v^{-1}(r(t))$ is a negative root therefore $w = tv <_B v$. □

3.5. **Interval partitions.** They form a natural subset of $NC(W, c)$ defined as follows.

Definition 3.9. Given a Coxeter element c, an *interval partition* is an element of $NC(W, c)$ whose associated parabolic group is standard, i.e. the interval partitions are the $w \in NC(W, c)$ such that $\Gamma(w) = W_J$ for some $J \subset S$. We denote by $INT(W, c)$ the set of interval partitions of $NC(W, c)$.

Equivalently, $w \in W$ is an interval partition if and only if $w \leqslant_B c$. Since a reduced word for c contains each simple reflection exactly once, it is easy to see that each subword is a reduced word, therefore the orders $\leqslant$ and $\leqslant_B$ coincide on $INT(W, c)$ and give it the structure of a Boolean lattice, isomorphic to the lattice of subwords of a reduced expression for c, or to the lattice of subsets of Δ, or of S.

One can check that, when W is the symmetric group with its canonical Coxeter generators and c is the cycle $(1 \dots n)$, the interval partitions coincide with the classical ones (cf. e.g. [30]).

Definition 3.10. For $w \in NC(W, c)$ we denote by $\underline{w}$ the largest interval partition below w and by $\overline{w}$ the smallest interval partition above w in $(NC(W, c), \leqslant)$.

The *relative Kreweras complement* of w with respect to $\overline{w}$ is $K(w, \overline{w}) = w^{-1}\overline{w}$.

Since the restriction to $INT(W, c)$ of the abolute order is a lattice order the existence and uniqueness of $\overline{w}$ and $\underline{w}$ is immediate. One can also characterize $\underline{w}$ by its associated simple system:

$$\Delta(\Gamma(\underline{w})) = \Delta(\Gamma(w)) \cap \Delta(W).$$

Analogously if $J \subset S$ is the support of w, then the subword γ of c consisting of elements of J is the unique Coxeter element in $NC(W, c) \cap W_J$ and one has $\gamma = \overline{w}$, moreover w has full support in $\Gamma(\gamma) = W_J$. It is easy to see that the map $w \mapsto \overline{w}$ is a lattice homomorphism.

There is also a characterization of $\overline{w}$ as a maximum, which we leave to the reader to check.

Proposition 3.11. *For $w \in NC(W, c)$ one has*

$$\overline{w} = \max\{\eta \in INT(W, c) \,|\, \eta \leqslant_B w\}.$$

We now consider the upper ideal $\{w \in NC(W, c) \,|\, w \geqslant \gamma\}$ for certain interval partitions γ.

Proposition 3.12. *Let $s \in S$ be a left descent of c (i.e. $sc <_B c$).*

i) Let $w \in NC(W,c)$ be such that $s \leqslant w$, then $sw \in W_{\langle s \rangle}$.

ii) The map $v \mapsto sv$ is a lattice homomorphism from $NC(W_{\langle s \rangle}, sc)$ to $NC(W,c)$ whose image is the subset $\{w \in NC(W,c) | w \geqslant s\}$.

There is a similar statement with right descents.

Proof. *i)* It follows from Proposition 2.7 that $sw \leqslant sc$ hence $sw \in NC(W_{\langle s \rangle})$.

ii) It is easy to see that if $v \in W_{\langle s \rangle}$ then $\ell_T(sv) = \ell_T(v) + 1$ which implies the result, using Proposition 2.7 to see that it is a homomorphism and the first statement to see that the map is surjective.

Finally the case of right descents follows by considering the order preserving isomorphism $w \mapsto w^{-1}$ from $NC(W,c)$ to $NC(W,c^{-1})$. □

Corollary 3.13. *Let a be an initial subword of c and b be a final subword with $\ell(a)+\ell(b) \leqslant \ell(c)$, then $ab \in INT(W,c)$ and the set $\{w \in NC(W,c) | ab \leqslant w\}$ is the image of the map $v \mapsto avb$ from $NC(W_{\langle ab \rangle}, a^{-1}cb^{-1})$ to $NC(W,c)$, which is a lattice homomorphism.*

Proof. By induction on $\ell(a) + \ell(b)$ using the preceding proposition. □

In the bipartite case, one can say more.

Proposition 3.14. *Let c be a bipartite Coxeter element then the complement map L restricts to an involution on $INT(W,c)$.*

Proof. It is easy to see that L corresponds to the complementation map if on identitifes $INT(W,c)$ with the set of subsets of Δ or S. □

Since L is an involutive anti-automorphism one can see that, if $\gamma \in INT(W,c)$ then L exchanges the sets $\{w | \overline{w} = \gamma\}$ and $\{w | \underline{w} = L(\gamma)\}$.

4. Two order relations on W

4.1. **Definition of the orders.** Consider a Coxeter system (W,S). For any covering relation of the absolute order $v \lessdot w$ in W one has $vw^{-1} \in T$, therefore v and w are comparable for the Bruhat order, namely one has $v <_B w$ if $\ell(v) < \ell(w)$ or $w <_B v$ if $\ell(w) < \ell(v)$. We introduce two order relations which encode this distinction by refining the absolute order.

Definition 4.1. For any covering relation $v \lessdot w$ in W define

$$v \sqsubset w \quad \text{if} \quad v <_B w,$$
$$v \ll w \quad \text{if} \quad w <_B v,$$

and extend these relations to order relations $\sqsubset$ and $\ll$ on W by transitive closure.

Since $\sqsubset$ and $\ll$ are included in $\lessdot$ (seeing relations as sets of pairs), they are acyclic and their transitive closures $\sqsubset$ and $\ll$ are partial orders which are included in $\leqslant$ and whose cover relations are $\sqsubset$ and $\ll$. This justifies the

previous definition. Also the rank function for $\leqslant$ serves as a rank function for $\sqsubset$ and $\ll$.

The order $\sqsubset$ was introduced by the second author in [21] on the set $NC(W,c)$, while $\ll$ was introduced by Belinschi and Nica in [6] in the case of classical noncrossing partitions, both with different definitions. We shall see below that they are specializations of our definitions.

Observe that, as an immediate consequence of the definitions, one has

$$\begin{aligned} v \sqsubset w \quad &\Longrightarrow \quad v \leqslant w,\ v \leqslant_B w, \\ v \ll w \quad &\Longrightarrow \quad v \leqslant w,\ w \leqslant_B v. \end{aligned}$$

The opposite implications, however, do not hold in general. Here is a counterexample in the symmetric group $\mathfrak{S}_5$: take (in cycle notation)

$$v = (2,4), \qquad w = (1,5)(2,3,4). \tag{10}$$

We have $v \leqslant w$, $\ell(v) = 3$ and $\ell(w) = 9$. There are two elements between v and w in the absolute order, which are $(1,5)(2,4)$ and $(2,3,4)$. Their respective Coxeter lengths are 10 and 2, therefore we have:

$$\begin{aligned} v = (2,4) \sqsubset (1,5)(2,4) \ll (1,5)(2,3,4) = w, \\ v = (2,4) \ll (2,3,4) \sqsubset (1,5)(2,3,4) = w. \end{aligned}$$

It follows that $v \not\sqsubset w$. On the other side, we have $v \leqslant_B w$ and a shortest path in the Bruhat graph is (an arrow with label t represents multiplication by t on the right):

$$(1,5)(2,3,4) \xrightarrow{(1,3)} (1,4,2,3,5) \xrightarrow{(2,5)} (1,4,2)(3,5) \xrightarrow{(3,5)} (1,4,2) \xrightarrow{(1,2)} (2,4).$$

The respective lengths of the elements in this path are 9, 8, 5, 4, 3. Note that v and w are both noncrossing partitions but the path goes through elements that are not noncrossing partitions.

Let $w \in W$ and let $\Gamma(w)$ be the associated parabolic subgroup. Since the orders $\sqsubset$ and $\ll$ can be defined using only the Bruhat graph and the absolute order, it follows from Proposition 2.11 that, on the set $\{v \in W \mid v \leqslant w\}$ the orders $\sqsubset$ and $\ll$ depend only on the Bruhat order on $\Gamma(w)$ (with its canonical system of generators).

In the sequel we shall mainly be interested in the restrictions of these order relations to $NC(W,c)$ for some standard Coxeter element c.

4.2. **The Kreweras complement.** Using the orders $\sqsubset$ and $\ll$ we can reformulate Proposition 3.5.

Proposition 4.2.

i) Let $v, w \in NC(W,c)$ with $v \lessdot w$, then we have:

$$v \ll w \Longrightarrow K(w) \sqsubset K(v).$$

ii) Let $v, w \in NC(W,c)$ with $v \lessdot w$, and suppose v has full support in $NC(W,c)$, then we have:

$$K(w) \sqsubset K(v) \Longrightarrow v \ll w. \tag{11}$$

Note that by taking the contraposition in $i)$ of Proposition 4.2, we see that the same result holds with K^{-1} instead of K. The situation is slightly different for $ii)$ and we need another argument. The map $\iota : w \mapsto w^{-1}$ leaves S and T invariant, preserves length, absolute length and support, moreover it sends $NC(W, c)$ to $NC(W, c^{-1})$. We have

$$(\iota \circ K_c)(w) = \iota(w^{-1}c) = c^{-1}w = c^{-1}\iota(w^{-1}) = (K_{c^{-1}}^{-1} \circ \iota)(w).$$

Applying the map ι to Equation (11) shows that $ii)$ also holds with K^{-1} instead of K. Using these remarks we can rephrase the proposition as follows.

Proposition 4.3. *Let $v, w \in NC(W, c)$ with $v \lessdot w$. Then at least one of the relations $v \sqsubset w$ or $K(w) \sqsubset K(v)$ holds. If both hold, neither v nor $K(w)$ have full support.*

There are similar statements for the map L in the bipartite case, which we leave to the reader.

4.3. **Characterization and properties of the order $\sqsubset$ on $NC(W, c)$.** We consider a Coxeter system (W, S) and a standard Coxeter element c.

Proposition 4.4. *Let $v, w \in NC(W, c)$ and denote by $\leqslant_{B_w}$ the Bruhat order on $\Gamma(w)$, then the three properties below are equivalent:*

- $i)$ $v \sqsubset w$,
- $ii)$ $v \leqslant w$ *and* $v \leqslant_{B_w} w$,
- $iii)$ $\Delta(\Gamma(v)) \subset \Delta(\Gamma(w))$.

Proof. If $v \sqsubset w$ then, as we saw, $v \leqslant w$ and $v \leqslant_{B_w} w$ so that $i)$ implies $ii)$.

Suppose now that $ii)$ holds. According to Proposition 3.3, write w as a product of the simple reflections of $\Gamma(w)$, say $w = \mathfrak{s}_1 \cdots \mathfrak{s}_k$. Any $v \leqslant_{B_w} w$ is the product of a subword $\mathfrak{s}_{i_1} \ldots, \mathfrak{s}_{i_l}$ and $\Delta(\Gamma(v)) = \{\mathfrak{s}_{i_1}, \ldots, \mathfrak{s}_{i_l}\} \subset \Delta(\Gamma(w))$ so that $ii)$ implies $iii)$.

Finally if $\Delta(\Gamma(v)) \subset \Delta(\Gamma(w))$ then by taking the generated groups one has $\Gamma(v) \subset \Gamma(w)$ and $v \leqslant w$. If $\ell_T(v) = \ell_T(w) - 1$ then $vw^{-1} \in T$ therefore either $v <_{B_w} w$ or $w <_{B_w} v$. Since $\Delta(\Gamma(v)) \subset \Delta(\Gamma(w))$ but $\Delta(\Gamma(v)) \neq \Delta(\Gamma(w))$ there exists a simple reflection $\mathfrak{s} \in \Delta(\Gamma(w)) \backslash \Delta(\Gamma(v))$. As v is a product of reflections in $\Delta(\Gamma(v))$ it follows that w cannot be obtained as a subword of any reduced decomposition of v, therefore $v <_{B_w} w$ and v is obtained by taking a subword of length $k-1$ of the reduced decomposition $w = \mathfrak{s}_1 \cdots \mathfrak{s}_k$. The proof now follows by induction on the cardinality of $\Delta(\Gamma(w)) \backslash \Delta(\Gamma(v))$, using the fact that the Bruhat graph of the subgroups $\Gamma(v)$ are obtained by restriction of the Bruhat graph of W (Proposition 2.11). $\square$

Property $iii)$ was the original definition of the order $\sqsubset$ on $NC(W, c)$ in [21]. Since any subset of $\Delta(\Gamma(w))$ is a simple system in $\Gamma(w)$ we see that the lower ideal $\{v : v \sqsubset w\}$ is isomorphic to the boolean lattice of order $\mathrm{rk}(w)$, in particular its cardinality is $2^{\mathrm{rk}(w)}$, moreover on this ideal the three order relations $\leqslant, \leqslant_{B_w}$ and $\sqsubset$ coincide.

Note that $\Delta(\Gamma(w)) \subset \Delta(\Gamma(v))$ if and only if $S(\Gamma(w)) \subset S(\Gamma(v))$ using the bijection between positive roots and reflections.

Let now c be a standard Coxeter element, since $\Gamma(c) = W$ and $\Delta(W) = S$ it follows that the interval partitions in $INT(W, c)$ are exactly the noncrossing partitions $w \in NC(W, c)$ satisfying $w \sqsubset c$.

Proposition 4.5. *Let $w \in NC(W, c)$ and $\gamma \in INT(W, c)$ then*

$$\gamma \leqslant w \Longleftrightarrow \gamma \sqsubset w.$$

Proof. If $\gamma \sqsubset w$ then obviously $\gamma \leqslant w$. If $\gamma \leqslant w$ then $\Gamma(\gamma) \subset \Gamma(w)$ therefore $S(\Gamma(\gamma)) \subset \Gamma(w)$. Since the elements of $S(\Gamma(\gamma))$ are simple reflections in W they are also simple reflections in $\Gamma(w)$ therefore $S(\Gamma(\gamma)) \subset S(\Gamma(w))$ and $\gamma \sqsubset w$ by *iii*) of Proposition 4.4. □

As is clear from its definition the order $\sqsubset$ is not invariant under conjugation and one cannot, as in the case of the abolute order, reduce the study of $\sqsubset$ on $NC(W, c)$ to the case where the Coxeter element c is bipartite. Actually we will see in Section 5 that, already for S_4, the order type of the poset $(NC(W, c), \sqsubset)$ does depend on the standard Coxeter element c.

4.4. **A simplification property.** The following result it is similar to Proposition 2.7.

Lemma 4.6. *Let $t \in T$ and $u, v \in NC(W, c)$ be such that $u \leqslant v \sqsubset vt \leqslant c$ then $u \sqsubset ut \leqslant vt$.*

Proof. First one has $u \leqslant ut \leqslant vt$ by Proposition 2.7. Observe that the statement of the lemma depends only on the absolute and the Bruhat orders inside the parabolic subgroup $\Gamma(vt)$ therefore we can assume that $vt = c$ is a standard Coxeter element. Since $ct \sqsubset c$ there exists $s \in S$ such that ct is a standard Coxeter element in $W_{\langle s \rangle}$. It follows that $u \in W_{\langle s \rangle}$ and $t \notin W_{\langle s \rangle}$ therefore $ut \notin W_{\langle s \rangle}$ and ut cannot be obtained as a subword of a reduced expression of u, thus $u \leqslant_B ut$ and $u \sqsubset ut$. □

4.5. $(NC(W, c), \sqsubset)$ **as a flag simplicial complex.** We now define a symmetric binary relation on the set T of reflections such that $(NC(W, c), \sqsubset)$ is the face poset of the associated flag simplicial complex. Recall that a simplicial complex is called a *flag* simplicial complex if, given vertices $v_1, \ldots, v_k$, the set $\{v_1, \ldots, v_k\}$ is a face of the complex if and only if for all $1 \leqslant i < j \leqslant n$, the set $\{v_i, v_j\}$ is a face. In order to define such a complex it is enough to a give the symmetric relation on the vertices: $\{v_i, v_j\}$ is a face.

Definition 4.7. Let $\between_c$ be the binary relation on T such that $t \between_c u$ if and only if

- t, u are c-noncrossing (see Definition 3.1),
- $\langle r(t) | r(u) \rangle \leqslant 0$.

Let $\Xi(W, c)$ denote the flag simplicial complex with vertex set T associated to the relation $\between_c$.

Theorem 4.8. *The map $w \mapsto S(\Gamma(w))$ is a bijection from $NC(W,c)$ onto $\Xi(W,c)$.*

Proof. The image of w is a subset of T and taking the subgroup generated by this subset gives $\Gamma(w)$. Since the map Γ, restricted to $NC(W,c)$, is injective it follows $w \mapsto S(\Gamma(w))$ is also injective. It remains to show that its image is $\Xi(W,c)$.

Since $S(\Gamma(w))$ is a simple system, the scalar product of any two of its elements is nonpositive. Moreover by Proposition 3.3, there is an indexing $S(\Gamma(w)) = \{\mathfrak{s}_1, \dots, \mathfrak{s}_k\}$ such that $\mathfrak{s}_1 \cdots \mathfrak{s}_k \leqslant c$, consequently $\mathfrak{s}_i\mathfrak{s}_j \leqslant c$ if $i < j$. It follows that $\mathfrak{s}_i \not\between_c \mathfrak{s}_j$ for all $i < j$, therefore $S(\Gamma(w)) \in \Xi(W,c)$.

It remains to show that the map is surjective. Let $X \in \Xi(W,c)$. Because $\langle r(t)|r(u)\rangle \leqslant 0$ for all $t, u \in X$, this is a simple system, i.e. $X = S(P)$ for some parabolic subgroup $P \subset W$. Now consider a directed graph G defined as follows:

- its vertex set is X,
- there is an edge from $t \in X$ to $u \in X$ if $\langle r(t)|r(u)\rangle < 0$ and $tu \leqslant c$.

It is well defined because, $\langle r(t)|r(u)\rangle < 0$ implies $tu \neq ut$ therefore, according to Remark 3.2, we cannot have both $tu \leqslant c$ and $ut \leqslant c$. The undirected version of G is the Coxeter graph of P (without labels on the edges). Since $P \subset W$ is finite, a classical argument implies that G is acyclic, so that it is possible to find an indexing $X = \{\mathfrak{s}_1, \dots, \mathfrak{s}_k\}$ such that the existence of a directed edge $\mathfrak{s}_i \to \mathfrak{s}_j$ implies $i < j$. It follows that $\mathfrak{s}_i\mathfrak{s}_j \leqslant c$ for all $1 \leqslant i < j \leqslant k$.

Let us consider the case $k = \#X = 3$, so suppose we have $X = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3\}$ with $\mathfrak{s}_1\mathfrak{s}_2 \leqslant c$, $\mathfrak{s}_1\mathfrak{s}_3 \leqslant c$, and $\mathfrak{s}_2\mathfrak{s}_3 \leqslant c$. Since $\mathfrak{s}_2\mathfrak{s}_3 \in NC(W,c)$ and this poset is a ranked lattice, the least upper bound of $\mathfrak{s}_2$ and $\mathfrak{s}_3$ is $\mathfrak{s}_2\mathfrak{s}_3$. On the other hand, by Proposition 2.7, we get $\mathfrak{s}_2, \mathfrak{s}_3 \leqslant \mathfrak{s}_1 c$ therefore $\mathfrak{s}_2\mathfrak{s}_3 \leqslant \mathfrak{s}_1 c$, and consequently $\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3 \leqslant c$, so that $X = S(\Gamma(\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3))$. The general case follows by induction on $\#X$ and this shows the surjectivity. □

4.6. **Chains in $NC(W,c)$.** A factorization

$$c = t_1 \dots t_n$$

where each t_i is a reflection is minimal. The number of such minimal factorizations of the cycle c is (Deligne's formula):

$$\frac{n!h^n}{|W|}.$$

One can interpret this result as the counting of maximal chains in $NC(W,c)$. This number can be obtained from the leading term in k of $\mathrm{Cat}^{(k)}(W)$ and the identity $\prod_i (e_i + 1) = |W|$.

A refined enumeration of maximal chains in $NC(W,c)$ using the relation $\sqsubset$ was obtained in [21].

Definition 4.9 ([21])**.** For each maximal chain $\varpi = (w_i)_{0\leqslant i\leqslant n}$ in $NC(W,c)$ (i.e. we have $\mathrm{rk}(w_i) = i$ and $w_i \lessdot w_{i+1}$), we define $\mathrm{nir}(\varpi)$ as the number of $i \in \{0,\dots,n-1\}$ such that $w_i \lessdot\!\!\lessdot w_{i+1}$, and

$$M(W,q) = \sum q^{\mathrm{nir}(\varpi)}$$

where we sum over ϖ maximal chain in $NC(W,c)$.

The second author showed in [21] that this polynomial is a rescaled version of *Fuß-Catalan numbers* $\mathrm{Cat}^{(k)}(W)$:

$$M(W,q) = n!(1-q)^n \,\mathrm{Cat}^{(\frac{q}{1-q})}(W). \tag{12}$$

Note that the inverse relation is

$$\mathrm{Cat}^{(k)}(W) = \frac{1}{n!}(1+k)^n M\big(W, \tfrac{k}{1+k}\big)$$

and it follows from (12) and (9) that there exists a formula in terms of the degrees of the group:

$$M(W,q) = \frac{n!}{|W|}\prod_{i=1}^{n}\big(d_i + q(h-d_i)\big). \tag{13}$$

We use here the degrees $d_i = e_i + 1$ rather than the exponents because the formula is more compact. The specialization $M(W,0) = n!$ is seen combinatorially: since interval partitions form a boolean lattice, the number of maximal chains in $INT(W,c)$ is $n!$.

Much more can be said in the case of the symmetric group. Let c be the long cycle $(1,2,3,\dots,n)$ in $\mathfrak{S}_n$. In [9], we gave a multivariate generating function for maximal chains $\pi_0 < \pi_1 < \cdots < \pi_{n-1}$ in the noncrossing partition lattice of the form

$$\sum_{\pi_0,\pi_1,\dots\pi_{n-1}} \overline{\mathrm{wt}}(\pi_0,\pi_1,\dots,\pi_{n-2}) = \prod_{i=1}^{n-2}(iX_i + n - i) \tag{14}$$

where the weight $\overline{\mathrm{wt}}$ is the product of the X_i over all $i \geqslant 1$ such that $\pi_i \nparallel \pi_{i+1}$.

Let $\underline{a} = (a_1,\dots,a_r)$ where $a_i \geqslant 2$. A factorization $c = z_1\dots z_r$ where z_i is a cycle of length a_i is said to be *of type* $\underline{a}$. This exists only if $\sum_{i=1}^{r}(a_i - 1) \geqslant n-1$ and the factorization is minimal in case of equality. Again one can interpret such factorizations as chains $\pi_0 < \pi_1 \dots < \pi_r$ (where $\pi_i = z_1\dots z_i$; $i \geqslant 1$) of a certain type in the lattice of noncrossing partitions. Using this interpretation and extending the definition of the weight $\overline{\mathrm{wt}}$ in an obvious way, we obtained the following generating function over the set $\mathcal{F}(\underline{a})$ of such chains:

Theorem 4.10. *Let $b_i = \sum_{j=1}^{i}(a_j - 1)$. We have:*

$$\sum_{z_1\dots z_r\in\mathcal{F}(\underline{a})} \overline{\mathrm{wt}}(z_1\dots z_r) = \prod_{i=1}^{r-1}\big(X_i b_i + n - b_i\big). \tag{15}$$

It would be interesting to investigate the existence of similar formulas for other types.

4.7. **Characterization and properties of the order $\ll$.** The following proposition is the analog, for the order $\ll$, of Proposition 4.4.

Proposition 4.11. *Let $v, w \in NC(W, c)$ and denote by $\leqslant_{B_w}$ the Bruhat order on $\Gamma(w)$, then the three properties below are equivalent:*

i) $v \ll w$,
ii) $v \leqslant w$ *and* $w \leqslant_{B_w} v$,
iii) $v \in \Gamma(w)$ *and* v *has full support as an element of* $\Gamma(w)$.

Proof. We have already noted that *i)* implies *ii)*.

If *ii)* holds then w can be obtained as a subword of a reduced decomposition of v in $\Gamma(w)$. Since w has full support in $\Gamma(w)$ it follows that v has full support in $\Gamma(w)$ and *iii)* holds.

Suppose now that *iii)* holds, thus that v has full support in $\Gamma(w)$. If $\ell_T(v) = \ell_T(w) - 1$ then either $w \leqslant_{B_w} v$ or $v \leqslant_{B_w} w$. Since w is a Coxeter element in $\Gamma(w)$ (cf. Proposition 3.3)) if $v \leqslant_{B_w} w$ then v cannot have full support. It follows that $w \leqslant_{B_w} v$ therefore $v \ll w$. If $\ell_T(v) = \ell_T(w) - l$, $l > 1$ let $x = K(v) = wv^{-1}$ be the Kreweras complement of v and let $x = \gamma_1 \dots \gamma_l$ be a reduced decomposition in $\Gamma(x) \subset \Gamma(w)$. The sequence $x_0 = e, x_1 = \gamma_1, \dots, x_l = x$ satifies $x_0 = e \leqslant_{B_x} x_1 \leqslant_{B_x} \dots \leqslant_{B_x} x_l = x$ and the Bruhat graph of $\Gamma(x)$ is the restriction of the Bruhat graph of $\Gamma(w)$ therefore $x_0 = e \leqslant_{B_w} x_1 \leqslant_{B_w} \dots \leqslant_{B_w} x_l = x$. Applying *ii)* of Proposition 4.2 and using induction on i, one has $v \ll x_{l-i}^{-1} w$ for all i. □

Let NC_n be the set of classical noncrossing partitions, corresponding to the symmetric group and the cycle $(1, \dots, n)$ as Coxeter element, then Belinschi and Nica [6] defined an order relation $\ll$ on NC_n by $\pi \ll \rho$ if $\pi \leqslant \rho$ and for every block B of ρ, the minimum and the maximum of B belong to the same block of π. It is easy to see that this agrees with our definition in this case.

Proposition 4.12. *The poset $(NC(W, c), \ll)$ has 2^n connected components. They are the lower ideals*

$$\{w \in NC(W, c) \ : \ w \ll \gamma\} = \{w \in NC(W, c) \ : \ \overline{w} = \gamma\} \tag{16}$$

where γ runs through the 2^n interval partitions of W. In particular, interval partitions are the maximal elements for the order $\ll$.

Proof. From the characterization of $\ll$ in Theorem 4.11, we can see that the two sets on both sides of (16) are the same. Clearly, each lower ideal considered here is connected, so it remains to show that they cannot be connected with each other. Let $v, w \in NC(W, c)$ with $v \ll w$, we thus have to show $\overline{v} = \overline{w}$, i.e. v and w have the same support. Since $v \ll w$, we have $w \leqslant_B v$ therefore $\mathrm{supp}(w) \subset \mathrm{supp}(v)$. If there exists $s \in \mathrm{supp}(v) \setminus \mathrm{supp}(w)$ then $w \in W_{\langle s \rangle}$ and $\Gamma(v) \subset W_{\langle s \rangle}$ which contradicts $v \in \Gamma(v) \subset \Gamma(w)$. □

Since c is itself an interval partition, the poset $(NCF(W,c), \ll)$ is one of the connected components described in the previous proposition. We can call it the *main connected component*. The other connected components are of similar nature, since they can be seen as the main connected component of a standard parabolic subgroup. It is therefore enough to study the properties of the main connected component.

Proposition 4.13. *Let $w \in NC(W,c)$, then the relative Kreweras complement $v \to v^{-1}\overline{w}$ defines a poset anti-isomorphism*

$$\big(\{v \in NC(W,c) \,:\, v \gg w\}, \ll\big) \longrightarrow \big(\{v \in NC(W,c) \,:\, v \sqsubset w^{-1}\overline{w}\}, \sqsubset\big). \tag{17}$$

Proof. If w has full support, i.e., $\overline{w} = c$, this follows from *ii*) of Proposition 4.2. Otherwise, w has full support as an element of the standard parabolic subgroup $\Gamma(\overline{w})$. Note that we have

$$\{v \in NC(W,c) \,:\, v \gg w\} \subset \Gamma(\overline{w})$$

by Proposition 4.12. Therefore we can apply Proposition 4.2 in the subgroup $\Gamma(\overline{w})$ which yields the result. □

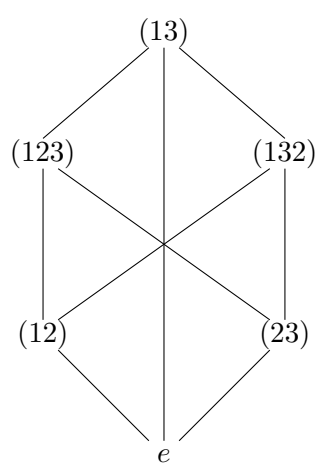
Corollary 4.14. *Upper ideals for $\ll$ are boolean posets.*

Proof. It was noted above the lower ideals for $\sqsubset$ are boolean. The result follows from the anti-isomorphism of the previous proposition, since boolean posets are anti-isomorphic to themselves. □

5. Examples

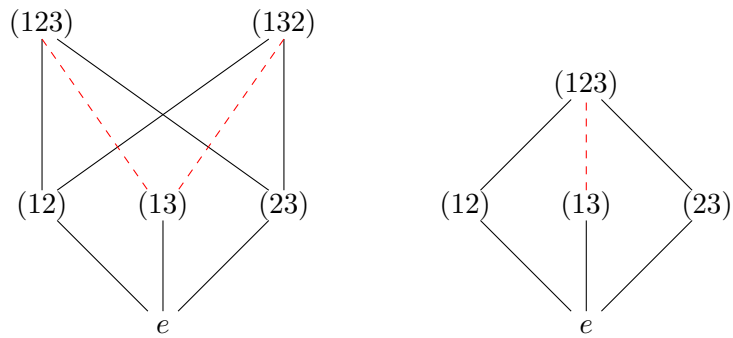
In this section we show a few pictures of the symmetric groups which illustrate the main properties of our objects.

5.1. The case of S_3.



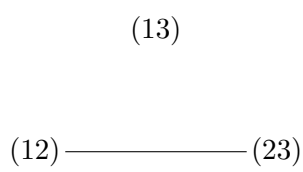
The Cayley graph of S_3

Permutations are denoted by their nontrivial cycles, e is the identity element. The Bruhat graph is obtained by orienting the edges downwards.



The orders $\sqsubset$ and $\ll$ on S_3 and on NC_3

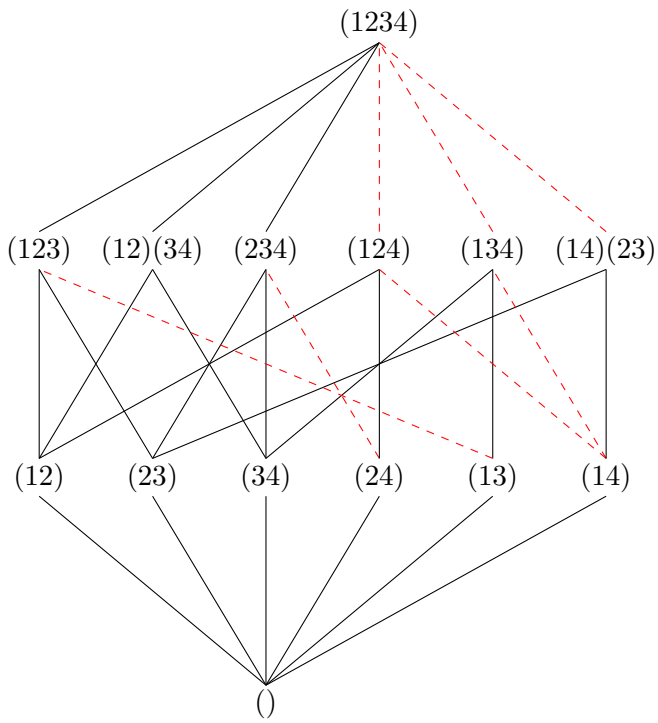
The cover relations of $\sqsubset$ are in black, those of $\ll$ are in red. Note that the underlying undirected graph on S_3 is the Cayley graph.



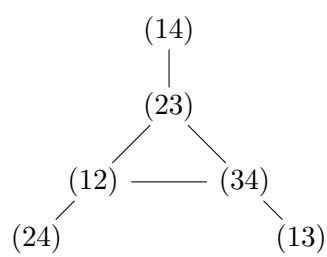
The graph of the relation $\between_c$

5.2. **The case of S_4.**

5.2.1. $c = (1234)$.

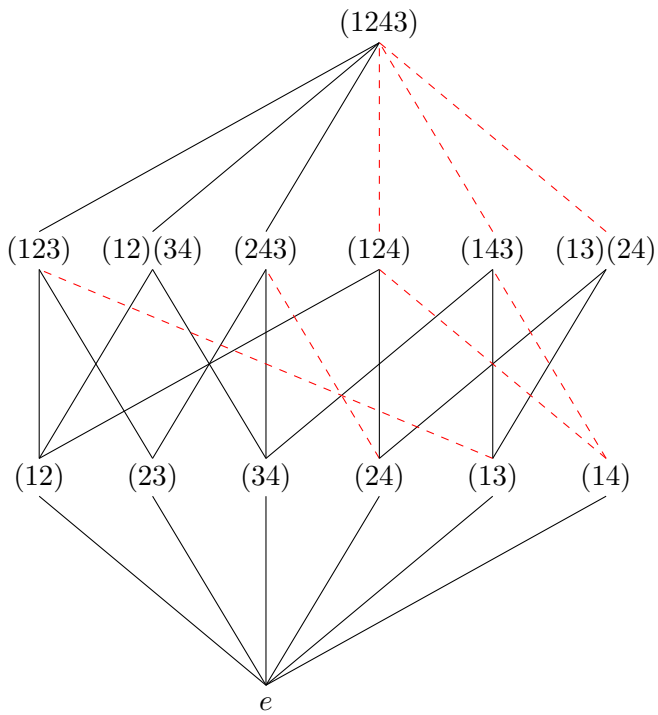


The order relations $\sqsubset$ and $\ll$ on NC_4

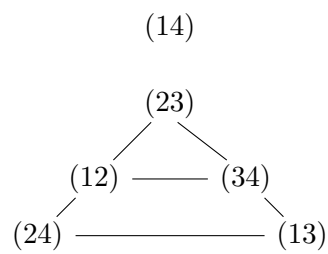


The graph of the relation $\between_c$

5.2.2. Here we change the Coxeter element for the bipartite element $c = (12)(34)(23) = (1243)$ and see that the graphs of $\between_c$ and of $\sqsubset$ for (1234) and (1243) are not isomorphic.



The order relations $\sqsubset$ and $\ll$ on $NC(S_4, (1243))$



The graph of the relation $\between_c$

6. Cluster complex, nonnesting partitions and Chapoton triangles

6.1. **The cluster complex.** The cluster complexes were introduced by Fomin and Zelevinsky in [19] as dual to generalized associahedra, in relation with cluster algebras. In this paper we will use the notion of c-clusters as defined by Reading [28, Section 7], following Marsh, Reineke and Zelevinsky [23].

Let $\Phi_{\geqslant -1} = (-\Delta) \cup \Pi$ be the set of *almost positive roots* (see [19]).

Proposition 6.1 ([28])**.** *For each $s \in S$, let σ_s denote the bijection from $\Phi_{\geqslant -1}$ to itself defined by:*

$$\sigma_s(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in (-\Delta)\backslash\{-r(s)\}, \\ s(\alpha) & \text{if } \alpha \in \Pi \cup \{-r(s)\}. \end{cases}$$

There exists a unique family of symmetric binary relations on $\Phi_{\geqslant -1}$ indexed by standard Coxeter elements, denoted by $\|_c$, such that:

- *if $\alpha, \beta \in -\Delta$, then $\alpha \|_c \beta$,*
- *if $\alpha \in -\Delta$ and $\beta \in \Pi$, then $\alpha \|_c \beta$ if and only if $r(-\alpha) \notin \mathrm{supp}(r(\beta))$,*
- *if $s \in S$ is a left descent of c, then $\alpha \|_c \beta \Longleftrightarrow \sigma_s(\alpha) \|_{scs} \sigma_s(\beta)$.*

Remark 6.2. Recall Remark 3.2 i) on the use of the symbol $\|_c$.

The definition of $\|_c$ is not fully explicit and uniqueness relies on Lemma 2.4. A more direct characterization of $\|_c$ is given in [13] using the subword complex. Another will be given in Proposition 6.6. Let us first review some results from [28, 29].

Definition 6.3 ([28])**.** The c-cluster complex $\Upsilon(W, c)$ is the flag simplicial complex on the vertex set $\Phi_{\geqslant -1}$ defined by the relation $\|_c$. A *c-cluster* is a maximal face of $\Upsilon(W, c)$.

This complex is pure, so that every c-cluster has dimension n. Clearly, the map σ_s provides an isomorphism between $\Upsilon(W, c)$ and $\Upsilon(W, scs)$ if $s \in S$ is a left descent of c. So the isomorphism type of $\Upsilon(W, c)$ does not depend on c, see [28, 29] for details.

The face generating function with respect to cardinality is:

$$\sum_{F \in \Upsilon(W,c)} x^{\#F} = \sum_{k=0}^{n} \mathrm{Nar}_k(W) x^{n-k}(1+x)^k. \tag{18}$$

In particular, taking the coefficient of x^n shows that the number of clusters is $\mathrm{Cat}(W)$. Moreover (18) implies that the integers $\mathrm{Nar}_k(W)$ are the entries of the h-vector of $\Upsilon(W, c)$.

Definition 6.4. The *positive part* of $\Upsilon(W, c)$, denoted by $\Upsilon^+(W, c)$, is the flag simplicial complex with vertex set Π and compatibility relation $\|_c$.

Clearly, $\Upsilon^+(W, c)$ is a (full) subcomplex of $\Upsilon(W, c)$. Note that it may happen that $\Upsilon^+(W, c_1)$ and $\Upsilon^+(W, c_2)$ are not isomorphic if c_1, c_2 are two different standard Coxeter elements. The face generating function with respect to cardinality is:

$$\sum_{F \in \Upsilon^+(W,c)} x^{\#F} = \sum_{k=0}^{n} \mathrm{Nar}_k^+(W) x^{n-k}(1+x)^k. \tag{19}$$

In particular, taking the coefficient of x^n shows that the number of positive c-clusters is $\mathrm{Cat}^+(W)$ and the integers $\mathrm{Nar}_k^+(W)$ are the entries of the h-vector of $\Upsilon^+(W, c)$.

We give a direct characterization of $\|_c$ in Proposition 6.6 below. It is an extension of Brady and Watt's results in the bipartite case [12] to any standard Coxeter element. We begin with a lemma.

Lemma 6.5. *Suppose $s \in S$ is a left descent of c, and let $t \in T\backslash\{s\}$. Then the following two conditions are equivalent:*

- $\langle r(s)|r(t)\rangle \geqslant 0$, *and s, t are c-noncrossing (see Definition 3.1),*
- $s \notin \mathrm{supp}(sts)$.

Proof. We begin by showing that the first condition implies the second one.
$\Longrightarrow$, case 1: $st \leqslant c$. By Proposition 3.12, we have $t \leqslant sc$ and $t \in W_{\langle s\rangle}$ therefore $s \notin \mathrm{supp}(t)$ and $\langle r(s)|r(t)\rangle \leqslant 0$. Since we already have $\langle r(s)|r(t)\rangle \geqslant 0$, we get $\langle r(s)|r(t)\rangle = 0$, which means $st = ts$. Then $s \notin \mathrm{supp}(sts)$ follows from $s \notin \mathrm{supp}(t)$ since $sts = t$.
$\Longrightarrow$, case 2: $ts \leqslant c$. We have $\ell_T(ts) = 2$ since $s \neq t$, therefore $s \leqslant ts$. By Proposition 3.12, we get $sts \leqslant sc \in W_{\langle s\rangle}$. It follows that $s \notin \mathrm{supp}(sts)$.
$\Longleftarrow$, case 1: $st = ts$. First note that we have $\langle r(s)|r(t)\rangle = 0$ in this case, so in particular it is $\geqslant 0$. Then, we have $sts = t$ so $s \notin \mathrm{supp}(t)$. It follows that $t \leqslant sc$ and $st \leqslant c$ by Proposition 3.12.
$\Longleftarrow$, case 2: $st \neq ts$. We have $sts \in W_{\langle s\rangle}$, so $ts = s(sts) \leqslant c$ by Proposition 3.12. Also, from $sts \in W_{\langle s\rangle}$ we get $\langle r(s)|r(sts)\rangle \leqslant 0$. Since s is in the orthogonal group, it preserves the scalar product and we get $\langle s(r(s))|s(r(sts))\rangle \leqslant 0$, hence $\langle -r(s)|r(t)\rangle \leqslant 0$ and $\langle r(s)|r(t)\rangle \geqslant 0$ as needed. □

Using the above lemma we can now give a more explicit characterization of the relation $\|_c$.

Proposition 6.6. *Let $\alpha, \beta \in \Pi$, then $\alpha \|_c \beta$ if and only if the following two conditions hold:*

- $r^{-1}(\alpha)$, $r^{-1}(\beta)$ *are c-noncrossing,*
- $\langle \alpha|\beta\rangle \geqslant 0$.

Proof. Let us introduce a family of binary relations $\hat{\|}_c$ on $\Phi_{\geqslant -1}$ by:

- if $\alpha, \beta \in \Pi$ one has $\alpha \hat{\|}_c \beta$ if and only if $\langle \alpha|\beta\rangle \geqslant 0$ and $r^{-1}(\alpha)$, $r^{-1}(\beta)$ are c-noncrossing,
- otherwise, $\alpha \hat{\|}_c \beta$ if and only if $\alpha \|_c \beta$.

We show that the relations $\hat{\|}_c$ satisfy the rules of $\|_c$ given in Definition 6.1. By uniqueness it follows that $\hat{\|}_c = \|_c$. The first two points in Definition 6.1 are obviously satisfied by the family $\hat{\|}_c$ therefore we just need to show that they satisfy the third point. Let s be a left descent of c and $\alpha, \beta \in \Phi_{\geqslant -1}$.
Case 1: $\alpha, \beta \in (-\Delta)\backslash\{-r(s)\}$. We have then $\sigma_s(\alpha) = \alpha$, $\sigma_s(\beta) = \beta$, it follows that $\alpha \hat{\|}_c \beta$ and $\sigma_s(\alpha) \hat{\|}_{scs} \sigma_s(\beta)$ both hold.
Case 2: $\alpha \in (-\Delta)\backslash\{r(s)\}$, $\beta = -r(s)$. One has $\alpha \hat{\|}_c \beta$, $\sigma_s(\alpha) = \alpha$, $\sigma_s(\beta) = -\beta > 0$ and $-\alpha \notin \mathrm{supp}(-\beta) = \{s\}$ therefore $\sigma_s(\alpha) \hat{\|}_{scs} \sigma_s(\beta)$.

Case 3: $\alpha, \beta \in \Pi\backslash\{r(s)\}$. In this case, one has $\sigma_s(\alpha) = s(\alpha)$ and $\sigma_s(\beta) = s(\beta)$. Since $s \in O(V)$ preserves the scalar product, $\langle\alpha|\beta\rangle = \langle\sigma_s(\alpha)|\sigma_s(\beta)\rangle$, moreover $r^{-1}(\sigma_s(\alpha)) = sr^{-1}(\alpha)s$ and $r^{-1}(\sigma_s(\beta)) = sr^{-1}(\beta)s$ therefore $r^{-1}(\sigma_s(\alpha))r^{-1}(\sigma_s(\beta)) = sr^{-1}(\alpha)r^{-1}(\beta)s$ and the conclusion follows since $\leqslant$ is invariant under conjugation.
Case 4: $\alpha = -r(s), \beta \in \Pi\backslash\{r(s)\}$. This case follows from Lemma 6.5 putting $s = r^{-1}(\alpha)$ and $t = sr^{-1}(\beta)s$.
Case 5: $\alpha = r(s), \beta \in \Pi\backslash\{r(s)\}$. Again use Lemma 6.5 for $s = r^{-1}(\alpha)$ and $t = r^{-1}(\beta)$.

Since the relations $\|_c$ and $\hat{\|}_c$ are symmetric we have covered all cases. □

By Proposition 3.3, if $w \leqslant c$ then w is a standard Coxeter element of $\Gamma(w)$ so that the complex $\Upsilon(\Gamma(w), w)$ is well defined. An interesting consequence of Proposition 6.6 is that for $t_1, t_2 \in \Pi(\Gamma(w))$, we have $\{t_1, t_2\} \in \Upsilon(\Gamma(w), w)$ if and only if $\{t_1, t_2\} \in \Upsilon(W, c)$. This will be used in Proposition 8.10 to show that any positive face in $\Upsilon(W, c)$ is a cluster in $\Upsilon(\Gamma(w), w)$ for some $w \leqslant c$, a result which is easily extended to all faces.

Note the similarity between Proposition 6.6 and Definition 4.7 therefore between the two flag simplicial complexes $\Upsilon^+(W, c)$ and $\Xi(W, c)$. Their vertex sets are respectively T and Π, and are in bijection via the map r, besides this the only difference is the required sign of the scalar product. Despite this they have very different properties: unlike $\Upsilon^+(W, c)$, the simplicial complex $\Xi(W, c)$ is not pure and its topology can be rather complicated.

Eventually, the following result will be useful:

Proposition 6.7. *If $F \in \Upsilon^+(W, c)$, there is an indexing $F = \{\alpha_1, \ldots, \alpha_k\}$ such that $r(\alpha_1) \cdots r(\alpha_k) \leqslant c$.*

Proof. It is sufficient to check this statement for $k = n$ and $r(\alpha_1) \cdots r(\alpha_n) = c$. This case follows from [13, Proposition 2.8].

Alternatively, this follows from the work of Brady and Watt [12]. In the case where c is a bipartite Coxeter element, [12, Note 3.3] implies the result. It remains to show that this is preserved under the moves $c \to scs$ as in Lemma 2.4. We omit details. □

6.2. **The relation $\|_c$ and the orders $\sqsubset$ and $\ll$.** In the simply laced case (i.e. types A_n, D_n and E_n in the classification), using Proposition 6.6 the compatibility relation $\|_c$ on Π can be completely rephrased in terms of the orders $\sqsubset$ and $\ll$, without using roots or even the group structure. This relies on three observations:

- Each rank 2 parabolic subgroup has type A_2 or A_1^2 in the classification. For any pair of distinct reflections in such a subgroup, we can take the product in some order to get a given Coxeter element. It means that the condition $tu \leqslant c$ or $ut \leqslant c$ for $t, u \in T$ is equivalent to the existence of $v \leqslant c$ such that $\mathrm{rk}(v) = 2$, and $t \leqslant v$, $u \leqslant v$.

- The condition $\langle r(t)|r(u)\rangle > 0$ means that $\{t, u\}$ is not the simple system of $\Gamma(v)$, so it is equivalent to the fact that either $t \ll v$ or $u \ll v$ (with v as above).
- The condition $\langle r(t)|r(u)\rangle = 0$ means $tu = ut$, it is equivalent to the fact that the interval $[e, tu]_{\leqslant}$ contains only e, t, u, tu.

It follows that $\alpha \parallel_c \beta$ is equivalent to:

- $r^{-1}(\alpha)$ and $r^{-1}(\beta)$ have a least upper bound v of rank 2,
- Either $[e, v]_{\leqslant} = \{e, r^{-1}(\alpha), r^{-1}(\beta), v\}$, or $r^{-1}(\alpha) \ll v$ or $r^{-1}(\beta) \ll v$.

6.3. **Nonnesting partitions.** When W is a Weyl group, we have an associated root poset, defined in terms of the *crystallographic root system* of W. We defined roots to be unit vectors, which is not convenient for crystallographic root system, so we only sketch the definition here and refer to [20, Chapter 2.9]. The idea is to allow to have roots of different lengths, so we consider positive numbers $(a_\alpha)_{\alpha\in\Pi}$ and assume that the vectors $a_\alpha\alpha$ satisfies, for all $\alpha, \beta \in \Pi$:

$$\frac{2\langle a_\alpha\alpha|a_\beta\beta\rangle}{\langle a_\beta\beta|a_\beta\beta\rangle} \in \mathbb{Z}.$$

Definition 6.8. The partial order $\preccurlyeq$ on Π is defined by $\alpha \preccurlyeq \beta$ if $a_\beta\beta - a_\alpha\alpha$ is a linear combination of simple roots with nonnegative coefficients and $(\Pi, \preccurlyeq)$ is called the *root poset* of W. We denote $NN(W)$ the set of antichains of $(\Pi, \preccurlyeq)$, and such an antichain is called a *nonnesting partition*.

In the case where W is not a Weyl group (i.e. one of $I_2(m)$, H_3, and H_4 in the classification), it is possible to define *ad hoc* posets having all the required properties of a root poset. See [1, Section 5.4.1] and [16] for more details on this subject. All our results are valid using these "noncrystallographic root posets".

Just as $NC(W, c)$ and $\Upsilon(W, c)$, the set $NN(W)$ is a flag simplicial complex. Its vertex set is Π, and two vertices are compatible if they are not comparable in the root poset. This complex is not pure.

Definition 6.9. The *support* of a nonnesting partition A is:

$$\mathrm{supp}(A) = \{\delta \in \Delta \,:\, \exists \alpha \in A,\ \delta \preccurlyeq \alpha\}.$$

It is similar to the notion of support a group element $w \in W$, so there should be no confusion. For example, note that for $A \in NN(W)$, we have $s \notin \mathrm{supp}(A)$ if and only if $\pi \in NN(W_{\langle s\rangle})$.

6.4. **Enumeration of full reflections.** The number of $t \in T$ with full support is $\mathrm{Nar}_1^+(W)$. Chapoton [15] obtained the formula

$$\mathrm{Nar}_1^+(W) = \frac{nh}{|W|}\prod_{i=2}^{n}(e_i - 1) \tag{20}$$

by a case by case verification, and he also conjectured a representation theoretical interpretation. We show that this formula can be obtained from the properties of $\ll$.

By Proposition 4.11, a reflection $t \in T$ is full if and only if $t \ll c$, which is equivalent to the existence of a maximal chain

$$e \sqsubset w_1 \ll w_2 \ll \cdots \ll w_n = c \tag{21}$$

with $t = w_1$ in $NC(W, c)$. Note that these maximal chains are those with the maximal number of $\ll$, as we have $e \sqsubset t$ for $t \in T$. So their number is the dominant coefficient in (13). Noticing that $d_n + q(h - d_n) = h$ is a constant, this dominant coefficient is:

$$\frac{n!h}{|W|} \prod_{i=1}^{n-1} (h - d_i). \tag{22}$$

The number of maximal chains as in (21) is also $(n-1)!$ times the number of full reflections: for a given w_1 the possible choices for $w_2, \ldots, w_n$ are the maximal chains in $[w_1, c]_{\ll}$ which is a boolean lattice of rank $n-1$ by Corollary 4.14 and their number is $(n-1)!$. To get the right hand side of (20) from (22) divided by $(n-1)!$, it remains only to use the equalities $d_i = e_i + 1$, $h - e_i = e_{n+1-i}$.

Fomin and Reading in [18, Section 13.4] asked for a better combinatorial way to relate full reflections with objects counted by Fuß-Catalan numbers (k-noncrossing partitions, generalized clusters, see [1]). Our derivation of the formula does not give a full answer but recasts the problem in a more general form: have a better combinatorial way to relate Fuß-Catalan objects and the generating function $M(w, q)$, thus explaining the relation (12).

6.5. **Chapoton triangles.** The F-, M- and H-triangles are polynomials in two variables defined respectively in terms of $\Upsilon(W, c)$, the Möbius function of $NC(W, c)$, and $NN(W)$. The $F = M$ and $H = M$ theorems state that these polynomials are related to each other by invertible substitutions (there is an $F = H$ theorem as an immediate consequence of the other two). They were conjectured by Chapoton in [15] and [14], respectively, and proved by Athanasiadis [2] and Thiel [31], respectively. See also [1] for generalizations.

Definition 6.10. Let μ denote the Möbius function of $NC(W, c)$. The M-triangle is the polynomial:

$$M(x, y) = \sum_{\substack{\alpha, \beta \in NC(W,c) \\ \alpha \leqslant \beta}} \mu(\alpha, \beta) x^{\mathrm{rk}(\alpha)} (-y)^{\mathrm{rk}(\beta) - \mathrm{rk}(\alpha)}.$$

The F-triangle is the polynomial:

$$F(x, y) = \sum_{F \in \Upsilon(W,c)} x^{\#(F \cap (-\Delta))} y^{\#(F \cap \Pi)}.$$

The H-triangle is the polynomial:

$$H(x, y) = \sum_{A \in NN(W)} x^{\#(A \cap \Delta)} y^{\#(A \cap (\Pi \setminus \Delta))}.$$

There exist slightly different conventions in the literature for these polynomials. Here and also for related polynomials in the sequel, we always take a convention ensuring that they have nonnegative integer coefficients, and total degree n.

The $F = M$ and $H = M$ theorems relates the three polynomials as follows:

$$F(x,y) = (1+y)^n H\Big(\tfrac{x}{1+y}, \tfrac{y}{1+y}\Big) = (1+x)^n M\Big(\tfrac{x}{1+x}, \tfrac{y-x}{1+x}\Big), \tag{23}$$

$$H(x,y) = (1-y)^n F\Big(\tfrac{x}{1-y}, \tfrac{y}{1-y}\Big) = (1+x-y)^n M\Big(\tfrac{x}{1+x-y}, \tfrac{y-x}{1+x-y}\Big), \tag{24}$$

$$M(x,y) = (1+y)^n H\Big(\tfrac{x}{1+y}, \tfrac{y+x}{1+y}\Big) = (1-x)^n F\Big(\tfrac{x}{1-x}, \tfrac{y+x}{1-x}\Big). \tag{25}$$

See [2, 14, 15, 31] for details. Note that all these relations suggest considering homogeneous polynomials in three variables, rather than only two variables. Then the relations become mere shifts of the variables. This idea will be used in Section 9.

Also, the polynomials satisfy a symmetry property. By the self-duality of $NC(W,c)$ one has

$$M(x,y) = x^n M(\tfrac{1}{x}, \tfrac{1}{y}). \tag{26}$$

and, using (23),

$$F(x,y) = (-1)^n F(-1-x, -1-y), \quad H(x-1,y) = y^n H(\tfrac{y}{x} - 1, \tfrac{1}{y}). \tag{27}$$

From the definition, it can be seen that these polynomials contains Catalan, positive Catalan, Narayana, and positive Narayana numbers as special cases. Let us also mention another interesting specialization: $H(-1,1)$ is the *double-positive Catalan number* introduced in [5], more generally the coefficients of $H(-1,y)$ are the *double-positive Narayana numbers*. See [5, Proposition 4.6] for the interpretation in terms of nonnesting partitions. Using (23), we have

$$H(-1,y) = (1-y)^n F\big(\tfrac{-1}{1-y}, \tfrac{1}{1-y}\big)$$

and the latter expression can be interpreted as the *local h-polynomial* of $\Upsilon^+(W,c)$, see [5, Remark 4.7] and Athanasiadis and Savvidou [3].

7. Intervals for $\sqsubset$ and $\ll$ and the cluster complex

7.1. Intervals for $\sqsubset$.

7.1.1. *Counting intervals.* If $v, w \in NC(W,c)$ and $v \sqsubset w$ we denote by $[v,w]_\sqsubset$ the interval consisting of all $x \in NC(W,c)$ such that $v \sqsubset x \sqsubset w$. Define the *height* of an interval $[v,w]_\sqsubset$ by $rk(w) - rk(v)$. The identity e is the smallest element in $NC(W,c)$ for $\sqsubset$ moreover, if w has rank r we have seen that the interval $[e,w]_\sqsubset$ is a boolean lattice with 2^r elements therefore, if $r \geqslant k$ the number of elements $v \sqsubset w$ with $\mathrm{rk}(w) - \mathrm{rk}(v) = k$ is equal to $\binom{r}{k}$. Since the number of elements of $NC(W,c)$ of rank r is $\mathrm{Nar}_r(W)$, we have the following (here n is the rank of W).

Proposition 7.1. *The number of intervals of height k for $\sqsubset$ is equal to*

$$\sum_{r=k}^{n} \mathrm{Nar}(W,r)\binom{r}{k}. \tag{28}$$

In the case of the symmetric group $\mathfrak{S}_n$, the total number of intervals for $\sqsubset$ is the *small Schröder number* s_n, defined by

$$\sum_{n\geqslant 0} s_n z^n = \frac{1+x-\sqrt{1-6x+x^2}}{4x}.$$

7.1.2. *Relation with the cluster complex and the associahedron.* We now relate $(NC(W,c),\sqsubset)$ with the cluster complex $\Upsilon(W,c)$ and the associahedron, a polytope whose face complex is dual to $\Upsilon(W,c)$ [19]. It is known that the h-vector of the associahedron is equal to the sequence of W-Narayana numbers, see [4] (or (18), using the duality). The next statement follows readily from (28) and the relation between the f- and h-vectors.

Proposition 7.2. *The number intervals of height k for $\sqsubset$ is equal to the number of faces of dimension k of the W-associahedron, or to the number of faces of cardinality $n-k$ of the cluster complex.*

For particular values of k there are bijective proofs of the equality in Proposition 7.2.

- For $k=n$: there is a unique interval of height n which is $INT(W,c)$ and a unique face of dimension n in the associahedron.
- For $k=0$: the number of vertices of the associahedron is equal to number of non-crossing partitions, or W-Catalan number, which is also the number of intervals of height 0 for $\sqsubset$. Bijective proofs of this fact have appeared in the litterature, see Section 8.
- For $k=n-1$: the number of intervals of height $n-1$ for $\sqsubset$ is equal to the number of almost positive roots, which is the number of vertices of the cluster complex. Indeed this is easy to check directly since these intervals fall into two categories:
  - the intervals $[e,\pi]$ where $\pi\in NC(W,c)$ has rank $n-1$. The Kreweras complement gives a bijection between this set and the elements of rank 1 in $NC(W,c)$ which are the reflections, in bijection with the positive roots.
  - the intervals $[s,c]_{\sqsubset}$ where $s\in S$ which are in bijection with the negative simple roots.

Observe that the study of the order $\sqsubset$ leads naturally to the notion of almost positive root, which is fundamental to the theory of clusters.

It would be interesting to give a bijective proof of Proposition 7.2 for other values of k. In the case of a bipartite Coxeter element, we will give, in Section 8.3, a bijection between faces of size $n-k$ of the cluster complex and intervals of height k for $\sqsubset$.

7.2. **Intervals for $\ll$.** In the case of the symmetric group $\mathfrak{S}_n$ with the standard Coxeter element $c = (1, 2, \dots, n)$, the number of intervals $v \ll w$ was computed in [24] and [26], its is the so-called *large Schröder number* S_{n-1} defined by

$$\sum_{n \geqslant 0} S_n z^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

They are related to small Schröder numbers by $S_n = 2s_n$ if $n \geqslant 1$.

In the general case we will again show a connection with the cluster complex. Refining the enumeration of intervals, we consider a two variable polynomial:

$$I(x, y) = \sum_{\substack{v, w \in NC(W,c) \\ v \ll w}} x^{\mathrm{rk}(v)} y^{\mathrm{rk}(w) - \mathrm{rk}(v)}. \tag{29}$$

It turns out to be related with the polynomial $M(x, y)$ as follows.

Theorem 7.3. *We have:*

$$I(x, y) = M(x - y, y), \qquad M(x, y) = I(x + y, y). \tag{30}$$

To prove that, first consider, for $w \in NC(W, c)$:

$$I_w(x) = \sum_{\substack{v \in NC \\ v \ll w}} x^{\mathrm{rk}(v)} \tag{31}$$

and

$$M_w(x) = (-1)^{\mathrm{rk}(w)} \sum_{\substack{v \in NC \\ v \leqslant w}} \mu(v, w)(-x)^{\mathrm{rk}(v)}. \tag{32}$$

Proposition 7.4. *We have $I_w(x + 1) = M_w(x)$.*

Proof. The case $w = c$ is sufficient, because we can see w itself as a standard Coxeter element of $\Gamma(w)$ and prove the identity in this subgroup. Then $I_c(x)$ is the rank generating function of $NCF(W, c)$, it is therefore given by the positive Narayana numbers and it remains to prove:

$$\sum_{k=1}^{n} \mathrm{Nar}_k^+ (1 + x)^k = (-1)^n \sum_{v \in NC(W,c)} \mu(v, c)(-x)^{\mathrm{rk}(v)}.$$

The right hand side is, using Kreweras complement:

$$\sum_{v \in NC(W,c)} \mu(e, K(v))(-x)^{\mathrm{rk}(v)} = \sum_{v \in NC(W,c)} \mu(e, v)(-x)^{n - \mathrm{rk}(v)} = x^n M(0, \tfrac{1}{x}).$$

This is also $x^n F(0, \frac{1}{x})$, via (23). The h-vector of $\Upsilon^+(W, c)$ is given by positive Narayana numbers, see (19), and we get the equality. $\square$

Proof of Theorem 7.3. Using Proposition 7.4, we have:

$$I(x,y) = \sum_{w \in NC(W,c)} I_w(\tfrac{x}{y}) y^{\mathrm{rk}(w)} = \sum_{w \in NC(W,c)} M_w(\tfrac{x}{y} - 1) y^{\mathrm{rk}(w)} = M(x-y, y).$$

□

Combining Equations (25) and (30), we obtain the relations between $I(x,y)$ and $H(x,y)$:

$$I(x,y) = (1+y)^n H\left(\frac{x-y}{1+y}, \frac{x}{1+y}\right),$$
$$H(x,y) = (1+x-y)^n I\left(\frac{y}{1+x-y}, \frac{y-x}{1+x-y}\right),$$

and the relations between $I(x,y)$ and $F(x,y)$:

$$I(x,y) = (1-x+y)^n F\left(\frac{x-y}{1-x+y}, \frac{x}{1-x+y}\right),$$
$$F(x,y) = (1+x)^n I\left(\frac{y}{1+x}, \frac{y-x}{1+x}\right).$$

In particular, $I(x,x) = F(0,x)$ gives the following:

Corollary 7.5. *We have*

$$\sum_{\substack{v,w \in NC(W,c) \\ v \ll w}} x^{\mathrm{rk}(w)} = \sum_{F \in \Upsilon^+(W,c)} x^{\#F}.$$

In particular, the number of intervals $v \ll w$ in $NC(W,c)$ is $\#\Upsilon^+(W,c)$.

We will give a bijective proof of the latter fact in the next section.

8. The bijection between positive faces of the cluster complex and intervals

In order to give a bijection between $\Upsilon^+(W,c)$ and intervals $v \ll w$ in $NC(W,c)$, an important ingredient is a bijection between clusters and noncrossing partitions. Such bijections have been described by Reading [28] and Athanasiadis et al. [4]. Although the one from [4] was only stated in the case of a bipartite Coxeter element, it turns out to be particularly adapted to the present situation as it can be rephrased in terms of the orders $\sqsubset$ and $\ll$. In fact, using properties of $\ll$ and $\sqsubset$ we will extend the bijection to cover the case of all standard Coxeter elements.

8.1. The bijection between clusters and noncrossing partitions.

We give here a bijection $\Psi_{W,c}$ between $NCF(W,c)$ and clusters in $\Upsilon^+(W,c)$. Note that it is straightforward to extend such a bijection to a bijection from $NC(W,c)$ to clusters in $\Upsilon(W,c)$: if $w \in NC(W,c)$ with $J = \mathrm{supp}(w)$, then the image of w is

$$\Psi_{W_J,\overline{w}}(w) \cup \{-r(s) \,:\, s \notin J\}.$$

Recall that $\overline{w}$ is the interval partition such that $\Gamma(\overline{w}) = W_J$ and it is indeed a standard Coxeter element of W_J. In the case of the bipartite Coxeter element, the bijection that we obtain is actually the inverse of the bijection in [4, Sections 5-6].

Definition 8.1. Let $w \in NC(W,c)$ then we define

$$\Phi_{W,c}(w) = \big(\mathrm{Inv}_R(w) \cap \Gamma(w)\big) \cup \big(\mathrm{Inv}_L(K(w)) \cap \Gamma(K(w))\big) \subset T,$$
$$\Psi_{W,c}(w) = r(\Phi_{W,c}(w)) \subset \Pi.$$

Let us write w and $K(w)$ as products of their associated simple generators as in Proposition 3.3:

$$w = \mathfrak{s}_1 \cdots \mathfrak{s}_k, \qquad K(w) = \mathfrak{s}_{k+1} \cdots \mathfrak{s}_n,$$

then one has

$$c = \mathfrak{s}_1 \cdots \mathfrak{s}_n,$$

and, using (2),

$$\Phi_{W,c}(w) = \{t_1, \dots, t_n\}$$

where:

$$t_i = \begin{cases} (\mathfrak{s}_k\mathfrak{s}_{k-1}\cdots\mathfrak{s}_{k+2-i})\mathfrak{s}_{k+1-i}(\mathfrak{s}_k\mathfrak{s}_{k-1}\cdots\mathfrak{s}_{k+2-i})^{-1} & \text{if } 1 \leqslant i \leqslant k, \\ (\mathfrak{s}_{k+1}\mathfrak{s}_{k+2}\cdots\mathfrak{s}_{i-1})\mathfrak{s}_i(\mathfrak{s}_{k+1}\mathfrak{s}_{k+2}\cdots\mathfrak{s}_{i-1})^{-1} & \text{if } k+1 \leqslant i \leqslant n. \end{cases}$$

It follows that

$$\Psi_{W,c}(w) = \{r(t_1), \dots, r(t_n)\} \subset \Pi.$$

Theorem 8.2. *The map $\Psi_{W,c}$ is a bijection from $NCF(W,c)$ to the set of c-clusters in $\Upsilon^+(W,c)$.*

First observe that, by a straightforward computation:

$$t_1 \cdots t_k = w,$$
$$t_{k+1} \cdots t_n = K(w),$$
$$t_1 \cdots t_n = c.$$

Lemma 8.3. *Let $w \in NCF(W,c)$ of rank k, t_i as above, and let $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$. Then we have $wt_i \sqsubset w$, $t_jK(w) \sqsubset K(w)$, and $w \ll wt_j$.*

Proof. An easy computation gives $wt_i = \mathfrak{s}_1 \cdots \mathfrak{s}_{k-i}\mathfrak{s}_{k+2-i} \cdots \mathfrak{s}_k$, whence the first inequality. The second is obtained similarly. The third one follows the second one by applying *ii*) of Proposition 4.2, using the fact that w has full support. □

Lemma 8.4. *Let i, j be such that either $1 \leqslant i < j \leqslant k$ or $k+1 \leqslant i < j \leqslant n$. Then we have:*

$$\langle r(t_i) | r(t_j) \rangle \geqslant 0.$$

Proof. We can focus on the case $1 \leqslant i < j \leqslant k$, the other one is obtained similarly but with $\mathfrak{s}_{k+1}, \dots, \mathfrak{s}_n$ instead of $\mathfrak{s}_1, \dots, \mathfrak{s}_k$. Using Proposition 2.1, we have

$$\begin{aligned} r(t_i) &= r(\mathfrak{s}_k \cdots \mathfrak{s}_{k+1-i} \cdots \mathfrak{s}_k) = \mathfrak{s}_k \cdots \mathfrak{s}_{k+2-i}\big(r(\mathfrak{s}_{k+1-i})\big) \\ &= -\mathfrak{s}_k \cdots \mathfrak{s}_{k+1-i}\big(r(\mathfrak{s}_{k+1-i})\big). \end{aligned}$$

The second equality follows from the fact that $\mathfrak{s}_{k+1-i} \notin \mathrm{Inv}_R(\mathfrak{s}_k \cdots \mathfrak{s}_{k+2-i})$ since they have disjoint support in $\Gamma(w)$. Similarly,

$$\begin{aligned} r(t_j) &= r(\mathfrak{s}_k \cdots \mathfrak{s}_{k+1-j} \cdots \mathfrak{s}_k) \\ &= \mathfrak{s}_k \cdots \mathfrak{s}_{k+1-i}\big(r(\mathfrak{s}_{k-i} \cdots \mathfrak{s}_{k-j} \cdots \mathfrak{s}_{k-i})\big). \end{aligned}$$

Since $\mathfrak{s}_k \cdots \mathfrak{s}_{k+1-i}$ preserves the scalar product, it remains to show that

$$\big\langle r(\mathfrak{s}_{k+1-i}) | r(\mathfrak{s}_{k-i} \cdots \mathfrak{s}_{k+1-j} \cdots \mathfrak{s}_{k-i}) \big\rangle \leqslant 0.$$

This follows from the fact that $\mathfrak{s}_1, \dots, \mathfrak{s}_k$ is a simple system, since the root $r(\mathfrak{s}_{k-i} \cdots \mathfrak{s}_{k+1-j} \cdots \mathfrak{s}_{k-i})$ is a positive linear combination of the roots $r(\mathfrak{s}_{k-i})$, $\dots$, $r(\mathfrak{s}_{k+1-j})$. □

Lemma 8.5. *Let $v \in NC(W, c)$ with $\mathrm{rk}(v) = n-2$, and let $x_1, x_2 \in T$ such that $c = vx_1x_2$. Suppose $v \sqsubset vx_1 \ll c$. Then $\langle r(x_1) | r(x_2) \rangle \geqslant 0$.*

Proof. The cardinality of $\mathrm{supp}(v)$ is at least $n-2$ since $\mathrm{rk}(v) = n-2$. We treat separately its possible values.

Case 1: $\#\,\mathrm{supp}(v) = n$.

Then $v \in NCF(W, c)$, so that we can apply Proposition 4.2 to the relation $v \sqsubset vx_1$. We get $x_2 \ll x_1x_2$, therefore x_1, x_2 is not a simple system in $\Gamma(x_1x_2)$, and $\langle r(x_1) | r(x_2) \rangle > 0$.

Case 2: $\#\,\mathrm{supp}(v) = n-1$.

Since v is not full, there exists an interval partition v' such that $v \lessdot v' \sqsubset c$. With $v' = vx_3$ and $c = v'x_4$, we have $vx_3x_4 = c$, $v \lessdot vx_3 \sqsubset c$, and $x_3x_4 = x_1x_2$. Since $\#\,\mathrm{supp}(v) = n-1 > \mathrm{rk}(v)$, we see that v is not an interval partition, so that $v \ll vx_3$. By Proposition 4.2, we get $K(vx_3) \sqsubset K(v)$, i.e., $x_4 \sqsubset x_3x_4$. Now by Lemma 4.6, $v \ll vx_3 \sqsubset vx_3x_4$ implies $x_3 \leqslant vx_3 \sqsubset vx_3x_4$ and $x_3 \sqsubset x_3x_4$ therefore x_3 and x_4 are the simple generators of $\Gamma(x_3x_4)$.

If $\{x_1, x_2\} \neq \{x_3, x_4\}$, we get $\langle r(x_1) | r(x_2) \rangle > 0$ by uniqueness of the simple system $\{x_3, x_4\}$. Otherwise, we have $x_1 = x_4$, indeed $x_1 \neq x_3$ since $v \sqsubset vx_1$ and $v \ll vx_3$. It follows that $x_2 = x_3$, and $x_3x_4 = x_1x_2$ becomes $x_2x_1 = x_1x_2$ so that $\langle r(x_1) | r(x_2) \rangle = 0$.

Case 3: $\#\,\mathrm{supp}(v) = n-2$.

To avoid multiple indices, assume that the simple reflections of W are indexed so that $c = s_1 \cdots s_n$. Let $J = \mathrm{supp}(v) = S \backslash \{i, j\}$ (with $i < j$) then v has length $\ell_T(v) = n-2$ and $v \in W_J$ therefore v is the Coxeter element of $W_J = \Gamma(v)$. It follows that v is an interval partition and $v = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_n$ (where s_i and s_j are omitted). Let

$$x_3 = s_n \cdots s_{j+1} s_{j-1} \cdots s_i \cdots s_{j-1} s_{j+1} \cdots s_n, \qquad x_4 = s_n \cdots s_j \cdots s_n.$$

then $vx_3x_4 = c$ and $x_3x_4 = x_1x_2$. By an argument similar to that in the previous lemma, we have

$$\langle r(x_3)|r(x_4)\rangle = \langle r(s_{j-1}\cdots s_i\cdots s_{j-1})|r(s_j)\rangle \leqslant 0$$

therefore x_3 and x_4 are the simple generators of $\Gamma(x_1x_2)$. The end of the proof is as in the previous case (here $x_1 \neq x_3$ because $vx_1 \ll c$ and $vx_3 \sqsubset c$). □

Lemma 8.6. *We have $\langle r(t_i)|r(t_j)\rangle \geqslant 0$ if $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$.*

Proof. By Lemma 8.3, we have $wt_i \sqsubset w \ll wt_j$. Then we can apply Lemma 8.5 in the subgroup $\Gamma(wt_j)$ to get the result. □

Proposition 8.7. *We have $\Psi(w) \in \Upsilon^+(W,c)$.*

Proof. We use the criterion in Proposition 6.6. Since $t_1\cdots t_n = c$, we have $t_it_j \leqslant c$ if $i < j$. The conditions on the scalar product are given by the previous lemmas. So $r(t_i) \parallel_c r(t_j)$ holds for $1 \leqslant i < j \leqslant n$ and the result follows. □

Let us now describe the inverse map. Let $F = \{t_1, \ldots, t_n\}$ be a face of the positive cluster complex then, by Proposition 6.7 we can assume that the t_i are ordered so that $t_1 \ldots t_n = c$ moreover, by Proposition 3.4 all orderings of the t_i such that this property holds true are obtained from this ordering by applying commutation relations among the t_i. As before we let $u_i = t_1 \ldots t_i$.

Lemma 8.8. *If $u_{i-1} \ll u_i \sqsubset u_{i+1}$ then $t_it_{i+1} = t_{i+1}t_i$.*

Proof. By Proposition 4.6 one has $t_i \sqsubset t_it_{i+1}$. Applying the Kreweras complement in $\Gamma(u_{i+1})$ and $i)$ of Proposition 4.2 we obtain $t_{i+1} \sqsubset t_it_{i+1}$. It follows that t_i, t_{i+1} form a simple system in $\Gamma(t_it_{i+1})$ and $\langle r(t_i)|r(t_{i+1})\rangle \leqslant 0$. Since $t_i \parallel_c t_{i+1}$ we have $\langle r(t_i)|r(t_{i+1})\rangle \geqslant 0$, it follows that $\langle r(t_i)|r(t_{i+1})\rangle = 0$ and t_i, t_{i+1} commute. □

Lemma 8.9. *Let $t, t' \in T$ be such that $tt' = t't$ and $w \preceq^{(1)} wt \preceq^{(2)} wt't$ then $w \preceq^{(2)} wt' \preceq^{(1)} wt't$, where $\preceq^{(1)}, \preceq^{(2)}$ denotes any combination of the orders $\sqsubset, \ll$.*

Proof. This follows from Corollary 2.2. □

Using lemmas 8.8 and 8.9 we can use commutation relations between the t_i to move all the $\ll$ to the right and assume that, for some k, one has

$$e \sqsubset u_1 \sqsubset u_2 \sqsubset \ldots \sqsubset u_k \ll u_{k+1} \ll \ldots \ll u_n = c.$$

Moreover, by Proposition 3.4, k and u_k are uniquely determined by this requirement. It is then easy to check that $\{t_1, \ldots, t_k\} = \mathrm{Inv}_R(u_k) \cap \Gamma(u_k)$ and $\{t_{k+1}, \ldots, t_n\} = \mathrm{Inv}_L(K(u_k)) \cap \Gamma(K(u_k))$ therefore $\Psi_{W,c}(u_k) = F$.

8.2. **The bijection between faces and intervals.** Consider a positive face $F = \{t_1, \dots, t_k\} \in \Upsilon^+(W, c)$. By Proposition 6.7 we can assume that the elements are indexed so that $t_1 \cdots t_k \leqslant c$.

Proposition 8.10. *Let $w = t_1 \cdots t_k$, then F is a cluster in $\Upsilon^+(\Gamma(w), w)$.*

Proof. First note that $\Upsilon^+(\Gamma(w), w)$ is well defined by Proposition 3.3. We have $t_i t_j \leqslant w$ if $i < j$ since $w = t_1 \cdots t_k$. Moreover, we have $\langle r(t_i) | r(t_j) \rangle \geqslant 0$ since $F \in \Upsilon^+(W, c)$. So $F \in \Upsilon^+(\Gamma(w), w)$ by Proposition 6.6, and it is a cluster since $\#F = \ell_T(w)$ is the rank of $\Gamma(w)$. □

Theorem 8.11. *With the notation as above, the map*

$$F \mapsto \left(\Psi^{-1}_{\Gamma(w),w}(F), w\right)$$

is a bijection from $\Upsilon^+(W, c)$ to the set of pairs $v, w \in NC(W, c)$ such that $v \ll w$.

Proof. First note that $\Psi^{-1}_{\Gamma(w),w}(F)$ is well defined by Proposition 8.10. By properties of the bijection Ψ, we have $\Psi^{-1}_{\Gamma(w),w}(F) \in NCF(\Gamma(w), w)$. By Proposition 4.11, this means $\Psi^{-1}_{\Gamma(w),w}(F) \ll w$.

We can describe the inverse bijection. To a pair $v, w \in NC(W, c)$ such that $v \ll w$, we associate $\Psi_{\Gamma(w),w}(v)$. Once we know that Ψ is a bijection, it is clear that we have two inverse bijections. □

The construction can be made more explicit. Let $u_i = t_1 \cdots t_i$ for $0 \leqslant i \leqslant k$. Up to some commutation among the t_i, we can assume

$$u_0 \boxminus \cdots \boxminus u_j \ll u_{j+1} \ll \cdots \ll u_k.$$

Then the image of F is (u_j, u_k). In the other direction, let $v, w \in NC(W, c)$ with $v \ll w$, $\mathrm{rk}(v) = j$ and $\mathrm{rk}(w) = k$. We write v and $v^{-1}w$ as a product of their associated simple reflections:

$$v = \mathfrak{s}_1 \cdots \mathfrak{s}_j, \qquad v^{-1}w = \mathfrak{s}_{j+1} \cdots \mathfrak{s}_k.$$

Then the inverse image of (v, w) is $\{t_1, \dots, t_k\}$ where we define $t_1, \dots, t_k$ as in the definition of Ψ:

$$t_i = \begin{cases} \mathfrak{s}_j \cdots \mathfrak{s}_{j+1-i} \cdots \mathfrak{s}_j & \text{if } 1 \leqslant i \leqslant j, \\ \mathfrak{s}_{j+1} \cdots \mathfrak{s}_{k+1+j-i} \cdots \mathfrak{s}_{j+1} & \text{if } j+1 \leqslant i \leqslant k. \end{cases}$$

Also, an immediate consequence of the construction is the following.

Definition 8.12. Let $F \in \Upsilon^+(W, c)$, and write $F = \{t_1, \dots, t_k\}$ such that $t_1 \cdots t_k \leqslant c$. Then we define (number of "square" relations):

$$\mathrm{sqr}(F) = \#\left\{i \ : 0 \leqslant i < k \text{ and } t_1 \cdots t_i \boxminus t_1 \cdots t_{i+1}\right\}.$$

This map is extended to $F \in \Upsilon(W, c)$ by requiring $\mathrm{sqr}(F) = \mathrm{sqr}(F \cap \Pi)$.

By Lemma 8.8 the number $\mathrm{sqr}(F)$ does not depend on the way we order F, as long as $t_1 \cdots t_k \leqslant c$.

Proposition 8.13. *We have:*

$$\sum_{F\in\Upsilon^+(W,c)} y^{\mathrm{sqr}(F)} z^{\#F} = \sum_{\substack{\alpha,\beta\in NC(W,c)\\ \alpha\ll\beta}} y^{\mathrm{rk}(\alpha)} z^{\mathrm{rk}(\beta)}.$$

However, this identity is not related to the $F = M$ theorem in a straightforward way, as the left hand side is seemingly unrelated to the polynomial $F(x, y)$. This will be clarified in Section 9.

8.3. **Bijection between faces of the cluster complex and intervals for $\sqsubset$ in the case of a bipartite Coxeter elements.** When $c = c_+c_-$ is a bipartite Coxeter element, one can give a bijection between the intervals for $\sqsubset$ and faces of the cluster complex. For this we need a preliminary result.

Let $u \ll v$ then, by applying Proposition 3.8, we get $Lv \sqsubset Lu$. It follows that $S(\Gamma(Lv)) \subset S(\Gamma(Lu))$. Let $w \in \Gamma(Lu)$ be the element corresponding to the set $S(\Gamma(Lu))\backslash S(\Gamma(Lv))$ then one has $w \sqsubset Lu$. Let us denote by $\psi(u, v)$ the pair $(x, y) = (w, Lu)$. It is clear that the pair (u, v) can be retrieved from (x, y) therefore the map ψ is injective.

Proposition 8.14. *The map ψ yields a bijection between intervals $u \ll v$ and intervals $x \sqsubset y$ such that $\underline{x} = e$.*

Proof. Let $u \ll v$ and $(w, Lu) = \psi(u, v)$, we prove that $\underline{w} = e$. Assume on the opposite that $\underline{w} \neq e$ then there exists $s \in S$ such that $s \in S(\Gamma(w)) \subset S(\Gamma(Lu))$ and $s \notin S(\Gamma(Lv)) = S(\Gamma(Lu))\backslash S(\Gamma(w))$. Since $s \leqslant Lu$ one has $sLu \leqslant Lu$ and $(Lu)s \leqslant Lu$. It follows easily that $u \in W_{\langle s\rangle}$ and $s \in S(\Gamma(v))$ which contradicts $u \ll v$. This proves that ψ maps intervals $u \ll v$ to intervals $x \sqsubset y$ with $\underline{x} = e$.

Conversely, let $x \sqsubset y$ be such that $\underline{x} = e$ and let z be the element of $\Gamma(y)$ corresponding to $S(\Gamma(y))\backslash S(\Gamma(x))$ then $z \sqsubset y$. Let us prove that $Ly \ll Lz$, it will follow from the construction that $(x, y) = \psi(Ly, Lz)$ and therefore that ψ is surjective. Let J be the support of Ly then one has $Ly \in W_J$ and one can write $y = c_+^{J^c}(c_+^J Lyc_-^J)c_-^{J^c}$ with obvious notations: e.g. c_+^J is the product of simple reflections in $J \cap S_+$, etc. Since $(c_+^J Lyc_-^J) \in W_J$ is follows that for $s \in J^c$ one has $s \in S(\Gamma(y))$ moreover since $\underline{x} = e$ one has $s \notin S(\Gamma(x))$ therefore $s \in S(\Gamma(z))$. It follows that $z = c_+^{J^c}\omega c_-^{J^c}$ with $\omega \in W_J$ moreover $\omega \sqsubset c_+^J Lyc_-^J$ since $z \sqsubset y$. Since Ly has full support in W_J we can apply *ii)* of Proposition 3.8 in the group W_J and conclude that $Lz = c_+^J\omega c_-^J \ll Ly$ as claimed. □

Let γ be an interval partition, corresponding to the parabolic subgroup W_J then one can write $\gamma = \gamma_+\gamma_-$ according to the bipartite decomposition. Let $v \in NC(W, c)$ be such that $\underline{v} = \gamma$, then $v = \gamma_+v'\gamma_-$ where $v' \in NC(W_{J^c}, L\gamma)$ moreover this gives a bijection between $NC(W_{J^c}, L\gamma)$ and the set of v such that $\underline{v} = \gamma$.

Using the bijection ψ^{-1} and composing with the bijection between intervals for $\ll$ and faces of the positive cluster complex we get a bijection

between intervals $v \sqsubset w$ for $\sqsubset$ with $\underline{v} = e$ and faces of the positive cluster complex $\Upsilon^+(W,c)$. It is an easy exercize to check that an interval of height k corresponds to a face of size $n-k$. This bijection is then easily extended to a bijection between intervals of $\sqsubset$ and faces of $\Upsilon(W,c)$ if $\underline{v} = \gamma$ add the set J to the face in $\Upsilon^+(W_{J^c}, c_{J^c})$.

9. Generalized $F = M$ and $H = M$ theorems

We show in this section that the relations between F-, H-, and M-polynomials can be proved and even generalized using the I-polynomial counting intervals for the order $\ll$. Here we consider homogeneous polynomials on variables $(x_s)_{s\in S}$ indexed by the simple reflections of W, and y, z. Note that the existence of multivariate analog of the identities was suggested by Armstrong [1, Open problem 5.3.5].

In general we denote $\underline{x}$ the set of x variables, leaving the index set implicit. Moreover $\underline{x} + A$ denote that all the x variables are shifted by A, and $\underline{x}$ is replaced by an expression to mean that all x variables are specialized to this expression, etc.

Let

$$\mathbf{I}(\underline{x}, y, z) = \sum_{\substack{\alpha,\beta\in NC(W,c)\\ \alpha\ll\beta}} \left(\prod_{s\in S\setminus \operatorname{supp}(\beta)} x_s\right) y^{\operatorname{rk}(\alpha)} z^{\operatorname{rk}(\beta)-\operatorname{rk}(\alpha)},$$

$$\mathbf{F}(\underline{x}, y, z) = \sum_{F\in\Upsilon(W,c)} \left(\prod_{\delta\in F\cap(-\Delta)} x_{r(\delta)}\right) \left(\frac{y}{z}\right)^{\operatorname{sqr}(F)} z^{\#F\cap\Pi},$$

$$\mathbf{M}(\underline{x}, y, z) = \sum_{\substack{\alpha,\beta\in NC(W,c)\\ \alpha\leqslant\beta}} \mu(\alpha,\beta) \left(\prod_{s\in S\setminus \operatorname{supp}(\beta)} x_s\right) y^{\operatorname{rk}(\alpha)} (-z)^{\operatorname{rk}(\beta)-\operatorname{rk}(\alpha)},$$

$$\mathbf{H}(\underline{x}, y, z) = \sum_{A\in NN(W)} \left(\prod_{\delta\in A\cap\Delta} \left(\frac{x_{r(\delta)}}{z}\right)\right) \left(\frac{y}{z}\right)^{\#A\cap(\Pi\setminus\Delta)} z^{\#\operatorname{supp}(A)}.$$

It is straightforward to check that they are all polynomials of total degree n. For example, note that we have $\operatorname{sqr}(f) \leqslant \#(F\cap\Pi)$ for $F \in \Upsilon(W,c)$ by definition of sqr, so the power of z in $\mathbf{F}(\underline{x}, y, z)$ is nonnegative, and we have $\operatorname{rk}(\alpha) \leqslant \#\operatorname{supp}(\alpha)$ for $\alpha \in NC(W,c)$ so the polynomials $\mathbf{I}(\underline{x}, y, z)$ and $\mathbf{M}(\underline{x}, y, z)$ have degree at most n.

The 2-variable polynomials from Section 6.5 are obtained as special cases, though by different specializations:

$$\begin{aligned} F(x,y) &= \mathbf{F}(x,y,y), \quad & H(x,y) &= \mathbf{H}(x,y,1),\\ M(y,z) &= \mathbf{M}(1,y,z), \quad & I(y,z) &= \mathbf{I}(1,y,z). \end{aligned} \tag{33}$$

Theorem 9.1. $\mathbf{F}(\underline{x}, y, z) = \mathbf{I}(\underline{x} + 1, y, z)$.

Proof. First, note that

$$\mathbf{F}(\underline{x}, y, z) = \sum_{J \subset S} \left(\prod_{s \in J} x_s \right) \mathbf{F}_{W_{S \setminus J}}(0, y, z).$$

Also, expanding the products in $\mathbf{I}(\underline{x}+1, y, z)$ gives:

$$\begin{aligned}
\mathbf{I}(\underline{x}+1, y, z) &= \sum_{\substack{\alpha, \beta \in NC(W,c) \\ \alpha \ll \beta}} \sum_{J \subset S \setminus \operatorname{supp}(\beta)} \left(\prod_{s \in J} x_s \right) y^{\operatorname{rk}(\alpha)} z^{\operatorname{rk}(\beta) - \operatorname{rk}(\alpha)} \\
&= \sum_{J \subset S} \sum_{\substack{\alpha, \beta \in NC(W,c),\ \alpha \ll \beta \\ \operatorname{supp}(\beta) \subset S \setminus J}} \left(\prod_{s \in J} x_s \right) y^{\operatorname{rk}(\alpha)} z^{\operatorname{rk}(\beta) - \operatorname{rk}(\alpha)} \\
&= \sum_{J \subset S} \left(\prod_{s \in J} x_s \right) \mathbf{I}_{W_{S \setminus J}}(1, y, z).
\end{aligned}$$

So it suffices to prove $\mathbf{F}(0, y, z) = \mathbf{I}(1, y, z)$, which is the content of Proposition 8.13. □

Theorem 9.2. $\mathbf{M}(\underline{x}, y, z) = \mathbf{I}(\underline{x}, y+z, z)$.

Proof. This follows from Proposition 7.4. In $I_\beta(\frac{y}{z}+1) = M_\beta(\frac{y}{z})$, multiply both sides by $\left(\prod_{s \in S \setminus \operatorname{supp}(\beta)} x_s\right) z^{\operatorname{rk}(\beta)}$ and sum over $\beta \in NC(W, c)$. □

Theorem 9.3. $\mathbf{H}(\underline{x}, y, z) = \mathbf{I}(\underline{x} - y + 1, y, z - 1)$.

Proof. First write the expansion:

$$\prod_{\delta \in A \cap \Delta} \left(\frac{x_{r(\delta)} + y}{z} \right) = \sum_{J \subset A \cap \Delta} \left(\frac{y}{z} \right)^{\#(A \cap \Delta) \setminus J} \prod_{\delta \in J} \left(\frac{x_{r(\delta)}}{z} \right).$$

It follows:

$$\begin{aligned}
\mathbf{H}(\underline{x} + y, y, z) &= \sum_{A \in NN(W)} \sum_{J \subset A \cap \Delta} \left(\prod_{\delta \in J} \left(\frac{x_{r(\delta)}}{z} \right) \right) \left(\frac{y}{z} \right)^{\#(A \setminus J)} z^{\# \operatorname{supp}(A)} \\
&= \sum_{J \subset \Delta} \left(\prod_{\delta \in J} x_{r(\delta)} \right) \sum_{\substack{A \in NN(W) \\ J \subset A}} \left(\frac{y}{z} \right)^{\#(A \setminus J)} z^{\# \operatorname{supp}(A) - \# J} \\
&= \sum_{J \subset \Delta} \left(\prod_{\delta \in J} x_{r(\delta)} \right) \mathbf{H}_{W_{S \setminus r^{-1}(J)}}(y, y, z).
\end{aligned}$$

In the proof of the previous theorem, we have seen that $\mathbf{I}(\underline{x}+1, y, z)$ admits a similar expansion on the x-variables. Consequently, $\mathbf{I}(\underline{x}+1, y, z-1)$ also admits a similar expansion, and it remains to show that $\mathbf{H}(y, y, z) = \mathbf{I}(1, y, z-1)$.

On one side we have:

$$\mathbf{H}(y,y,z) = \sum_{A\in NN(W)} y^{\#A} z^{\#\operatorname{supp}(A)-\#A} = \sum_{I\subset S}\sum_{k=1}^{\#I} \operatorname{Nar}_k^+(W_I) y^k z^{\#I-k}.$$

On the other side,

$$\begin{aligned}\mathbf{I}(1,y,z-1) &= \sum_{\substack{\alpha,\beta\in NC(W,c)\\ \alpha\ll\beta}} y^{\operatorname{rk}(\alpha)}(z-1)^{\operatorname{rk}(\beta)-\operatorname{rk}(\alpha)}\\ &= \sum_{\alpha\in NC(W,c)} y^{\operatorname{rk}(\alpha)} z^{\#\operatorname{supp}(\alpha)-\operatorname{rk}(\alpha)},\end{aligned}$$

where the last equality follows from the binomial theorem, since the β such that $\beta \gg \alpha$ form a boolean lattice whose maximal element is $\overline{\alpha}$, by Proposition 4.13 and Corollary 4.14. This sum can again be expressed in terms of the numbers $\operatorname{Nar}_k^+(W_I)$, so that $\mathbf{H}(y,y,z) = \mathbf{I}(1,y,z-1)$. □

By combining the previous theorems, we get relations between **F**-, **H**-, and **M**-polynomials:

$$\mathbf{F}(\underline{x},y,z) = \mathbf{M}(\underline{x}+1,y-z,z), \tag{34}$$

$$\mathbf{F}(\underline{x},y,z) = \mathbf{H}(\underline{x}+y,y,z+1), \tag{35}$$

$$\mathbf{H}(\underline{x},y,z) = \mathbf{M}(\underline{x}-y+1,y-z+1,z-1). \tag{36}$$

The next property is best seen on the **H**-polynomial.

Proposition 9.4. $\mathbf{H}(\underline{x}-1,y,z)$ *is homogeneous of degree* n.

Proof. An element $A\in NN(W)$ can be uniquely written $A'\cup A''$ where $A'\cap\Delta=\varnothing$ and $A''\subset\Delta\backslash\operatorname{supp}(A')$. Note that $\#\operatorname{supp}(A)=\#\operatorname{supp}(A')+\#A''$. For a fixed A', any subset $A''\subset\Delta\backslash\operatorname{supp}(A')$ is valid, so that the sum over A'' factorizes and gives:

$$\begin{aligned}&\sum_{\substack{A\in NN(W)\\ A\cap(\Pi\backslash\Delta)=A'}} \left(\prod_{\delta\in A\cap\Delta}\left(\frac{x_{r(\delta)}}{z}\right)\right)\left(\frac{y}{z}\right)^{\#A\cap(\Pi\backslash\Delta)} z^{\#\operatorname{supp}(A)}\\ &\qquad = \left(\prod_{\delta\in\Delta\backslash\operatorname{supp}(A')}(1+x_{r(\delta)})\right)\left(\frac{y}{z}\right)^{\#A'} z^{\#\operatorname{supp}(A')}.\end{aligned}$$

After a change of variables $\underline{x}\to\underline{x}-1$, the latter expression is homogeneous. Since $\mathbf{H}(\underline{x}-1,y,z)$ is obtained by summing over A', it is too. □

This homogeneity implies that the 3-variable polynomial $\mathbf{H}(x,y,z)$ can be expressed in terms of the 2-variable polynomial $H(y,z)=\mathbf{H}(1,y,z)$. It also implies that the **F**-, **I**-, and **M**-polynomials also become homogeneous after suitable shifts in the variables. So the other 3-variable polynomials can also be expressed in terms of their 2-variable specialization from Section 6.5.

To recover the $F = M$ identity in Section 6.5 from the present results, first use the homogeneity of $\mathbf{F}(x-1, y, z-1)$ (obtained from the previous proposition together with (35)) to write:

$$\mathbf{F}(x-1, y, z-1) = x^n \mathbf{F}(0, \tfrac{y}{x}, \tfrac{z}{x} - 1),$$

then the substitution $(x, z) \to (x+1, y+1)$ gives:

$$\mathbf{F}(x, y, y) = (1+x)^n \mathbf{F}(0, \tfrac{y}{1+x}, \tfrac{y-x}{1+x}) = (1+x)^n \mathbf{M}(1, \tfrac{x}{1+x}, \tfrac{y-x}{1+x})$$

where the last equality comes from (34). Then using (33), we recover the relation between $F(x, y)$ and $M(x, y)$ in (23). Similarly, the homogeneity of $\mathbf{H}(\underline{x} - 1, y, z)$ gives:

$$\mathbf{H}(x-1, y, z) = z^n \mathbf{H}(\tfrac{x}{z} - 1, \tfrac{y}{z}, 1). \tag{37}$$

Then the substitution $(x, z) \to (x+y+1, y+1)$ gives:

$$\mathbf{H}(x+y, y, y+1) = (1+y)^n \mathbf{H}(\tfrac{x}{1+y}, \tfrac{y}{1+y}, 1).$$

Using (35) on the left hand side, then (33), we recover the relation between $F(x, y)$ and $H(x, y)$ in (23).

Proposition 9.5. $\mathbf{H}(\underline{x} - 1, y, z)$ *is symmetric in* y *and* z.

Proof. Similar to the expansions we have seen in this section, we have:

$$\mathbf{H}(\underline{x}, y, z) = \sum_{J \subset S} \left(\prod_{s \in J} x_s \right) \mathbf{H}_{W_{S \setminus J}}(0, y, z).$$

So we can just prove that $\mathbf{H}(0, y, z)$ is symmetric. By (37), we have

$$\mathbf{H}(0, y, z) = z^n \mathbf{H}(\tfrac{1}{z} - 1, \tfrac{y}{z}, 1) = z^n H(\tfrac{1}{z} - 1, \tfrac{y}{z}).$$

By (27), the latter is equal to $y^n H(\frac{1}{y} - 1, \frac{z}{y})$, which is precisely the same up to exchanging y and z. □

It is natural to look for an involution on nonnesting partitions that would prove the symmetry of $\mathbf{H}(\underline{x}, y, z)$. This is the subject of conjectures by Panyushev [27, Conjecture 6.1], see also [15, Section 5]. To our knowledge this is still an open problem.

Acknowledgement

We thank Cesar Ceballos for explanations about the subword complex.

References

[1] D. Armstrong: *Generalized noncrossing partitions and combinatorics of Coxeter groups.* Mem. Amer. Math. Soc. 202 (2009), no. 949.

[2] C.A. Athanasiadis: *On some enumerative aspects of generalized associahedra.* Europ. J. Combin. 28 (2007), 1208–1215.

[3] C.A. Athanasiadis, C. Savvidou: *The local h-vector of the cluster subdivision of a simplex.* Sém. Lothar. Combin. 66(2012), Art. B66c.

[4] C.A. Athanasiadis, T. Brady, J. MacCammond, C. Watt: *h-vectors of generalized associahedra and noncrossing partitions.* Int. Math. Res. Not. 2006, Art. ID 69705, 28 pp.

[5] E. Barnard, N. Reading: *Coxeter-biCatalan combinatorics.* J. Algebr. Comb. (2017).

[6] S.T. Belinschi, A. Nica: *η-series and a Boolean Bercovici-Pata bijection for bounded k-tuples.* Adv. Math. 217 (2008), 1–41.

[7] D. Bessis: *The dual braid monoid.* Ann. Sci. École Norm. Sup. 36 (2003), 647–683.

[8] P. Biane: *Some properties of crossings and partitions.* Discrete Math. 175 (1997), no. 1-3, 41–53.

[9] P. Biane, M. Josuat-Vergès: *Minimal factorizations of a cycle: a multivariate generating function.* FPSAC 2016, DMTCS Proc. BC, 2016, 239–250

[10] A. Björner, F. Brenti: *Combinatorics of Coxeter Groups.* Graduate Text in Mathematics, Vol. 237, Springer Verlag, 2005.

[11] T. Brady, C. Watt: *$K(\pi,1)$'s for Artin groups of finite type*, in Proceedings of the conference on geometric and combinatorial group theory, Part I (Haifa 2000), Geom. Dedicata 94 (2002), 225–250.

[12] T. Brady, C. Watt: *Non-crossing partition lattices in finite real reflection groups.* Trans. Amer. Math. Soc. 360 (2008), no. 4, 1983–2005.

[13] C. Ceballos, J.-P. Labbé, C. Stump: *Subword complexes, cluster complexes, and generalized multi-associahedra.* J. Alg. Combin. 39(1), 2014, 17–51.

[14] F. Chapoton: *Enumerative properties of generalized associahedra.* Sém. Lothar. Combin. 51(2004), Art. B51b.

[15] F. Chapoton: *Sur le nombre de réflexions pleines dans les groupes de coxeter finis.* Bull. Belgian Math. Soc. 13 (2006), 585–596.

[16] M. Cuntz, C. Stump: *On root posets for noncrystallographic root systems.* Math. Comp. 84, 2015.

[17] M. Dyer: *On the "Bruhat graph" of a Coxeter system.* Compos. Math. 78(2) (1991), 185–191.

[18] S. Fomin, N. Reading: *Generalized cluster complexes and Coxeter combinatorics.* Int. Math. Res. Notices 2005, no. 44, 2709–2757.

[19] S. Fomin, A. Zelevinsky: *Cluster algebras. II. Finite type classification.* Invent. Math. 154 (2003), no. 1, 63–121.

[20] J.E. Humphreys: *Reflection Groups and Coxeter Groups.* Cambridge University Press, Cambridge, 1990.

[21] M. Josuat-Vergès: *Refined enumeration of noncrossing chains and hook formulas.* Ann. Combin. 19(3) (2015), 443–460.

[22] G. Kreweras: *Sur les partitions non croisées d'un cycle.* Disc. Math. 1(4) (1972), 333–350.

[23] R. Marsh, M. Reineke, A. Zelevinsky: *Generalized associahedra via quiver representations.* Trans. Amer. Math. Soc. 355 (2003), no. 10, 4171–4186.

[24] A. Nica: *Non-crossing linked partitions, the partial order $\ll$ on $NC(n)$, and the S-transform.* Proc. Amer. Math. Soc. 138 (2010), no. 4, 1273–1285.

[25] A. Nica, R. Speicher: *Lectures on the combinatorics of free probability.* London Mathematical Society Lecture Note Series, 335. Cambridge University Press, Cambridge, 2006.

[26] P. Petrullo, D. Senato: *Explicit formulae for Kerov polynomials.* J. Algebr. Comb. 33 (2011), 141–151.

[27] D. Panyushev: *Ad-nilpotent ideals of a Borel subalgebra: generators and duality.* J. Algebra 274 (2004), 822–846.

[28] N. Reading: *Clusters, Coxeter-sortable elements and noncrossing partitions.* Trans. Amer. Math. Soc. 359(12) (2007), 5931–5958.

[29] N. Reading, D. Speyer: *Cambrian fans.* J. Eur. Math. Soc. 11(2) (2009), 407–447.

[30] R. Speicher, R. Woroudi : *Boolean convolution*. Fields Institute Communications, Vol. 12 (D. Voiculescu, ed.), AMS, 1997, pp. 267–279.

[31] M. Thiel: *On the H-triangle of generalised nonnesting partitions*. Europ. J. Combin. 39 (2014), 244–255.

Laboratoire d'Informatique Gaspard Monge, Université Paris-Est Marne-la-Vallée, CNRS