Lattès maps and the interior of the bifurcation locus

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Abstract

We study the phenomenon of robust bifurcations in the space of holomorphic maps of $\mathbb{P}^2(\mathbb{C})$. We prove that any Lattès example of sufficiently high degree belongs to the closure of the interior of the bifurcation locus. In particular, every Lattès map has an iterate with this property. To show this, we design a method creating robust intersections between the limit set of a particular type of iterated functions system in $\mathbb{C}^2$ with a well-oriented complex curve. Then we show that any Lattès map of sufficiently high degree can be perturbed so that the perturbed map exhibits this geometry.

Contents

1 Introduction

1.1 Context

In the article [15], Mañé, Sad and Sullivan, and independently Lyubich in [14], introduced a relevant notion of stability for holomorphic families $(f_\lambda)_{\lambda\in\Lambda}$ of rational mappings of degree d on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, parameterized by a complex manifold Λ. The family $(f_\lambda)_{\lambda\in\Lambda}$ is J-stable in a connected open subset $\Omega \subset \Lambda$ if in Ω the dynamics is structurally stable on the Julia set J. It can be shown that this is equivalent to the fact that periodic repelling points stay repelling points inside the given family. The bifurcation set is the complementary of the locus of stability. A remarkable fact is that the J-stability locus is dense in Λ for every such family. Moreover, parameters with preperiodic critical points are dense in the bifurcation locus.

In higher dimension, less is known. We will only discuss the 2-dimensional case in this paper. The research in this field mostly takes inspiration from two different types of maps with different behaviour : polynomial automorphisms of $\mathbb{C}^2$ and holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$. Knowledge about bifurcations of polynomial automorphisms is growing quickly. Let us quote the work of Dujardin and Lyubich ([10]) which introduces a satisfactory notion of stability and shows that homoclinic tangencies, which are the 2-dimensional counterpart of preperiodic critical points, are dense in the bifurcation locus.

From now on, we are interested in the case of holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$. The natural generalization of the one-dimensional theory was designed by Berteloot, Bianchi and Dupont in [3]. Their notion of stability is as follows : let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic maps of degree d on $\mathbb{P}^2(\mathbb{C})$ where Λ is simply connected. Then the following assertions are equivalent:

1. The function on Λ defined by the sum of Lyapunov exponents of the equilibrium measure $\mu_{f_\lambda} : \lambda \mapsto \chi_1(\lambda) + \cdots + \chi_k(\lambda)$ is pluriharmonic on Λ.
2. The sets $(J^*(f_\lambda))_{\lambda\in\Lambda}$ move holomorphically in a weak sense, where $J^*(f_\lambda)$ is the support of the measure μ_{f_λ}.
3. There is no (classical) Misiurewicz bifurcation in Λ.
4. Repelling periodic points contained in $J^*(f_\lambda)$ move holomorphically over Λ.

If these conditions are satisfied, we say that $(f_\lambda)_{\lambda\in\Lambda}$ is J^*-stable. If $(f_\lambda)_{\lambda\in\Lambda}$ is not J^*-stable at a parameter λ_0, we will say that a bifurcation occurs at λ_0.

A major difference with the one-dimensional case is the existence of open sets of bifurcations. Recently, several works have shown the existence of persistent bifurcations near well-chosen maps. By [3], to obtain open subsets in the bifurcation locus, it is enough to create a persistent intersection between the postcritical set and a hyperbolic repeller contained in J^*. Dujardin gives in [9] two mechanisms leading to such persistent intersections. The first one is based on topological considerations and the second uses the notion of blender, which is a hyperbolic set with very special fractal properties. Both enable to get persistent bifurcations near maps of the form $(z, w) \mapsto (p(z), w^d + \kappa)$. The results of Dujardin have been improved by Taflin in [18]. Taflin shows that if p and q are two polynomials of degree bounded by d such that p is a polynomial corresponding to a bifurcation in the space of polynomials of degree d, then the map (p, q) can be approximated by polynomial skew products having an iterate with a blender and then by open sets of bifurcations. Note that the idea of blender arised in the work of Bonatti and Diaz on real diffeomorphisms ([6]) and already appeared in holomorphic dynamics in the work of the author ([5]).

Lattès maps are holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$ which are semi-conjugate to an affine map on some complex torus $\mathbb{T}$ (see [11] for a classification and [4] for

a characterisation of Lattès maps in terms of the maximal entropy measure). It is natural to be interested in these maps in the context of bifurcation theory because their Julia set is equal to the whole projective space $\mathbb{P}^2(\mathbb{C})$. This property seems to have a great potential to create persistent intersection between the postcritical set and the Julia set even after perturbation. Berteloot and Bianchi proved in [2] that the Hausdorff dimension of the bifurcation locus near a Lattès map is equal to that of the parameter space.

1.2 Main result

Dujardin asked in [9] if it was possible to find open sets of bifurcations near any Lattès map. In this article we give a partial answer to this question. Here is our main result :

Theorem. *For every two-dimensional complex torus $\mathbb{T}$, there is an integer d (depending on the torus $\mathbb{T}$) such that every Lattès map defined on $\mathbb{P}^2(\mathbb{C})$ of degree $d' > d$ induced by an affine map on $\mathbb{T}$ is in the closure of the interior of the bifurcation locus in $Hol_{d'}$.*

Let us remark that the degree d is unknown (the situation here is similar to Buzzard's article [7]). Moreover, d depends on the torus $\mathbb{T}$. This is due to the necessity of making only holomorphic perturbations. Let us also point out that it is always possible to find Lattès maps of arbitrarily high degree associated to a given torus $\mathbb{T}$ (see the remark after Proposition 3.1.6). As an immediate consequence of the theorem we get :

Corollary. *For every Lattès map L of degree d, there is an integer $n(L)$ such that for every $n \geq n(L)$, the iterate L^n is in the closure of the interior of the bifurcation locus in Hol_{d^n}.*

1.3 Outline of proof

To prove this result, we create persistent intersections between the postcritical set and a hyperbolic repeller contained in the Julia set. Our proof has two main parts : first, we create a toy-model which allows to obtain intersections between the limit set of some particular type of IFS, called correcting IFS, and a curve that is "well-oriented". Then, in a second time, we perturb the Lattès map to create both the correcting IFS and the well-oriented curve inside the postcritical set. This construction exhibits properties somehow similar to the *blenders* of Bonatti-Diaz ([6]), with the difference that the covering property holds at the level of the tangent maps of the IFS (see also the notion of *parablenders* appeared in the work of Berger ([1])).

In a first part, we develop an intersection principle (see Proposition 2.1.6). A grid of balls G in $\mathbb{C}^2$ is the union of a finite number of balls regularly located at N^4 vertices of a lattice defined by a $\mathbb{R}$-basis of $\mathbb{C}^2$. If we consider a line $\mathfrak{C}$, a pigeonhole argument ensures that if $\mathfrak{C}$ is well oriented and G has a sufficient number of balls $N = N(r)$ (where r is the relative size of a ball compared to the mesh of the grid) then $\mathfrak{C}$ intersects a ball of G. We consider a class of IFS such that each inverse branch is very close to a homothety. When we iterate them, a drift can appear : the iterates become less and less conformal. Our class of IFS (called correcting IFS) is designed so that they have the property of correcting themselves from the drift. A linear correction principle is given in Proposition 2.1.10. In subsections 2.2 and 2.3, we treat the case of a curve close to a line and an IFS close to be linear. Our interest in such IFS is that any well-oriented quasi-line $\mathfrak{C}$ intersects the limit set of a correcting IFS. To prove this result, which is Proposition 2.3.1, we ensure that at each step the quasi-line $\mathfrak{C}$ intersects a

grid of ball G^j which is dynamically defined with the inverse branches of the IFS. Then we use inductively the intersection and the correction principles to ensure that at the next step, $\mathfrak{C}$ intersects a grid of balls G^{j+1} with bounded drift. The intersection of the grids G^j is in the limit set, so we produce an intersection between $\mathfrak{C}$ and the limit set of the IFS. Since the property of being correcting is open, this intersection is persistent.

In the second part, we make three successive perturbations of a Lattès map L, denoted by L', L'' and L''', in such a way that L''' has a robust bifurcation. We work in homogenous coordinates and do explicit perturbations of the following form :

$$[P_1 : P_2 : P_3] \to [P_1 + R_1 P_3 : P_2 + R_2 P_3 : P_3]$$

where R_1 and R_2 are rational maps. An important technical point (Proposition 3.2.1) is that we can choose the coordinates so that P_3 splits. Then if R_1 and R_2 are well chosen the degree does not change. The first perturbation L' (Propositions 4.4.4 and 4.4.5) is intended to create a correcting IFS in a ball $\mathfrak{B}$ in $\mathbb{C}^2$. Another important technical point is that we can find some critical point c which is preperiodic, with associated periodic point p_c such that both the preperiod n_c and the period n_{pc} of the preperiodic critical orbit are bounded independently of L (see Proposition 3.3.1). Then we want to create a well-oriented quasi-line inside the postcritical set which intersects $\mathfrak{B}$. The second perturbation L'' in Lemma 4.5.10 ensures that the postcritical set at p_c is not singular. The third and last perturbation L''' is given in Lemma 4.5.11. It is intended to control the differential at p_c. This allows us to fix the orientation of the postcritical set at p_c and then we use the linear dynamics of the Lattès map L on the torus $\mathbb{T}$ in order to propagate this geometric property up to $\mathfrak{B}$ (see Proposition 4.5.3). Note that the periodic point need not lie in $\mathfrak{B}$. At this stage we have both a correcting IFS and a well-oriented quasi-line so we are in position to conclude in section 5.

1.4 Plan

In section 2, we develop the theory of intersection between a quasi-line and the limit set of a correcting IFS : the intersection principle and the correction principle are stated in subsection 2.1 and we prove the intersection result in subsection 2.3. In section 3, we provide background on Lattès maps and prove a few properties which will be useful later. Some complications arise from Lattès maps whose linear part is not the identity. In section 4, we develop the perturbative argument. After giving some preliminaries (subsection 4.1) and fixing many constants (subsections 4.2 and 4.3), we create a correcting IFS in subsection 4.4. In subsection 4.5, we create a well oriented curve inside the postcritical set. Finally, we conclude in section 5 by applying the formalism of subsection 2.3 to the perturbed map L'''.

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2 Intersecting a curve and the limit set of an IFS

2.1 Linear model

In this section, we will work with an IFS, whose maps are small perturbations of homotheties of the form $\frac{1}{a} \cdot \mathrm{Id}$ with $a \in \mathbb{R}^*$ and $|a| > 1$. This IFS will be obtained by perturbating a Lattès map and its limit set will have persistent intersections with a curve.

Definition 2.1.1. *Given a $\mathbb{R}$-basis $(u_1, u_2, u_3, u_4) \in (\mathbb{C}^2)^4$, a point $o \in \mathbb{C}^2$, an integer N and $r \in (0,1)$, by a grid of balls we mean the union of the balls of radius $r.\min_{1\leq i\leq 4} ||u_i||$ centered at the points $o + iu_1 + ju_2 + ku_3 + klu_4$ where $-N \leq i, j, k, l \leq N$. We will denote it by $G = (u, o, N, r)$. The middle part of G is the set $\{o + xu_1 + yu_2 + zu_3 + wu_4, 0 \leq |x|, |y|, |z|, |w| \leq \frac{N}{2}\}$. The hull of G is the set $\{o + xu_1 + yu_2 + zu_3 + wu_4, 0 \leq |x|, |y|, |z|, |w| \leq N\}$. The size of G is $size(G) = 2N \cdot \max_{1\leq i\leq 4} ||u_i||$.*

In the following, the parameter r will be bounded from below and we will let $\max_{1\leq i\leq 4} ||u_i|| \to 0$ so that the radius of the balls $r.\min_{1\leq i\leq 4}||u_i||$ will tend to 0. The integer N will be taken sufficiently large to satisfy some conditions depending on the degree of the Lattès map. Herebelow the notions of "opening" and "slope" are relative to the standard euclidean structure of $\mathbb{C}^2$.

Notation 2.1.2. *For a non zero vector $w \in \mathbb{C}^2$ and $\theta > 0$, we will denote $C^{w,\theta}$ the cone of opening θ centered at w.*

Notation 2.1.3. *For any quadruple of non zero vectors w_1, w_2, w_3, w_4 in $\mathbb{C}^2$, we will denote $\overline{w} = \overline{(w_1, w_2, w_3, w_4)}$ its projection onto $\mathbb{P}(\mathbb{R}^8)$. For any matrix $U \in \mathrm{GL}_2(\mathbb{C})$, we simply denote by $U\cdot$ the induced action on $\mathbb{P}(\mathbb{R}^8)$.*

Definition 2.1.4. *The middle part of a ball (resp. the $\frac{3}{4}$-part) is the ball of same center and $\frac{1}{2}$ times its radius (resp. $\frac{3}{4}$ times its radius).*

Definition 2.1.5. *A holomorphic curve C is a (ε, w)-quasi-line if C is a graph upon a disk in $\mathbb{C} \cdot w$ of slope bounded by ε relative to the projection onto w. A (ε, w)-quasi-diameter of a ball $\mathbb{B}$ is a (ε, w)-quasi-line C intersecting the ball of same center as $\mathbb{B}$ and of radius $\frac{1}{10}$ times the radius of $\mathbb{B}$.*

Here is our "intersection principle" :

Proposition 2.1.6. *For every $u \in (\mathbb{C}^2)^4$, $r > 0$, $\eta > 0$ and $w_0 \in \mathbb{C}^2$, there exists a neighborhood $\mathcal{N}(u)$ of $\overline{u}$ in $\mathbb{P}(\mathbb{R}^8)$, there exists $\theta > 0$, $N(r) > 0$ and a vector $w \in \mathbb{C}^2$ with $||w - w_0|| < \eta$ such that the following property (P) holds :*

(P) For every grid of balls $G = (u, o, N, r)$ such that $\overline{u} = \overline{(u_1, u_2, u_3, u_4)} \in \mathcal{N}(u)$ and $N > N(r)$, for every (θ, w)-quasi-line of direction in $C^{w,2\theta}$ intersecting the middle part of the grid of balls G, there is a non empty intersection between the line and the middle part of one of the balls of the grid.

Moreover, property (P) stays true for w' sufficiently close to w.

Proof. Let us first prove the result in the case of a line intersecting the grid of balls. After composition by a real linear isomorphism if necessary, we can suppose $(u_1, u_2, u_3, u_4) = (e_1, ie_1, e_2, ie_2)$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ so that the centers of the balls of the grid have integer coordinates. Let us take $w_1 = \frac{\alpha_1}{\beta} e_1 + \frac{\alpha_2}{\beta} ie_1 + \frac{\alpha_3}{\beta} e_2 + \frac{\alpha_4}{\beta} ie_2$ such that $||w_1 - w_0|| < \eta$ with rational coordinates $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \in \mathbb{Z}$. We take $m = \lfloor \frac{10}{r} \rfloor$. Then, let us take the vector $w = w_1 + \frac{1}{m\beta} e_1 + \frac{1}{m^2\beta} ie_1 + \frac{1}{m^3\beta} e_2 + \frac{1}{m^4\beta} ie_2$ and $N > 10\beta m^5 = N(r)$. We can increase m if necessary so that w satisfies $||w - w_0|| < \eta$.

Lemma 2.1.7. *There is a non empty intersection between any line of direction in $C^{w,2\theta}$ intersecting the middle part of the grid of balls G and the middle part of one of the balls of the grid of balls if θ is sufficiently small.*

Proof. We divide each mesh of the lattice into m^4 hypercubes. To each of these hypercubes, we can assign the quadruple of integers given by the coordinates of a given corner. Taking new coordinates by making a translation if necessary, we can

suppose that the union of the middle parts of the balls of the lattice contains the union of the hypercubes whose four coordinates are all equal to 0 modulo m. Let us take a point x_0 of the line inside the middle part of the lattice, and for every $k \in \mathbb{N}$, we denote : $x_k = x_0 + k \cdot w$. Then, we have that :

$$\lfloor x_{k+\beta m,1} \rfloor \equiv \lfloor x_{k,1} \rfloor + 1 \pmod{m} \text{ and } \lfloor x_{k+\beta m^2,2} \rfloor \equiv \lfloor x_{k,2} \rfloor + 1 \pmod{m}$$

$$\lfloor x_{k+\beta m^3,3} \rfloor \equiv \lfloor x_{k,3} \rfloor + 1 \pmod{m} \text{ and } \lfloor x_{k+\beta m^4,4} \rfloor \equiv \lfloor x_{k,4} \rfloor + 1 \pmod{m}$$

Since $N > 10\beta m^5 = N(r)$, the previous relations imply there exists some x_n which intersects some hypercube of integer coordinates congruent to $(0,0,0,0)$ inside the grid of balls. This implies that the line intersects the middle part of one of the balls of the grid.

□

This intersection persists for any line of direction in $C^{w,2\theta}$ and for any $\overline{(u_1,u_2,u_3,u_4)}$ in a small neighborhood $\mathcal{N}(u)$ of $\overline{u}$. Then, the result stays true if we take (θ, w)-quasi-lines for θ sufficiently small since property in (P) is open for the C^1 topology and w' sufficiently close to w.

□

The following corollary gives the same conclusion as the previous result but this time with more than one possible direction for the quadruple of vectors of the lattice.

Corollary 2.1.8. *For every finite subgroup $\mathcal{M} \subset \mathrm{Mat}_2(\mathbb{C})$, for every $u \in (\mathbb{C}^2)^4$, there exists a neighborhood $\mathcal{N}(u)$ of $\overline{u}$ in $\mathbb{P}(\mathbb{R}^8)$ such that for every $r > 0$, there exists $\theta > 0$, $N(r) > 0$ and a vector $w \in \mathbb{C}^2$ such that the following property (P) holds :*

(P) For every $U \in \mathcal{M}$, for every grid of balls $G = (u, o, N, r)$ such that $\overline{u} \in \mathcal{N}(u) \cup U \cdot \mathcal{N}(u) \cup \cdots \cup U^{ord(U)-1} \cdot \mathcal{N}(u)$ and $N > N(r)$, for every (θ, w)-quasi-line of direction in $C^{w,2\theta}$ intersecting the middle part of the grid of balls G, there is a non empty intersection between the line and the middle part of one of the balls of the grid.

Moreover, this proposition remains true for w' sufficiently close to w.

Proof. We just have to apply $\mathrm{ord}(\mathcal{M})$ times Proposition 2.1.6. □

Notation 2.1.9. *We will denote by $\mathrm{Mat}_2(\mathbb{C})$ the metric space of $(2,2)$ complex matrices with the distance induced by the norm $||.|| = ||.||_{2,2}$.*

In the following, x will be a real positive parameter. We remind that in a first reading it is advised to assume that $U = I_2$. The following proposition is the "linear correction principle" we discussed in the introduction.

Proposition 2.1.10. *For every finite subgroup $\mathcal{M} \subset \mathrm{Mat}_2(\mathbb{C})$, there exists an integer $n > 0$, $(n+1)$ balls $V^0, V^1, ..., V^n \subset \mathrm{Mat}_2(\mathbb{C})$ such that for every $0 < x < 1$, there exists a neighborhood $\mathcal{U}_x$ of I_2 in $\mathrm{GL}_2(\mathbb{C})$, two open sets $\mathcal{U}'_x \subset \mathcal{U}''_x \subset \mathrm{GL}_2(\mathbb{C})$ which are union of balls $\mathcal{U}'_x = \bigcup_{1 \leq p \leq n} (\mathcal{U}'_x)^p$ and $\mathcal{U}''_x = \bigcup_{1 \leq p \leq n} (\mathcal{U}''_x)^p$ such that : $(\mathcal{U}'_x)^p \subset (\mathcal{U}''_x)^p$ for each $1 \leq p \leq n$ with the following properties :*

(i) If $M \in \mathcal{U}_x$, $U \in \mathcal{M}$ and $j \in \mathbb{N}$, then for every $M_0 \in (x \cdot V^0)$:

$$U^j M U (I_2 + M_0) \in U^{j+1} \cdot \mathcal{U}'_x$$

(ii) If $M \in \mathcal{U}'_x$, $U \in \mathcal{M}$ and $j \in \mathbb{N}$, then there exist two integers $1 \leq p, p' \leq n$ such that $M \in (\mathcal{U}'_x)^p$ with the property that for every $M_0 \in (x \cdot V^0)$ and for every $M_{p'} \in (x \cdot V^{p'})$, we have :

$$U^j M U (I_2 + M_0) \in U^{j+1} \cdot (\mathcal{U}''_x)^p$$
$$U^j M U (I_2 + M_0) U (I_2 + M_{p'}) \in U^{j+2} \cdot \mathcal{U}_x$$

Proof. We consider the vector space $\mathrm{Mat}_2(\mathbb{C}) \simeq \mathbb{R}^8$. Let us consider a covering of the sphere of center 0 of radius r (which will be chosen later) $S(0,r)$ by n balls $B(X_i, \frac{1}{20}r)$ of radius $\frac{1}{20}r$. The following geometrical lemma is trivial :

Lemma 2.1.11. *For every $1 \leq p \leq n$, $X \in B(X_i, \frac{1}{10}r)$, we have : $||X - X_i|| < \frac{1}{2}r$*

Now, let us call $\mathcal{U}_1 = B(I_2, \frac{9}{10}r)$, $(\mathcal{U}_1')^p = B(I_2 + X_i, \frac{1}{20}r)$ and $(\mathcal{U}_1'')^p = B(I_2 + X_i, \frac{1}{10}r)$ for each $1 \leq p \leq n$, $\mathcal{U}_1' = \bigcup_{1 \leq p \leq n} (\mathcal{U}_1')^p$ and $\mathcal{U}_1'' = \bigcup_{1 \leq p \leq n} (\mathcal{U}_1'')^p$. Increasing the number n of open sets $(\mathcal{U}_1')^p$ if necessary, we can suppose that for every $U \in \mathcal{M}$ and for each $p \leq n$, there exists $p' \leq n$ such that $(\mathcal{U}_1')^p \cdot U = U \cdot (\mathcal{U}_1')^{p'}$ and $(\mathcal{U}_1'')^p \cdot U = U \cdot (\mathcal{U}_1'')^{p''}$.

Lemma 2.1.12. *There exists $r_0 > 0$ such that if $r < r_0$, for every $Y \in (\mathcal{U}_1'')^p$: $Y(I_2 - X_i) \in B(I_2, \frac{1}{2}r)$.*

Proof. The first assertion is an easy consequence of the previous lemma. Then, the Taylor formula gives us that at 0 at the first order in X :

$$(I_2 + X)(I_2 - X_i) = I_2 + X - X_i + O(r^2)$$

Then, if r is sufficiently small, the following lemma implies that for every $X \in B(X_i, \frac{1}{10}r)$:

$$(I_2 + X)(I_2 - X_i) \in B(I_2, \frac{1}{2}r)$$

This means that for every $Y \in (\mathcal{U}_1'')^p$: $Y(I_2 - X_i) \in B(I_2, \frac{1}{2}r)$.

□

Now, it is clear it is possible to take a sufficiently small balls $V^0, V^1, ..., V^n$ centered at $0, -X_1, ..., -X_n$ such that :
- If $M \in \mathcal{U}_1$, then for every $M_0 \in V^0$, we have : $M(I_2 + H_0) \in \mathcal{U}_1'$
- If $M \in \mathcal{U}_1'$, then there exists $1 \leq p \leq n$ such that $M \in (\mathcal{U}_1')^p$ and for every $M_0 \in V^0$, we have : $M(I_2 + H_0) \in (\mathcal{U}_1'')^p$.

The previous lemma implies that if $M \in (\mathcal{U}_1')^p$ and $M_0 \in V^0$ are such that $M(I_2 + M_0) \in (\mathcal{U}_1'')^p$, then for every $M_p \in V^p$, we have that : $M(I_2 + M_0)(I_2 + M_p) \in \mathcal{U}_1$. Then, properties (i) and (ii) are verified for $x = 1$ and $U = I_2$. For each $0 < x < 1$, let us take the balls $x \cdot V^0, x \cdot V^1, ..., x \cdot V^n \subset \mathrm{Mat}_2(\mathbb{C})$ and let us apply the homothety of factor x of center I_2 to the sets $\mathcal{U}_1, \mathcal{U}_1', \mathcal{U}_1''$, $(\mathcal{U}_1')^p$ and $(\mathcal{U}_1'')^p$ to get the sets $\mathcal{U}_x, \mathcal{U}_x'$, $\mathcal{U}_x''$, $(\mathcal{U}_x')^p$ and $(\mathcal{U}_x'')^p$ such that properties (i) and (ii) are verified for $x < 1$ and $U = I_2$.

Let us now suppose that $U \neq I_2$. The two inclusions $U^j M U(I_2 + M_0) \in U^{j+1} \cdot \mathcal{U}_x'$ and $U^j M U(I_2 + M_0) \in U^{j+1} \cdot (\mathcal{U}_x'')^p$ are still true by reducing V^0 a finite number of times if necessary. Let us take $p \leq n$ and $p' \leq n$ such that $(\mathcal{U}_1')^p \cdot U = U \cdot (\mathcal{U}_1')^{p'}$ and $M_{p'} \in V^{p'}$. Then :

$$U^j \cdot (\mathcal{U}_x'')^p \cdot U(I_2 + M_{p'}) = U^j \cdot U \cdot (\mathcal{U}_x'')^{p'} \cdot (I_2 + M_{p'})$$

$$= U^{j+1} \cdot (\mathcal{U}_x'')^{p'} \cdot (I_2 + M_{p'}) \subset U^{j+1} \cdot \mathcal{U}_x$$

This implies that for every $M_0 \in (x \cdot V^0)$ and for every $M_{p'} \in (x \cdot V^{p'})$, we have $U^j M U(I_2 + M_0) \in U^{j+1} \cdot (\mathcal{U}_x'')^p$ and $U^j M U(I_2 + M_0) U(I_2 + M_{p'}) \in U^{j+2} \cdot \mathcal{U}_x$, which concludes the proof of the result.

□

Let us point out the following obvious result for later reference.

Proposition 2.1.13. *For every $u \in (\mathbb{C}^2)^4$, there exists a positive number $x(u) > 0$ such that for every $0 < x < x(u)$, for every $M \in \mathcal{U}_x''$, then $M \cdot \overline{u}$ belongs to $\mathcal{N}(u)$.*

2.2 Quasi-linear model

Here we slightly perturb the linear maps we used before but we show we can keep results on persistent intersections. Let us recall that the integer n was defined in Proposition 2.1.10. Let us remind that $\mathcal{M} \subset \mathrm{Mat}_2(\mathbb{C})$ is a finite subgroup.

Definition 2.2.1. *Let f be a linear map defined on an open subset of $\mathbb{C}^2$. We say that f is of type (x,p) for any $0 \leq p \leq n$ if f can be written $f = \frac{1}{a_f}(A_f + h)$ with $a_f \in \mathbb{C}^*$, $A_f \in \mathcal{M}$ and $h \in x \cdot V^p$ (where V^p was defined in Proposition 2.1.10). The modulus $|a_f|$ is called the contraction factor of f.*

Let f be a smooth map defined on an open subset of $\mathbb{C}^2$. We say that f is quasi-linear of type (x,p) if $f = \tilde{f} + \frac{1}{a_f}\varepsilon_1$ with $\tilde{f} = \frac{1}{a_f}(A_f + h)$ linear of type (x,p) and we have :

$$||\varepsilon_1||_{C^2} < \frac{1}{1000} \cdot ||h||$$

The following can be seen as a consequence of Proposition 2.1.10 in the quasi-linear setting. Remember that $x(u) > 0$ was defined in Proposition 2.1.13.

Proposition 2.2.2. *Let $\mathcal{M} \subset \mathrm{Mat}_2(\mathbb{C})$ be a finite subgroup of unitary matrices. Reducing $x(u)$ if necessary, for every grid of balls $G = (u, o, N, r)$, for every quasi-linear map f of type (x,p) such that $x < x(u)$ and $2|a_f| \cdot size(G) \cdot ||f||_{C^2} < \frac{r}{2}$, there is a grid of balls $G' = (u', o', N, r')$ included inside $f(G)$ with $\overline{u'} = (Df)_o \cdot \overline{u}$ and :*

$$r' = r - 2|a_f| \cdot size(G) \cdot ||f||_{C^2}$$

Proof. We just have to take $o' = f(o)$, $\overline{u'} = (Df)_o \cdot \overline{u}$ with Df_o linear of type (x,p). Remind that by definition, the size of G is $\mathrm{size}(G) = 2N \cdot \max_{1 \leq i \leq 4} ||u_i||$. When $||f||_{C^2} = 0$, the image of G under f is a grid of balls G' of same relative size $r' = r$, each of ball of G' is the image of a ball of G under $f = \frac{1}{a_f}(A_f + h)$. Since A_f is unitary and h is of type (x,p), reducing $x(u) > 0$ (independently of f) if necessary, we have that the radius of a ball of G' is between $\frac{1}{2|a_f|}$ and $\frac{2}{|a_f|}$ times the radius of a ball of G.

If $||f||_{C^2} \neq 0$, the image of each ball of G under f still contains a ball of G' but this time by the Taylor formula there is an additive term smaller than $\mathrm{size}(G) \cdot ||f||_{C^2}$ in the differential of f. Then the relative size r' is such that :

$$r' \geq r - \frac{\mathrm{size}(G) \cdot ||f||_{C^2}}{\frac{1}{2|a_f|}} \geq r - 2|a_f| \cdot \mathrm{size}(G) \cdot ||f||_{C^2}$$

□

2.3 Intersecting a curve and the limit set of an IFS in $\mathbb{C}^2$

In this subsection, we give an abstract condition ensuring the existence of an intersection (robust by construction) between a holomorphic curve in $\mathbb{C}^2$ and the limit set of an IFS. This will be the model for robust bifurcations near Lattès maps. Remind that n was defined in Proposition 2.1.10, N, w, θ in Proposition 2.1.6. Remind that the middle part and the $\frac{3}{4}$-part of a ball were defined in Definition 2.1.4, the middle part and the hull of a grid of balls were defined in Definition 2.1.1. In the following, for a holomorphic map $\mathcal{G}$ defined on a (closed) ball $\mathfrak{B} \subset \mathbb{C}^2$, we will denote $||\mathcal{G}||_{C^2} = \max_{\mathfrak{B}} ||D^2\mathcal{G}||$.

Proposition 2.3.1. *Let $(\mathcal{G}_1, ..., \mathcal{G}_q)$ be a IFS given by q maps defined on a ball $\mathfrak{B} \subset \mathbb{C}^2$ of radius $R > 0$ satisfying the following properties :*

1. $\bigcup_{1\leq j\leq q}\mathcal{G}_j(\mathfrak{B})$ *contains a grid of balls* $G^1=(u^1,o^1,n_G,r^1)$ *with* $q=(2n_G+1)^4$ *such that each* $\mathcal{G}_j(\mathfrak{B})$ *contains a ball of* G^1
2. *the contraction factor of the IFS* $(\mathcal{G}_1,...,\mathcal{G}_q)$ *is* $|a|\geq 2$
3. *there exist* $(n+1)$ *balls* $\mathfrak{B}_0,\mathfrak{B}_1,...,\mathfrak{B}_n\subset\mathfrak{B}$ *of radius larger than* $\nu\cdot R$ *(*$0<\nu<1$*), such that the* $\frac{3}{4}$*-parts of* $\mathfrak{B}_0,\mathfrak{B}_1,...,\mathfrak{B}_n$ *are included in the hull of* G^1*, and satisfying the following property : for each* $1\leq j\leq q$ *such that* $\mathcal{G}_j(\mathfrak{B})\subset\mathfrak{B}_p$*,* $\mathcal{G}_j$ *is quasi-linear of type* (x,p) *with* $x<x(u^1)$ *and :* $\mathcal{G}_j=\tilde{g}_j+\frac{1}{a}\varepsilon_j=\frac{1}{a}(A+h_j)+\frac{1}{a}\varepsilon_j$ *with :* $||\varepsilon_j||_{C^2}<\frac{1}{1000}\cdot||h_j||$*. Moreover,* $\bigcup_{1\leq j\leq q}\mathcal{G}_j(\mathfrak{B}_p)$ *contains a grid of balls* $\Gamma_p^1=(u^1,o_p^1,n_G,s^1)$ *for each* $0\leq p\leq n$ *with* $s^1\geq\nu\cdot r^1$
4. $n_G>\frac{10}{\nu}\cdot N(\frac{\nu\cdot r^1}{10})$
5. $|a|\cdot R\cdot\max_{1\leq j\leq q}(||\mathcal{G}_j||_{C^2})<\frac{\nu\cdot r^1}{100}$

Let $\mathfrak{C}$ *be a* (θ,w)*-quasi-line of direction in* $C^{w,2\theta}$ *such that* $\mathfrak{C}$ *intersects the middle part of* G^1*.*

Then $\mathfrak{C}$ *intersects the limit set of the IFS* $(\mathcal{G}_1,...,\mathcal{G}_q)$*.*

When the conditions 1 to 5 are satisfied, we summarize them by saying that $(\mathcal{G}_1,...,\mathcal{G}_q)$ is a correcting IFS. The proposition will be the immediate consequence of the following lemma :

Lemma 2.3.2. *There exist* $(n+2)$ *sequences of grids* $(G^j)_{j\geq1}=(u^j,o^j,n_G,r^j)_{j\geq1}$ *and* $(\Gamma_p^j)_{j\geq1}=(u^j,o_p^j,n_G,s^j)_{j\geq1}$ *with* $0\leq p\leq n$ *such that we have the following properties :*

1. *For every* $j>1$*,* G^j *is included inside a ball of* G^{j-1} *and for every* $j>1$*, there are* $i_1,...,i_{j-1}\leq q$ *such that :* $G^j\subset(\mathcal{G}_{i_1}\circ...\circ\mathcal{G}_{i_{j-1}})(G^1)$
2. *For every* $j>1$*,* $0\leq p\leq n$ *:* $\Gamma_p^j\subset(\mathcal{G}_{i_1}\circ...\circ\mathcal{G}_{i_{j-1}})(\Gamma_p^1)$
3. *For every* $j\geq1$*,* $D(\mathcal{G}_{i_1}\circ...\circ\mathcal{G}_{i_{j-1}})_{o^1}\in A^{j-1}\cdot\mathcal{U}_x''$ *and for* $j\geq2$ *:*

$$r^j\geq r^1-2|a|\cdot R\cdot\max_{1\leq j\leq q}(||\mathcal{G}_j||_{C^2})\sum_{l\geq0}^{j-2}\frac{1}{|a|^l}\ \text{ and }\ s^j\geq s^1-2|a|\cdot R\cdot\max_{1\leq j\leq q}(||\mathcal{G}_j||_{C^2})\sum_{l\geq0}^{j-2}\frac{1}{|a|^l}$$

4. *For every* $j>1$*, there exists* $1\leq p_j\leq n$ *such that the quasi-line* $\mathfrak{C}$ *intersects the middle part of a ball of* $\Gamma_{p_j}^j$

Proof. The proof of this lemma is based on an induction procedure. We begin by an initialisation called Case 0 where we pick the grids of balls at the first level G^1 and Γ_p^1 for $0\leq p\leq n$. We intersect for the first time the quasi-line $\mathfrak{C}$ with a ball and we construct the grids at the second level. Case 0 is somewhat different from the rest of the demonstration because we do not not control the initial position of $\mathfrak{C}$. Then, Case 1 has to be thought as the most frequent situation : $\mathfrak{C}$ intersects a grid of balls whose geometry is good enough, and we can intersect $\mathfrak{C}$ with a new grid whose geometry is very close to the previous one. Then, it may happen a time when the geometry of this grid is too deformed. Then we apply a "correction" (Cases 2 and 3), which leads back to Case 1.

Case 0 : initialization

By hypothesis, $\bigcup_{1\leq j\leq q}\mathcal{G}_j(\mathfrak{B})$ contains a grid of balls $G^1=(u^1,o^1,n_G,r^1)$ and similarly $\bigcup_{1\leq j\leq q}\mathcal{G}_j(\mathfrak{B}_p)$ contains a grid of balls $\Gamma_p^1=(u^1,o_p^1,n_G,s^1)$ for each $0\leq p\leq n$ with $s^1\geq\nu\cdot r^1$. So, for the first step $j=1$, the $(n+2)$ grids of balls are already constructed.

By Corollary 2.1.8, $\mathfrak{C}$ intersects in its middle part a ball of Γ_0^1 : indeed, Γ_0^1 is a grid of balls such that $\overline{u^1} \in A \cdot \mathcal{U}_x \subset A \cdot \mathcal{U}_x''$, we have $s^1 \geq \nu \cdot r^1 > \frac{\nu \cdot r^1}{10}$ and $n_G > \frac{10}{\nu} \cdot N(\frac{\nu \cdot r^1}{10}) > N(\frac{\nu \cdot r^1}{10})$ (beware that the matrix A corresponds to the matrix denoted by U in Corollary 2.1.8). Then it intersects the middle of the ball $\mathcal{G}_{i_1}(\mathfrak{B})$ of G^1 which contains this ball of Γ_0^1. According to Proposition 2.2.2, there exists a grid of balls $G^2 = (u^2, o^2, n_G, r^2)$ included in $\mathcal{G}_{i_1}(G^1)$. We have $\overline{u^2} \in A \cdot \mathcal{N}(u^1)$ and $r^2 \geq r^1 - 2|a| \cdot \text{size}(G^1) \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2}) \geq r^1 - 2|a| \cdot R \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2}) > \frac{\nu \cdot r^1}{10}$. Applying Proposition 2.2.2 to the grids of balls Γ_p^1 $(0 \leq p \leq n)$, there exist $(n+1)$ grids of balls $\Gamma_p^2 = (u^2, o_p^2, n_G, s^2)$ included in $\mathcal{G}_{i_1}(\Gamma_p^1)$ for $0 \leq p \leq n$. We have : $\overline{u^2} \in A \cdot \mathcal{N}(u^1)$ and $s^2 \geq s^1 - 2|a| \cdot \text{size}(\Gamma_p^1) \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2}) \geq s^1 - 2|a| \cdot R \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2}) > \frac{\nu \cdot r^1}{10}$.

Let us now suppose by induction that the $(n+2)$ sequences of grids of balls satisfying (1),(2),(3) and (4) are constructed up to step j with the additional properties that $\mathfrak{C}$ intersects in its middle part a ball of $\Gamma_{p_{j-1}}^{j-1}$ and that the following property is verified :

(Q) For every i such that $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}} \circ \mathcal{G}_i)(G^1) \subset \Gamma_{p_{j-1}}^{j-1}$, we have :

$$D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}} \circ \mathcal{G}_i)_{o^1} \in A^j \cdot \mathcal{U}_x''$$

Let us construct the grids of balls at the next step. The proof is inductive, at each step of the proof we are in one of the three cases we are going to discuss, which differ by two parameters. We have a quasi-line intersecting a grid of balls and we have to make a different choice to intersect a ball corresponding to one of the $(n+1)$ types of differentials we introduced earlier. Note that after Case 0, we will necessarily be in Case 1.

Case 1 : $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}})_{o^1} \in A^{j-1} \cdot \mathcal{U}_x$ and $p_{j-1} = 0$

By construction, $\mathfrak{C}$ intersects in its middle part a ball B_0^{j-1} of the grid of balls Γ_0^{j-1}. Since $\Gamma_0^j = (u^j, o_0^j, n_G, s^j)$ is a grid of balls such that $\Gamma_0^j \subset (\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}})(\Gamma_0^1)$ and $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}})_{o^1} \in A^{j-1} \cdot \mathcal{U}_x$, we have according to Proposition 2.1.13 that $\overline{u^j} \in A^{j-1} \cdot \mathcal{N}(u^1)$. The relative size of $\mathfrak{B}_0$ compared to $\mathfrak{B}$ is equal to ν, the $\frac{3}{4}$-part of $\mathfrak{B}_0$ is included in the hull of G^1 and $n_G > \frac{10}{\nu} \cdot N(\frac{\nu \cdot r^1}{10})$. Then it is possible to take an union of balls of Γ_0^j included in B_0^{j-1} which form a grid of balls Γ' of basis u^j, relative size s^j and with $(\frac{1}{10}\nu \cdot n_G)^4$ balls. By construction, we can take Γ' such that $\mathfrak{C}$ intersects the middle part of Γ'. Since $\overline{u^j} \in A^{j-1} \cdot \mathcal{N}(u^1)$, $s^j > \frac{\nu \cdot r^1}{10}$ and $\frac{1}{10}\nu \cdot n_G > N(\frac{\nu \cdot r^1}{10})$ we have according to Corollary 2.1.8 that $\mathfrak{C}$ intersects in its middle part a ball of Γ_0^j. This ball is included inside $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})(\mathfrak{B})$ with $\mathcal{G}_{i_j}$ quasi-linear of type 0.

In particular, $\mathfrak{C}$ intersects the middle part of a grid of balls $G^{j+1} \subset (\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})(G^1)$. According to Propositions 2.1.10, 2.2.2 and Property (Q), G^{j+1} is a grid of balls $G^{j+1} = (u^{j+1}, o^{j+1}, n_G, r^{j+1})$ with $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot \mathcal{U}_x'$ and :

$$r^{j+1} \geq r^j - 2|a| \cdot \text{size}(G^j) \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2}) \geq r^j - 2|a| \cdot \frac{R}{|a|^{j-1}} \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2})$$

$$r^{j+1} \geq r^1 - 2|a| \cdot R \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2}) \sum_{l \geq 1}^{j-2} \frac{1}{|a|^l} > \frac{\nu \cdot r^1}{10}$$

Still according to Propositions 2.1.10, 2.2.2 and Property (Q), there exist $(n+1)$ grids of balls Γ_p^{j+1} (for $0 \leq p \leq n$) included in $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})(\Gamma_p^1)$ such that :

$$s^{j+1} \geq s^j - 2|a| \cdot \text{size}(\Gamma_p^j) \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2}) \geq s^j - 2|a| \cdot \frac{R}{|a|^{j-1}} \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j||_{C^2})$$

$$s^{j+1} \geq s^1 - 2|a| \cdot R \cdot \max_{1\leq j\leq q}(||\mathcal{G}_j||_{C^2}) \sum_{l\geq 1}^{j-2} \frac{1}{|a|^l} > \frac{\nu \cdot r^1}{10}$$

The grids of balls G^{j+1} and Γ_p^{j+1} (for $0 \leq p \leq n$) satisfy (1),(2),(3),(4). In particular, $\mathfrak{C}$ intersects in its middle part a ball of Γ_0^j.

Since $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot \mathcal{U}'_x$ and $p_j = 0$, by Proposition 2.1.10, if $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j} \circ \mathcal{G}_i)(G^1) \subset \Gamma_{p_j}^j$ for some i, then $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j} \circ \mathcal{G}_i)(G^1)$ is a grid of balls $(u^{j,i}, o^{j,i}, n_G, r^{j,i})_j$ such that $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j} \circ \mathcal{G}_i)_{o^1} \in A^{j+1} \cdot \mathcal{U}''_x$, this means that (Q) is verified.

Then, after Case 1 and according to Proposition 2.1.10, only two cases can occur. If $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot \mathcal{U}_x$ and we can apply Case 1 once again. If $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot (\mathcal{U}'_x - (\mathcal{U}'_x \cap \mathcal{U}_x))$, there exists $1 \leq p \leq n$ such that $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot (\mathcal{U}')^p$. In this case, we are going to "correct" the next grids in a procedure given by Cases 2 and 3.

Case 2: $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}})_{o^1} \in A^{j-1} \cdot (\mathcal{U}'_x)^p$ and $p_{j-1} = 0$

By construction, $\mathfrak{C}$ intersects in its middle part a ball of the grid of balls Γ_0^{j-1}. We have according to Proposition 2.1.13 that $\overline{u^j} \in A^{j-1} \cdot \mathcal{N}(u^1)$. Then, using the same argument as in Case 1, we have according to Corollary 2.1.8 that $\mathfrak{C}$ intersects in its middle part a ball of $\Gamma_{p_j}^j$ included inside $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}})(\mathfrak{B})$ where $\mathcal{G}_{i_{j-1}}$ is quasi-linear of type 0 and $p_j = p'$ is chosen according to Proposition 2.1.10. In particular, $\mathfrak{C}$ intersects the middle part of a grid of balls $G^{j+1} \subset (g_{i_1} \circ ... \circ g_{i_j})(G^1)$. According to Propositions 2.1.10 and 2.2.2, G^{j+1} is a grid of balls $G^{j+1} = (u^{j+1}, o^{j+1}, n_G, r^{j+1})$ with $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot (\mathcal{U}''_x)^p$, there exist $(n+1)$ grids of balls Γ_p^{j+1} (for $0 \leq p \leq n$) included in $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})(\Gamma_p^1)$ and r^{j+1}, s^{j+1} satisfy the inequalities of property 3. The grids of balls G^{j+1} and Γ_p^{j+1} (for $0 \leq p \leq n$) satisfy (1),(2),(3),(4).

Since $|a| \cdot R \cdot \max_{1\leq j\leq q}(||\mathcal{G}_j||_{C^2}) < \frac{\nu \cdot r^1}{100}$ we have for every $0 \leq p \leq n$, for every $j \geq 1$ the following bounds : $r^j, s_p^j > \frac{\nu \cdot r^1}{10}$. Since $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot (\mathcal{U}'_x)^p$ and $p_j = p'$ is chosen according to Proposition 2.1.10 (see Proposition 2.1.10 for the definition of p'), we have for every i such that $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j} \circ g_i)(G^1) \subset \Gamma_{p_j}^j$ that $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j} \circ \mathcal{G}_i)(G^1)$ is a grid of balls $(u^{j,i}, o^{j,i}, n_G, r^{j,i})_j$ such that $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j} \circ \mathcal{G}_i)_{o^1} \in A^{j+1} \cdot \mathcal{U}_x$, this means that (Q) is verified.

After Case 2, it follows from Proposition 2.1.10 that necessarily the two conditions of the following Case 3 are satisfied.

Case 3: $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}})_{o^1} \in A^{j-1} \cdot (\mathcal{U}''_x)^p$ and $p_{j-1} \neq 0$

Induction shows that p_{j-1} had been chosen to get special composition properties (see Case 2, beware that the number denoted here by p_{j-1} corresponds to the number denoted by "p_j" in Case 2), let us pick $p_j = 0$. By construction, $\mathfrak{C}$ intersects in its middle part a ball of the grid of balls $\Gamma_{p_{j-1}}^{j-1}$. Once again : $\overline{u^j} \in A^{j-1} \cdot \mathcal{N}(u^1)$ and we have according to Corollary 2.1.8 that $\mathfrak{C}$ intersects in its middle part the ball of Γ_0^j included inside $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})(\mathfrak{B})$ with $\mathcal{G}_{i_j}$ quasi-linear of type p_{j-1}. In particular, $\mathfrak{C}$ intersects the middle part of a grid of balls $G^{j+1} \subset (\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})(G^1)$. Once again, we can construct grids of balls G^{j+1} and Γ_p^{j+1} (for $0 \leq p \leq n$) which satisfy (1),(2),(3),(4) but this time with $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot \mathcal{U}_x$. In particular, $\mathfrak{C}$ intersects in its middle part a ball of $\Gamma_{p_j}^j$. Moreover, Proposition 2.1.10 still insures that (Q) is verified.

Since $D(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})_{o^1} \in A^j \cdot \mathcal{U}_x$, we are now in Case 1 once again.

□

Proof of Proposition 2.3.1. According to the previous lemma, for each $j \geq 1$, $\mathfrak{C}$ intersects $\Gamma^j_{p_j} \subset (\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_{j-1}})(\Gamma^1_{p_j})$. But $\Gamma^1_{p_j} \subset \bigcup_{1 \leq j \leq q} \mathcal{G}_j(\mathfrak{B}_{p_j}) \subset \bigcup_{1 \leq j \leq q} \mathcal{G}_j(\mathfrak{B})$. This shows that for each $j \geq 1$, there exist $q \geq i_1, ..., i_j \geq 1$ such that $\mathfrak{C}$ intersects $(\mathcal{G}_{i_1} \circ ... \circ \mathcal{G}_{i_j})(\mathfrak{B})$. This implies that $\mathfrak{C}$ intersects the limit set of the IFS $(\mathcal{G}_1, ..., \mathcal{G}_q)$. □

3 Preliminaries on Lattès maps

3.1 Definitions

Definition 3.1.1. *A Lattès map is a holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$ which is semi-conjugate to an affine map on the torus. For such a map, we have the following commutative diagram :*

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\mathcal{L}} & \mathbb{T} \\ \Pi \downarrow & & \downarrow \Pi \\ \mathbb{P}^2(\mathbb{C}) & \xrightarrow[L]{} & \mathbb{P}^2(\mathbb{C}) \end{array}$$

where $\mathbb{T}$ is a complex torus of dimension 2, Π is a ramified covering of the projective space $\mathbb{P}^2(\mathbb{C})$ by the torus $\mathbb{T}$ and $\mathcal{L}$ is an affine map.

Proposition 3.1.2. *The periodic points of any Lattès map are dense in $\mathbb{P}^2(\mathbb{C})$. The Julia set of any Lattès map is equal to $\mathbb{P}^2(\mathbb{C})$.*

Notation 3.1.3. *In the following, for every $\tau \in \mathbb{C}$ such that $Im(\tau) > 0$, we will denote $L(\tau)$ the lattice in $\mathbb{C}$ given by : $L(\tau) = \mathbb{Z} + \tau \cdot \mathbb{Z}$ and by $L^2(\tau)$ the associated product lattice $L^2(\tau) = L(\tau) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + L(\tau) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We also let $\xi = e^{i\frac{2\pi}{6}}$.*

The following proposition can be found in [13].

Proposition 3.1.4. *If an affine map on a torus $\mathbb{T}$ induces a Lattès map L on $\mathbb{P}^2(\mathbb{C})$, then the torus $\mathbb{T}$ is of the form $\mathbb{C}^2/\Lambda$ where Λ is one of the six following lattices and the projection $\Pi : \mathbb{T} \to \mathbb{P}^2(\mathbb{C})$ is given (in some affine chart for Cases 1,2,3,4) by the following formulas :*

Case 1 $\Lambda = L^2(\tau), (x, y) \mapsto [\wp(x) + \wp(y) : \wp(x)\wp(y) : 1]$

Case 2 $\Lambda = L^2(\xi), (x, y) \mapsto [\wp'(x) + \wp'(y) : \wp'(x)\wp'(y) : 1]$

Case 3 $\Lambda = L^2(i), (x, y) \mapsto [\wp^2(x) + \wp^2(y) : \wp^2(x)\wp^2(y) : 1]$

Case 4 $\Lambda = L^2(\xi), (x, y) \mapsto [(\wp')^2(x) + (\wp')^2(y) : (\wp')^2(x)(\wp')^2(y) : 1]$

Case 5 $\Lambda = L^2(i), (x, y) \mapsto [(\wp(x)\wp(y) + e_1^2)^2 : (\wp(x) + \wp(y))^2 : (\wp(x)\wp(y) - e_1^2)^2]$

Case 6 $\Lambda = L(\tau) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + L(\tau) \cdot \begin{pmatrix} \xi^2 \\ \xi \end{pmatrix}, (x, y) \mapsto [\wp'(x_1) - \wp'(y_1) : \wp(x_1) - \wp(y_1) : \wp'(x_1)\wp(y_1) - \wp(x_1)\wp'(y_1)]$

where $\tau \in \mathbb{H}$, $e_1 = \wp(\frac{1}{2})$ and (x_1, y_1) is the function of (x, y) given by :

$$x_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + y_1 \begin{pmatrix} \xi^2 \\ \xi \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

In the following, we will denote π the projection from $\mathbb{C}^2$ to $\mathbb{T}^2 = \mathbb{C}^2/\Lambda$.

Definition 3.1.5. *A Ueda map is a holomorphic map on $\mathbb{P}^2(\mathbb{C})$ such that there exists a Lattès example $\tilde{L}$ on $\mathbb{P}^1(\mathbb{C})$ such that we have :*

$$L \circ \eta = \eta \circ (\tilde{L}, \tilde{L})$$

where η is the map between $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^2(\mathbb{C})$ which is just the projectivization of $(x, y) \mapsto (x + y, xy)$, given by :

$$\eta : ([x : x'], [y : y']) \mapsto [xy' + x'y : xy : x'y']$$

Such a map L is semi-conjugate to an affine map on the complex torus $\mathbb{T}$ and is a Lattès map.

Lattès maps corresponding to Cases 1,2,3 and 4 of Proposition 3.1.4 are Ueda maps.

The following is an easy consequence of Propositions 3.1 to 3.6 of [11].

Proposition 3.1.6. *Let Λ, Π be one of the lattices and associated coverings defined in Proposition 3.1.4. There exists a finite group of unitary matrices $G_{Lattès} = G_{Lattès}(\Lambda, \Pi)$ of finite order such that every Lattès map has its linear part of the form aA where $a \in \mathbb{C}^*$, $|a| \geq 1$ and $A \in G_{Lattès}$.*

Remark. *Here, the scaling factor a takes discrete values. Moreover, arbitrarily large values of $|a|$ can be obtained (it can be easily seen by taking the composition of a Lattès map with itself). The equality of the two topological degrees gives : $(d')^2 = |a|^4 \cdot |det(A)|^2$ where d' is the algebraic degree of L.*

Since according to the previous result, there are only finitely many possible linear parts A for a Lattès map (up to multiplication by the factor a) which are all of finite order, we can define the following integer.

Definition 3.1.7. *We denote by $ord_{Lattès}$ the product of all the orders of the possible linear parts A for a Lattès map.*

It can be found in [11] that $\mathrm{ord}_{\mathrm{Lattès}}$ is equal to $6^2 \cdot 8^2 \cdot 12 \cdot 24$. In a first reading, we encourage the reader to consider only the case where the linear part of the Lattès map is equal to Id. In the other cases, the dynamical ideas are the same but with a few additional technicalities from algebra. In particular, it is sufficient in order to prove in some cases the corollary of the main result (see subsection 1.2).

3.2 An algebraic property of Lattès maps

The goal of this subsection is to prove the following result.

Proposition 3.2.1. *For every torus $\mathbb{T}$, there exists an integer $i = i(\mathbb{T}^2)$ such that for any $k > 0$, there exists an integer $d_k > 0$ such that for any Lattès map L of algebraic degree $d > d_k$, coming from an affine map on $\mathbb{T}$, there exists a homogenous change of coordinates φ on $\mathbb{P}^2(\mathbb{C})$ such that : $\varphi^{-1} \circ L \circ \varphi$ is a holomorphic endomorphism of $\mathbb{P}^2(\mathbb{C})$ of the form $[\overline{P}_1 : \overline{P}_2 : \overline{P}_3]$ where the polynomial $\overline{P}_3$ is a product of irreducible factors $\overline{P}_{3,j}$ such that at least k factors $\overline{P}_{3,j}$ are of degree bounded by i.*

Definition 3.2.2. *Let v be a vector of $\mathbb{C}^2$ which belongs to a lattice Λ and $v_0 \in \mathbb{C}^2$. We suppose that the action of Λ upon $\mathbb{C} \cdot v$ by translation is cocompact. Let $\mathbb{T}^2 = \mathbb{C}^2/\Lambda$ and $\pi : \mathbb{C}^2 \to \mathbb{T}^2$ be the natural projection. Then, then we say that $\pi(v_0 + \mathbb{C} \cdot v)$ is a compact line of the torus $\mathbb{T}$ of direction v. It is compact and $\pi(v_0 + \mathbb{C} \cdot v) \simeq \pi(v_0) + \mathbb{C}/\Lambda' \cdot v$ for some subgroup $\Lambda' \subset \Lambda$. The family of compact lines of the torus $\mathbb{T}$ of direction v is the family of all the compact lines of the torus of direction v obtained by varying v_0.*

Let us point out the fact that $v \in \Lambda$ is not sufficient to conclude that the action of Λ upon $\mathbb{C} \cdot v$ by translation is cocompact.

Proposition 3.2.3. *Let Λ, Π be one of the lattices and associated coverings defined in Proposition 3.1.4. Let v be a vector of $\mathbb{C}^2$ which belongs to Λ such that the action of Λ upon $\mathbb{C} \cdot v$ by translation is cocompact. The family of images under Π of compact lines of direction v on the torus $\mathbb{T}$ is a family of algebraic curves of $\mathbb{P}^2(\mathbb{C})$ of degree bounded by $i = i(v, \mathbb{T}^2)$.*

Proof. Let $\mathcal{F}$ be the family of images of compact lines of direction v on the torus $\mathbb{T}$ under Π. The family $\mathcal{F}$ is a holomorphic compact family of compact curves so that by the GAGA principle it is an algebraic family of curves and in particular their degree is bounded by some $i = i(v, \mathbb{T}^2)$ □

Proposition 3.2.4. *Let Λ, Π be one of the lattices and associated coverings defined in Proposition 3.1.4. Then, there exists a line δ in $\mathbb{P}^2(\mathbb{C})$ such that $\Pi^{-1}(\delta)$ contains at least one compact line $\mathcal{D}$ of $\mathbb{T}$.*

Proof. In each case, the following compact lines are convenient for δ and we give the preimage compact lines $\mathcal{D}$. The first four cases cover the case of a Ueda map.

Case 1 : $\delta = \{Y = 0\}$ Indeed, $Y = 0$ if and only if $\wp(x)\wp(y) = 0$. $\Pi^{-1}(\{Y = 0\})$ is an union of compact lines of the torus of the form $\{x_0\} \times \mathbb{T}^1$ and $\mathbb{T}^1 \times \{y_0\}$ where the x_0, y_0 are in $\wp^{-1}(\{0\})$.

Case 2: $\delta = \{Y = 0\}$ Indeed, $Y = 0$ if and only if $\wp'(x)\wp'(y) = 0$. $\Pi^{-1}(\{Y = 0\})$ is an union of compact lines of the torus of the form $\{x_0'\} \times \mathbb{T}^1$ and $\mathbb{T}^1 \times \{y_0'\}$ where the x_0', y_0' are in $(\wp')^{-1}(\{0\})$.

Case 3: $\delta = \{Y = 0\}$ Indeed, $Y = 0$ if and only if $\wp^2(x)\wp^2(y) = 0$. $\Pi^{-1}(\{Y = 0\})$ is an union of compact lines of the torus of the form $\{x_0\} \times \mathbb{T}^1$ and $\mathbb{T}^1 \times \{y_0\}$ where the x_0, y_0 are in $\wp^{-1}(\{0\})$.

Case 4: $\delta = \{Y = 0\}$ Indeed, $Y = 0$ if and only if $(\wp'(x))^2(\wp'(y))^2 = 0$. $\Pi^{-1}(\{Y = 0\})$ is an union of compact lines of the torus of the form $\{x_0'\} \times \mathbb{T}^1$ and $\mathbb{T}^1 \times \{y_0'\}$ where the x_0', y_0' are in $(\wp')^{-1}(\{0\})$.

Case 5: $\delta = \{X = Z\}$ Indeed, $X = Z$ if and only if $(\frac{\wp(x)\wp(y)+e_1^2}{\wp(x)\wp(y)-e_1^2})^2 = 1$, this means if and only if $4e_1^2\wp(x)\wp(y) = 0$. $\Pi^{-1}(\{X = Z\})$ is an union of compact lines of the torus of the form $\{x_0\} \times \mathbb{T}^1$ and $\mathbb{T}^1 \times \{y_0\}$ where the x_0, y_0 are in $\wp^{-1}(\{0\})$.

Case 6: $\delta = \{Z = 0\}$ Indeed, $Z = 0$ if and only if $\wp'(x_1)\wp(y_1) - \wp(x_1)\wp'(y_1) = 0$. $\Pi^{-1}(\{Z = 0\})$ contains the compact line of the torus $\{x_1 = y_1\}$ (in the coordinates x_1, y_1).

In all the cases, the preimage of δ by Π contains a compact line of the torus.

□

Proposition 3.2.5. *If an affine map $\mathcal{L}$ of linear part aA on a torus $\mathbb{T}$ induces a Lattès map L on $\mathbb{P}^2(\mathbb{C})$ and $\mathcal{D}$ is the preimage under Π of the compact line δ given by Proposition 3.2.4, then the preimage of $\mathcal{D}$ under $\mathcal{L}$ is a finite union of compact lines of the torus. Moreover, the number of possible directions is finite. For each $k > 0$, there exists $d_k > 0$ such that for every Lattès map L of algebraic degree greater than d_k induced by an affine map $\mathcal{L}$ on $\mathbb{T}$, there exist at least k distinct irreducible components of degree bounded by i which are preimages of δ by L.*

Proof. From Proposition 3.1.6, we know that the linear part of $\mathcal{L}$ is of the form aA with $A \in G_{\text{Lattès}}$. Let $v \in \Lambda$ be the direction of $\mathcal{L}$. Since $(aA)^{-1}(v) \in \Lambda$ (because $\mathcal{L}$ induces a Lattès map), any preimage of $\mathcal{D}$ by $\mathcal{L}$ is a compact line of the torus. $G_{\text{Lattès}}$ is finite and so the possible number of directions is finite. Let $\mathcal{D}''$ be a preimage of $\mathcal{D}$ by $\mathcal{L}$.

We have the following straightforward property : for every lines Δ_1, Δ_2 of direction v in $\mathbb{C}^2$, if $\pi(\Delta_1) = \pi(\Delta_2)$, then every two points respectively in Δ_1 and Δ_2 are joined by a vector which lies in $(0,1) \cdot v + (0,1) \cdot i \cdot v + \Lambda$. Let us take $\lambda_1, \lambda_2 \in \Lambda$ non $\mathbb{R}$-colinear such that $A^{-1}\lambda_1$ and $A^{-1}\lambda_2$ are not colinear to $\mathcal{D}''$. Then, there is some constant $a_k > 0$ such that if $|a| > a_k$, then at least $100k$ vectors in the set $\mathbb{Z} \cdot (aA)^{-1}(\lambda_1) + \mathbb{Z} \cdot (aA)^{-1}(\lambda_2)$ won't belong to $(0,1) \cdot v + (0,1) \cdot i \cdot v + \Lambda$ and then at least $100k$ images of $\mathcal{D}''$ by translations of vectors in $\mathbb{Z} \cdot (aA)^{-1}(\lambda_1) + \mathbb{Z} \cdot (aA)^{-1}(\lambda_2)$ are distinct preimages of $\mathcal{D}$ by $\mathcal{L}$.

Their images under Π are irreducible components of degree bounded by i. If $|a| > a_k$, at least k (this term k is not optimal and we get it by projection of the previous $100k$ lines) of them are distinct preimages of δ by L. But $|a| > a_k$ if $\deg(L)$ is superior to some value $d_{k,\Lambda,A}$. Then, it suffices to take for d_k the maximal value of $d_{k,\Lambda,A}$ when varying Λ and A in the finite sets they belong (see Propositions 3.1.4 and 3.1.6). □

Proof of Proposition 3.2.1. Let δ be a line in $\mathbb{P}^2(\mathbb{C})$ as in Proposition 3.2.4. The result is a consequence of Proposition 3.2.5 because after a suitable change of coordinates, we can take $\delta = \{Z = 0\}$. Then $\{\overline{P}_3 = 0\}$ contains at least the k irreducible components of degree bounded by i which are preimages of δ by L.

□

3.3 A periodic orbit in the postcritical set

Remind that the integer $\text{ord}_{\text{Lattès}}$ was defined in Definition 3.1.7. Beware that in the following, the period of a periodic point is the exact period.

Proposition 3.3.1. *There exists an integer $K > 0$ such that for every Lattès map L defined on $\mathbb{P}^2(\mathbb{C})$, there exists a point c in the critical curve of L which is sent after n_c iterations on a periodic point p_c of period n_{pc} such that :*

1. *$n_c + n_{pc} \leq K$*
2. *n_{p_c} is a multiple of $ord_{Lattès}$*

Proof. Let us start with the case of one dimensional Lattès maps.

Lemma 3.3.2. *Let $\tilde{L}$ be a one-dimensional Lattès map. There exists a critical point $\tilde{c}$ of L which is sent after $\tilde{n}_c \leq 12$ iterations on a periodic point $\tilde{p}_c$ of period $\tilde{n}_{pc} \leq 12$.*

Proof. The Lattes map $\tilde{L}$, according to Lemma 3.4 of [16], is such that the postcritical set $P_{\tilde{L}}$ of $\tilde{L}$ is entirely included inside the set of critical values of the covering Θ of $\mathbb{P}^1(\mathbb{C})$ by the complex torus $\mathbb{T}^1$. This implies that every critical point of $\tilde{L}$ is sent after one iteration inside the set of the critical values of Θ. Moreover, let us bound from above the number of critical values. This number c_r is bounded from above by the number of critical points (counted with multiplicity). Still according to [16], Θ can only be a covering of orders $\text{ord}(\Theta) = 2, 3, 4$ or 6. The Riemann-Hürwitz formula gives us that : $\chi(\mathbb{T}^1) = \text{ord}(\Theta)\chi(\mathbb{P}^1(\mathbb{C})) - c_r$ which implies $c_r = 2 \cdot \text{ord}(\Theta)$. In particular, this means that the image of every critical point $\tilde{c}$ of $\tilde{L}$ is sent after $\tilde{n}_c \leq 12$ iterations on a periodic point $\tilde{p}_c$ of period $\tilde{n}_{pc} \leq 12$. □

Lemma 3.3.3. *Let L be a Ueda map. There exists a point c in the critical curve of L which is sent after $n_c \leq 12$ iterations on a periodic point p_c of period $n_{pc} \leq 24 \cdot ord_{Lattès}$ which is a multiple of $ord_{Lattès}$. In particular, we have : $n_c + n_{pc} \leq 12 + 24 \cdot ord_{Lattès}$.*

Proof. We take a critical point $\tilde{c}$ of $\tilde{L}$ given by the previous lemma. We take a periodic point $\tilde{p}$ of $\tilde{L}$ of period $2 \cdot \mathrm{ord}_{\mathrm{Lattès}}$ (it can be found in [12] that such a point actually exists because any rational map on $\mathbb{P}^1(\mathbb{C})$ of degree greater than 2 has a point of strict period $2 \cdot \mathrm{ord}_{\mathrm{Lattès}} > 4$). Then the point $c = \eta(\tilde{c}, \tilde{p})$ is a critical point of L. It is sent after $n_c \leq 12$ iterations on a periodic orbit $\eta(\tilde{p}_c, \tilde{L}^{n_c}(\tilde{p}))$. The period of $\tilde{p}_c$ is $\tilde{n}_{pc} \leq 12$ and $\tilde{p}$ is of period $2 \cdot \mathrm{ord}_{\mathrm{Lattès}}$. This implies that in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, the periodic point $(\tilde{p}_c, \tilde{L}^{n_c}(\tilde{p}))$ for $(\tilde{L}, \tilde{L})$ is of period a multiple of $2 \cdot \mathrm{ord}_{\mathrm{Lattès}}$ bounded by $24 \cdot \mathrm{ord}_{\mathrm{Lattès}}$. Since the map η is a two-covering, in $\mathbb{P}^2(\mathbb{C})$, the periodic point $\eta(\tilde{p}_c, \tilde{L}^{n_c}(\tilde{p}))$ for L is of period n_{pc} which is a multiple of $\mathrm{ord}_{\mathrm{Lattès}}$ bounded by $24 \cdot \mathrm{ord}_{\mathrm{Lattès}}$. Then $n_c + n_{pc} \leq 12 + 24 \cdot \mathrm{ord}_{\mathrm{Lattès}}$. □

Let us now prove 3.3.1. It was shown in [11] (Theorems 4.2 and 4.4) that :

1. Either one map in $\{L, L^2, L^3\}$ is a Ueda map. In this first case, the previous lemma shows that one of the maps in $\{L, L^2, L^3\}$ has a point of its critical curve which is sent after at most 12 iterations onto a periodic orbit of period a multiple of $\mathrm{ord}_{\mathrm{Lattès}}$ bounded by $24 \cdot \mathrm{ord}_{\mathrm{Lattès}}$. This implies that there exists a critical point of L which is sent after n_c iterations onto a periodic orbit of period n_{pc} which is a multiple of $\mathrm{ord}_{\mathrm{Lattès}}$ with $n_c + n_{pc} \leq 3 \cdot (12 + 24 \cdot \mathrm{ord}_{\mathrm{Lattès}})$.
2. One of the maps L^k in $\{L, L^2, L^3, L^6\}$ is preserving an algebraic web associated to a smooth cubic (see [8] for this notion). This implies (see the remark after Theorem A in [8]) that the critical set of L^k is sent after one iteration into the set of critical values of Π which is a curve $\mathcal{PC}$. In this second case, we have that $L^k(\mathcal{PC}) \subset \mathcal{PC}$ and L^k induces by restriction a map on $\mathcal{PC}$. Taking the normalization of $\mathcal{PC}$ if necessary, we can suppose that $\mathcal{PC}$ is regular. There are two possibilities. Either $\mathcal{PC}$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ and L^k induces a rational map so it has a periodic point of period $\mathrm{ord}_{\mathrm{Lattès}}$(again, it can be found in [12] that such a point actually exists). Either $\mathcal{PC}$ is isomorphic to a complex torus and L^k induces a multiplication on this torus which also has a periodic point of period $\mathrm{ord}_{\mathrm{Lattès}}$. In both cases, we see that L has a critical point which is sent after at most 6 iterations on a point of period a multiple of $\mathrm{ord}_{\mathrm{Lattès}}$ bounded by $6 \cdot \mathrm{ord}_{\mathrm{Lattès}}$.

Then, taking $K = \max(3 \cdot (12 + 24 \cdot \mathrm{ord}_{\mathrm{Lattès}}), 6 + 6 \cdot \mathrm{ord}_{\mathrm{Lattès}})$, the proof of the proposition is done.

□

4 Perturbations of Lattès maps

4.1 Some useful lemmas

In this subsection, we prove two lemmas about complex analysis. The constants which are involved in these lemmas will be fixed in the two next subsections.

Lemma 4.1.1. *For every $m > 0$, for every ball $\tilde{\mathbb{B}}$, for every $1 > f_1 > 0$, $1 > f_2 > 0$, there exist constants $\rho = \rho(m, \tilde{\mathbb{B}}) > 0$, $\sigma = \sigma(m, \tilde{\mathbb{B}}) > 0$ such that for rational function h of degree equal to m, there exists a ball $\mathbb{B} \subset \tilde{\mathbb{B}} \subset \mathbb{C}^2$ of radius larger than ρ such that*

$$\forall z \in \mathbb{B}, \frac{||Dh(z)||}{|h(z)|} \leq \sigma \tag{1}$$

$$\forall (z, z') \in \mathbb{B}^2, \frac{|h(z)|}{|h(z')|} \leq 1 + f_1 \tag{2}$$

$$\forall (z, z') \in \mathbb{B}^2, \arg(h(z)) - \arg(h(z')) \leq f_2 \tag{3}$$

The proposition will be a consequence of the following lemma.

Lemma 4.1.2. *For every $m > 0$, for every ball $\tilde{\mathbb{B}}$, there exist constants $\overline{\rho} = \overline{\rho}(m, \tilde{\mathbb{B}}) > 0$, $\tau = \tau(m, \tilde{\mathbb{B}}) > 0$ such that for rational function h of degree m, there exists a ball $\overline{\mathbb{B}} \subset \tilde{\mathbb{B}} \subset \mathbb{C}^2$ of radius larger than $\overline{\rho}$ such that :*

$$\frac{inf_{\overline{\mathbb{B}}}|h|}{sup_{\overline{\mathbb{B}}}|h|} \geq \tau$$

Proof. Let us denote $\mathcal{R}_{\text{norm}}$ the set of rational maps of degree m which can be written $h = \frac{h_1}{h_2}$ where h_1 and h_2 are two polynomials whose coefficients (a_{ij}) and (b_{ij}) are such that : $\max(a_{ij}) = \max(b_{ij}) = 1$. $\mathcal{R}_{\text{norm}}$ is a compact set. For a given $h \in \mathcal{R}_{\text{norm}}$, since $h \neq 0$, there exists $\overline{\rho}_h > 0$, $\tau_h > 0$, a ball $\overline{\mathbb{B}}_h \subset \tilde{\mathbb{B}} \subset \mathbb{C}^2$ of radius $\overline{\rho}_h$ such that :

$$\frac{\inf_{\overline{\mathbb{B}}_h}|h|}{\sup_{\overline{\mathbb{B}}_h}|h|} \geq \tau_h$$

The constants $\overline{\rho}_h$ and τ_h can be chosen locally constant for rational functions in $\mathcal{R}_{\text{norm}}$ near h. Since $\mathcal{R}_{\text{norm}}$ is compact, if we choose $\overline{\rho} = \overline{\rho}(n, \tilde{\mathbb{B}})$ the minimum of the $\overline{\rho}_h$ and $\tau = \tau(n, \tilde{\mathbb{B}})$ the minimum of the τ_h for a finite covering of $\mathcal{R}_{\text{norm}}$, we have : for rational map $h \in \mathcal{R}_{\text{norm}}$ of degree m, there exists a ball $\overline{\mathbb{B}} \subset \tilde{\mathbb{B}} \subset \mathbb{C}^2$ of radius larger than $\overline{\rho}$ such that :

$$\frac{\inf_{\overline{\mathbb{B}}}|h|}{\sup_{\overline{\mathbb{B}}}|h|} \geq \tau$$

Since every rational map h of degree m can be written $h = C^{ste} \cdot \tilde{h}$ with $\tilde{h} \in \mathcal{R}_{\text{norm}}$, the result is true for every rational map of degree m.

□

Proof of Proposition 4.1.1. We fix such a ball $\overline{\mathbb{B}}$. Up to multiplying h by a constant, which does not affect (1), we can suppose that $|h|_\infty = 1$. We denote $\frac{1}{\tau} = \sigma$. Then by the Cauchy inequality we have : $\frac{||Dh(z)||}{|h(z)|} \leq \frac{1}{\tau} = \sigma$, this is (1). Then (2) and (3) are simple consequences of (1). The proposition is proven.

□

Let us point out for further the following interpolation result. Remind that the integer n was defined in Proposition 2.1.10.

Lemma 4.1.3. *Let us take n balls $\mathcal{V}_1, ..., \mathcal{V}_n \subset \text{Mat}_2(\mathbb{C})$. There exists an integer $\tilde{d} = \tilde{d}(\mathcal{V}_1, ..., \mathcal{V}_n)$ and two real numbers $1 > f_1 > 0$ and $1 > f_2 > 0$ such that for every $\xi > 0$, there exists a constant $\nu = \nu(n, \xi) > 0$ such that : for every ball $\mathbb{B} \subset \mathbb{C}^2$ of radius bounded by 1, for every $\theta_0 \in \mathbb{R}$, there exist a polynomial map $H = H(\mathcal{V}_1, ..., \mathcal{V}_n, \mathbb{B}, \theta_0)$ of $\mathbb{C}^2$ of degree $\tilde{d}$ and $(n+1)$ balls $\mathbb{B}_0, ..., \mathbb{B}_n \subset \mathbb{B}$ of radius greater than $\nu \cdot rad(\mathbb{B})$ such that on each $\mathbb{B}_j$:*

$$\forall t \in (1 - f_1, 1 + f_1), \forall \theta \in (\theta_0 - f_2, \theta_0 + f_2) : -e^{i\theta} \cdot t \cdot DH \in \mathcal{V}_j \text{ and } 2 \cdot |H|_\infty < \xi$$

Proof. We call $\tilde{v}_1, ..., \tilde{v}_n$ the centers of the balls $\mathcal{V}_1, ..., \mathcal{V}_n \subset \text{Mat}_2(\mathbb{C})$. Let us take the ball $\mathbb{B} = \mathbb{B}(0, 1)$. For a given $\theta_0 \in [0, 2\pi]$, there exists H having its differentials at n points $p_i \in \mathbb{B}(0, 1)$ satisfying $H(p_i) = 0$ and $DH_{p_i} = e^{-i.\theta_0} \cdot \tilde{v}_j$ by interpolation. Taking sufficiently small balls $\mathbb{B}_1, ..., \mathbb{B}_n$ of radius ν around the points p_i, this gives the result for a given $\theta_0 \in [0, 2\pi]$ and $t = 1$. Moreover, since the required condition are open, H can be taken uniform on a small interval of values of θ and a small interval $(1 - f_1, 1 + f_1)$ of values of t. Then $\tilde{d}$, ν and f_1 can be taken locally constant in θ. Since $[0, 2\pi]$ is compact, we take the maximal value of $\tilde{d}$ and the minimal values of ν and f_1 on a finite covering of $[0, 2\pi]$ by intervals where $\tilde{d}$, ν and f_1 can be taken constant on each interval of the covering. In particular, since this covering is finite, there exists $f_2 > 0$ such that for each $\theta_0 \in [0, 2\pi]$, it is possible to find constant H, m, ν, f_1 for every $\theta \in (\theta_0 - f_2, \theta_0 + f_2)$. This gives us the result for the fixed ball

$\mathbb{B}(0,1)$. Then, the result follows for any ball $\mathbb{B}(\gamma,r)$ with $r \leq 1$ by taking the map $\tilde{H} = (r\cdot\mathrm{Id}+\gamma)\circ H\circ(\frac{1}{r}\cdot\mathrm{Id}-\gamma)$. It is easy to check that : $-e^{i\theta_0}\cdot t\cdot D\tilde{H} = -e^{i\theta_0}\cdot t\cdot DH \in \mathcal{V}_j$ and $2\cdot|\tilde{H}|_\infty < 2\cdot r\cdot|H|_\infty < \xi$. □

4.2 Fixing the constants relative to the torus $\mathbb{T}$ and the matrix of the linear part A

In the two next subsections, we fix some notation and define a certain number of constants and objects in the following specified order. As a guide for the reader objets denoted in roman letters are relative to $\mathbb{P}^2(\mathbb{C})$, and gothic letters are relative to the torus.

1. We fix a torus $\mathbb{T}$ and euclidean coordinates $\pi : \mathbb{C}^2 \to \mathbb{T}$. We fix the projection $\Pi : \mathbb{T} \to \mathbb{P}^2(\mathbb{C})$ as in Proposition 3.1.4. We fix the group $G_{\text{Lattès}} = G_{\text{Lattès}}(\mathbb{T},\Pi)$ given by Proposition 3.1.6.
2. We fix a Fubini-Study metric $||.||_{FS}$ on $\mathbb{P}^2(\mathbb{C})$.
3. We fix the matrix of the linear part A with $A \in G_{\text{Lattès}}$. We fix a line δ as in Proposition 3.2.5. We fix affine coordinates $[z_1,z_2,z_3]$ on $\mathbb{P}^2(\mathbb{C})$ as in Proposition 3.2.1 in which $\delta = \{z_3 = 0\}$. In the following, we dehomogenize by working in the chart $\{[z_1,z_2,z_3] : z_3 \neq 0\}$ on $\mathbb{P}^2(\mathbb{C})$.
4. We first need a proposition.

   Notation 4.2.1. *We will denote : $V^j = v_j + r_j\cdot B(0,1)$ (remind the balls V^j were defined in Proposition 2.1.10). $\frac{1}{4}\cdot V^j$ will denote the ball of same center as V^j and with quarter of radius.*

   We fix $\mathfrak{p}_0 \in \mathbb{T}$ such that $\Pi(\mathfrak{p}_0) \in \{[z_1,z_2,z_3] : z_3 \neq 0\}$ and $D\Pi_{\mathfrak{p}_0}$ is invertible. There exist invertible matrices $M_1,...,M_n$ such that for every j :

   $$(D\Pi^{-1})_{\Pi(\mathfrak{p}_0)}\cdot\Big(D\Pi_{\mathfrak{p}_0}\cdot A\cdot(D\Pi^{-1})_{\Pi(\mathfrak{p}_0)}\Big)^{-1}\cdot M_j\cdot\Big(D\Pi_{\mathfrak{p}_0}\cdot A\cdot(D\Pi^{-1})_{\Pi(\mathfrak{p}_0)}\Big)^{-1}\cdot D\Pi_{\mathfrak{p}_0} = v_j$$

   Then, by continuity we have :

   Lemma 4.2.2. *There exists a ball $\tilde{\mathfrak{B}} = \tilde{\mathfrak{B}}(\mathbb{T},A) \subset \mathbb{T}$ (remind we have fixed euclidean coordinates on $\mathbb{T}$) where Π is invertible such that $\Pi(\tilde{\mathfrak{B}}) \Subset \{[z_1,z_2,z_3] : z_3 \neq 0\}$, a constant $\sigma' = \sigma'(\mathbb{T},A) > 0$ and n balls $\mathcal{V}_1,...,\mathcal{V}_n \subset \mathrm{Mat}_2(\mathbb{C})$ with :*

   $$\forall \mathfrak{p}_i \in \tilde{\mathfrak{B}}, (D\Pi^{-1})_{\Pi(\mathfrak{p}_1)}\cdot\Big(D\Pi_{\mathfrak{p}_2}\cdot A\cdot(D\Pi^{-1})_{\Pi(\mathfrak{p}_3)}\Big)^{-1}\cdot\mathcal{V}_j\cdot\Big(D\Pi_{\mathfrak{p}_4}\cdot A\cdot(D\Pi^{-1})_{\Pi(\mathfrak{p}_5)}\Big)^{-1}\cdot D\Pi_{\mathfrak{p}_6} \in \frac{1}{4}\cdot V^j$$

   Lemma 4.2.3. *Reducing $\tilde{\mathfrak{B}}$ if necessary, there exists a constant $\sigma' = \sigma'(\mathbb{T},A) > 0$ such that for every w with $||w|| = 1$, we have :*

   $$\inf_{\mathfrak{p}\in\tilde{\mathfrak{B}}} ||D(\Pi\circ A\circ\Pi^{-1})_{\Pi(\mathfrak{p})}(w)|| \geq \sigma'\cdot(\sup_{\mathfrak{p}\in\tilde{\mathfrak{B}}}||D\Pi_{\mathfrak{p}}||)\cdot||A||\cdot(\sup_{\mathfrak{p}\in\tilde{\mathfrak{B}}}||(D\Pi^{-1})_{\Pi(\mathfrak{p})}||)$$

   Proof. We take :

   $$\sigma' = \frac{1}{2}\cdot\frac{\inf(\mathrm{Sp}((D(\Pi\circ A\circ\Pi^{-1})_{\Pi(\mathfrak{p}_0)})}{(\sup_{\mathfrak{p}\in\tilde{\mathfrak{B}}}||D\Pi_{\mathfrak{p}}||)\cdot||A||\cdot(\sup_{\mathfrak{p}\in\tilde{\mathfrak{B}}}||(D\Pi^{-1})_{\Pi(\mathfrak{p})}||)}$$

   and the condition holds reducing the size of the ball $\tilde{\mathfrak{B}}$ around $\mathfrak{p}_0$ if necessary. □

   We fix such a ball $\tilde{\mathfrak{B}}$, a constant $\sigma' > 0$ and n balls $\mathcal{V}_1,...,\mathcal{V}_n \subset \mathrm{Mat}_2(\mathbb{C})$ of centers $\tilde{v}_1,...,\tilde{v}_n$.
5. We will use the following notation :

   Notation 4.2.4. *In the following, we still denote $||M||$ the norm $||.||_{2,2}$ of a fixed matrix. We will denote $||D\Pi|| = \sup_{\mathfrak{p}\in\tilde{\mathfrak{B}}}||D\Pi_{\mathfrak{p}}||$ and $||D\Pi^{-1}|| = \sup_{\mathfrak{p}\in\tilde{\mathfrak{B}}}||(D\Pi^{-1})_{\Pi(\mathfrak{p})}||$.*

6. We fix $\tilde{\mathbb{B}} = \tilde{\mathbb{B}}(\mathbb{T}, A, \tilde{\mathfrak{B}})$ a ball included in $\Pi(\tilde{\mathfrak{B}})$. There exists some constant $\iota > 0$ such that for every ball $\mathbb{B} \subset \tilde{\mathbb{B}}$ of radius r, $\Pi^{-1}(\mathbb{B}) \cap \tilde{\mathfrak{B}}$ contains a ball of radius $\iota \cdot r$. We fix such a constant ι. We take the restriction of $||.||_{FS}$ on $\tilde{\mathbb{B}}$. Since $\tilde{\mathbb{B}} \Subset \{[z_1, z_2, z_3] : z_3 \neq 0\}$, this restriction is equivalent to the euclidean metric on $\{[z_1, z_2, z_3] : z_3 \neq 0\}$.
7. We fix the integer $m = \tilde{d}$ and the reals $f_1, f_2 > 0$ given by Lemma 4.1.3 associated to the balls $\mathcal{V}_1, ..., \mathcal{V}_n \subset \mathrm{Mat}_2(\mathbb{C})$.
8. We fix the constants $\rho = \rho(m) > 0$, $\sigma = \sigma(m) > 0$ given by Lemma 4.1.1 associated to the integer $m = \tilde{d}$, the ball $\tilde{\mathbb{B}}$ and the two reals f_1, f_2.
9. We take a constant $\xi = \xi(\mathbb{T}, \Pi, \mathcal{V}_i, A, \sigma, \sigma')$ satisfying the following inequality :

$$0 < \xi < \frac{1}{4} \cdot \min_{1 \leq j \leq n} r_j \cdot \min\left(\frac{\sigma' \cdot ||A||^2}{2||\Pi^{-1}||_{C^2} \cdot ||D\Pi||}, \frac{||A||^2}{2\sigma \cdot ||D\Pi||^3 \cdot ||D\Pi^{-1}||^3}\right)$$

10. From 4.2.7 and 4.2.8 Lemma 4.1.3 gives us a new constant $\nu = \nu(m, \xi) > 0$.
11. Corollary 2.1.8 gives us a constant $N(\frac{\nu\rho}{10})$.
12. We fix a constant d^1 defined as follows. Let us point out that for any Lattès map L of algebraic degree d' coming from an affine map $\mathcal{L}$ on $\mathbb{T}$, of linear part aA, the equality of the two topological degrees gives : $(d')^2 = |a|^4 \cdot |\det(A)|^2$. There are $(d')^2 = |a|^4 \cdot |\det(A)|^2$ disjoint preimages of the torus $\mathbb{T}$ by the affine map $\mathcal{L}$ of volume $\frac{\mathrm{vol}(\mathbb{T})}{|a|^4 \cdot |\det(A)|^2}$. Let us denote vol_r the volume of a ball of radius r. Let us take d^1 such that both $(d^1)^2 \cdot \frac{\mathrm{vol}_{\iota \cdot \rho}}{10 \cdot \mathrm{vol}(\mathbb{T})} > (\frac{1}{\nu} \cdot N(\frac{\nu\rho}{10}))^4$ and $(d^1)^2 \cdot \geq 100 \cdot \max_{A \in G_{\text{Lattès}}} |\det(A)|^2$. In particular, this last condition implies that for any Lattès map of algebraic degree $d' \geq d^1$, we have $|a| \geq 2$.
13. i was defined in Proposition 3.2.1 and K in Proposition 3.3.1, we fix $n_H = E(\frac{m+2K}{i}) + 1$. We fix $d^2 = d_{n_H+100}$ (this integer was also defined in Proposition 3.2.1). We fix $d^3 = 2K$.
14. We fix $d = \max(d^1, d^2, d^3)$.

4.3 Fixing the constants relative to the Lattès map

15. Let L be a Lattes map $L = [\overline{P}_1 : \overline{P}_2 : \overline{P}_3]$ of degree $d' > d$ associated to an affine map on $\mathbb{T}$ of linear part aA.
16. According to Proposition 3.2.1, in the coordinates $[z_1, z_2, z_3]$ which were fixed in 3. we have that :

$$\overline{P}_3(z_1, z_2, z_3) = \prod_{1 \leq j \leq J} \overline{P}_{3,j}(z_1, z_2, z_3)$$

with $P_{3,j}$ irreducible and $\deg(\overline{P}_{3,i}) \leq i = i(\mathbb{T}^2)$ for $J \geq j \geq J - m - 2K + 1$ (remind that K was defined in Proposition 3.3.1). In plain words, the last factors of the product have degree bounded by a constant i depending only on the chosen torus $\mathbb{T}$. We will consider the restriction of L to $\{[z_1, z_2, z_3] : z_3 \neq 0\} \cap L^{-1}(\{[z_1, z_2, z_3] : z_3 \neq 0\})$. We have :

$$L(z_1, z_2) = \left(\frac{\overline{P_1}(z_1, z_2, 1)}{\overline{P_3}(z_1, z_2, 1)}, \frac{\overline{P}_2(z_1, z_2, 1)}{\overline{P_3}(z_1, z_2, 1)}\right)$$

We denote $P_i(z_1, z_2) = \overline{P}_i(z_1, z_2, 1)$.
17. There exists a periodic point p_c of period n_{pc} (which is a multiple of $\mathrm{ord}_{\text{Lattès}}$) which belongs to the postcritical set of L, according to Proposition 3.3.1, we fix it once for all. We call c the point of the critical curve such that p_c is in the

orbit of c and we have $n_c + n_{pc} \leq K$ according to Proposition 3.3.1, where K is independent of the choice of $\mathbb{T}$ and L. Since p_c is repelling, we can suppose that c is the only critical point in $\{c, L(c), ..., p_c, ..., L^{n_{pc}-1}\}$. We choose homogenous polynomials of degree 1 denoted by $Q_1, ..., Q_{n_c+n_{pc}-1}$ such that :

$$Q_1(c) = Q_2(L(c)) = ... = Q_{n_c}(p_c) = Q_{n_c+n_{pc}-1}(L^{n_{pc}-1}(p_c)) = 0 \qquad (4)$$

It is possible to take these polynomials such that at least one of the coefficients of z_1 and z_2 is non equal to 0 so we take the polynomials with this property.

18. Putting
$$\tilde{P}_3(z_1, z_2) = \prod_{J-2(n_c+n_{pc})-m+1 \leq j \leq J} P_{3,j}(z_1, z_2),$$
let us denote by h the rational function defined by :
$$h(z_1, z_2) = \frac{\prod_{1 \leq j \leq n_c+n_{pc}-1}(Q_j(z_1, z_2, 1))^2}{\tilde{P}_3(z_1, z_2)}$$

19. We denote : $h(\Pi(\mathfrak{p}_0)) = |h(\Pi(\mathfrak{p}_0))|e^{i\theta_1}$.

20. We choose the ball $\mathbb{B} \subset \tilde{\mathbb{B}}$ of radius larger than ρ according to Lemma 4.1.1 applied to the ball $\tilde{\mathbb{B}}$ chosen in 4.2.6 and to the the constants m, f_1, f_2 chosen in 4.2.7. We pick a ball $\mathfrak{B} \subset \Pi^{-1}(\mathbb{B}) \cap \tilde{\mathfrak{B}}$. According to 4.2.6, $\mathfrak{B}$ can be taken with its radius equal to $\iota \cdot \rho$ and this bound on its radius (not the ball itself, but the bound on its radius) is independent of L. Since $d' \geq d \geq d^1$ with $(d^1)^2 \cdot \frac{\text{vol}_{\iota\cdot\rho}}{10\cdot\text{vol}(\mathbb{T})} > (\frac{1}{\nu} \cdot N(\frac{\nu\rho}{10}))^4$, there are at least $(\frac{1}{\nu} \cdot N(\frac{\nu\rho}{10}))^4$ preimages of $\mathfrak{B}$ by the affine map $\mathcal{L}$ inside $\mathfrak{B}$ which form a grid of balls.

21. We fix the polynomial map $H = H(\mathcal{V}_1, ..., \mathcal{V}_n, \mathbb{B}, \theta_0)$ of $\mathbb{C}^2$ of degree $m = \tilde{d}$ and $(n+1)$ balls $\mathbb{B}_0, ..., \mathbb{B}_n \subset \mathbb{B}$ given by Lemma 4.1.3 and corresponding to this ball $\mathbb{B}$ and the value $\theta_0 = \theta_1 - 2\arg(a)$ where θ_1 was defined in 4.3.19. Each of them has its radius larger than ν times the radius of $\mathbb{B}$. We take $(n+1)$ balls $\mathfrak{B}_0 \subset \mathfrak{B} \cap \Pi^{-1}(\mathbb{B}_0), ..., \mathfrak{B}_n \subset \mathfrak{B} \cap \Pi^{-1}(\mathbb{B}_n)$ of radius $\iota \cdot (\nu \cdot \text{rad}(\mathbb{B}))$. Then the quotient $\frac{\text{rad}(\mathfrak{B}_j)}{\text{rad}(\mathfrak{B})}$ is equal for each $j \in \{1, ..., n\}$ to $\frac{\iota\cdot(\nu\cdot\text{rad}(\mathbb{B}))}{\text{rad}(\mathfrak{B})} = \frac{\iota\cdot(\nu\cdot\rho)}{\iota\cdot\rho} = \nu$. Let us point out that this bound on the radius is still independent of L.

4.4 Creating a correcting IFS

Notation 4.4.1. *In the following we construct three holomorphic families of holomorphic maps of $\mathbb{P}^2(\mathbb{C})$ which are successive perturbations of L : $L' = L'_{\varepsilon_1}$, $L'' = L''_{\varepsilon_1,\varepsilon_2}$ and $L''' = L'''_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{D}$. We have $L'_0 = L$, $L''_{\varepsilon_1,0} = L'_{\varepsilon_1}$ and $L'''_{\varepsilon_1,\varepsilon_2,0} = L''_{\varepsilon_1,\varepsilon_2}$. We often forget the ε_i and just denote L', L'', L''' for simplicity when there is no risk of confusion.*

Notation 4.4.2. *We consider the $q = q(d)$ preimages of $\Pi(\mathfrak{B})$ under L included inside $\Pi(\mathfrak{B})$ and the corresponding local inverses $(g_j)_{1\leq j\leq q}$ which form an IFS. We denote by $(\mathcal{G}_j)_{1\leq j\leq q}$ the corresponding maps on $\mathfrak{B}$. For further perturbations L', L'', L''' of L, we consider the analogous objects and we call them $(g'_j)_{1\leq j\leq q}$, $(g''_j)_{1\leq j\leq q}$, $(g'''_j)_{1\leq j\leq q}$ and $(\mathcal{G}'_j)_{1\leq j\leq q}$, $(\mathcal{G}''_j)_{1\leq j\leq q}$, $(\mathcal{G}'''_j)_{1\leq j\leq q}$.*

Notation 4.4.3. *In the following, we will consider the continuation $p(L')$ (resp. $p(L''), p(L''')$) of the periodic point p_c. This one is well defined according to the implicit function Theorem since p_c is repelling. In fact, for the successive perturbations that we will consider, we will always have $p(L') = p(L'') = p(L''') = p_c$.*

Proposition 4.4.4. *Let L be a Lattès map of degree $d' > d$ coming from an affine map on $\mathbb{T}$, of linear part aA. Let $L = (\frac{P_1}{P_3}, \frac{P_2}{P_3})$ be the expression of L in the chart*

$\{[z_1, z_2, z_3] : z_3 \neq 0\}$ *defined in 4.2.3. Then the family of rational maps* $(L'_{\varepsilon_1})_{\varepsilon_1}$ *where* $L' = L'_{\varepsilon_1} = (\frac{P'_1}{P_3}, \frac{P'_2}{P_3})$ *defined by :*

$$P'_1(z_1, z_2) = P_1(z_1, z_2) + \varepsilon_1 h(z_1, z_2) P_3(z_1, z_2) H_1(z_1, z_2) \tag{5}$$

$$P'_2(z_1, z_2) = P_2(z_1, z_2) + \varepsilon_1 h(z_1, z_2) P_3(z_1, z_2) H_2(z_1, z_2) \tag{6}$$

where h *was defined in 4.3.18,* H *in 4.3.21 and* $\varepsilon_1 \in \mathbb{D}$ *is such that :*

1. *For every* $\varepsilon_1 \in \mathbb{D}$, $L' = L'_{\varepsilon_1}$ *extends to a holomorphic map of* $\mathbb{P}^2(\mathbb{C})$ *of the same degree as* L *and* $(L'_{\varepsilon_1})_{\varepsilon_1}$ *is a holomorphic family of holomorphic maps of* $\mathbb{P}^2(\mathbb{C})$
2. $p(L') = p_c$ *is periodic for* L' *and is in the forward orbit of* c *:* $p_c = (L'_{\varepsilon_1})^{n_c}(c)$ *and* $(L'_{\varepsilon_1})^{n_{pc}}(p_c) = p_c$. *Moreover* $D(L'_{\varepsilon_1})_c = D(L)_c, \cdots, D(L'_{\varepsilon_1})_{(L'_{\varepsilon_1})^{n_{pc}-1}(p_c)} = DL_{L^{n_{pc}-1}(p_c)}$ *for every* $\varepsilon_1 \in \mathbb{D}$

Proof. Let first remark that since P_3 admits at least $n_H = (E(\frac{m+2K)}{i}) + 1)$ factors of degree bounded by i, the degrees of hP_3H_1 and hP_3H_2 are bounded by $\deg(P_1) = \deg(P_2)$. Since the property of being a holomorphic mapping is open, L' is a holomorphic mapping for sufficiently small values of ε_1. For simplicity we will suppose that this is true for $\varepsilon_1 \in \mathbb{D}$ after rescaling if necessary. Since ε_1 is just a linear factor, $(L'_{\varepsilon_1})_{\varepsilon_1}$ is a holomorphic family of holomorphic maps of $\mathbb{P}^2(\mathbb{C})$. Thus item 1 is proven. Item 2 is a consequence of the quadratic terms Q_j^2 in h (see 4.3.18).

□

Proposition 4.4.5. *Let a Lattès map* L *of degree* $d' > d$ *coming from an affine map on* $\mathbb{T}$, *of linear part* aA. *We are working in the chart* $\{[z_1, z_2, z_3] : z_3 \neq 0\}$ *defined in 4.2.3. In this chart,* $L = (\frac{P_1}{P_3}, \frac{P_2}{P_3})$. *Let* L' *as in Proposition 4.4.4. Then there exists* $t > 0$ *such that for every* $0 \leq p \leq n$, *for every real* $0 < \varepsilon_1 < 1$, *there exists a ball* $\mathfrak{B}_p \subset \mathfrak{B} \subset \mathbb{C}^2$ *of radius* $rad(\mathfrak{B}_p) \geq \nu \cdot rad(\mathfrak{B})$ *and a neighborhood* $\mathcal{X}_{\varepsilon_1}$ *of* L' *in* $Hol_{d'}$ *such that for every* $L''' \in \mathcal{X}_{\varepsilon_1}$, *if* j *is such that* $\mathcal{G}'''_j(\mathfrak{B}) \subset \mathfrak{B}_p$ *then* $\mathcal{G}'''_j$ *is of type* $(t\varepsilon_1, p)$.

Proof. In the following, we omit the index j on g_j, g'_j, $\mathcal{G}_j$ and $\mathcal{G}'_j$ and we take $0 < \varepsilon_1 < 1$. Let us remind we work in the chart : $[z_1, z_2, z_3] \mapsto (\frac{z_1}{z_3}, \frac{z_2}{z_3})$ on $\mathbb{P}^2(\mathbb{C})$. We first show the result for L'. We have for every $\mathfrak{p} \in \mathfrak{B} \cap \mathcal{G}'(\mathfrak{B})$:

$$D\mathcal{G}'_{\mathfrak{p}} - D\mathcal{G}_{\mathfrak{p}} = D\Pi^{-1}_{g'(\Pi(\mathfrak{p}))} \cdot Dg'_{\Pi(\mathfrak{p})} \cdot D\Pi_{\mathfrak{p}} - D\Pi^{-1}_{g(\Pi(\mathfrak{p}))} \cdot Dg_{\Pi(\mathfrak{p})} \cdot D\Pi_{\mathfrak{p}} =$$

$$(D\Pi^{-1}_{g'(\Pi(\mathfrak{p}))} - D\Pi^{-1}_{g(\Pi(\mathfrak{p}))}) \cdot Dg'_{\Pi(\mathfrak{p})} \cdot D\Pi_p + D\Pi^{-1}_{g(\Pi(\mathfrak{p}))} \cdot (Dg'_{\Pi(\mathfrak{p})} - Dg_{\Pi(\mathfrak{p})}) \cdot D\Pi_{\mathfrak{p}}$$

with :

$$Dg'_{\Pi(\mathfrak{p})} - Dg_{\Pi(\mathfrak{p})} = \left(I_2 + (DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot D(\varepsilon_1.h.H)_{g(\Pi(\mathfrak{p}))}\right)^{-1} \cdot (DL_{g(\Pi(\mathfrak{p}))})^{-1} - (DL_{g(\Pi(\mathfrak{p}))})^{-1}$$

$$= -(DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot \varepsilon_1 D(h.H)_{g(\Pi(\mathfrak{p}))} (DL_{g(\Pi(\mathfrak{p}))})^{-1} + o(\varepsilon_1)$$

$$D(h.H)_{g(\Pi(\mathfrak{p}))} = h(g(\Pi(\mathfrak{p}))) \cdot DH_{g(\Pi(\mathfrak{p}))} + H(g(\Pi(\mathfrak{p}))) \cdot Dh_{g(\Pi(\mathfrak{p}))}$$

Then we have :

$$Dg'_{\mathfrak{p}} - Dg_{\mathfrak{p}} = \eta_1 + \eta_2 - D\Pi^{-1}_{g(\Pi(\mathfrak{p}))} \cdot (DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot \varepsilon_1 \cdot h(g(\Pi(\mathfrak{p}))) \cdot DH_{g(\Pi(\mathfrak{p}))} \cdot (DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot D\Pi_{\mathfrak{p}} + o(\varepsilon_1)$$

where $\eta_1 = (D\Pi^{-1}_{g'(\Pi(\mathfrak{p}))} - D\Pi^{-1}_{g(\Pi(\mathfrak{p}))}) \cdot Dg'_{\Pi(\mathfrak{p})} \cdot D\Pi_{\mathfrak{p}}$ and :

$$\eta_2 = -D\Pi^{-1}_{g(\Pi(\mathfrak{p}))} \cdot (DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot \varepsilon_1 \cdot H(g(\Pi(\mathfrak{p}))) \cdot Dh_{g(\Pi(\mathfrak{p}))} \cdot (DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot D\Pi_{\mathfrak{p}}$$

Lemma 4.4.6. *For any* $\mathfrak{p} \in \mathfrak{B} \cap \mathcal{G}'(\mathfrak{B})$ *we have :*

$$-D\Pi^{-1}_{g(\Pi(\mathfrak{p}))} \cdot (DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot \varepsilon_1 \cdot h(g(\Pi(\mathfrak{p}))) \cdot DH_{g(\Pi(\mathfrak{p}))} \cdot (DL_{g(\Pi(\mathfrak{p}))})^{-1} \cdot D\Pi_{\mathfrak{p}} \in \frac{1}{4|a|^2} \cdot \varepsilon_1 \cdot |h(\Pi(\mathfrak{p}_0))| \cdot V^j$$

Proof. This is due to the fact that H has been taken so that $-\frac{h(\Pi(\mathfrak{p}))}{a^2} \cdot D(H)_{g(\Pi(\mathfrak{p}))}$ belongs to $\frac{|h(\Pi(\mathfrak{p}_0))|}{|a|^2} \cdot \mathcal{V}_j$ and by the definition of $\mathcal{V}_j$ (see Lemma 4.2.2). □

Lemma 4.4.7. *We have :* $||\eta_1|| < \frac{1}{4|a|^2} \cdot \varepsilon_1 \cdot |h(\Pi(\mathfrak{p}_0))| \cdot \min_{1 \leq j \leq n} r_j$

Proof. Since $2 \cdot |H|_\infty \leq \xi$ and by Lemma 4.2.3 for every $\mathfrak{p} \in \mathfrak{B} \cap \mathcal{G}'(\mathfrak{B})$ we have :

$$\begin{aligned} ||g'(\Pi(\mathfrak{p})) - g(\Pi(\mathfrak{p}))|| &\leq \frac{\varepsilon_1 \cdot |h(g(\Pi(\mathfrak{p})))| . \xi}{\inf_{||w||=1} ||D(\Pi \circ aA \circ \Pi^{-1})(w)||} \\ &< \frac{\varepsilon_1 \cdot |h(g(\Pi(\mathfrak{p})))| \cdot \xi}{\sigma' \cdot |a| \cdot ||A|| \cdot ||D\Pi|| \cdot ||D\Pi^{-1}||} \end{aligned}$$

Then we have :

$$\begin{aligned} ||\eta_1|| &= ||(D\Pi^{-1}_{g'(\Pi(\mathfrak{p}))} - D\Pi^{-1}_{g(\Pi(\mathfrak{p}))}) \cdot Dg'_{\Pi(\mathfrak{p})} \cdot D\Pi_{\mathfrak{p}}|| \\ &\leq ||D\Pi^{-1}_{g'(\Pi(\mathfrak{p}))} - D\Pi^{-1}_{g(\Pi(\mathfrak{p}))}|| \cdot ||Dg'_{\Pi(\mathfrak{p})}|| \cdot ||D\Pi|| \\ &< \frac{\varepsilon_1 \cdot |h(g(\Pi(\mathfrak{p})))| \cdot \xi}{\sigma' \cdot |a| ||A|| \cdot ||D\Pi|| \cdot ||D\Pi^{-1}||} \cdot ||\Pi^{-1}||_{C^2} \cdot ||Dg'_{\Pi(\mathfrak{p})}|| \cdot ||D\Pi|| \\ &< \frac{\varepsilon_1 \cdot 2|h(\Pi(\mathfrak{p}_0))| \cdot \xi}{\sigma' \cdot |a| \cdot ||A|| \cdot ||D\Pi|| \cdot ||D\Pi^{-1}||} \cdot ||\Pi^{-1}||_{C^2} \cdot (\frac{1}{|a| \cdot ||A||} \cdot ||D\Pi|| \cdot ||D\Pi^{-1}||) \cdot ||D\Pi|| \\ &< \frac{1}{4|a|^2} \cdot \varepsilon_1 \cdot |h(\Pi(\mathfrak{p}_0))| \cdot \min_{1 \leq j \leq n} r_j \end{aligned}$$

by the inequality on ξ. □

Lemma 4.4.8. *We have :* $||\eta_2|| < \frac{1}{4|a|^2} . \varepsilon_1 . |h(\Pi(\mathfrak{p}_0))| . \min_{1 \leq j \leq n} r_j ,$

Proof.

$$\begin{aligned} ||\eta_2|| &\leq ||D\Pi^{-1}|| \cdot ||DL^{-1}|| \cdot \varepsilon_1 \cdot ||H|| \cdot ||Dh|| \cdot ||DL^{-1}|| \cdot ||D\Pi|| \\ &\leq ||D\Pi^{-1}|| \cdot \Big(\frac{1}{|a| \cdot ||A||} \cdot ||D\Pi|| \cdot ||D\Pi^{-1}||\Big)^2 \cdot \varepsilon_1 \cdot \xi \cdot ||Dh|| \cdot ||D\Pi|| \\ &= \frac{1}{|a|^2} \cdot \varepsilon_1 \cdot \xi \cdot \frac{1}{||A||^2} \cdot ||D\Pi||^3 \cdot ||D\Pi^{-1}||^3 \cdot ||Dh|| \\ &\leq \frac{1}{4|a|^2} \cdot \varepsilon_1 \cdot |h(\Pi(\mathfrak{p}_0))| \cdot \min_{1 \leq j \leq n} r_j \end{aligned}$$

by the inequality on ξ and because : $\forall z \in \mathfrak{B}, ||Dh(z)|| \leq \sigma \cdot |h(z)| \leq \sigma \cdot 2|h(\Pi(\mathfrak{p}_0))|$. □

The three previous lemmas imply that on $\mathfrak{B} \cap \mathcal{G}'(\mathfrak{B})$, $D\mathcal{G}'_{\mathfrak{p}} - D\mathcal{G}_{\mathfrak{p}}$ belongs to $t \cdot V^j$ with $t = \frac{\varepsilon_1 \cdot |h(\Pi(\mathfrak{p}_0))|}{|a|^2}$. Then by continuity, for a given ε_1 (and then a given L'), there exists a neighborhood $\mathcal{X}_{\varepsilon_1}$ of L' in $\mathrm{Hol}_{d'}$ such that for every sufficiently small perturbation $L''' \in \mathcal{X}_{\varepsilon_1}$ of L', if j is such that $\mathcal{G}'''_j(\mathfrak{B}) \subset \mathfrak{B}_p$ then $\mathcal{G}'''_j$ is of type $(t\varepsilon_1, p)$. This proves the first item of Proposition 4.4.5. The proof of Proposition 4.4.5 is complete.

□

4.5 Well oriented postcritical set

Notation 4.5.1. *We fix* $\mathfrak{pc}$ *a point of* $\Pi^{-1}(p_c)$.

Notation 4.5.2. *We denote by PCrit(L) the postcritical set of L, this is the set* $PCrit(L) = \bigcup_{n\geq 0}(L)^n(Crit(L))$ *where Crit(L) is the critical set of L. The notation will be the same for perturbations* L', L'', L''.

We pick a vector w_1 and a value θ given by Corollary 2.1.8 corresponding to the 4-tuple u of the four vectors of a basis of Λ. Still according to Corollary 2.1.8, there exists an open set of admissible values for w_1 so we choose to take it in the following way. The map $\mathcal{L}^{n_{pc}\cdot \mathrm{ord}(A)}$ is an affine map on the torus $\mathbb{T}$ of linear part $a^{n_{pc}\cdot \mathrm{ord}(A)} \cdot A^{n_{pc}\cdot \mathrm{ord}(A)} = a^{n_{pc}\cdot \mathrm{ord}(A)} \cdot I_2^{n_{pc}} = a^{n_{pc}\cdot \mathrm{ord}(A)} \cdot I_2$ with $|a| \geq 2$ (see 3.2.12). Points with dense forward orbit for $\mathcal{L}^{n_{pc}\cdot \mathrm{ord}(A)}$ are dense in $\mathbb{T}$. Moreover, since n_{pc} divides $n_{pc} \cdot \mathrm{ord}(A)$, $\mathfrak{pc}$ is a fixed point of $\mathcal{L}^{n_{pc}\cdot \mathrm{ord}(A)}$. We pick w_1 such that $\mathfrak{pc} + w_1$ is a point of dense forward orbit for $\mathcal{L}^{n_{pc}\cdot \mathrm{ord}(A)}$. Since the linear part of $\mathcal{L}^{n_{pc}\cdot \mathrm{ord}(A)}$ is $a^{n_{pc}\cdot \mathrm{ord}(A)} \cdot I_2$ and $\mathfrak{pc}$ is a fixed point of $\mathcal{L}^{n_{pc}\cdot \mathrm{ord}(A)}$, we have that the whole forward orbit of $\mathfrak{pc}+w_1$ by $\mathcal{L}^{n_{pc}\cdot \mathrm{ord}(A)}$ is contained in the line going through $\mathfrak{pc}$ and $\mathfrak{pc}+w_1$. In particular, this line is dense in the torus $\mathbb{T}$. We pick w_2 such that (w_1, w_2) is a basis of $\mathbb{C}^2$ and w_2 is not tangent to $\Pi^{-1}(\mathrm{PCrit}(L))$ at $\mathfrak{pc}$.

Here is the main result of this subsection :

Proposition 4.5.3. *Let* $\mathfrak{B}$ *be as in 4.3.20. There exists a neighborhood* $\mathcal{W}(L)$ *of L in* $Hol_{d'}$ *such that : every map* $L' = L'_{\varepsilon_1}$ *as in Proposition 4.4.4 is accumulated by maps* $L'''_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = L'''$ *in* $\mathcal{W}(L)$ *such that there exists a component* $\Gamma \subset \Pi^{-1}(PCrit(L'''))$ *whose restriction to* $\mathfrak{B}$ *is a* (θ, w_1)*-quasi-diameter (remind that this notion was defined in Definition 2.1.5).*

The following lemma is well known.

Lemma 4.5.4. *Let* $\mathfrak{L}$ *be a linear automorphism of* $\mathbb{C}^2$ *and* $\Gamma \subset \mathbb{C}^2$ *a complex submanifold through 0 such that :*

1. *the eigenvalues* λ, μ *of* $\mathfrak{L}$ *are such that* $|\lambda| > |\mu| > 1$. *Let* w_λ *and* w_μ *be the respective eigenvectors.*
2. w_μ *is transverse to* Γ *at 0*

Then, $(\mathfrak{L}^k(\Gamma))_{k\geq 0}$ *converges uniformly to the line* $\mathbb{C} \cdot w_\lambda$ *in the* C^1*-topology.*

Proof. We can take $w_\lambda = e_1$ and $w_\mu = e_2$. The eigenvector w_μ of μ is transverse to Γ at 0. Then locally Γ is a graph γ over a small disk $\mathbb{D}_\gamma \subset \mathbb{D} : \{(t, \gamma(t)) : t \in \mathbb{D}_\gamma)\}$. For every $k \in \mathbb{N}$, $\mathfrak{L}^k(\{(t, \gamma(t)) : t \in \mathbb{D}_\gamma)\}) = \{(\lambda^k \cdot t, \mu^k \cdot \gamma(t)) : t \in \mathbb{D}_\gamma\}$. Since $|\lambda| > 1$, for large k, we have $\mathbb{D} \subset \lambda^k \cdot \mathbb{D}_\gamma$. Then $\mathfrak{L}^k(\{(t, \gamma(t)) : t \in \mathbb{D}_\gamma)\})$ contains $\{(s, \mu^k \cdot \gamma(\frac{s}{\lambda^k})) : s \in \mathbb{D}\}$. But there exists $C_\gamma > 0$ such that $|\gamma(t)| < C_\gamma \cdot t$ near 0. Then $\mu^k \cdot \gamma(\frac{s}{\lambda^k}) < C_\gamma \cdot (\frac{\mu}{\lambda})^k$ converges uniformly to 0 on $\mathbb{D}_\gamma$. Then, for every $\theta' > 0$, there exists k such that $\mathfrak{L}^k(\{(t, \gamma(t)) : t \in \mathbb{D}_\gamma)\})$ contains $\{(s, \tilde{\gamma}(s)) : s \in \mathbb{D}\} = \{(s, \mu^k \cdot \gamma(\frac{s}{\lambda^k}) : s \in \mathbb{D}\}$ with $|\tilde{\gamma}(s)| \leq \theta'$. Then by the Cauchy inequality this implies that $|(\tilde{\gamma})'(s)| \leq \theta'$.

□

Notation 4.5.5. *We denote for every* $(\lambda, \mu) \in (\mathbb{C}^*)^2$ *by* $Diag_{\lambda,\mu}$ *the following map from* $\mathbb{C}^2$ *to* $\mathbb{C}^2$ *:*

$$Diag_{\lambda,\mu} : (z_1, z_2) \mapsto (\lambda \cdot z_1, \mu \cdot z_2)$$

Lemma 4.5.6. *The linear part* $A \in G_{Lattès}$ *of L is diagonalizable.*

Proof. Since $G_{\mathrm{Lattès}}$ is a finite group, we have that A is of finite order. In particular, $A^{\mathrm{ord}(G_{\mathrm{Lattès}})} = I_2$. Then $R(A) = 0$ where $R(X) = X^{\mathrm{ord}(G_{\mathrm{Lattès}})} - 1$ has simple roots. Then A is diagonalizable. □

Lemma 4.5.7. *Let $(f_\varepsilon)_{\varepsilon\in\mathbb{D}^3}$ be a holomorphic family of holomorphic germs defined in a neighborhood $\mathcal{U}$ of 0 such that for every $\varepsilon \in \mathbb{D}^3$, $D(f_\varepsilon)_0$ is diagonalizable and 0 is a repelling fixed point for f_ε. We denote by $\lambda(\varepsilon)$ and $\mu(\varepsilon)$ the eigenvalues of f_ε at 0 and w_λ and w_μ be associated eigenvectors. We suppose that in the family $(f_\varepsilon)_{\varepsilon\in\mathbb{D}^3}$, $w_\lambda = w_1$ and $w_\mu = w_2$ are constant. We suppose that $|\lambda(0)|^2 > |\mu(0)| \geq |\lambda(0)|$. Then there exists a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of 0 and a neighborhood $\mathcal{V}$ of 0 in $\mathbb{D}^3$ such that for every $\varepsilon \in \mathcal{V}$, f_ε is holomorphically linearizable in $\mathcal{U}'$: there exists a holomorphic map φ_{f_ε} defined on $\mathcal{U}'$ such that :*

$$Diag_{\lambda(\varepsilon),\mu(\varepsilon)} \circ \varphi_{f_\varepsilon} = \varphi_{f_\varepsilon} \circ f_\varepsilon$$

Moreover, φ_{f_ε} varies continuously with ε in the C^0 topology.

The proof will be based on the following well known result (see Theorem 6.2.3 in [17]).

Proposition 4.5.8. *Let F be an invertible map with repulsive fixed point 0. Suppose that the eigenvalues λ, μ of DF_0 satisfy the condition $|\lambda|^2 > |\mu| \geq |\lambda| > 1$. Then F is holomorphically conjugate to $Diag_{\lambda,\mu}$.*

For the resonant case we use the following lemma :

Lemma 4.5.9. *Let F be an invertible map in a neighborhood of 0 with a repelling fixed point at 0. Let us denote the eigenvalues of DF_0 by λ, μ. Let us suppose that φ_1 and φ_2 are two holomorphic maps conjugating F to $Diag_{\lambda,\mu}$. Then $\varphi_1^{-1} \circ \varphi_2$ is linear.*

Proof. Let us write $\chi = \varphi_1^{-1} \circ \varphi_2 = (\chi^1, \chi^2)$ and $\chi^j(z) = \sum_{k\geq 1} \chi_k^j \cdot z^k$ where $z^k = z_1^{k_1} \cdot z_2^{k_2}$ and $j \in \{1, 2\}$. We have that χ commutes with $L_{\lambda,\mu}$. Then :

$$\lambda \cdot \sum_{|k|\geq 1} \chi_k^1 \cdot z^k = \sum_{|k|\geq 1} \chi_k^1 \cdot (\mathrm{Diag}_{\lambda,\mu}(z))^k \text{ and } \mu \cdot \sum_{|k|\geq 1} \chi_k^2 \cdot z^k = \sum_{|k|\geq 1} \chi_k^2 \cdot (\mathrm{Diag}_{\lambda,\mu}(z))^k$$

In particular, since $\lambda, \mu \neq 1$ this implies that $\chi_k^1 = \chi_k^2 = 0$ for every $|k| > 1$.

□

We now prove Lemma 4.5.7.

Proof of Lemma 4.5.7. We take a neighborhood $\mathcal{V}$ of 0 in $\mathbb{D}^3$ such that for every $\varepsilon \in \mathcal{V}$ we have that $|\lambda(\varepsilon)|^2 < |\mu(\varepsilon)|$ and $|\mu(\varepsilon)|^2 < |\lambda(\varepsilon)|$. It is a consequence of Theorem 6.2.3 of [17] (this result goes back to Poincaré) that for every $\varepsilon \in \mathcal{V}$, f_ε is holomorphically linearizable at 0 in some neighborhood $\mathcal{U}'_\varepsilon$ of 0. We show here that the linearizing map φ_{f_ε} varies continuously with ε in the C^0 topology. This will imply in particular that the neighborhood $\mathcal{U}'_\varepsilon$ can be taken uniform $\mathcal{U}'$ in ε.

For this we follow the proof of Theorem 6.2.3 of [17] and its notations. The proof is divided into 3 steps.

The first step itself is divided into two steps. The first one is a linear change of coordinates that we will denote by φ_{lin}. φ_{lin} is not unique but it becomes unique if w_1 is sent on e_1 and w_2 is sent on e_2. Thus this map $\varphi_{lin} = \varphi_{lin}(f_\varepsilon)$ is uniquely defined and varies continuously in the C^0 topology. The second one is a change of coordinates that we will denote $\varphi_1(z) = (\varphi_1^1(z_1), z_2)$ such that $\varphi_1^1(z_1) = z_1 + \sum_{k=0}^{+\infty} \frac{1}{\lambda^{k+1}} \cdot A(f_\varepsilon^k(z_1))$. There exists a constant K such that $|A(z_1)| \leq K|z_1|^2$ for every map f_ε with $\varepsilon \in \mathcal{V}$ (reducing $\mathcal{V}$ if necessary). Since φ_1^1 is the sum of a normally convergent series whose terms all vary continuously, φ_1^1 and then $\varphi_1 \circ \varphi_{lin}$ vary continuously in the C^0 topology. After these two changes of coordinates, f_ε is reduced to the form $(z_1, z_2) \mapsto (\lambda \cdot z_1, g(z_1, z_2))$ with g varying continuously.

In the second step, one defines some infinite product $\gamma(z) = \prod_{k=0}^{+\infty}(1 + B(F^n(z))$. We have $|B(z)| \leq K'|z|^2$ and reducing $\mathcal{V}$ if necessary, we can suppose this estimate is true for every $\varepsilon \in \mathcal{V}$. Then γ is normally convergent and varies continuously. The map ψ such that $\frac{\partial \psi}{\partial z_2}$ then still varies continuously, just as $\varphi_2(z) = (z_1, \psi(z))$. After this third change of coordinates, f_ε is reduced to the form $(z_1, z_2) \mapsto (\lambda \cdot z_1, \mu \cdot z_2 + h(z_1))$ with h varying continuously.

Finally, the last change of coordinates φ_3 is of the form $(z_1, \eta(z))$ with $\eta(z) = z_2 + q(z_1)$ and $q(z_1) = q_1 \cdot z_1 + q_2 \cdot z_1^2 + ...$ with $q_j = \frac{h_j}{\mu - \lambda^j}$ for each $j \geq 2$. Since h varies continuously, so do the coefficients q_j for $j \geq 2$. Since $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1 \circ \varphi_{lin}$ for every $\varepsilon \in \mathcal{V}$, we have that q_1 is uniquely defined by Lemma 4.5.9 and varies continuously. Finally, q and then η and φ_3 vary continuously. This implies that for every $\varepsilon \in \mathcal{V}$, f_ε is holomorphically linearizable by $\varphi_{f_\varepsilon} = \varphi_3 \circ \varphi_2 \circ \varphi_1 \circ \varphi_{lin}$. Moreover, φ_{f_ε} varies continuously in the C^0 topology. □

Remind that c is a point of $\mathbb{P}^2(\mathbb{C})$ which was defined in 4.3.17 and that the notation $p(L'')$ was introduced at the beginning of section 4.4. In the following lemma, we perturb $L' = L'_{\varepsilon_1}$ into $L'' = L''_{\varepsilon_1,\varepsilon_2}$ to ensure that the critical point c is not singular.

Lemma 4.5.10. *There exists a holomorphic family of holomorphic maps of $\mathbb{P}^2(\mathbb{C})$ denoted by $(L''_{\varepsilon_1,\varepsilon_2})_{(\varepsilon_1,\varepsilon_2)\in\mathbb{D}^2}$ such that :*

1. *for every $\varepsilon_1 \in \mathbb{D}$, $L''_{\varepsilon_1,0} = L'_{\varepsilon_1}$*
2. *p_c is in the postcritical set of $L''_{\varepsilon_1,\varepsilon_2}$ and* the postcritical set is not singular at p_c for $\varepsilon_2 \neq 0$
3. *$p(L''_{\varepsilon_1,\varepsilon_2}) = p_c$ is periodic for $L''_{\varepsilon_1,\varepsilon_2}$ and is in the forward orbit of c : $p_c = (L''_{\varepsilon_1,\varepsilon_2})^{n_c}(c)$ and $(L''_{\varepsilon_1,\varepsilon_2})^{n_{pc}}(p_c) = p_c$*
4. *$D((L''_{\varepsilon_1,\varepsilon_2})^{n_{pc}})_{p_c} = D(L^{n_{pc}})_{p_c}$*

Proof. We first make an invertible linear change of coordinates so that in the new coordinates $[x_1, x_2, x_3]$, the point c is equal to $[0, 0, 1]$ and the point $L(c)$ is in the chart $\{x_3 \neq 0\}$. We choose homogenous polynomials of degree 1 in the variables x_1, x_2, x_3 denoted by $R_2, \ldots, R_{n_c+n_{pc}-1}$ such that :

$$R_2(L(c)) = \cdots = R_{n_c-1}(L^{n_c-1}(c)) = R_{n_c}(p_c) = \cdots = R_{n_c+n_{pc}-1}(L^{n_{pc}-1}(p_c)) = 0 \tag{7}$$

$$R_2(c) \neq 0, \cdots, R_{n_c-1}(c) \neq 0 \ , \ R_{n_c}(c) \neq 0, \cdots, R_{n_c+n_{pc}-1}(c) \neq 0 \tag{8}$$

This is possible since $c, \ldots, L^{n_{pc}-1}(p_c)$ are distinct. We denote :

$$\gamma_1(x_1, x_2) = \prod_{2 \leq k \leq n_c+n_{pc}-1} (R_k(x_1, x_2, 1))^2$$

In $\{x_3 \neq 0\}$, the critical set of L' is the set $\{\mathrm{Jac}(P') = 0\}$ where $\mathrm{Jac}(P')$ is equal to $\frac{\partial P'_1}{\partial x_1} \cdot \frac{\partial P'_2}{\partial x_2} - \frac{\partial P'_1}{\partial x_2} \cdot \frac{\partial P'_2}{\partial x_1}$. The critical set at c is not singular if the gradient of the map $(x_1, x_2) \mapsto \frac{\partial P'_1}{\partial x_1} \cdot \frac{\partial P'_2}{\partial x_2} - \frac{\partial P'_1}{\partial x_2} \cdot \frac{\partial P'_2}{\partial x_1}$ is non zero at c, in particular if :

$$\frac{\partial}{\partial x_1}(\mathrm{Jac}(P'))(c) = \frac{\partial^2 P'_1}{\partial x_1^2} \cdot \frac{\partial P'_2}{\partial x_2} + \frac{\partial P'_1}{\partial x_1} \cdot \frac{\partial^2 P'_2}{\partial x_1 \partial x_2} - \frac{\partial^2 P'_1}{\partial x_1 \partial x_2} \cdot \frac{\partial P'_2}{\partial x_1} - \frac{\partial P'_1}{\partial x_2} \cdot \frac{\partial^2 P'_2}{\partial x_1^2} \neq 0$$

If this is the case, there is nothing to do and we can take $L'' = L'$. Let us suppose this is not so. We distinguish two cases.

First case : we suppose that $\frac{\partial P_2'}{\partial x_2} \neq 0$. For every $\varepsilon_2 \in \mathbb{C}$ we consider the following perturbation of L' defined by $L''_{\varepsilon_1,\varepsilon_2} = L'' = (\frac{P_1''}{P_3''}, \frac{P_2''}{P_3''})$ with $P_2'' = P_2'$, $P_3'' = P_3'$ and :

$$P_1''(x_1, x_2) = P_1'(x_1, x_2) + \varepsilon_2 \cdot \gamma_1(x_1, x_2) \cdot x_1^2$$

Because of the choice of the degree d in 4.2.13 and 4.2.14, we have $\deg(P_1'') \leq \deg(P_1')$. Since the property of being a holomorphic mapping is open, L'' is a holomorphic mapping on $\mathbb{P}^2(\mathbb{C})$ for sufficiently small values of ε_2. Then, item 1 is obvious. Because of the quadratic terms R_k^2 in the definition of γ_1, c stays preperiodic (with the same periodic orbit $p(L''_{\varepsilon_1,\varepsilon_2}) = p_c$) for $L''_{\varepsilon_1,\varepsilon_2}$, this implies item 3. Still because of the quadratic terms R_k^2 in γ_1 we have that $D_{L(c)}(L''_{\varepsilon_1,\varepsilon_2}) = D_{L(c)}(L'_{\varepsilon_1}) = D_{L(c)}L$, $\cdots$, $D_{(L)^{n_{pc}-1}(p_c)}(L''_{\varepsilon_1,\varepsilon_2}) = D_{(L)^{n_{pc}-1}(p_c)}(L'_{\varepsilon_1}) = D_{(L)^{n_{pc}-1}(p_c)}L$, so we both have that p_c is in the postcritical set of $L''_{\varepsilon_1,\varepsilon_2}$ and that item 4 is true. Moreover we have $D_cL'' = D_cL$ so c is still critical. The only second order partial derivative which depends on ε_2 is : $\frac{\partial^2 P_1''}{\partial x_1^2}(c) = \frac{\partial^2 P_1'}{\partial x_1^2}(c) + 2 \cdot \gamma_1(c) \cdot \varepsilon_2$ with $\gamma_1(c) \neq 0$. Then the map $\varepsilon_2 \mapsto \frac{\partial}{\partial x_1}(\mathrm{Jac}(P''))(c)$ is an affine map in ε_2 of non zero coefficient equal to $2 \cdot \gamma_1(c) \cdot \frac{\partial P_2'}{\partial x_2}$. Then, it is non zero for $\varepsilon_2 \in \mathbb{D}^*$. This implies that the critical set is not singular at c. Then there is a component of the postcritical set at c which is not singular. Since $DL_c, \cdots, DL_{L^{n_{pc}}(p_c)}$ are not singular, there is a component of the postcritical set at p_c which is not singular. Thus item 2 is true.

Second case : we suppose that $\frac{\partial P_2'}{\partial x_2} = 0$. For every $\varepsilon_2 \in \mathbb{C}$ we consider the following perturbation of L' defined by $L''_{\varepsilon_1,\varepsilon_2} = L'' = (\frac{P_1''}{P_3''}, \frac{P_2''}{P_3''})$ with $P_3'' = P_3'$ and :

$$P_1''(x_1, x_2) = P_1'(x_1, x_2) + \varepsilon_2 \cdot \gamma_1(x_1, x_2) \cdot x_1$$

$$P_2''(x_1, x_2) = P_2'(x_1, x_2) + \varepsilon_2 \cdot \gamma_1(x_1, x_2) \cdot x_1 x_2$$

Because of the choice of the degree d in 4.2.13 and 4.2.14, we have $\deg(P_1'') \leq \deg(P_1')$ and $\deg(P_2'') \leq \deg(P_2')$. Since the property of being a holomorphic mapping is open, L'' is a holomorphic mapping for sufficiently small values of ϵ_2. Then, item 1 is obvious. As in the first case, items 3 and 4 are true and p_c stays postcritical. We have :

$$D_cL'' = D_cL' + \varepsilon_2 \cdot \gamma_1(c) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since at the point c, we have both $\frac{\partial P_2'}{\partial x_2} = 0$ and $\mathrm{Jac}(P')(c) = \frac{\partial P_1'}{\partial x_1} \cdot \frac{\partial P_2'}{\partial x_2} - \frac{\partial P_1'}{\partial x_2} \cdot \frac{\partial P_2'}{\partial x_1} = 0$, this implies that we still have $\mathrm{Jac}(P'')(c) = 0$ and the point c is still critical. The only second order partial derivative which depends on ε_2 is : $\frac{\partial^2 P_2''}{\partial x_1 \partial x_2}(c) = \frac{\partial^2 P_2'}{\partial x_1 \partial x_2}(c) + \gamma_1(c) \cdot \varepsilon_2$ with $\gamma_1(c) \neq 0$. Then the map $\varepsilon_2 \mapsto \frac{\partial}{\partial x_1}(\mathrm{Jac}(P''))(c)$ is a polynomial of degree 2 in ε_2 of non zero coefficient of degree 2 equal to $(\gamma_1(c))^2$. Then, rescaling if necessary, it is non zero for $\varepsilon_2 \in \mathbb{D}^*$. As in case 1, we conclude that item 2 is satisfied. This concludes the proof of the proposition.

□

Remind that w_1 and w_2 were defined just at the beginning of this subsection. The notation $p(L'')$ was introduced at the beginning of section 4.4. In the following lemma, we perturb the periodic orbit p_c in such a way that we can choose the two eigenvalues at this periodic point .

Lemma 4.5.11. *There exists a holomorphic family of holomorphic maps of $\mathbb{P}^2(\mathbb{C})$ denoted by $(L'''_{\varepsilon_1,\varepsilon_2,\varepsilon_3})_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in\mathbb{D}^3}$ such that :*

1. *for every $\varepsilon_1, \varepsilon_2 \in \mathbb{D}$, $L'''_{\varepsilon_1,\varepsilon_2,0} = L''_{\varepsilon_1,\varepsilon_2}$*

2. *$p(L''') = p_c$ is periodic for L''' ($(L''')^{n_{p_c}}(p_c) = p_c$) and is in the postcritical set of L''' ($p_c = (L''')^{n_c}(c)$)*
3. *if $\varepsilon_3 > 0$, then the eigenvalues λ, μ of the periodic point p_c are such that : $|\mu|^2 > |\lambda| > |\mu|$*
4. *the eigenvector w_μ associated to μ at p_c is equal to $D\Pi_{\mathfrak{pc}}(w_2)$ and then transverse to the postcritical set at p_c*
5. *the eigenvector w_λ associated to λ at p_c is equal to $D\Pi_{\mathfrak{pc}}(w_1)$*

Proof. We first make an invertible linear change of coordinates so that in the new coordinates $[y_1, y_2, y_3]$, the point p_c is equal to $[0, 0, 1]$ and the point $L(p_c)$ is in the chart $\{y_3 \neq 0\}$. We choose homogenous polynomials of degree 1 in the variables y_1, y_2, y_3 denoted by $S_1, \dots, S_{n_c-1}, S_{n_c+1}, \dots, S_{n_c+n_{pc}-1}$ such that :

$$S_1(c) = S_2(L(c)) = \cdots = S_{n_c-1}(L^{n_c-1}(c)) = S_{n_c+1}(L(p_c)) = \cdots = S_{n_c+n_{pc}-1}(L^{n_{pc}-1}(p_c)) = 0 \tag{9}$$

$$S_1(p_c) \neq 0 \ , \ S_2(p_c) \neq 0, \cdots, S_{n_c-1}(p_c) \neq 0 \ , \ S_{n_c+1}(p_c) \neq 0, \cdots, S_{n_c+n_{pc}-1}(p_c) \neq 0 \tag{10}$$

This is possible since $c, \dots, L^{n_{pc}-1}(p_c)$ are distinct. We denote :

$$\gamma_2(y_1, y_2) = \prod_{j \neq n_c} (S_j(y_1, y_2, 1))^2$$

For every $\varepsilon_3, \kappa_i \in \mathbb{C}$ we consider the following perturbation of L'' defined by $L'''_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = L''' = (\frac{P_1'''}{P_3'''}, \frac{P_2'''}{P_3'''})$ with $P_3''' = P_3''$ and :

$$P_1'''(y_1, y_2) = P_1''(y_1, y_2) + \varepsilon_3 \cdot \gamma_2(y_1, y_2) \cdot (\kappa_1 y_1 + \kappa_2 y_2)$$

$$P_2'''(y_1, y_2) = P_2''(y_1, y_2) + \varepsilon_3 \cdot \gamma_2(y_1, y_2) \cdot (\kappa_3 y_1 + \kappa_4 y_2)$$

Because of the choice of the degree d in 4.2.13 and 4.2.14, we have $\deg(P_1''') \leq \deg(P_1'')$ and $\deg(P_2''') \leq \deg(P_2'')$. Since the property of being a holomorphic mapping is open, L''' is a holomorphic mapping on $\mathbb{P}^2(\mathbb{C})$ for sufficiently small values of ε_3. Then, item 1 is obvious. Then, because of the quadratic terms S_j^2 in γ_2, it is clear that c stays preperiodic (with the same periodic orbit p_c) for L''' and $D_{L(p_c)}L''' = D_{L(p_c)}L$, $\cdots$, $D_{L^{n_{pc}-1}(p_c)}L''' = D_{L^{n_{pc}-1}(p_c)}L$. This shows item 2. In the chart $\{y_3 \neq 0\}$, we have :

$$D_{p_c}L''' = D_{p_c}L'' + \varepsilon_3 \cdot \gamma_2(p_c) \cdot \begin{pmatrix} \kappa_1' & \kappa_2' \\ \kappa_3' & \kappa_4' \end{pmatrix}$$

with $\gamma_2(p_c) \neq 0$. We have $D_{L(p_c)}L''' = D_{L(p_c)}L$, $\cdots$, $D_{L^{n_{pc}-1}(p_c)}L''' = D_{L^{n_{pc}-1}(p_c)}L$. We also have the equality : $D_{p_c}L \cdot \ldots \cdot D_{L^{n_{pc}-1}(p_c)}L = a^{n_{p_c}} \cdot I_2$ because the period n_{p_c} is a multiple of the order of A (see Proposition 3.3.1). Then we have :

$$D_{p_c}(L''')^{n_{p_c}} = a^{n_{p_c}} \cdot \left(I_2 + \varepsilon_3 \cdot \gamma_2(p_c) \cdot \begin{pmatrix} \kappa_1' & \kappa_2' \\ \kappa_3' & \kappa_4' \end{pmatrix} \cdot (D_{p_c}L)^{-1} \right)$$

Let us denote by M the matrix whose two columns are $D\Pi_{\mathfrak{pc}}(w_1)$ and $D\Pi_{\mathfrak{pc}}(w_2)$. We choose :

$$\begin{pmatrix} \kappa_1' & \kappa_2' \\ \kappa_3' & \kappa_4' \end{pmatrix} = \frac{1}{\gamma_2(p_c)} \cdot M \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot M^{-1} \cdot (D_{p_c}L)$$

Then :

$$D_{p_c}(L''')^{n_{p_c}} = a^{n_{p_c}} \cdot M \cdot \begin{pmatrix} 1+\epsilon_3 & 0 \\ 0 & 1 \end{pmatrix} \cdot M^{-1}$$

This equality implies that items 3,4 and 5 are satisfied and this ends the proof of the proposition.

□

We are now able to prove Proposition 4.5.3.

Proof of Proposition 4.5.3 . We consider the holomorphic family of holomorphic maps $(L'''_{\varepsilon_1,\varepsilon_2,\varepsilon_3})_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in\mathbb{D}^3}$. According to Lemma 4.5.11, every $L'''_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is diagonalizable and admits w_1 and w_2 as eigenvectors. Then according to Lemma 4.5.7, we can take some uniform open set set $\mathbb{B}_{lin} \subset \mathbb{P}^2(\mathbb{C})$, some ball B_{lin}, such that there exists $\varphi_{L'''}$ defined on $\mathbb{B}_{lin}$ with values in $B_{lin} \subset \mathbb{C}^2$ such that L''' is linearizable by $\varphi_{L'''} : \mathbb{B}_{lin} \mapsto B_{lin}$. Moreover $\varphi_{L'''}$ varies continuously with L'''. We denote by $\mathfrak{B}_{lin}$ some ball in $\Pi^{-1}(\mathbb{B}_{lin}) \subset \mathbb{T}$.

Lemma 4.5.12. *Let Γ' be the diameter of $\mathfrak{B}_{lin}$ of direction w_1. Then there exists n_0 such that $\bigcup_{1\leq n\leq n_0} \mathcal{L}^n(\Gamma')$ contains a $(0, w_1)$-quasi-diameter of $\mathfrak{B}$.*

Proof. $\bigcup_{1\leq n\leq +\infty} \mathcal{L}^n(\Gamma')$ is dense in $\mathbb{T}$ by the choice of w_1. Then there exists n_0 such that $\bigcup_{1\leq n\leq n_0} \mathcal{L}^n(\Gamma')$ contains a $(0, w_1)$-quasi-diameter of $\mathfrak{B}$.

□

From Lemma 4.5.9, we know that $\varphi_L \circ \Pi$ is linear. Rewriting this result in $\mathbb{P}^2(\mathbb{C})$ we have :

Corollary 4.5.13. *Let Γ'' be the diameter of B_{lin} of direction $(\varphi_L \circ \Pi)(w_1)$. Then there exists n_0 such that $\Pi^{-1}(\bigcup_{1\leq n\leq n_0} L^n(\varphi_L^{-1}(\Gamma'')) \cap \mathbb{B})$ contains a $(0, w_1)$-quasi-diameter of $\mathfrak{B}$.*

By continuity of $L''' \mapsto \varphi_{L'''}$ (see Lemma 4.5.7), we have the following perturbation result :

Corollary 4.5.14. *There exists $\theta' > 0$, some neighborhood $\mathcal{W}_2(L)$ of L in $Hol_{d'}$ and an integer n_0 such that for every $(\theta', (\varphi_L \circ \Pi)(w_1))$-quasi-diameter Γ'' of B_{lin}, for every $L''' \in \mathcal{W}_2(L)$, we have that $\Pi^{-1}(\bigcup_{1\leq n\leq n_0}(L''')^n(\varphi_{L'''}^{-1}(\Gamma'')) \cap \mathbb{B})$ contains a (θ, w_1)-quasi-diameter of $\mathfrak{B}$.*

Remind that w_2 is not tangent to $\Pi^{-1}(\mathrm{PCrit}(L))$ at $\mathfrak{pc}$. We can take a neighborhood $\mathcal{W}_3(L)$ of L such that every map in $\mathcal{W}_3(L)$ for which $\mathfrak{pc}$ is in the postcritical set still satisfies this condition. We consider $\mathcal{W}(L) = \mathcal{W}_1(L) \cap \mathcal{W}_2(L) \cap \mathcal{W}_3(L)$. Since the conclusions 1,2,3 and 4 of Lemma 4.5.11 are satisfied, according to Lemma 4.5.4, there exists a disk $\tilde{\Gamma}$ included in the postcritical set of L''' such that $\Pi^{-1}(\tilde{\Gamma})$ contains a (θ', w_1)-quasi-diameter of $\mathfrak{B}_{lin}$ (remind θ' was defined in Lemma 4.5.14). According to Lemma 4.5.14, $\bigcup_{1\leq n\leq n_0}(\mathcal{L}''')^n(\Pi^{-1}(\tilde{\Gamma}))$ contains a (θ, w_1)-quasi-diameter of $\mathfrak{B}$ so the conclusion follows. □

5 Proof of the main result

We consider the perturbations L''' in $\mathcal{W}(L)$ as in the previous subsection and such that $L''' = L'''_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \in \mathcal{X}_{\varepsilon_1}$ (the neighborhood $\mathcal{X}_{\varepsilon_1}$ of L_{ε_1} was introduced in Proposition 4.4.5, all maps in $\mathcal{X}_{\varepsilon_1}$ have a correcting IFS). Let us consider the union of all the sets $\mathcal{G}'''_j(\mathfrak{B}) \subset \mathfrak{B}$ for $1 \leq j \leq q$. Reducing $\mathcal{W}(L)$ if necessary, by continuity it contains a grid of balls $G^1 = (u^1, o^1, n_G, r^1)$ with $r^1 \geq \frac{\iota\rho}{2}$ (see 4.3.20).

Proposition 5.0.15. *There exists an integer d (depending only from $\mathbb{T}$) such that for every Lattès map L inducing an affine map on $\mathbb{T}$ of linear part aA, every map L''' as given in Proposition 4.5.3 is such that :*

1. *$\bigcup_{1\leq j\leq q} \mathcal{G}'''_j(\mathfrak{B})$ contains a grid of balls $G^1 = (u^1, o^1, n_G, r^1)$ with $q = (2n_G+1)^4$ such that each $\mathcal{G}'''_j(\mathfrak{B})$ contains a ball of G^1*
2. *the contraction factor of the IFS $(\mathcal{G}'''_1, ..., \mathcal{G}'''_q)$ is $|a| \geq 2$*

3. *there exist* $(n+1)$ *balls* $\mathfrak{B}_0, \mathfrak{B}_1, ..., \mathfrak{B}_n \subset \mathfrak{B}$ *of relative size larger than* ν*, such that the* $\frac{3}{4}$*-parts of* $\mathfrak{B}_0, \mathfrak{B}_1, ..., \mathfrak{B}_n$ *are included in the hull of* G^1*, and satisfying the following property : for each* $1 \leq j \leq q$ *such that* $\mathcal{G}_j'''(\mathfrak{B}) \subset \mathfrak{B}_p$*,* $\mathcal{G}_j'''$ *is quasi-linear of type* (x,p) *with* $x < x(u^1)$ *and :* $\mathcal{G}_j''' = \tilde{g}_j + \frac{1}{a}\varepsilon_j = \frac{1}{a}(A + h_j) + \frac{1}{a}\varepsilon_j$ *with :* $||\varepsilon_j||_{C^2} < \frac{1}{1000} \cdot ||h_j||$*. Moreover,* $\bigcup_{1 \leq j \leq q} \mathcal{G}_j'''(\mathfrak{B}_p)$ *contains a grid of balls* $\Gamma_p^1 = (u^1, o_p^1, n_G, s^1)$ *for each* $0 \leq p \leq n$ *with* $s^1 \geq \nu \cdot r^1$
4. $n_G > \frac{10}{\nu} \cdot N(\frac{\nu \cdot r^1}{10})$
5. $|a| \cdot R \cdot \max_{1 \leq j \leq q}(||\mathcal{G}_j'''||_{C^2}) < \frac{\nu \cdot r^1}{100}$
6. *there exists a* (θ, w)*-quasi-diameter of* $\mathfrak{B}$ *inside* $\Pi^{-1}(PCrit(L'''))$

Proof. The first item was stated before the proposition. The second one comes from 4.2.12 and the fourth one from 4.3.20. The fifth one can be obtained from a reduction of $\mathcal{W}(L)$ if necessary. The last one is a consequence of Proposition 4.5.3. We show the third item. The existence of the balls $\mathfrak{B}_p$ of relative size ν is a consequence of Proposition 4.4.5. The inclusions $\mathfrak{B}_0, \mathfrak{B}_1, ..., \mathfrak{B}_n \subset \mathfrak{B}$ and the inequality on n_G ensure that there are sufficiently many $\mathcal{G}_j'''(\mathfrak{B})$ so that the $\frac{3}{4}$-parts of $\mathfrak{B}_0, \mathfrak{B}_1, ..., \mathfrak{B}_n$ are included in the hull of G^1. Let us now consider the $(n+1)$ sets $\bigcup_{j \leq q} \mathcal{G}_j'''(\mathfrak{B}_p) \subset \mathfrak{B}$ for $0 \leq p \leq n$. Reducing $\mathcal{W}(L)$ a last time if necessary, by continuity each of them contains a grid of balls $\Gamma_p^1 = (u^1, o_p^1, n_G, s^1)$ with $s^1 \geq \frac{\nu \cdot r^1}{2}$. The property stated in item 3 is also a consequence of Proposition 4.4.5. □

The intersection $\bigcap_{j \geq 1} \mathcal{G}_j(\Pi(\mathfrak{B}))$ is in the Julia set of L. Since $\bigcap_{j \geq 1} \mathcal{G}_j(\Pi(\mathfrak{B}))$ is a basic repeller, it is a consequence of Lemma 2.3 of [9] that $\bigcap_{j \geq 1} \mathcal{G}_j'''(\Pi(\mathfrak{B}))$ is in the Julia set of L''' for sufficiently small perturbations L''' of L. According to Proposition 2.3.1 (beware that the maps $\mathcal{G}_j'''$ in our case correspond to the maps $\mathcal{G}_j$ of the proposition), we can conclude this gives us persistent intersections between the Julia set and the postcritical set. This is true for every L''' defined as before and we know that L is accumulated by such maps inside $\mathrm{Hol}_{d'}$. By [2] (see Proposition 2.5 of [9] for a result in our case), we know that persistent intersections between the postcritical set and a hyperbolic repeller inside the Julia set imply the presence of open sets inside the bifurcation locus. Since they are only finitely many $A \in G_{\text{Lattès}}$ for a given torus $\mathbb{T}$, d is well defined. This proves the final result.

References

[1] Pierre Berger. Generic family with robustly infinitely many sinks. *Invent. Math.*, 205(1):121–172, 2016.

[2] François Berteloot and Fabrizio Bianchi. Perturbations d'exemples de Lattès et dimension de Hausdorff du lieu de bifurcation. *to appear on the Journal de Mathématiques Pures et Appliquées*, 2016.

[3] François Berteloot, Fabrizio Bianchi, and Christophe Dupont. Dynamical stability and Lyapunov exponents for holomorphic endomorphisms of $\mathbb{P}^2$. *to appear on the Annales Scientifiques de l'ENS.*

[4] François Berteloot and Christophe Dupont. Une caracterisation des exemples de Lattès par leur mesure de Green. *Comment. Math. Helv.*, 80(3), 2005.

[5] Sebastien Biebler. Persistent homoclinic tangencies and infinitely many sinks for residual sets of automorphisms of low degree in $\mathbb{C}^3$. *arxiv:1611.02011.*

[6] Christian Bonatti and Lorenzo J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math. (2)*, 143(2):357–396, 1996.

[7] Gregery T. Buzzard. Infinitely many periodic attractors for holomorphic maps of 2 variables. *Ann. of Math. (2)*, 145(2):389–417, 1997.

[8] Marius Dabija and Matthias Jonsson. Algebraic webs invariant under endomorphisms. *Publ. Math.*, pages 137–148, 2010.

[9] Romain Dujardin. Non density of stability for holomorphic mappings on $\mathbb{P}^k$. *Journal de l'Ecole Polytechnique*, 4:813–843, 2017.

[10] Romain Dujardin and Mikhail Lyubich. Stability and bifurcations for dissipative polynomial automorphisms of $\mathbb{C}^2$. *Invent. Math.*, 200(2):439–511, 2015.

[11] Rong Feng. Lattes maps on $\mathbf{P}^2$. *J. Math.Pures Appl.*, pages 636–650, 2010.

[12] I.N.Baker. Fixpoints of polynomials and rational functions. *J. London Math. Soc.*, pages 615–622, 1964.

[13] Jyoichi Kaneko, Syoshi Tokugana, and Masaaki Yoshida. Complex crystallographic groups II. *J. Math. Soc. Japan*, 34:595–605, 1982.

[14] Mikhail Lyubich. An analysis of stability of the dynamics of rational functions. *Teoriya Funk., Funk. Anal. Prilozh.*, 42:72–81, 1984.

[15] Ricardo Mañe, Paulo Sad, and Dennis Sullivan. On the dynamics of rational maps. *Ann. Sci. Ecole Norm. Sup.*, 16:193–217, 1983.

[16] John Milnor. On Lattès maps, in : Dynamics on the Riemann sphere. *European Math. Soc.*, pages 9–43, 2006.

[17] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda. *Holomorphic dynamics*, volume 66. Cambridge Studies in advanced mathematics, 2000.

[18] Johan Taflin. Blenders near polynomial product maps of $\mathbb{C}^2$. *Arxiv:1702.02115v2*, 2017.

LAMA, UMR8050, UNIVERSITE PARIS-EST MARNE LA VALLEE, 5 BOULEVARD DESCARTES, 77454 CHAMPS SUR MARNE, FRANCE

E-mail address : `sebastien.biebler@u-pem.fr`