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NUMERICAL APPROXIMATIONS OF MCKEAN ANTICIPATIVE BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS ARISING IN INITIAL MARGIN REQUIREMENTS *

A. Agarwal[1], S. De Marco[2], E. Gobet[3], J. G. López-Salas[4], F. Noubiagain[5] and A. Zhou[6]

Abstract. We introduce a new class of anticipative backward stochastic differential equations with a dependence of McKean type on the law of the solution, that we name MKABSDE. We provide existence and uniqueness results in a general framework with relaxed regularity assumptions on the parameters. We show that such stochastic equations arise within the modern paradigm of derivative pricing where a central counterparty (CCP) demands each member to deposit variation and initial margins to cover their exposure. In the case when the initial margin is proportional to the Conditional Value-at-Risk (CVaR) of the contract price, we apply our general result to obtain existence and uniqueness of the price as a solution of a MKABSDE. We also provide several linear and non-linear approximations, which we solve using different numerical methods.

Keywords: non-linear pricing, CVaR initial margins, anticipative BSDE, weak non-linearity.
MSC2000: 60H30, 65C05, 65C30

Résumé. Nous introduisons une nouvelle famille d'équations différentielles stochastiques rétrogrades anticipatives ayant une dépendance par rapport à la loi de la solution, que nous appelons MKABSDE. Ces équations apparaissent dans le contexte moderne de la valorisation de dérivés en présence d'appels de marge de la part d'une chambre de compensation. Nous démontrons un résultat d'existence et unicité sous des hypothèses relativement faibles sur les coefficients de l'équation. Dans le cas oú les appels de marge sont proportionnels à la VaR conditionnelle (CVaR) du prix du contrat, notre résultat général entraîne l'existence et unicité pour le prix en tant que solution d'une MKABSDE. Nous considérons plusieurs approximations linéaires et non-linéaires de cette équation, que nous abordons avec différentes méthodes numériques.

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1. Initial margin and McKean Anticipative BSDE (MKABSDE)

1.1. Financial context and motivation

The paradigm of linear risk-neutral pricing of financial contracts has changed in the last few years, influenced by the regulators. Nowadays, banks and financial institutions have to post collateral to a central counterparty (CCP, also called clearing house) in order to secure their positions. Everyday, the CCP asks each member to post a certain amount according to the exposure of their Over-the-Counter (OTC) contracts. The variation and initial margin deposits correspond to collaterals in order to cover respectively a new contract at inception and the daily change in its market value on the one hand, and the possible mark-to-market loss during the liquidation period in case of default on the other hand (see, for example, [Bas15] for details). In this work we focus only on the initial margin requirement (IM for short), and we investigate how it affects the valuation and hedging of the contract. As stated in [Bas15, p.11 3(d)], *"IM protects the transacting parties from the potential future exposure that could arise from future changes in the mark-to-market value of the contract during the time it takes to close out and replace the position in the event that one or more counterparties default. The amount of initial margin reflects the size of the potential future exposure. It depends on a variety of factors, [...] the expected duration of the contract closeout and replacement period, and can change over time."* In this work, we will consider IM deposits that are proportional to the Conditional Value-at-Risk (CVaR) of the contract price over a future period of length Δ (typically $\Delta = 1$ week or 10 days, standing for the closeout and replacement period). We focus on CVaR rather than Value-at-Risk (VaR) due to its pertinent properties; it is indeed well established that CVaR is a coherent risk measure whereas VaR is not [ADEH99].

We make some distinctions in our analysis according to the way the *contract price* is computed in the presence of IM. While [Bas15] refers to a mark-to-market value of the contract that can be seen as an *exogenous* value, we investigate the case where this value is *endogenous* and is given by the value of the hedging portfolio including the additional IM costs. By doing so, we introduce a new non-linear pricing rule, that is: the value of the hedging portfolio V_t together with its hedging component π_t solve a stochastic equation including a term depending on the law of the solution (due to the CVaR). We justify that this problem can be seen as a new type of anticipative Backward Stochastic Differential Equation (BSDE) with McKean interaction [McK66]. From now on, we refer to this kind of equation as MKABSDE, standing for McKean Anticipative BSDE; the subsection 1.2 gives a toy example of such a model. In Section 2, we derive stability estimates for these MKABSDEs, under general Lipschitz conditions, and prove existence and uniqueness results. In Section 3, we verify that these results can be applied to a general complete Itô market [KS98], accounting for IM requirements. Then, we derive some approximations based on classical non-linear BSDEs whose purpose is to quantify the impact of choosing the reference price for IM as exogenous or endogenous, and to compare with the case without IM. Essentially, in Theorem 3.1 we prove that the hedging portfolio with exogenous or endogenous reference price for IM coincide up to order 2 in Δ when Δ is small (which is compatible with Δ equal to few days). Section 4 is devoted to numerical experiments: we solve the different approximating BSDEs using finite difference methods in dimension 1, and nested Monte Carlo and regression Monte Carlo methods in higher dimensions.

1.2. An example of anticipative BSDE with dependence in law

We start with a simple financial example with IM requirements, in the case of a single tradable asset. A more general version with a multidimensional Itô market will be studied in Section 3. Let us assume that the price of a tradable asset, denoted S, evolves accordingly to a geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \tag{1.1}$$

where $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+$ and W is an one-dimensional Brownian motion.

In the classical financial setting (see, for example, [MR05]), consider the situation where a trader wants to sell a European option with maturity $T > 0$ and payoff $\Phi(S_T)$, and to hedge it dynamically with risky and riskless assets S and S^0, where $S^0_t = e^{rt}$ for $t \in [0, T]$ and r is a risk-free interest rate. We denote by (V, π),

the value of the self-financing portfolio and π the amount of money invested in the risky asset, respectively. In order to ensure the replication of the payoff at maturity, the couple (V, π) should solve the following stochastic equation

$$\begin{cases} \mathrm{d}V_t &= r\left(V_t - \pi_t\right)\mathrm{d}t + \pi_t \dfrac{\mathrm{d}S_t}{S_t},\ t \in [0,T], \\ V_T &= \Phi(S_T). \end{cases} \tag{1.2}$$

Eq (1.2) is a BSDE since the terminal condition of V is imposed. Because all the coefficients are linear in V and π, (1.2) is a linear BSDE (see [KPQ97] for a broad overview on BSDEs and their applications in finance).

Accounting for IM requirement will introduce an additional cost in the above self-financing dynamics. We assume that the required deposit is proportional to the CVaR of the portfolio over Δ days (typically $\Delta = 10$ days) at the risk-level α (typically $\alpha = 99\%$). The funding cost for this deposit is determined by an interest rate R.[1] Therefore, the IM cost can be modelled as an additional term in the dynamics of the self-financing portfolio as

$$\mathrm{d}V_t = \left(r\left(V_t - \pi_t\right) - R\ \mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}\left(V_t - V_{t+\Delta}\right)\right)\mathrm{d}t + \pi_t \frac{\mathrm{d}S_t}{S_t}, \tag{1.3}$$

where the CVaR of a random variable L, conditional on the underlying sigma-field $\mathcal{F}_t$ at time t, is defined by (see [RU00])

$$\mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(L) = \inf_{x\in\mathbb{R}} \mathbb{E}\left[\frac{(L-x)^+}{1-\alpha} + x \middle| \mathcal{F}_t\right]. \tag{1.4}$$

Since $V_{t+\Delta}$ may be meaningless as t gets close to T, in (1.3) one should consider $V_{(t+\Delta)\wedge T}$ instead. Rewriting (1.3) in integral form together with the replication constraint, we obtain a BSDE

$$V_t = \Phi(S_T) + \int_t^T \left(-r\left(V_s - \pi_s\right) - \mu\pi_s + R\ \mathbf{CVaR}^{\alpha}_{\mathcal{F}_s}\left(V_s - V_{(s+\Delta)\wedge T}\right)\right)\mathrm{d}s - \int_t^T \pi_s \sigma \mathrm{d}W_s,\ t \in [0,T]. \tag{1.5}$$

The conditional CVaR term is anticipative and non-linear in the sense of McKean [McK66], for it involves the law of future variations of the portfolio conditional to the knowledge of the past. This is an example of McKean Anticipative BSDE, which we study in broader generality in Section 2.

Coming back to the financial setting, (V, π) stands for a valuation rule which treats the IM adjustment as endogenous (in the sense that CVaR is computed on V itself). One could alternatively consider that CVaR is related to an exogenous valuation (the so-called *mark-to-market*), for instance the one due to (1.2) (assuming that (1.2) models the market evolution of the option price). Later in Section 3, we give quantitative error bounds between these different valuation rules. Without advocating one with respect to the other, we rather compare their values and estimate (theoretically and numerically) how well one of their output prices approximates the others. As a consequence, these results may serve as a support for banks and regulators for improving risk management and margin requirement rules.

1.3. **Literature review on anticipative BSDEs and comparison with our contribution**

BSDEs were introduced by Pardoux and Peng [PP90]. Since then, the theoretical properties of BSDEs with different generators and terminal conditions have been extensively studied. The link between Markovian BSDEs and partial differential equations (PDEs) was studied in [PP92]. Under some smoothness assumptions, [PP92] established that the solution of the Markovian BSDE corresponds to the solution of a semi-linear parabolic PDE. In addition, several applications in finance have been proposed, in particular by El Karoui and co-authors [KPQ97] who considered the application to European option pricing in the constrained case. In fact, [KPQ97]

[1] This interest rate corresponds to the difference of a funding rate minus the interest rate paid by the CCP for the deposit, typically $R \approx 3\%$

showed that, under some constraints on the hedging strategy, the price of a contingent claim is given by the solution of a non-linear convex BSDE.

Recently, a new class of BSDEs called anticipated BSDEs (ABSDEs for short) was introduced by Peng and Yang [PY09]. The main feature of this class is that the generator includes not only the value of the solution at the present, but also at a future date. As in the classical theory of BSDEs, there exists a duality between these ABSDEs and stochastic differential delay equations. In [PY09] the existence, uniqueness and a comparison theorem for the solution is provided under a kind of Lipschitz condition which depends on the conditional expectation. One can also find more general formulations of ABSDE in Cheredito and Nam [CN17]. As in the case of classical BSDEs, the question of weakening the Lipschitz condition considered in [PY09] has been tackled by Yang and Elliott [YE13], who extended the existence theorem for ABSDEs from Lipschitz to continuous coefficients, and proved that the comparison theorem for anticipated BSDEs still holds. They also established a minimal solution.

At the same time, Buckdahn and Imkeller [BI09] introduced the so-called time-delayed BSDEs (see also Delong and Imkeller [DI09, DI10]). As opposed to the ABSDEs of [PY09], in this case the generator depends on the values of the solution at the present and at past dates, weighted with a time delay function. Assuming that the generator satisfies a certain kind of Lipschitz assumption depending on a probability measure, Delong and Imkeller [DI10] proved the existence and uniqueness of a solution for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of the generator. These authors also showed that, when the generator is independent of y and for a small delay, existence and uniqueness hold for an arbitrary Lipschitz constant. Later, Delong and Imkeller [DI12] provided an application of time-delayed BSDEs to problems of pricing and hedging, and portfolio management. This work focuses on participating contracts and variable annuities, which are worldwide life insurance products with capital protections, and on claims based on the performance of an underlying investment portfolio.

More recently, Crépey *et al.* [CESS17] have worked in a setting which is close to the problem we tackle here, introducing an application of ABSDEs to the problem of computing different types of valuation adjustments (XVAs) for derivative prices, related to funding (X=F), capital (X=F) and credit risk (X=C). In particular, they focus on the case where the initial margins of an OTC contract can be funded directly with the economic capital of the bank involved in the trade, giving rise to different terms in the price evolution equation. The connection of economic capital and funding valuation adjustment leads to an ABSDE, whose anticipated part consists of a conditional risk measure of the martingale increment of the solution over a future time period. These authors have showed that the system of ABSDEs formed by the FVA and the KVA processes is well-posed. Mathematically, the existence and uniqueness of the solution to the system is established through the convergence of Picard iterations.

Inspired by the dynamics of the self-financial portfolio in 1.5, we consider a new type of ABSDEs (the McKean ABSDEs) where the generator depends on the value of the solution, but also on the law of the whole trajectory between the present and a future date, possibly up to maturity. We state a priori estimates on the differences between the solutions of two such MKABSDEs. Based on these estimates, we derive existence and uniqueness of the solution to a MKABSDE via a fixed-point theorem.

2. A general McKean Anticipative BSDE

In order to give meaning to (1.5) and to more general (multidimensional) cases such as eq (3.2) below, we now introduce a general mathematical setup for studying existence and uniqueness of solutions.

2.1. Notation

Let $T > 0$ be the finite time horizon and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a d-dimensional Brownian motion, where $d \geq 1$. We denote $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by W, completed with the $\mathbb{P}$-null sets of $\mathcal{F}$. Let $t \in [0, T], \beta \geq 0$ and $m \in \mathbb{N}^*$. We will make use of the following notations:

- For any $a = (a_1, ..., a_m) \in \mathbb{R}^m$, $|a| = \sqrt{\sum_{i=1}^m a_i^2}$.

- Given a process $(x_s)_{s\in[0,T]}$, we set $x_{t:T} := (x_s)_{s\in[t,T]}$.
- $L^2_T(\mathbb{R}^m) = \left\{\mathbb{R}^m\text{-valued } \mathcal{F}_T\text{-measurable random variables } \xi \text{ such that } \mathbb{E}\left[|\xi|^2\right] < \infty\right\}$.
- $\mathbb{H}^2_{\beta,T}(\mathbb{R}^m) = \left\{\mathbb{R}^m\text{-valued and } \mathcal{F}\text{-adapted stochastic processes } \varphi \text{ such that } \mathbb{E}\left[\int_0^T e^{\beta t}|\varphi_t|^2 \mathrm{d}t\right] < \infty\right\}$. For $\varphi \in \mathbb{H}^2_{\beta,T}(\mathbb{R}^m)$, we define $||\varphi||_{\mathbb{H}^2_{\beta,T}} = \sqrt{\mathbb{E}\left[\int_0^T e^{\beta t}|\varphi_t|^2 \mathrm{d}t\right]}$.
- $\mathbb{S}^2_{\beta,T}(\mathbb{R}^m) = \left\{\text{Continuous processes } \varphi \in \mathbb{H}^2_{\beta,T}(\mathbb{R}^m) \text{ such that } \mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}|\varphi_t|^2\right] < \infty\right\}$. For $\varphi \in \mathbb{S}^2_{\beta,T}(\mathbb{R}^m)$, we define $||\varphi||_{\mathbb{S}^2_{\beta,T}} = \sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}|\varphi_t|^2\right]}$.

Note that $\mathbb{H}^2_{\beta,T}(\mathbb{R}^m) = \mathbb{H}^2_{0,T}(\mathbb{R}^m)$ and $\mathbb{S}^2_{\beta,T}(\mathbb{R}^m) = \mathbb{S}^2_{0,T}(\mathbb{R}^m)$, for any $\beta \geq 0$. The additional degree of freedom given by the parameter β in the definition of the space norm will be useful when deriving a priori estimates (see Lemma 2.2).

2.2. **Main result**

Our aim is to find a pair of processes $(Y,Z) \in \mathbb{S}^2_{0,T}(\mathbb{R}) \times \mathbb{H}^2_{0,T}\left(\mathbb{R}^d\right)$ satisfying

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s, \Lambda_s\left(Y_{s:T}\right)\right) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \qquad t \in [0,T], \tag{2.1}$$

for a certain mapping $\Lambda_t(\cdot)$ to be defined below. We call Equation (2.1) McKean Anticipative BSDE (MKAB-SDE) with parameters (f, Λ, ξ). In order to obtain existence and uniqueness of solutions, we require that the mappings f and Λ satisfy some suitable Lipschitz properties (specified below), and that the terminal condition ξ be square integrable.

Assumption (S)**.** *For any $y, z, \lambda \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $f(\cdot, y, z, \lambda)$ is a $\mathcal{F}$-adapted stochastic process with values in $\mathbb{R}$ and there exists a constant $C_f > 0$ such that almost surely, for all $(s, y_1, z_1, \lambda_1), (s, y_2, z_2, \lambda_2) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$,*

$$|f(s, y_1, z_1, \lambda_1) - f(s, y_2, z_2, \lambda_2)| \leq C_f\left(|y_1 - y_2| + |z_1 - z_2| + |\lambda_1 - \lambda_2|\right).$$

Moreover, $\mathbb{E}\left[\int_0^T |f(s,0,0,0)|^2 \mathrm{d}s\right] < \infty$.

Assumption (A)**.** *For any $X \in \mathbb{S}^2_{0,T}(\mathbb{R})$, $(\Lambda_t(X_{t:T}))_{t\in[0,T]}$ defines a stochastic process that belongs to $\mathbb{H}^2_{0,T}(\mathbb{R})$. There exist a constant $C_\Lambda > 0$ and a family of measures $(\nu_t)_{t\in[0,T]}$ on $\mathbb{R}$ such that for every $t \in [0,T]$, ν_t has support included in $[t,T]$, $\nu([t,T]) = 1$, and for any $y^1, y^2 \in \mathbb{S}^2_{0,T}(\mathbb{R})$, we have*

$$\left|\Lambda_t\left(y^1_{t:T}\right) - \Lambda_t\left(y^2_{t:T}\right)\right| \leq C_\Lambda \mathbb{E}\left[\int_t^T \left|y^1_s - y^2_s\right| \nu_t(\mathrm{d}s) \Big| \mathcal{F}_t\right], \mathrm{d}t \otimes \mathrm{d}\mathbb{P} \text{ a.e. }.$$

Moreover, there exists a constant $\kappa > 0$ such that for every $\beta \geq 0$ and every continuous path $x : [0,T] \to \mathbb{R}$,

$$\int_0^T e^{\beta s} \int_s^T |x_u| \nu_s(\mathrm{d}u) \mathrm{d}s \leq \kappa \sup_{t\in[0,T]} e^{\beta t}|x_t|.$$

We will say that a function $\tilde{f}$ (resp. a mapping $\tilde{\Lambda}$) satisfies Assumption (S) (resp. (A)) if that assumption holds for the choice $f = \tilde{f}$ (resp. $\Lambda = \tilde{\Lambda}$). We can now give the main result of this section.

Theorem 2.1. *Under Assumptions (S) and (A), for any terminal condition $\xi \in L^2_T(\mathbb{R})$ the BSDE (2.1) has a unique solution $(Y,Z) \in \mathbb{S}^2_{0,T}(\mathbb{R}) \times \mathbb{H}^2_{0,T}\left(\mathbb{R}^d\right)$.*

2.3. **Proof of Theorem 2.1**

The proof uses classical arguments. We first establish apriori estimates in the same spirit as in [KPQ97] on the solutions to the BSDE. Then for a suitable constant $\beta \geq 0$, we use Picard's fixed point method in the space $\mathbb{S}^2_{\beta,T}(\mathbb{R}) \times \mathbb{H}^2_{\beta,T}(\mathbb{R}^d)$ to obtain existence and uniqueness of a solution to Equation (2.1).

Lemma 2.2. *Let $(Y^1, Z^1), (Y^2, Z^2) \in \mathbb{S}^2_{0,T}(\mathbb{R}) \times \mathbb{H}^2_{0,T}(\mathbb{R}^d)$ be solutions to MKABSDE (2.1) associated respectively to the parameters (f^1, Λ^1, ξ^1) and (f^2, Λ^2, ξ^2). We assume that f^1 satisfies Assumption (S) and that Λ^1 satisfies Assumption (A). Let us define $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$, $\delta\xi := \xi^1 - \xi^2$. Finally, let us define for $s \in [0,T]$,*

$$\delta_2 f_s = f^1\left(s, Y^2_s, Z^2_s, \Lambda^2\left(Y^2_{s:T}\right)\right) - f^2\left(s, Y^2_s, Z^2_s, \Lambda^2\left(Y^2_{s:T}\right)\right), \quad \text{and} \quad \delta_2\Lambda_s = \Lambda^1_s\left(Y^2_{s:T}\right) - \Lambda^2_s\left(Y^2_{s:T}\right).$$

Then there exists a constant $C > 0$ such that for $\mu > 0$, we have for β large enough

$$||\delta Y||^2_{\mathbb{S}^2_{\beta,T}} \leq C\left(e^{\beta T}\mathbb{E}\left[|\delta\xi|^2\right] + \frac{1}{\mu^2}\left(||\delta_2 f||^2_{\mathbb{H}^2_{\beta,T}} + C_{f^1}||\delta_2\Lambda||^2_{\mathbb{H}^2_{\beta,T}}\right)\right),$$
$$||\delta Z||^2_{\mathbb{H}^2_{\beta,T}} \leq C\left(e^{\beta T}\mathbb{E}\left[|\delta\xi|^2\right] + \frac{1}{\mu^2}\left(||\delta_2 f||^2_{\mathbb{H}^2_{\beta,T}} + C_{f^1}||\delta_2\Lambda||^2_{\mathbb{H}^2_{\beta,T}}\right)\right).$$

Proof. The proof is based on similar arguments used in [KPQ97]. Let us use the decomposition:

$$\begin{aligned}
&|f^1(s, Y^1_s, Z^1_s, \Lambda^1_s(Y^1_{s:T})) - f^2(s, Y^2_s, Z^2_s, \Lambda^2_s(Y^2_{s:T}))| \\
&\qquad \leq \left|f^1(s, Y^1_s, Z^1_s, \Lambda^1_s(Y^1_{s:T})) - f^1(s, Y^2_s, Z^2_s, \Lambda^2_s(Y^2_{s:T}))\right| \\
&\qquad + \left|f^1(s, Y^2_s, Z^2_s, \Lambda^2_s(Y^2_{s:T})) - f^2(s, Y^2_s, Z^2_s, \Lambda^2_s(Y^2_{s:T}))\right| \\
&\qquad \leq C_{f^1}\left(|\delta Y_s| + |\delta Z_s| + |\Lambda^1_s(Y^1_{s:T}) - \Lambda^2_s(Y^2_{s:T})|\right) + |\delta_2 f_s| \\
&\qquad \leq C_{f^1}\left(|\delta Y_s| + |\delta Z_s| + |\Lambda^1_s(Y^1_{s:T}) - \Lambda^1_s(Y^2_{s:T})| + |\delta_2\Lambda_s|\right) + |\delta_2 f_s|.
\end{aligned}$$

By Itô's lemma on the process $t \to e^{\beta t}|\delta Y_t|^2$, where $\beta \geq 0$, and using the previous inequality, we have that

$$\begin{aligned}
&e^{\beta t}|\delta Y_t|^2 + \beta\int_t^T e^{\beta s}|\delta Y_s|^2 \mathrm{d}s + \int_t^T e^{\beta s}|\delta Z_s|^2 \mathrm{d}s \\
&= e^{\beta T}|\delta\xi|^2 + 2\int_t^T e^{\beta s}\delta Y_s\left(f^1(s, Y^1_s, Z^1_s, \Lambda^1_s(Y^1_{s:T})) - f^2(s, Y^2_s, Z^2_s, \Lambda^2_s(Y^2_{s:T}))\right)\mathrm{d}s - 2\int_t^T e^{\beta s}\delta Y_s\delta Z_s \mathrm{d}W_s \\
&\leq e^{\beta T}|\delta\xi|^2 + 2\int_t^T e^{\beta s}|\delta Y_s|\left(C_{f^1}\left(|\delta Y_s| + |\delta Z_s| + |\Lambda^1_s(Y^1_{s:T}) - \Lambda^1_s(Y^2_{s:T})| + |\delta_2\Lambda_s|\right) + |\delta_2 f_s|\right)\mathrm{d}s \\
&- 2\int_t^T e^{\beta s}\delta Y_s\delta Z_s \mathrm{d}W_s. \qquad (2.2)
\end{aligned}$$

Applying Young's inequality with $\lambda, \mu \neq 0$, we have

$$\begin{aligned}
&2|\delta Y_s|\left(C_{f^1}(|\delta Z_s| + |\Lambda^1_s(Y^1_{s:T}) - \Lambda^1_s(Y^2_{s:T})| + |\delta_2\Lambda_s|) + |\delta_2 f_s|\right) \\
&\qquad \leq \frac{C_{f^1}}{\lambda^2}|\delta Z_s|^2 + \lambda^2 C_{f^1}|\delta Y_s|^2 + \frac{C_{f^1}}{\lambda^2}|\Lambda^1_s(Y^1_{s:T}) - \Lambda^1_s(Y^2_{s:T})|^2 + \lambda^2 C_{f^1}|\delta Y_s|^2 \\
&\qquad + \frac{C_{f^1}}{\mu^2}|\delta_2\Lambda_s|^2 + \mu^2 C_{f^1}|\delta Y_s|^2 + \frac{1}{\mu^2}|\delta_2 f_s|^2 + \mu^2|\delta Y_s|^2 \\
&\qquad \leq \left(\mu^2 + C_{f^1}(\mu^2 + 2\lambda^2)\right)|\delta Y_s|^2 + \frac{C_{f^1}}{\lambda^2}|\delta Z_s|^2 + \frac{C_{f^1}}{\lambda^2}|\Lambda^1_s(Y^1_{s:T}) - \Lambda^1_s(Y^2_{s:T})|^2
\end{aligned}$$

$$+ \frac{C_{f^1}}{\mu^2}|\delta_2\Lambda_s|^2 + \frac{1}{\mu^2}|\delta_2 f_s|^2.$$

Then plug this bound into (2.2) to get

$$\begin{aligned}
&e^{\beta t}|\delta Y_t|^2 + \beta\int_t^T e^{\beta s}|\delta Y_s|^2\mathrm{d}s + \int_t^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s \\
&\quad\le e^{\beta T}|\delta\xi|^2 + \Big(\mu^2 + C_{f^1}(2+\mu^2+2\lambda^2)\Big)\int_t^T e^{\beta s}|\delta Y_s|^2\mathrm{d}s + \frac{C_{f^1}}{\lambda^2}\int_t^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s \\
&\quad+ \frac{C_{f^1}}{\lambda^2}\int_t^T e^{\beta s}|\Lambda_s^1(Y_{s:T}^1) - \Lambda_s^1(Y_{s:T}^2)|^2\mathrm{d}s + \frac{C_{f^1}}{\mu^2}\int_t^T e^{\beta s}|\delta_2\Lambda_s|^2\mathrm{d}s + \frac{1}{\mu^2}\int_t^T e^{\beta s}|\delta_2 f_s|^2\mathrm{d}s \\
&\quad- 2\int_t^T e^{\beta s}\delta Y_s\delta Z_s\mathrm{d}W_s. \qquad (2.3)
\end{aligned}$$

Choosing $\lambda^2 > C_{f^1}$ and

$$\beta \ge \mu^2 + C_{f^1}(2+\mu^2+2\lambda^2), \qquad (2.4)$$

we get from (2.3) that

$$\begin{aligned}
\mathbb{E}\left[\int_t^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s\right] &\le \frac{\lambda^2}{\lambda^2 - C_{f^1}}\mathbb{E}\left[e^{\beta T}|\delta\xi|^2 + \frac{C_{f^1}}{\lambda^2}\int_t^T e^{\beta s}|\Lambda_s^1(Y_{s:T}^1) - \Lambda_s^1(Y_{s:T}^2)|^2\mathrm{d}s\right] \\
&\quad+ \frac{\lambda^2}{\lambda^2 - C_{f^1}}\mathbb{E}\left[\frac{C_{f^1}}{\mu^2}\int_t^T e^{\beta s}|\delta_2\Lambda_s|^2\mathrm{d}s + \frac{1}{\mu^2}\int_t^T e^{\beta s}|\delta_2 f_s|^2\mathrm{d}s\right].
\end{aligned}$$

Here we have used that the stochastic integral in (2.3) is a true martingale, by invoking $\delta Y \in \mathbb{S}^2_{0,T}(\mathbb{R})$, $\delta Z \in \mathbb{H}^2_{0,T}(\mathbb{R}^d)$, the computations like in (2.7) and a localization procedure. From (2.3) we also have that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}|\delta Y_t|^2 + \left(1 - \frac{C_{f^1}}{\lambda^2}\right)\int_0^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s\right] \\
&\le \mathbb{E}\Bigg[e^{\beta T}|\delta\xi|^2 + \frac{C_{f^1}}{\lambda^2}\int_0^T e^{\beta s}|\Lambda_s^1(Y_{s:T}^1) - \Lambda_s^1(Y_{s:T}^2)|^2\mathrm{d}s + \frac{C_{f^1}}{\mu^2}\int_0^T e^{\beta s}|\delta_2\Lambda_s|^2\mathrm{d}s + \frac{1}{\mu^2}\int_0^T e^{\beta s}|\delta_2 f_s|^2\mathrm{d}s \\
&\quad+ 2\sup_{t\in[0,T]}\left|\int_t^T e^{\beta s}\delta Y_s\delta Z_s\mathrm{d}W_s\right|\Bigg]. \qquad (2.5)
\end{aligned}$$

As Λ^1 satisfies Assumption (A), the Jensen inequality yields that

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^{\beta s}|\Lambda_s^1(Y_{s:T}^1) - \Lambda_s^1(Y_{s:T}^2)|^2\mathrm{d}s\right] &\le C_\Lambda^2\mathbb{E}\left[\int_0^T e^{\beta s}\int_s^T |\delta Y_u|^2\nu_s(\mathrm{d}u)\mathrm{d}s\right] \\
&\le \kappa C_\Lambda^2\mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}|\delta Y_t|^2\right]. \qquad (2.6)
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, there exists a positive constant C_1 such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T e^{\beta s}\delta Y_s\delta Z_s\mathrm{d}W_s\right|\right] \le C_1\mathbb{E}\left[\left(\int_0^T e^{2\beta s}|\delta Y_s|^2|\delta Z_s|^2\mathrm{d}s\right)^{1/2}\right]$$

$$\leq C_1 \mathbb{E}\left[\left(\sup_{s\in[0,T]} e^{\beta s}|\delta Y_s|^2\right)^{1/2}\left(\int_0^T e^{\beta s}|\delta Z_s|^2 \mathrm{d}s\right)^{1/2}\right]. \tag{2.7}$$

Therefore, by Young's inequality with $\gamma > 0$, we have

$$\begin{aligned}
2\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T e^{\beta s}\delta Y_s \delta Z_s \mathrm{d}W_s\right|\right] &\leq \frac{C_1}{\gamma^2}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}|\delta Y_t|^2\right] + \gamma^2 C_1 \mathbb{E}\left[\int_0^T e^{\beta s}|\delta Z_s|^2 \mathrm{d}s\right] \\
&\leq \frac{C_1}{\gamma^2}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}|\delta Y_t|^2\right] + \frac{\gamma^2 C_1 \lambda^2}{\lambda^2 - C_{f^1}}\mathbb{E}\Bigg[e^{\beta T}|\delta\xi|^2 + \frac{C_{f^1}}{\mu^2}\int_t^T e^{\beta s}|\delta_2\Lambda_s|^2 \mathrm{d}s \\
&\quad + \frac{1}{\mu^2}\int_t^T e^{\beta s}|\delta_2 f_s|^2 \mathrm{d}s + \frac{C_{f^1}}{\lambda^2}\int_t^T e^{\beta s}|\Lambda_s^1(Y_{s:T}^1) - \Lambda_s^1(Y_{s:T}^2)|^2 \mathrm{d}s\Bigg].
\end{aligned} \tag{2.8}$$

Combining Inequalities (2.5)–(2.8) leads to

$$\begin{aligned}
&\left(1 - \frac{C_1}{\gamma^2} - \frac{\kappa C_{f^1} C_\Lambda^2}{\lambda^2} - \frac{\kappa C_{f^1} C_1 C_\Lambda^2 \gamma^2}{\lambda^2 - C_{f^1}}\right)\mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}|\delta Y_t|^2\right] + \left(1 - \frac{C_{f^1}}{\lambda^2}\right)\mathbb{E}\left[\int_0^T e^{\beta s}|\delta Z_s|^2 \mathrm{d}s\right] \\
&\leq \left(1 + \frac{\gamma^2 C_1 \lambda^2}{\lambda^2 - C_{f^1}}\right)\left(\mathbb{E}\left[e^{\beta T}|\delta\xi|^2\right] + \frac{1}{\mu^2}\mathbb{E}\left[C_{f^1}\int_0^T e^{\beta s}|\delta_2\Lambda_s|^2 \mathrm{d}s + \int_0^T e^{\beta s}|\delta_2 f_s|^2 \mathrm{d}s\right]\right).
\end{aligned}$$

Let us define the continuous function Γ by

$$\Gamma(\gamma,\lambda) = 1 - \frac{C_1}{\gamma^2} - \frac{\kappa C_{f^1} C_\Lambda^2}{\lambda^2} - \frac{\kappa C_{f^1} C_1 C_\Lambda^2 \gamma^2}{\lambda^2 - C_{f^1}}$$

for any $\gamma > 0$ and any $\lambda > 0$ with $\lambda^2 > C_{f^1}$. Observe that if we set $\gamma(\lambda) = \sqrt{\lambda}$ with $\lambda > 0$, we have $\lim_{\lambda\to\infty}\Gamma(\gamma(\lambda),\lambda) = 1$, so there exist λ, γ large enough such that $\Gamma(\gamma,\lambda) > 0$. For such a choice of γ and λ, we then obtain the announced result with the constant

$$C = \frac{1 + \frac{C_1\gamma^2\lambda^2}{\lambda^2 - C_{f^1}}}{\min\left(\Gamma(\gamma,\lambda), 1 - \frac{C_{f^1}}{\lambda^2}\right)}.$$

Recall that β is large enough according to λ (see inequality (2.4)). □

Proof of Theorem 2.1. We use the previous apriori estimates in the case where (Y^1, Z^1) and (Y^2, Z^2) solve respectively the BSDEs

$$\begin{aligned}
Y_t^1 &= \xi + \int_t^T f_s\left(U_s^1, V_s^1, \Lambda_s\left(U_{s:T}^1\right)\right)\mathrm{d}s - \int_t^T Z_s^1 \mathrm{d}W_s, \\
Y_t^2 &= \xi + \int_t^T f_s\left(U_s^2, V_s^2, \Lambda_s\left(U_{s:T}^2\right)\right)\mathrm{d}s - \int_t^T Z_s^2 \mathrm{d}W_s.
\end{aligned}$$

Here, $(U^1, V^1), (U^2, V^2) \in \mathbb{S}_{0,T}^2(\mathbb{R}) \times \mathbb{H}_{0,T}^2(\mathbb{R}^d)$ are given processes. Therefore $f_s\left(U_s^1, V_s^1, \Lambda_s\left(U_{s:T}^1\right)\right)$ and $f_s\left(U_s^2, V_s^2, \Lambda_s\left(U_{s:T}^2\right)\right)$ define processes in $\mathbb{H}_{0,T}^2(\mathbb{R})$ owing to the Assumptions (S) and (A). Therefore, the existence and uniqueness of (Y^1, Z^1) and (Y^2, Z^2) in $\mathbb{S}_{0,T}^2(\mathbb{R}) \times \mathbb{H}_{0,T}^2(\mathbb{R}^d)$ as solutions of standard BSDEs is automatic (see [KPQ97, Theorem 2.1, Proposition 2.2]).

In addition, the process $Y^1 - Y^2$ is then solution to the BSDE $Y_t^1 - Y_t^2 = \int_t^T \delta_2 f_s \mathrm{d}s - \int_t^T \left(Z_s^1 - Z_s^2\right) \mathrm{d}W_s$, where the driver $\delta_2 f_s = f_s\left(U_s^1, V_s^1, \Lambda_s\left(U_{s:T}^1\right)\right) - f_s\left(U_s^2, V_s^2, \Lambda_s\left(U_{s:T}^2\right)\right)$ does not depend on Y_s^1 nor Y_s^2. Using Lemma 2.2 for $C_f = 0$ and $\mu > 0$, we have that for $\beta > 0$ large enough,

$$||\delta Y||^2_{\mathbb{S}^2_{\beta,T}} + ||\delta Z||^2_{\mathbb{H}^2_{\beta,T}} \leq \frac{C}{\mu^2} ||\delta_2 f||^2_{\mathbb{H}^2_{\beta,T}}.$$

Moreover,

$$\begin{aligned} ||\delta_2 f||^2_{\mathbb{H}^2_{\beta,T}} &= \mathbb{E}\left[\int_0^T e^{\beta s} |f_s\left(U_s^1, V_s^1, \Lambda_s\left(U_{s:T}^1\right)\right) - f_s\left(U_s^2, V_s^2, \Lambda_s\left(U_{s:T}^2\right)\right)|^2 \mathrm{d}s\right] \\ &\leq 3C_f^2 \mathbb{E}\left[\int_0^T e^{\beta s}\left(|\delta U_s|^2 + |\delta V_s|^2 + |\Lambda_s\left(U_{s:T}^1\right) - \Lambda_s\left(U_{s:T}^2\right)|^2\right) \mathrm{d}s\right]. \end{aligned}$$

As we have that $||\delta U||^2_{\mathbb{H}^2_{\beta,T}} \leq T||\delta U||^2_{\mathbb{S}^2_{\beta,T}}$, and

$$\mathbb{E}\left[\int_0^T e^{\beta s} |\Lambda_s\left(U_{s:T}^1\right) - \Lambda_s\left(U_{s:T}^2\right)|^2 \mathrm{d}s\right] \leq C_\Lambda^2 \mathbb{E}\left[\int_0^T e^{\beta s} \int_s^T |\delta U_u|^2 \nu_s\left(\mathrm{d}u\right) \mathrm{d}s\right] \leq \kappa C_\Lambda^2 ||\delta U||^2_{\mathbb{S}^2_{\beta,T}}.$$

We obtain that

$$||\delta Y||^2_{\mathbb{S}^2_{\beta,T}} + ||\delta Z||^2_{\mathbb{H}^2_{\beta,T}} \leq \frac{3CC_f^2}{\mu^2}\left(\left(\kappa C_\Lambda^2 + T\right) ||\delta U||_{\mathbb{S}^2_{\beta,T}} + ||\delta V||^2_{\mathbb{H}^2_{\beta,T}}\right).$$

We now choose $\mu^2 > 3CC_f^2(\kappa C_\Lambda^2 + T + 1)$, and obtain that for β large enough, the mapping $\phi : (U, V) \to (Y, Z)$ is a contraction in the space $\mathbb{S}^2_{\beta,T}(\mathbb{R}) \times \mathbb{H}^2_{\beta,T}(\mathbb{R}^d)$. Hence, we get existence and uniqueness of a solution to the BSDE (2.1). □

3. The Case of CVaR initial margins

In this section, we apply the previous results on MKABSDE to equation (1.5) and to its generalizations (with respect to the dimension of S, the underlying dynamic model and the terminal condition) that will be defined below. Beyond usual existence and uniqueness results, our aim is to analyse related approximations, obtained when CVaR is evaluated using Gaussian expansions (justified as $\Delta \to 0$, see Theorem 3.1).

3.1. A well posed problem

Let us consider a general Itô market with d tradable assets [KS98, Chapter 1]. The riskless asset S^0 (money account) follows the dynamics $\frac{\mathrm{d}S_t^0}{S_t^0} = r_t \mathrm{d}t$, and we have d risky assets $\left(S^1, ..., S^d\right)$ following

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_t^i \mathrm{d}t + \sum_{j=1}^d \sigma_t^{ij} \mathrm{d}W_t^j, \ S_0^i = s_0^i \in \mathbb{R}, \ 1 \leq i \leq d. \tag{3.1}$$

The processes $r, \mu := \left(\mu^i\right)_{1 \leq i \leq d}, \sigma := \left(\sigma^{ij}\right)_{1 \leq i,j \leq d}$ are $\mathcal{F}$-adapted stochastic processes with values respectively in $\mathbb{R}, \mathbb{R}^d$, and the set of matrices of size $d \times d$. Moreover, we assume that $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ a.e., the matrix σ_t is invertible and the processes r and $\sigma^{-1}\left(\mu - r\mathbf{1}\right)$ are uniformly bounded, where we define the column vector $\mathbf{1} := (1, ..., 1)^\top \in \mathbb{R}^d$. For a path-dependent payoff ξ paid at maturity T, the dynamics of the hedging portfolio

(V,π) with CVaR initial margin requirement (over a period $\Delta>0$) is given by

$$V_t=\xi+\int_t^T\left(-r_sV_s+\pi_s\left(r_s\mathbf{1}-\mu_s\right)+R\,\mathbf{CVaR}^{\alpha}_{\mathcal{F}_s}\left(V_s-V_{(s+\Delta)\wedge T}\right)\right)\mathrm{d}s-\int_t^T\pi_s\sigma_s\mathrm{d}W_s.$$

Here π is a row vector whose ith coordinate consists of the amount invested in ith asset. The derivation is analogous to that of Section 1.2. This equation rewrites, with the variables $(V,Z=\pi\sigma)$,

$$V_t=\xi+\int_t^T\left(-r_sV_s+Z_s\sigma_s^{-1}\left(r_s\mathbf{1}-\mu_s\right)+R\,\mathbf{CVaR}^{\alpha}_{\mathcal{F}_s}\left(V_s-V_{(s+\Delta)\wedge T}\right)\right)\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s. \tag{3.2}$$

Existence and uniqueness of a solution to the above MKABSDE are consequences of Theorem 2.1.

Corollary 3.1. *For any square integrable terminal condition ξ, the CVaR initial margin problem (3.2) is well posed with a unique solution $(V,Z)\in\mathbb{S}^2_{\beta,T}(\mathbb{R})\times\mathbb{H}^2_{\beta,T}(\mathbb{R}^d)$ for any $\beta\geq 0$.*

Proof. The driver of the BSDE has the form

$$f\left(t,v,z,\lambda\right)=-r_tv+z\sigma_t^{-1}\left(r_t\mathbf{1}-\mu_t\right)+\lambda,\ t\geq 0,\ v,\lambda\in\mathbb{R},\ z\in\mathbb{R}^d,$$

and we also introduce the functional

$$\Lambda_t\left(X_{t:T}\right):=R\,\mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}\left(X_t-X_{(t+\Delta)\wedge T}\right)=R\inf_{x\in\mathbb{R}}\mathbb{E}\left[\frac{\left(X_t-X_{(t+\Delta)\wedge T}-x\right)^+}{1-\alpha}+x\mid\mathcal{F}_t\right],\ t\in[0,T],\ X\in\mathbb{S}^2_{0,T}(\mathbb{R}).$$

Since r and $\sigma^{-1}\left(\mu-r\mathbf{1}\right)$ are uniformly bounded, f clearly satisfies Assumption (S). We now check that Λ satisfies Assumption (A). For $X\in\mathbb{S}^2_{0,T}(\mathbb{R})$ and $x\in\mathbb{R}$, we have

$$\mathbb{E}\left[X_t-X_{(t+\Delta)\wedge T}\middle|\mathcal{F}_t\right]\leq\inf_{x\in\mathbb{R}}\mathbb{E}\left[\frac{(X_t-X_{(t+\Delta)\wedge T}-x)^+}{1-\alpha}+x\middle|\mathcal{F}_t\right]\leq\mathbb{E}\left[\frac{(X_t-X_{(t+\Delta)\wedge T})^+}{1-\alpha}\middle|\mathcal{F}_t\right], \tag{3.3}$$

where for the left hand side (l.h.s.) we use the fact that as $\alpha\in(0,1)$, for $z,x\in\mathbb{R}$, $\frac{(z-x)^+}{1-\alpha}+x\geq z$, and for the right hand side (r.h.s.), we upper bound the infimum with the value taken at $x=0$. As it is easy to check that both the l.h.s. and the r.h.s. of (3.3) belong to $\mathbb{H}^2_{0,T}$, we conclude that $\Lambda_t\left(X\right)\in\mathbb{H}^2_{0,T}(\mathbb{R})$. Now, let $X^1,X^2\in\mathbb{S}^2_{0,T}$. Then, we have that

$$\begin{aligned}\left|\Lambda_t\left(X^1\right)-\Lambda_t\left(X^2\right)\right|&\leq R\mathbb{E}\left[\left|\frac{X^1_t-X^2_t-\left(X^1_{(t+\Delta)\wedge T}-X^2_{(t+\Delta)\wedge T}\right)}{1-\alpha}\right|\mid\mathcal{F}_t\right]\\&\leq\frac{R}{1-\alpha}\mathbb{E}\left[\int_t^T\left|X^1_s-X^2_s\right|\nu_t\left(\mathrm{d}s\right)\mid\mathcal{F}_t\right],\end{aligned}$$

where for the first inequality, we use the fact that $\left|\inf_{x\in\mathbb{R}}g^1(x)-\inf_{x\in\mathbb{R}}g^2(x)\right|\leq\sup_{x\in\mathbb{R}}\left|g^1(x)-g^2(x)\right|$ for any functions $g^1,g^2:\mathbb{R}\to\mathbb{R}$ and the 1-Lipschitz property of the positive part function, and for the second inequality, $\nu_t(\mathrm{d}s):=\delta_t(\mathrm{d}s)+\delta_{(t+\Delta)\wedge T}(\mathrm{d}s)$, where for $u\geq 0$, δ_u is the Dirac measure on $\{u\}$. Moreover, for $\beta\geq 0$,

$$\int_0^T e^{\beta s}\int_s^T\left|X^1_u\right|\nu_s\left(\mathrm{d}u\right)=\int_0^T e^{\beta t}\left(\left|X^1_t\right|+\left|X^1_{(t+\Delta)\wedge T}\right|\right)\mathrm{d}t\leq 2\sup_{t\in[0,T]}e^{\beta t}\left|X^1_t\right|,$$

so Assumption (S) holds with $\kappa=2$ and $C_\Lambda=\frac{R}{1-\alpha}$. We finally apply Theorem 2.1 to complete the proof. □

3.2. Approximation by standard BSDEs when $\Delta \ll 1$

The numerical solution of (3.2) is challenging in full generality. In fact, it is a priori more difficult than solving a standard BSDE, for which we can employ, for example, regression Monte-Carlo methods (see e.g. [GT16] and references therein). In this work, we take advantage of the fact that Δ is small (recall $\Delta =$ one week or 10 days) in order to provide handier approximations of (V, Z), given in terms of standard non-linear or linear BSDEs. Below we define these different BSDEs and provide the error estimates of such approximations.

At the lowest order in the parameter $\sqrt{\Delta}$, for $s \in [0, T]$, formally we have that, conditionally to $\mathcal{F}_s$,

$$V_s - V_{(s+\Delta)\wedge T} \approx -\int_s^{(s+\Delta)\wedge T} Z_u \mathrm{d}W_u \overset{(d)}{=} -|Z_s|\sqrt{(s+\Delta)\wedge T - s} \times G,$$

where we freeze the process Z at current time s and $G \overset{(d)}{=} \mathcal{N}(0,1)$ is independent from $\mathcal{F}_s$. This is an approximation of CVaR using the "Delta" of the portfolio (see [GHS00, Section 2]). Plugging this approximation into (3.2), and defining

$$C_\alpha := \mathbf{CVaR}^\alpha(\mathcal{N}(0,1)) = \left.\frac{e^{-x^2/2}}{(1-\alpha)\sqrt{2\pi}}\right|_{x=\mathcal{N}^{-1}(\alpha)}, \tag{3.4}$$

we obtain a standard non-linear BSDE

$$V_t^{NL} = \xi + \int_t^T \left(-r_s V_s^{NL} + Z_s^{NL}\sigma_s^{-1}(r_s\mathbf{1} - \mu_s) + R\,C_\alpha\sqrt{(s+\Delta)\wedge T - s}|Z_s^{NL}|\right)\mathrm{d}s - \int_t^T Z_s^{NL}\mathrm{d}W_s. \tag{3.5}$$

Seeing V^{NL} as a function of the small parameter Δ appearing in the driver, and making an expansion at the orders 0 and 1 w.r.t. $\sqrt{\Delta}$ by following the expansion procedure in [GP15], we obtain two linear BSDEs, respectively (V^{BS}, Z^{BS}) and (V^L, Z^L) where

$$V_t^{BS} = \xi + \int_t^T \left(-r_s V_s^{BS} + Z_s^{BS}\sigma_s^{-1}(r_s\mathbf{1} - \mu_s)\right)\mathrm{d}s - \int_t^T Z_s^{BS}\mathrm{d}W_s, \tag{3.6}$$

$$V_t^{L} = \xi + \int_t^T \left(-r_s V_s^{L} + Z_s^{L}\sigma_s^{-1}(r_s\mathbf{1} - \mu_s) + RC_\alpha\sqrt{(s+\Delta)\wedge T - s}|Z_s^{BS}|\right)\mathrm{d}s - \int_t^T Z_s^{L}\mathrm{d}W_s. \tag{3.7}$$

Let us comment on these different models.

- The simplest equation is (V^{BS}, Z^{BS}), corresponding to the usual linear valuation rule [KPQ97, Theorem 1.1] without IM requirement. When the model is a one-dimensional geometric Brownian motion and $\xi = (S_T - K)_+$, the solution is given by the usual Black-Scholes formula.
- The second simplest equation is (V^L, Z^L) where the IM cost is computed using the "Delta" of an exogenous reference price given by the simplest pricing rule (V^{BS}, Z^{BS}) without IM. This is still a linear BSDE but its simulation is not simple though, since one needs to know Z^{BS} to simulate (V^L, Z^L). We use a nested Monte-Carlo procedure in our experiments.
- The third equation is (V^{NL}, Z^{NL}) where the IM cost is computed using the "Delta" of the endogenous price (V^{NL}, Z^{NL}) itself.

Existence and uniqueness of a solution to the BSDEs (3.5), (3.6) and (3.7) are direct consequences of [PP90], as the respective drivers satisfy standard Lipschitz properties and the processes r and $\sigma^{-1}(r\mathbf{1} - \mu)$ are bounded.

Proposition 3.1. *The standard BSDEs (3.5), (3.6) and (3.7) have a unique solution in the L_2-space $\mathbb{S}^2_{0,T} \times \mathbb{H}^2_{0,T}$, and their norms are uniformly bounded in $\Delta \leq T$.*

The main result of this part is the following theorem.

Theorem 3.1. *Define the L_2 time-regularity index of Z^{NL} by*

$$\mathcal{E}^{NL}(\Delta) := \frac{1}{\Delta}\mathbb{E}\left[\int_0^T \int_t^{(t+\Delta)\wedge T} \left|Z_s^{NL} - Z_t^{NL}\right|^2 \mathrm{d}s\mathrm{d}t\right]. \tag{3.8}$$

We always have $\sup_{0<\Delta\leq T}\mathcal{E}^{NL}(\Delta) < +\infty$. Moreover, there exist constants $K_1, K_2, K_3 > 0$, independent from Δ, such that

$$||V^L - V^{BS}||^2_{\mathbb{S}^2_{0,T}} + ||Z^L - Z^{BS}||^2_{\mathbb{H}^2_{0,T}} \leq K_1\Delta, \tag{3.9}$$

$$||V^{NL} - V^L||^2_{\mathbb{S}^2_{0,T}} + ||Z^{NL} - Z^L||^2_{\mathbb{H}^2_{0,T}} \leq K_2\Delta^2, \tag{3.10}$$

$$||V - V^{NL}||^2_{\mathbb{S}^2_{0,T}} + ||Z - Z^{NL}||^2_{\mathbb{H}^2_{0,T}} \leq K_3\Delta\left(\Delta + \mathcal{E}^{NL}(\Delta)\right). \tag{3.11}$$

In addition, we have

$$\mathcal{E}^{NL}(\Delta) = O(\Delta), \tag{3.12}$$

and thus $||V - V^{NL}||^2_{\mathbb{S}^2_{0,T}} + ||Z - Z^{NL}||^2_{\mathbb{H}^2_{0,T}} = O(\Delta^2)$ provided that the additional sufficient conditions below are fulfilled:

(i) *the terminal condition is a Lipschitz functional of S, that is, $\xi = \Phi(S_{0:T})$ for some functional Φ satisfying*

$$|\Phi(x_{0:T}) - \Phi(x'_{0:T})| \leq C_\Phi \sup_{t\in[0,T]} |x_t - x'_t|,$$

for any continuous paths $x, x' : [0,T] \to \mathbb{R}^d$;

(ii) *the coefficients r, σ, μ are constant.*

Let us remark that the results from [Zha04] used in the proof of estimate (3.12) and consequently the estimate (3.12) itself should also hold under (i) and the following more general assumptions:

(iii) the processes r, σ, μ are Markovian, i.e. $r_t = \hat{r}(t, S_t)$, $\sigma_t^{ij} = \hat{\sigma}^{ij}(t, S_t)$ and $\mu_t^i = \hat{\mu}^i(t, S_t)$ for some deterministic functions $\hat{r}, \hat{\mu}^i, \hat{\sigma}^{ij}$;

(iv) the functions $x \to \hat{\mu}^i(x)x_i$, $x \to \hat{\sigma}^{ij}(x)x_i$ are globally Lipschitz in $(t, x) \in [0,T]\times\mathbb{R}^d$, for any $1 \leq i, j \leq d$;

(v) the functions $\hat{r}$ and $\hat{\sigma}^{-1}(\hat{r}\mathbf{1} - \hat{\mu})$ are globally Lipschitz in $(t, x) \in [0,T] \times \mathbb{R}^d$.

As mentioned above, we may expect that $\mathcal{E}^{NL}(\Delta) = O(\Delta)$ also under (i)-(iii)-(iv)-(v), so that (V, Z) and (V^{NL}, Z^{NL}) are very close to each other. These approximation results illustrate that there is a significative difference (at the order of $\sqrt{\Delta}$) between valuation with or without initial margin cost (see (3.9)); however, the other valuation rules yield comparable values as soon as $\Delta \ll 1$ (see (3.10)-(3.11)).

3.3. **Proof of Theorem 3.1**

▷ *Estimate on $\mathcal{E}^{NL}(\Delta)$.* We start with a deterministic inequality. For any positive function Ψ and any $\beta \geq 0$, we have

$$\int_0^T e^{\beta t}\left(\int_t^{(t+\Delta)\wedge T} \Psi_s \mathrm{d}s\right)\mathrm{d}t \leq \Delta\int_0^T e^{\beta s}\Psi_s \mathrm{d}s. \tag{3.13}$$

Indeed the left hand side of (3.13) can be written as

$$\int_0^T\int_0^T e^{\beta t}\Psi_s \mathbf{1}_{t\leq s\leq (t+\Delta)\wedge T}\mathrm{d}s\mathrm{d}t = \int_0^T \Psi_s\left(\int_0^T e^{\beta t}\mathbf{1}_{t\leq s\leq (t+\Delta)\wedge T}\mathrm{d}t\right)\mathrm{d}s, \tag{3.14}$$

which readily gives the announced result. Using $(a+b)^2 \leq 2a^2 + 2b^2$ and (3.13) with $\beta = 0$ gives

$$\mathcal{E}^{NL}(\Delta) \leq \frac{2}{\Delta}\mathbb{E}\left[\int_0^T \int_t^{(t+\Delta)\wedge T} (|Z_s^{NL}|^2 + |Z_t^{NL}|^2)\mathrm{d}s\mathrm{d}t\right] \leq 4\mathbb{E}\left[\int_0^T \left|Z_t^{NL}\right|^2 \mathrm{d}t\right], \tag{3.15}$$

which is uniformly bounded in Δ (Proposition 3.1).

We now derive finer estimates that reveal the L_2 time-regularity of Z^{NL} under the extra assumptions (i)-(ii). In this Markovian setting, we know that Z^{NL} has a càdlàg version (see [Zha04, Remark (ii) after Lemma 2.5]). Then, introduce the equidistant times $t_i = i\Delta$ for $0 \leq i \leq n := \lfloor \frac{T}{\Delta} \rfloor$ and $t_{n+1} = T$. We claim that

$$\mathcal{E}^{NL}(\Delta) \leq 4\sum_{i=0}^{n} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \left|Z_s^{NL} - Z_{t_i}^{NL}\right|^2 + \left|Z_s^{NL} - Z_{t_{i+1}}^{NL}\right|^2 \mathrm{d}s\right]. \tag{3.16}$$

With this result at hand, the estimate (3.12) directly follows from an application of [Zha04, Theorem 3.1]. To get (3.16), set $\varphi^-(s)$ and $\varphi^+(s)$ for the grid times before and after s. Then, we write

$$\begin{aligned}
\int_0^T \int_t^{(t+\Delta)\wedge T} |Z_s^{NL} - Z_t^{NL}|^2 \mathrm{d}s\mathrm{d}t &\leq 2\sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \int_t^{(t+\Delta)\wedge T} (|Z_s^{NL} - Z_{t_{i+1}}^{NL}|^2 + |Z_t^{NL} - Z_{t_{i+1}}^{NL}|^2)\mathrm{d}s\mathrm{d}t \\
&\leq 2\int_0^T \int_t^{(t+\Delta)\wedge T} \left(|Z_s^{NL} - Z_{\varphi^-(s)}^{NL}|^2 + |Z_s^{NL} - Z_{\varphi^+(s)}^{NL}|^2\right) \mathrm{d}s\mathrm{d}t + 2\Delta \int_0^T |Z_t^{NL} - Z_{\varphi^+(t)}^{NL}|^2 \mathrm{d}t \\
&\leq 4\Delta \left(\int_0^T |Z_t^{NL} - Z_{\varphi^-(t)}^{NL}|^2 \mathrm{d}t + \int_0^T |Z_t^{NL} - Z_{\varphi^+(t)}^{NL}|^2 \mathrm{d}t\right)
\end{aligned}$$

where we have used (3.13) with $\beta = 0$. The inequality (3.16) readily follows.

▷ *Proof of* (3.9). This error estimate is related to the difference of two linear BSDEs. The drivers of V^{BS} and V^L are respectively $f^{BS}(s, y, z, \lambda) = -r_s y + z\sigma_s^{-1}(r_s\mathbf{1} - \mu_s)$, and

$$f^L(s, y, z, \lambda) = -r_s y + z\sigma_s^{-1}(r_s\mathbf{1} - \mu_s) + RC_\alpha\sqrt{(s+\Delta)\wedge T - s}|Z_s^{BS}|,$$

for $s \in [0, T]$, $v, \lambda \in \mathbb{R}$ and $z \in \mathbb{R}^d$, hence

$$\left(f^L - f^{BS}\right)(s, y, z, \lambda) = RC_\alpha\sqrt{(s+\Delta)\wedge T - s}|Z_s^{BS}|.$$

By Lemma 2.2, we obtain that for $\mu > 0$, β large enough and $K_1 = \frac{C}{\mu^2}(RC_\alpha)^2 ||Z^{BS}||_{\mathbb{H}^2_{\beta,T}}$,

$$||V^L - V^{BS}||^2_{\mathbb{S}^2_{0,T}} + ||Z^L - Z^{BS}||^2_{\mathbb{H}^2_{0,T}} \quad \leq \quad ||V^L - V^{BS}||^2_{\mathbb{S}^2_{\beta,T}} + ||Z^L - Z^{BS}||^2_{\mathbb{H}^2_{\beta,T}} \leq K_1\Delta.$$

We are done with (3.9).

▷ *Proof of* (3.10). Then, as $\xi \in L^2$, as the processes $r, \sigma^{-1}(\mu - r\mathbf{1})$ are bounded and as the non-linear term

$$t, z \in [0, T] \times \mathbb{R}^d \to \mathbb{R}C_\alpha\sqrt{(t+\Delta)\wedge T - t}|z|$$

is Lipschitz in the variable z, uniformly in time, we obtain Inequality (3.10) as an application of [GP15, Theorem 2.4], for which assumptions $\mathbf{H.1} - \mathbf{H.3}$ are satisfied.

▷ *Proof of* (3.11). Using computations similar to those in the proof of Lemma 2.2, we obtain existence of a constant $C>0$ such that for $\mu>0$ and β large enough,

$$||V-V^{NL}||^2_{\mathbb{S}^2_{\beta,T}}+||Z-Z^{NL}||^2_{\mathbb{H}^2_{\beta,T}} \leq \frac{C}{\mu^2}\left|\left| \mathbf{CVaR}^{\alpha}_{\mathcal{F}_\cdot}\left(V^{NL}_\cdot - V^{NL}_{(\cdot+\Delta)\wedge T}\right) - \mathbf{CVaR}^{\alpha}_{\mathcal{F}_\cdot}\left(-\int_\cdot^{(\cdot+\Delta)\wedge T} Z^{NL}_\cdot \mathrm{d}W_s\right)\right|\right|^2_{\mathbb{H}^2_{\beta,T}}.$$

As the CVaR function is subadditive [RU00], we have that given A,B two random variables, and $t\in[0,T]$, $\mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(A) \leq \mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(B) + \mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(A-B)$. Inverting the roles of A and B, we obtain that

$$\begin{aligned} 0 \leq \left|\mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(A) - \mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(B)\right| &\leq \max\left(\mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(A-B), \mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(B-A)\right) \\ &\leq \frac{1}{1-\alpha}\left(\mathbb{E}\left[(A-B)^+ \mid \mathcal{F}_t\right] + \mathbb{E}\left[(B-A)^+ \mid \mathcal{F}_t\right]\right) = \frac{\mathbb{E}\left[|A-B| \mid \mathcal{F}_t\right]}{1-\alpha}, \end{aligned}$$

where for the last inequality, we have used that for $U\in\{A-B, B-A\}$, $\inf_{x\in\mathbb{R}}\mathbb{E}\left[\frac{(U-x)^+}{1-\alpha}+x \mid \mathcal{F}_t\right] \leq \mathbb{E}\left[\frac{U^+}{1-\alpha} \mid \mathcal{F}_t\right]$. We then have that

$$\left|\mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(A) - \mathbf{CVaR}^{\alpha}_{\mathcal{F}_t}(B)\right|^2 \leq \frac{1}{(1-\alpha)^2}\mathbb{E}\left[(A-B)^2 \mid \mathcal{F}_t\right].$$

Setting, for $t\in[0,T]$, $A_t = -\int_t^{(t+\Delta)\wedge T} Z^{NL}_t \mathrm{d}W_s$ and $B_t = V^{NL}_t - V^{NL}_{(t+\Delta)\wedge T}$ and using the previous inequality, we obtain that

$$||V-V^{NL}||^2_{\mathbb{S}^2_{\beta,T}} + ||Z-Z^{NL}||^2_{\mathbb{H}^2_{\beta,T}} \leq \frac{C}{\mu^2(1-\alpha)^2}\mathbb{E}\left[\int_0^T e^{\beta t}\left(V^{NL}_t - V^{NL}_{(t+\Delta)\wedge T} + \int_t^{(t+\Delta)\wedge T} Z^{NL}_t \mathrm{d}W_s\right)^2 \mathrm{d}t\right].$$

We use the following decomposition,

$$\begin{aligned} V^{NL}_t - V^{NL}_{(t+\Delta)\wedge T} + \int_t^{(t+\Delta)\wedge T} Z^{NL}_t \mathrm{d}W_s &= \int_t^{(t+\Delta)\wedge T}\left(-r_s V^{NL}_s + Z^{NL}_s \sigma_s^{-1}(r_s\mathbf{1}-\mu_s) + R\lambda C_\alpha\sqrt{(s+\Delta)\wedge T - s}|Z^{NL}_s|\right)\mathrm{d}s \\ &\quad - \int_t^{(t+\Delta)\wedge T}\left(Z^{NL}_s - Z^{NL}_t\right)\mathrm{d}W_s =: \Pi_1(t) - \Pi_2(t), \end{aligned}$$

so that $||V-V^{NL}||^2_{\mathbb{S}^2_{\beta,T}} + ||Z-Z^{NL}||^2_{\mathbb{H}^2_{\beta,T}} \leq \frac{2C}{\mu^2(1-\alpha)^2}\mathbb{E}\left[\int_0^T e^{\beta t}\left(\Pi_1^2(t)+\Pi_2^2(t)\right)\mathrm{d}t\right]$. By Jensen's inequality and the inequality (3.13), we get

$$\begin{aligned} \mathbb{E}\left[\int_0^T e^{\beta t}\Pi_1^2(t)\mathrm{d}t\right] &\leq 3\Delta\mathbb{E}\left[\int_0^T e^{\beta t}\int_t^{(t+\Delta)\wedge T}\left(\left(-r_s V^{NL}_s\right)^2 + \left(Z^{NL}_s\sigma_s^{-1}(r_s\mathbf{1}-\mu_s)\right)^2 + \left(R\lambda C_\alpha\sqrt{\Delta}|Z^{NL}_s|\right)^2\right)\mathrm{d}s\mathrm{d}t\right] \\ &\leq 3\Delta^2\left(|r|^2_\infty + |\sigma^{-1}(r\mathbf{1}-\mu)|^2_\infty + (RC_\alpha)^2\right)\mathbb{E}\left[\int_0^T e^{\beta t}(|V^{NL}_t|^2 + |Z^{NL}_t|^2)\mathrm{d}t\right]. \end{aligned}$$

By invoking the uniform estimate of Proposition 3.1, we finally obtain that $\mathbb{E}\left[\int_0^T e^{\beta t}\left(\Pi_1(t)\right)^2\mathrm{d}t\right] \leq \tilde{K}_1\Delta^2$ for some $\tilde{K}_1$. Moreover, using Ito's isometry, we have that

$$\mathbb{E}\left[\int_0^T e^{\beta t}\Pi_2^2(t)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{\beta t}\int_t^{(t+\Delta)\wedge T}\left|Z^{NL}_s - Z^{NL}_t\right|^2\mathrm{d}s\mathrm{d}t\right] \leq e^{\beta T}\Delta\mathcal{E}^{NL}(\Delta).$$

Gathering all the previous arguments leads to the (3.11). The proof of Theorem 3.1 is completed. □

4. Numerical Examples

In the absence of numerical methods to estimate the solution of the McKean Anticipative BSDE (3.2) in full generality, we rather solve numerically the BSDE approximations (3.5) or (3.7) as discussed in Section 3.2. For this purpose, when the dimension d is greater than one, we use the Stratified Regression Multistep-forward Dynamical Programming (SRMDP) scheme developed in [GLSTV16]. In our numerical tests in this section, we set the coefficients of the model (3.1) to be constant (multi-dimensional geometric Brownian motion) and we take $\mu^i = r$. Observe that setting $R = 0$ reduces the original BSDE to the linear equation (3.6). This will serve us as a benchmark value in order to measure the impact of Initial Margins (IM).

4.1. Finite difference method for (V^{NL}, Z^{NL}) in dimension 1

In order to check the validity of our results, we first obtain a benchmark when $d = 1$ by solving the semi-linear parabolic PDE related to the BSDE (3.5) when $\xi = \Phi(S_T)$, see [PR14]. By an application of Itô's lemma, the semi-linear PDE is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + C_\alpha R\,\sigma\sqrt{(t+\Delta)\wedge T - t}\left|S\frac{\partial V}{\partial S}\right| - r\,V = 0, \quad (t,S) \in [0,T)\times\mathbb{R}^+, \tag{4.1}$$

$$V(T,S) = \Phi(S), \quad S \in \mathbb{R}^+, \tag{4.2}$$

and $(V_t^{NL}, Z_t^{NL}) = (V(t,S_t), \frac{\partial V}{\partial S}(t,S_t)\sigma S_t)$.

Remark 4.1. *If $\Phi(S) = \max(S-K,0)$ or $\max(K-S,0)$ for some $K > 0$, i.e., either a call or a put option payoff, we expect the gradient $\frac{\partial V}{\partial S}$ to have a constant sign. In such a case, the PDE (4.1)–(4.2) becomes linear and in fact has an explicit solution, given by a Black-Scholes formula with time-dependent continuous dividend yield $d(t) = -C_\alpha R\,\sigma\sqrt{(t+\Delta)\wedge T - t}\ \mathrm{sign}(\frac{\partial V}{\partial S})$.*

We use a classical finite difference methods to solve (4.1)-(4.2) (see, for example, [AP05]). First, we perform a change of variable, $x = \ln S$, so that the PDE can be rewritten in the following form for the function $v(t,x) := V(t,e^x)$:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \Big(r - \frac{1}{2}\sigma^2\Big)\frac{\partial v}{\partial x} + C_\alpha R\sigma\sqrt{(t+\Delta)\wedge T - t}\left|\frac{\partial v}{\partial x}\right| - r\,v = 0, \quad (t,x)\in[0,T)\times\mathbb{R}, \tag{4.3}$$

$$v(T,x) = \Phi(e^x), \quad x \in \mathbb{R}. \tag{4.4}$$

We denote the finite difference domain by $D = [0,T]\times[x_{\min}, x_{\max}]$ with $-\infty < x_{\min} < x_{\max} < \infty$. The domain D is approximated with a uniform mesh $\mathcal{D} = \big\{(t^n, x_i) : n = 0,1,\dots,N,\ i = 0,1,\dots,M\big\}$, where $t^n := n\Delta t$ and $x_i := x_{\min} + i\Delta x$. Here, for N time intervals, $\Delta t = T/N$ and $\Delta x = (x_{\max} - x_{\min})/M$ for M spatial steps. Furthermore, we denote $v(t^n, x_i) = v_i^n$. Next, consider the following finite difference derivative approximations under the well-known ω-scheme, i.e., we replace v_i^n by $\omega v_i^n + (1-\omega)v_i^{n+1}$, where $\omega \in [0,1]$ is a constant parameter, such that

$$\frac{\partial v}{\partial t}(t^n,x_i) \approx \frac{v_i^{n+1} - v_i^n}{\Delta t}, \quad \frac{\partial v}{\partial x}(t_n,x_i) \approx \omega\frac{v_{i+1}^n - v_i^n}{\Delta x} + (1-\omega)\frac{v_{i+1}^{n+1} - v_i^{n+1}}{\Delta x},$$
$$\frac{\partial^2 v}{\partial x^2}(t_n,x_i) \approx \omega\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2} + (1-\omega)\frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{(\Delta x)^2}.$$

The choice $\omega = 0.5$ corresponds to Crank-Nicolson method. We also "linearize" the non-linear term by treating it as explicit, i.e., at any time t_n we take $\left|\frac{\partial v}{\partial x}(t^n,x_i)\right| \approx \left|\frac{\partial v}{\partial x}(t^{n+1},x_i)\right|$.

The substitution of finite difference derivative approximations in (4.3)-(4.4) along with the "linearization" step, leads to the following tridiagonal linear system at each time step $n = N-1, \dots, 0$ which can be solved by Thomas algorithm [YG73]: $Av^n = b^{n+1}$, with nonzero coefficients of the tridiagonal matrix $A = (a_{i,j})$ given by

$$\begin{aligned} &a_{0,0} = 1, && a_{0,1} = 0, && a_{M,M-1} = 0, \quad a_{M,M} = 1, \\ &a_{i,i} = 1 + 2\theta\omega + \kappa\omega + \rho\omega, && a_{i,i+1} = -\theta\omega - \kappa\omega, && a_{i-1,i} = -\theta\omega, \quad i = 1, \dots, M-1, \end{aligned}$$

and the time dependent vector b^{n+1} as:

$$\begin{aligned} b_0^{n+1} = v_0^{n+1}, \quad b_M^{n+1} = v_M^{n+1}, b_i^{n+1} &= \theta(1-\omega)v_{i-1}^{n+1} + (1 - 2\theta(1-\omega) - \kappa(1-\omega) - \rho(1-\omega))v_i^{n+1} \\ &+ (\theta(1-\omega) + \kappa(1-\omega))v_{i+1}^{n+1} + \beta^n |v_{i+1}^{n+1} - v_i^{n+1}|, \quad i = 1, \dots, M-1, \end{aligned}$$

where v_0^{n+1}, v_M^{n+1} are given by the boundary conditions and the remaining constants are defined as below

$$\theta = \frac{\sigma^2 \Delta t}{2(\Delta x)^2}, \quad \kappa = \frac{(r - \frac{1}{2}\sigma^2)\Delta t}{\Delta x}, \quad \rho = r\Delta t, \quad \beta^n = \frac{C_\alpha R \sigma \Delta t \sqrt{(t_n + \Delta) \wedge T - t_n}}{\Delta x}.$$

The ith coordinate of vector v^n is the approximation of the value $v(t^n, x_i)$.

We set the model parameters as $T = 1$, $\sigma = 0.25$, $r = 0.02$, $\alpha = 0.99$ and $\Delta = 0.02$ (1 week) and consider three different options – call, put and butterfly, for different strikes. We set $R = 0.02$ when accounting for IM and $R = 0$ otherwise. The finite difference space domain is taken as $[\ln(10^{-6}), \ln(4K)]$ while for SRMDP algorithm we take $[-5, 5]$ to be the space domain. Furthermore, for finite difference scheme, $N = 10^3$ and $M = 10^6$. For **LP0** version of SRMDP algorithm, the number of hypercubes are 2800, the number of time steps are 50 and the number of simulations per hypercube are 2500. In Figure 1 and Table 1, we present the results for implied volatilities, prices and deltas of several call options including not only *at the money* strike but also *in* and *out of the money* strikes. First, we compute the values using the classical Black-Scholes formula (B-S $R = 0$) in order to allow the reader to assess the impact of taking into account IM. Next, we solve the non-linear BSDE using the three discussed method:; the exact Black-Scholes formula where IM is considered as a time dependent dividend yield (BS $R = 0.02$) (see Remark 4.1), the finite difference method (FD) and SRMDP algorithm. For the last method, we compute 95% confidence intervals for price and delta of the options. We also present the results of several put options in Figure 2 and Table 2. In any case, we observe that the IM has a significant impact on the Implied Volatility of option prices (about 20-30 bps for usual prices). Finally, we consider butterfly options with payoff function

$$\Phi(S_T) = (S_T - (K-2))^+ - 2\,(S_T - K)^+ + (S_T - (K+2))^+ .$$

This derivative product involves three options with different strikes, the investor buys a call option with low strike price $K - 2$, buys a call option with high strike price $K + 2$ and sells two call options with strike price K. Note that the sign of the first derivative of option price (delta) of a butterfly option varies with the value of the underlying asset, therefore explicit Black-Scholes formula is not available when IM is also taken into consideration ($R = 0.02$). The results are presented in Figure 3 and Table 3. In all the three cases, we observe that SRMDP algorithm provides good accuracy when compared to the true values and finite difference estimates.

4.2. Variance reduction for solving (V^{NL}, Z^{NL}) using (V^{BS}, Z^{BS})

In order to asses the impact of using $R > 0$ on the solution of the BSDE (3.5), in the case of European call and put options in one dimension, it is better to solve the BSDE difference $(V_t^{DF}, Z_t^{DF}) = (V_t^{NL} - V_t^{BS}, Z_t^{NL} - Z_t^{BS})$ which has a reduced variance in the algorithm. Note that for a call option

$$Z_t^{BS} = \sigma S_t \Phi(d_1), \quad d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

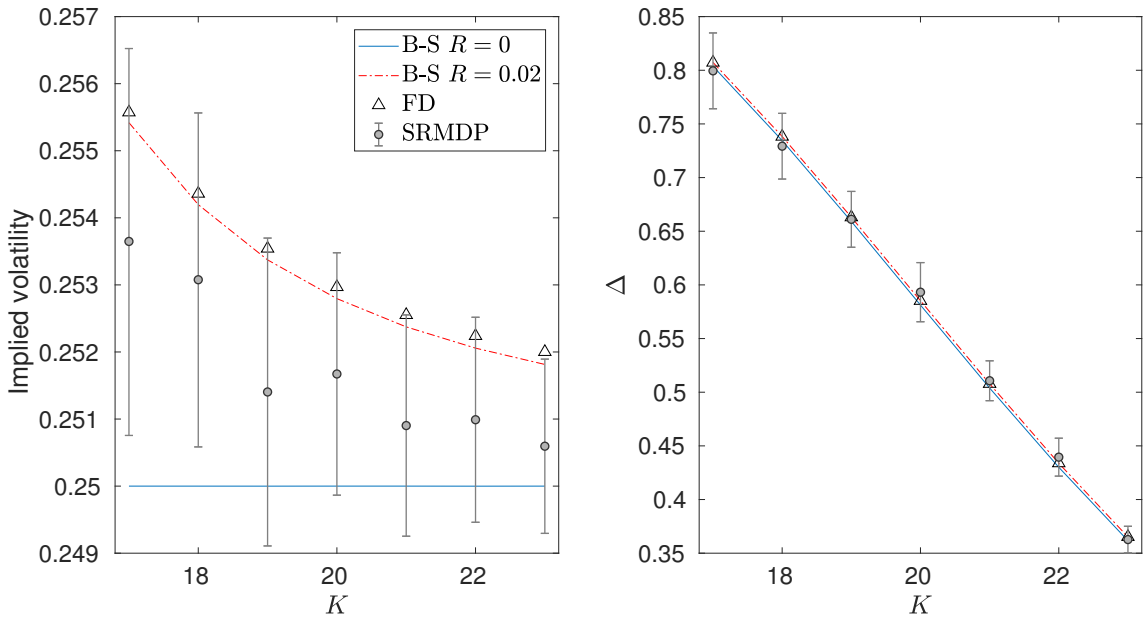

FIGURE 1. Implied volatility and delta for call options with spot value $S_0 = 20$ and different strikes K

	B-S $R = 0$		B-S $R = 0.02$		FD		SRMDP	
K	B-S$(0, S_0)$	$\Delta(0, S_0)$	B-S$(0, S_0)$	$\Delta(0, S_0)$	$V(0, S_0)$	$\nabla V(0, S_0)$	95% CI $V_0^{NL}(S_0)$	95% CI $Z_0^{NL}(S_0)/(\sigma S_0)$
17	3.9534	0.8037	3.9835	0.8073	3.9844	0.8072	$[3.9575, 3.9897]$	$[0.7641, 0.8347]$
18	3.2795	0.7345	3.3071	0.7383	3.3082	0.7382	$[3.2833, 3.3161]$	$[0.6986, 0.7598]$
19	2.6863	0.6592	2.7111	0.6631	2.7123	0.6630	$[2.6797, 2.7134]$	$[0.6350, 0.6871]$
20	2.1741	0.5812	2.1959	0.5852	2.1973	0.5852	$[2.1730, 2.2012]$	$[0.5656, 0.6207]$
21	1.7398	0.5039	1.7587	0.5079	1.7601	0.5078	$[1.7338, 1.7601]$	$[0.4920, 0.5292]$
22	1.3777	0.4301	1.3939	0.4338	1.3953	0.4338	$[1.3734, 1.3975]$	$[0.4218, 0.4571]$
23	1.0805	0.3617	1.0941	0.3651	1.0954	0.3652	$[1.0752, 1.0946]$	$[0.3503, 0.3750]$

TABLE 1. Price and delta for call options with spot value $S_0 = 20$ and different strikes K.

where Φ is the standard Gaussian cumulative distribution function. Therefore $|Z_t^{NL}| = |Z_t^{DF} + \sigma S_t \Phi(d_1)|$. Then, the BSDE for the difference (V_t^{DF}, Z_t^{DF}) in the case of a call option[2] is given by:

$$V_t^{DF} = 0 + \int_t^T \left(-rV_s^{DF} + C_\alpha R\sqrt{(s+\Delta)\wedge T - s}|Z_s^{DF} + \sigma S_s \Phi(d_1)| \right) \mathrm{d}s - \int_t^T Z_s^{DF} \mathrm{d}W_s.$$

In Table 4, the BSDE (V_t^{DF}, Z_t^{DF}) is solved for several call and put options using the SRMDP algorithm. Besides, exact solutions (ES) are computed through the difference between Black-Scholes formula where IM's contribution is considered as a time-dependent dividend yield and the classical Black-Scholes formula with $R = 0$.

[2]For a put option an analogous BSDE can be written taking into account that $Z_t^{BS} = \sigma S_t(\Phi(d_1) - 1)$.

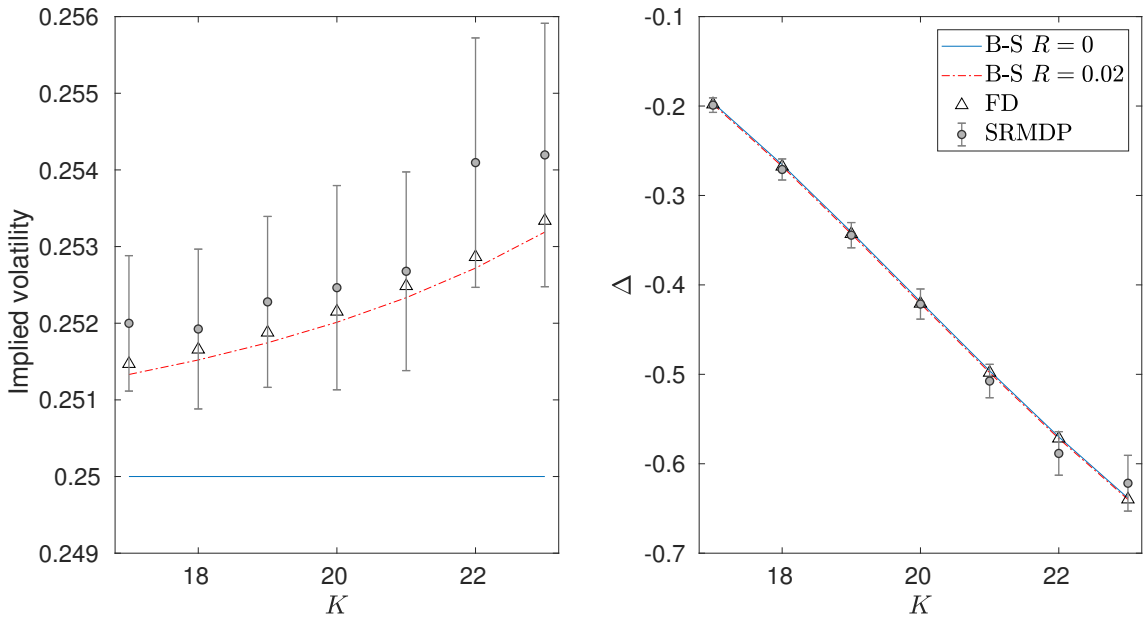

FIGURE 2. Implied volatility and delta for put options with spot value $S_0 = 20$ and different strikes K.

	B-S $R = 0$		B-S $R = 0.02$		FD		SRMDP	
K	B-S$(0, S_0)$	$\Delta(0, S_0)$	B-S$(0, S_0)$	$\Delta(0, S_0)$	$V(0, S_0)$	$\nabla V(0, S_0)$	95% CI $V_0^{NL}(S_0)$	95% CI $Z_0^{NL}(S_0)/(\sigma S_0)$
17	0.6168	-0.1963	0.6241	-0.1980	0.6249	-0.1981	$[0.6229, 0.6328]$	$[-0.2071, -0.1908]$
18	0.9231	-0.2655	0.9331	-0.2675	0.9340	-0.2676	$[0.9289, 0.9426]$	$[-0.2827, -0.2592]$
19	1.3101	-0.3408	1.3229	-0.3429	1.3239	-0.3430	$[1.3186, 1.3350]$	$[-0.3585, -0.3303]$
20	1.7781	-0.4188	1.7938	-0.4209	1.7949	-0.4209	$[1.7869, 1.8077]$	$[-0.4383, -0.4046]$
21	2.3240	-0.4961	2.3426	-0.4981	2.3438	-0.4981	$[2.3350, 2.3557]$	$[-0.5262, -0.4888]$
22	2.9421	-0.5699	2.9635	-0.5718	2.9646	-0.5717	$[2.9615, 2.9871]$	$[-0.6128, -0.5641]$
23	3.6251	-0.6383	3.6490	-0.6400	3.6501	-0.6398	$[3.6436, 3.6695]$	$[-0.6529, -0.5906]$

TABLE 2. Price and delta for put options with spot value $S_0 = 20$ and different strikes K.

Once again these tests allow us to demonstrate that SRMDP algorithm provides accurate results in one dimension.

4.3. Nested Monte Carlo for computing (V^L, Z^L) in dimension 1

As discussed in Section 3.2, we can further approximate the solution of non-linear BSDE V^{NL} with a linear BSDE V^L with portfolio weight $Z = Z^{BS}$. In this case, we have an explicit stochastic representation for Z_t^{BS} given as follows

$$Z_t^{BS} = \frac{\partial V_t^{BS}}{\partial S}(\sigma S_t)^{-1},$$

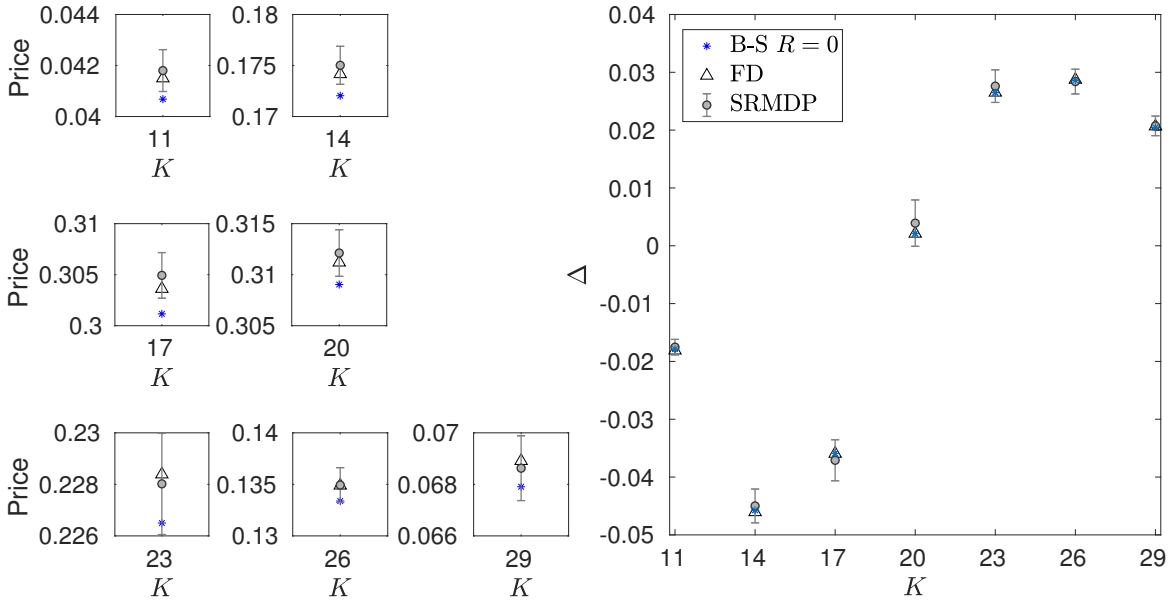

FIGURE 3. Price and delta for butterfly options with spot value $S_0 = 20$ and different strikes K.

	B-S $R = 0$		FD		SRMDP	
K	B-S$(0, S_0)$	$\Delta(0, S_0)$	$V(0, S_0)$	$\nabla V(0, S_0)$	95% CI $V_0^{NL}(S_0)$	95% CI $Z_0^{NL}(S_0)/(\sigma S_0)$
11	0.0407	-0.0178	0.0415	-0.0181	$[0.0410, 0.0426]$	$[-0.0189, -0.0162]$
14	0.1720	-0.0457	0.1742	-0.0461	$[0.1732, 0.1769]$	$[-0.0479, -0.0421]$
17	0.3012	-0.0359	0.3036	-0.0359	$[0.3027, 0.3072]$	$[-0.0406, -0.0336]$
20	0.3090	0.0021	0.3112	0.0021	$[0.3098, 0.3144]$	$[-0.0001, \ 0.0079]$
23	0.2265	0.0265	0.2284	0.0265	$[0.2261, 0.2300]$	$[\ 0.0248, \ 0.0304]$
26	0.1334	0.0286	0.1349	0.0287	$[0.1333, 0.1366]$	$[\ 0.0263, \ 0.0305]$
29	0.0679	0.0205	0.0689	0.0207	$[0.0674, 0.0699]$	$[\ 0.0190, \ 0.0224]$

TABLE 3. Price and delta for butterfly options with spot value $S_0 = 20$ and different strikes K.

	ES: B-S $R = 0.02$ − B-S $R = 0$		SRMDP	
	B-S$(0, S_0)$	$\Delta(0, S_0)$	95% CI $V_0^{DF}(S_0)$	95% CI $Z_0^{DF}(S_0)/(\sigma S_0)$
Call, $K = 17$	0.0302	0.0036	$[0.0302, 0.0304]$	$[\ 0.0033, \ 0.0037]$
Call, $K = 20$	0.0218	0.0040	$[0.0218, 0.0219]$	$[\ 0.0038, \ 0.0042]$
Call, $K = 23$	0.0136	0.0035	$[0.0136, 0.0137]$	$[\ 0.0033, \ 0.0036]$
Put, $K = 17$	0.0074	-0.0017	$[0.0074, 0.0075]$	$[-0.0020, -0.0016]$
Put, $K = 20$	0.0157	-0.0021	$[0.0157, 0.0158]$	$[-0.0023, -0.0019]$
Put, $K = 23$	0.0239	-0.0016	$[0.0239, 0.0240]$	$[-0.0018, -0.0015]$

TABLE 4. SRMDP algorithm for BSDE (V_t^{DF}, Z_t^{DF}).

where $V_t^{BS}(s) := \mathbb{E}[e^{-r(T-t)}\Phi(S_T)|S_t = s]$. Then, we use the likelihood ratio method of Broadie and Glasserman [BG96] to find out the derivative and get

$$Z_t^{BS}(s) = \frac{\partial}{\partial s}\mathbb{E}[e^{-r(T-t)}\Phi(X_T)|S_t = s](\sigma s)^{-1} = \mathbb{E}\left[e^{-r(T-t)}\Phi(S_T)\frac{W_T - W_t}{(T-t)}(\sigma s)\Big|S_t = s\right](\sigma s)^{-1}$$
$$= \mathbb{E}\left[e^{-r(T-t)}\big(\Phi(S_T) - \Phi(S_t)\big)\frac{W_T - W_t}{(T-t)}\Big|S_t = s\right]. \tag{4.5}$$

Therefore, in linear BSDE V^L with portfolio weight Z^{BS}, we have

$$V_0^L = \mathbb{E}\left[e^{-rT}\Phi\left(S_T\right) + \int_0^T e^{-rs}\left(RC_\alpha|Z_s^{BS}|\sqrt{(s+\Delta)\wedge T - s}\right)ds\right]$$
$$= \mathbb{E}\left[e^{-rT}\Phi\left(S_T\right) + Te^{-rU}\left(RC_\alpha|Z_U^{BS}|\sqrt{(U+\Delta)\wedge T - U}\right)\right],$$

where U is a uniformly distributed independent random variable on $[0, T]$ and Z_s^{BS} is given as in (4.5). By once again using the likelihood ratio method, we get the following formula

$$Z_0^L = \mathbb{E}\Big[e^{-rT}\Phi\left(S_T\right)\frac{W_T}{T} + Te^{-rU}\left(RC_\alpha|Z_U^{BS}|\sqrt{(U+\Delta)\wedge T - U}\right)\frac{W_U}{U}\Big].$$

We solve the linear BSDE V^L by taking advantage of finite difference method and Nested Monte Carlo algorithm (Nested MC) for different payoffs (calls, puts and butterfly options) where we use the same model parameters as earlier. In Nested Monte Carlo algorithm, we estimate Z^{BS} as in (4.5) using 100 independent inner sample paths for each outer Monte Carlo sample path. The results are presented in Table 5. We observe that as the Nested Monte Carlo algorithm results are accurate in one dimension, the algorithm provides an alternative to compute the estimates for (V^L, Z^L) in higher dimensions.

	FD		Nested MC	
			95% CI	95% CI
	$V(0, S_0)$	$\nabla V(0, S_0)$	$V_0^L(S_0)$	$Z_0^L(S_0)/(\sigma S_0)$
Call, $K = 17$	3.9843	0.8072	$[3.9796, 3.9856]$	$[0.8059, 0.8082]$
Call, $K = 20$	2.1971	0.5852	$[2.1931, 2.1979]$	$[0.5843, 0.5862]$
Call, $K = 23$	1.0953	0.3653	$[1.0924, 1.0958]$	$[0.3644, 0.3659]$
Put, $K = 17$	0.6249	-0.1981	$[0.6233, 0.6249]$	$[-0.1983, -0.1977]$
Put, $K = 20$	1.7950	-0.4209	$[1.7937, 1.7966]$	$[-0.4216, -0.4205]$
Put, $K = 23$	3.6502	-0.6398	$[3.6468, 3.6511]$	$[-0.6407, -0.6393]$
Butterfly, $K = 11$	0.0414	-0.0181	$[0.0412, 0.0415]$	$[-0.0181, -0.0180]$
Butterfly, $K = 20$	0.3112	0.0021	$[0.3112, 0.3119]$	$[0.0021, 0.0022]$
Butterfly, $K = 29$	0.0689	0.0206	$[0.0686, 0.0690]$	$[0.0206, 0.0207]$

TABLE 5. Nested MC algorithm for BSDE (V_t^L, Z_t^L).

4.4. **Basket options in higher dimensions**

In this section we solve the non-linear BSDE in high dimensions using SRMDP algorithm. In this setting, traditional full grid methods like finite difference are not able to tackle the problem for dimension greater than 3.

We consider call option on a basket of d assets where the asset process is modelled by multi-dimensional geometric Brownian motion with constant correlation $\rho_{ij} = \rho = 0.75$ for $i \neq j$ and constant volatility $\sigma_0 = 0.25$. The full-rank volatility matrix σ in model (3.1) is then given by

$$\sigma\sigma^\top = \Sigma \text{ where } \Sigma := (\Sigma_{ij})_{1\leq i,j\leq d} \text{ with } \Sigma_{ij} = \sigma_0^2\rho, i \neq j \text{ and } \Sigma_{ii} = \sigma_0^2.$$

Then, $A_0 := \left(\left((\sigma^1 S_0^1)^\top, \ldots, (\sigma^d S_0^d)^\top\right)^\top\right)^{-1}$ where σ^i is the ith row of σ. The payoff is given by

$$\Phi(S_T^1, \ldots, S_T^d) = \left(\sum_i p^i S_T^i - K\right)^+.$$

The option expiration is set to $T = 1$ year and the interest rate $r = 0.02$. We suppose that weights $p_i = \frac{1}{d}$ for all i. The strike price K equals 20 and the initial values of the assets $S_0 = (S_0^1, \ldots, S_0^d)$ are specified in Table 6. The rest of the model parameters are the same as earlier. In this table, we present prices and deltas for different basket options with several underlyings. In the first column, classical crude Monte Carlo values are shown (MC $R = 0$, IM was not considered). In the second column SRMDP values are displayed taking into account IM.

	MC ($R = 0$)		SRMDP ($R = 0.02$)	
	95% CI	95% CI	95% CI	95% CI
S_0	$V_0^{BS}(S_0)$	$Z_0^{BS}(S_0)A_0$	$V_0^{NL}(S_0)$	$Z_0^{NL}(S_0)A_0$
$(18, 20)$	$[1.5102, 1.5113]$	$[-0.0685, -0.0682]$	$[1.5015, 1.5468]$	$[-0.0772, -0.0649]$
		$[0.6237, 0.6245]$		$[0.6297, 0.6556]$
$(18, 20, 22)$	$[2.0067, 2.0081]$	$[-0.4676, -0.4671]$	$[1.9915, 2.0447]$	$[-0.4756, -0.4641]$
		$[0.3813, 0.3817]$		$[0.3873, 0.4167]$
		$[0.7435, 0.7443]$		$[0.7725, 0.7882]$
$(16, 18, 20, 22)$	$[1.4470, 1.4481]$	$[-0.6589, -0.6582]$	$[1.4677, 1.5090]$	$[-0.6689, -0.6182]$
		$[0.1062, 0.1064]$		$[0.0962, 0.1374]$
		$[0.4334, 0.4338]$		$[0.4234, 0.4628]$
		$[0.6093, 0.6100]$		$[0.5943, 0.6310]$
$(16, 18, 20, 22, 24)$	$[1.9672, 1.9676]$	$[-1.0467, -1.0455]$	$[1.9928, 2.0692]$	$[-1.0767, -1.0242]$
		$[-0.0855, -0.0852]$		$[-0.1155, -0.0752]$
		$[0.3342, 0.3347]$		$[0.3042, 0.3467]$
		$[0.5655, 0.5662]$		$[0.5355, 0.5762]$
		$[0.7039, 0.7047]$		$[0.6839, 0.7167]$

TABLE 6. Prices and deltas for the basket call option.

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