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BIFURCATIONS IN FAMILIES OF POLYNOMIAL SKEW PRODUCTS

MATTHIEU ASTORG AND FABRIZIO BIANCHI

We initiate a parametric study of families of polynomial skew products, i.e., polynomial endomorphisms of \mathbb{C}^2 of the form $F(z, w) = (p(z), q(z, w))$ that extend to endomorphisms of $\mathbb{P}^2(\mathbb{C})$. Our aim is to study and give a precise characterization of the *bifurcation current* and the *bifurcation locus* of such a family. As an application, we precisely describe the geometry of the bifurcation current near infinity, and give a classification of the hyperbolic components. This is the first study of a bifurcation locus and current for an explicit and somehow general family in dimension larger than 1.

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1. INTRODUCTION

A polynomial skew product in two complex variables is a polynomial endomorphism of \mathbb{C}^2 of the form $F(z, w) = (p(z), q(z, w))$ that extends to an endomorphism of $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$. The dynamics of these maps was studied in detail in [Jon99]. Despite (and actually because of) their specific form, they have already provided examples of dynamical phenomena not displayed by one-dimensional polynomials, such as wandering domains [ABD⁺16], or Siegel disks in the Julia set [Bia16b]. They have also been used to construct examples of non dynamical Green currents [Duj16b] or stable manifolds dense in \mathbb{P}^2 [Taf17]. In this paper we address the question of understanding the dynamical stability of such maps. In order to do this, let us first introduce the framework for our work and previous known results.

A *holomorphic family* of endomorphisms of \mathbb{P}^k is a holomorphic map $f: M \times \mathbb{P}^k \rightarrow M \times \mathbb{P}^k$ of the form $f(\lambda, z) = (\lambda, f_\lambda(z))$. The complex manifold M is the *parameter space* and we require that all f_λ have the same degree. In dimension $k = 1$, the study of *stability* and *bifurcation* within such families was initiated by Mané-Sad-Sullivan [MSS83] and Lyubich [Lyu83] in the 80s. They proved that many possible definitions of stability are actually equivalent, allowing one to decompose the parameter space of any family of rational maps into a *stability locus* and a *bifurcation locus*. In 2000, by means of the *Lyapunov function*, DeMarco [DeM01, DeM03] constructed a natural *bifurcation current* precisely supported on the bifurcation locus. This allowed for the start of a pluripotential study of the bifurcations of rational maps.

The simplest and most fundamental example we can consider is the quadratic family. We have a parameter space $M = \mathbb{C}$ and, for every $\lambda \in M$, we consider the quadratic polynomial $f_\lambda(z) = z^2 + \lambda$. In this case, it is possible to prove that the bifurcation locus is precisely the boundary of the *Mandelbrot set* (notice that, in this context, the equality between the bifurcation measure and the harmonic measure for the Mandelbrot set is due to N. Sibony, see [Sib81]). The study of this particular family is of fundamental importance for all the theory, since, by a result of McMullen [McM00] (see also [Lei90]), copies of the Mandelbrot set are dense in the bifurcation locus of any family of rational maps. Understanding the geometry of the Mandelbrot set is still today a major research area.

The theory by Mané-Sad-Sullivan, Lyubich and De Marco was recently extended to any dimension by Berteloot, Dupont, and the second author [BBD15, Bia16a], see Theorems 2.6 and 2.8 below (the second, for an adapted version of the main result in our context). Despite the quite precise understanding of the relation between the various phenomena related to stability and bifurcation (motion of the repelling cycles, Lyapunov function, Misiurewicz parameters), apart from specific examples ([BT17]) or near special parameters ([BB16, Duj16a, Taf17]), we still miss a concrete and somehow general family whose bifurcations can be explicitly exhibited and studied. This paper aims at providing a starting point for this study for any family of polynomial skew products. More precisely, we will establish *equidistribution properties* towards the bifurcation current (Section 3), study the possible *hyperbolic components* (Sections 7 and 8) and give a precise description of the *accumulation of the bifurcation locus at infinity* (Section 6). This will be achieved by means of precise formulas for the Lyapunov exponents, which will in turn allow for a precise *decomposition of the bifurcation locus* (Section 4).

Let us now be more specific, and enter more into the details of our results and techniques. While many of the results apply to more general families, we will mainly focus on the family of *quadratic skew products*. It is not difficult to see (Lemma 5.1) that the dynamical study of this family can be reduced to that of the family

$$(z, w) \mapsto (z^2 + d, w^2 + az^2 + bz + c).$$

It is also quite clear (see also Section 4) that the bifurcation of this family consists of two parts: the bifurcation locus associated to the polynomial family $z^2 + d$, and a part corresponding to “vertical” bifurcation in the fibres. We thus fix the base polynomial $p(z) = z^2 + d$ and consider the bifurcations associated with the other three parameters. Our first result is the following basic decomposition of both the bifurcation locus and current. This is essentially based on formulas for the Lyapunov exponents of a polynomial skew product due to Jonsson [Jon99] (see Theorem 2.2). Our result is actually more general than the situation described above, and holds in any family of polynomial skew products, see Section 4.

Theorem A. *Let $F_\lambda(z, w) = (p(z), q_\lambda(z, w))$ be a family of polynomial skew products of degree d . Then*

$$(1) \quad T_{\text{bif}} = \int_{z \in J_p} T_{\text{bif}, z} \mu_p = \lim_{N \rightarrow \infty} \frac{1}{d^N} \sum_{z \in \text{Per}_N(p)} T_{\text{bif}}(Q_z^N)$$

and

$$(2) \quad \text{Bif}(F) = \bigcup_{z \in J_p} \text{Bif}_z = \overline{\bigcup_N \bigcup_{z \in \text{Per}_N(p)} \text{Bif } Q_z^N}.$$

Here, we denote by Bif_z (respectively $T_{\text{bif}, z}$) the bifurcation locus (respectively current) associated to the non autonomous iteration of the polynomials $Q_{\lambda, z}^n := q_{\lambda, p^n(z)} \circ \cdots \circ q_{\lambda, p(z)} \circ q_{\lambda, z}$. These are defined respectively as the non-normality locus for the iteration of some critical point in the fibre at z , and the Laplacian of the corresponding sum of the Green function evaluated at the critical points (see Section 4.1). In the case of a periodic point z , we are actually considering iterations of the return maps to the fibre. The result follows combining the above mentioned result by Jonsson, the equidistribution of the periodic points with respect to the equilibrium measure, and the characterization of the bifurcation locus as the closure of the *Misiurewicz parameters* ([BBD15], and Theorem 2.8 below).

Another classical way to approximate the bifurcation current is by seeing this as a limit of currents detecting dynamically interesting parameters. A first example of this is a theorem by Levin [Lev82] (see also [Lev90]) stating that the centres of the hyperbolic components of the Mandelbrot set equidistribute the bifurcation current, which is supported on its boundary. This means that the bifurcation current detects the asymptotic distribution in the parameter space of the parameters possessing a periodic critical point of period n . This result was later generalized in order to cover any family of polynomials [BB11, Oku14], the distribution of parameters with a cycle of any given multiplier [BB11, BG15b, Gau16] and also the distribution of preperiodic critical point [DF08, FG15]. See also [GOV17] for the most recent account and results in this direction for families of rational maps.

In our situation, we can prove the following equidistribution result, giving the convergence for the parameters admitting a periodic point with *vertical* multiplier η (the study of “horizontal” multipliers naturally gives the bifurcation current for the base p). We also have a more general statement (see Theorem 3.4 and Corollary 3.5) valid for any family of endomorphism of \mathbb{P}^k . This gives also a first parametric equidistribution result for holomorphic dynamical systems in several complex variables.

Theorem B. *Let F be a holomorphic family of polynomial skew-products of degree $d \geq 2$ over a fixed base p , and parametrized by $\lambda \in M$. For all $\eta \in \mathbb{C}$ outside of a polar subset, we have*

$$\frac{1}{d^{2n}}[\text{Per}_n^v(\eta)] \rightarrow T_{\text{bif}},$$

where

$$\text{Per}_n^v(\eta) := \{\lambda \in M : \exists(z, w) \text{ such that } F_\lambda^n(z, w) = (z, w) \text{ and } \frac{\partial F_\lambda^n}{\partial w}(z, w) = \eta\}.$$

The equidistribution result stated above plays a crucial role in the next step of our investigation: the description of the accumulation of the bifurcation locus at infinity. Notice that the study of degenerating dynamical systems, and in particular of their Lyapunov exponents, is an active current area of research, see for instance [Fav16]. In the following result not only we describe this accumulation locus, but we quantify this accumulation by means of the equilibrium measure of the base polynomials. An analogous result for quadratic rational map is proved in [BG15a].

Theorem C. *Let $F_{abc}(z, w) = (p(z), w^2 + az^2 + bz + c)$. Then the bifurcation current extends to a current \hat{T}_{bif} on \mathbb{P}^3 and*

$$\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2] = \int_{z \in J_p} [E_z] \mu_p = \pi_* \left(\frac{1}{2} \int_z ([\{z\} \times \mathbb{P}^1] + [\mathbb{P}^1 \times \{z\}]) \mu_p(z) \right)$$

where $E_z := \{[a, b, c] \in \mathbb{P}_\infty^2 : az^2 + bz + c = 0\}$ and $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}_\infty^2$ is the map that associates to (x, y) the point $[a, b, c]$ such that x, y are the roots of $aX^2 + bX + c$.

A similar result also applies for subfamilies of F_{abc} . In particular, for the subfamily given by $a = 0$, we have the following description. Notice that this is not only a by-product of the previous theorem, but actually a main step in the proof. Indeed, the current in Theorem C will be constructed by slicing it with respect to lines corresponding to special subfamilies. Tools from the theory of horizontal currents as developed by Dinh and Sibony also play a crucial role in our proof.

Theorem C’. *Let $F_{bc}(z, w) = (p(z), w^2 + bz + c)$. Then the bifurcation current extends to a current \hat{T}_{bif} on \mathbb{P}^2 and*

$$\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^1] = (\pi_0)_* \mu_p$$

where $\pi_0 : \mathbb{C} \rightarrow \mathbb{P}_\infty^1$ is given by $\pi_0(z) = [-1, z]$.

All the results presented until now are related to the description of the bifurcation locus. Our last result concerns stable components, and in particular *unbounded* hyperbolic components. It follows from the results above that the stability of a polynomial skew product is determined by the behaviour of the critical points of the form $(z, 0)$, with $z \in J(p)$. We prove that there exists a natural decomposition of the parameter space: a (compact) region \mathcal{C} , where all these critical points have bounded orbit, a region \mathcal{D}

where all the critical points $(z, 0)$ escape under iteration, and a region \mathcal{M} where maps admit critical points with either behaviour.

In particular, components in \mathcal{D} can be thought of as the analogous in this situation of the complement of the Mandelbrot set. By the description above, the accumulation of the bifurcation locus at infinity is given by the image under π of the points in $\mathbb{P}^1 \times \mathbb{P}^1$ with at least one component in the Julia set of p . Thus, \mathcal{D} in particular consists of components accumulating on the image of $\mathcal{F}(p) \times \mathcal{F}(p)$ by the map π of Theorem C (here $\mathcal{F}(p)$ denotes the Fatou set of p). Let \mathcal{D}' be the set of these components. Our last result gives a complete classification of these components, essentially stating that distinct couples of Fatou components are associated to distinct hyperbolic components.

Theorem D. *Let $F_{abc}(z, w) = (p(z), w^2 + az^2 + bz + c)$ such that the Julia set of p is locally connected. Let \mathcal{D} be the set of parameters (a, b, c) such that all critical points $(z, 0)$ escape to infinity, and \mathcal{D}' the set of the components in \mathcal{D} accumulating on some point of $\pi(\mathcal{F}(p) \times \mathcal{F}(p))$. There is a natural bijection between \mathcal{D}' and the (non ordered) pairs of connected components of the Fatou set $\mathcal{F}(p)$ of p .*

As above, we state also the analogous result for the simpler case of the family $a = 0$.

Theorem D'. *Let $F_{bc}(z, w) = (p(z), w^2 + bz + c)$ such that the Julia set of p is locally connected. Let \mathcal{D} be the set of parameters (b, c) such that all critical points $(z, 0)$ escape to infinity, and \mathcal{D}' the set of the components in \mathcal{D} accumulating on some point of $\pi(\mathcal{F}(p))$. There is a natural bijection between \mathcal{D}' and the connected components of the Fatou set $\mathcal{F}(p)$ of p .*

Theorem D is proved by exhibiting distinct topologies for the Julia sets of maps in components associated to distinct Fatou components (see Theorem 7.13). In Section 8 we also study the problem of the existence of unbounded hyperbolic components in \mathcal{M} . By adapting an example by Jonsson, we can prove that such a component can actually exist (see Propositions 8.5 and 8.8).

The paper is organized as follows. After the presentation of the necessary preliminaries on polynomial skew products and bifurcations in several variables, the exposition is divided in two parts. In the first we prove Theorems A and B, giving the approximations for the bifurcation current that we need in the sequel. In this part, we do not need to restrict to quadratic skew products, and actually, as mentioned above, we can prove an equidistribution formula for any family of endomorphisms of \mathbb{P}^k . Then, in the second part, we focus on quadratic skew products, and in particular on the study of their parameter space near infinity (proving Theorems C and D).

2. PRELIMINARIES: POLYNOMIAL SKEW PRODUCTS AND BIFURCATIONS

2.1. Polynomial skew products. In this section we collect notations, definitions, and results concerning the dynamics of polynomial skew products that we will need through all the paper. Unless otherwise stated, all the results are due to Jonsson [Jon99].

Definitions and notations. We consider here a polynomial skew product of the form

$$F(z, w) = (p(z), q(z, w))$$

of algebraic degree $d \geq 2$. Recall that we require F to be extendible to \mathbb{P}^2 . This means that p has degree d , and that the coefficient of w^d in the second coordinate is non zero.

We will assume that this is 1. The second coordinate will be also written as $q_z(w)$, to emphasize the variable w and see the map q as a family of polynomials depending on z . We shall denote by z_1, \dots, z_n, \dots the orbit of a point $z \in \mathbb{C}$ under the base polynomial p . In this way, we can write

$$F^n(z, w) = (z_n, q_{z_{n-1}} \cdots q_{z_1} \circ q_z(w)).$$

We shall denote by $Q_z^n(w)$ the second coordinate of F^n .

We will be mainly interested in the recurrent part of the dynamics. In particular, we restrict in the following to the points of the form (z, w) , with $z \in K_p$ (we denote by J_p and K_p the Julia set and the filled Julia set of p , respectively). The family $\{Q_z^n\}_n$ gives rise to a (non autonomous) iteration on the fibre $\{z\} \times \mathbb{C}$. Since $z \in K_p$, the orbit of (z, w) is bounded if and only if the orbit of w under the non autonomous iteration of Q_z^n is bounded. We then define K_z as the points in \mathbb{C} (to be thought of as the fibre over z) with bounded orbit under this iteration. The set J_z is the boundary of K_z . Notice that J_z and K_z can be defined for any point $z \in \mathbb{C}$, but only detect the boundedness of the second coordinate of the orbit.

Green functions and Julia set. A standard way to detect the boundedness of an orbit (or its escape rate) is by means of a dynamical Green function. In our setting, given $(z, w) \in \mathbb{C}^2$, we can consider three possible Green functions:

- (1) the *Green function* of (z, w) , defined as $G(z, w) := \lim_{n \rightarrow \infty} d^{-n} \log^+ |F^n(z, w)|$;
- (2) the *horizontal Green function* $G_p(z, w) = G_p(z)$, where $G_p(z) = \lim_{n \rightarrow \infty} |p^n(z)|$ is the Green function of the base polynomial p , and
- (3) for $z \in \mathbb{C}$, the *vertical Green function* $G_z(w) := G(z, w) - G_p(z, w)$.

The last function detects the escape rate of the sequence $Q_z^n(w)$. The set K_z is then the zero level of the function G_z . In the case of points $z \in K_p$, the Green function reduces to the vertical Green function. By means of the Green functions, it is immediate to deduce that the maps $z \mapsto K_z$ and $z \mapsto J_z$ are respectively upper and lower semicontinuous with respect to the Hausdorff topology.

Another useful feature of the Green function is that it allows to construct the *equilibrium measure* for the map F . The construction is now classical and proceeds as follows (see for instance [FS95, DS10]). First of all, we consider the *Green current* $T := dd^c G$ (on \mathbb{P}^2). This is positive, closed (1,1)-current. Its support is precisely the non normality locus of the sequence of iterates of F . Since the potential of T is continuous (actually Hölder continuous) it is possible to consider the wedge product $\mu := T \wedge T$. This is a positive measure, which turns out to be invariant, of constant Jacobian, ergodic, mixing. It detects the distribution of periodic points and preimages of generic points. Its support is the *Julia set* J of the map F . In the case of polynomial skew products, we have the following structure result for J .

Theorem 2.1. *Let F be a polynomial skew product. Then*

$$J(F) = \overline{\bigcup_{z \in J_p} \{z\} \times J_z}.$$

Moreover, $J(F)$ coincides with the closure of the repelling points.

Notice that the last assertion in this result is known not to hold for general endomorphisms of \mathbb{P}^2 ([HP94, FS01]).

Lyapunov exponents. Since the equilibrium μ is ergodic, by Oseledec Theorem we can associate two Lyapunov exponents to it. The idea, again now classical, is the following. There exists an invariant splitting $E_1 \oplus E_2$ of the tangent space at (μ -almost all) points $x \in J$, depending measurably on x . The differential of F acts on these sub bundles E_i , and there exists constants λ_1, λ_2 such that $\|DF^n\|_{E_i} \sim e^{n\lambda_i}$. More precisely, for μ -almost every x and every $v \in E_i(x)$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|DF_x^n(v)\| = \lambda_i.$$

The constants λ_i are called the Lyapunov exponents of the ergodic system (J, F, μ) . In the case of polynomial skew products, we have the following explicit formulas for the Lyapunov exponents (for more general formulas, valid for any regular polynomial, see [BJ00]).

Theorem 2.2. *Let F be a polynomial skew product. Then the equilibrium measure of F admits the two Lyapunov exponents*

$$L_p = \log d + \sum_{z \in C_p} G_p(z) \text{ and } L_v = \log d + \int \left(\sum_{w \in C_z} G(z, w) \right) \mu_p,$$

where C_p and μ_p are the critical set and the equilibrium measure of p and C_z is the critical set of q_z .

Notice, in particular, that the Lyapunov exponent L_p coincides with the Lyapunov exponent of the system (J_p, p, μ_p) . We use the notation L_p and L_v (*vertical*) because we will not assume any ordering between these two quantities.

Hyperbolicity and vertical expansion. We conclude this section by introducing an adapted notion of hyperbolicity, particularly useful in the study of polynomial skew products. Recall that an endomorphism F of \mathbb{P}^k is *hyperbolic* or *uniformly expanding on the Julia set* if there exist constants $c > 0, K > 0$ such that, for every $x \in J$ and $v \in T_x \mathbb{P}^k$, we have $\|DF_x^n(v)\|_{\mathbb{P}^k} \geq cK^n$ (with respect for instance to the standard norm on \mathbb{P}^k). In the case of polynomial skew products, this condition in particular forces the base polynomial p to be hyperbolic. Since we will be mainly concerned with the vertical dynamics over the Julia set of p , the following definition gives an analogous notion of hyperbolicity, more suitable to our purposes.

Given an invariant set Z for p (we shall primarily use $Z = J_p$) we set

- $C_Z := \bigcup_{z \in Z} \{z\} \times C_z$ for the *critical set over Z* ,
- $D_Z := \bigcup_{n \geq 1} f^n C_Z =: \bigcup_{z \in Z} \{z\} \times D_{Z,z}$ for the *postcritical set over Z* , and
- $J_Z := \bigcup_{z \in Z} \{z\} \times J_z =: \bigcup_{z \in Z} \{z\} \times J_z^*$ for the *Julia set over Z* .

When dropping the index Z , we mean that we are considering $Z = J_p$. We then have the following definition.

Definition 2.3. *Let $F(z, w) = (p(z), q(z, w))$ be a polynomial skew product and $Z \subset \mathbb{C}$ such that $p(Z) \subset Z$. We say that F is *vertically expanding over Z* if there exist constants $c > 0$ and $K > 1$ such that $(Q_z^n)'(w) \geq cK^n$ for every $z \in Z$, $w \in J_z$ and $n \geq 1$.*

For polynomials on \mathbb{C} , hyperbolicity is equivalent to the fact that the closure of the postcritical set is disjoint from the Julia set. In our situation, we have the following analogous characterization.

Theorem 2.4. *Let $F(z, w) = (p(z), q(z, w))$ be a polynomial skew product. Then the following are equivalent:*

- (1) F is vertically expanding on Z ;
- (2) $D_Z \cap J_Z = \emptyset$.

The previous results allows one to give a similar characterization of hyperbolicity for polynomial skew products. Notice that the same result is not known for general endomorphisms of \mathbb{P}^k .

Theorem 2.5. *Let $F(z, w) = (p(z), q(z, w))$ be a polynomial skew product. Then the following are equivalent:*

- (1) F is hyperbolic;
- (2) $D \cap J = \emptyset$;
- (3) p is hyperbolic, and F is vertically expanding over J_p .

2.2. Stability, bifurcations, and hyperbolicity. In this paper we will be concerned with stability and bifurcation of polynomial skew product of \mathbb{C}^2 (extendible to \mathbb{P}^2). The definition and study of these for endomorphisms of projective spaces of any dimension is given in [BBD15, Bia16a]. For a general presentation of this and related results, see also [BB17].

Theorem 2.6 ([BBD15, Bia16a]). *Let F be a holomorphic family of endomorphisms of \mathbb{P}^k of degree $d \geq 2$. Then the following are equivalent:*

- (1) asymptotically, the repelling cycles move holomorphically;
- (2) there exists an equilibrium lamination for the Julia sets;
- (3) $dd^c L = 0$;
- (4) there are no Misiurewicz parameters.

The holomorphic motion of the repelling cycles is defined as in dimension 1 (see e.g. [Ber13, Duj11], or [BBD15, Definition 1.2] in this context). The *asymptotically essentially* means that we can follow $d^{kn} - o(d^{kn})$ out of the $\sim d^{kn}$ repelling points, see [Bia16a, Definition 1.3]. This condition can be improved to the motion of all repelling cycles contained in the Julia set if $k = 2$, or if the family is an open set in the family of *all* endomorphisms of a given degree. See also [Ber17] for another description of the asymptotic bifurcations of the repelling cycles. L denotes the sum of the Lyapunov exponents, which is a psh function on the parameter space. Thus, $dd^c L$ is a positive closed (1,1) current on the parameter space. Finally, we just mention that the *equilibrium lamination* is a weaker notion of holomorphic motion, that provides an actual holomorphic motion for a full measure (for the equilibrium measure) subset of the Julia set (see [BBD15, Definition 1.4]). Since it will be used in the sequel, we give the precise definition of the last concept in the theorem above. This is a generalization to any dimension of the notion of non-persistently preperiodic (to a repelling cycle) critical point.

Definition 2.7. *Let f be a holomorphic family of endomorphisms of \mathbb{P}^k and let C_f be the critical set of the map $f(\lambda, z) = (\lambda, f_\lambda(z))$. A point λ_0 of the parameter space M is called a Misiurewicz parameter if there exist a neighbourhood $N_{\lambda_0} \subset M$ of λ_0 and a holomorphic map $\sigma: N_{\lambda_0} \rightarrow \mathbb{C}^k$ such that:*

- (1) for every $\lambda \in N_{\lambda_0}$, $\sigma(\lambda)$ is a repelling periodic point;

- (2) $\sigma(\lambda_0)$ is in the Julia set J_{λ_0} of f_{λ_0} ;
- (3) there exists an n_0 such that $(\lambda_0, \sigma(\lambda_0))$ belongs to some component of $f^{n_0}(C_f)$;
- (4) $\sigma(N_{\lambda_0})$ is not contained in a component of $f^{n_0}(C_f)$ satisfying 3.

In view of Theorem 2.6, it makes sense to define the *bifurcation locus* as the support of the *bifurcation current* $T_{\text{bif}} := dd^c L$. If any (and thus all of the) conditions in Theorem 2.6 hold, we say that the family is *stable*.

Since we will be mainly concerned with families of polynomial skew products in dimension 2, we cite an adapted version of the result above in our setting.

Theorem 2.8 ([BBD15]). *Let F be a holomorphic family of polynomial skew products of degree $d \geq 2$. Then the following are equivalent:*

- (1) *the repelling cycles move holomorphically;*
- (2) *there exists an equilibrium lamination for the Julia sets;*
- (3) $dd^c L = dd^c(L_p + L_v) = 0$;
- (4) *there are no Misiurewicz parameters.*

As mentioned above, since the dimension is $k = 2$, we can promote the first condition in Theorem 2.6 to the motion of all the repelling cycles contained in the Julia set. By the result of Jonsson (Theorem 2.1), we then know that polynomial skew products cannot have repelling points outside the Julia set.

Let us now consider a *hyperbolic parameter* for a family of endomorphisms of \mathbb{P}^2 . A natural question is to ask whether all the corresponding stability component consists of hyperbolic parameters. While this is not known in general, by Theorem 2.5 it follows that this is true for polynomial skew products.

Lemma 2.9. *Let F be a stable family of polynomial skew products. If there exists λ_0 such that F_{λ_0} is hyperbolic, then F_λ is hyperbolic for every λ in the parameter space.*

Proof. Let $M \ni \lambda_1 \neq \lambda_0$ be any parameter. Assume that λ_1 is not hyperbolic. By Theorem 2.5, this implies that the postcritical hypersurfaces $f^n(C_{f_{\lambda_1}})$ have an accumulation point z_1 on the Julia set of f_{λ_1} .

Since the family is stable, by Theorem 2.8 the repelling points move holomorphically. Denote by \mathcal{R} the collection of the holomorphic graphs in the product space $M \times \mathbb{P}^k$. Since repelling points are dense in the Julia set, there exists a sequence of points r_n with $r_n \in \mathcal{R}$ converging to z_1 . By compactness, there exist a graph γ over M which is an accumulation point for the sequence of the holomorphic motions of the repelling points r_n . Since the family is stable there are no Misiurewicz parameters. Thus, all these motions do not create transverse intersections with the postcritical set. This, by Hurwitz theorem, implies that the graph γ is contained in the closure of the postcritical set. This gives the desired contradiction, since this cannot be true at the hyperbolic parameter in λ_0 . \square

Remark 2.10. *By the same argument (and using Theorem 2.4) we can prove that if a parameter in a stable component is vertically expanding, the same is true for all the parameter in the component. An improvement of this fact will be given in Lemma 7.3*

For families of polynomial skew products, it thus makes sense to talk about *hyperbolic components* (respectively *vertically expanding components*), i.e., stable components whose elements are (all) hyperbolic (respectively, vertically expanding). We will characterize

and study some components of this kind in Sections 7 and 8. We conclude this section with a result relating the existence of non hyperbolic component for families of polynomial skew products to the same question for the (possibly non autonomous) iteration of polynomial maps in dimension 1.

Proposition 2.11. *If a polynomial family of skew products has a non hyperbolic stable component, then some one-dimensional (possibly non autonomous) polynomial family has a non hyperbolic stable component.*

The proof of this proposition exploits the characterization of the stability of a skew product family with respect to the stability of the dynamics on the periodic fibres, and will be given in Section 4.3.

Part 1. Approximations for the bifurcation current

3. EQUIDISTRIBUTION RESULTS IN THE PARAMETER SPACE (THEOREM B)

Our aim in this section is to prove Theorem 3.12. We will actually prove a more general result (see Theorem 3.4 and Corollary 3.5), valid for any family of endomorphisms of \mathbb{P}^2 . We shall later specialise to our setting of polynomial skew products (see Section 3.2).

3.1. A general equidistribution result for endomorphisms of \mathbb{P}^k . Let us begin with a rather general equidistribution result that holds for families of endomorphisms of \mathbb{P}^k , $k \geq 1$. Let M be a complex manifold, and let

$$\begin{aligned} M \times \mathbb{P}^k &\rightarrow \mathbb{P}^k \\ (\lambda, z) &\mapsto f_\lambda(z) \end{aligned}$$

be a holomorphic map, defining a holomorphic family of endomorphisms of \mathbb{P}^k . Assume that for all $n \in \mathbb{N}^*$ there exists at least one parameter $\lambda \in M$ such that for all periodic points of exact period n for f_λ ,

$$\det(Df_\lambda^n(z) - \text{Id}) \neq 0.$$

Let

$$\widetilde{\text{Per}}_n^J = \{(\lambda, \eta) \in M \times \mathbb{C} : \exists z \in \mathbb{P}^k \text{ of exact period } n \text{ for } f_\lambda, \text{ and } \text{Jac} f_\lambda^n(z) = \eta\}$$

where we denote by Jac the determinant of the Jacobian matrix, and let Per_n^J be the closure of $\widetilde{\text{Per}}_n^J$ in $M \times \mathbb{C}$. As we will see below, Per_n^J is an analytic hypersurface in $M \times \mathbb{C}$.

Proposition 3.1. *There exists a sequence of holomorphic maps $P_n : M \times \mathbb{C} \rightarrow \mathbb{C}$ such that:*

- (1) *For all $\lambda \in M$, $P_n(\lambda, \cdot)$ is a monic polynomial of degree $\delta_n \sim \frac{d^{nk}}{n}$*
- (2) *$P_n(\lambda, \eta) = 0$ if and only if $(\lambda, \eta) \in \text{Per}_n^J$.*

Moreover, if $(\lambda, \eta) \in \text{Per}_n^J \setminus \widetilde{\text{Per}}_n^J$, there exists $z \in \mathbb{P}^k$ and $m < n$ dividing n such that $f_\lambda^m(z) = z$, $\text{Jac}(f_\lambda^m)(z) = \eta$, and 1 is an eigenvalue of $Df_\lambda^m(z)$.

Proof. Set

$$\mathcal{P}_n := \{(\lambda, z) \in M \times \mathbb{P}^k, f_\lambda^n(z) = z\}.$$

It is an analytic hypersurface of $M \times \mathbb{P}^k$. Let $\varphi_n : \mathcal{P}_n \rightarrow \mathbb{C}$ be defined by

$$\varphi_n(\lambda, z) = \det(Df_\lambda^n(z) - \text{Id}).$$

It is a holomorphic map on the regular set of \mathcal{P}_n . Now let Z_n be the projection of $\varphi_n^{-1}(0)$ on M . By Remmert's mapping theorem, Z_n is an analytic subset of M .

Since we assumed that M does not have persistent parabolic cycles, Z_n cannot be all of M , so it is a proper analytic subset of M . Now define

$$\Omega_n := M \setminus \bigcup_{k \leq n} Z_k$$

and

$$\begin{aligned} p_n : \Omega_n \times \mathbb{C} &\rightarrow \mathbb{C} \\ (\lambda, \eta) &\mapsto \prod_{z \in P_n(\lambda)} (w - \text{Jac } f_\lambda^n(z)) \end{aligned}$$

where $P_n(\lambda)$ denotes the set of periodic points of exact period n for f_λ .

Note that Ω_n is an open and dense subset of M , since it is the complement of a closed proper analytic subset of M . By the implicit function theorem and the definition of Ω_n , the periodic points move holomorphically in Ω_n , i.e., for any $\lambda_0 \in \Omega_n$ and any $z \in P_n(\lambda_0)$, there is a neighbourhood U of λ_0 in Ω_n and a holomorphic map $U \ni \lambda \mapsto z_\lambda \in P_n(\lambda)$. This implies that p_n is holomorphic on $\Omega_n \times \mathbb{C}$, as it is clearly holomorphic with respect to w . Since it is locally bounded, Riemann's extension theorem implies that it is in fact holomorphic on all of $M \times \mathbb{C}$.

Now notice that for all $\lambda \in M$, n divides the multiplicity of every root w of the polynomial $p_n(\lambda, \cdot)$. Indeed, if $z \in P_n(\lambda)$ is such that $w = \text{Jac } f_\lambda^n(z)$, then it is also the case for the other points of the cycle, namely the $f^m(z)$, $0 \leq m \leq n-1$. So, for every $\lambda \in M$, there is a unique monic polynomial map $P_n(\lambda, \cdot)$ such that $P_n(\lambda, \cdot)^n = p_n(\lambda, \cdot)$. Since p_n is globally holomorphic, so is P_n , and for all λ , $P_n(\lambda, \cdot)$ is a monic polynomial of degree $\delta_n \sim \frac{d^{nk}}{n}$.

Let us now analyse the zero set of P_n : it is the same as the zero set of p_n . It is clear from the definitions that if $(\lambda, \eta) \in \text{Per}_n^J$ then $p_n(\lambda, \eta) = 0$, and that conversely, if $p_n(\lambda, \eta) = 0$ and $\lambda \in \Omega_n$, then $(\lambda, \eta) \in \text{Per}_n^J$. Let us now assume that $p_n(\lambda, \eta) = 0$ and that $\lambda \notin \Omega_n$. Since Ω_n is dense in M , there exists a sequence $\lambda_i \rightarrow \lambda$ with $\lambda_i \in \Omega_n$. Since we have $\lim_{i \rightarrow \infty} p_n(\lambda_i, \eta) = 0$, there exists a sequence $(z_i)_{i \in \mathbb{N}}$ of points in $E(\lambda_i)$ such that $\text{Jac}(f_{\lambda_i}^n)(z_i) - \eta$ converges to zero. This means that (λ, η) is in the closure of $\widetilde{\text{Per}_n^J}$, thus proving the second item.

Let us now prove the last claim: we assume that $(\lambda, \eta) \in \text{Per}_n^J \setminus \widetilde{\text{Per}_n^J}$. Let $(z_i)_{i \in \mathbb{N}}$ and $(\lambda_i)_{i \in \mathbb{N}}$ be as above. By compactness of \mathbb{P}^k , up to extracting a subsequence we may assume that z_i converges to some $z \in \mathbb{P}^k$, which must satisfy $f_\lambda^n(z) = z$ and $\text{Jac}(f_\lambda^n)(z) = \eta$. Since $(\lambda, \eta) \notin \widetilde{\text{Per}_n^J}$, the exact period of z is some integer $m < n$ dividing n . By the implicit function theorem, if 1 were not an eigenvalue of $Df_\lambda^n(z)$, then for parameters close enough to λ there would be only one periodic point of period

dividing n near z . But that would contradict the fact that $z_i, f_{\lambda_i}^m(z_i), \dots, f_{\lambda_i}^{n/m-1}(z_i)$ each are distinct points of period n for f_{λ_i} that are all converging to z as $i \rightarrow \infty$. \square

For $\eta \in \mathbb{C}$, we denote by $\text{Per}_n^J(\eta)$ the set defined by

$$\text{Per}_n^J(\eta) := \{\lambda \in \mathbb{C} : (\lambda, \eta) \in \text{Per}_n^J\}.$$

The set $\text{Per}_n^J(\eta)$ is an analytic hypersurface of M .

Define now

$$\begin{aligned} L_n : M \times \mathbb{C} &\rightarrow \mathbb{C} \\ (\lambda, \eta) &\mapsto \frac{1}{d^{nk}} \log |P_n(\lambda, \eta)| \end{aligned}$$

By the Lelong-Poincaré equation, we have that $dd_{\lambda, \eta}^c L_n = \frac{1}{d^{nk}} [\text{Per}_n^J]$, where $\frac{1}{d^{nk}} [\text{Per}_n^J]$ is the (normalized) current of integration on Per_n^J . Likewise, we have

$$dd_{\lambda}^c L_n(\cdot, \eta) = \frac{1}{d^{nk}} [\text{Per}_n^J(\eta)].$$

Let also $L : M \rightarrow \mathbb{R}^+$ be the sum of the Lyapunov exponents of f_{λ} with respect to its equilibrium measure μ_{λ} . We recall here a useful result by Berteloot-Dupont-Molino (see also [BD17]).

Lemma 3.2 ([BDM08], Lemma 4.5). *Let f be an endomorphism of \mathbb{P}^k of algebraic degree $d \geq 2$. Let $\varepsilon > 0$ and let $R_n^{\varepsilon}(f)$ be the set of repelling periodic points z of exact period n for f , such that $|\frac{1}{n} \log |\text{Jac} f^n(z)| - L(f)| \leq 2\varepsilon$. Then for n large enough, $\text{card } R_n^{\varepsilon}(f) \geq d^{nk}(1 - \varepsilon)^3$.*

Note that in particular, this implies that $\text{card } R_n^{\varepsilon}(f) \sim_{n \rightarrow \infty} \text{card } E_n(f)$, where $E_n(f)$ is the set of periodic points of exact period n .

Theorem 3.3 ([BDM08]). *Let f be an endomorphism of \mathbb{P}^k of algebraic degree $d \geq 2$ and let $\varepsilon > 0$. Then*

$$\frac{1}{\text{card } R_n^{\varepsilon}(f)} \sum_{z \in R_n^{\varepsilon}(f)} \log |\text{Jac} f^n(z)| \rightarrow L(f).$$

Actually, the statement appearing in [BDM08] involves an average on the set $R_n(f)$ of all repelling cycles of period n instead of $R_n^{\varepsilon}(f)$, but using Lemma 3.2, it is not difficult to see that the two statements are equivalent.

Recall the definition of an Axiom A endomorphism $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$. Let Ω_f denote the non-wandering set, i.e.

$$\Omega_f := \{z \in \mathbb{P}^k : \forall U \text{ neighbourhood of } z, \exists n \in \mathbb{N}^* \text{ s.t. } f^n(U) \cap U \neq \emptyset\}.$$

We say that f is Axiom A if periodic points are dense in Ω_f , and Ω_f is hyperbolic.

We now state the main convergence result of this section.

Theorem 3.4. *Assume that there is at least one parameter $\lambda_0 \in M$ such that f_{λ_0} is Axiom A and $\{\text{Jac}(f_{\lambda_0}^n)(z) : f_{\lambda_0}^n(z) = z \text{ and } n \in \mathbb{N}\}$ is not dense in \mathbb{C} . We have $L_n \rightarrow L$, the convergence taking place in $L_{\text{loc}}^1(M \times \mathbb{C})$.*

By taking dd^c on both sides, we obtain the following equidistribution result as a corollary.

Corollary 3.5. *Under the same assumptions as in Theorem 3.4, for any $\eta \in \mathbb{C}$ outside of a polar set we have that*

$$\frac{1}{d^{nk}} [\text{Per}_n^J(\eta)] \rightarrow T_{\text{bif}},$$

the convergence taking place in the sense of currents.

In order to prove the convergence in Theorem 3.4, in the spirit of [BB09] we first study the convergence of suitable modifications of the potentials L_n .

Definition 3.6. *We define the functions L_n, L_n^+ and L_n^r as follows.*

$$\begin{aligned} L_n(\lambda, \eta) &= \frac{1}{d^{nk}} \log |P_n(\lambda, \eta)| \\ L_n^+(\lambda, \eta) &= \frac{1}{d^{nk}} \sum_{z \in E_n(\lambda)} \log^+ |\eta - \eta_n(z, \lambda)|, \text{ where } \eta_n(z, \lambda) := \text{Jac} f_\lambda^n(z) \\ L_n^r(\lambda) &= \frac{1}{2\pi d^{nk}} \int_0^{2\pi} \log |P_n(\lambda, r e^{it})| dt \end{aligned}$$

Lemma 3.7. *For any $\eta \in \mathbb{C}$, the sequence of maps $L_n^+(\cdot, \eta)$ converges pointwise and L_{loc}^1 to L on M .*

Proof. Fix $\eta \in \mathbb{C}$, and let $\varepsilon > 0$. We have:

$$|L_n^+(\lambda, \eta)| \leq \frac{\text{card } E_n(\lambda)}{d^{nk}} \sup_{z \in \mathbb{P}^k} \|Df_\lambda(z)\|$$

which is locally bounded from above. Moreover:

$$\begin{aligned} |L_n^+(\lambda, \eta)| &= \frac{1}{d^{nk}} \left(\sum_{z \in R_n^\varepsilon(\lambda)} \log |\eta - \eta_n(z, \lambda)| + \sum_{z \in E_n(\lambda) - R_n^\varepsilon(\lambda)} \log^+ |\eta - \eta_n(z, \lambda)| \right) \\ &= \frac{1}{d^{nk}} \sum_{z \in R_n^\varepsilon(\lambda)} \log |\eta_n(z, \lambda)| + O((L(\lambda) - \varepsilon)^{-n}) \\ &\quad + \frac{\text{card}(E_n(\lambda) - R_n^\varepsilon(\lambda))}{d^{nk}} \log(|\eta| + (L(\lambda) + \varepsilon)^n) \\ &= \frac{1}{d^{nk}} \sum_{z \in R_n^\varepsilon(\lambda)} \log |\eta_n(z, \lambda)| + o(1) \\ &= L(\lambda) + o(1). \end{aligned}$$

In the last two equalities, we used Theorem 3.3 and Lemma 3.2. Therefore the sequence of maps L_n^+ converges pointwise to $(\lambda, \eta) \mapsto L(\lambda)$ on $M \times \mathbb{C}$, and since the L_n 's are plurisubharmonic functions that are locally uniformly bounded from above, by Hartogs lemma, the convergence also happens in L_{loc}^1 . \square

Lemma 3.8. *For any $r > 0$, the sequence of maps L_n^r converges pointwise and L_{loc}^1 to L on M .*

Proof. First notice that, for every $a \in \mathbb{C}$, we have

$$\log \max(|a|, r) = \frac{1}{2\pi} \int_0^{2\pi} \log |a - r e^{it}| dt.$$

Therefore,

$$\begin{aligned}
L_n^r(\lambda) &= \frac{1}{2\pi d^{nk}} \int_0^{2\pi} \log \prod_{z \in E_n(\lambda)} |re^{it} - \eta_n(z, \lambda)| dt \\
&= \frac{1}{2\pi d^{nk}} \sum_{z \in E_n(\lambda)} \int_0^{2\pi} \log |re^{it} - \eta_n(z, \lambda)| dt \\
&= \frac{1}{d^{nk}} \sum_{z \in E_n(\lambda)} \log \max(r, |\eta_n(z, \lambda)|) \\
&= \frac{1}{d^{nk}} \sum_{\eta_n(z, \lambda) > r} \log |\eta_n(z, \lambda)| + O\left(\frac{1}{d^{nk}} \text{card} \{z \in E_n(\lambda) : |\eta_n(z, \lambda)| \leq r\}\right) \\
&= L(\lambda) + o(1),
\end{aligned}$$

which gives the pointwise convergence. Moreover, $L_n^r(\lambda)$ is uniformly locally bounded from above, according to Lemma 3.7 and since $L_n^r \leq L_n^+$. Therefore, by Lebesgue's dominated convergence theorem it converges L_{loc}^1 to L . \square

Proof of Theorem 3.4. First, note that the sequence L_n does not converge to $-\infty$. Indeed, by assumption there is $\eta_0 \in \mathbb{C}$ and $r > 0$ such that no cycle of f_{λ_0} has a Jacobian in $\mathbb{D}(\eta_0, r)$. Moreover, since f_{λ_0} is Axiom A, its cycles move holomorphically for λ near λ_0 , which implies that $(\lambda_0, \eta_0) \notin \overline{\bigcup_{n \in \mathbb{N}^*} \text{Per}_n}$. Therefore the sequence $L_n(\lambda_0, \eta_0)$ does not converge to $-\infty$.

Let $\varphi : M \times \mathbb{C} \rightarrow \mathbb{R}$ be a psh function such that a subsequence L_{n_j} converges L_{loc}^1 to φ . Let $(\lambda_0, \eta_0) \in M \times \mathbb{C}$. We have to prove that $\varphi(\lambda_0, \eta_0) = L(\lambda_0)$.

First, let us prove that $\varphi(\lambda_0, \eta_0) \leq L(\lambda_0)$. Let $\varepsilon > 0$ and B_ε be the ball of radius ε centered at (λ_0, η_0) in $M \times \mathbb{C}$. Using the submean inequality and the L_{loc}^1 convergence of L_n^+ , we have

$$\varphi(\lambda_0, \eta_0) \leq \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \varphi \leq \frac{1}{|B_\varepsilon|} \lim_j \int_{B_\varepsilon} L_{n_j} \leq \frac{1}{|B_\varepsilon|} \lim_j \int_{B_\varepsilon} L_{n_j}^+ \leq \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} L.$$

Then letting $\varepsilon \rightarrow 0$, we have that $\varphi(\lambda_0, \eta_0) \leq L(\lambda_0)$, which gives the desired inequality.

Now let us prove the opposite inequality. Here we assume additionally that $\eta_0 \neq 0$. Let $r_0 = |\eta_0|$, and let us first notice that

$$(3) \quad \text{for almost every } \theta \in S^1, \quad \limsup_j L_{n_j}(\lambda_0, r_0 e^{i\theta}) = L(\lambda_0).$$

Indeed, for any $t \in S^1$ we have

$$\limsup_j L_{n_j}(\lambda_0, r_0 e^{it}) \leq \limsup_j L_{n_j}^+(\lambda_0, r_0 e^{it}) = L(\lambda_0)$$

and by Fatou's lemma and the pointwise convergence of $L_n^{r_0}$ we get

$$\begin{aligned}
L(\lambda_0) &= \lim L_n^{r_0}(\lambda_0) = \limsup_j \frac{1}{2\pi} \int_0^{2\pi} L_{n_j}(\lambda_0, r_0 e^{it}) dt \\
&\leq \frac{1}{2\pi} \limsup_j \frac{1}{2\pi} \int_0^{2\pi} L_{n_j}(\lambda_0, r_0 e^{it}) dt
\end{aligned}$$

which proves (3).

Suppose now to obtain a contradiction that $\varphi(\lambda_0, \eta_0) < L(\lambda_0)$. Since L is continuous and φ is upper semi-continuous, there is $\varepsilon > 0$ and a neighborhood V_0 of (λ_0, η_0) such that for all $(\lambda, \eta) \in V_0$,

$$\varphi(\lambda, \eta) - L(\lambda) < -\varepsilon.$$

We may assume without loss of generality that $V_0 = B_0 \times \mathbb{D}(\eta_0, \gamma)$, where B_0 is a ball containing λ_0 . Hartogs' Lemma then gives

$$\limsup_j \sup_{V_0} L_{n_j} - L \leq \sup_{V_0} \varphi - L \leq -\varepsilon.$$

But this contradicts (3).

Therefore, we have proved that any convergent subsequence of L_n in the L_{loc}^1 topology of $M \times \mathbb{C}$ must agree with L on $M \times \mathbb{C}^*$, and since $M \times \{0\}$ is negligible, this proves that L_n converges L_{loc}^1 to L on $M \times \mathbb{C}$. \square

3.2. Equidistribution for polynomial skew-products. We now adapt the general equidistribution result above and its proof to get an equidistribution statement adapted to the case of a family $(f_\lambda)_{\lambda \in M}$ of polynomial skew-product endomorphisms of \mathbb{P}^2 . Since the construction is very similar to the one above, we will omit part of the proofs. The main difference with the above, more general case is that in the setting of skew-products, we can get a more precise description of the case of cycles with parabolic eigenvalues that mirrors the classical one-dimensional case.

We assume that $(f_\lambda)_{\lambda \in M}$ is a holomorphic family of endomorphisms of \mathbb{P}^2 of the form

$$f_\lambda(z, w) = (p(z), q_\lambda(z, w)),$$

where $p : \mathbb{C} \rightarrow \mathbb{C}$ is a degree d polynomial and for all $z \in \mathbb{C}$, for all $\lambda \in M$, $q_\lambda(z, \cdot)$ is also a degree d polynomial. Recall that $Q_{z, \lambda}^n$ is defined by

$$f_\lambda^n(z, w) = (p^n(z), Q_{z, \lambda}^n(w)).$$

Proposition 3.9. *There exists a sequence of holomorphic maps $P_n^v : M \times \mathbb{C} \rightarrow \mathbb{C}$ such that:*

- (1) *For all $\lambda \in M$, $P_n^v(\lambda, \cdot)$ is a monic polynomial*
- (2) *If $\eta \neq 1$, then $P_n^v(\lambda, \eta) = 0$ if and only if there exists $(z, w) \in \mathbb{C}^2$ that is periodic of exact period n , and $(Q_z^n)'(w) = \eta$;*
- (3) *If $\eta = 1$, then $P_n^v(\lambda, \eta) = 0$ if and only if there exists $(z, w) \in \mathbb{C}^2$ such that (z, w) is periodic of exact period m dividing n for f_λ , and $(Q_{z, \lambda}^m)'(w)$ is a primitive $\frac{n}{m}$ -th root of unity.*

Proposition 3.9 is slightly more precise than its counterpart Proposition 3.1. This time, we are going to deduce it directly from the corresponding one-dimensional result, see [BB11, Theorem 2.1] (see also [Ber13, Theorem 2.3.1]).

Theorem 3.10. *Let $(q_\lambda)_{\lambda \in M}$ be a holomorphic family of degree d rational maps, and let $k \in \mathbb{N}$. There is a holomorphic map $P_k : M \times \mathbb{C} \rightarrow \mathbb{C}$ such that:*

- (1) *For all $\lambda \in M$, $P_k(\lambda, \cdot)$ is a monic polynomial*
- (2) *If $\eta \neq 1$, then $P_k(\lambda, \eta) = 0$ if and only if there exists $w \in \mathbb{C}$ that is periodic of exact period k , and $(q_\lambda^k)'(w) = \eta$.*
- (3) *If $\eta = 1$, then $P_k(\lambda, \eta) = 0$ if and only if there exists $w \in \mathbb{C}$ such that w is periodic of exact period m dividing k for f_λ , and $(q_\lambda^m)'(w)$ is a primitive $\frac{k}{m}$ -th root of unity.*

Proof of Proposition 3.9. By assumption, there exists a polynomial p of degree d such that for any $\lambda \in M$, f_λ is of the form $f_\lambda(z, w) = (p(z), q_\lambda(z, w))$. For any $n \in \mathbb{N}$, let $P_n(p)$ denote the set of periodic points for p of exact period n . Let

$$P_n^v(\lambda, \eta) := \prod_{m|n} \prod_{z \in E_m(p)} P_{z, \frac{n}{m}}^v(\lambda, \eta)$$

where $P_{z, \frac{n}{m}}^v : M \times \mathbb{C} \rightarrow \mathbb{C}$ is the map given by Theorem 3.10 for the family of degree d^m polynomials $\{Q_{z, \lambda}^m : \lambda \in M\}$ with $k := \frac{n}{m}$. It is straightforward to check that P_n^v satisfies the required properties. \square

Definition 3.11. For any $\eta \in \mathbb{C}$, we set:

$$\text{Per}_n^v(\eta) := \{\lambda \in M : P_n^v(\lambda, \eta) = 0\}.$$

Theorem 3.12. Let M be a holomorphic family of polynomial skew-products of \mathbb{C}^2 of degree $d \geq 2$ over a fixed base p . For all $\eta \in \mathbb{C}$ outside of a polar subset, we have

$$\frac{1}{d^{2n}} [\text{Per}_n^v(\eta)] \rightarrow T_{\text{bif}}.$$

Proof. Let

$$L_n^v(\lambda, \eta) := \frac{1}{d^{nk}} \log |P_n^v(\lambda, \eta)|.$$

Similarly to the proof of Corollary 3.5, in order to prove Theorem 3.12, it is enough to prove that $L_n^v \rightarrow L_v$ in $L_{\text{loc}}^1(M \times \mathbb{C})$. Indeed, in the family $(f_\lambda)_{\lambda \in M}$, the exponent L_p is constant so $T_{\text{bif}} = dd^c L_v$ (see Theorems 2.2 and 2.8). Set

- (1) $L_n^{v,+}(\lambda, \eta) = \frac{1}{d^{nk}} \sum_{(z,w) \in E_n(\lambda)} \log^+ |\eta - \eta_n(z, w, \lambda)|$, where $\eta_n(z, w, \lambda) := (Q_{z,\lambda})'(w)$
- (2) $L_n^{v,r}(\lambda) = \frac{1}{2\pi d^{nk}} \int_0^{2\pi} \log |P_n^v(\lambda, re^{it})| dt$.

The desired convergence of L_n^v follows from the convergences of $L_n^{v,+}$ and $L_n^{v,r}$ to L^v . The proof of these last two, as well as the deduction of the convergence of L_n^v , is an adaptation of the methods of the previous section, see the proofs of Lemmas 3.7, 3.8 and Theorem 3.4 respectively. \square

4. LYAPUNOV EXPONENTS AND FIBER-WISE BIFURCATIONS (THEOREM A)

The goal of this section is to prove Theorem A. This will provide another kind of approximation of the bifurcation current (and locus), as an average of one-dimensional bifurcation phenomena. We will at first consider the non-autonomous dynamics associated to the fibre at any point of the Julia set of the base polynomial. We will define bifurcations for such (non-autonomous) dynamical systems, and interpret them with respect to the non normality of the critical orbit. Then, we will establish various approximation formulas for the bifurcation current, which will be seen as average of the bifurcation currents associate to the bifurcation of the various fibres. This will provide a proof for (1) in Theorem A, and so also of an inclusion of (2). By exploiting the previous parts and the characterization of the bifurcation locus as the closure of *Misiurewicz parameters*, we will then establish the opposite inclusions, thus completing the proof of Theorem A. As a consequence of our formulas, we will also prove Proposition 2.11. We conclude this section with some explicit examples of families, where we can prove that all the one-dimensional iterations bifurcate exactly on the same subset of the parameters space (and with the same bifurcation current).

4.1. Non autonomous bifurcations. In this section we consider a polynomial family

$$(p(z), q_{\lambda,z}(w))$$

and in particular the non-autonomous iteration associated to a fibre z , for some $z \in J_p$. Recall that the Green function (restricted to the fiber over $z \in J_p$) is given by $G_{\lambda,z}(w) = \lim_{n \rightarrow \infty} \frac{1}{n} \log^+ \|Q_{\lambda,z}^n(w)\|$. This is a psh function on the parameter space. The following result gives the equivalence between critical stability and harmonicity of the associated Green function.

Proposition 4.1. *Let $c(\lambda)$ be a (marked) critical point of $q_{\lambda,z}$. The following are equivalent:*

- (1) *the family $Q_{\lambda,z}^n(c(\lambda))$ is normal;*
- (2) *the current $dd_{\lambda}^c G_{\lambda}(z, c(\lambda))$ is zero.*

Proof. The argument is now standard. We briefly review it for completeness. If the sequence $Q_{\lambda,z}^n(c)$ is normal, we have two cases: it may diverge to ∞ , locally uniformly in the parameter space, or, up to subsequence, converge to some holomorphic function. In the first case, G is the local uniform limit of pluriharmonic function, and is thus pluriharmonic. In the second case, G is equal to 0. In both cases, $dd^c G_{\lambda,c} = 0$.

Let us now assume that $dd^c G_{\lambda,c} = 0$. Again, we have two cases: G is identically zero, or always positive. In the first case, the sequence $Q_{\lambda,z}^n(c)$ is uniformly bounded. In the second, it diverges (locally uniformly with the parameter). In both cases, the family is normal. \square

Definition 4.2. *We denote by $B_{z,c}$, $T_{\text{bif},z,c}$ and $\text{Bif}_{z,c}$ the boundedness locus, the bifurcation current and the bifurcation current associated to a marked critical point c in the fibre z . These are defined as follows:*

$$\begin{aligned} B_{z,c} &:= \{ \lambda : G(\lambda, z, c) = 0 \}, \\ T_{\text{bif},z,c} &:= dd_{\lambda}^c G(\lambda, z, c(\lambda)), \\ \text{Bif}_{z,c} &:= \text{Supp } T_{\text{bif},z,c}. \end{aligned}$$

Analogously, B_z , Bif_z and $T_{\text{bif},z}$ will be the unions (or the sum) of the sets (currents) above, for c critical point for q_z .

It is immediate to check that $\text{Bif}_{z,c} = \partial B_{z,c}$. The following lemma gives some basic semicontinuity properties of these sets that we shall need in the sequel. We will give a more precise description of these sets, for the family of quadratic skew products, in Theorem 5.5.

Lemma 4.3. *Let $M' \Subset M$ be a compact subset of the parameter space. Then:*

- (1) *the set $M' \cap B_z$ varies upper semicontinuously with z ;*
- (2) *the set $M' \cap \text{Bif}_z$ varies lower semicontinuously with z .*

Proof. We drop the restriction to M' to simplify the notation. We fix a critical point in the fibre z . We will prove that the sets $B_{z,c} = \{ G(\lambda, z, c) = 0 \}$ and $\text{Bif}_{z,c} = \partial B_{z,c}$ are upper and lower semicontinuous, respectively. The first assertion is immediate, since the 0 locus of a continuous function is closed (this is the same reason for which the filled Julia set of a polynomial depends upper semicontinuously with the polynomial).

The continuity of G also implies the continuity of the family of currents $T_{bif,z,c}$. The family $\text{Bif}_{z,c} = \text{Supp } T_{bif,z,c}$ is thus lower semicontinuous. \square

We conclude this section proving a first part of Theorem A. The reverse inclusion will be proved in Section 4.3.

Proposition 4.4. *Let $F_\lambda(z, w) = (p(z), q_\lambda(z, w))$ be a family of polynomial skew products of degree d . Then*

$$T_{\text{bif}} = \int_{z \in J_p} T_{\text{bif},z} \mu_p \text{ and } \text{Bif}(F) \subseteq \cup_{z \in J_p} \text{Bif}_z.$$

Proof. The first formula follows from the formula for the vertical Lyapunov exponent in Theorem 2.2. The inclusion is an immediate consequence of the first formula and the fact that $z \mapsto \text{Bif}_z$ is lower semicontinuous. \square

4.2. Approximations for the bifurcation current: periodic fibres. In this section we characterize the Lyapunov exponents of a skew product map by means of the Green functions of the return maps of the periodic vertical fibers. This allows us to approximate the bifurcation current by means of the bifurcation currents of these return maps.

We are given a family of skew-products of \mathbb{C}^2 , extendible to \mathbb{P}^2 , of the form

$$F(\lambda, z, w) = (\lambda, F_\lambda(z, w)) = (p_\lambda(z), q(\lambda, z, w)) = (p(z), q_{\lambda,z}(w)).$$

Notice that in this section we explicitly allow $p(z)$ to depend on λ . By Theorem 2.2 we know that the two Lyapunov exponents of F_λ are equal to

$$L_p(\lambda) = L(p) = \log d + \sum_{z \in C_{p_\lambda}} G_{p_\lambda}(z)$$

$$L_v(\lambda) = \log d + \int \left(\sum_{w \in C_{\lambda,z}} G_\lambda(z, w) \right) \mu_{p_\lambda},$$

where C_{p_λ} and μ_{p_λ} are the critical set and the equilibrium measure of p_λ and $C_{\lambda,z}$ is the critical set of $q_{\lambda,z}$.

By [Pha05, Theorem 2.2] (see also [DS10, Theorem 2.50]), the sum $L(\lambda) = L_v(\lambda) + L_p(\lambda)$ and the maximum $\max(L_p(\lambda), L_v(\lambda))$ are psh function. It is in general not true that the smallest Lyapunov exponent, namely $\min(L_p(\lambda), L_v(\lambda))$ is psh.

In our situation, we are interested in the two functions L_p and L_v . The first one is clearly psh, since it is the Lyapunov function of a polynomial family on \mathbb{C} . The first thing we are going to prove is an approximation formula for L_v , from which we shall deduce that also this exponent is psh (this is obvious if p is constant, since the sum is psh).

Remark 4.5. *This will in particular prove that there is a way to parametrize both the Lyapunov exponents in such a way that they depend psh on the parameter, something which is not known for general families.*

We shall denote by $\mathcal{R}_N(\lambda) \subset \mathcal{P}_N(\lambda)$ the two sets

$$\mathcal{P}_N(\lambda) := \{ z \in \mathbb{C} : p_\lambda^n(z) = z \}$$

$$\mathcal{R}_N(\lambda) := \{ z \in \mathbb{C} : p_\lambda^n(z) = z, |(p_\lambda^n)'(z)| > 1 \}$$

By a well-known theorem of Fatou, for every λ there exists a $N(\lambda)$ such that every cycle of exact length at least $N(\lambda)$ is repelling. This implies that $\frac{\#\mathcal{P}_N(\lambda)}{\#\mathcal{R}_N(\lambda)} \rightarrow 1$ for $N \rightarrow \infty$.

Proposition 4.6. *We have*

$$L_v(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{d^N} \sum_{z \in \mathcal{P}_N(\lambda)} \sum_{w \in C_{\lambda,z}} G_\lambda(z, w) = \lim_{N \rightarrow \infty} \frac{1}{d^N} \sum_{z \in \mathcal{R}_N(\lambda)} \sum_{w \in C_{\lambda,z}} G_\lambda(z, w)$$

where the convergence is pointwise and in $L^1_{loc}(M)$. In particular, $L_v(\lambda)$ is psh.

Proof. From the theorem of Fatou mentioned above, the two limits are equal. So, we compute only one of the two, namely the second one. Recall the basic approximation formula for the equilibrium measure of the polynomial p :

$$\frac{1}{d^N} \sum_{z \in \mathcal{R}_N(\lambda)} \delta_z \rightarrow \mu_p.$$

Moreover, notice that the function $G_\lambda(z, w)$ is continuous in all its three variables. The desired (pointwise) convergence thus follows.

Notice now that both sequences are locally uniformly bounded: this follows from the fact that G is continuous and $\#\mathcal{P}_N(\lambda), \#\mathcal{R}_N(\lambda) \leq d^N$, for every $N \in \mathbb{N}$. It thus suffices to prove that the first sequence consists of psh functions to get that there exist L^1_{loc} limits for the sequence. By the previous part, the only possible limit will then be $L_v(\lambda)$ proving the statement. In order to do so, since G is psh, it suffices to notice that the set C_N given by

$$C_N := \{ (\lambda, z, w) : z \in \mathcal{P}_N(\lambda), w \in C_{\lambda,z} \}$$

is an analytic subset of $M \times \mathbb{C}^2$. The assertion follows. \square

The previous proposition shows in particular that $dd^c L_v =: T_{\text{bif}}^v$ is a well-defined closed positive (1,1)-current on M . In particular, we have $dd^c L = dd^c L_p + dd^c L_v = T_{\text{bif}}^p + T_{\text{bif}}^v$. Since $\text{Bif}(p) \equiv \text{Supp } dd^c L_p$, we can thus see the bifurcation locus $\text{Bif}(F)$ as a union (non necessarily disjoint)

$$\text{Bif}(F) = \text{Bif}(p) \cup \text{Bif}(q)$$

where we denoted $\text{Bif}(q) := \text{Supp } T_{\text{bif}}^v = \text{Supp } dd^c L_v$. The next step consists in getting a better understanding of the set $\text{Bif}(q) \setminus \text{Bif}(p)$. We start with a Lemma that shows how the Lyapunov exponent can be seen as an average of the exponents of the return maps on the repelling fibers.

Lemma 4.7. *We have*

$$(4) \quad L_v(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{Nd^N} \sum_{z \in \mathcal{R}_N(\lambda)} \sum_{w \in C(Q_{\lambda,z}^N)} G_{Q_{\lambda,z}^N}(w)$$

where the convergence is pointwise and in $L^1_{loc}(M)$.

Proof. By the previous proposition, we only have to prove that, for any λ and N ,

$$(5) \quad \sum_{z \in \mathcal{R}_N(\lambda)} \sum_{w \in C_{\lambda,z}} G_\lambda(z, w) = \frac{1}{N} \sum_{z \in \mathcal{R}_N(\lambda)} \sum_{w \in C(Q_{\lambda,z}^N)} G_{Q_{\lambda,z}^N}(w).$$

First, notice that, for every $\lambda \in M$, $z \in \mathcal{R}_N(\lambda)$ (not necessarily of exact period N) and $w \in \mathbb{C}$ we have that $G_\lambda(z, w) = G_{Q_{\lambda,z}^N}(w)$. Indeed,

$$\begin{aligned}
(6) \quad G_\lambda(z, w) &= \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \|F^n(z, w)\| \\
&= \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \|Q_{\lambda,z}^n(w)\| \\
&= \lim_{n \rightarrow \infty} \frac{1}{(d^N)^{n/N}} \log^+ \|(Q_{\lambda,z}^N)^{n/N}(w)\| \\
&= G_{Q_{\lambda,z}^N}(w)
\end{aligned}$$

where to get the second line we used the assumption that $(p^n(z))_n$ is bounded (since $z \in \mathcal{R}_N(\lambda)$) and in the last one that the degree of $Q_{\lambda,z}^N$ is d^N . So, the left hand side of (5) is equal to

$$\sum_{z \in \mathcal{R}_N(\lambda)} \sum_{w \in C_{\lambda,z}} G_{Q_{\lambda,z}^N}(w).$$

We are thus left with checking that, for a given skew product $f(z, w) = (p(z), q_z(w))$, for every N -periodic point z of p , we have

$$\sum_{j=0}^{N-1} \sum_{w \in C_{p^j(z)}} G_{Q_{p^j(z)}^N}(w) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{w \in C(Q_{p^j(z)}^N)} G_{Q_{p^j(z)}^N}(w).$$

Let us first describe the critical set of $Q_{p^j(z)}^N$, that we denote by C^j . Since $Q_{p^j(z)}^N$ is by definition equal to $q_{p^{j-1}(z)} \circ \cdots \circ q_z \circ q_{p^{N-1}(z)} \circ \cdots \circ q_{p^{j+1}(z)} \circ q_{p^j(z)}$, we have $C^j = C(Q_{p^j(z)}^N) = \bigcup_{i=0}^{N-1} C_i^j$, where

$$\begin{aligned}
C_0^j &= C(q_{p^j(z)}) \\
C_1^j &= q_{p^j(z)}^{-1} C(q_{p^{j+1}(z)}) \\
C_2^j &= q_{p^j(z)}^{-1} q_{p^{j+1}(z)}^{-1} C(q_{p^{j+2}(z)}) = [Q_{p^j(z)}^2]^{-1} C(q_{p^{j+2}(z)}) \\
&\vdots \\
C_{N-1}^j &= q_{p^j(z)}^{-1} q_{p^{j+1}(z)}^{-1} \cdots q_{p^{j-2}(z)} C(q_{p^{j-1}(z)}) = [Q_{p^j(z)}^{N-1}]^{-1} C(q_{p^{j-1}(z)})
\end{aligned}$$

(each term C_i^j is to be thought of as a subset of the fiber over $p^k(z)$ which is the preimage of the critical set of $q_{p^{k+l \pmod N}(z)}$ by $Q_{p^k(z)}^l$). So, it suffices to prove that, for any $0 \leq j, i \leq N-1$, we have

$$\sum_{w \in C_0^j = C(q_{p^j(z)})} G_{Q_{p^j(z)}^N}(w) = \sum_{w \in C_{j-i}^i = [Q_{p^i(z)}^{j-i}]^{-1}(C(q_{p^j(z)}))} G_{Q_{p^i(z)}^N}(w),$$

where $j-i$ has to be taken modulo N . But this follows from the fact that the Green function G of f satisfies the property that $G(f(\cdot)) = dG(\cdot)$. Indeed, for points $(p^l(z), w)$ in the fiber $\{p^l(z)\} \times \mathbb{C}$, we have $G_f(p^l(z), w) = G_{Q_{p^l(z)}^N}(w)$ (by (6)), and moreover

$\{p^i(z)\} \times C_{j-1}^i$ contains exactly d^{j-i} preimages by F^{j-i} of any point in $\{p^j(z)\} \times C_0^j$ (out of the total $d^{2(j-i)}$, since we do not consider preimages other than the ones contained in the fiber over $p^j(z)$). So, for each $w \in C_0^j$ in the left sum, with value $G_{Q_{p^j(z)}^N}(w)$, there are d^{j-i} preimages $w_1, \dots, w_{d^{j-i}}$ in the right sum, each one with value $G_{Q_{p^i(z)}^N}(w_l) = G_{Q_{p^j(z)}^N}(w)/d^{j-i}$. The assertion follows. \square

Since the convergence to L_v happens in $L_{loc}^1(M)$, it would be tempting to take the dd^c on both sides of (4) and thus see $dd^c L_v$ as an average of bifurcation currents for the families $Q_{z,\lambda}^N$. The problem here is that z depends on λ (it will be the solution $z(\lambda)$ of some equation $p_\lambda^N z(\lambda) = z(\lambda)$), in general in a non-globally holomorphic way (and, moreover, it may stop to be repelling on $\text{Bif}(p)$).

This interpretation is actually possible over a stable component for p (i.e., on a component $\Omega \subset M \setminus \text{Bif}(p)$), where we can holomorphically follow the repelling cycles for p , and thus in particular get well-defined families $(\lambda, Q_{\lambda,z(\lambda)}^N)$. We shall denote as usual by $\text{Bif}(Q_{\lambda,z(\lambda)}^N)$ the bifurcation locus of one such family.

Corollary 4.8. *Assume $dd^c L_p \equiv 0$. Denote by $\rho_{j,N}(\lambda)$, $1 \leq j \leq n(N) \sim d^N$ the holomorphic motions of the repelling cycles of p . Then*

$$\begin{aligned}
 dd_\lambda^c L_v(\lambda) &= \lim_{N \rightarrow \infty} \frac{1}{Nd^N} \sum_{1 \leq j \leq n(N)} \sum_{w \in C(Q_{\lambda,\rho_{j,N}(\lambda)}^N)} dd_\lambda^c G_{Q_{\lambda,\rho_{j,N}(\lambda)}^N}(w) \\
 &= \lim_{N \rightarrow \infty} \frac{1}{Nd^N} \sum_{1 \leq j \leq n(N)} T_{\text{bif}}(Q_{\lambda,\rho_{j,N}(\lambda)}^N).
 \end{aligned}$$

The above results allow to get the following description of the bifurcation locus for a family of skew-products. We shall prove the converse inclusion in the next section.

Corollary 4.9. *We have*

$$(7) \quad \text{Bif}(F) \subseteq \text{Bif}(p) \cup \bigcup_{\Omega \text{ cc of } M \setminus \text{Bif}(p)} \overline{\bigcup_{N \geq 1} \bigcup_{1 \leq j \leq n_\Omega(N)} \text{Bif}(Q_{\rho_{j,N}^\Omega}^N)}$$

where the $\rho_{j,N}^\Omega$ are the holomorphic motions of the N -repelling cycles for p on the connected component $\Omega \subset M \setminus \text{Bif}(p)$ and $Q_{\rho_{j,N}^\Omega}^N$ denotes the 1-dimensional family $(\lambda, w) \mapsto (\lambda, Q_{\lambda,\rho_{j,N}^\Omega}^N(w))$, whose parameter space is Ω .

Remark 4.10. *Notice in particular that, since the above inclusion comes from a limit, it is still true if we substitute the union on N with a union over $N \geq N_0$, for any $N_0 \in \mathbb{N}$. In particular, this gives*

$$\text{Bif}(F) \subseteq \text{Bif}(p) \cup \bigcup_{\Omega \text{ cc of } M \setminus \text{Bif}(p)} \bigcap_{N_0 \in \mathbb{N}} \overline{\bigcup_{N \geq N_0} \bigcup_{1 \leq j \leq n_\Omega(N)} \text{Bif}(Q_{\rho_{j,N}^\Omega}^N)}.$$

Remark 4.11. *The formulas for the bifurcation current established in this and the previous section may be used to study self intersections of this current, in the spirit of [BB07]. These should be related with stronger bifurcations, and higher codimension*

equidistribution phenomena (see for instance [Duj11] for an account of this for rational maps). We postpone this study to a later work.

4.3. A decomposition for the bifurcation locus. We now study better the relation between the stability of a family of skew-products and the normality of critical orbits. This gives us the converse inclusion in (7). We start with critical points in the repelling periodic fibres.

Proposition 4.12. *Assume $dd^c L_p \equiv 0$. Denote by $\rho_{j,N}(\lambda)$, $1 \leq j \leq n(N) \sim d^N$ the holomorphic motions of the repelling cycles of p . Then for every N, j , $1 \leq j \leq n(N)$, we have*

$$(8) \quad \text{Bif}(F) \supseteq \text{Bif}(Q_{\rho_{j,N}}^N)$$

where $Q_{\rho_{j,N}}^N$ denotes the 1-dimensional family $(\lambda, w) \mapsto (\lambda, Q_{\lambda, \rho_{j,N}}^N(w))$.

Proof. Given N and j as in the statement, let us consider a parameter $\lambda_0 \in \text{Bif}(Q_{\rho_{j,N}}^N)$. There thus exists a parameter λ_1 nearby which is Misiurewicz for the family $Q_{\rho_{j,N}}^N$, i.e., there exist a critical point c for $Q_{\rho_{j,N}(\lambda_1)}^N$, a number $N_0 \geq 1$ and a N_1 -periodic repelling point w for $Q_{\rho_{j,N}(\lambda_1)}^N$ such that $Q_{\rho_{j,N}(\lambda_1)}^{N N_0}(c) = w$, and the relation $Q_{\rho_{j,N}(\lambda)}^{N N_0}(c(\lambda)) = w(\lambda)$ does not hold for every λ near λ_1 (where $c(\lambda)$ and $w(\lambda)$ are the local holomorphic motions of c and w as a critical point and as a N_1 -periodic repelling point). The point $(\rho_{j,N}(\lambda_1), c)$ is in particular critical also for F , and the point $(\rho_{j,N}(\lambda_1), w)$ is NN_1 -periodic and repelling for F . So, we only have to check that there does not exist any holomorphic map $\lambda \rightarrow (z(\lambda), \tilde{c}(\lambda)) \in C(F_\lambda)$ such that $(z(\lambda_1), \tilde{c}(\lambda_1)) = (\rho_{j,N}(\lambda_1), c)$ and the relation $F_\lambda^{N N_0}((z(\lambda), \tilde{c}(\lambda))) = (\rho_{j,N}(\lambda), w(\lambda))$ holds persistently in a neighbourhood of λ_1 . First of all, by the finiteness of the p^{N_0} -preimages of $\rho_{j,N}(\lambda_1)$, up to restricting ourselves to a small neighbourhood of this point, we can assume that every F^{N_0} -preimage of $(\rho_{j,N}(\lambda), w(\lambda))$ belongs to the fiber of $\rho_{j,N}(\lambda)$, too. In this way, any persistent critical relation must happen in the fibers of $\rho_{j,N}(\lambda)$. This is excluded, since the parameter is Misiurewicz for the restricted family. \square

By Proposition 4.12 and Corollary 4.9 we immediately get the following description of $\text{Bif}(F)$.

Corollary 4.13. *We have*

$$(9) \quad \text{Bif}(F) = \text{Bif}(p) \cup \bigcup_{\Omega \text{ cc of } M \setminus \text{Bif}(p)} \overline{\bigcup_{N \geq 1} \bigcup_{1 \leq j \leq n_\Omega(N)} \text{Bif}(Q_{\rho_{j,N}^\Omega}^N)}.$$

From Remark 4.10 we thus get the following: given any connected component Ω of $M \setminus \text{Bif}(p)$, any \bar{N} and j , with $1 \leq j \leq n^\Omega(\bar{N})$, we have

$$\text{Bif}(Q_{\rho_{j,\bar{N}}^\Omega}^{\bar{N}}) = \bigcap_{N_0 \in \mathbb{N}} \overline{\bigcup_{N \geq N_0} \bigcup_{1 \leq j \leq n_\Omega(N)} \text{Bif}(Q_{\rho_{j,N}^\Omega}^N)}.$$

We can now complete the proof of Theorem A, and also prove Proposition 2.11

End of the proof of Theorem A. Equation (1) follows from Proposition 4.4 and Corollary 4.8. The equality between $\text{Bif}(F)$ and the union of the bifurcation loci associated to the periodic fibres is given by Corollary 4.13. Because of Proposition 4.4, the only

thing left to do is to prove the inclusion $\text{Bif}_z \subset \text{Bif}(F)$, for any $z \in J_p$. For z periodic and repelling, as mentioned above, this follows from Corollary 4.13. For a generic z , this then follows from the lower semicontinuity of $z \mapsto \text{Bif}_z$ (when restricted to any compact subset of M), see Lemma 4.3. \square

Remark 4.14. *In [Ber17] Berteloot observes that, in dimension 1, the bifurcation of one repelling cycle in an open set forces almost all the repelling cycles to bifurcate (in the sense that the cardinality of non-bifurcating n -cycles is negligible with respect to the number of all the cycles, for $n \rightarrow \infty$). Our result gives the (weaker) conclusion that, for families of polynomial skew products, the bifurcation of a periodic fibre forces a positive quantity ($\sim \alpha d^n$) of other periodic fibre to bifurcate.*

Proof of Proposition 2.11. Let F be a family of polynomial skew products. If the base polynomial p is not stable and admits some non hyperbolic component, the assertion is trivial. Let us thus assume for simplicity that the base polynomial is independent from the parameter. By Theorem A, in a stable component for F all the fibres are stable. Thus, if F admits a stable and non hyperbolic component, the same is true for the family associated to some (possibly non periodic) fibre. \square

Question 4.15. *It would be interesting to know if the existence of a non hyperbolic stable component for a family of polynomial skew products implies the existence of such a component for a family of (autonomous) polynomials on \mathbb{C} . This would be true if, given a family of skew products, we knew that the hyperbolicity of all the return maps on the periodic fibres implies the hyperbolicity of all the fibres.*

4.4. Bifurcations and preimages of fibres. By arguments similar to the ones above, by means of the equidistribution of preimages of generic points we can establish the following further approximation approximation of the bifurcation current and locus. We assume for simplicity that p is constant, otherwise the formula takes a form similar to that in Equation (9).

Proposition 4.16. *Let $F_\lambda(z, w) = (p(z), q_\lambda(z, w))$ be a holomorphic family of polynomial skew products of degree $d \geq 2$. Let $z \in J_p$.*

$$T_{\text{bif}} = \lim_{N \rightarrow \infty} \frac{1}{d^N} \sum_{y: p^N(y)=z} T_{\text{bif},y} \text{ and } \text{Bif}(F) = \overline{\bigcup_{N \in \mathbb{N}} \bigcup_{y: p^N(y)=z} \text{Bif } y}.$$

In case of p non constant, but only stable, we consider the preimages, for every λ , of the motion $z(\lambda)$ of some point in the Julia set of p_λ .

4.5. An explicit example. In [DH08], DeMarco and Hruska study polynomial skew products of the form $(z^2, w^2 + \lambda z)$. They also remark that every such map is semi-conjugated to the map $(z^2, w^2 + \lambda)$. It is then natural to expect that the bifurcation locus – and current – of this family are the same as the quadratic family $w^2 + \lambda$. Here we prove that this is indeed the case.

Proposition 4.17. *The bifurcation locus of the family $(\lambda, z, w) \mapsto (\lambda, z^2, w^2 + \lambda z)$ is the Mandelbrot set \mathcal{M}_2 , i.e., the bifurcation locus of the 1-dimensional family $(\lambda, w) \mapsto (\lambda, w^2 + \lambda)$.*

Proof. By the results (and with the same notations) of this section, we shall prove that $\text{Bif}_z = \mathcal{M}_2$, for every $z \in S^1$. Recall that the Mandelbrot set is given by the $\lambda \in \mathbb{C}$ such that the sequence

$$\Lambda_0 = \lambda \quad \Lambda_{n+1} = \Lambda_n^2 + \Lambda_n$$

diverges. Given a point $z \in S^1$, let us compute the set \mathcal{B}_z . Denote as usual $z_0 := z$ and $z_j := z^{2^j}$. It is then immediate to prove by induction that

$$q_{z_j} \circ \cdots \circ q_{z_0}(z, 0) = \Lambda_j z_j.$$

This proves that the orbit of $(z, 0)$ diverges if and only if the sequence Λ_j diverges, that is, if $\lambda \in \mathcal{M}_2$.

The induction above is trivial for $j = 0$ and we then have

$$\begin{aligned} q_{z_j} \circ \cdots \circ q_{z_0}(z, 0) &= (q_{z_{j-1}} \circ \cdots \circ q_{z_0}(z, 0))^2 + \lambda z_j = (\Lambda_{j-1} z_{j-1})^2 + \lambda z_j \\ &= (\Lambda_{j-1}^2 + \lambda) z_j = \Lambda_j z_j, \end{aligned}$$

which completes the proof. \square

It is not difficult to prove, using the same idea, that also the bifurcation current associated to any $z \in S^1$ is exactly the equilibrium measure on the Mandelbrot set. Moreover, the same method as above applies to the family $(z, w) \mapsto (z^2, w^2 + \lambda z^2)$.

Part 2. Quadratic skew products: the bifurcations at infinity

5. QUADRATIC SKEW PRODUCTS

We now specialize to endomorphisms of \mathbb{P}^2 that are quadratic polynomial skew-products, i.e. maps of the form

$$(p(z), Az^2 + Bzw + Cw^2 + Dz + Ew + F),$$

where p is a quadratic polynomial.

Lemma 5.1. *Every quadratic skew product with $p(z)$ as first component is affinely conjugated to a map of the form*

$$(p(z), w^2 + az^2 + bz + c).$$

Proof. Notice that we necessarily have $C \neq 0$ in order to extend the endomorphism to \mathbb{P}^2 . The assertion follows by conjugating with a change of variables of the form $(z, w) \mapsto (z, \alpha z + \beta w + \gamma)$. \square

Notice that, by a further change of variable in z , we can also assume that $p(z)$ is of the form $z^2 + d$.

Remark 5.2. *Many of the results in this second part actually hold for the families of polynomial skew-product of the form*

$$(z, w) \mapsto (p(z), w^d + a_d z^d + \dots a_1 z + a_0)$$

for any degree $d \geq 2$. However, while for $d = 2$ this family gives a parametrization of the full family of degree 2 skew products, the same is not true for $d > 2$.

From here on, we will therefore consider the space $\mathbf{Sk}(p, 2)$ of quadratic skew-products over the base p as identified with \mathbb{C}^3 . We will also work with the compactification of $\mathbf{Sk}(p, 2)$ as \mathbb{P}^3 , and denote by \mathbb{P}_∞^2 the hyperplane at infinity.

Notice that, for all maps in $\mathbf{Sk}(p, 2)$, the fiber at any z contains a unique critical point, $w = 0$. By the results of the previous section, in order to understand the stability of the family we then need to study the Green function at the points of the form $(z, 0)$, with $z \in J_p$. This leads to the following definition. Recall that the set \mathcal{B}_z (Definition 4.2) in this situation denotes the parameters such that $(z, 0)$ has bounded orbit, or, equivalently, such that the Green function at $(z, 0)$ is zero.

Definition 5.3. *We partition the parameter space $\mathbf{Sk}(p, 2)$ into the following three subsets:*

- (1) $\mathcal{C} := \{\lambda \in \mathbb{C}^3 : \forall z \in J_p, G(z, 0) = 0\} = \bigcap_{z \in J_p} \mathcal{B}_z$
- (2) $\mathcal{D} := \{\lambda \in \mathbb{C}^3 : \forall z \in J_p, G(z, 0) > 0\} = \bigcap_{z \in J_p} \mathcal{B}_z^c$
- (3) $\mathcal{M} := \mathbb{C}^3 \setminus (\mathcal{C} \cup \mathcal{D})$.

Note that in the case where J_p is connected, \mathcal{C} is (in restriction to our family) what in [Jon99] is called the *connectedness locus*, meaning the set of parameters such that J_p is connected, and for all $z \in J_p$, J_z is connected. The set \mathcal{C} is closed and the set \mathcal{D} is open. It follows from [Jon99] that \mathcal{D} is in fact a union of vertically expanding components (hyperbolic, if p is hyperbolic). As we will see below, \mathcal{C} is bounded (Corollary 5.8) but \mathcal{M} and \mathcal{D} are not (and actually contain unbounded hyperbolic components, see Sections 7 and 8).

It has been recently proved by Dujardin [Duj16a] and Taffin [Taf17] that some polynomial skew-products are in the interior of the bifurcation locus, a phenomenon that contrasts with the one-variable situation. We note here that such behaviour can only occur in \mathcal{M} : indeed, parameters in \mathcal{D} are vertically expanding hence in the stability locus. As for parameters in \mathcal{C} , note that any connected component of $\hat{\mathcal{C}}$ is a stable component. Concerning possible non-hyperbolic components in \mathcal{C} or \mathcal{M} , see Proposition 2.11 above.

Our main goal in the remaining part of this section is proving the following theorem, describing the accumulation of the sets \mathcal{B}_z , Bif_z and Bif at infinity (in the parameter space) from a topological point of view. In the next Section we will improve this theorem in a quantitative way, leading to Theorem C.

Definition 5.4. *Given $z \in J_p$, we denote by E_z the set $\{[a, b, c] \in \mathbb{P}_\infty^2 : az^2 + bz + c = 0\}$. We set $E := \bigcup_{z \in J_p} E_z$.*

Theorem 5.5. *In the family $\mathbf{Sk}(p, 2)$, the following hold.*

- (1) *For every $z \in J_p$, the cluster set at infinity of \mathcal{B}_z and Bif_z is exactly E_z .*
- (2) *The cluster set at infinity of Bif is exactly E .*

It will be useful to fix the following notations. For $\lambda := (a, b, c) \in \mathbb{C}^3$ and $p(z) = z^2 + d$, let

$$f_\lambda(z, w) = (p(z), w^2 + az^2 + bz + c).$$

For $\lambda \in \mathbb{C}^3$, let $\rho_\lambda(z_0) := az_0^2 + bz_0 + c$ and $\|\rho_\lambda\|_\infty := \sup_{z \in J_p} |az^2 + bz + c|$. We then have the following elementary lemma.

Lemma 5.6. *Assume that $G_\lambda(z_0, 0) = 0$. Then $|\rho_\lambda(z_0)| \leq 2\sqrt{\|\rho_\lambda\|_\infty}$. In particular, the cluster set of B_{z_0} in \mathbb{P}_∞^2 is included in E_{z_0} .*

Proof. For $n \in \mathbb{N}$, let $z_n := p^n(z_0)$ and $\rho_n := az_n^2 + bz_n + c$. Let $w_n := Q_{z_0}^{n+1}(0)$: then $w_0 = \rho_0$ and $w_{n+1} = w_n^2 + \rho_n$. Therefore we have $|w_{n+1}| \geq |w_n|^2 - \|\rho_\lambda\|_\infty$, and since by assumption $(w_n)_{n \in \mathbb{N}}$ is bounded, we must have for all $n \in \mathbb{N}$ that $|w_n| \leq 2\sqrt{\|\rho_\lambda\|_\infty}$. The result follows by taking $n = 0$. \square

Proof of Theorem 5.5. By Lemma 5.6, the cluster set of \mathcal{B}_z is included in E_z . We thus prove the opposite inclusion. We first consider z such that $z = p^n(z)$. Since E_z is an irreducible curve (more precisely, a projective line), it is enough to note that there is a component C of $\text{Per}_n(0)$ such that for all $\lambda \in C$, $f_\lambda^n(z, 0) = (z, 0)$. Indeed, that component C intersects the plane at infinity in some (1 dimensional) hypersurface that is contained in E_z and is therefore equal to E_z . Moreover, it is clear that $C \subset \mathcal{B}_z$.

Let us now pick any (non necessarily periodic) $z \in J_p$. Let D be any complex line in \mathbb{P}^3 that intersects \mathbb{P}_∞^2 at the point $[0 : 0 : 1]$. Then the set

$$D \cap \{(a, b, c) \in \mathbb{C}^3 : aX^2 + bX + c \text{ has a root in } J_p\}$$

is compact, and for any $y \in J_p$, $\mathcal{B}_y \cap D$ is compact. Lemma 4.3 implies that the map $J_p \ni y \mapsto \mathcal{B}_y \cap D$ is upper semicontinuous. Therefore, if we pick a sequence $z_n \rightarrow z$ of periodic points for p , we have that

$$\limsup_{n \rightarrow \infty} (\mathcal{B}_{z_n} \cap D) \subset \mathcal{B}_z \cap D.$$

Since this is true for any complex line D intersecting \mathbb{P}_∞^2 at $[0 : 0 : 1]$, we also have

$$\limsup_{n \rightarrow \infty} \overline{\mathcal{B}_{z_n}} \subset \overline{\mathcal{B}_z},$$

where $\overline{\mathcal{B}_z}$ denotes the closure of \mathcal{B}_z in \mathbb{P}^3 . Therefore the cluster set of \mathcal{B}_z contains E_z , and the first assertion is proved. The assertion for the boundary easily follows.

Let us now prove the second assertion. Observe that $\text{Bif} \subset \bigcup_{z \in J_p} \mathcal{B}_z$. Therefore, the cluster set at infinity of Bif is contained in the cluster set of $\bigcup_{z \in J_p} \mathcal{B}_z$, which a priori might be larger than the union of cluster sets of \mathcal{B}_z ; but the estimate from Lemma 5.6 implies that this is not the case. Indeed, let $(a_n, b_n, c_n)_{n \in \mathbb{N}}$ be a sequence of points in $\bigcup_{z \in J_p} \mathcal{B}_z$ going to infinity, and converging to $[a, b, c] \in \mathbb{P}_\infty^2$. For each n there is at least one $z_n \in J_p$ such that $(a_n, b_n, c_n) \in \mathcal{B}_{z_n}$, and thus, by Lemma 5.6,

$$|a_n z_n^2 + b_n z_n + c_n| \leq 2 \sqrt{\sup_{z \in J_p} |a_n z^2 + b_n z + c_n|}.$$

Since $(a, b, c) \mapsto \sup_{z \in J_p} |az^2 + bz + c|$ is a vector space norm on \mathbb{C}^3 , there is some constant $C_p > 0$ such that for all $(a, b, c) \in \mathbb{C}^3$,

$$\frac{1}{C_p} \|(a, b, c)\|_\infty \leq \sup_{z \in J_p} |az^2 + bz + c| \leq C_p \|(a, b, c)\|_\infty$$

and therefore, setting $M_n := \|(a_n, b_n, c_n)\|_\infty$, we have

$$\left| \frac{a_n}{M_n} z_n^2 + \frac{b_n}{M_n} z_n + \frac{c_n}{M_n} \right| \leq 2C_p \sqrt{\frac{1}{M_n}}.$$

Passing to the limit, we conclude that $z \mapsto az^2 + bz + c$ must vanish at least once on J_p .

This takes care of one inclusion. Now let us prove that the cluster of the bifurcation locus on the plane at infinity contains the set E . Take $[a, b, c] \in E$, assume it is a point in \mathbb{P}_∞^2 such that $az^2 + bz + c = 0$, where $z \in J_p$. By Theorem A, we know that $\partial\mathcal{B}_z \subset \text{Bif}$ (here, the boundary is taken in \mathbb{C}^3). By the first item, we know that $\partial\mathcal{B}_z$ accumulates on \mathbb{P}_∞^2 to the set $\{[a, b, c] : az^2 + bz + c = 0\}$; this concludes the proof. \square

We can also explicitly describe the cluster set for 2-dimensional algebraic subfamilies. We state the description for the subfamily given by $a = 0$.

Corollary 5.7. *In the 2-dimensional subfamily given by $a = 0$, the cluster of Bif on the line \mathbb{P}_∞^1 at infinity is an affine copy of the Julia set J_p of p .*

Proof. In the subfamily $a = 0$, if $[b : c]$ is in the line at infinity, then $[b : c]$ is accumulated by the bifurcation locus if and only if $bX + c$ has a root in J_p , that is if and only if $-\frac{c}{b} \in J_p$. \square

A similar description holds for the two-dimensional subfamilies given by prescribing a solution to the polynomial $az^2 + bz + c$. More specifically, given $\omega \in \mathbb{C}$, for the families given by $\{(a, b, c) : a\omega^2 + b\omega + c = 0\}$. We will come back to this point in Section 6, when describing the cluster set from the quantitative point of view of currents.

We also have the following immediate consequence of Theorem 5.5.

Corollary 5.8. *Let $z_1, z_2, z_3 \in J_p$ be three distinct points. Then $\mathcal{B}_{z_1} \cap \mathcal{B}_{z_2} \cap \mathcal{B}_{z_3}$ is compact. In particular, \mathcal{C} is compact.*

Proof. If $(a : b : c) \in \mathbb{P}_\infty^2$ were accumulated by $\mathcal{B}_{z_1} \cap \mathcal{B}_{z_2} \cap \mathcal{B}_{z_3}$, then $aX^2 + bX + c$ would have z_1, z_2, z_3 as roots, and we would have $a = b = c = 0$, which is impossible. So $\mathcal{B}_{z_1} \cap \mathcal{B}_{z_2} \cap \mathcal{B}_{z_3}$ is closed and bounded in \mathbb{C}^3 . In particular, $\mathcal{C} = \bigcap_{z \in J_p} \mathcal{B}_z$ is compact. \square

6. THE BIFURCATION CURRENT AT INFINITY (THEOREM C)

The goal of this section is to prove Theorem C, that improves the results of the previous section by describing the accumulation at infinity of the bifurcation current from a quantitative point of view. Recall that we are considering the family

$$(p(z), w^2 + az^2 + bz + c)$$

where p is a polynomial of degree 2, and a, b, c are three complex parameters. First of all, we prove that we can extend the bifurcation current to the compactification \mathbb{P}^3 of the parameter space (see also [BG15a] for an analogous result for quadratic rational maps).

Lemma 6.1. *There exists a positive closed $(1, 1)$ -current \hat{T}_{bif} on \mathbb{P}^3 whose mass equals 1 and such that*

- (1) $\hat{T}_{\text{bif}}|_{\mathbb{C}^2} = T_{\text{bif}}$;
- (2) for a generic $\eta \in \mathbb{C}$, the sequence $4^{-n}[\text{Per}_n^J(\eta)]$ converges to \hat{T}_{bif} in the sense of currents of \mathbb{P}^3 ;
- (3) for a generic $\eta \in \mathbb{C}$, the sequence $4^{-n}[\text{Per}_n^v(\eta)]$ converges to \hat{T}_{bif} in the sense of currents of \mathbb{P}^3 .

Proof. The existence of \hat{T}_{bif} follows by an application of Skoda-El Mir Theorem. Indeed, by the equidistribution results in Section 3, the mass of T_{bif} on \mathbb{C}^2 is 1. We thus can trivially extend T_{bif} to \mathbb{P}^3 , and the mass of the extension still satisfies $\|\hat{T}_{\text{bif}}\| = 1$.

We now promote the equidistribution of $[\text{Per}_n(\eta)]$ to T_{bif} on \mathbb{C}^3 to an equidistribution to \hat{T}_{bif} on \mathbb{P}^3 (we denote by $\text{Per}_n(\eta)$ both Per_n^J and Per_n^v , the proof is the same). First recall (see Section 3) that the $\text{Per}_n(\eta)$ are actually algebraic surfaces on \mathbb{P}^3 , of mass $\sim 4^n$. Thus, the sequence $4^{-n}[\text{Per}_n(\eta)]$ gives a sequence of uniformly bounded (in mass) positive closed currents. We have to prove that any limit of this sequence coincides with \hat{T}_{bif} . Let us denote by T a cluster of the sequence. By Siu's decomposition Theorem, we have $T = S + \alpha[\mathbb{P}_\infty^2]$, where S has no mass on \mathbb{P}_∞^2 . It follows from the description of the accumulation of the bifurcation locus given in Section 5 that $\alpha = 0$. Moreover, we have $S = T_{\text{bif}}$ on \mathbb{C}^3 . This complete the proof. \square

Remark 6.2. *The result above also applies to any subfamily of the family F_{abc} whose (extended) parameter space is a subvariety in \mathbb{P}^3 .*

A description of the accumulation of the support of \hat{T}_{bif} at infinity can be easily deduced from Theorem 5.5.

Lemma 6.3. *For any algebraic subfamily of F_{abc} , the cluster set at infinity of the support of \hat{T}_{bif} is included in the set*

$$\{[a, b, c]: az^2 + bz + c = 0 \text{ for some } z \in J_p\} = \cup_{z \in J_p} E_z = E.$$

We actually have the equality in the Lemma above, unless the accumulation at infinity of the family is precisely given by some E_z .

Our goal here is to prove the following strengthening of the statement above: for the family F_{abc} , the intersection between \hat{T}_{bif} and the current of integration $[\mathbb{P}_\infty^2]$ on the hyperplane at infinity is well defined, and that its support is precisely the set E . We will also precisely characterize this intersection current, by means of the equilibrium measure of p . We first specialize to the family given by $a = 0$. We will see later how to move to the general setting.

6.1. The family $(p(z), w^2 + bz + c)$. As observed in Corollary 5.7, for $a = 0$ the set given by Lemma 6.3 reduces to the set of $[b, c]$ satisfying $az + b = 0$ for some $z \in J_p$. This set is an affine copy of J_p . We denote it by $J_{p,\infty}$. Let $\mu_{p,\infty}$ denote the corresponding equilibrium measure on $J_{p,\infty}$. We shall prove that the intersection of the current \hat{T}_{bif} with $[\mathbb{P}_\infty^1]$ is well defined and given precisely by $\mu_{p,\infty}$. This will be achieved by means of the following lemma.

Lemma 6.4. *Let $F_{bc}(z, w) = (p(z), w^2 + bz + c)$. For a generic $\eta \in \mathbb{D}$ we have*

$$4^{-n}[\text{Per}_n^v(\eta)] \wedge [\mathbb{P}_\infty^1] \rightarrow \mu_{p,\infty}.$$

Proof. By the equidistribution of the periodic points of p towards μ_p , it is enough to prove that

$$[\text{Per}_n^v(\eta)] \wedge [\mathbb{P}_\infty^1] \sim 2^n \sum_{p^n(y)=y} \delta_y.$$

Here, by abuse of notation, we denote by $y \in \mathbb{P}_\infty^1$ the point corresponding to $[b, c]$ with $y = -c/b$. Also, we think of the current on the left hand side as a measure on \mathbb{P}_∞^1 . Notice that, again by the equidistribution of periodic points, we know that it is the same

to count the y 's in the right hand side sum with or without multiplicity as solutions of p .

First, we prove that the support of $[\text{Per}_n^v(\eta)] \wedge [\mathbb{P}_\infty^1]$ is contained in the union of the solution of $p^n(y) = y$. Indeed, every $\text{Per}_n^v(\eta)$ is contained in the boundedness locus B_z of some fibre z of period (dividing) n (since a periodic cycle of vertical multiplier $\eta \in \mathbb{D}$ attracts a critical point). By the description of the set \mathcal{B}_z given in Section 5, this set precisely clusters at the corresponding point in $J_{p,\infty}$.

To conclude, we prove that at every point $y \in J_{p,\infty}$ corresponding to a fibre z of period n the Lelong number of $[\text{Per}_n^v(\eta)] \wedge [\mathbb{P}_\infty^1]$ at y is $\sim 2^n$. This means finding $\sim 2^n$ local components (with multiplicity) of $\text{Per}_n^v(\eta)$ accumulating on y . Since the total mass of $[\text{Per}_n^v(\eta)] \wedge [\mathbb{P}_\infty^1]$ is $\sim 4^n$ and the cardinality of n -periodic points for p is $\sim 2^n$, it is enough to find at least $\sim 2^n$ components for every y (which necessarily cluster at $y \in J_{p,\infty}$). Since the return map of the fibre corresponding to y is of degree 2^n , this follows since the mass of $\text{Per}_1(\eta)$ in this one-dimensional family is 2^n . \square

We can now prove our main result concerning this family (giving Theorem C').

Theorem 6.5. *The measure $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^1]$ exists and is equal to $\mu_{p,\infty}$.*

Remark 6.6. *With $p(z) = z^2 - 2$, Theorem 6.5 implies that $\mu_{p,\infty}$ is the equilibrium measure on the interval $[-2, 2]$. In [BG15a], the authors prove that the corresponding object for the two-dimensional family of rational maps of degree 2 is also a measure supported on the interval $[-2, 2]$. However, in that case the measure has positive Lelong numbers at some points of its support (it is actually totally atomic), while in our case it is absolutely continuous with respect to the Lebesgue measure of the interval.*

Proof of Theorem 6.5. We first prove that the intersection in the statement exists, and then we prove that it coincides with $\mu_{p,\infty}$.

The good definition of the intersection follows the same argument as in [BG15a, Lemma 4.3]. We give it for completeness, also to highlight that a different approach will be needed when considering the complete family. We take any complex line L intersecting \mathbb{P}_∞^1 in a point disjoint from $J_{p,\infty}$. The complement of this line is a copy of \mathbb{C}^2 . Since the set $J_{p,\infty}$ is compact in this copy of \mathbb{C}^2 , we can define the intersection here by means of [Dem97, Proposition 4.1]. We then trivially extend this intersection as zero on the line L .

Remark 6.7. *When considering the full family, with the three-dimensional parameter space, we cannot find a line in \mathbb{P}_∞^2 disjoint from E (and thus decompose \mathbb{P}^3 as the union of \mathbb{C}^3 and a hyperplane disjoint from E) and apply the argument above.*

We now prove that $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^1] = \mu_{p,\infty}$. By Lemma 6.4, it is enough to prove that

$$4^{-n}[\text{Per}_n^v(\eta)] \wedge [\mathbb{P}_\infty^1] \rightarrow \hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^1].$$

The idea is the following: the main obstacle in getting the convergence above would be that some components of $\text{Per}_n^v(\eta)$ become more and more tangent to \mathbb{P}_∞^1 as $n \rightarrow \infty$ (possibly with some multiple of the plane at infinity in their cluster set). But this cannot happen, because of Lemma 5.6.

A way to make the above precise is to use the theory of horizontal currents developed by Dinh-Sibony and Pham (see also [Duj07, Definition 2.1 and Definition-Proposition 2.2]). We recall that a closed positive $(1, 1)$ -current in the product $\mathbb{D} \times \mathbb{D}$ is *horizontal* if

its support is contained in a set of the form $\mathbb{D} \times K$, for some K compact in \mathbb{D} . We will make use of the following result (see [DS06, Theorem 2.1], and also [Pha05, Theorem A.2] for the case where φ is not necessarily smooth).

Theorem 6.8 (Dinh-Sibony). *Let \mathcal{R} be a closed positive horizontal $(1, 1)$ -current on $\mathbb{D} \times \mathbb{D}$, with support contained in $\mathbb{D} \times K$. Then the slice \mathcal{R}_z of \mathcal{R} is well defined for every $z \in \mathbb{D}$. The slices are measures on \mathbb{D} , supported in K , of constant mass. If φ is a smooth psh function on $\mathbb{D} \times \mathbb{D}$ then the function $z \mapsto \langle \mathcal{R}_z, \varphi(z, \cdot) \rangle$ is psh.*

By the description of the cluster set of the \mathcal{B}_z 's given in Section 5, we can find a biholomorphic image of a polydisc $\Delta \subset \mathbb{P}^2$ such that the following hold (by abuse of notation, we think of the polydisc directly in \mathbb{P}^2):

- (1) $\{0\} \times \mathbb{D} \subset \mathbb{P}_\infty^1$;
- (2) there exists $K \Subset \mathbb{D}$ such that $\text{supp} \hat{T}_{\text{bif}} \cap \Delta \subset \mathbb{D} \times K$ and $\text{supp}[\text{Per}_n^v(0)] \cap \Delta \subset \mathbb{D} \times K$ for every n .

Indeed, suppose this is not true. We then find points in $\text{Per}_n^v(0)$ accumulating some point in $\mathbb{P}_\infty^1 \setminus J_{p,\infty}$. Since all the $\text{Per}_n^v(0)$ cluster on $J_{p,\infty}$, this contradicts Lemma 5.6

With this setting, we see that all the $[\text{Per}_n^v(\eta)]$ and \hat{T}_{bif} are (uniformly) horizontal currents on Δ . The convergence above can thus be rephrased as a convergence for the slices at 0:

$$4^{-n}[\text{Per}_n^v(\eta)]_0 \rightarrow \left(\hat{T}_{\text{bif}} \right)_0.$$

By standard arguments, the convergence can be tested against smooth psh tests. By Theorem 6.8 above we know that, for every φ smooth and psh in Δ , the functions $u_n(z) := 4^{-n}[\text{Per}_n^v(\eta)]_z(\varphi(z, \cdot))$ and $u(z) := \left(\hat{T}_{\text{bif}} \right)_z(\varphi(z, \cdot))$ are psh. We claim that $u_n \rightarrow u$ in L_{loc}^1 . This is true because the convergence of $4^{-n}[\text{Per}_n^v(0)]$ to \hat{T}_{bif} implies that of $\varphi 4^{-n}[\text{Per}_n^v(\eta)]$ to $\varphi \hat{T}_{\text{bif}}$ in the product space Δ . Since the projection on the first coordinate of Δ is continuous, we have $u_n \rightarrow u$ as distributions. Thus, by [Hör07, Theorem 3.2.12], we have $u_n \rightarrow u$ in L_{loc}^1 . This also implies that $u_n \rightarrow u$ almost everywhere.

Now, by Hartogs' Lemma the L_{loc}^1 limit of a sequence of psh function is greater than or equal to the pointwise limit. In our case, the pointwise limit of the u_n is given by $u'(z) = \langle \lim_{n \rightarrow \infty} [\text{Per}_n^v(\eta)]_z, \varphi \rangle$. Since $u'(0) = \langle \mu_{p,\infty}, \varphi \rangle$, we just need to prove that $u'(0) \geq u(0)$. Since u is psh and $u = u'$ almost everywhere, we have a sequence of $z_m \in \mathbb{D}$ converging to 0 and such that $u(z_m) = u'(z_m) \rightarrow u(0)$. It is then enough to prove that the limit of the $u'(z_m)$ is equal to $u'(0)$, i.e., that

$$\left\langle \left(\hat{T}_{\text{bif}} \right)_{z_m}, \varphi \right\rangle \rightarrow \langle \mu_{p,\infty}, \varphi \rangle.$$

Since $u(z_m) = u'(z_m)$, every limit of the slice measures on the left hand side is an invariant measure supported on the Julia set on the slice at 0 (corresponding to $J_{p,\infty}$). Let ν be any such limit. It is enough to prove that $\nu = \mu_{p,\infty}$. Suppose this is not the case. Lemma 6.9 below gives a contradiction with the fact that $\langle \mu, \psi \rangle \leq \langle \nu, \psi \rangle$ for every psh function ψ , as proved in the previous part. This completes the proof. \square

Lemma 6.9. *Let p be any polynomial on \mathbb{C} , μ its equilibrium measure and ν any invariant measure supported on the Julia set of p . If $\mu \neq \nu$ there exists a subharmonic function ψ on \mathbb{C} such that $\langle \mu, \psi \rangle > \langle \nu, \psi \rangle$.*

Proof. Let p_ν and p_μ be the respective logarithmic potentials of ν and μ , that is, $p_\mu(z) = \int_{\mathbb{C}} \log |z - w| d\mu(w)$ and similarly for ν . Recall that the energy of a compactly supported Radon probability measure m is defined by $I(m) = \int_{\mathbb{C}} p_m(z) dm(z)$. Since μ is the equilibrium measure of the Julia set of p , it is known (see for instance [Ran95]) that $I(\mu) > I(\nu)$ for every $\nu \neq \mu$. Therefore there must exist z_0 such that $p_\mu(z_0) > p_\nu(z_0)$ (recall that the potential of μ is constant on its support). Now we set $\psi(z) = \log |z - z_0|$. By definition of p_μ and p_ν , we then have $\langle \mu, \psi \rangle > \langle \nu, \psi \rangle$. Thus, ψ has the required property. \square

6.2. The general case. We now describe the intersection of the bifurcation current \hat{T}_{bif} with the hyperplane at infinity \mathbb{P}_∞^2 in the full family. The idea and steps will be as follows:

- (1) we can define the intersection when restricted to (almost) every line in the hyperplane at infinity (this is done essentially by the same argument as above);
- (2) we can use the previous partial intersections to prove the existence of the intersection of \hat{T}_{bif} with the integration current on the hyperplane at infinity $[\mathbb{P}_\infty^2]$;
- (3) we will lift the defined current to the space $\mathbb{P}^1 \times \mathbb{P}^1$, where the coordinates stand for the solutions of the polynomial $az^2 + bz + c$ associated with $[a, b, c] \in \mathbb{P}_\infty^2$;
- (4) in these coordinates, we can give a precise description and an explicit formula for the current.

First of all, we consider any two-dimensional subfamily given by an hyperplane $\alpha a + \beta b + \gamma c = 0$ in the parameter space (a, b, c) , satisfying the condition

$$(10) \quad [\alpha, \beta, \gamma] \neq [z^2, z, 1] \text{ for any } z \in J_p.$$

This means that the hyperplane at infinity of the family is different from any line E_z (see Section 5) corresponding to any $z \in J_p$. The following Lemma is proved in essentially the same way as for the family $a = 0$. In particular, this measure can still be identified with the equilibrium measure of the polynomial p .

Lemma 6.10. *For any family satisfying (10), the intersection of the current \hat{T}_{bif} with the hyperplane at infinity is a well defined positive measure, whose support coincides with the intersection between E and the line $\alpha a + \beta b + \gamma c = 0$.*

We can now consider the full family. First of all, we prove that the intersection of $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2]$ is well defined.

Lemma 6.11. *For the family F_{abc} the intersection $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2]$ is well defined.*

Proof. Since the support of T_{bif} only clusters on $E = \cup_{z \in J_p} E_z$, we need only prove the statement in a neighbourhood of E . Take a point $[a_0, b_0, c_0] \in E$. There exist z_0 and z_1 (not necessarily distinct) such that $[a_0, b_0, c_0] \in E_{z_0}, E_{z_1}$ but $[a_0, b_0, c_0] \notin E_z$ for every $z \neq z_0, z_1$. To prove that the intersection is well defined, we prove that $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2]$ has locally bounded mass near $[a_0, b_0, c_0]$. We fix local coordinates x, y such that

- (1) $[a_0, b_0, c_0]$ becomes the origin;
- (2) the coordinate axis are transversal to both E_{z_0} and E_{z_1} at the origin.

Lemma 6.10 above implies that the intersection $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2] \wedge [L]$ is well defined for lines L parallel (or almost parallel) to the x and y axis. Since all these intersections are

measures with uniformly bounded mass, the intersection between $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2]$ and the currents $\int_{x \in I} [L_x]$ and $\int_{y \in I} [L_y]$ are well defined, where I is a small open neighbourhood of 0, L_x the line $\{x = \text{constant}\}$, L_y the line $\{y = \text{constant}\}$ and the integrations are against the standard Lebesgue measure. This implies that the intersections between $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2]$ and respectively $dx \wedge id\bar{x}$ and $dy \wedge id\bar{y}$ are of locally bounded mass, and thus well defined. This implies the statement. \square

In order to describe the intersection given by Lemma 6.11, it is useful to consider a change of coordinates on the hyperplane at infinity. More specifically, consider the map given by

$$\begin{aligned} \pi : \mathbb{P}^1 \times \mathbb{P}^1 &\rightarrow \mathbb{P}^2 \\ (x, y) &\mapsto [a, b, c] \end{aligned}$$

with a, b, c such that x and y are the two solutions of $aX^2 + bX + c$. The map π is clearly well defined on the quotient $\mathbb{P}^1 \times \mathbb{P}^1 / \sim$ given by $(x, y) \sim (y, x)$. By abuse of notation, notice that $a = 0$ corresponds to x or y being ∞ . Consider now $z \in J(p)$ and the corresponding set $E_z \subset \mathbb{P}_\infty^2$. Recall that this is given by the $[a, b, c]$ satisfying $az^2 + bz + c = 0$. Thus, its lift to $\mathbb{P}^1 \times \mathbb{P}^1$ by the map π is given by $([\mathbb{P}^1] \times \{z\}) \cup (\{z\} \times [\mathbb{P}^1])$. We can now conclude the proof of Theorem C.

Theorem 6.12. *For the family F_{abc} we have*

$$\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2] = \int_z [E_z] \mu_p(z).$$

Proof. Since $\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2]$ is well defined, we can lift it to a positive closed current on $\mathbb{P}^1 \times \mathbb{P}^1$. In order to prove the statement, we will prove that

$$\pi^* \left(\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2] \right) = \left(\frac{1}{2} \int_z ([\{z\} \times \mathbb{P}^1] + [\mathbb{P}^1 \times \{z\}]) \mu_p(z) \right)$$

For every η , the lift to $\mathbb{P}^1 \times \mathbb{P}^1$ of the intersection at infinity of the current $[\text{Per}_n^v(\eta)]$ is given by an average of currents of the form

$$[\{z\} \times \mathbb{P}^1] + [\mathbb{P}^1 \times \{z\}]$$

with z such that $p^n(z) = z$. This implies that

$$\pi^* \left(\hat{T}_{\text{bif}} \wedge [\mathbb{P}_\infty^2] \right) = \left(\int_z ([\{z\} \times \mathbb{P}^1] + [\mathbb{P}^1 \times \{z\}]) \nu(z) \right)$$

for some measure ν on J_p . We can thus find ν by considering a slice of the current above by a complex line. In particular, we can consider the line corresponding to $a = 0$, and the assertion follows from Theorem 6.5. \square

7. UNBOUNDED HYPERBOLIC COMPONENTS IN \mathcal{D} (THEOREM D)

In this section we study unbounded components in \mathcal{D} . It follows from the above characterization of the accumulation set of the bifurcation locus at infinity (see Theorems 5.5 and 6.12) that we can associate to every (unordered) pair of components of the Fatou set of p a vertically expanding component in \mathcal{D} (hyperbolic, if p is hyperbolic). Here we prove that all these components are distinct, i.e., we cannot have a component in \mathcal{D} accumulating points in two distinct components of the Fatou set at infinity. This is done by studying the topology of the Julia sets for parameters near infinity. Indeed,

since these components are vertically expanding, the Julia sets of maps in the same components must share the same topology (see Lemma 7.3). It thus suffices to exhibit distinct topologies for different cluster points at infinity. For the sake of exposition, we shall first treat the case of the family $(z^2, w^2 + bz + c)$ (thus corresponding to fixing the polynomial $p(z) = z^2$ and considering the slice $a = 0$ in the parameter space), and then see how to handle the general case.

7.1. The family $(z^2, w^2 + bz + c)$. When $p(z) = z^2$, we have two possible components Δ_0 and Δ_∞ near infinity (the first one corresponding to the unit disk, and the second one to $\{z \in \mathbb{C} : |z| > 1\}$). Recall that z here corresponds to $-c/b$. The component Δ_∞ thus contains trivial products $(z, w) \mapsto (z^2, w^2 + c)$ for c large enough, while Δ_0 contains skew-products of the form $(z, w) \mapsto (z^2, w^2 + bz)$ for b large enough. Hence all we have to do is to prove that a map of the form $(z^2, w^2 + bz)$ cannot be in the same stability component as $(z^2, w^2 + c)$, for b and c large enough. Let us first set some notations.

Definition 7.1. Let Σ denote a copy of the standard Cantor set that is invariant under $w \mapsto -w$, and let S be the suspension given by $S := ([0, 1] \times \Sigma) / \sim$, where $(0, w) \sim (1, -w)$.

Here, it is immediate to see that the Julia set of a product $(z, w) \mapsto (z^2, w^2 + c)$ is homeomorphic to $S^1 \times \Sigma$. The desired result thus follows from the following topological description of the Julia set of skew-products in Δ_0 .

Proposition 7.2. For $b \in \mathbb{C}$ sufficiently large, the Julia set of the map $(z, w) \mapsto (z^2, w^2 + bz)$ is homeomorphic to the suspension S . In particular, it is not homeomorphic to $S^1 \times \Sigma$.

For the proof, see [DH08, Lemma 5.5]. In this particular case, for any $w \in \mathbb{C}$ the curve $S_w := \{(e^{it}, e^{2it}w) : t \in [0, 2\pi]\}$ is mapped to $S_{g(w)}$, from which Proposition 7.2 follows without difficulty. In the next section, we treat a more general family in which the situation is not so explicit, using a similar but more technical type of argument.

7.2. The general case. We now consider the complete family $(z, w) \mapsto (p(z), w^2 + az^2 + bz + c)$. The distinction of the unbounded components in \mathcal{D} will be done by means of the following lemma.

Lemma 7.3. Let $F_\lambda(z, w) = (p(z), q_\lambda(z, w))$ be a family of polynomial skew products defined on some parameter space Λ . Let Z be a compact invariant set for p . Assume that F_{λ_0} is uniformly vertically expanding on $J_Z = \cup_{z \in Z} \{z\} \times J_z$. Then F_{λ_0} is structurally stable on J_Z .

Proof. We follow the classical one dimensional construction of the conjugation valid for hyperbolic polynomial maps, see e.g., [BH]. For ease of notation, we write F_0 for F_{λ_0} and assume that $\lambda \in \mathbb{D}$.

By uniform expansiveness and continuity, there exist ε and $C > 1$ such that, for every λ sufficiently small and every $(z, w) \in J_Z(F_0)$ we have $|q'_{\lambda, z}(w')| > C > 1$ for every $w' \in B(w, \varepsilon)$. This implies that, denoting by (z_n, w_n) the orbit of (z, w) under F_0 , we have $q_{\lambda, z}(B(w_n, \varepsilon)) \supset B(w_{n+1}, C'\varepsilon)$ for some $1 < C' < C$. It follows that the

diameter of $B(w_{n+1}, \varepsilon)$ inside $q_{\lambda,z}(B(w_n, \varepsilon))$ is uniformly bounded from above and that, if $x, y \in B(w_n, \varepsilon)$ and $q_{\lambda,z}(x), q_{\lambda,z}(y) \in B(w_{n+1}, \varepsilon)$, then

$$d_{B(w_{n+1}, \varepsilon)}(q_{\lambda,z}(x), q_{\lambda,z}(y)) > C'' d_{B(w_n, \varepsilon)}(x, y).$$

for some uniform constant $C'' > 1$. Thus, the intersection

$$B(w, \varepsilon) \cap q_{\lambda,z}^{-1}(B(w_1, \varepsilon)) \cap \cdots \cap q_{\lambda,z_{n-1}}^{-1} \circ \cdots \circ q_{\lambda,z}^{-1}(B(w_n, \varepsilon))$$

consists of a single point. Denote it by $h(z, w)$. Then, $q_{z,\lambda} \circ h(z, w) = h(z_1, q_{z,0}(w))$.

The map h constructed above is continuous. Since it is the identity for z , we check the continuity in w . A basis of open neighbourhoods of (z, w) in the vertical fibre at z is given by the intersections $\cap_{i=0}^n (Q_{z,0}^n)^{-1}(B(w_n, \varepsilon))$. These open sets are sent to the corresponding open sets $\cap_{i=0}^n (Q_{z,\lambda}^n)^{-1}(B(w_n, \varepsilon))$. This proves continuity. Since we can start the construction at a different λ near 0, the map h is invertible and thus a homeomorphism. \square

Remark 7.4. Notice that Z in the previous statement may contain critical points. This does not interfere with the construction, which is done fibre by fibre.

Recall that, by Remark 2.10, it makes sense to speak about *vertically expanding* components of the stability locus. This notion coincides with hyperbolicity in case of an hyperbolic base p .

Recall from the previous section that the rational map $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ is defined by $\pi(x, y) = [1, -x - y, xy]$, so that if $\pi([x, y]) = [a, b, c]$ then x and y are the roots of the polynomial $aX^2 + bX + c$. By the description of the accumulation at infinity of the bifurcation locus, it follows that every point $[a, b, c] \in \mathbb{P}_\infty^2$ such that the two roots of $aX^2 + bX + c$ belong to the Fatou set of p is accumulated by a (unique) component of \mathcal{D} . Moreover, the component is the same if we move the two roots in the respective components of the Fatou set of p . If we denote by \mathcal{S}_p the set of unordered pairs of Fatou components of p , this immediately gives the following result (where $\pi_0(\mathcal{D})$ denotes the set of the connected components of \mathcal{D}).

Proposition 7.5. *The map $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}_\infty^2$ descends to a well-defined map $\omega : \mathcal{S}_p \rightarrow \pi_0(\mathcal{D})$.*

Let us now restrict our attention to the image \mathcal{D}' of the map ω above (thus corresponding to unordered pairs of Fatou components of p). Our result below completes the classification of these components, for p with locally connected Julia set.

Theorem 7.6. *Assume that J_p is locally connected. Then $\omega : \mathcal{S}_p \rightarrow \mathcal{D}'$ is bijective.*

Since ω is surjective on \mathcal{D}' , all it remains is to prove that it is injective. The rest of the section is devoted to that task.

Definition 7.7. *Let $C \subset \mathbb{C}$ be a topological circle, and $\tilde{C} \subset C \times \mathbb{C}$ a second topological circle. We will say that \tilde{C} winds n times above C if the projection $\pi_1 : \tilde{C} \rightarrow C$ is an unbranched covering of degree n .*

The next proposition (see [DH84]) is the reason why we assume J_p to be locally connected:

Proposition 7.8. *Let $K \subset \mathbb{C}$ be a full, compact, locally connected set. Then for every connected component U_i of \hat{K} , its closure \bar{U}_i is homeomorphic to a closed disk.*

Since J_p is locally connected, this implies in particular the boundary of every bounded Fatou component of p is a Jordan curve, which we will need in the proof below.

Let $F(z, w) = (p(z), w^2 + az^2 + bz + c)$. Let U, V be Fatou components of p with $p(U) = V$, and let s be the number of roots of $aX^2 + bX + c$ lying in U , counted with multiplicity. Let C be a simple closed curve in $\partial V \times \mathbb{C}$. Let $\hat{C} := F^{-1}(C) \cap (\partial U \times \mathbb{C})$.

Definition 7.9. *Let $F(z, w) = (p(z), w^2 + az^2 + bz + c) \in \mathcal{D}$ and let $r(F) := \inf_{z \in J_p} |az^2 + bz + c|$. Let $\gamma: [0, 1] \rightarrow \mathbb{C}^2$ be a simple closed curve, given by $\gamma(t) = (\gamma_z(t), \gamma_w(t))$. We say that γ is admissible if for all $t \in [0, 1]$, $|\gamma_w(t)| < r(F)$.*

Note that in particular, if $C \subset K(F)$ and $F \in \mathcal{D}$, then C is admissible.

Lemma 7.10. *Let $[a, b, c] \in \mathbb{P}_\infty^2$ be such that the roots of $aX^2 + bX + c$ are in the Fatou set of p . For $T \in \mathbb{C}$ with $|T|$ large enough, the map $F_T := (z, w) \mapsto (p(z), w^2 + T(az^2 + bz + c))$ satisfies the following properties:*

- (1) *If C is an admissible curve, then so is every component of $F_T^{-1}(C)$;*
- (2) *There exists $0 < R < r(F_T)$ such that for all $z \in J_p$, $K_z \subset \mathbb{D}(0, R)$.*

Proof. Let $R(F_T) := \sup_{z \in J_p} |Taz^2 + Tz + Tc|$. Then there exists a positive constant $\alpha = \alpha(a, b, c, p)$ such that for all $T \in \mathbb{C}$, we have $\frac{1}{\alpha}|T| \leq r(F_T) \leq R(F_T) \leq \alpha|T|$. Moreover, for $|T|$ large enough F_T lies in \mathcal{D} , and for all $z \in J_p$, we have $K_z \subset \mathbb{D}(0, 2\sqrt{R(F_T)})$ by Lemma 5.6. Therefore we may take $R := 2\sqrt{R(F_T)}$ for item (2). For item (1), observe that if $(z, w) \in F^{-1}(C)$ and C is admissible, then $|w| = O(\sqrt{|T|})$ and therefore any component of $F^{-1}(C)$ is also admissible. \square

Lemma 7.11. *Assume that $F \in \mathcal{D}$, the roots of $aX^2 + bX + c$ are in the Fatou set of p and that $\|(a, b, c)\|_\infty$ is large enough so that Lemma 7.10 holds. Assume that C winds once above ∂V and is admissible. Then*

- (1) *if $s = 0$ or $s = 2$, \hat{C} has two connected components C_1 and C_2 , and their linking number is equal to $\frac{s}{2}$. Both components wind once above ∂U ;*
- (2) *if $s = 1$, then \hat{C} is connected and winds twice above ∂U .*

Notice the assumptions imply, in particular, that the boundary of V is a Jordan curve.

Proof. Let $\delta \in \{1, 2\}$ be the degree of $p: U \rightarrow V$ ($\delta = 1$ if U contains no critical point of p , and $\delta = 2$ otherwise). Let $\gamma: \mathbb{R}/\mathbb{Z} \rightarrow C$ defined by $\gamma(t) := (\gamma_V(t), \gamma_w(t))$ be a parametrization of C . Let $\gamma_1: \mathbb{R} \rightarrow \mathbb{C}^2$ be a lift by F of $t \mapsto \gamma(\delta t)$. We can define a parametrization of ∂V by $p \circ \gamma_U(t) = \gamma_V(\delta t)$ for all $t \in \mathbb{R}/\mathbb{Z}$. So, the map γ_1 is of the form

$$\gamma_1(t) = (\gamma_U(t), w_t)$$

and w_t satisfies the equation

$$w_t^2 = \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c).$$

Observe that the curve $t \mapsto \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c)$ turns s times around $w = 0$. We now distinguish between the cases $s \in \{0, 2\}$ or $s = 1$.

- (1) If $s \in \{0, 2\}$, since the curve $t \mapsto \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c)$ turns an even number of times around $w = 0$ as t goes from 0 to 1, we have $w_1 = w_0$ by monodromy. Therefore $\gamma_1(1) = \gamma_1(0)$, and $\gamma_1(\mathbb{R})$ is a closed loop winding once above ∂U . Since $F : \hat{C} \rightarrow C$ has degree 2δ , and $F : \gamma_1(\mathbb{R}) \rightarrow C$ has degree δ , there is a second lift $\gamma_2 : \mathbb{R} \rightarrow C_2$ parametrizing a second connected component of \hat{C} . Moreover, γ_2 has the form

$$\gamma_2(t) = (\gamma_U(t), -w_t)$$

and therefore the linking number of C_1 and C_2 is given by the number of turns around $w = 0$ of $t \mapsto w_t$ as t varies from 0 to 1, namely $\frac{s}{2}$.

- (2) If $s = 1$: now the curve $t \mapsto \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c)$ turns exactly once around $w = 0$ as t goes from 0 to 1. Therefore, by monodromy we have $w_1 = -w_0$ and $w_2 = w_0$. This means that the support of $\gamma_1(\mathbb{R})$ is a curve that winds twice above ∂U . Moreover, as $\gamma_1(\mathbb{R}) \cap \{z = \gamma_U(0)\} = \{(\gamma_U(0), \pm w_0)\}$, the degree of $F : \gamma_1(\mathbb{R}) \rightarrow C$ is 2δ and therefore $\hat{C} = \gamma_1(\mathbb{R})$.

□

Lemma 7.12. *Assume that $F \in \mathcal{D}$, the roots of $aX^2 + bX + c$ are in the Fatou set of p and that $\|(a, b, c)\|_\infty$ is large enough so that Lemma 7.10 holds. Assume that C winds twice above ∂V . Then \hat{C} has two connected components C_1 and C_2 . Both are curves that wind twice above ∂U and their linking number is s .*

Proof. The proof is similar to that of the previous Lemma. Let $\delta \in \{1, 2\}$ be the degree of $p : U \rightarrow V$. Since C winds twice above ∂V , it has a parametrization $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow C$ of the form $\gamma(t) = (\gamma_V(2t), \gamma_w(t))$, where for all $t \in \mathbb{R}$, $\gamma_w(t + \frac{1}{2}) \neq \gamma_w(t)$. As before, let $\gamma_1 : \mathbb{R} \rightarrow \hat{C}$ be a lift by F of $t \mapsto \gamma(\delta t)$. Then γ_1 has the form

$$\gamma_1(t) = (\gamma_U(2t), w_t),$$

and $t \mapsto w_t$ satisfies the equation

$$w_t^2 = \gamma_w(t) - (a\gamma_U(2t)^2 + b\gamma_U(2t) + c).$$

Note that $a\gamma_U(1)^2 + b\gamma_U(1) + c = a\gamma_U(0)^2 + b\gamma_U(0) + c$ but $\gamma_w(\frac{1}{2}) \neq \gamma_w(0)$, hence $w_{\frac{1}{2}} \neq w_0$. Also note that as t varies from 0 to 1, the loop $t \mapsto \gamma_w(t) - (a\gamma_U(2t)^2 + b\gamma_U(2t) + c)$ turns $2s$ times around $w = 0$. Therefore by monodromy, we have $w_1 = w_0$, so that $\gamma_1(1) = \gamma_1(0)$ and $\gamma_1(\mathbb{R})$ is a closed loop that winds twice above ∂U .

Again, the degree of $F : \hat{C} \rightarrow C$ is 2δ , and the degree of $F : \gamma_1(\mathbb{R}) \rightarrow C$ is only δ . Moreover, $w_{\frac{1}{2}} \neq -w_0$ (since $w_{\frac{1}{2}}^2 \neq w_0^2$), and therefore $\gamma_2 : \mathbb{R} \rightarrow C$ defined by $\gamma_2(t) = (\gamma_U(2t), -w_t)$ parametrizes a second and different component of \hat{C} . For degree reasons, \hat{C} is exactly equal to $C_1 \cup C_2$, where C_i is the support of $\gamma_i(\mathbb{R})$. Each C_i is a loop winding twice above ∂U , and $C_1 \cap C_2 = \emptyset$ since for all $t \in \mathbb{R}$, $w_t \neq 0$. Therefore the C_i are the connected components of \hat{C} . Moreover, since $\gamma_1(t) = (\gamma_U(2t), w_t)$ and $\gamma_2(t) = (\gamma_U(2t), -w_t)$, the linking number of C_1 and C_2 is given by the number of times that $t \mapsto w_t$ turns around $w = 0$ as t varies from 0 to 1, namely s . □

On our way to prove Theorem 7.6, we will need the following topological description of the Julia sets of maps in \mathcal{D} , which has independant interest.

Theorem 7.13. *Assume that J_p is locally connected. Let Σ be the standard Cantor set and S be its suspension, as in Definition 7.1.*

- (1) *If there is a (bounded) periodic Fatou component of p containing exactly one root of $aX^2 + bX + c$ (counted with multiplicity), then for any bounded Fatou component U of p , $J_{\partial U}$ is homeomorphic to S .*
- (2) *If there is a Fatou component of p containing 2 roots of $aX^2 + bX + c$ (counted with multiplicity), then $J_2(F)$ is homeomorphic to $J_p \times \Sigma$.*
- (3) *If both roots of $aX^2 + bX + c$ lie in the basin of infinity of p , then $J_2(F)$ is homeomorphic to $J_p \times \Sigma$.*
- (4) *If the only root of $aX^2 + bX + c$ not in the basin of infinity of p is in a component V that is not periodic, then for all components W in the prehistory of V we have that $J_{\partial W}$ is homeomorphic to S , and for all other components U we have that $J_{\partial U}$ is homeomorphic to $S^1 \times \Sigma$.*

Proof. If p has no periodic bounded Fatou component, then by Sullivan's theorem p only has the basin of infinity as a Fatou component. In that case, there can be only one component in \mathcal{D}' , which is that containing product maps; therefore $J_2(F)$ is homeomorphic to $J_p \times \Sigma$. From now on, we assume that p has a cycle of bounded Fatou components. Let $R > 0$ be as in lemma 7.10, let U_0 be a bounded periodic Fatou component for p of period $m \in \mathbb{N}^*$, and let $W_0 := \partial U_0 \times \mathbb{D}(0, R)$. By the maximum modulus principle, U is simply connected and since J_p is locally connected, ∂U is a Jordan curve (see [DH84]). Let $U_i := p^{m-i}(U_0)$ be a cyclic numbering of the cycle of components containing U_0 , with $i = 0, \dots, m-1$, so that $p(U_{i+1}) = U_i$.

- (1) Assume first that each component in the forward orbit of U contains either zero or two roots of $aX^2 + bX + c$. Since W_0 is homotopic to a curve winding once above ∂U_0 , by Lemma 7.11, $W_1 := F^{-1}(W_0) \cap (\partial U_1 \times \mathbb{C})$ is homotopic to two disjoint curves, each winding once above ∂U_{m-1} . Therefore, W_1 is a disjoint union of the interior of two solid tori, each winding once above ∂U_{m-1} . Letting $W_n := F^{-1}(W_0) \cap (\partial U_n \times \mathbb{C})$, we therefore get by induction that W_n is a disjoint union of the interior of 2^n solid tori, each winding once above ∂U_n . Since $W_m \subseteq W_0$, we get that $\bigcap_{n \in m\mathbb{N}} W_n$ is homeomorphic to $S^1 \times \Sigma$.
- (2) Assume now that there exists a component in the cycle containing U_0 (we may assume without loss of generality that it is U_0 itself) that contains exactly one root of $aX^2 + bX + c$. We proceed as before, letting $W_0 := \partial U_0 \times \mathbb{D}(0, R)$ and $W_n := F^{-n}(W_0) \cap (\partial U_n \times \mathbb{C})$. This time, Lemma 7.12 implies that W_1 is homotopic to a double winding curve above ∂U_1 . Therefore W_1 is the interior of a double winding solid torus, and for all $n \geq 1$, W_n is the disjoint union of the interior of 2^{n-1} solid tori, each winding twice above ∂U_n . Therefore, $J_{\partial U_0} = \bigcap_{n \in m\mathbb{N}} W_n$ is homeomorphic to the suspension S .

To conclude the proof of Theorem 7.13, notice that if U, V are two Fatou components of p such that $p(U) = V$, and $J_{\partial V}$ is homeomorphic to either $S^1 \times \Sigma$ or S , then Lemmas 7.11 and 7.12 allow us to determine the topology of $J_{\partial U}$. More precisely, letting $s \in \{0, 1, 2\}$ be the number of roots of $aX^2 + bX + c$ contained in U , we have the following

- (1) If $s = 0$ or $s = 2$: then $J_{\partial U}$ is homeomorphic to $J_{\partial V}$
- (2) If $s = 1$: then $J_{\partial U}$ is homeomorphic to S .

Since every Fatou component of p is preperiodic to U_0 by Sullivan's theorem, the rest of the proof follows. \square

We are now ready to prove the injectivity of ω .

Proof of Theorem 7.6. Recall that by Lemma 7.3, if $(F_t)_{t \in [0,1]}$ is an arc in \mathcal{D} , there is an isotopy $h_t : J_2(F_0) \rightarrow J_2(F_t)$ such that

$$(11) \quad h_t \circ F_0 = F_t \circ h_t.$$

In particular, if F_0, F_1 are two skew-products in the same connected component of \mathcal{D} , then they can be joined by an arc and therefore their Julia sets are isotopic. Let $F_0, F_1 \in \mathcal{D}$, with $F_i(z, w) = (p(z), w^2 + a_i z^2 + b_i z + c)$, $i \in \{0, 1\}$. For any Fatou component U of p , let $s_i(U)$ be the number of roots of $a_i X^2 + b_i X + c_i$ counted with multiplicity contained in U . It is enough to prove that if $J_2(F_0)$ is isotopic to $J_2(F_1)$ then for any bounded Fatou component U of p , we have $s_0(U) = s_1(U)$.

Let U be a bounded Fatou component of p and let $z \in \partial U$, $w \in J_z(F_0)$. Assume that there exists an isotopy $(h_t)_{t \in [0,1]}$ as in (11). By Theorem 7.13, there exists a unique closed simple curve C_0 passing through $F_0(z, w)$ and contained in $J_2(F_0) \cap (\partial V \times \mathbb{C})$, where $V := p(U)$. That curve winds either once or twice above ∂V . Let $C_1 := h_1(C_0)$ and $\hat{C}_i := F_i^{-1}(C_i) \cap (\partial U \times \mathbb{C})$, where $i \in \{0, 1\}$. Since the number of connected components of \hat{C}_i and their linking number are invariant under isotopy, Lemmas 7.11 and 7.12 imply that $s_0(U) = s_1(U)$. Since this is true for any bounded Fatou component U of p , the map ω is injective and the proof is finished. \square

8. UNBOUNDED HYPERBOLIC COMPONENTS IN \mathcal{M}

We have provided in the previous section a complete classification of unbounded components of \mathcal{D} accumulating on $\mathbb{P}_\infty^2 \setminus E$. In this section we adapt an interesting example ([Jon99, Example 9.6]) to construct unbounded hyperbolic components in \mathcal{M} . For the sake of notation, we start setting the following definition, motivated by Corollary 5.8.

Definition 8.1. *Let p be a quadratic polynomial. Let $z_1, z_2 \in J_p$ (possibly with $z_1 = z_2$). We say that a hyperbolic component $U \subset \mathbf{Sk}(p, 2)$ is of type $\{z_1, z_2\}$ if for all $z \in J_p$, $G(z, 0) = 0$ if and only if $z = z_1$ or $z = z_2$. We may write $\{z_1\}$ instead of $\{z_1, z_1\}$.*

The following theorem provides a basic classification of unbounded hyperbolic components in \mathcal{M} . While for \mathcal{D} we looked for a correspondence with (couples of) points in the Fatou set of p , for \mathcal{M} we see that a natural correspondence exists with (couples of) points in the Julia set of p .

Theorem 8.2. *Let p be a quadratic polynomial and $U \subset \mathbf{Sk}(p, 2)$ be an unbounded hyperbolic component in \mathcal{M} . Then there are $z_1, z_2 \in J_p$ such that U is either of type $\{z_1\}$ or of type $\{z_1, z_2\}$. Moreover, if U is of type $\{z_1\}$ then z_1 must be periodic for p , and if it is of type $\{z_1, z_2\}$ then either both z_1 and z_2 are periodic or one is preperiodic to the other.*

Proof. By Theorem A for any $f_1, f_2 \in U$ and $z \in J_p$, we have that $(z, 0)$ has a bounded orbit for f_1 if and only if it has a bounded orbit for f_2 . Since U is unbounded, Corollary 5.8 implies that there are at most two points $z_1, z_2 \in J_p$ such that $(z_i, 0)$ has bounded

orbit, and since U is a component in \mathcal{M} there is at least a $z \in J_p$ such that $(z, 0)$ has bounded orbit. Therefore there are $z_1, z_2 \in J_p$ (possibly with $z_1 = z_2$) such that U is of type $\{z_1, z_2\}$. In order to prove the remaining claims of the theorem, we will use the following lemma.

Lemma 8.3. *Let f be a polynomial skew-product that is vertically expanding above J_p . Let $z \in J_p$ and V be a connected component of \mathring{K}_z . There exists $n \in \mathbb{N}^*$ such that $f^n(V)$ contains a critical point for f .*

We refer to [DH08, Proposition 3.8] for a proof of this fact. It is stated there in the case of an Axiom A polynomial skew-product but the proof only uses vertical expansion over J_p .

Assume first that U is of type $\{z\}$, and let V be the connected component of \mathring{K}_z containing $(z, 0)$. By Lemma 8.3, there is $n \in \mathbb{N}^*$ such that $f^n(V)$ contains a critical point. But since all critical points $(y, 0)$, $y \in J_p$ escape if $y \neq z$, this means that $f^n(V) = V$ and $(z, 0) \in V$. In particular, we must have $p^n(z) = z$. Similarly, if U is of type $\{z_1, z_2\}$, let V_i denote the component of \mathring{K}_{z_i} containing $(z_i, 0)$ ($1 \leq i \leq 2$). By Lemma 8.3, there are $n_1, n_2 \in \mathbb{N}^*$ such that $f^{n_i}(V_i)$ is either V_1 or V_2 , from which the result follows. \square

We are now ready to give examples of all three possibilities of unbounded hyperbolic components in \mathcal{M} . We will need the following elementary lemma, following from Section 5.

Lemma 8.4. *Let $z_1, z_2 \in J_p$ with $z_1 \neq z_2$ and assume that U is a hyperbolic unbounded component of type $\{z_1, z_2\}$. Then the cluster of U on \mathbb{P}_∞^2 is exactly $\{(1 : -z_1 - z_2 : z_1 z_2)\}$.*

The following is an adaptation of [Jon99, Example 9.6].

Proposition 8.5. *Let $p(z) := z^2 - 2$, and let $g_t(z, w) := (p(z), w^2 + t(z + 1)(2 - z))$. Then for all $t > 0$ large enough,*

- (1) g_t is hyperbolic;
- (2) for all $z \in J_p \setminus \{-1, 2\}$, the critical point $(z, 0)$ escapes to infinity;
- (3) the critical points $(-1, 0)$ and $(2, 0)$ are fixed.

Proof. Observe that for all $z \in J_p$, $R := 3\sqrt{t}$ is an escape radius (i.e., $K_z \subset \mathbb{D}(0, 3\sqrt{t})$) and $|w| \geq 3\sqrt{t}$ implies that $|Q_z(w)| > 3\sqrt{t}$. Set

$$A_t := \{z \in [-2, 2] : |t(z + 1)(z - 2)| \geq 3\sqrt{t}\}.$$

Claim 8.6. *For $t > 0$ large enough and for any $z \in J_p \setminus \{-1, 2\}$ there exists $n \geq 0$ such that $p^n(z) \in A_t$.*

Proof of Claim 8.6. Notice that p is semi-conjugated on J_p to the doubling map on \mathbb{R}/\mathbb{Z} via the map $\varphi : \mathbb{R}/\mathbb{Z} \rightarrow J_p$ given by $\varphi(x) = 2 \cos(2\pi x)$. Note that $\varphi([0]) = 2$ and $\varphi([\frac{1}{3}]) = -1$. We start by proving the following statement: let $\varepsilon > 0$ and let

$$\tilde{A} := \left(\varepsilon, \frac{1}{3} - \varepsilon\right) \cup \left(\frac{2}{3} + \varepsilon, 1 - \varepsilon\right) \subset \mathbb{R}/\mathbb{Z}.$$

Then for any $\theta \in \mathbb{R}/\mathbb{Z} \setminus \{[0], [\frac{1}{3}], [\frac{1}{2}], [\frac{2}{3}]\}$, there exists $n \in \mathbb{N}$ such that $2^n \theta \in \tilde{A}$. To see that this last statement holds, first note that if $\theta \in [-\varepsilon, \varepsilon]$ and $\theta \neq 0$ then eventually

$2^n\theta \in \tilde{A}$ (provided ε is small enough, say $\varepsilon < \frac{1}{8}$). So, setting $I := (\frac{1}{3} - \varepsilon, \frac{2}{3} + \varepsilon)$, we may assume that $2^n\theta \in I \cup [\frac{1}{2}, 0]$ for all n . Thus, apart from the point $\theta = \frac{1}{2}$ (which is sent to the fixed point 0), we have that $2^{n+1}\theta$ belongs to I for all n . Thus, $2^n\theta$ must remain forever in a small neighborhood of $\{\frac{1}{3}, \frac{2}{3}\}$, and therefore that $\theta \in \{\frac{1}{3}, \frac{2}{3}\}$.

From this it follows that for small enough $\delta > 0$, we have that for any $z \in J_p \setminus \{-2, -1, 2\}$ there exists $n \in \mathbb{N}$ such that $p^n(z) \in (-1 + \delta, 2 - \delta)$. Since for $t > 0$ large enough we have that $t(z+1)(z-2) > 3\sqrt{t}$ for $z \in (-1 + \delta, 2 - \delta)$, the claim is proved (notice that $-2 \in A_t$ for t large enough). \square

Claim 8.7. *The following three assertions hold for t large enough, for $\delta > 0$ small enough (allowed to depend on t).*

(1) $\{|\operatorname{Im}(w)| \leq \delta\} \cap K_z = \emptyset$ for all $z \in J_p$ such that $|z+1| > \delta$ and $|z-2| > \delta$.

Set $U_\delta := \{|\operatorname{Im}(w)| \leq \delta, |\operatorname{Re}(w)| \leq \frac{1}{3}\}$ and $U'_\delta := \{|\operatorname{Im}(w)| \leq \delta, |\operatorname{Re}(w)| \leq \frac{1}{4}\}$. Then

(2) for all $z \in J_p$ such that $|z+1| < \delta$ or $|z-2| < \delta$, we have $q_z(U_\delta) \subset U'_\delta$;

(3) for all $z \in J_p \setminus \{-1, 2\}$, we have $U_\delta \cap K_z = \emptyset$.

Proof of Claim 8.7. Let us prove each item separately.

(1) Since K is closed, it is enough to prove that for all $z \in J_p \setminus \{-1, 2\}$, $\{\operatorname{Im}(w) = 0\} \cap K_z = \emptyset$. Fix $z \in J_p \setminus \{-1, 2\}$ and $w \in \mathbb{R}$. For t large enough, by Claim 8.6 there is some $n \geq 0$ such that $p^n(z) \in A_t$. Set $w_n := Q_z^n(w) \in \mathbb{R}$. Then $Q_z^{n+1}(w) = w_n^2 + t(z+1)(2-z) \geq t(2-z)(z+1) \geq 3\sqrt{t}$ and therefore $f^n(z, w) \notin K$, hence $(z, w) \notin K$.

(2) Let $t > 0$ large enough for Claim 8.6 to hold. Let $\delta > 0$ be given by the previous item. Note that for all $z \in J_p$ such that either $|z-2| \leq \delta$ or $|z+1| \leq \delta$, we have $|t(z+1)(z-2)| \leq 4t\delta$.

Fix $z \in J_p$ as in the statement and $w \in U_\delta$. Set $w_1 := q_z(w)$. Then,

$$\begin{cases} \operatorname{Re}(w_1) = \operatorname{Re}(w)^2 - \operatorname{Im}(w)^2 + t(2-z)(z+1) \\ \operatorname{Im}(w_1) = 2\operatorname{Im}(w)\operatorname{Re}(w) \end{cases}$$

and therefore

$$\begin{cases} |\operatorname{Re}(w_1)| \leq \frac{1}{9} + \delta^2 + 4t\delta < \frac{1}{4} \\ |\operatorname{Im}(w_1)| \leq 2\delta\frac{1}{3} \leq \delta \end{cases}$$

provided that δ is small enough. The assertion follows.

(3) For each z as in the statement and $w \in U_\delta$, by means of Claim 8.6 and iterating the second item we find a smallest $n \geq 1$ such that $p^n(z) \in A_t$ and $Q_z^n(w) \in U_\delta$. By the first item, $f^n(z, w) \notin K$; so $(z, w) \notin K$, and the proof is completed. \square

Let us now return to the proof of Proposition 8.5. Item 3 is trivial. Item 2 follows immediately from the last item of Claim 8.7. In order to prove that g_t is indeed hyperbolic, we prove that the post-critical set does not accumulate on J (see Theorem 2.5). Since the critical set over J_p is given by $[-2, 2] \times \{0\}$, it is enough to prove that

$$\text{for every } z \in [-2, 2] \text{ we have } d(F^n(z, 0), J) > \delta \text{ for every } n \geq 0.$$

where δ is as in Claim 8.7. We can assume that $\delta < \frac{1}{12}$ and that the distance between J and $J_p \times \{w \geq 3\sqrt{t}\}$ is also larger than δ .

Notice that item 3 of Claim 8.7 and the lower semicontinuity of J_z imply that we have $J \cap ([-2, 2] \times U_\delta) = \emptyset$. Thus, the claim is true for $n = 0$. Since $(2, 0)$ and $(-1, 0)$ are fixed, the claim is true for these two points. Moreover, the claim holds for every $z \in A_t$, since by definition $|q_z(0)| = |t(z+1)(z-2)| \geq 3\sqrt{t}$.

Fix then any other $z \in J_p$ and set $(z_n, w_n) := (p^n(z), Q_z^n(0))$. Notice that $w_n \in \mathbb{R}$. By Claim 8.6, there exists n such that $z_n \in A_t$. By the first item of Claim 8.7, it is then enough to prove that $d((z_j, w_j), J) \geq \delta$ for $1 \leq j < n$. But the second item of Claim 8.7 implies that $w_j \in \mathbb{R} \cap U'_\delta$. Since $J \cap ([-2, 2] \times U_\delta) = \emptyset$, the assertion follows. \square

Proposition 8.8. *Let $p(z) = z^2 - 2$. There are unbounded hyperbolic components in $Sk(p)$ of type $\{-1, 2\}$, $\{2\}$, and $\{-2, 2\}$.*

Notice the the component $\{-1, 2\}$ corresponds to two periodic points for p , while $\{-2, 2\}$ corresponding to a periodic point and a corresponding preperiodic point.

Proof. According to Proposition 8.5, the maps g_t are all hyperbolic, and since $t \mapsto g_t$ is a continuous, unbounded path in $\mathbf{Sk}(p, 2)$, they all belong to the same hyperbolic component which is unbounded and of type $\{-1, 2\}$. The existence of components of type $\{2\}$ and $\{-2, 2\}$ follows from considering skew-products of respective forms $(z, w) \mapsto (z^2 - 2, w^2 + t(2 - z))$ and $(z, w) \mapsto (z^2 - 2, w^2 + t(z + 2)(2 - z))$, and adapting Proposition 8.5 to those cases. \square

Question 8.9. *Let p be any quadratic polynomial, and $z_1, z_2 \in J_p$ be periodic points. Is it true that there exists in $\mathbf{Sk}(p, 2)$ an unbounded hyperbolic component of type $\{z_1, z_2\}$? The method that we used here in the case $p(z) = z^2 - 2$ relies crucially on the real structure of J_p , and cannot readily be adapted to the case where p is not real.*

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