Asymptotic stabilization of stationnary shock waves using a boundary feedback law

Vincent Perrollaz

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Abstract

In this paper we consider scalar conservation laws with a convex flux. Given a stationnary shock, we provide a feedback law acting at one boundary point such that this solution is now asymptotically stable in L^1-norm in the class of entropy solution.

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1 Generalities and previous results.

Scalar conservation laws in one dimension are equations of the form

$$u_t + (f(u))_x = 0, \tag{1}$$

where $u : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$.

They are used, for instance, to model traffic flow or gas networks, but their importance also lies in being a first step in the understanding of systems of conservation laws $u : \mathbb{R} \to \mathbb{R}^d$. Those systems of equations model a huge number of physical phenomena: gas dynamics, electromagnetism, magneto-hydrodynamics, shallow water theory, combustion theory... see [15, Chapter2].

For equations such as (1), the Cauchy problem on the whole line is well posed in small time in the framework of classical solutions and with a $\mathcal{C}^1$ initial value. However those solutions generally blow up in finite time: shock waves appear. Hence to get global in time results, a weaker notion of solution is called for.

In [29] Oleinik proved that given a flux $f \in \mathcal{C}^2$ such that $f'' > 0$ and any $u_0 \in L^\infty(\mathbb{R})$ there exists a unique weak solution to:

$$u_t + (f(u))_x = 0, \qquad x \in \mathbb{R} \quad \text{and} \quad t > 0, \tag{2}$$

$$u(0,.) = u_0, \tag{3}$$

satisfying the additional condition:

$$\frac{u(t, x+a) - u(t,x)}{a} \leq \frac{E}{t} \qquad \text{for} \quad x \in \mathbb{R}, \quad t > 0, \quad \text{and} \quad a > 0. \tag{4}$$

Here E depends only on the quantities $\inf(f'')$ and $\sup(f')$ taken on $[-||u_0||_{L^\infty}, ||u_0||_{L^\infty}]$ and not on u_0.

Later in [22], Kruzkov extended this global result to the multidimensional problem, with a C^1 flux $f : \mathbb{R} \to \mathbb{R}^n$ not necessarily convex:

$$u_t + \operatorname{div}(f(t,x,u)) = g(t,x,u), \qquad \text{for } t > 0 \quad \text{and } x \in \mathbb{R}^n. \tag{5}$$

This time the weak entropy solution is defined as satisfying the following integral inequality: for all real numbers k and all non-negative functions $\phi \in \mathcal{C}^1(\mathbb{R}^2)$

$$\iint_{\mathbb{R}^2} |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u) - f(k))\nabla\phi + \operatorname{sgn}(u-k)g(t,x,u)\phi dt dx \\ + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx \geq 0. \tag{6}$$

The initial boundary value problem for equation (1) is also well posed as shown by Leroux in [24] for the one dimensional case with BV data, by Bardos, Leroux and Nédélec in [8] for the multidimensional case with C^2 data and later by Otto in [30] (see also [27]) for L^∞ data. However the meaning of the boundary condition is quite intricate and the Dirichlet condition may not be fulfilled pointwise a.e. in time. We will go into further details later.

Before describing in detail our particular problem, let us recall a few general facts on general control systems. Consider such a system :

$$\begin{cases} \dot{X} = F(X,U), \\ X(t_0) = X_0, \end{cases} \tag{7}$$

(X being the state of the system belongs to the space $\mathcal{X}$ and U the so called control belongs to the space $\mathcal{U}$), we can consider two classical problems (among others) in control theory.

1. First the exact controllability problem which consists, given two states X_0 and X_1 in $\mathcal{X}$ and a positive time T, in finding a certain function $t \in [0,T] \mapsto U(t) \in \mathcal{U}$ such that the solution to (7) satisfies $X(T) = X_1$.

2. If $F(0,0) = 0$, the problem of asymptotic stabilization by a stationary feedback law asks to find a function of the state $X \in \mathcal{X} \mapsto U(X) \in \mathcal{U}$, such that for any state X_0 a maximal solution $X(t)$ of the closed loop system:

$$\begin{cases} \dot{X}(t) = F(X(t), U(X(t))), \\ X(t_0) = X_0, \end{cases} \tag{8}$$

   is global in time and satisfies additionally:

$$\forall R > 0,\ \exists r > 0 \quad \text{such that } ||X_0|| \leq r \quad \Rightarrow \quad \forall t \in \mathbb{R},\ ||X(t)|| \leq R, \tag{9}$$

$$X(t) \underset{t \to +\infty}{\to} 0. \tag{10}$$

The asymptotic stabilization property might seem weaker than exact controllability : for any initial state X_0, we can find T and $U(t)$ such that the solution to (7) satisfies $X(T) = 0$ in this way we stabilize 0 in finite time. However this method suffers from a lack of robustness with respect to perturbation: with any error on the model, or on the initial state, the control may not act properly anymore since at most we reach a close neighbourhood of the state 0. But if that stationnary state is unstable we then deviate significantly. This motivates the problem of asymptotic stabilization by a stationary feedback law which is more robust. Indeed in the case of perturbations, once we deviate enough from 0, the control acts up again and drive us toward 0. An additionnal property garanteeing a good robustness with respect to perturbations is the existence of a Lyapunov functionnal. In finite dimension it is often the case that if we can find a feedback function $\mathcal{U}$ stabilizing the stationnary state, we can find another one for which we additionnally have a Lyapunov function.

We are interested in the controllability properties of (1) when we use the boundary data as controls. In the framework of entropy solutions, some results exist for the exact controllability problem problem, see [6], [7], [1] [4], [5], [3], [11], [18], [19], [17], [21], [25], [26], [31]. See also [20] for a related problem.

Once we look at the problem of asymptotic stabilization in a classical framework the litterature is huge see the book [9] for an up to date bibliography. In the framework of entropy solution however the only existing articles (known to the author) are [13], [10] and [32]. Furthermore in those articles the goal is to stabilize a stationnary state which is actually regular. The entropy framework is only used to garantee more stability. In this paper we aim to stabilize results particular to the entropy framework : stationnary shock waves.

2 Discussion on the problem and on the proofs

Let us present rather informally and in the simpler case of Burgers' equation the problems we are interested in, the kind of result we want to obtain and the idea behind the proofs.

Burgers' equation is the simplest equation of type (1), it reads

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0. \tag{11}$$

If we look at the regular stationnary states it is clear that we have the constant states indexed by $\mathbb{R}$. For any real number $k \in \mathbb{R}$ the function u_k defined by

$$\forall x \in \mathbb{R}, \qquad u_k(x) := k, \tag{12}$$

is obviously a solutions of (11).

If we consider a stationnary entropy solution u, it is clear, since it is a weak solution that u^2 is a constant. Futhermore using Oleinik's estimate (4) which is valid for any time t since u is stationnary that u is actually decreasing. In the end we see that the family of such solutions is described by a positive real number k and a real number p through

$$\forall x \in \mathbb{R}, \qquad u_{k,p}(x) := \begin{cases} k & \text{if } x < p, \\ -k & \text{if } x > p. \end{cases} \tag{13}$$

Of course we see that at the discontinuity $x = p$, the Rankine-Hugoniot condition holds

$$0 = \frac{\frac{k^2}{2} - \frac{(-k)^2}{2}}{k - (-k)} = \frac{f(k) - f(-k)}{k - (-k)}.$$

Now let us look at the stability of those stationnary solutions.

- For the family u_k defined by (12), using the results of [15] Chapter 11 section 8, if we consider an initial data u_0 and a number $\epsilon > 0$ such that

  $$\forall x \in \mathbb{R}, \qquad |u_k(0, x) - u_0(x)| \leq \epsilon,$$

  then we have for the solution u of (11) corresponding to u_0

  $$\forall x \in \mathbb{R}, \qquad |u_k(t, x) - u_(t, x)| \leq \epsilon.$$

  So we have stability (though not asymptotic stabilization) of u_k in the L^∞ setting.

- Let us now consider (11) on the interval $(0, L)$ with additionnal boundary conditions

  $$\begin{cases} u(t, 0) = \alpha, \\ u(t, L) = \beta, \end{cases} \tag{14}$$

  once again let us mention that we cannot expect those boundary conditions to hold for a.e. time t. This is related to the presence of boundary layers at the borders, we will make a precise statement on the sense of the boundary conditions in the next part.

  It can be shown using generalized characteristics (see [32]) that if $k \neq 0$, $\alpha = \beta = k$ there exists a time T such that for $u_0 \in L^\infty(0, L)$ then the entropy solution u satisfy

  $$\forall t \geq T, \qquad \forall x \in (0, L), \qquad u(t, x) = k.$$

This is enough to show the asymptotic stabilization in $L^\infty(0,L)$ (and of course also in L^1) toward u_k.

As for robustness result, let us suppose that $\alpha, \beta > 0$ then we have a time $T > 0$ such that for any initial data u_0 the entropy solution u satisfies

$$\forall t \geq T, \qquad \forall x \in (0,L), \qquad u(t,x) = \alpha,$$

so as long as α is close to k we still have some reasonnable asymptotics.

- On the other hand for $k > 0$ if we look at the family $(u_{k,p})_{p\in(0,L)}$ it is clear that all those solutions satisfy (14) with $\alpha = k$ and $\beta = -k$. Since

$$||u_{k,p} - u_{k,p'}||_{L^1(0,L)} = 2|p - p'| \cdot k,$$

we already see that we cannot expect asymptotic stabilization for this family in L^1. In the L^∞ setting we have the following result from [28], if $\alpha = k$ and $\beta = -k$ there exists a time $T > 0$ such that for any initial data $u_0 \in L^\infty$ there exists $p \in (0,L)$ such that the entropy solution to (11),(14) satisfies

$$\forall t \geq T, \qquad \forall x \in (0,L), \qquad u(t,x) = u_{k,p}(x),$$

but the position p of the singularity does depend on u_0, so we basically cannot expect asymptotic stabilization in L^∞, though simple stability may still hold.

As far as robustness is concerned, it can be shown (using the results on generalized characteristics of [32]) for instance that if $\alpha > k$ and $\beta = -k$ then we have a time $T > 0$ such that for any initial data u_0 the entropy solution u satisfies

$$\forall t \geq T, \qquad \forall x \in (0,L), \qquad u(t,x) = \beta,$$

and even starting from $u_{k,p}$ we go far from it in L^∞ and in L^1.

For a more precise discussion of the above see [28].

Following the previous results, the goal is now, given a stationnary state $u_{k,p}$ to provide a feedback law for the boundary conditions such that $u_{k,p}$ is asymptotically stable for the semigroup.

To that end the idea is (very roughly) the following. According to the results of [28] we can expect that if we inject $\alpha = k$ and $\beta = -k$ in the system after some time we get a stationnary shock wave $u_{k,p'}$, now we want to move the singularity from p' to p, to that end we oserve the value of $u(t,.)$ at p if it is k then $p' < p$ and so the singularity needs to move to the right, so we modify α to be a bit more than k, after some time the trace to the left of the singularity will be this state so using the rankine Hugoniot condition the singularity will move with positive speed. Of course with $p' > p$ we set α a bit less than k so after some time the singularity will move to the left.

In practice there are multiple difficulties when we want to implement the above strategy.

1. Since we are in feedback form with no access to t, we cannot wait for the profile to be a $u_{k,p'}$ before using the second strategy which basically reduces the dynamic to a 1d phenomenon.

2. The time it takes for the inbound α to get to the singularity depends on the position of the singularity and of the state α. So basically we expect than rather than some scalar ODE on the position of the singularity we end up with a delayed differential equation with a delay depending on the solution itself.

3. We will get some kind of oscillatory phenomenon of the singularity around the goal p, we need to make sure that there is some kind of "damping".

4. The regularity will be $L^\infty_t \mathrm{BV}_x \cap \mathrm{Lip}_t L^1_x$ so we need some kind of filtered value of $u(t,.)$ near p.

Let us discuss now the content of the following sections. In Section 3 we will provide the main result and some definitions necessary for it. In Section 4 we provide the remaining definitions necessary for the proof. In 5, we will show that the closed loop system does have a unique solution which depends continuously of the initial data. In Section 6 we will provide results on generalized characteristics in particular their interactions with the boundary. They will be our main tool to study the solutions. In Section 7 we will prove the main result using a Lemma on delayed differential equations which itself is proved in A.

3 Main Result

Definition 1. *In the whole paper we will suppose the following fixed.*

- *The flux $f : \mathbb{R} \to \mathbb{R}$ will be a $\mathcal{C}^2$ uniformly convex function, so in particular*

$$\lim_{u \to \pm\infty} f(u) = +\infty.$$

We will additionnally suppose that

$$\min f = f(0) = 0,$$

but this is not restrictive since given a and b the flux change

$$\tilde{f}(u) := f(a+u) - b,$$

sends entropy solution on entropy solution.

- *Given a positive number m we can now define the numbers $u_l(m)$ and $u_r(m)$ satisfying*

$$u_l(m) < u_r(m), \qquad f(u_l(m)) = f(u_r(m)) = m.$$

- *We can now define another family of stationnary solutions. Let us consider $m > 0$ and $\alpha \in (0, L)$ we define*

$$\forall (t,x) \in \mathbb{R} \times (0,L), \qquad \bar{u}_{\alpha,m}(t,x) := \begin{cases} u_l(m) & \text{if } x < \alpha, \\ u_r(m) & \text{if } x \geq \alpha. \end{cases} \tag{15}$$

Proof. Since $\frac{f(u_l(m))-f(u_r(m))}{u_l(m)-u_r(m)} = 0$ the Rankine-Hugoniot condition is satisfied and thus $\bar{u}_{\alpha,m}$ is indeed a weak solution.

Since $u_l(m) < u_r(m)$ and f is convex the following entropy condition is also satisfied. For any $k \in (u_l(m), u_r(m))$,

$$\frac{f(u_l(m)) - f(k)}{u_l(m) - k} \geq \frac{f(u_l(m)) - f(u_r(m))}{u_l(m) - u_r(m)} \geq \frac{f(k) - f(u_r(m))}{k - u_r(m)}.$$

□

To describe the feedback law we will need the following functions. We suppose that we are given an interval $[0, L]$ and a position $\alpha \in (0, L)$.

Definition 2. *Let us consider three positive numbers $\epsilon,\ \delta, \nu$. (Those will be parameters to be tuned later on) We will suppose that $[\alpha - \delta, \alpha + \delta] \subset (0, L)$ and define the functions.*

$$\forall z \in \mathbb{R}, \qquad \mathcal{A}_{\epsilon,\nu}(z) := \begin{cases} -\epsilon & \text{if } z \leq -\nu, \\ \epsilon \frac{z}{\nu} & \text{if } -\nu \leq z \leq \nu\ . \\ \epsilon & \text{if } \nu \leq z \end{cases} \tag{16}$$

$$\forall u \in L^1(0, L), \qquad \mathcal{O}_{\alpha,\delta}(u) := \frac{1}{2\delta} \int_{\alpha-\delta}^{\alpha+\delta} (u(x) - \bar{u}_{\alpha,m}) dx. \tag{17}$$

We will now be interested in the solutions the following closed loop system

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(t, 0)" = "u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t, .))), \\ u(t, L)" = "u_r(m), \\ u(0, x) = u_0(x) \end{cases} \tag{18}$$

Theorem 1. *Given L, α, m and δ we can find ϵ and ν small enough such that given $u_0 \in \mathrm{BV}(0, L)$ the system (18) has a unique entropy solution u. Furthermore there are constants $C, M > 0$ such that*

$$\forall t \geq 0, \qquad ||u(t, .) - \bar{u}_{\alpha,m}||_{L^1(0,L)} \leq M e^{-Ct} ||u_0 - \bar{u}_{\alpha,m}||_{L^1(0,L)}. \tag{19}$$

Remark 1.

- *In the proofs we will precise the way ϵ and ν must be chosen.*
- *Note that we have chosen to act at the left boundary but the same result would hold with an action at the right boundary.*
- *The convexity of f is however crucial to the analysis.*

4 Entropy solution and Boundary conditions

We need to precise the sense in which we consider the solutions since we have both regularity problems and overdetermined boundary conditions (see [15] for a general exposition). We will

follow [23] and [8]. (one can also look at [2] and [12] for more general and up to date results) We will need the notations

$$\forall (a,b) \in \mathbb{R}^2, \qquad I(a,b) := [\min(a,b), \max(a,b)].$$

$$\forall z \in \mathbb{R}, \qquad \mathrm{sgn}(z) := \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 3. *We say that a function $u \in L^\infty([0,+\infty); \mathrm{BV}(0,L))$ is an entropy solution of (18) when for any number $k \in \mathbb{R}$ and any positive function $\phi \in \mathcal{C}^1_c(\mathbb{R}^2)$ we have*

$$\begin{aligned} \int_0^{+\infty} \int_0^L |u(t,x)-k| \partial_t \phi(t,x) + \mathrm{sgn}(u(t,x)-k)(f(u(t,x))-f(k)) \partial_x \phi(t,x) dx dt \\ \int_0^{+\infty} \mathrm{sgn}(u_r(m)-k)(f(k)-f(u(t,L^-)))\phi(t,L) dt \\ - \int_0^{+\infty} \mathrm{sgn}(u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t,.))) - k)(f(k)-f(u(t,0^+)))\phi(t,0) dt \\ + \int_0^L |u_0(x)-k| \phi(0,x) dx \geq 0 \end{aligned} \quad (20)$$

Let us be more explicit on the sense in which the boundary conditions hold.

Definition 4. *For $u \in \mathbb{R}$ we define $\mathrm{Adm}_l(u)$ and $\mathrm{Adm}_r(u)$to be*

$$\mathrm{Adm}_l(u) := \begin{cases} \{z \in \mathbb{R} \ : \ f'(z) \leq 0\} & \text{if } f'(u) \leq 0 \\ \{z \in \mathbb{R} \ : \ f'(z) < 0 \quad \text{and } f(z) \geq f(u)\} \cup \{u\} & \text{if } f'(u) > 0 \end{cases}$$

$$\mathrm{Adm}_r(u) := \begin{cases} \{z \in \mathbb{R} \ : \ f'(z) \geq 0\} & \text{if } f'(u) \geq 0 \\ \{z \in \mathbb{R} \ : \ f'(z) > 0 \quad \text{and } f(z) \geq f(u)\} \cup \{u\} & \text{if } f'(u) < 0 \end{cases}$$

At the right boundary we ask that for almost all time $t \geq 0$

$$u(t,L^-) \in \mathrm{Adm}_r(u_r(t)),$$

which means

$$\forall k \in I(u(t,L^-), u_r(m)), \qquad \mathrm{sgn}(u(t,L^-)-u_r(m))(f(u(t,L^-))-f(k)) \geq 0, \quad (21)$$

At the left boundary we ask that for almost all time $t \geq 0$

$$u(t,0^+) \in \mathrm{Adm}_l(u_l(t)),$$

$$\begin{aligned} \forall k \in I\big(u(t,0^+), u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t,.)))\big), \\ \mathrm{sgn}\Big(u(t,0^+) - (u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t,.))))\Big)(f(u(t,0^+))-f(k)) \leq 0, \end{aligned} \quad (22)$$

Remark 2. *The formulation in term of admissibility set depends on the convexity of f while (21) and (22) are more general.*

5 Existence and Uniqueness

In this part we consider a fixed $u_0 \in \mathrm{BV}(0,L)$ and we want to show the existence and uniqueness of a solution to the closed loop system (18).

Let us first recall the following result from [23], [8].

Theorem 2. *Given any time $T > 0$ and functions $u_0 \in \mathrm{BV}(0,L)$, $v_l \in \mathrm{BV}_{loc}(0,+\infty)$ and $v_r \in \mathrm{BV}_{loc}(0,+\infty)$ there exists a unique entropy solution $v \in L^\infty_{loc}((0,+\infty);\mathrm{BV}(0,L)) \cap \mathrm{Lip}_{loc}(\mathbb{R}^+;L^1(0,L))$ to*

$$\begin{cases} \partial_t v + \partial_x f(v) = 0, \\ v(t,0) = v_l(t), \\ v(t,L) = v_r(t), \\ v(0,x) = v_0(x). \end{cases} \tag{23}$$

Once again we interpret a solution of (23) to mean

$$\begin{aligned} \int_0^{+\infty}\int_0^L |v(t,x)-k|\partial_t\phi(t,x) + \mathrm{sgn}(v(t,x)-k)(f(v(t,x))-f(k))\partial_x\phi(t,x)dxdt \\ \int_0^{+\infty} \mathrm{sgn}(v_r(t)-k)(f(k)-f(v(t,L^-)))\phi(t,L)dt \\ -\int_0^{+\infty} \mathrm{sgn}(v_l(t)-k)(f(k)-f(v(t,0^+)))\phi(t,0)dt \\ + \int_0^L |v_0(x)-k|\phi(0,x)dx \geq 0, \end{aligned}$$

for any number k and any positive function $\phi \in \mathcal{C}^1(\mathbb{R}^2)$.

Definition 5. *Given a function $z \in L^\infty(\mathbb{R}^+) \cap \mathrm{Lip}(\mathbb{R}^+)$ we use the previous result to get $u \in L^\infty_{loc}(\mathbb{R}^+;\mathrm{BV}(0,L)) \cap \mathrm{Lip}_{loc}(\mathbb{R}^+;L^1(0,L))$ the solution to*

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(t,0) = u_l(m) - \mathcal{A}_{\epsilon,\nu}(z(t)), \\ u(t,L) = u_r(m), \\ u(0,x) = u_0(x). \end{cases} \tag{24}$$

We will now define the operator $\mathcal{F}$ by

$$\forall t \geq 0, \qquad \mathcal{F}(z)(t) := \mathcal{O}_{\alpha,\delta}(u(t,.)). \tag{25}$$

We now recall another result from [28].

Proposition 1. *If we consider initial data v_0, w_0 in $\mathrm{BV}(0,L)$ and boundary data v_l, v_r, w_l and w_r in $\mathrm{Lip}([0,T])$, the solutions v and w of*

$$\begin{cases} \partial_t v + \partial_x f(v) = 0, \\ v(t,0) = v_l(t), \\ v(t,L) = v_r(t), \\ v(0,x) = v_0(x), \end{cases} \qquad \begin{cases} \partial_t w + \partial_x f(w) = 0, \\ w(t,0) = w_l(t), \\ w(t,L) = w_r(t), \\ w(0,x) = w_0(x), \end{cases} \tag{26}$$

satisfy

$$\forall T>0, \qquad \int_0^L (v(T,x)-w(T,x))^+dx \leq \int_0^L (v_0(x)-w_0(x))^+dx + \int_0^T (v_l(t)-w_l(t))^+ + (v_r(t)-w_r(t))^+dt. \quad (27)$$

Where we used

$$\forall r\in\mathbb{R}, \qquad r^+ := \max(0,r).$$

Proof. This is just a particular case of Theorem 2.4 in [28]. □

Proposition 2. *The space $L^\infty(\mathbb{R}^+)\cap \mathrm{Lip}(\mathbb{R}^+)$ is stable under $\mathcal{F}$. Furthermore $\mathcal{F}$ has a unique fixed point on this space.*

Proof.

- We note that using the definition of $\mathcal{A}_{\epsilon,\nu}$ we get

$$\forall t\geq 0, \qquad \mathcal{A}_{\epsilon,\nu}(z(t))\in[-\epsilon,\epsilon],$$

therefore with

$$C := \max\left(||u_0||_{L^\infty(0,L)}, |u_l(m)|+\epsilon, |u_r(m)|\right),$$

we see that the constant function C (resp. $-C$) is solution of the system which is greater (resp. smaller) than u on the boundary so using Proposition 1 we see that we have

$$\forall (t,x)\in\mathbb{R}^+\times[0,L], \qquad -C\leq u(t,x)\leq C.$$

- For the next part of the result we use $k=\pm C$ in the definition of an entropy solution with a test function ϕ which has a support in $(0,+\infty)\times(0,L)$ to get

$$\int_0^{+\infty}\int_0^L u(t,x)\partial_t\phi(t,x)+f(u(t,x))\partial_x\phi(t,x)dxdt=0. \quad (28)$$

A classical density argument shows that the equation above is still admissible if ϕ is just Lipschitz.

Now given a time T positive numbers h and θ we define

$$\phi_\theta(t,x) := \psi_\theta(t)\kappa_\theta(x),$$

with

$$\psi_\theta(t) := \begin{cases} 0 & \text{if } t\leq T-\theta \\ \frac{t-T-\theta}{\theta} & \text{if } T-\theta\leq t\leq T \\ 1 & \text{if } T\leq t\leq T+h \\ \frac{T+h+\theta-t}{\theta} & \text{if } T+h\leq t\leq T+h+\theta \\ 0 & \text{otherwise,} \end{cases}$$

$$\kappa_\theta(x) := \begin{cases} 0 & \text{if } x\leq \alpha-\delta-\theta \\ \frac{x-\alpha+\delta}{\theta} & \text{if } \alpha-\delta-\theta\leq x\leq \alpha-\delta \\ 1 & \text{if } \alpha-\delta\leq x\leq \alpha+\delta \\ \frac{\alpha+\delta+\theta-x}{\theta} & \text{if } \alpha+\delta\leq x\leq \alpha+\delta+\mu \\ 0 & \text{otherwise,} \end{cases}.$$

Taking $\theta \to 0$ in (28) we obtain

$$\int_{\alpha-\delta}^{\alpha+\delta} u(T,x)dx - \int_{\alpha-\delta}^{\alpha+\delta} u(T+h,x)dx + \int_{T}^{T+h} f(u(t,\alpha-\delta)) - f(u(t,\alpha+\delta))dt = 0.$$

Using the definition of $\mathcal{O}_{\alpha,\delta}$ (17), the L^∞ bound and the convexity of f we now get

$$|\mathcal{F}(z)(T+h) - \mathcal{F}(z)(T)| \leq h\frac{\max(f(C), f(-C))}{2\delta}.$$

- Consider y and z two Lipschitz bounded functions. Let us call u and v the entropy solution involved in the definitions of $\mathcal{F}(y)$ and $\mathcal{F}(z)$. Using Proposition 1 we get

$$\begin{aligned}
|\mathcal{F}(y)(T) - \mathcal{F}(z)(T)| &\leq \frac{1}{2\delta}\int_0^L |u(T,x) - v(T,x)|dx \\
&\leq \frac{1}{2\delta}\int_0^T |\mathcal{A}_{\epsilon,\nu}(y(t)) - \mathcal{A}_{\epsilon,\nu}(z(t))|dt \\
&\leq \frac{1}{2\delta}\int_0^T \frac{\epsilon}{\nu}|y(t) - z(t)|dt \\
&\leq \frac{\epsilon}{2\delta\nu}T||y-z||_{L^\infty(0,T)}.
\end{aligned}$$

  This is enough to show that $\mathcal{F}$ is continuous with respect to the uniform convergence on any compact. But $\mathcal{F}$ takes value on a set which is uniformly bounded with equilipschitz functions, and is therefore a compact set for this precise topology. We can apply Schauder fixed point Theorem. (see [33])

- Let us now consider two such fixed points y and z. The previous calculation but gives

$$\forall T \geq 0, \qquad |y(T) - z(T)| \leq \frac{\epsilon T}{2\delta\nu}||y-z||_{L^\infty(0,T)}.$$

  For continuous functions $t \mapsto ||.||_{L^\infty(0,t)}$ is continuous and nondecreasing so if we define

$$T^* := \sup\{T \geq 0 \ : \ ||y-z||_{L^\infty(0,T)} = 0\},$$

  we see that if $\frac{\epsilon T}{2\delta\nu} < 1$ we have $T \leq T^*$, therefore

$$T^* \geq \frac{2\delta\nu}{\epsilon}.$$

  If we suppose that $T^* < +\infty$, since y is equal to z on $[0,T^*]$ so are u and v but then applying Proposition 1 with $u(T,.)$ as initial data we have with the same calculation as before

$$\forall T \geq T^*, \qquad |y(T) - z(T)| \leq \frac{\epsilon(T-T^*)}{2\delta\nu}||y-z||_{L^\infty(T^*,T)}.$$

  but if $\frac{(T-T^*)\epsilon}{2\delta\nu} < 1$ we see that $||y-z||_{L^\infty(0,T)} = 0$ so $T \leq T^*$ which is absurd therefore $T^* = +\infty$ and the fixed point of $\mathcal{F}$ is unique.

□

6 Generalized Characteristics and the boundary

We describe in this section a technical tool that will be used extensively in the following to study the local properties of the solution of the closed loop system. We begin by recalling a few definitions and results from [14]. We will refer in this section to the system

$$\begin{cases} \partial_t u + \partial_x(f(u)) = 0 & \text{on } (0,+\infty)\times(0,L), \\ u(0,.) = u_0 & \text{on } (0,L), \\ \operatorname{sgn}(u(t,L^-) - u_r(t))(f(u(t,L^-)) - f(k)) \geq 0 & \forall k \in I(u_r(t), u(t,L^-)),\ dt\ a.e., \\ \operatorname{sgn}(u(t,0^+) - u_l(t))(f(u(t,0^+)) - f(k)) \leq 0 & \forall k \in I(u_l(t), u(t,0^+)),\ dt\ a.e., \end{cases} \tag{29}$$

where **only for this section** u_l and u_r are two regulated functions of time thus defined on $\mathbb{R}^+$, $u_0 \in \mathrm{BV}(0,L)$ and u is the unique entropy solution.

Remark 3. *If the boundary condition at $x = 0$ in (29) is satisfied at time t it means that*

- *either $u(t,0^+) = u_l(t)$*
- *or for any state $k \in I(u_l(t), u(t,0^+))$ we have*

$$\frac{f(u(t,0^+) - f(k)}{u(t,0^+) - k} \leq 0,$$

which means that any wave generated by the Riemann problem between $u_l(t)$ and $u(t,0^+)$ leaves the domain.

the same kind of interpretation holds for the boundary condition at $x = L$.

Following [14] we introduce the notion of generalized characteristic.

Definition 6.

- *If γ is an absolutely continuous function defined on an interval $(a,b) \subset \mathbb{R}^+$ and with values in $(0,L)$, we say that γ is a generalized characteristic of (29) if:*

$$\dot\gamma(t) \in I(f'(u(t,\gamma(t)^-)), f'(u(t,\gamma(t)^+))) \qquad dt\ a.e..$$

This is the classical characteristic ODE taken in the weak sense of Filippov [16].

- *A generalized characteristic γ is said to be genuine on (a,b) if:*

$$u(t,\gamma(t)^+) = u(t,\gamma(t)^-) \quad dt\ a.e..$$

We recall the following results from [14].

Theorem 3.

- *For any (t,x) in $(0,+\infty)\times(0,L)$ there exists at least one generalized characteristic γ defined on (a,b) such that $a < t < b$ and $\gamma(t) = x$.*
- *If γ is a generalized characteristics defined on (a,b) then for almost all t in (a,b):*

$$\dot\gamma(t) = \begin{cases} f'(u(t,\gamma(t)) & \text{if } u(t,\gamma(t)^+) = u(t,\gamma(t)^-), \\ \frac{f(u(t,\gamma(t)^+)) - f(u(t,\gamma(t)^-))}{u(t,\gamma(t)^+) - u(t,\gamma(t)^-)} & \text{if } u(t,\gamma(t)^+) \neq u(t,\gamma(t)^-). \end{cases}$$

- *If γ is a genuine generalized characteristics on (a,b) (with $\gamma(a),\gamma(b) \in (0,L)$), then there exists a C^1 function v defined on (a,b) such that:*

$$\begin{gathered} u(b,\gamma(b)^+) \leq v(b) \leq u(b,\gamma(b)^-), \\ u(t,\gamma(t)^+) = v(t) = u(t,\gamma(t)^-) \quad \forall t \in (a,b), \\ u(a,\gamma(a)^-) \leq v(a) \leq u(a,\gamma(a)^+). \end{gathered} \tag{30}$$

Furthermore (γ, v) satisfy the classical ODE equation:

$$\begin{cases} \dot{\gamma}(t) = f'(v(t)), \\ \dot{v}(t) = 0, \end{cases} \qquad \forall t \in (a,b). \tag{31}$$

- *Two genuine characteristics may intersect only at their endpoints.*
- *If γ_1 and γ_2 are two generalized characteristics defined on (a,b), then we have:*

$$\forall t \in (a,b), \quad (\gamma_1(t) = \gamma_2(t) \Rightarrow \forall s \geq t,\ \gamma_1(s) = \gamma_2(s)).$$

- *For any (t,x) in $\mathbb{R}^+ \times (0,L)$ there exist two generalized characteristics χ^+ and χ^- called maximal and minimal and associated to v^+ and v^- by (31), such that if γ is a generalized characteristic going through (t,x) then*

$$\begin{gathered} \forall s \leq t, \qquad \chi^-(s) \leq \gamma(s) \leq \chi^+(s), \\ \chi^+ \text{ and } \chi^- \text{ are genuine on } \{s < t\}\ , \\ v^+(t) = u(t,x^+) \qquad \text{and} \qquad v^-(t) = u(t,x^-). \end{gathered}$$

Note that in the previous theorem, every property dealt only with the interior of $\mathbb{R}^+ \times [0,L]$. The following result describe the influence of the boundary conditions on the generalized characteristics.

Proposition 3. *Let u be the unique entropy solution of (29) and consider χ a genuine characteristic on an interval $[a,b]$ such that*

$$\forall t \in (a,b], \qquad \chi(t) \in (0,L),$$

then we know from the Theorem above that there is a constant $v \in \mathbb{R}$ such that

$$\forall t \in [a,b], \qquad \dot{\chi}(t) = f'(v)$$

and

$$\forall t \in (a,b), \qquad u(t,\gamma(t)) = v.$$

But then we have

$$\chi(a) = 0 \Rightarrow u_l(a^+) \leq v \leq u_l(a^-), \tag{32}$$

$$\chi(a) = L \Rightarrow u_r(a^-) \leq v \leq u_r(a^+). \tag{33}$$

(Where the existence of the limits is an hypothesis)

The main difficulty in the proof comes from the fact that the boundary conditions in (29) are satisfied only for almost all times. Before the proof let us begin with a lemma.

Lemma 1. *Consider $0 \leq t_0 < t_1$, $0 \leq x_A < x_B \leq L$ and $0 \leq x_C < x_D \leq L$. We introduce*

$$s_l := \frac{x_C - x_A}{t_1 - t_0}, \qquad s_r = \frac{x_D - x_B}{t_1 - t_0}.$$

We then have

$$\begin{aligned}\int_{x_A}^{x_B} u(t_0,x)dx - \int_{x_C}^{x_D} u(t_1,x)dx \\ + \int_{t_0}^{t_1} |f(u(t,(x_A + s_l(t-t_0))^+)) - s_l u(t,(x_A + s_l(t-t_0))^+)] \\ - [f(u(t,(x_B + s_r(t-t_0))^-)) - s_r u(t,(x_B + s_r(t-t_0))^-)]dt = 0.\end{aligned}$$

Note that by letting x_A tend to x_B we have the following equality wih $x_A = x_B$.

$$\begin{aligned}- \int_{x_C}^{x_D} u(t_1,x)dx \\ + \int_{t_0}^{t_1} [f(u(t,(x_A + s_l(t-t_0))^+)) - s_l u(t,(x_A + s_l(t-t_0))^+)] \\ - [f(u(t,(x_A + s_r(t-t_0))^+)) - s_r u(t,(x_A + s_r(t-t_0))^+)]dt = 0.\end{aligned}$$

Proof. We define

$$\chi_l(t) := x_A + (t-t_0)s_l, \qquad \chi_r(t) := x_B + s_r(t-t_0).$$

We can see that

$$\forall t \in [t_0, t_1], \qquad 0 \leq \chi_l(t) < \chi_r(t) \leq L.$$

Of course we also have

$$\chi_l(t_1) = x_C, \qquad \chi_r(t) = x_D.$$

We will now define for $\epsilon > 0$ small enough

$$\rho_\epsilon(t) := \begin{cases} 0 & \text{if } t \leq t_0 \\ \frac{t-t_0}{\epsilon} & \text{if } t_0 \leq t \leq t_0 + \epsilon \\ 1 & \text{if } t_0 + \epsilon \leq t \leq t_1 - \epsilon \\ \frac{t_1 - t}{\epsilon} & \text{if } t_1 - \epsilon \leq t \leq t_1 \\ 0 & \text{if } t_1 \leq t \end{cases}$$

it is clear that ρ_ϵ is Lipshitz continuous and has support $[t_0, t_1]$. We also need

$$\phi_\epsilon(t,x) := \begin{cases} 0 & \text{if } x \leq \chi_l(t) + \epsilon \\ \frac{x - (\chi_l(t) + \epsilon)}{\epsilon} & \text{if } \chi_l(t) + \epsilon \leq x \leq \chi_l(t) + 2\epsilon \\ 1 & \text{if } \chi_l(t) + 2\epsilon \leq x \leq \chi_r(t) - 2\epsilon \\ \frac{\chi_r(t) - \epsilon - x}{\epsilon} & \text{if } \chi_r(t) - 2\epsilon \leq x \leq \chi_r(t) - \epsilon \\ 0 & \text{if } \chi_r(t) - \epsilon \leq x \end{cases}$$

Now one can see that the function defined by

$$\forall (t,x) \in \mathbb{R}^2, \qquad \phi_\epsilon(t,x) := \rho_\epsilon(t)\phi_\epsilon(t,x)$$

is Lipschitz and has compact support in $(0,+\infty)\times(0,L)$ so one can use it has a test function in the weak formulation of the equation that is

$$\int_{\mathbb{R}^2} u(t,x)\partial_t\psi(t,x) + f(u(t,x))\partial_x\psi(t,x)dtdx = 0.$$

But then letting $\epsilon \to 0^+$ and remembering that u is Lipshitz in time with value in L^1 we get the result. □

Proof of Proposition 3. We will prove the two inequalities of (32) independently, (33) is a simple adaptation and so left to the reader.

- Since $f'(v) > 0$ the estimate is obvious if $f'(u_l(a^+)) \leq 0$ so we can suppose $f'(u_l(a^+)) > 0$ but then it implies that for δ small enough

  $$\forall t \in [a, a+\delta], \qquad f'(u_l(t)) > 0$$

  and using the definition of Adm_l in this case we see that for almost all $t \in [a, a+\delta]$ we have $f(u(t,0^+)) \geq f(u_l(t))$.

  For $\epsilon > 0$ small enough we apply Lemma 1 to $t_0 = a$, $t_1 = a + \frac{\epsilon}{f'(v)}$, $x_A = x_B = 0$, $x_C = 0$ and $x_D = \epsilon$ to obtain

  $$-\int_0^\epsilon u(a + \frac{\epsilon}{f'(v)}, x)dx + \int_a^{a+\frac{\epsilon}{f'(v)}} f(u(t,0^+)) - [f(v) - vf'(v)]dt = 0.$$

  Using the previous inequality we get that for $\epsilon < f'(v)\delta$

  $$-\int_0^\epsilon u(a + \frac{\epsilon}{f'(v)}, x)dx + \int_a^{a+\frac{\epsilon}{f'(v)}} f(u_l(t)) - [f(v) - vf'(v)]dt \leq 0,$$

  but now f is convex for

  $$f(u_l(t)) - f(v) + vf'(v) \geq f'(v)u_l(t),$$

  so we actually have

  $$-\int_0^\epsilon u(a + \frac{\epsilon}{f'(v)}, x)dx + \int_a^{a+\frac{\epsilon}{f'(v)}} f'(v)u_l(t)dt \leq 0. \tag{34}$$

  If we apply Lemma 1 to $t_0 = a + \frac{\epsilon}{f'(v)}$, $t_1 = b$, $x_A = 0$, $x_B = \epsilon$, $x_C = f'(v)(b-a) - \epsilon$ and $x_D = f'(v)(b-a)$ we get

  $$\int_0^\epsilon u(a + \frac{\epsilon}{f'(v)}, x)dx - \int_{f'(v)(b-a)-\epsilon}^{f'(v)(b-a)} u(b,x)dx + \int_{a+\frac{\epsilon}{f'(v)}}^b [f(u(t,(f'(v)(t-a)-\epsilon)^+) - f'(v)u(t,(f'(v)(t-a)-\epsilon)^+)] - [f(v) - f'(v)v]dt = 0.$$

  And the convexity of f implies

  $$[f(u(t,(f'(v)(t-a)-\epsilon)^+) - f'(v)u(t,(f'(v)(t-a)-\epsilon)^+)] - [f(v) - f'(v)v] \geq 0,$$

so we actually have

$$\int_0^{\epsilon} u(a+\frac{\epsilon}{f'(v)},x)dx - \int_{f'(v)(b-a)-\epsilon}^{f'(v)(b-a)} u(b,x)dx \leq 0. \tag{35}$$

But now adding (34) and (35) we end up with

$$\int_{f'(v)(b-a)-\epsilon}^{f'(v)(b-a)} u(b,x)dx \geq f'(v)\int_a^{a+\frac{\epsilon}{f'(v)}} u_l(t)dt,$$

and finally dividing by ϵ and taking $\epsilon \to 0^+$ we obtain the left inequality of (32).

- For the right inequality of (32). We will proceed in three steps.
  - Step 1, using Lemma 1 we get for $c \in (a,b)$

$$\int_a^c f(u(t,\chi(t))) - \dot{\chi}(t)u(t,\chi(t)) - f(u(t,\chi(t)+\epsilon)) + \dot{\chi}(t)u(t,\chi(t)+\epsilon)dt \\ \int_0^{\epsilon} u(a,x)dx - \int_{\chi(c)}^{\chi(c)+\epsilon} u(c,x)dx = 0,$$

    using the properties of χ we get

$$\int_a^c f(v) - f'(v)v - f(u(t,\chi(t)+\epsilon)) + f'(v)u(t,\chi(t)+\epsilon)dt \\ \int_0^{\epsilon} u(a,x)dx - \int_{\chi(c)}^{\chi(c)+\epsilon} u(c,x)dx = 0,$$

    using the convexity of f we get

$$\int_0^{\epsilon} u(a,x)dx \geq \int_{\chi(c)} \chi(c) + \epsilon u(c,x)dx,$$

    dividing by ϵ and letting $\epsilon \to 0$ we get

$$u(a,0^+) \geq u(c,\chi(c)^+) = v.$$

  - Step 2, since $f'(v) > 0$ and f' is increasing we have $f'(u(a,0^+)) > 0$ so for some point $\bar{x} \in (0,L)$ arbitrarily close to 0 we have $f'(u(a,x_0)) > 0$ and considering the minimal backward characteristic γ through $(t,\bar{x})$ we have $\dot{\gamma}(t) = f'(\bar{v}) > 0$ for some $\bar{v}$, therefore if $\bar{x}$ is close enough to 0 we have a time $c \in (0,a)$ such that $\gamma(c) = 0$. If we consider now a time $t \in (c,a)$ should we have $f'(u(t,0^+)) < 0$ then for x close enough to 0 we have both

$$f'(u(t,x)) < 0 \qquad \text{and} \qquad x < \gamma(t),$$

    but then the maximal backward characteristic through (t,x) will necessarily cross γ in (c,t) which is not possible thanks to Theorem 3. We can thus conclude that for any time $t \in (c,a)$

$$f'(u(t,0^+)) \geq 0.$$

But then using since the boundary condition at 0 in (29) holds for almost all time t we see that

$$u(t,0^+) = u_l(t), \qquad dt \ a.e. \quad \text{in } (c,a).$$

And also $f'(u_l(a^-)) \geq 0$.

– Step 3, let us consider $u_i > u_l(a^-)$. Using Step 2 we can see that $f'(u_i) > 0$. Furthermore for a small $\delta > 0$ we get

$$\forall t \in (a-\delta, a), \qquad f'(u_i) > f'(u_l(t)).$$

For $\epsilon > 0$, denote by a_ϵ and χ_ϵ the time and curve defined by

$$a_\epsilon := a - \frac{\epsilon}{f'(u_i)}, \qquad \forall t \in (a_\epsilon, a), \quad \chi_\epsilon(t) := \epsilon - f'(u_i)(a-t).$$

We have $\chi_\epsilon(a_\epsilon) = 0$ so using Lemma 1 on the triangle of vertices $(a,0)$, (a,ϵ) and $(a_\epsilon, 0)$ we get

$$-\int_0^\epsilon u(a,x)dx + \int_{a_\epsilon}^a f(u(t,0^+)) - f(u(t,\chi_\epsilon(t)^-) + f'(u_i)u(t,\chi_\epsilon(t)^-)dt = 0.$$

Using the result of the previous step we have then

$$-\int_0^\epsilon u(a,x)dx + \int_{a_\epsilon}^a f(u_l(t)) - f(u(t,\chi_\epsilon(t)^-) + f'(u_i)u(t,\chi_\epsilon(t)^-)dt = 0.$$

but since for ϵ small enough $a_\epsilon \geq a - \delta$ we have $f'(u_l(t)) > 0$ and $u_l(t) < u_i$. This means that $f(u_l(t)) \leq f(u_i)$ so we have

$$-\int_0^\epsilon u(a,x)dx + \int_{a_\epsilon}^a f(u_i) - f(u(t,\chi_\epsilon(t)^-) + f'(u_i)u(t,\chi_\epsilon(t)^-)dt \geq 0.$$

But f is convex therefore

$$f(u_i) - f(u(t,\chi_\epsilon(t)^-) + f'(u_i)u(t,\chi_\epsilon(t)^-) \leq f'(u_i)u_i,$$

and so

$$-\int_0^\epsilon u(a,x)dx + (a-a_\epsilon)f'(u_i)u_i \geq 0,$$

dividing by ϵ and taking $\epsilon \to 0^+$ we end up with

$$-u(a,0^+) + u_i \geq 0,$$

so using the result of Step 1 we can conclude

$$u_i \geq u(a,0^+) \geq v,$$

But u_i was arbitrarily close to $u_l(a^-)$ so as announced

$$v \leq u_l(a^-).$$

□

7 Asymptotic Stabilization

In this section u will be a given solution to the closed loop system (18). We will show estimates (19).

Lemma 2. *Consider T_1 given by the following definition*

$$A_{m,\epsilon} := \frac{f(u_l(m)-\epsilon)}{2}, \qquad T_1 := \max\left(\frac{L}{f'(u_l(A_{m,\epsilon}))}, \frac{L}{-f'(u_r(A_{m,\epsilon}))}\right).$$

there exist two Lipschitz functions $\beta_1, \beta_2 : (T_1, +\infty) \to (0, L)$ such that if we consider $(\bar{t}, \bar{x}) \in (T_1, +\infty) \times (0, L)$, then we have the alternatives

$$0 < \bar{x} < \beta_1(\bar{t}) \qquad \Rightarrow \qquad u(\bar{t}, \bar{x}^{\pm}) \in [u_l(m) - \epsilon, u_l(m) + \epsilon], \tag{36}$$

$$\beta_1(\bar{t}) < \bar{x} < \beta_2(\bar{t}) \qquad \Rightarrow \qquad -\frac{L}{\bar{t}} \leq f'(u(\bar{t}, \bar{x}^{\pm})) \leq \frac{L}{\bar{t}} \tag{37}$$

$$\beta_2(\bar{t}) < \bar{x} < L \qquad \Rightarrow \qquad u(\bar{t}, \bar{x}^{\pm}) = u_r(m). \tag{38}$$

Proof. We will proceed in mutliple steps.

- We consider $(\bar{t}, \bar{x}) \in (0, +\infty) \times (0, L)$. Using Theorem 3 we get the minimal backward characteristics γ. We call $[a, b]$ its maximal domain of definition. Following Theorem 3 and the maximality of $[a, b]$ we see that we have

$$(\gamma(a) = 0 \quad \text{and } a > 0) \quad \text{or } (\gamma(a) = L \quad \text{and } a > 0) \quad \text{or } a = 0.$$

  - In the first case, using Theorem 2, we have $u \in \mathrm{Lip}([0, \bar{t}]; L^1(0, L))$ therefore the boundary data at $x = 0$

$$t \mapsto u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t, .))),$$

    is Lipschitz. Using Proposition 3 we have then

$$u(\bar{t}, \bar{x}^-) = u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(a, .))) \in [u_l(m) - \epsilon, u_l(m) + \epsilon],$$

    given the definition of $\mathcal{A}_{\epsilon,\nu}$.
  - In the second case, Proposition 3 gives directly

$$u(\bar{t}, \bar{x}^-) = u_r(m).$$

  - Finally in the last case we have $a = 0$ and

$$\forall t \in [0, b], \qquad \dot{\gamma}(t) = f'(u(\bar{t}, \bar{x}^-)),$$

    thus

$$\gamma(\bar{t}) - \gamma(0) = f'(u(\bar{t}, \bar{x}^-))\bar{t},$$

    which means (since $\bar{x} = \gamma(\bar{t})$)

$$f'(u(\bar{t}, \bar{x})) = \frac{\bar{x} - \gamma(0)}{\bar{t}}.$$

Using $0 \leq \gamma(0) \leq L$ we get

$$\frac{\bar{x} - L}{\bar{t}} \leq f'(u(\bar{t}, \bar{x}^-)) \leq \frac{\bar{x}}{\bar{t}}. \tag{39}$$

which implies

$$-\frac{L}{\bar{t}} \leq f'(u(\bar{t}, \bar{x}^-)) \leq \frac{L}{\bar{t}}.$$

Now using Theorem 3 we know that genuine characteristics do not cross. Therefore given $\bar{t}$ the set of $\bar{x}$ for which we are in first case, second case or third case are connected therefore intervals, they form a partition of $[0, L]$. And from a geometrical viewpoint it is obvious that from the left to the right we have points from the first case, points from the last case and points from the second case.

- At this point we have indeed constructed two functions β_1 and β_2 such that (36), (37) and (38) hold for $\bar{x}^-$.

  Since if $0 < c < \bar{x} < d < 1$ we have

  $$u(\bar{t}, \bar{x}^+) = \lim_{\epsilon \to 0^+} u(\bar{t}, (\bar{x} + \epsilon)^-),$$

  (36), (37), (38) and (39) also hold for $\bar{x}^+$.

  Note that using (39) we get for any $t > 0$

  $$\beta_1(t) = 0 \qquad \Rightarrow \qquad -\frac{L}{t} \leq f'(u(t, 0^+)) \leq 0.$$

  We have on one hand

  $$u(t, 0^+) \leq 0 < u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t, .))),$$

  and using Remark 3 we can deduce

  $$f(u(t, 0^+)) \geq f(u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t, .)))),$$

  which implies

  $$f(u(t, 0^+)) \geq f(u_l(m) - \epsilon).$$

  On the other hand, if $t \geq T_1$, we have using the definition of T_1

  $$f'(u_r(A_{m,\epsilon})) \leq -\frac{L}{T_1} \leq f'(u(t, 0^+)) \leq 0,$$

  which implies that

  $$u_r(A_{m,\epsilon}) \leq u(t, 0^+) \leq 0,$$

  and therefore

  $$f(u(t, 0^+)) \leq f(u_r(A_{m,\epsilon})) = A_{m,\epsilon} = \frac{f(u_l(m) - \epsilon)}{2} < f(u_l(m) - \epsilon).$$

  which is contradictory. And we can deduce that

  $$\forall t \geq T_1, \qquad \beta_1(t) > 0.$$

In the same way, using (39) we get for any $t > 0$

$$\beta_2(t) = L \qquad \Rightarrow \qquad 0 \leq f'(u(t, L^-)) \leq \frac{L}{t}.$$

On one hand we get

$$u(t, L^-) \geq 0 > u_r(m),$$

and using Remark 3 we have in particular

$$f(u(t, L^-)) \geq f(u_r(m)) = m,$$

On the other hand, if $t \geq T_1$ we have using the definition of T_1

$$f'(u_l(A_{m,\epsilon})) \geq \frac{L}{T_1} \geq \frac{L}{t} \geq f'(u(t, L^-)) \geq 0,$$

and therefore

$$u_l(A_{m,\epsilon}) \geq u(t, L^-) \geq 0.$$

We can then obtain

$$f(u(t, L^-)) \leq f(u_l(A_{m,\epsilon})) = A_{m,\epsilon} = \frac{f(u_l(m) - \epsilon)}{2} < f(u_l(m) - \epsilon) < f(u_l(m)) = m,$$

which is contradictory. So we can conclude that

$$\forall t \geq T_1, \qquad \beta_2(t) < 0.$$

- It remains to prove that β_1 and β_2 are Lipschitz functions. To this end let us first remark that those functions are uniquely defined through our previous requirements.

  Now consider $\bar{t} \in (T_1, +\infty)$. Then $\bar{x} := \beta_1(\bar{t}) \in (0, L)$, so we have a unique forward characteristic through $(\bar{t}, \bar{x})$ let us call it γ_1, defined on a certain interval $[\bar{t}, c]$ with $c > \bar{t}$. Let us fix $t \in (\bar{t}, c)$.

  If we choose $x \in (0, \gamma_1(t))$ if we consider γ_2 the minimal backward characteristic through (t, x), it is defined maximally on an interval $[b, t]$. By uniqueness of forward characteristic we have

  $$\forall s \in [\max(\bar{t}, b), t], \qquad \gamma_2(s) < \gamma_1(s).$$

  We have two alternatives.

  - But then if $b > \bar{t}$ we have $\gamma_2(b) = 0$ and $b > 0$ therefore $x < \beta_1(t)$.
  - If on the other hand we have $b \leq \bar{t}$ then $\gamma_2(\bar{t}) < \gamma_1(\bar{t}) = \bar{x} = \beta_1(t)$. But then γ_2 is also the minimal backward characteristic through $(\bar{t}, \gamma_2(\bar{t}))$ and thus $b > 0$ and $\gamma_2(b) = 0$ therefore $x < \beta_1(t)$.

  In the end we have proved

  $$\forall x \in (0, \gamma_1(t)), \qquad x < \beta_1(t),$$

  therefore we have $\gamma_1(t) \leq \beta_1(t)$.

  If we choose $x \in (\gamma_1(t), L)$ and consider γ_3 the minimal backward characteristic through (t, x) defined maximally on $[b, t]$. Using the uniqueness of forward characteristic we have

  $$\forall x \in [\max(b, \bar{t}), t], \qquad \gamma_3(s) > \gamma_1(s).$$

  We have two alternatives.

– If $b > \bar{t}$ we have $\gamma_3(b) = L$ and $b > 0$ therefore $x > \beta_2(t) \geq \beta_1(t)$.

– If on the other hand $b \leq \bar{t}$ we have $\gamma_3(\bar{t}) > \gamma_1(\bar{t}) = \beta_1(\bar{t})$, but then γ_3 is also the minimal backward characteristic through $(\bar{t}, \gamma_3(\bar{t}))$ and by construction of β_1 and β_2 we have can conclude $x > \beta_1(t)$.

So we have proved

$$\forall x \in (\gamma_1(t), L), \qquad x > \beta_1(t),$$

which implies $\gamma_1(t) \geq \beta_1(t)$. Since we already had the other inequality $\beta_1 = \gamma_1$. But this means that β_1 being a generalized characteristic is Lipschitz.

The same argument works for β_2.

□

Lemma 3. *There exists a time T_2 independent of the initial data u_0 (see (40) for an exact formula) and a Lipschitz function $\beta : [T_2, +\infty[\to (0, L)$ satisfying*

$$\forall t \geq T_2, \quad \forall x \in (0, L), \qquad \begin{cases} x < \beta(t) \Rightarrow u_l(m) - \epsilon \leq u(t, x^+) \leq u_l(m) + \epsilon \\ x > \beta(t) \Rightarrow u(t, x^-) = u_r(m) \end{cases}$$

Proof. We just need to show the existence of $T_2 > 0$ independant of u_0 such that

$$\forall t \geq T_2, \qquad \beta_1(t) = \beta_2(t).$$

Let us suppose $\beta_1(t) < \beta_2(t)$ for $t \in [T_1, T]$.

- Using the definition of β_1 and looking at the minimal backward characteristics through $(t, \beta_1(t))$ we get

  $$u_l(m) - \epsilon \leq u(t, \beta_1(t)^-) \leq u_l(m) + \epsilon,$$

  and considering the maximal one

  $$f'(u_r(A_{m,\epsilon})) \leq -\frac{L}{T_1} \leq -\frac{L}{t} \leq f'(u(t, \beta_1(t)^+)) \leq \frac{L}{t} \leq \frac{L}{T_1} \leq f'(u_l(A_{m,\epsilon})).$$

  But then

  $$u_r(A_{m,\epsilon}) \leq u(t, \beta_1(t)^+) \leq u_l(A_{m,\epsilon}) < u_l(m) - \epsilon.$$

  Furthermore Theorem 3 grants for almost all $t \in (T_1, T)$

  $$\dot{\beta_1}(t) = \frac{f(u(t, \beta_1(t)^-)) - f(u(t, \beta_1(t)^+))}{u(t, \beta_1(t)^-) - u(t, \beta_1(t)^+)}.$$

  Now remark that for w, z the formula

  $$\frac{f(z) - f(w)}{z - w} = \int_0^1 f'(\theta w + (1 - \theta) z) d\theta,$$

  show that this function is increasing in both variables therefore

  $$\dot{\beta_1}(t) \geq \frac{f(u_l(m) - \epsilon) - f(u_r(A_{m,\epsilon}))}{u_l(m) - \epsilon - u_r(A_{m,\epsilon})} =: c_1 > 0.$$

- Using the definition of β_2 and looking at the maximal backward characteristics through $(t, \beta_2(t))$ we get
$$u_r(m) = u(t, \beta_2(t)^+),$$
and considering the minimal one
$$f'(u_r(A_{m,\epsilon})) \leq -\frac{L}{T_1} \leq -\frac{L}{t} \leq f'(u(t, \beta_2(t)^-)) \leq \frac{L}{t} \leq \frac{L}{T_1} \leq f'(u_l(A_{m,\epsilon})).$$
But then
$$u_r(m) < u_r(A_{m,\epsilon}) \leq u(t, \beta_2(t)^-) \leq u_l(A_{m,\epsilon}).$$
Furthermore Theorem 3 grants for almost all $t \in (T_1, T)$
$$\dot{\beta}_2(t) = \frac{f(u(t, \beta_2(t)^-)) - f(u(t, \beta_2(t)^+))}{u(t, \beta_2(t)^-) - u(t, \beta_2(t)^+)}.$$
and as before
$$\dot{\beta}_2(t) \leq \frac{f(u_l(A_{m,\epsilon})) - f(u_r(m))}{u_l(A_{m,\epsilon}) - u_r(m)} =: -c_2 < 0.$$

- We have then $\beta_1(T_1) \geq 0$ for almost all $t \in (T_1, T)$, $\dot{\beta}_1(t) \leq c_1$, so
$$\beta_1(T) \geq c_1(T - T_1).$$
In the same way we obtain
$$\beta_2(t) \leq L - c_2(T - T_1).$$
But we had supposed $\beta_1(T) \leq \beta_2(T)$ so
$$T \leq T_1 + \frac{L}{c_1 + c_2} =: T_2. \tag{40}$$

We have thus shown that
$$\forall t \geq T_2, \qquad \beta_1(t) = \beta_2(t).$$

□

From this result, we get multiple properties.

Remark 4. *We have the following*
$$\forall t \geq T_2, \quad \forall x \in (0, \beta(t)), \qquad u_l(m) - \epsilon \leq u(t, x) \leq u_l(m) + \epsilon$$
and combined with Definition 4 this implies
$$u(t, 0^+) = u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(t, .)) \quad dt \text{ a.e.}$$
We also have
$$\forall t \geq T_2, \quad \forall x \in (\beta(t), L), \qquad u(t, x) = u_r(m),$$

We can then deduce using (17)

$$\forall t \geq T_2, \quad -\frac{u_l(m)-u_r(m)}{2} \leq \mathcal{O}_{\alpha,\delta}(u(t,.)) \leq \frac{u_l(m)-u_r(m)}{2} + \epsilon,$$

$$\forall t \geq T_2, \qquad \big(\beta(t) > \alpha+\delta \qquad \Rightarrow \qquad \mathcal{O}_{\alpha,\delta}(u(t,.)) \geq \frac{u_l(m)-u_r(m)}{2} - \epsilon\big),$$

$$\forall t \geq T_2, \qquad \big(\beta(t) < \alpha-\delta, \qquad \Rightarrow \qquad \mathcal{O}_{\alpha,\delta}(u(t,.)) = \frac{u_r(m)-u_l(m)}{2}\big),$$

And finally using Theorem 3

$$\forall t \geq T_2, \qquad \tilde{c}_{\epsilon,m} \leq \dot{\beta}(t) \leq \bar{c}_{\epsilon,m}.$$

where we have defined

$$\tilde{c}_{\epsilon,m} := \frac{f(u_l(m)-\epsilon) - f(u_r(m))}{u_l(m)-\epsilon-u_r(m)} < 0 \tag{41}$$

and

$$\bar{c}_{\epsilon,m} := \frac{f(u_l(m)+\epsilon) - f(u_r(m))}{u_l(m)+\epsilon-u_r(m)} > 0. \tag{42}$$

And note that $\bar{c}_{\epsilon,m}$ and $\tilde{c}_{\epsilon,m}$ tend to 0 when $\epsilon \to 0$, independantly of ν, α, δ.

Lemma 4. *Consider θ given by*

$$\theta := \max\left(\frac{u_l(m)}{u_l(m)-u_r(m)-\epsilon}, \frac{\epsilon-u_r(m)}{u_l(m)+\epsilon-u_r(m)}\right) \in (0,1) \tag{43}$$

(Note that as ϵ tends to 0, θ tends to a limit strictly less than 1.)

Then for $t \geq T_2$,

$$\beta(t) \geq \alpha+\theta\delta \qquad \Rightarrow \qquad \mathcal{O}_{\alpha,\delta}(u(t,.)) \geq \frac{u_l(m)-\epsilon}{2},$$

$$\beta \leq \alpha-\theta\delta \qquad \Rightarrow \qquad \mathcal{O}_{\alpha,\delta}(u(t,.)) < \frac{u_r(m)}{2}.$$

Proof. Let us first recall that $u_r(m) < 0 < u_l(m) - \epsilon$.

Note that if $\alpha-\delta < \beta(t) < \alpha+\delta$ and we introduce

$$z := \frac{\beta(t)-\alpha}{\delta} \in (-1,1),$$

- we have using the definition of $\mathcal{O}_{\alpha,\delta}$ (17)

$$\begin{aligned}\mathcal{O}_{\alpha,\delta}(u(t,.)) &\geq \frac{\beta(t)-(\alpha-\delta)}{2\delta}(u_l(m)-\epsilon) + \frac{\alpha+\delta-\beta(t)}{2\delta}u_r(m) - \frac{u_l(m)+u_r(m)}{2} \\ &= \frac{\beta(t)-\alpha}{\delta}\frac{u_l(m)-u_r(m)-\epsilon}{2} + \frac{\delta(u_l(m)-\epsilon+u_r(m))}{2\delta} - \frac{u_l(m)+u_r(m)}{2} \\ &= z\frac{u_l(m)-u_r(m)-\epsilon}{2} - \frac{\epsilon}{2}.\end{aligned}$$

but it is clear that this last term is increasing and equals $\frac{u_l(m)-\epsilon}{2}$ for z equal to

$$\theta_1 := \frac{u_l(m)}{u_l(m)-u_r(m)-\epsilon} \in (0,1).$$

- we also have

$$\begin{aligned}\mathcal{O}_{\alpha,\delta}(u(t,.)) &\leq \frac{\beta(t)-(\alpha-\delta)}{\delta}\frac{u_l(m)+\epsilon}{2}+\frac{\alpha+\delta-\beta(t)}{\delta}\frac{u_r(m)}{2}-\frac{u_l(m)+u_r(m)}{2}\\ &= \frac{\beta(t)-\alpha}{\delta}\frac{u_l(m)-u_r(m)+\epsilon}{2}+\frac{\delta(u_l(m)+\epsilon+u_r(m)}{2\delta}-\frac{u_l(m)+u_r(m)}{2}\\ &= z\frac{u_l(m)-u_r(m)+\epsilon}{2}+\frac{\epsilon}{2}.\end{aligned}$$

  But it is clear that this last term is increasing and equals $\frac{u_r(m)}{2}$ for z equal to

$$\theta_2 := -\frac{\epsilon-u_r(m)}{u_l(m)+\epsilon-u_r(m)} \in (-1,0).$$

- So we have

$$\beta(t) \geq \alpha+\theta_1\delta \qquad \Rightarrow \mathcal{O}_{\alpha,\delta}(u(t,.)) \geq \frac{u_l(m)-\epsilon}{2},$$

$$\beta(t) \leq \alpha+\theta_2\delta \qquad \Rightarrow \mathcal{O}_{\alpha,\delta}(u(t,.)) \leq \frac{u_r(m)}{2},$$

  And also by a simple calculation

$$-1 < \theta_2 \leq \theta_1 < 1.$$

  So taking $\theta := \max(|\theta_1|,|\theta_2|)$, we have indeed $\theta \in (0,1)$ and

$$\alpha-\theta\delta \leq \alpha+\theta_2\delta < \alpha+\theta_1\delta \leq \alpha+\theta\delta.$$

- Finally the cases $\beta(t) < \alpha-\delta$ and $\beta(t) > \alpha+\delta$ are obvious consequences of the previous calculations.

□

Lemma 5. *There exists ϵ_0 such that given $\nu_0 := \frac{u_l(m)-u_r(m)}{2}$, for any $\nu > \nu_0$ and any $\epsilon \in (0, \min(\epsilon_0, \nu - \frac{u_l(m)-u_r(m)}{2}))$ there exists T_3 independant of u_0 (see (48) for the exact formula) satisfying*

$$\forall t \geq T_3, \qquad \alpha-\delta < \beta(t) < \alpha+\delta.$$

Proof. Let us consider $\epsilon_1 := \nu - \frac{u_l(m)-u_r(m)}{2}$. Then using (41), (42) and (43), we know that

$$\frac{\bar{c}_{\epsilon,m}}{(1-\theta)f'(u_l(m)-\epsilon)} \underset{\epsilon\to 0}{\to} 0,$$

and

$$\frac{\tilde{c}_{\epsilon,m}}{(1-\theta)f'(u_r(m))} \underset{\epsilon\to 0}{\to} 0.$$

So there exists $\epsilon_0 < \epsilon_1$ such that

$$\forall \epsilon \in (0,\epsilon_0), \qquad \begin{cases} \frac{\bar{c}_{\epsilon,m}}{(1-\theta)f'(u_l(m)-\epsilon)} < \frac{\delta}{L} \\ \frac{\tilde{c}_{\epsilon,m}}{(1-\theta)f'(u_r(m))} < \frac{\delta}{L}. \end{cases} \tag{44}$$

With such a choice of parameters let us show the result. We will proceed in multiple steps.

- Let us suppose that for an interval $[a,b] \subset [T_2, +\infty[$ we have

$$\forall t \in [a,b], \qquad \beta(t) \geq \alpha + \theta\delta.$$

Using Lemma 4 we have

$$\forall t \in [a,b], \qquad \mathcal{O}_{\alpha,\delta}(u(t,.)) \geq \frac{u_l(m)-\epsilon}{2} > 0.$$

But thanks to $\nu > \nu_0$ and $\epsilon < \epsilon_1$ we deduce

$$\forall t \in [a,b], \qquad \mathcal{A}_{\epsilon,\nu}\left(\mathcal{O}_{\alpha,\delta}(u(t,.))\right) \geq \frac{\epsilon}{\nu}\frac{u_l(m)-\epsilon}{2} > 0.$$

But then using Remark 4 we have for almost any $t \in [a,b]$

$$0 < u_l(m) - \epsilon \leq u(t,0^+) \leq u_l(m) - \frac{\epsilon}{\nu}\frac{u_l(m)-\epsilon}{2} < u_l(m). \tag{45}$$

Now let us suppose that $[a + \frac{L}{f'(u_l(m)-\epsilon)}, b]$ non empty and consider a time $\bar{t}$ in the interval. Looking at the minimal backward characteristic through $(\bar{t}, \beta(\bar{t}))$ and using Lemmas 3 and 2 we have

$$u(\bar{t}, \beta(\bar{t})^-) = u(\bar{t} - \frac{\beta(\bar{t})}{f'(u(\bar{t}, \beta(\bar{t})^-))}, 0^+).$$

But clearly using Lemma 3 and 2 we have $u(\bar{t}, \beta(\bar{t})^-) \geq u_l(m) - \epsilon$ so

$$0 \leq \frac{\beta(\bar{t})}{f'(u(\bar{t}, \beta(\bar{t})^-))} \leq \frac{L}{f'(u_l(m)-\epsilon)},$$

so we have

$$a \leq \bar{t} - \frac{\beta(\bar{t})}{f'(u(\bar{t}, \beta(\bar{t})^-))} \leq b.$$

From this and Proposition 3 we deduce

$$u(\bar{t}, \beta(\bar{t})^-) \leq u_l(m) - \frac{\epsilon}{\nu}\frac{u_l(m)-\epsilon}{2}.$$

But looking at the maximal backward characteristic trhough $(bt, \beta(\bar{t})$ and using Lemmas 3 and 2 we get

$$u(\bar{t}, \beta(\bar{t})^+) = u_r(m).$$

Using Theorem 3 we have shown that if

$$b - a \geq \frac{L}{f'(u_l(m)-\epsilon)},$$

then for almost any time t of the interval $[a + \frac{L}{f'(u_l(m)-\epsilon)}, b]$

$$\dot{\beta}(t) \leq \frac{f(u_l(m) - \frac{\epsilon}{\nu}\frac{u_l(m)-\epsilon}{2}) - f(u_r(m))}{u_l(m) - \frac{\epsilon}{\nu}\frac{u_l(m)-\epsilon}{2} - u_r(m)} := \tilde{d}_{\epsilon,m} < 0.$$

But since β is confined inside $(\alpha+\theta\delta, L)$ on $[a,b]$ we require

$$\tilde{d}_{\epsilon,m}(b-a-\frac{L}{f'(u_l(m)-\epsilon)}+L \geq \alpha+\theta\delta,$$

which is in fact

$$b-a \leq \frac{L}{f'(u_l(m)-\epsilon)}+\frac{\alpha+\theta\delta-L}{\tilde{d}_{\epsilon,m}}. \tag{46}$$

- The same method show that if

$$\forall t \in [a,b], \qquad \beta(t) \leq \alpha-\theta\delta,$$

then for almost any $t \in [a,b]$, we have

$$u(t,0^+) \geq u_l(m)-\frac{\epsilon}{\nu}\frac{u_r(m)}{2} > u_l(m).$$

Then should we have

$$a+\frac{\alpha-\theta\delta}{f'(u_l(m)-\epsilon)} \leq \bar{t} \leq b,$$

we have

$$u(\bar{t},\beta(\bar{t}^-)) \geq u_l(m)-\frac{\epsilon}{\nu}\frac{u_r(m)}{2},$$

and then

$$\dot{\beta}(t) \geq \bar{d}_{\epsilon,m} := \frac{f(u_l(m)-\frac{\epsilon}{\nu}\frac{u_r(m)}{2})-f(u_r(m))}{u_l(m)-\frac{\epsilon}{\nu}\frac{u_r(m)}{2}-u_r(m)} > 0.$$

And in the end because β is supposed to be confined to $[0,\alpha-\theta\delta]$ for $t \in [a,b]$ we have the restriction

$$b-a \leq \frac{\alpha-\theta\delta}{f'(u_l(m)-\epsilon)}+\frac{\alpha-\theta\delta}{\bar{d}_{\epsilon,m}}, \tag{47}$$

- To conclude this part, we have showed that if we define

$$T_3 = T_2+\max\left(-\frac{L-\alpha-\theta\delta}{\tilde{d}_{\epsilon,m}}-\frac{L}{f'(u_l(m)-\epsilon)}, \frac{\alpha-\theta\delta}{f'(u_l(m)-\epsilon)}+\frac{\alpha-\theta\delta}{\bar{d}_{\epsilon,m}}\right). \tag{48}$$

(see (46) and (47)) then β cannot be continuously in $(0,\alpha-\theta\delta)$ or $(\alpha+\theta\delta)$ on $[T_2,T_3]$. Since β is Lipschitz we have a time $\bar{t} \in [T_2,T_3]$ such that

$$\beta(\bar{t}) \in (\alpha-\theta\delta, \alpha+\theta\delta).$$

- Let us now consider an hypothetical time $b \geq T_3$ such that

$$\beta(b) \geq \alpha+\delta.$$

Using the previous result we can define

$$a := \sup\{t \in [T_2,b] \ : \ \beta(a) = \alpha+\theta\delta\}.$$

We have then $\beta(a) = \alpha + \theta\delta$, and

$$\forall t \in [a, b], \qquad \beta(t) \geq \alpha + \theta\delta.$$

But thanks to Remark 4 we also know

$$\forall t \in [a, b], \qquad \dot{\beta}(t) \leq \bar{c}_{\epsilon,m},$$

therefore

$$\alpha + \delta - \alpha - \theta\delta \leq \beta(b) - \beta(a) \leq \bar{c}_{\epsilon,m}(b - a).$$

Therefore

$$b - a \geq \frac{(1-\theta)\delta}{\bar{c}_{\epsilon,m}}.$$

And thanks to (44) we get

$$b - a > \frac{L}{f'(u_l(m) - \epsilon)}.$$

But then for a time t in the (non empty) interval $(a + \frac{L}{f'(u_l(m)-\epsilon)}, b)$ we have considering the minimal backward characteristic

$$u(t, \beta(t)^-) = u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(s, .))),$$

for s such that

$$\frac{\beta(t)}{t - s} = f'(u(t, \beta(t)^-),$$

and thus

$$f'(u_l(m) - \epsilon) \leq \frac{L}{t - s},$$

but then

$$s \geq t - \frac{L}{f'(u_l(m) - \epsilon)} \geq b - \frac{L}{f'(u_l(m) - \epsilon)} > a.$$

Since $\beta(s) \geq \alpha + \theta\delta$ we also have thanks to Lemma 4 and $\epsilon < \epsilon_1$

$$u(t, \beta(t)^-) \leq u_l(m) - \frac{\epsilon}{\nu}\frac{u_l(m) - \epsilon}{2} < 0.$$

Now thanks to Theorem 3 we can conclude that for almost any $t \in (a + \frac{L}{f'(u_l(m)-\epsilon)}, b)$ we have

$$\dot{\beta}(t) = \frac{f(u(t, \beta(t)^-)) - f(u_r(m))}{u(t, \beta(t)^-) - u_r(m)} < 0.$$

But then

$$\beta(a + \frac{L}{f'(u_l(m) - \epsilon)} > \beta(b) \geq \alpha + \delta,$$

and once again

$$\beta(a + \frac{L}{f'(u_l(m) - \epsilon)} - \beta(a) \leq \bar{c}_{\epsilon,m}\frac{L}{f'(u_l(m) - \epsilon)},$$

but we also have

$$\beta(a + \frac{L}{f'(u_l(m) - \epsilon)} - \beta(a) \geq \alpha + \delta - \alpha - \theta\delta = (1 - \theta)\delta,$$

so we end up with the inequality

$$(1-\theta)\delta \leq \bar{c}_{\epsilon,m}\frac{L}{f'(u_l(m)-\epsilon)},$$

which rewritten

$$\frac{\bar{c}_{\epsilon,m}}{(1-\theta)f'(u_l(m)-\epsilon)} \geq \frac{\delta}{L}$$

is incompatible with (44).

In the end we have shown that

$$\forall b \geq T_3, \qquad \beta(b) < \alpha + \delta.$$

- The same method grants

$$\forall b \geq T_3, \qquad \beta(b) > \alpha - \delta.$$

□

Lemma 6. *If we call S the function*

$$\forall t > 0, \qquad S(t) := \frac{1}{2\delta}\int_{\alpha-\delta}^{\alpha+\delta}(u(t,x) - \bar{u}_{\alpha,m}(x))dx,$$

then for ν sufficiently large (see formula (50)) one can find a $\mathcal{C}^0$ function $\tau : [T_3, +\infty) \to [\frac{\alpha-\delta}{f'(u_l(m)+\epsilon)}, \frac{\alpha-\delta}{f'(u_l(m)-\epsilon)}]$ such that for any time $t \geq T_3$

$$\dot{S}(t) = \frac{f(u_l(m) - \frac{\epsilon}{\nu}S(t-\tau(t))) - f(u_r(m))}{2\delta}.$$

Proof. • We have seen in Remark 3 that

$$\forall t \geq T_3, \qquad -\frac{u_l(m)-u_r(m)}{2} \leq S(t) \leq \frac{u_l(m)-u_r(m)}{2} + \epsilon. \tag{49}$$

So thanks to our choices of $\epsilon < \nu - \frac{u_l(m)-u_r(m)}{2}$ we have

$$\mathcal{A}_{\epsilon,\nu}(S(t)) = \frac{\epsilon}{\nu}S(t).$$

It is classical that S is Lipschitz and satisfies for almost all t

$$\dot{S}(t) = \frac{f(u(t,\alpha-\delta)) - f(u(t,\alpha+\delta))}{2\delta}.$$

Now using the previous Lemmas we have

$$\forall t \geq T_3, \qquad u(t,\alpha+\delta) = u_r(m),$$

and

$$u(t,\alpha-\delta) = u(s,0^+),$$

with

$$\frac{\alpha-\delta}{t-s} = f'(u(t,\alpha-\delta)).$$

But we also have thanks to Proposition 3

$$u(s,0^+) = u_l(m) - \mathcal{A}_{\epsilon,\nu}(\mathcal{O}_{\alpha,\delta}(u(s,.))).$$

We end up with

$$\dot{S}(t) = \frac{f(u_l(m) - \mathcal{A}_{\epsilon,\nu}(S(t-\tau(t))) - f(u_r(m))}{2\delta} = \frac{f(u_l(m) - \frac{\epsilon}{\nu}S(t-\tau(t)) - f(u_r(m))}{2\delta},$$

with

$$\tau(t) = t - s = \frac{\alpha - \delta}{f'(u(t,\alpha-\delta))}.$$

And we already see that

$$\frac{\alpha-\delta}{f'(u_l(m)+\epsilon} \leq \tau(t) \leq \frac{\alpha-\delta}{f'(u_l(m)-\epsilon)},$$

thanks to the previous Lemmas.

- All that remains is to prove the regularity of the delay τ. Since at this point it is not even clear that τ is continuous. Thanks to the finite propagation speed, a point of discontinuity in time of τ (thus of $u(t,\alpha-\delta)$) is also a point of discontinuity in space. Let us consider t such that

$$u(t,(\alpha-\delta)^-) > u(t,(\alpha-\delta)^+).$$

Considering the extremal backward characteristics using Theorem 3, Proposition 3 and Remark 3 we get two times $t - \frac{\alpha-\delta}{f'(u_l(m)-\epsilon)} \leq t_1 < t_2 < t$ such that

$$\begin{cases} u(t,(\alpha-\delta)^-) = u(t_2,0^+), & \frac{\alpha-\delta}{t-t_2} = f'(u(t_2,0^+)) \\ u(t,(\alpha-\delta)^+) = u(t_1,0^+), & \frac{\alpha-\delta}{t-t_1} = f'(u(t_1,0^+)) \end{cases}$$

We have therefore

$$(t-t_2)f'(u_l(m) - \mathcal{A}_{\epsilon,\nu}(S(t_2))) = (t-t_1)f'(u_l(m) - \mathcal{A}_{\epsilon,\nu}(S(t_1))).$$

Now thanks to (49), (16) and the choices $\epsilon < \nu - \frac{u_l(m)-u_r(m)}{2}$, we have in fact

$$(t-t_2)f'(u_l(m) - \frac{\epsilon}{\nu}S(t_2)) = (t-t_1)f'(u_l(m) - \frac{\epsilon}{\nu}S(t_1)).$$

Now we introduce the function G defined on $[t - \frac{\alpha-\delta}{f'(u_l(m)-\epsilon)}, t]$ by

$$G(r) := (t-r)f'(u_l(m) - \frac{\epsilon}{\nu}S(r)).$$

It is clearly Lipschitz and since f' is $\mathcal{C}^1$ we can use the chain rule to get almost everywhere

$$G'(r) = -f'(u_l(m) - \frac{\epsilon}{\nu}S(r)) - (t-r)\frac{\epsilon}{\nu}\dot{S}(r)f''(u_l(m) - \frac{\epsilon}{\nu}S(r)).$$

But we have

$$f'(u_l(m) - \frac{\epsilon}{\nu}S(r)) \geq f'(u_l(m) - \epsilon),$$

and

$$\left|\dot{S}(r)(t-r)f''(u_l(m)-\frac{\epsilon}{\nu}S(r))\right| \leq \frac{\max(f(u_l(m)+\epsilon)-f(u_r(m)),f(u_r(m))-f(u_l(m)-\epsilon))}{2\delta} \times \frac{\alpha-\delta}{f'(u_l(m)-\epsilon)} \max_{w\in[u_l(m)-\epsilon,u_l(m)+\epsilon]} f''(w).$$

Let us call $M_{m,\epsilon,\delta}$ the righthand side, which is independant of ν then if

$$\frac{\epsilon}{\nu} < \frac{f'(u_l(m)-\epsilon)}{M_{m,\epsilon,\delta}}, \tag{50}$$

we actually have $G'(r) < 0$ and then $G(t_1) \neq G(t_2)$ which is contradictory. Thus τ is actually continuous.

□

Lemma 7. *For ν sufficiently large (see (51)) and ϵ satisfying the previous conditions, we have constants $C, M_1 > 0$ independant of u_0 such that*

$$\forall t \geq T_3, \qquad |S(t)| \leq M_1 e^{-Ct} \sup_{s\in[0,T_3]} |S(s)|.$$

Proof. We just show that we can apply Proposition 4 proved in the Appendix for $t \in [T_3, +\infty)$.

Thanks to the previous Lemmas, we have indeed S is $\mathcal{C}^1$ and satisfying

$$\dot{S}(t) = g(S(t-\tau(t)))$$

with τ continuous and

$$g(z) = \frac{f'(u_l(m)-\frac{\epsilon}{\nu}z)-f(u_r(m))}{2\delta}.$$

Now thanks to Remark 3 we have

$$-\frac{u_l(m)-u_r(m)}{2} \leq S(t) \leq \frac{u_l(m)-u_r(m)}{2}+\epsilon.$$

The delay satisfies

$$\frac{\alpha-\delta}{f'(u_l(m)+\epsilon)} \leq \tau(t) \leq \frac{\alpha-\delta}{f'(u_l(m)-\epsilon)}.$$

Finally the function satifies $g(0) = 0$ and its derivatives is given by

$$g'(z) = \frac{\epsilon}{2\delta\nu} f''(u_l(m)-\frac{\epsilon}{\nu}z).$$

So using the uniform convexity of f we get

$$-\frac{M\epsilon}{2\delta\nu} \leq g'(z) \leq -\frac{m\epsilon}{2\delta\nu},$$

using

$$m := \min_{z\in[u_l(m)-\epsilon,u_l(m)+\epsilon]} f''(z) > 0, \qquad M := \max_{z\in[u_l(m)-\epsilon,u_l(m)+\epsilon]} f''(z).$$

We conclude by observing that condition (60) of the Appendix becomes in our case

$$\frac{3(\alpha-\delta)M\epsilon}{2\delta f'(u_l(m)-\epsilon)} < \nu. \tag{51}$$

□

Lemma 8. *With the previous choices of parameters, we have for T_4 given by*

$$T_4 = T_3 + \frac{L}{f'(u_l(m) - \epsilon)},$$

two constants M_2 and M_3 such that

$$\forall t \geq T_4, \qquad |\beta(t) - \alpha| \leq M_2 e^{-Ct} \sup_{s\in[0,T_4]} |S(s)|, \tag{52}$$

and

$$\forall t \geq T_4, \quad \forall x < \beta(t), \qquad |u(t,x) - u_l(m)| \leq M_3 e^{-Ct} \sup_{s\in[0,T_4]} |S(s)|. \tag{53}$$

Proof. We will proceed in multiple steps.

- Using the previous Lemma and the boundary conditions we have

$$\forall t \geq T_3, \qquad |u(t,0^+) - u_l(m)| \leq \min\left(\epsilon, \frac{\epsilon M_1}{\nu} e^{-Ct} \sup_{s\in[0,T_3]} |S(s)|\right).$$

  For $t \geq T_3 + \frac{L}{f'(u_l(m)-\epsilon)}$ and $x < \beta(t)$ looking at the minimal backward characteristic and using Theorem 3 and Proposition 3 we get

$$|u(t,x) - u_l(m)| \leq M_2 e^{-Ct} \sup_{s\in[0,T_3]} |S(s)|$$

  with

$$M_2 := \frac{\epsilon M_1 e^{C\frac{\alpha-\delta}{f'(u_l(m)-\epsilon)}}}{\nu}.$$

  And since $T_4 > T_3$, (52) is now obvious.

- Now consider $t \geq T_4$. Let us suppose that $\beta(t) \geq \alpha$, we have

$$\begin{aligned} S(t) &\geq \frac{\alpha + \delta - \beta(t)}{2\delta} u_r(m) - \frac{u_l(m) + u_r(m)}{2} \\ &\quad + \frac{\beta(t) - \alpha + \delta}{2\delta} \left(u_l(m) - \min\left(\epsilon, M_2 e^{-Ct} \sup_{s\in[0,T_4]} |S(s)|\right)\right) \\ &= (\beta(t) - \alpha)\frac{u_l(m) - u_r(m) - \epsilon}{2\delta} - \frac{M_2 e^{-Ct} \sup_{s\in[0,T_4]} |S(s)|}{2}. \end{aligned}$$

  And so

$$0 \leq \beta(t) - \alpha \leq M_3 e^{-Ct} \sup_{s\in[0,T_4]} |S(s)|,$$

  with

$$K_2 := \frac{\frac{M_2}{2} + M_1}{u_l(m) - u_r(m) - \epsilon}.$$

  The case of $\alpha \geq \beta(t)$ can be treated in the same way.

□

Proof. of Theorem 1.

We just need to write for $t \geq T_4$

$$\begin{aligned}\int_0^L |u(t,x) - \bar{u}_{\alpha,m}(x)|dx &= \int_0^{\min(\alpha,\beta(t))} |u(t,x) - \bar{u}_{\alpha,m}(x)|dx \\ &\quad + \int_{\min(\alpha,\beta(t))}^{\max(\alpha,\beta(t))} |u(t,x) - \bar{u}_{\alpha,m}(x)|dx \\ &\quad + \int_{\max(\alpha,\beta(t))}^{L} |u(t,x) - \bar{u}_{\alpha,m}(x)|dx \\ &\leq \min(\alpha,\beta(t))M_2 e^{-Ct} \sup_{s\in[0,T_3]} |S(s)| \\ &\quad + |\beta(t) - \alpha| 2\max(-u_r(m), u_l(m) + \epsilon) + 0 \\ &\leq (LM_2 + 2\max(-u_r(m), u_l(m) + \epsilon)M_3)e^{-Ct} \sup_{s\in[0,T_4]} |S(s)|.\end{aligned}$$

The conclusion then comes from the independance of all the constants from the initial data. And

$$\sup_{s\in[0,T_4]} |S(s)| \leq C \int_0^L |u_0(x) - \bar{u}_{\alpha,m}(x)|dx,$$

since the semigroup is continuous in L^1. □

A A result on delayed differential equations

Proposition 4. *Let us consider a function $\theta \in \mathcal{C}^1(\mathbb{R}^+)$, a constant $T > 0$ and a function g such that*

$$\forall t \geq T > 0, \qquad \dot{\theta}(t) = g(\theta(t - \tau(t))).$$

We will suppose the following

- *There exists a positive real number M such that*

$$\forall t \geq 0, \qquad -M \leq \theta(t) \leq M. \tag{54}$$

- *We have two positive real numbers τ_m and τ_M such that*

$$\forall t \geq 0, \qquad \tau_m \leq \tau(t) \leq \tau_M. \tag{55}$$

- *The function τ is continuous.*
- *We have two positive numbers c and ϵ such that*

$$\forall u \in [-M, M], \qquad -\epsilon \leq g'(u) \leq -c < 0. \tag{56}$$

- *The origin is stationnary*

$$g(0) = 0. \tag{57}$$

- *The following condition holds*

$$\epsilon(\tau_m + \tau_M) \leq 1. \tag{58}$$

Then if we define

$$\forall t \geq T, \qquad B(t) := \max_{s \in [t-3\tau_M, t]} |\theta(t)|,$$

we have the following conclusions.

- *If the following condition holds*

$$\epsilon(\tau_m + \tau_M) \leq 1, \tag{59}$$

  then M is non decreasing.

- *If the following holds*

$$\epsilon(2\tau_M + \tau_m) < 1 \tag{60}$$

  then M satisfies

$$\forall t \geq T, \qquad B(t + 3\tau_M) \leq KB(t), \tag{61}$$

  for K given by

$$K = \frac{1 + \epsilon(2\tau_M + \tau_m)c\tau_M}{1 + c\tau_M} < 1.$$

- *And from those properties we get*

$$\forall t \geq t_0 \qquad |\theta(t)| \leq e^{\frac{\ln(K)}{3\tau_M}(t-t_0)} B(t_0). \tag{62}$$

Proof. Let us begin by pointing out that using the definition of B, properties (57) and (56) of g and properties (55) of τ we have that for any time t, the function θ is $\epsilon B(t)$-Lipschitz on $[t - 2\tau_M, t + \tau_m]$.

- We will now show that B is non increasing. Consider a fixed positive time t. We have three alternatives.

  If $\forall s \in [t - \tau_M, t], \qquad \theta(s) > 0$, then we have

$$\theta(t) > 0, \qquad \text{and} \quad \forall s \in [t, t + \tau_m], \qquad \theta'(s) < 0,$$

  but then θ is decreasing on $[t, t + \tau_m]$ so for $s \in [t, t + \tau_m]$ either

$$0 \leq \theta(s) \leq \theta(t) \leq B(t),$$

  or $\theta(s) < 0$ in which case we have $s_0 \in [t, s]$ such that $\theta(s_0)$ but then

$$|\theta(s)| = |\theta(s) - \theta(s_0)| \leq \epsilon B(t)|s - s_0| \leq \epsilon\tau_m B(t) \leq B(t).$$

  In both case we got (thanks to (59))

$$\forall s \in [t, t + \tau_m], \qquad |\theta(s)| \leq B(t),$$

  and therefore

$$\forall s \in [t, t + \tau_m], \qquad B(s) \leq B(t).$$

  If $\forall s \in [t - \tau_M, t], \qquad \theta(s) > 0$, the symmetrical argument show that

$$\forall s \in [t, t + \tau_m], \qquad B(s) \leq B(t).$$

Finally if we have $s_0 \in [t-\tau_M, t]$ such that $\theta(s_0) = 0$ then we have

$$|\theta(s)| = |\theta(s) - \theta(s_0)| \leq \epsilon B(t)|s - s_0| \leq \epsilon(\tau_M + \tau_m)B(t) \leq B(t).$$

and therefore using (59)

$$\forall s \in [t, t+\tau_m], \qquad B(s) \leq B(t).$$

- We will now prove (61). We consider a positive time t which will be fixed. Let us consider α a positive number. We will consider once again three alternatives.
  - We suppose here that

    $$\forall s \in [t-2\tau_M, t], \qquad \theta(s) \geq \alpha.$$

    Using (56) we thus have

    $$\forall s \in [t-\tau_M, t+\tau_m], \qquad \dot{\theta}(s) \leq -c\alpha,$$

    but then we can deduce using $\theta(t-\tau_M) \leq B(t)$ that

    $$\forall s \in [t, t+\tau_m], \qquad \theta(s) \leq B(t) - c\alpha\tau_M.$$

    We now use the Lipschitz constant of θ to get

    $$\forall s \in [t, t+\tau_m], \qquad \theta(s) \geq \theta(t) - \epsilon B(t)(s-t) \geq \alpha - \epsilon B(t)\tau_m,$$

    Combining the previous estimates we get

    $$\forall s \in [t, t+\tau_m], \qquad |\theta(s)| \leq \max(\epsilon B(t)\tau_m - \alpha, B(t) - c\alpha\tau_M).$$

  - We suppose here that

    $$\forall s \in [t-2\tau_M, t], \qquad \theta(s) \leq -\alpha.$$

    Using (56) we thus have

    $$\forall s \in [t-\tau_M, t+\tau_m], \qquad \dot{\theta}(s) \geq c\alpha,$$

    but then we can deduce using $\theta(t-\tau_M) \geq -B(t)$ that

    $$\forall s \in [t, t+\tau_m], \qquad \theta(s) \geq -B(t) + c\alpha\tau_M.$$

    We use the Lipschitz constant for θ to get

    $$\forall s \in [t, t+\tau_m], \qquad \theta(s) \leq \theta(t) + \epsilon B(t)(s-t) \leq -\alpha + \epsilon B(t)\tau_m,$$

    Combining the previous estimates we get

    $$\forall s \in [t, t+\tau_m], \qquad |\theta(s)| \leq \max(\epsilon B(t)\tau_m - \alpha, B(t) - c\alpha\tau_M).$$

– The last case is now obviously

$$\exists s_0 \in [t - 2\tau_M, t], \qquad -\alpha \leq \theta(s_0) \leq \alpha.$$

But then we have using the Lipschitz constant of θ

$$\forall s \in [t, t + \tau_m], \qquad |\theta(s)| \leq |\theta(s_0)| + \epsilon B(t)|s - s_0| \leq \alpha + \epsilon B(t)(2\tau_M + \tau_m).$$

We can sum up the previous estimates by

$$\forall s \in [t, t + \tau_m], \qquad |\theta(s)| \leq \max(\alpha + \epsilon B(t)(2\tau_M + \tau_m), \epsilon B(t)\tau_m - \alpha, B(t) - c\alpha\tau_M).$$

But it is clear that

$$\forall \alpha \geq 0, \qquad \epsilon B(t)\tau_m - \alpha \leq \alpha + \epsilon B(t)(2\tau_M + \tau_m),$$

thus we have in fact

$$\forall s \in [t, t + \tau_m], \qquad |\theta(s)| \leq \max(\alpha + \epsilon B(t)(2\tau_M + \tau_m), B(t) - c\alpha\tau_M).$$

And we can now minimize the righthandside with respect to α. Since the functions are affine (one increasing the other decreasing) the corresponding α satisfies

$$\alpha + \epsilon B(t)(2\tau_M + \tau_m) = B(t) - c\alpha\tau_M,$$

which is

$$\alpha = \frac{1 - \epsilon(2\tau_M + \tau_m)}{1 + c\tau_m}.$$

and so we end up with

$$\forall s \in [t, t + \tau_m], \qquad |\theta(s)| \leq \left(\frac{1 + c\tau_M\epsilon(2\tau_M + \tau_m)}{1 + c\tau_M}\right) B(t).$$

Finally by bootstrapping the result and using the fact that B is non increasing we have

$$\forall s \in [t, t + 3\tau_M], \qquad |\theta(s)| \leq \left(\frac{1 + c\tau_M\epsilon(2\tau_M + \tau_m)}{1 + c\tau_M}\right) B(t),$$

which is as announced

$$\forall t \geq T, \qquad B(t + 3\tau_M) \leq KB(t).$$

- To get (62) we consider $t > t_0$ and denote N the integer satisfying

$$t - 3(N + 1)\tau_M \leq t_0 \leq t - 3N\tau_M \Leftrightarrow N \leq \frac{t - t_0}{3\tau_M} \leq N + 1.$$

We then have

$$\begin{aligned} B(t) &\leq KB(t - 3\tau_M) \\ &\leq K^2 B(t - 2(3\tau_M)) \\ &\leq K^N B(t - N(3\tau_M)) \\ &\leq e^{\ln(K)N} B(t_0) \\ &\leq e^{\frac{\ln(K)}{3\tau_M}(t - t_0)} B(t_0). \end{aligned}$$

□

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