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Attractors for two dimensional waves with homogeneous Hamiltonians of degree 0

Yves Colin de Verdière ^{*} &
Laure Saint-Raymond [†]

January 16, 2018

Abstract

The density stratification in an incompressible fluid is responsible for the propagation of internal waves. In domains with topography, these waves exhibit interesting features. In particular, numerical and lab experiments show that, in two dimensions, for generic forcing frequencies, these waves concentrate on attractors. The goal of this paper is to analyze mathematically this behavior, using tools from spectral theory and microlocal analysis.

The same results apply also to inertial waves in rotating fluids.

1 Motivation

The mathematical problem which will be discussed in this paper is motivated by the physical observation that, in presence of topography, forcing internal waves leads to the formation of singular geometric patterns, which accumulate most of the energy.

Our goal is to provide a mathematical explanation for this phenomenon, beyond the ray approximation which fails at large wavelengths (in particular at the forced wavelengths).

1.1 2D internal waves in a domain with topography

The physical system we would like to describe is an inviscid incompressible fluid, stratified in density with profile $\bar{\rho}$ and submitted to a stationary forcing. Assuming that the forcing is weak and that the stratification $\bar{\rho}$ is stable so that

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the departure from equilibrium remains small, the equations can be linearized and state:

$$\begin{cases} \bar{\rho}\partial_t\delta u = -\delta\rho g e_3 - \nabla P + F e^{-i\omega_0 t}, \\ \operatorname{div}(\delta u) = 0, \\ \partial_t\delta\rho + \delta u \cdot \nabla\bar{\rho} = 0 \end{cases}$$

denoting $\delta\rho$ the fluctuation of density, and δu the fluctuation of velocity. The last term is a periodic forcing term which could be, in experiments, a boundary condition. Note that the pressure P is well-defined by the incompressibility condition if we add a zero-flux condition on the boundary $\delta u \cdot n_{|\partial\Omega} = 0$.

The first important feature of this system is the fact that

(P0) the propagator has a symbol which is homogeneous of degree 0.

Replacing formally the derivative ∂_{x_j} by ip_j in the previous system, we can diagonalize the matrix and find indeed the two eigenvalues iH_{\pm} with

$$H_{\pm}(x, p) = \pm N(x_3) \frac{|p_3|}{|p|}$$

where $N = (-g\partial_{x_3}(\log\bar{\rho}))^{1/2}$ denotes the Brunt-Vaisala frequency.

Of course, if the propagation holds in a rectangular box (with horizontal and vertical boundaries) and that N is constant, one has a Fourier basis for

$$\{u \in L^2(\Omega) / \operatorname{div} u = 0, \quad n \cdot u_{|\partial\Omega} = 0\}$$

and the propagator is just a Fourier multiplicator, with explicit eigenvalues. This is however a non generic situation, which happens only when the domain has symmetries. In general, the zero-flux condition is incompatible with any decomposition on special functions. In the lab experiment [3] for instance, the symmetry breaking is obtained by introducing a sloping boundary.

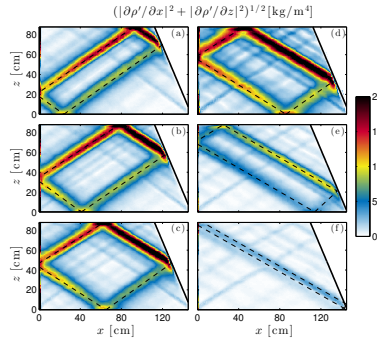


Figure 1: Experimental observation of attractors.

From the PhD thesis of C. Brouzet [3], under the supervision of T. Dauxois (ENS Lyon, 2016)

In this 2D trapezoidal geometry, it has been observed (see [12, 9, 3]) that the displacement of particles due to the internal waves in the fluid is strongly inhomogeneous, and that the energy concentrates on attractors. Furthermore, some branches are more energetic than others, which seems to indicate that there is a focusing mechanism due to the reflection on the slope. The geometry of these attractors, especially the number of branches, depends on the forcing frequency and on the slope.

With numerical simulations, the long time behavior can be investigated systematically. Three scenarios appear depending on the slope and on the forcing frequency, which are the convergence to a limit cycle, the concentration in a corner, or some mixing. Note that, both at the experimental and numerical levels, there is no difference between a complicated attractor and an ergodic behavior.

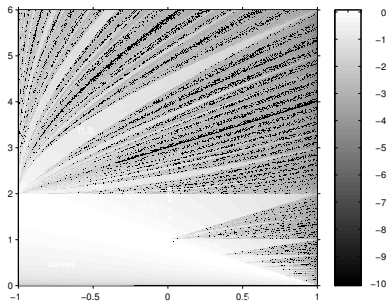


Figure 2: Greyscale : Lyapunov exponent of the trajectories.

White regions : attractors. Black regions : no pattern emerges from the ray tracing.

From L. Maas, D. Benielli, J. Sommeria, F. Lam [12]

1.2 Consistency with semi-classical predictions

These observations are consistent with the predictions of the semiclassical approximation, i.e. with the dynamics governing the propagation of wave packets when the wavelength is much smaller than the size of the domain (and the typical length of inhomogeneities if any). For a linear stratification $N \equiv 1$, the Hamiltonian equations state

$$\begin{cases} \frac{dx_1}{dt} = -\frac{p_3^2}{|p|^3}, & \frac{dp_1}{dt} = 0, \\ \frac{dx_3}{dt} = \frac{p_3 p_1}{|p|^3}, & \frac{dp_3}{dt} = 0. \end{cases}$$

Note that the group velocity is orthogonal to the wavenumber p , and that its modulus is conversely proportional to $|p|$, which are typical features from dynamics with homogeneous Hamiltonian of degree 0.

This dynamical equations are supplemented with reflection conditions at the boundary

$$\begin{cases} \frac{(p'_1)^2}{|(p')^2|} = \frac{p_1^2}{|p|^2} = \cos^2 \phi \text{ coming from the conservation of energy} \\ (p'_1 - p_1) \cos \alpha - (p'_3 - p_3) \sin \alpha = 0 \text{ for the phase to be constant on the slope} \end{cases}$$

where α denotes the angle between the horizontal and the slope

$$x_1 \sin \alpha + x_3 \cos \alpha = C,$$

and ϕ is the angle between the direction of propagation and the vertical (note that the direction of propagation is orthogonal to p). The combination of these two conditions gives a polynomial equation (intersection of a degenerate hyperbola and a line), which has two roots : one wave is incoming and the other is outgoing. Note that the energy does not fix the modulus of the wavenumber. This mode of propagation is therefore very different from the well-known dynamics of acoustic or electromagnetic waves.

Forgetting the group velocity for the moment, we can use the following reduction to represent the trajectories. Specular reflection on the horizontal and vertical boundaries is equivalent to free propagation in a domain which is extended by symmetry:

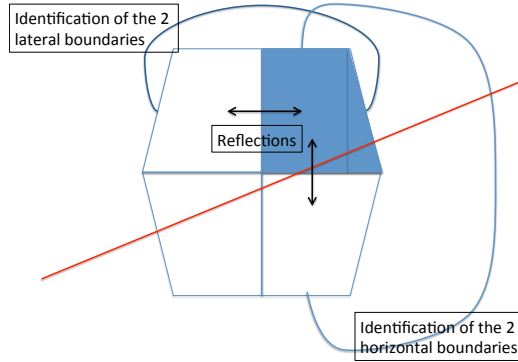


Figure 3: Rays have a constant direction in the extended domain.

When a trajectory exits the domain on the left (resp. on the right), it re-enters on the right (resp. on the left) at the symmetric point. One can then identify the points of the boundary, and get a (non smooth) dynamics on a torus, along a fixed vector field. We have actually four copies of this dynamics, corresponding to the four possible directions given by $p_1^2/|p|^2 = \cos^2 \phi$.

In other words, this construction shows that

(P1) each energy shell can be represented as a foliation of an oriented torus.

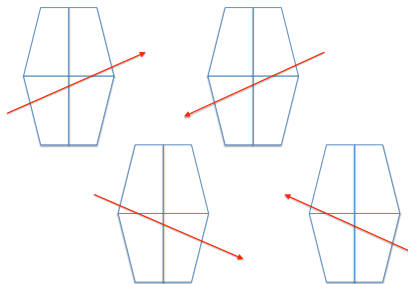


Figure 4: Foliation corresponding to the energy surface $\cos \phi$

The long time behaviour of the dynamics can be characterized by exhibiting a Poincaré section, and looking at the Poincaré return map. If $\pi/2 - \phi > \alpha$, a Poincaré section is $x_3 = 0$ and it is easy to see that all trajectories will focus on a corner, which is a sink point of the foliation. If $\pi/2 - \phi < \alpha$, a Poincaré section is the lateral boundary $x_1 = x_1^+(x_3)$. In the sequel, we will concentrate on this second case.

For all values of α , $\phi > \pi_2 - \alpha$, the Poincaré return map $f_{\alpha, \phi}$ is an orientation preserving homeomorphism of the circle \mathbb{S}^1 . Such a map f admits a crucial dynamical invariant $\rho(f) \in \mathbb{R}/\mathbb{Z}$, referred to as the rotation number, which has been introduced by Poincaré. It can be defined for instance as follows : for any $c, c' \in \mathbb{S}^1$

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{j \in [0, n-1] / f^j(c') \in [c, f(c)]\}|.$$

(we refer to the first chapter of [5] for a brief presentation of the combinatorial theory of Poincaré.) The main properties of the rotation number $\rho(f)$ are:

- when the rotation number $\rho(f)$ is rational, f has periodic points (having all the same period) and any orbit is asymptotic to a periodic orbit.
- when the rotation number $\rho(f)$ is irrational, f is semi-conjugated to a rotation g of angle $\rho(f)$, i.e. there exists a monotone map h such that $h \circ f = g \circ h$. Note that, in general h is not a bijection: the inverse image of some point may be an interval. In [6], Denjoy proved that the map h is an homeomorphism as soon as f is smooth enough.

For generic families of C^2 circle diffeomorphisms depending on one parameter, it has been proved by Arnold and Hermann (see Chapter 4 of [5]) that the rotation number is locally constant (as a function of this parameter), precisely when it is rational: this phenomenon is known as *frequency locking*. The set of parameters for which the rotation number is irrational is a Cantor set possibly with non zero Lebesgue measure (devil's staircase). This provides bifurcation diagrams with Arnold's tongues very similar to Fig 2. Unfortunately this theory does not apply here as the Poincaré return map is only piecewise affine. However it is clear that fixed points of the (iterated) return map are stable under

small perturbations of α or ϕ . We therefore preserve the *band structure of the bifurcation diagram* for rational rotation numbers.

Remark 1.1. *Note that, in the smooth case, the degeneracy at the ends of the bands is due to the fact that the derivative of the Poincaré return map tends to 1, while in the affine case, we expect fixed points either to disappear or to bifurcate on three fixed points, because of a discontinuity of the derivative.*

We will focus here on angles ϕ_0 which lie in the interior of a band, i.e. such that

(P2) there exists an open set I containing ϕ_0 on which $\rho_{\alpha,\phi} \in \mathbb{Q}$ is constant and the periodic points are hyperbolic.

All trajectories of the semiclassical dynamics with energy $\omega_0 = |\cos \phi_0|$ then converge to some attractor, corresponding to some fixed point of the Poincaré return map f_{α,ϕ_0} . Note that this convergence is relatively slow. We indeed have that the return time, i.e. the time t_n needed to run through the approximate cycle (from $x_3^{(n)}$ to $x_3^{(n+1)}$) is exponentially increasing with n

$$c \left(\frac{\cos(\phi - \alpha)}{\cos(\phi + \alpha)} \right)^n \leq |t_n| \leq C \left(\frac{\cos(\phi - \alpha)}{\cos(\phi + \alpha)} \right)^n .$$

The distance to the limit cycle is exponentially decreasing with n

$$|x_3^{(n)} - \bar{x}_3| \leq C \left(\frac{\cos(\phi + \alpha)}{\cos(\phi - \alpha)} \right)^n .$$

In particular, the distance to the limit cycle is of the order of $O(1/t)$.

Numerical computations show that the limit cycle obtained from the ray tracing coincides with the attractor which is observed in the lab experiments [3, 2]. However the semiclassical approximation does not seem a priori to apply in this situation since there is no scale separation between the wavelengths of the forcing and the size of the domain. What we expect is that this scale separation is generated by the dynamics itself since $|p(t)| \rightarrow \infty$ linearly as $t \rightarrow \infty$ along all trajectories. The idea is then to quantify this escape, and to deduce scattering and spectral properties for the wave operator.

Remark 1.2. *In the specific case of a 2D fluid with linear stratification (N constant), characteristics play also a role when solving the equation for generalized eigenfunctions*

$$(H - \omega_0)u = 0$$

so they are somehow related to the spectral structure of H (see [13]). But it is still not clear how they are involved in the evolution problem

$$\frac{1}{i} \partial_t u + Hu = f e^{-i\omega_0 t} .$$

In particular, at this stage, there is no difference between stable and unstable cycles.

Remark 1.3. *Inertial waves, which propagate under the combined effects of rotation (Coriolis force) and incompressibility, exhibit exactly the same behavior. Their dispersion relation is indeed*

$$\omega = \pm f \frac{p_3}{|p|}.$$

Only the polarization conditions for the reflected waves are a little bit more complicated to obtain.

2 Main results

2.1 Mathematical setting

The mathematical model we consider in this paper reproduces the important features of the previous physical system, but with more regularity in order that techniques of pseudo-differential calculus can be used.

More precisely, we consider a general scalar equation of the form

$$\frac{1}{i} \partial_t u + H u = f e^{-i\omega_0 t} \tag{2.1}$$

on a 2D torus X equipped with a smooth density dq , where $f e^{-i\omega_0 t}$ is a periodic smooth forcing, and H is a bounded self-adjoint operator on $L^2(X, dq)$ satisfying the following assumptions.

(M0) H is a pseudo-differential operator with a smooth principal symbol $h : T^*X \setminus \{0\} \rightarrow \mathbb{R}$ homogeneous of degree 0 and a vanishing subprincipal symbol.

For a nice introduction to pseudo-differential operators and Fourier Integral Operators, we refer to [7]. Let us recall for the sake of completeness that the principal symbol h and sub-principal symbol h_{-1} of a pseudo-differential operator H of order 0 are defined in terms of the action of H on fast oscillating functions:

$$\langle H(ae^{i\tau S}) | ae^{i\tau S} \rangle_{L^2(X, dq)} = \int_X |a(q)|^2 (h(q, S'(q)) + \tau^{-1} h_{-1}(q, S'(q))) dq + O(\tau^{-2})$$

The function h (resp. h_{-1}) is a smooth homogeneous function of degree 0 (resp. -1) on $T^*X \setminus 0$ which is well defined independently of the choice of local coordinates. Note however that the subprincipal symbol h_{-1} depends on the measure dq .

The assumption that h is homogeneous implies that any energy shell $\Sigma_\omega = h^{-1}(\omega)$ is conic. We further assume that the energy shell Σ_{ω_0} is non degenerate, i.e. that $dh \neq 0$ on Σ_{ω_0} . This is a generic assumption on the frequency ω_0 thanks to Sard Theorem.

We now introduce some additional geometrical and dynamical assumptions in order to avoid singularities. Part of these will be removed in [4].

Denote by Z the oriented manifold of dimension 2 which is the quotient of the conic energy shell Σ_{ω_0} by the positive homotheties $(q, p) \mapsto (q, \lambda p)$. We can think of Z as the boundary at infinity of the energy shell. The behaviour of the Hamiltonian dynamics as $t \rightarrow \infty$ is driven by some geometrical objects defined on Z . We will assume for simplicity that:

(M1) The energy shell Σ_{ω_0} is nondegenerate and the canonical projection $\pi : Z \rightarrow X$ is a finite covering of degree n .

It means that, for each $q \in X$, $\#\pi^{-1}(q) = n$ and the points in $\pi^{-1}(q)$ are smoothly dependent on q . This is the natural extension of the condition (P1) where $n = 4$.

Moreover Z is equipped with a 1D-foliation, denoted by \mathcal{F} , defined as follows: the Hamiltonian vector field X_h is tangent to Σ_{ω_0} and homogeneous of degree -1 , this implies that the oriented direction of X_h induces a field of oriented directions on Z .

We claim that this foliation is non singular. Assume that it is not the case. The foliation is singular at the points where X_h is parallel to the cone direction. Denoting $q = (x, y)$ and $p = (\xi, \eta)$, the projection of the foliation on X is generated by the vectors $(\partial_\xi h, \partial_\eta h)$. Therefore the foliation \mathcal{F} is singular if and only if this vector vanishes. But then the tangent space to Σ_{ω_0} , which is defined by the non trivial equation $\partial_x h dx + \partial_y h dy = 0$, does not project in a surjective way on the tangent space of X . This contradicts our assumption.

We also retrieve the property that the wave number p is orthogonal (for the duality) to the direction of the projection of X_h on X . This is due to the Euler relation $\xi \partial_\xi h + \eta \partial_\eta h = 0$. Another way to interpret this last condition is to say that the cones generated by the leaves of \mathcal{F} are Lagrangian. Note that the foliation is on Z . The projection on X is a multiple foliation, meaning that at each point of X there is n distinct oriented directions. We could assume for simplicity that the projections of each connected component of Z onto X are diffeomorphisms. If it is not the case, it will be enough to consider a finite covering of X .

The last assumption is on the dynamics of \mathcal{F} :

(M2) The foliation \mathcal{F} is Morse-Smale,

which is a generic condition, to be compared to (P2). Let us recall what it means:

- there is a finite number of compact leaves (diffeomorphic to circles), also called cycles in the sequel.
- These compact leaves are hyperbolic (the corresponding linear Poincaré maps are expanding or contracting).

- And all other leaves are accumulating only along two of the previous closed leaves at $\pm\infty$.

Note that conditions (M1) and (M2) are stable under small perturbations, therefore are still verified for energy levels close to ω_0 .

Remark 2.1. *The regularity assumption is encoded in (M0). In particular, at this stage, even though we can capture the effect of the zero flux condition for internal waves in a model without boundary (see Fig. 3), this model is not smooth enough to enter in this class of operators.*

If the boundary is a polygon, we have seen that the Poincaré section is only piecewise affine. This difficulty could be removed by considering a smooth domain, which leads to a foliation with singular points (corresponding to critical angles). But our results can actually be extended to singular foliations (see [4]).

A more serious difficulty comes from the fact that the wavelength jumps at each return time, which means that there is no smooth normal form which conjugates the geometric object given by the foliation and the semiclassical dynamics.

To tackle the original problem, we would probably need to introduce another covering to take into account this additional complexity.

In this abstract setting, we can now formulate our result describing the long time behavior of the forced system.

Theorem 2.1. *Consider the forced equation*

$$\frac{1}{i}\partial_t u + Hu = fe^{-i\omega_0 t}, \quad u|_{t=0} = 0$$

where the pseudo-differential operator H and the forcing frequency ω_0 satisfy (M0)(M1)(M2) and f is smooth.

If ω_0 is not an eigenvalue of H , then the solution to previous equation can be decomposed in a unique way as

$$u(t)e^{i\omega_0 t} = u_\infty + b(t) + \epsilon(t)$$

where

- $u_\infty = (H - \omega_0 - i0)^{-1}f$ belongs to the Sobolev spaces $\mathcal{H}^{-1/2-0}$ and is not in L^2 except if it vanishes;
- $b(t)$ is a bounded function with values in L^2 whose time Fourier transform vanishes near 0;
- $\epsilon(t)$ tends to 0 in $\mathcal{H}^{-1/2-0}$.

The singular support of u_∞ is contained in the union of the projections of the stable closed leaves of \mathcal{F} . More precisely the wavefront set $WF(u_\infty)$ is contained in the Lagrangian cones generated by the stable cycles of \mathcal{F} .

Furthermore, the energy $\|u(t)\|_{L^2(X, dq)}^2$ grows linearly except if u_∞ vanishes.

We will give in Theorem 5.1 a much more precise description of u_∞ as a Lagrangian state (or Fourier integral distribution) associated to the previous conic Lagrangian manifolds.

Roughly speaking, the wave front set $WF(u_\infty)$ characterizes the singularities of the generalized function u_∞ , not only in space, but also with respect to its Fourier transform at each point. In more familiar terms, $WF(u_\infty)$ tells not only where the function u_∞ is singular (which is already described by its singular support), but also how or why it is singular, by being more exact about the direction in which the singularity occurs.

More precisely, if u is a Schwartz distribution on \mathbb{R}^d , $(q_0, p_0) \notin F(u)$ means that there exists a test function a with $a(q_0) \neq 0$ so that the Fourier transform of au is fast decaying in p in some conic neighborhood of p_0 , or in other words that there exists a pseudo-differential operator E elliptic at the point (q_0, p_0) so that Eu is smooth. This definition is completely intrinsic, and can be extended for a smooth manifold X .

Remark 2.2. *Note that there is no assumption regarding a scale separation. This result is about the spectral structure of H and the localisation of the wave-front set. It is of course related to the semiclassical analysis presented in the previous paragraph, but it is not a simple consequence of it.*

The analysis does not rely on explicit computations as the method of partial pressures used in [13] to obtain rays as the characteristics of some transport equations. What we use here is only the geometric structure of the foliation, which is generic for a homogeneous Hamiltonian of degree 0 in dimension 2.

2.2 Strategy of the proof

What we expect to show is that the singularities of the solution to the forced equation (2.1) are localized on the stable closed cycles of the classical dynamics. What we need to understand here is therefore the relation between the evolution $\exp(-itH)$ and the dynamics of the Hamiltonian h where h is the principal symbol of H .

In semi-classical analysis, the fundamental remark is that the symbolic calculus provides good approximations as one has the identity

$$[\varepsilon\partial_q, \varphi(q)] = O(\varepsilon).$$

One can indeed consider that the derivation operators (quantified by Fourier variables) and the multiplication operators (quantified by spatial variables) commute up to *smaller error terms*, and thus use computations in the phase space T^*X to show that the energy associated to (2.1) propagates along the characteristics of h (introducing for instance the Wigner transform).

The point of view of pseudo-differential calculus is a little bit different. The heart of the theory relies on the fact that the principal symbol h of a pseudo-differential operator H , even though it does not determine H completely, contains almost all the information, up to *smoother error terms*. More precisely,

one defines classes of pseudo-differential operators according to the order m of their symbol

$$S_m = \{\sigma = \sigma(p, q) \in C^\infty(T^*X \setminus X \times \{0\}) / |\partial_q^\alpha \partial_p^\beta \sigma| \leq C_{\alpha, \beta} (1 + |p|)^{m - |\beta|}\}.$$

Then, if H is a pseudodifferential operator of principal symbol h and order m , we can define another operator $Op^W(h)$ using the Weyl quantization

$$Op^W(h)u(q) := \frac{1}{(2\pi)^2} \int e^{i(q-q') \cdot p} h\left(\frac{q+q'}{2}, p\right) u(q') dq' dp$$

and compare both operators : we find that $H - Op^W(h)$ is of order at most $m - 1$.

Furthermore, one can prove that the symbol of the commutator $i[H_1, H_2]$ is the Lie bracket $\{h_1, h_2\}$, so that

$$\text{order}([H_1, H_2]) \leq \text{order}(H_1) + \text{order}(H_2) - 1.$$

By iterating this kind of estimates, we then see that the symbolic calculus allows to obtain expansions up to any regularizing order. Of course, in general, one cannot hope to get an explicit description of the solution to (2.1) by these microlocal techniques, but one expects at least to localize singularities.

More precisely, in our problem, we will be able to import three types of informations

- (i) the dynamics associated to h has hyperbolic cycles by assumption (M2). This ‘‘attraction’’ may be expressed with a Lyapunov functional, also called escape function (which is typically $\pm \xi_3$ in the example of the previous paragraph, depending on the leaf which is considered).

By quantization of this escape function, we get a conjugation operator D for H , which allows to extend the resolvent of H on the real axis outside the discrete spectrum. In particular, this provides interesting informations on the spectrum of H . This theory, which goes back to Mourre and which is also referred to as ‘‘limit absorption principle’’ or ‘‘commutator method’’, will be presented in Section 3.

- (ii) Some attractors of the dynamics associated to h are stable, other are unstable. A refinement of the previous argument based on escape functions, combined with a spectral representation of the solutions to (2.1), shows that the energy cannot concentrate on unstable cycles (see Section 4).
- (iii) In the neighborhood of (stable) hyperbolic cycles, the dynamics has a universal behavior (obtained by choosing a suitable system of coordinates linearising the Poincaré return map). By quantization of this change of variables, we get a normal form operator. Then, we can prove that, modulo this transformation, generalized eigenfunctions of H (i.e. solutions to $Hv = \omega v$) solve locally a universal ordinary differential equation. This determines in particular the singular behavior of the solution $u(t)$ as $t \rightarrow +\infty$ close to the attractors. This precise description of the limit distribution including its wavefront set is given in Section 5.

3 Escape function and Mourre theory

The starting point of our analysis is to get the spectral structure of H close to a non degenerate ω_0 (in the sense of assumptions (M1)(M2)), using the conjugate operator method.

3.1 A short review on Mourre theory

Let H be a self-adjoint operator on some Hilbert space, say L^2 . Here we further assume that H is bounded, as it is the case in the application we have in mind.

Definition 3.1. *Let D be a self-adjoint (unbounded) operator.*

- We say that H is n -smooth with respect to D if the iterated brackets $B_1 := i[H, D]$ and $B_k := [B_{k-1}, D]$ are bounded up to $k = n$.
- We define also the Sobolev scale \mathcal{H}^s ($s \in \mathbb{R}$) associated to D by

$$\mathcal{H}^s = \{u \in L^2 / (1 + D^2)^{s/2} u \in L^2\}.$$

The main result of the conjugate operator theory (see [14, 11]) is the following theorem :

Theorem 3.1 (Mourre). *Let us assume that H is n -smooth with respect to D , and that we have the following commutator estimate : for $\chi, \psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^+)$ with $\chi\psi = \psi$,*

$$\chi(H)B_1\chi(H) \geq \alpha\psi(H) + K \quad \text{for some compact operator } K. \quad (3.1)$$

Then, for any closed interval $I \subset \text{supp}(\psi)$

- (i) H has a finite set $\sigma_p(H)$ of eigenvalues in I ;
- (ii) the resolvent $(H - z)^{-1}$ defined for $\Im(z) \neq 0$ admits boundary values at the points $\omega \in I \setminus \sigma_p(H)$ in the space $O_s := L(\mathcal{H}^s, \mathcal{H}^{-s})$ for $s > 1/2$.
- (iii) these boundary values $(H - \omega \pm i0)^{-1}$ admits continuous derivatives of order n in the spaces O_s with $s > n - 1/2$.

Proof. For the sake of completeness, we give here a sketch of proof of a slightly simpler result, which turns out to be quite simple in our case since H and B_1 are bounded operators in L^2 .

- The first step is to prove that the discrete spectrum in the interval I is finite. If it is not, there exists a sequence of orthonormal eigenfunctions ϕ_n with $H\phi_n = \omega_n\phi_n$. By the commutator estimate (3.1), we then get

$$0 = \langle \phi_n, B_1\phi_n \rangle \geq \alpha\|\phi_n\|^2 + \langle \phi_n, K\phi_n \rangle.$$

Since the ϕ_n are orthogonal, $\phi_n \rightharpoonup 0$ weakly in L^2 , and since K is compact,

$$\lim_{n \rightarrow \infty} \langle \phi_n, K\phi_n \rangle = 0.$$

We obtain a contradiction.

- We then choose ω_0 which is not an eigenvalue of H . Define P_δ to be the spectral projector of H on $[\omega_0 - \delta, \omega_0 + \delta]$. As $\delta \rightarrow 0$, P_δ converges weakly to 0, and since K is compact; $KP_\delta \rightarrow 0$. One can then find δ small enough so that $\pm P_\delta KP_\delta \leq \alpha/2P_\delta^2$, from which we deduce that

$$P_\delta B_1 P_\delta \geq \frac{\alpha}{2} P_\delta^2. \quad (3.2)$$

In the sequel we will remove the subscript δ and call P this spectral projection.

We then define $BB^* = PB_1P$ and

$$G_z(\varepsilon) = (H - z - i\varepsilon BB^*)^{-1}, \quad F_z(\varepsilon) = |D + i|^{-\nu} G_z(\varepsilon) |D + i|^{-\nu},$$

for some $\nu > 1/2$. Since H is self-adjoint, these operators are defined and bounded for any z with $\Im z \geq 0$, and we obviously have

$$\|G_z(\varepsilon)\| \leq \frac{C}{\varepsilon}, \quad \|F_z(\varepsilon)\| \leq \frac{C}{\varepsilon}.$$

Using (3.2) and the fact that $\Im z \geq 0$, we also have that

$$\begin{aligned} \|PG_z(\varepsilon)|D + i|^{-\nu}\|^2 &= \||D + i|^{-\nu} \bar{G}_z(\varepsilon) P^2 G_z(\varepsilon) |D + i|^{-\nu}\| \\ &\leq \frac{2}{\alpha\varepsilon} \||D + i|^{-\nu} \bar{G}_z(\varepsilon) (\varepsilon BB^* + \Im z) G_z(\varepsilon) |D + i|^{-\nu}\| \\ &\leq \frac{1}{\alpha\varepsilon} \||D + i|^{-\nu} (\bar{G}_z(\varepsilon) - G_z(\varepsilon)) |D + i|^{-\nu}\| \\ &\leq \frac{2}{\alpha\varepsilon} \||D + i|^{-\nu} G_z(\varepsilon) |D + i|^{-\nu}\| \leq \frac{2}{\alpha\varepsilon} \|F_z(\varepsilon)\| \end{aligned}$$

Finally, using the spectral localization, we have that for any z such that $\Re z \in [\omega_0 - \delta/2, \omega_0 + \delta/2]$

$$\|(I - P)G_z(\varepsilon)|D + i|^{-\nu}\| \leq C_\delta. \quad (3.3)$$

In particular, one has

$$\|G_z(\varepsilon)|D + i|^{-\nu}\| \leq C_\delta \left(\frac{\|F_z(\varepsilon)\|}{\varepsilon} \right)^{\frac{1}{2}}$$

and by interpolation for $0 \leq \eta < \nu$

$$\|G_z(\varepsilon)|D + i|^{-\eta}\| \leq C_\delta \left(\frac{1}{\varepsilon} \right)^{1 - \frac{\eta}{\nu}} \left(\frac{\|F_z(\varepsilon)\|}{\varepsilon} \right)^{\frac{\eta}{2\nu}}. \quad (3.4)$$

- Let us now look at the derivative of $F_z(\varepsilon)$ with respect to ε .

$$\frac{dF_z(\varepsilon)}{d\varepsilon} = |D + i|^{-\nu} G_z(\varepsilon) P [D, H] P G_z(\varepsilon) |D + i|^{-\nu}.$$

A straightforward computation shows that

$$P[D, H]P = [D, H - z - i\varepsilon BB^*] + [D, i\varepsilon BB^*] + (P - I)[D, H]P + [D, H](P - I)$$

from which we get

$$\begin{aligned} \frac{dF_z(\varepsilon)}{d\varepsilon} &= |D + i|^{-\nu} (-DG_z(\varepsilon) + G_z(\varepsilon)D) |D + i|^{-\nu} \\ &\quad + \varepsilon |D + i|^{-\nu} G_z(\varepsilon) [D, iB^*B] G_z(\varepsilon) |D + i|^{-\nu} \\ &\quad + |D + i|^{-\nu} G_z(\varepsilon) (P - I) [D, H] P G_z(\varepsilon) |D + i|^{-\nu} \\ &\quad + |D + i|^{-\nu} G_z(\varepsilon) [D, H] (P - I) G_z(\varepsilon) |D + i|^{-\nu} \end{aligned}$$

From the a priori estimates (3.3)(3.4), we then deduce that the worst term is the first one

$$\left\| \frac{dF_z(\varepsilon)}{d\varepsilon} \right\| \leq C \left(\frac{1}{\varepsilon} \right)^{\frac{1}{2\nu}} \|F_z(\varepsilon)\|^{1 - \frac{1}{2\nu}}.$$

We then obtain that $\|F_z(\varepsilon)\|$ has no singularity as $\varepsilon \rightarrow 0$ since $2\nu > 1$.

This proves that $(H - z)^{-1}$ defined for $\Im(z) > 0$ admits boundary values at the points $\omega \in [\omega_0 - \delta/2, \omega_0 + \delta/2]$ in the space $O_1 := L(\mathcal{H}^1, \mathcal{H}^{-1})$.

The additional regularity is obtained by looking at higher order derivative of $F_z(\varepsilon)$. □

3.2 Escape function for the classical dynamics

As explained in paragraph 2.2, the existence of a conjugate operator for a pseudo-differential operator H of principal symbol h is related to the fact that the Hamiltonian dynamics of h admits an escape function. In what follows, we will construct D as a pseudo-differential operator of degree 1 with principal symbol d and the commutator identity (3.1) will follow from an estimate on the principal symbol of $B_1 = i[H, D]$ which is $b_1 = \{h, d\}$. The property (3.1) will indeed be obtained from Gårding's inequality (see [8] pp 129–136)

Theorem 3.2 (Gårding). *Let B be a self-adjoint pseudo-differential operator of degree 0 on a compact manifold, with principal symbol $b > 0$. Then*

$$B \geq cId + R$$

for some $c > 0$ and some operator R of order -1 .

The construction of the escape function d is therefore the heart of the proof. We want to construct a function $d : \Sigma_{\omega_0} \rightarrow \mathbb{R}$, called an escape function, homogeneous of degree 1, so that the Poisson bracket $\{h, d\}$, which is homogeneous of degree 0, is strictly positive. The function d can then be extended smoothly to energies close to ω_0 .

Proposition 3.2. *Under the assumptions (M0), (M1) and (M2), there exists a global escape function d for the Hamiltonian h :*

$$\{h, d\} > 0 \text{ on } T^*X \setminus 0.$$

We start by constructing a normal form of the foliation on each basin.

Lemma 3.3. *Let γ be a hyperbolic closed leaf of the foliation \mathcal{F} with Lyapunov exponent $e^{-2\pi\lambda} \neq 1$. Denote by B_γ the basin of attraction (or “repulsion”) of γ . There exists a diffeomorphism of B_γ on $(\mathbb{R}/2\pi\mathbb{Z})_x \times \mathbb{R}_y$ so that the foliation is given by $dy + \lambda y dx = 0$ oriented by $dx > 0$.*

Proof. Up to a change of orientation, we can assume that γ is a stable cycle.

- The first step is to construct the normal form close to the cycle. Let S be a local Poincaré section transverse to γ . By definition, the Poincaré return map P sends S on itself. By Sternberg’s linearization theorem for 1D maps [19], there is a chart $(I, 0) \subset (\mathbb{R}_y, 0)$ of S so that

$$P(y) = \mu y \text{ with } 0 < \mu = e^{-2\pi\lambda} < 1.$$

We then choose the normalisation of the vector field V tangent to the foliation \mathcal{F} so that the return time is 2π . And we denote by x the coordinate starting from 0 along S and so that $V.x = 1$. By definition, $x \in \mathbb{R}/2\pi\mathbb{Z}$. This construction provides a foliation \mathcal{F}_0 given by $dy + \lambda y dx = 0$ and oriented by $dx > 0$.

We have now two periodic foliations \mathcal{F} and \mathcal{F}_0 : they agree at $x = 0$ and $x = 2\pi$. Thus, on each section $x = \text{constant}$, there exists a unique diffeomorphism sending the first onto the second one. One checks that it is smooth. In other words, the normal \mathcal{F}_0 is a re-parametrization of \mathcal{F} , which encodes the geometry of the trajectories.

- The second step is then to extend the normal form globally in the basin. We will use here ideas from scattering theory introduced by Neslon [16].

We first choose the normalization of a generator of the foliation so that it extends smoothly the vector field V defined near γ . We denote by $U(t)$ the flow of V on B , and by $U_0(t)$ the flow of $V_0 = \partial_x - \lambda y \partial_y$ on $B_0 := (\mathbb{R}/2\pi\mathbb{Z})_x \times \mathbb{R}_y$. Both flows are complete.

Let us then consider the map $W : B \rightarrow B_0$ defined by

$$W(q) = \lim_{t \rightarrow +\infty} U_0(-t)U(t)q.$$

The limit clearly exists because W is the identity near γ and for any $q \in B$, $U(t)q \rightarrow \gamma$ as $t \rightarrow +\infty$. Both flows are therefore conjugated by W . \square

Remark 3.4. *Note that for any $q \in X$ outside the closed leaves, q belongs both to the basin of attraction of a stable cycle, and to the basin of repulsion of an unstable cycle. The foliation \mathcal{F} at q is therefore conjugated to two foliations \mathcal{F}_0 with different λ .*

As a corollary of the previous Lemma, we obtain the local expression of the Hamiltonian h :

Lemma 3.5. *Let B_γ be the basin of attraction (or “repulsion”) of the hyperbolic closed leaf γ of the foliation \mathcal{F} , and denote by (x, y, ξ, η) the coordinates associated to the normal form introduced in Lemma 3.3.*

Then there exists a conic neighborhood of the cone $\Gamma_\gamma \subset \Sigma_{\omega_0}$ generated by B_γ defined by

$$U_\gamma := \{(x, y, \xi, \eta) \mid |\xi| < c\eta\} \subset T^*X \setminus 0$$

such that the Hamiltonian h can be written locally on U_γ

$$h(x, y, \xi, \eta) - \omega_0 = \Phi^2(x, y, \xi, \eta) \left(\frac{\xi}{\eta} - \lambda y \right)$$

for some non vanishing function Φ homogeneous of degree 0.

Proof. The orthogonal of the foliation \mathcal{F}_0 is given by $\xi - \lambda y \eta = 0$. Hence $\eta \neq 0$ on the cone generated by B_γ . Without loss of generality, we choose $\eta > 0$ on the cone Γ_γ generated by γ .

Any homogeneous function vanishing on Γ_γ is the product of

$$h_0(x, y, \xi, \eta) = \frac{\xi}{\eta} - \lambda y$$

by a non vanishing multiplier, which is homogeneous of degree 0 because of the assumption (M0).

Moreover, in order to get the right stability or unstability according to the sign of λ , this multiplier has to be non negative. Defining Φ as its square root gives the expected formula. \square

Proof of Proposition 3.2. Equipped with these preliminary results, we can construct the escape function.

- The first step is to define local escape functions on the basin of attraction B_γ , by using the explicit formula of the Hamiltonian h in the coordinates associated with the normal form. For any $k > 0$, we can define a smooth function $\phi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\phi \equiv 1$ on $[-k, k]$ and satisfying $y\phi'(y) \leq 0$ on \mathbb{R} . We then set

$$d_\gamma := \lambda \eta \phi(y).$$

A straightforward computation shows that

$$\{\Phi^2 h_0, d_\gamma\} = \Phi^2 \{h_0, d_\gamma\} + h_0 \{\Phi^2, d_\gamma\} = \lambda^2 \Phi^2 (\phi - y\phi')$$

on the cone of $\Gamma_\gamma \subset \Sigma_{\omega_0}$ generated by B_γ .

Hence the function d_γ satisfies

$$\begin{aligned} \{h, d_\gamma\} &\geq 0 \text{ on } \Sigma_{\omega_0}, \\ \{h, d_\gamma\} &> 0 \text{ on } \Gamma_{\gamma, k} \end{aligned}$$

where $\Gamma_{\gamma,k} \subset \Sigma_{\omega_0}$ is the cone generated by $\{(x,y) \in B_0 \mid |y| < k\}$.

- The global escape function is obtained as a sum of local escape functions. From the assumption (M2), we know that Σ_{ω_0} is a finite union of B_γ . Each of these B_γ has a normal form, and the corresponding cone $\Gamma_\gamma \subset \Sigma_{\omega_0}$ can be covered by the unions of $\Gamma_{\gamma,k}$ for all $k \in \mathbb{N}^*$

$$\Sigma_{\omega_0} \subset \bigcup_{\gamma \text{ closed leaf}} \bigcup_{k \in \mathbb{N}^*} \Gamma_{\gamma,k}.$$

Since the quotient Z of Σ_{ω_0} by positive dilations is compact, we can extract a finite covering

$$\Sigma_{\omega_0} \subset \bigcup_{\gamma \text{ closed leaf}} \Gamma_{\gamma,k_\gamma}.$$

By adding the corresponding local escape functions, we then get a function d such that $\{h, d\} > 0$ on Σ_{ω_0} . \square

Remark 3.6. For any $z \in \Sigma_{\omega_0}$ which is not in the union of cones generated by the unstable leaves, the Hamiltonian trajectory $\phi_t(z)$ converge as $t \rightarrow +\infty$ to the infinity of the cones generated by the stable leaves, which are invariant conic Lagrangian submanifolds of $T^*X \setminus 0$.

The dynamics on these cones is spiraling: in polar coordinates (x, y) , we get

$$\dot{x} = \Phi^2 \frac{1}{\eta}, \quad \dot{y} = \Phi^2 \lambda y \eta \quad \dot{\eta} = \Phi^2 \lambda$$

or equivalently

$$y = y_0 e^{-\lambda x}, \quad \eta = \eta_0 e^{\lambda x}.$$

4 Quasi-resonances and long time behaviour

Mourre theory provides informations on the spectral decomposition of H close to a non degenerate ω_0 (in the sense of assumptions (M1)(M2)). We will now use this (local) spectral representation of H to describe the long time behaviour of the solution u to the forced equation (2.1).

In the sequel, we further assume that ω_0 is not an eigenvalue of H .

4.1 Functional representation of the solution to (2.1)

The asymptotic behaviour of $u(t)$ is described in the following Proposition.

Proposition 4.1. Under the previous assumptions on H and ω_0 , the solution to (2.1) can be decomposed as

$$u(t) = e^{-i\omega_0 t} (H - \omega_0 - i0)^{-1} f + b(t) + \epsilon(t)$$

where $b(t)$ is a bounded oscillating function of t with values in L^2 whose inverse Fourier transform is supported in $\sigma_{pp}(H) \cup \sigma_{\text{sing}}(H)$, and $\epsilon(t)$ tends to 0 in $\mathcal{H}^{-1/2-0}$.

This decomposition is obtained from a general result on some oscillating integrals :

Lemma 4.2. *Let $\mathcal{B}, \mathcal{B}_0 \subset \mathcal{B}$ be two Banach spaces. Let $\nu = \nu_{ac} + \nu_{sing}$ be a compactly supported Radon measure on \mathbb{R} with values in \mathcal{B} such that*

- *the absolutely continuous part $\nu_{ac} = m(s)ds$ with m of class C^1 near 0*
- *the singular part ν_{sing} has values in \mathcal{B}_0 and is supported outside from 0.*

Then the singular integral $I(t) = \langle (1 - e^{-its})s^{-1} | d\nu(s) \rangle$ can be decomposed in a unique way as

$$I(t) = I_\infty + b(t) + \epsilon(t)$$

where $I_\infty = \langle (s - i0)^{-1} | d\nu(s) \rangle \in \mathcal{B}$, $\epsilon(t) \rightarrow 0$ in \mathcal{B} as $t \rightarrow \infty$, and b is bounded with values in \mathcal{B}_0 and inverse Fourier transform supported on $\text{supp}(\nu_{sing})$.

Proof. The proof is actually a simple calculation on distributions. Let $a > 0$ be such that

$$[-a, a] \cap \text{supp}(\nu_{sing}) = \emptyset.$$

Since ν is compactly supported, there is no issue of convergence at infinity. We split the integral in two parts :

$$I_1(t) = \int_{|s| < a} \frac{1 - e^{-its}}{s} d\nu_{ac} + \int_{|s| \geq a} \frac{1}{s} d\nu_{ac}$$

$$I_2(t) = - \int_{|s| \geq a} \frac{e^{-its}}{s} d\nu_{ac} + \int_{|s| \geq a} \frac{1 - e^{-its}}{s} d\nu_{sing}.$$

We check that the second term in $I_2(t)$ satisfies the properties required for $b(t)$, while the first term in $I_2(t)$ tends to 0 thanks to the Riemann-Lebesgue Theorem.

We have then to study $I_1(t)$. As m is of class C^1 near 0, we have that $m(s) = m(0) + sm_1(s)$ so that

$$I_1(t) = \int_{|s| < a} \frac{1 - \cos ts}{s} (m(0) + sm_1(s)) ds + i \int_{|s| < a} \frac{\sin ts}{s} (m(0) + sm_1(s)) ds + \int_{|s| \geq a} \frac{1}{s} d\nu_{ac}$$

The second integral converges to $i\pi m(0)$. Using the parity of \cos and the Riemann-Lebesgue theorem, we obtain that the first integral tends to

$$\int_{|s| < a} \frac{m(s) - m(0)}{s} ds = \text{p.v.} \left(\int_{|s| < a} \frac{m(s)}{s} ds \right).$$

Adding the third term, we can remove the truncation.

In the limit, we get finally

$$\text{p.v.} \left(\int \frac{m(s)}{s} ds \right) + i\pi m(0) = \langle (s - i0)^{-1} | m \rangle$$

by the Sokhotski-Plemelj theorem. □

Proof of Proposition 4.1. . The solution u of (2.1) is given by

$$u(t) = \frac{e^{-it\omega_0} - e^{-itH}}{H - \omega_0} f$$

Let us then introduce the spectral decomposition of f

$$f = \int d\nu(\omega_0 + s).$$

We then have that

$$u(t) = \int \frac{e^{-it\omega_0} - e^{-its}}{s - \omega_0} d\nu(\omega_0 + s) = \int \frac{1 - e^{-its}}{s} d\nu(s).$$

Theorem 3.1 provides the existence of a closed interval I which contains 0 such that

- (i) H has no eigenvalue in I ;
- (ii) the resolvent $(H - \omega_0 - z)^{-1}$ defined for $\Im(z) \neq 0$ admits boundary values at the points $\omega \in I$, with continuous derivatives of order 1 in the spaces $O_{1/2+0}$.

Using the regularity of the resolvent on I , we get that $\nu \in C^1(I, \mathcal{H}^{-1/2-0})$ is given by

$$\nu(\lambda) = 12i\pi((H - \omega_0 - \lambda - i0)^{-1}f - (H - \omega_0 - \lambda + i0)^{-1}f).$$

In other words, ν satisfies the assumptions of the Lemma 4.2 with $\mathcal{B} = O_{-1/2-0}$ and $\mathcal{B}_0 = L^2$. The decomposition of the oscillating integral obtained in Lemma 4.2 gives therefore the expected behavior for $u(t)$.

Let us insist on the fact that $u(t)$ stays in L^2 for any finite time. Its L^2 norm will in general converge to $+\infty$ as $t \rightarrow \infty$. This asymptotics says that $u(t)e^{i\omega_0 t}$ converges in $\mathcal{H}^{-1/2-0}$ as $t \rightarrow +\infty$, modulo a bounded function with values in L^2 . \square

Remark 4.3. *Note that the transfer of energy via quasi-resonances is a little bit different from the classical resonance process. The point is that the generalized eigenfunctions corresponding to the continuous part of the spectrum have infinite energy, it is therefore impossible that they emerge in finite time even though the forcing is concentrated on one frequency.*

The quasi-resonant process is governed by the following scheme

- (i) *the system pumps energy exciting a small band of frequencies around the forcing frequency;*
- (ii) *this band shrinks gradually as time increases;*

(iii) as t tends to infinity, the energy grows linearly in time and becomes infinite: what we observe looks more and more like a generalized eigenfunction. If μ is the spectral measure of f defined by

$$\int_{\mathbb{R}} \chi(\lambda) d\mu(\lambda) := \langle \chi(H)f | f \rangle,$$

we get

$$\|u(t)\|^2 = 4 \int_J \frac{\sin^2(\omega_0 - \lambda)t/2}{(\omega_0 - \lambda)^2} d\mu(\lambda)$$

and hence the behaviour of the energy is determined by the behaviour of the spectral measure of f near ω_0 . In our case, we have $d\mu(\lambda) = |a(\lambda)|^2 d\lambda$ so that

$$\|u(t)\|^2 = 4\pi |a(\omega_0)|^2 t + o(t). \quad (4.1)$$

4.2 Energy carried by the unstable cycles

A refinement of the conjugate operator method allows actually to prove that there exists a direction of propagation for the evolution $\exp(-itH)$. We indeed have the following theorem

Theorem 4.1 (Mourre). *Let us assume that H is at least 2-smooth with respect to D , and that the commutator estimate (3.1) holds. Denote by P_D^- the spectral projectors of D on \mathbb{R}^- . Then,*

$$\sup_{\lambda \in I} \|P_D^-(H - \lambda - i0)^{-1} |D + i|^{-\nu}\| \leq C \text{ for } \nu > 1.$$

Sketch of proof. The arguments here are quite similar to those used in the proof of Theorem 3.1. We define for $\Im(z) > 0$, $\epsilon > 0$ and $\nu > 1$

$$G_z(\epsilon) = (H - z - i\epsilon BB^*)^{-1}, \quad F_z(\epsilon) = P_D^- e^{\epsilon D} G_z(\epsilon) |D + i|^{-\nu}.$$

so that

$$\|F_z(\epsilon)\| \leq \frac{C}{\epsilon}.$$

- Let us now look at the derivative of $F_z(\epsilon)$ with respect to ϵ .

$$\frac{dF_z(\epsilon)}{d\epsilon} = P_D^- e^{\epsilon D} (DG_z(\epsilon) - G_z(\epsilon)P[H, D]PG_z(\epsilon)) |D + i|^{-\nu}.$$

A straightforward computation shows that

$$\begin{aligned} \frac{dF_z(\epsilon)}{d\epsilon} &= P_D^- e^{\epsilon D} G_z(\epsilon) D |D + i|^{-\nu} \\ &\quad + \epsilon P_D^- e^{\epsilon D} G_z(\epsilon) [D, iB^*B] G_z(\epsilon) |D + i|^{-\nu} \\ &\quad + P_D^- e^{\epsilon D} G_z(\epsilon) (P - I) [D, H] P G_z(\epsilon) |D + i|^{-\nu} \\ &\quad + P_D^- e^{\epsilon D} G_z(\epsilon) [D, H] (P - I) G_z(\epsilon) |D + i|^{-\nu} \end{aligned}$$

Note that the exponential term introduces a shift, so that the derivative D acts only on the left in the first term. From (3.3)(3.4), we then get estimates for the three first terms.

- The difficulty here is to get a control on the last term, and more precisely to prove that $[D, H](P - I)G_z(\varepsilon)|D + i|^{-\nu}$ is a bounded operator from L^2 to \mathcal{H}^η for some $\eta > 0$.

We start from the following identity

$$\begin{aligned} (P - I)G_z(\varepsilon)|D + i|^{-\nu} &= (P - I)G_z(0)(1 + i\varepsilon B^* B G_z(\varepsilon))|D + i|^{-\nu} \\ &= (P - I)G_z(0)|D + i|^{-\nu} + o(\varepsilon^{1/2}) \end{aligned}$$

where the remainder is estimated by (3.4).

We then have to prove that $|D|^s [D, H](P - I)G_z(0)|D + i|^{-\nu}$ is a bounded operator. The idea is to use the fact that, for any bounded operator C and for any $\eta \in [0, 1[$

if $[C, D]$ is bounded, then $|D|^\eta C |D + i|^{-1}$ is bounded.

In our case, $C = [D, H](P - I)G_z(0)$ is indeed a bounded operator, and the commutator can be expressed in terms of $[D, H]$ and $[D, [D, H]]$ which are bounded as well.

- We finally end up as previously with an inequality of the type

$$\left\| \frac{dF_z(\varepsilon)}{d\varepsilon} \right\| \leq C \left(\frac{1}{\varepsilon} \right)^{1 - \frac{\eta}{2\nu}} \|F_z(\varepsilon)\|^{\frac{\eta}{2\nu}}.$$

We then obtain that $\|F_z(\varepsilon)\|$ has no singularity as $\varepsilon \rightarrow 0$.

□

A corollary of this theorem is that the energy cannot concentrate on the unstable cycles.

Proposition 4.4. *Denote by u the solution of the forced equation (2.1). Let γ be an unstable cycle in Σ_{ω_0} , B_γ its basin of repulsion. Denote by Q a localization operator whose microlocal support lies in a conic neighborhood of the cone Γ generated by B_γ . Then $Qu_\infty \in L^2$.*

Proof. Going back to the construction of the escape function close to γ , we see that in local coordinates

$$d_\gamma(x, y, \xi, \eta) = \lambda \eta \phi_\gamma(y)$$

with $\lambda < 0$. In particular d_γ is strictly negative near Γ .

In other words, this means that

$$Q = QP_D^-.$$

modulo smoothing operators.

We then conclude that for any $\nu > 1$

$$\begin{aligned} Qu_\infty &= QP_D^-(H - \lambda - i0)^{-1}f \\ &= QP_D^-(H - \lambda - i0)^{-1}|D + i|^{-\nu}(|D + i|^\nu f) \in L^2(X), \end{aligned}$$

which concludes the proof. \square

Remark 4.5. *By using a stronger version of Theorem 4.1 (see [15] Corollary I.3, Equation (II)), one can get a description of the evolution, and not only of the resolvent. In particular, one can prove that $u(t)$ stays bounded in L^2 near the unstable cycles as $t \rightarrow +\infty$.*

5 Precise description of u_∞

Thanks to the normal form defined in Lemma 3.3, one can actually obtain a very precise description of u_∞ near any closed leaf γ . In particular this gives almost explicit formula for the wavefront.

Theorem 5.1. *Under the assumptions (M0), (M1) and (M2), for any $f \in C^\infty(X)$, the distribution $u_\infty = (H - \omega_0 - i0)^{-1}f$ is smooth outside the projections on X of the stable closed leaves. If γ is such a stable closed leave generating a cone Γ , then u_∞ is, microlocally near Γ , a Fourier integral distribution of order 0, whose conic Lagrangian manifold is Γ and whose principal symbol is a non-homogeneous symbol of order 0 on Γ invariant by the dynamics of X_h .*

In more explicit terms, denote by (x, y, ξ, η) the coordinates associated with the normal form on the basin B_γ so that γ projects onto $y = 0$. Let $\chi \in C_c^\infty(\mathbb{R})$ be a test function which is identically equal to 1 near 0. Then the Fourier transform with respect to y of χu_∞ satisfies

$$\widehat{u_\infty}(x, \eta) \equiv \mathbf{1}_{\eta \geq 0} \sum_{j=0}^{\infty} u_j(x, \eta)$$

where u_j is a symbol of degree $-j$. Furthermore the half density $u_0 \sqrt{dx d\eta}$ is invariant by the restriction of the Hamiltonian dynamics X_h to Γ .

Once again the challenge here is to transfer informations we have on the classical Hamiltonian dynamics, especially the local representation of h near the closed leaves, to get informations on the solutions to $(H - \omega_0)v \in C^\infty(X)$. We will then need a counterpart of the normal form at the level of operators.

5.1 Pseudo-differential normal form

Let γ be a cycle on Σ_{ω_0} , and Γ the cone associated to γ . In coordinates associated with the normal form defined in Lemma 3.3,

$$\Gamma := \{(x, y, \xi, \eta) / \xi = \lambda y \eta \text{ and } \eta > 0\} \subset T^*(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}) \setminus 0$$

By Lemma 3.5, there exists a conic neighborhood U of the cone generated by the basin B_γ

$$U := \{(x, y, \xi, \eta) \mid |\xi| < c\eta\} \subset T^*X \setminus 0 \quad (5.1)$$

such that the principal symbol of H can be written locally on U

$$\begin{aligned} h(x, y, \xi, \eta) - \omega_0 &= \Phi^2(x, y, \xi, \eta)h_0(x, y, \xi, \eta) \\ \text{with } h_0(x, y, \xi, \eta) &= \frac{\xi}{\eta} - \lambda y \text{ for some } \lambda \neq 0. \end{aligned}$$

We further know that the sub-principal symbol of H vanishes.

We would therefore like to define a reference operator H_0 of principal symbol h_0 and zero subprincipal symbol. The difficulty is that h_0 is not an admissible symbol, being not smooth at $\eta = 0$. We therefore choose for H_0 any pseudo-differential operator of degree 0 whose full symbol is h_0 in the cone U and which is elliptic outside U . Note that the symbol of H_0 cannot be real valued since the sign of h_0 changes on Γ .

We then have the following pseudo-differential normal form result.

Proposition 5.1. *Let H be a self-adjoint pseudo-differential operator, with principal symbol $h = h_0\Phi^2$ in U and vanishing sub-principal symbol. There exists an elliptic pseudo-differential operator A of principal symbol $1/\Phi$ so that*

$$A^*HA - H_0 = R$$

where R is a smoothing operator when acting on functions which are microlocalized on U , i.e. its full symbol is fast decaying in U .

Proof. We proceed by induction. It is enough to consider symbols in U . In what follows \equiv_U means that the difference is smoothing in U .

Defining $A_0 = \text{Op}^W(1/\Phi)$, we get first

$$A_0^*HA_0 \equiv_U H_0 + P_2$$

for some pseudo-differential operator P_2 of order -2. This uses the fact that the sub-principal symbol of $A_0^*HA_0$ vanishes; this follows from the identity which extends the formula for the principal symbol of the commutator used in Section 3.2 :

$$\text{sub}(PQ) = \text{sub}(P)\sigma_p(P) + \sigma_p(Q)\text{sub}(P) + \frac{1}{2i}\{\sigma_p(P), \sigma_p(Q)\}$$

and the fact that $\text{sub}(A_0) = 0$ since we have used the Weyl quantization.

We then have to find, for any integer $n \geq 1$, some self-adjoint pseudo-differential operators A_n and P_{n+2} of respective orders $-n$ and $-(n+2)$ so that

$$e^{-iA_n}(H_0 + P_{n+1})e^{iA_n} \equiv_U H_0 + P_{n+2}$$

This gives the following co-homological equation in U for the principal symbol a_n of A_n

$$\{h_0, a_n\} + p_{n+1} = 0 \text{ in } U.$$

In order to solve this equation, we decompose a_n and p_{n+1} into Fourier series with respect to x . As we expect a_n to be homogeneous of degree $-n$, we set

$$a_n = \eta^{-n} \sum_{k \in \mathbb{Z}} \alpha_{n,k} \left(y, \frac{\xi}{\eta} \right) e^{ikx}, \quad p_{n+1} = \eta^{-(n+1)} \sum_{k \in \mathbb{Z}} \rho_{n,k} \left(y, \frac{\xi}{\eta} \right) e^{ikx},$$

we get, using the variable $s = \xi/\eta$:

$$s(\partial_y \alpha_{n,k} + \lambda \partial_s \alpha_{n,k}) + (\lambda n - ik) \alpha_{n,k} = \rho_{n,k}$$

This singular differential equation admits a unique smooth solution in $\mathbb{R}_{y,s}^2$ given by

$$\alpha_{n,k}(y, s) = \frac{1}{\lambda} \int_0^1 \tau^{n-1-\frac{ik}{\lambda}} \rho_{n,k} \left(y + \frac{(\tau-1)s}{\lambda}, \tau s \right) d\tau$$

Furthermore, since the partial derivatives of $\alpha_{n,k}$ are controlled by those of $\rho_{n,k}$ uniformly in k , we obtain that the solution a_n has the same regularity as p_{n+1} . \square

5.2 Solutions of $(H - \omega_0)u \in C^\infty$ near the closed leaves

The general theory tells us that the wavefront set of any solution to $(H - \omega_0)u \in C^\infty$ is contained in the characteristic manifold Σ_{ω_0} and is invariant by the dynamics of X_h . The idea is then to use the reference operator H_0 to get an explicit formula for the singularity.

Proposition 5.2. *Any distribution u so that $(H - \omega_0)u$ is smooth, is given, microlocally near each closed cycle γ in the normal form chart, by an expansion of the form*

$$u = A \left(\sum_{k \in \mathbb{Z}} v_k (y + i0)^{-1+ik/\lambda} e^{ikx} \right)$$

where A is an elliptic pseudo-differential operator of degree 0, and the growth of (v_k) is controlled by (5.2).

In particular, such a solution is smooth microlocally near a closed cycle γ as soon as it is microlocally L^2 near that cycle.

In order to establish this result, the first step is to solve (microlocally) the equation $H_0 v \in C^\infty$ for the reference operator H_0 .

Lemma 5.3. *Any solution of $H_0 v \in C^\infty$, whose wavefront set is contained in the set U defined by (5.1), admits the following expansion, up to smooth terms,*

$$v(x, y) \equiv \sum_{k \in \mathbb{Z}} v_k (y + i0)^{-1+ik/\lambda} e^{ikx}.$$

Proof. We know already that the wave front set of v is included in the characteristic manifold $\xi = \lambda y \eta$ intersected with U . Composing on the left by $D := \partial_y$, we get that v satisfies $(\partial_x - \lambda(y\partial_y + 1))v \in C^\infty$.

Using a Fourier decomposition with respect to x , we are then reduced to solve

$$(ik - \lambda(y\partial_y + 1)) v_k = -\lambda g_k$$

for some smooth g_k .

This is a simple linear ordinary differential equation the solution of which is given, for $y > 0$, by

$$v_k(y) = \int_0^1 s^{-ik/\lambda} g_k(ys) ds + l_k^+ y_+^{-1+ik/\lambda}$$

A similar expression holds for $y < 0$. The first term is smooth if g_k is.

Summing the Fourier expansions and using again the Sokhotski-Plemelj theorem, we get

$$v \equiv \sum_k v_k^+(y+i0)^{-1+ik/\lambda} e^{ikx} + \sum_k v_k^-(y-i0)^{-1+ik/\lambda} e^{ikx}$$

Now, since the Fourier transform of the second sum is supported by $\eta < 0$, it should vanish because of the wavefront set assumption. \square

We then need to characterize the admissible sequences (v_k) .

Lemma 5.4. *The sum $T := \sum v_k(y+i0)^{-1+ik/\lambda} e^{ikx}$ defines a tempered distribution if and only if*

$$v_k e^{\pi(k/\lambda)-} \text{ is of polynomial growth.} \quad (5.2)$$

Furthermore, if $T \in L_{loc}^2$ near $y = 0$, the sequence (v_k) is identically zero.

Proof. We recall that the Fourier transform extends to tempered distributions (see [10], section 2.3). In particular we have, for $\alpha \in \mathbb{R} \setminus 0$:

$$\mathcal{F}((y+i0)^{-1+i\alpha})(\eta) = -\frac{2\pi i e^{-\alpha\pi/2}}{\Gamma(1-i\alpha)} \eta_+^{-i\alpha} \equiv \gamma_\alpha \eta_+^{-i\alpha}$$

We know also that

$$|\Gamma(1-i\alpha)| = \sqrt{\frac{\pi\alpha}{\sinh \pi\alpha}}$$

so that

$$|\gamma_\alpha| = 2\pi e^{-\alpha\pi/2} \sqrt{\frac{\sinh \pi\alpha}{\pi\alpha}} \sim_{\alpha \rightarrow \infty} \sqrt{\frac{2\pi}{|\alpha|}} e^{\pi\alpha-}.$$

The condition for T to be a tempered distribution is that the series of its Fourier coefficients grows at most polynomially, which is exactly condition (5.2).

Since none of the elementary functions $(y+i0)^{-1+i\alpha}$ is in L_{loc}^2 , the only way that $T \in L_{loc}^2$ near $y = 0$ is that all Fourier coefficients vanish. \square

Equipped with this characterization of the solutions to $H_0 v \in C^\infty$, we can now deduce the structure of singularities in u_∞ using the normal form.

Proof of Proposition 5.2. Let u be a solution to $(H - \omega_0)u \in C^\infty$. We know that $WF(u) \subset \Sigma_{\omega_0}$.

Using a microlocal partition of unity on Σ_{ω_0} , we can decompose

$$u = \sum_{\gamma} u_{\gamma}$$

where u_{γ} is a solution to $(H - \omega_0)u_{\gamma} \in C^\infty$, microlocalised on the cone Γ_{γ} generated by the cycle γ . We have then $WF(u_{\gamma}) \subset \Gamma_{\gamma}$.

By Proposition 5.1, there exists an operator A elliptic on a conic neighborhood of Γ_{γ} such that the equation $Hu_{\gamma} \in C^\infty$ rewrites

$$H_0 A^{-1} u_{\gamma} \in C^\infty, \quad WF(A^{-1} u_{\gamma}) \subset \Gamma_{\gamma}$$

We get then, from Lemma 5.3 and 5.4

$$A^{-1} u \equiv \sum_k v_k (y + i0)^{-1+ik/\lambda} e^{ikx}$$

□

5.3 Proof of Theorem 5.1

Let $u_\infty = (H - \omega_0 - i0)^{-1} f$ for some $f \in C^\infty(X)$.

• By Proposition 4.4, we get that u_∞ is L^2 near the unstable cycles. Then, by Proposition 5.2, we deduce that u_∞ is smooth near the unstable cycles.

Remark 5.5. *Note that the same argument shows that any L^2 eigenfunction of H is actually smooth.*

• Since the wavefront set of u_∞ is invariant by the dynamics, and that u_∞ is smooth near the unstable cycles, we further obtain that u_∞ is smooth in the unstable basins. In view of the general form of the solutions of $(H - \omega_0)u \in C^\infty$ on the cone generated by any basin B_{γ} , this regularity implies that (v_k) is fast decaying.

• Any distribution microlocalized on U of the form $v(x, y) = \sum_k v_k (y + i0)^{-1+ik/\lambda} e^{ikx}$, with fast decaying coefficients $(v_k e^{\pi(k/\lambda)-})$ is a Fourier integral distribution associated to Γ .

This follows easily by taking the y -Fourier transform $\hat{v}(x, \eta) = \sum \gamma_{k/\lambda} v_k \eta_+^{-ik/\lambda} e^{ikx}$ which is a symbol of degree 0 in η .

Then applying the operator A keep that property.

Remark 5.6. *Note that, once we know that the wavefront set is located on the cones generated by stable orbits, this singular behavior could be also obtained by semiclassical arguments, or by boundary layer techniques.*

6 Conclusion

We have shown that the phenomenon of concentration of the energy on attractors is a very general feature of forced dynamics governed by a pseudo-differential operator with principal symbol homogeneous of degree 0, associated to a non singular foliation.

In our work [4] in preparation, we extend this work to more singular situations. We keep the assumptions that Σ_{ω_0} is nondegenerate, and that it satisfies a Morse-Smale property which is crucial to build escape functions and apply Mourre theory. But we remove the assumption that the projection π is a finite covering by using suitable canonical transformations and their quantizations as in [20, 21]. We can then admit singular foliations with singular hyperbolic points which are foci or nodes (but not saddle points for the moment). Actually, these singular points and the corresponding generic normal forms are already studied in the context of implicit differential equations (see [1]).

However, even extended to singular foliations, this general theory fails to apply to the specific situation of internal waves observed in lab experiments :

- we would indeed need to understand how to catch the effects of boundaries which create discontinuities in the frequency space;
- furthermore it would be necessary to add viscous effects, which are no more negligible when the wavelengths become large (see [18, 17]).

These questions are major challenges for the mathematical analysis.

Another interesting perspective comes from the following remark. Although it is linear, the system we have studied here exhibit the main features of wave turbulence. We have indeed proved that

- with a stationary forcing at wavelengths $O(1)$, small scales will be excited;
- the energy cascade can be read on the frequency distribution of the singular part of the solution. We got that the y - Fourier transform of the limit is not going to 0.

These properties come clearly from the quasi-resonant mechanism, or in other words to the fact that the spectrum of the operator is continuous. This could be an indication for the study of turbulence due to some weak coupling of waves.

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