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Explicit expression of the microscopic renormalized energy for a pinned Ginzburg-Landau functional

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Abstract

We get a new expression of the microscopic renormalized energy for a pinned Ginzburg-Landau type energy [with or without magnetic field] modeling small impurities. This is done by obtaining a sharp decomposition for the minimal energy of a Dirichlet type functional with an L^∞ -weight.

In particular we get an explicit expression of the microscopic renormalized energy for a circular diluted impurity. We proceed also to the minimization of this renormalized energy in some cases.

Keywords. Ginzburg-Landau type energy, pinning, renormalized energy
Mathematics Subject Classification (2000). 49K20 35J66 35J50

1 Introduction

1.1 Main results

The goal of this article is to give an explicit formula for a *microscopic renormalized energy* in the context of the study of a *pinned Ginzburg-Landau type energy*.

This renormalized energy allows to know the location of vorticity defects inside small impurities in an heterogenous superconductor. The microscopic renormalized energy may be defined *via* an auxiliary minimization problem involving unimodular maps.

The study of this auxiliary problem is the heart of this work. The main result of this article is the following theorem:

Theorem 1. *Let*

- $\omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ be a smooth bounded simply connected open set s.t. $0 \in \omega$,
- $N \in \mathbb{N}^*$ and $(\omega^N)^* := \{(z_1, \dots, z_N) \in \omega^N \mid z_i \neq z_j \text{ for } i \neq j\}$,
- $B \in (0; 1)$, $b \in [B; B^{-1}]$ and $\alpha \in L^\infty(\mathbb{R}^2, [B^2; B^{-2}])$ be s.t. $\alpha \equiv b^2$ in ω .

For $\mathbf{d} \in \mathbb{Z}^N$ and $\mathbf{z} \in (\omega^N)^*$, we write for $R > 1$ and small $\rho \in (0; 1)$, $\mathcal{D}_{R,\rho,\mathbf{z}} := B(0, R) \setminus \bigcup_i \overline{B(z_i, \rho)}$ and $\mathcal{E}_{\mathbf{d}}(\mathcal{D}_{R,\rho,\mathbf{z}}) := \{u \in H^1(\mathcal{D}_{R,\rho,\mathbf{z}}, \mathbb{S}^1) \mid \deg(u) = \mathbf{d}\}$.

Then there exist

- $f : (R_0, \infty) \rightarrow \mathbb{R}^+$ which satisfies $B^2\pi \ln(R) - C_{\omega,B} \leq f(R) \leq B^{-2}\pi \ln(R) + C_{\omega,B}$ [with $C_{\omega,B}$ is a constant depending only on ω & B and $R_0 > 1$ is sufficiently large],
- $$W^{\text{micro}} : \begin{array}{ccc} (\omega^N)^* \times \mathbb{Z}^N & \rightarrow & \mathbb{R} \\ (\mathbf{z}, \mathbf{d}) & \mapsto & W^{\text{micro}}(\mathbf{z}, \mathbf{d}) \end{array}$$

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s.t. for $\mathbf{d} \in \mathbb{Z}^N$ and $\mathbf{z} \in (\omega^N)^\star$, when $R \rightarrow \infty$ and $\rho \rightarrow 0^+$, we have

$$\inf_{u \in \mathcal{E}_d(\mathcal{D}_{R,\rho,\mathbf{z}})} \frac{1}{2} \int_{\mathcal{D}_{R,\rho,\mathbf{z}}} \alpha |\nabla u|^2 = \left(\sum_{i=1}^N d_i \right)^2 f(R) + b^2 \pi \sum_{i=1}^N d_i^2 |\ln \rho| + W^{\text{micro}}(\mathbf{z}, \mathbf{d}) + o(1). \quad (1)$$

[Note that the *degree* of a function is defined in Section 2.2].

Remark 1. 1. The map $W^{\text{micro}} : (\omega^N)^\star \times \mathbb{Z}^N \rightarrow \mathbb{R}$ depends only on α , ω and N .

2. The function $f(\cdot)$ is defined by $f(R) := \inf_{\substack{v \in H^1(B_R \setminus \bar{\omega}, \mathbb{S}^1) \\ \deg(v)=1}} \frac{1}{2} \int_{B_R \setminus \bar{\omega}} \alpha |\nabla v|^2$.

In the diluted circular case, *i.e.*, the set ω is the unit disk \mathbb{D} and when $\alpha \equiv 1$ outside ω , we may obtain an explicit expression for W^{micro} .

Proposition 2. *If ω is the unit disk \mathbb{D} and $\alpha = \begin{cases} b^2 & \text{if } x \in \omega \\ 1 & \text{if } x \notin \omega \end{cases}$, then the microscopic renormalized energy with N vortices $(\mathbf{z}, \mathbf{d}) = \{(z_1, d_1), \dots, (z_N, d_N)\}$ is*

$$W^{\text{micro}}(\mathbf{z}, \mathbf{d}) = -b^2 \pi \left[\sum_{i \neq j} d_i d_j \ln |z_i - z_j| + \frac{1-b^2}{1+b^2} \left(\sum_{j=1}^N d_j^2 \ln(1 - |z_j|^2) + \sum_{i \neq j} d_i d_j \ln |1 - z_i \bar{z}_j| \right) \right].$$

Remark 3. Section 7 is dedicated to the case of the weight considered in Proposition 2. Proposition 2 is proved Section 7.4. The minimization of the renormalized energy W^{micro} in this situation is presented in some particular cases Section 7.5.

1.2 Motivations

Theorem 1 may have several applications. For us, the main motivation appears in the study of a pinned Ginzburg-Landau type energy modeling a superconductor with impurities.

The superconductivity phenomenon is an impressive property that appears on some materials called *superconductors*. When a superconductor is cooled below a critical temperature, it carries electric currents without dissipation [no electrical resistance] and expels magnetic fields from its body [Meissner effect].

But if the conditions imposed on the material are too strong [*e.g.* a strong magnetic field] then the superconductivity properties may be destroyed: the material has a classical behavior in some areas of the material. These areas are called *vorticity defects*.

The present work gives informations for type II superconductors which are characterized by the possible coexistence of vorticity defects with areas in a superconducting state. This state is called the *mixed state*. Namely, for an increasing intensity of the magnetic field, the vorticity defects appear first with a small number and look like disks with small radii of order $\varepsilon > 0$. Usually, the mathematical studies for type II superconductors take place in the extrem type II case by considering $\varepsilon \rightarrow 0$. See [SS07] for a rigorous and quite complete presentation of these facts.

In an homogeneous superconductor, the vorticity defects arrange themselves into triangular Abrikosov lattice. In the presence of a current, vorticity defects may move, generating dissipation, and destroying zero-resistance state. A way to prevent this motion is to trap the vorticity defects in small areas called *pinning sites*. In practice, pinning sites are often impurities which are present in a non perfect sample or intentionally introduced by irradiation, doping of impurities.

In order to prevent displacements in the superconductor, the key idea is to consider very small impurities. The heart of this article is to answer to the following question: *Once the vorticity defects are trapped by small impurities, what is their locations inside the impurities [microscopic location] ?*

The mathematical theory of the superconductivity knew an increasing popularity with the pioneering work of Bethuel, Brezis and Hélein [BBH94]&[BBH93]. They studied the minimizers of

the simplified Ginzburg-Landau energy

$$E_\varepsilon : H^1(\Omega, \mathbb{C}) \rightarrow \mathbb{R}^+ \\ u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2$$

submitted to a Dirichlet boundary condition in the asymptotic $\varepsilon \rightarrow 0$. In their works, Ω is a bounded simply connected domain which is a cross section of an homogenous superconducting cylinder $\Omega \times \mathbb{R}$.

In this simplified model, a map u which minimizes E_ε [under boundary conditions] models the state of the superconductor in the mixed state. The superconducting area is the set $\{|u| \simeq 1\}$ and the vorticity defects are the connected components of $\{|u| \simeq 0\}$. Due to a fast phase transition one may see the vorticity defects as $\{|u| \leq 1/2\}$. We mention that a quantization of the vorticity defects may be done by observing the *degree* of a minimizers around their boundaries. In this context we say that z is a *vortex* of u when it is an isolated zero of u with a non zero degree. With this model we recover the basic description of the vorticity defects as small discs with radii of order of ε centered at a vortex. A Dirichlet boundary condition [with a non zero degree] mimics the application of a magnetic field by forcing the presence of vorticity defects. More realistic models including the presence of a magnetic field were intensively studied. Despite the present work applies in these magnetic models [see [Dos19]], in order to motivate our main results, for sake of simplicity of the presentation, we focus on the simple model ignoring the magnetic field.

A part of the main results of [BBH94] concerns *quantization & location* of the vorticity defects and an asymptotic estimate of the energy of a minimizer. All these results are related with the crucial notion of *renormalized energy*. The [Dirichlet] boundary condition used in [BBH94] is not physical [non gauge invariant]. But one may modify the renormalized energy [by replacing the Dirichlet boundary condition with a degree condition], after this modification the renormalized energy plays a role in a more realistic model with no boundary condition and with a magnetic field [see *e.g.* [Dos19] where this fact is highlighted].

One may modify the above model in order to consider a superconducting cylinder with impurities. This is done with the help of a *pinning term* $a : \Omega \rightarrow \mathbb{R}^+$ by considering the functional

$$E_\varepsilon^{\text{pinned}} : H^1(\Omega, \mathbb{C}) \rightarrow \mathbb{R}^+ \\ u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (a^2 - |u|^2)^2 \cdot$$

There are a lot of works which deal with a such energy. Some variants are studied in the literature with the function a which is "smooth" or piecewise constant; independent of ε or depending on ε ... See the Introduction of [Dos15] for a more complete presentation of this models.

In order to present the interpretation of the pinning term, we focus on the case of a pinning term $a : \Omega \rightarrow \mathbb{R}$ piecewise constant. Say, for some $b \in (0; 1)$ we have $a(\Omega) = \{1; b\}$ and $a^{-1}(\{b\})$ is a smooth compact subset of Ω whose connected components represent the impurities. A possible interpretation of a such pinning term is an heterogeneity in temperature. Letting T_c be the critical temperature below which superconductivity appears, if $T_1 < T_c$ is the temperature in $a^{-1}(\{1\})$, then $T_b = (1 - b^2)T_c + b^2T_1$ is the temperature in $a^{-1}(\{b\})$. Here the impurities are "heat" areas [note that $T_1 < T_2 < T_c$]. See Section 2.2 of the Introduction of [Dos10].

In order to consider "small" impurities we need to use an ε -dependent pinning term [$a_\varepsilon : \Omega \rightarrow \{b; 1\}$ with b independent of ε]. Then we may model *shrinking* impurities: the diameter of the connected components of $a^{-1}(\{b\})$ tend to 0. In [Dos13], the case of *diluted impurities* without magnetic field is considered and, in [Dos19], diluted impurities for a magnetic energy is treated.

The diluted case is the case where the impurities are small and the inter-distance between two impurities is very larger than their diameters. It is proved that the vorticity defects are trapped by the impurities [*pinning effect*]. And when vorticity defects are included in an impurity ω_0 then their location inside ω_0 [the *microscopic location*] depends only on their number, b and the form of ω_0 [see [DM11] and [Dos13] for the case without magnetic field and [Dos19] for the case with magnetic field]. It minimizes a microscopic renormalized energy W^{micro} . In particular the the renormalized energy W^{micro} is the same using the simplified model ignoring the magnetic field or

the magnetic model. And thus in both models the microscopic location of the vorticity defects is the same.

In [Dos15]-[Section 2] it is explained in detailed the link between the minimization problem considered in Theorem 1 and the microscopic location of vortices in a diluted case.

Remark 4. 1. In [DM11], the existence and the role of W^{micro} was established. But its expression was not really explicit.

2. In particular, in the case of an impurity which is a disk containing a unique vortex, it was expected that the limiting location is the center of the disc. The expression of W^{micro} obtained in [DM11] does not allow to get this result easily. This result was obtained from scratch in [Dos15]. This result is obvious with the explicit expression obtained in Proposition 2.

3. Theorem 1 has a more general scope than needed. Indeed:

i. In Theorem 1, the points z_i 's correspond to the location of the vortices inside an impurity. The weight α is close to a_ε^2 rescaled at the size of the impurity.

Essentially, in the diluted case, we have to consider $\alpha = \begin{cases} 1 & \text{outside } \omega \\ b^2 & \text{in } \omega \end{cases}$ where ω is the form of the impurity.

ii. With the help of the main result of [DM11], [Dos13] and [Dos19], in order to study W^{micro} in the context of a pinned Ginzburg-Landau type function [with or without magnetic field], we may focus on the case $d_i = 1$ for $i \in \{1, \dots, N\}$. But, since the minimization problem considered in Theorem 1 is of its self-interest we treat the case of general degrees.

4. If

- $\omega \subset Y := (-1/2; 1/2] \times (-1/2; 1/2]$ is as in Theorem 1,
- $\alpha = \begin{cases} 1 & \text{in } Y \setminus \omega \\ b^2 & \text{in } \omega \end{cases}$,
- α is 1-periodic,

then W^{micro} [given in Theorem 1] should govern the limiting location of vortices inside an impurity for the periodic non diluted case. But, there is no result which asserts that in the non diluted case the microscopic location of the vortices may be studied with this minimization problem. [Despite we believe that in the non diluted periodic case the microscopic location of vortices should be given by minimal configurations of W^{micro} with degree 1]

2 Notation and basic properties

2.1 General notation

2.1.1 Set and number

- For $z \in \mathbb{C}$, $|z|$ is the modulus of z , $\text{Re}(z) \in \mathbb{R}$ is the real part of z , $\text{Im}(z) \in \mathbb{R}$ is the imaginary part of z and \bar{z} is the conjugate of z .
- " \wedge " stands for the vectorial product in \mathbb{C} , i.e., $z_1 \wedge z_2 = \text{Im}(\bar{z}_1 z_2)$, $z_1, z_2 \in \mathbb{C}$.
- For $z \in \mathbb{C}$ and $r > 0$, $B(z, r) = \{\tilde{z} \in \mathbb{C} \mid |z - \tilde{z}| < r\}$. When $z = 0$ we simply write $B_r := B(0, r)$ and, in the particular case $z = 0$ & $r = 1$, we write $\mathbb{D} = B(0, 1)$.
- For a set $A \subset \mathbb{R}^2 \simeq \mathbb{C}$, we let \bar{A} be the closure of A and ∂A be the boundary of A ; in particular we write $\mathbb{S}^1 = \partial \mathbb{D}$ for the unit circle.

2.1.2 Asymptotic

- In this article $R > 1$ is a large number and $\rho \in (0; 1)$ is a small number. We are essentially interested in the asymptotic $R \rightarrow \infty$ and $\rho \rightarrow 0^+$.
- The notation $o_R(1)$ [resp. $o_\rho(1)$] means a quantity depending on R [resp. ρ] which tends to 0 when $R \rightarrow +\infty$ [resp. $\rho \rightarrow 0^+$]. When there is no ambiguity we write $o(1)$.
- The notation $o[f(R)]$ [resp. $o[f(\rho)]$] means a quantity $g(R)$ [resp. $g(\rho)$] s.t. $\frac{g(R)}{f(R)} \rightarrow 0$ when $R \rightarrow +\infty$ [resp. $\frac{g(\rho)}{f(\rho)} \rightarrow 0$ when $\rho \rightarrow 0$]. When there is no ambiguity we write $o(f)$.
- The notation $\mathcal{O}[f(R)]$ [resp. $\mathcal{O}[f(\rho)]$] means a quantity $g(R)$ [resp. $g(\rho)$] s.t. $\frac{g(R)}{f(R)}$ [resp. $\frac{g(\rho)}{f(\rho)}$] is bounded [independently of the variable] when R is large [resp. $\rho > 0$ is small]. When there is no ambiguity we write $\mathcal{O}(f)$.

2.2 Function and degree

The functions we consider are essentially defined on *perforated domains*:

Definition 5. We say that $\mathcal{D} \subset \mathbb{R}^2$ is a *perforated domain* when $\mathcal{D} = \Omega \setminus \cup_{i=1}^P \bar{\omega}_i$ where $P \in \mathbb{N}^*$ and $\Omega, \omega_1, \dots, \omega_P$ are smooth simply connected bounded open sets s.t. for $i \in \{1, \dots, P\}$ we have $\bar{\omega}_i \subset \Omega$ and, for $i \neq j$, $\bar{\omega}_i \cap \bar{\omega}_j = \emptyset$. If $P = 1$ we say that \mathcal{D} is an annular type domain.

In this article the test functions stand in the standard Sobolev space of order 1 with complex values modeled on L^2 , denoted by $H^1(\Omega, \mathbb{C})$, where Ω is a smooth open set.

Our main interest is based on unimodular map, *i.e.*, the test functions are \mathbb{S}^1 -valued. Thus we focus on maps lying in $H^1(\Omega, \mathbb{S}^1) := \{u \in H^1(\Omega, \mathbb{C}) \mid |u| = 1 \text{ a.e in } \Omega\}$

We let $\text{tr}_{\partial\Omega} : H^1(\Omega, \mathbb{C}) \rightarrow H^{1/2}(\partial\Omega, \mathbb{C})$ be the surjective *trace operator*. Here $H^{1/2}(\partial\Omega, \mathbb{C})$ is the *trace space*. For Γ a connected component of $\partial\Omega$ and $u \in H^1(\Omega, \mathbb{C})$, $\text{tr}_\Gamma(u)$ is the restriction of $\text{tr}_{\partial\Omega}(u)$ to Γ .

For $\Gamma \subset \mathbb{R}^2$ a Jordan curve and $g \in H^{1/2}(\Gamma, \mathbb{S}^1)$, the degree (winding number) of g is defined as $\text{deg}_\Gamma(g) := \frac{1}{2\pi} \int_\Gamma g \wedge \partial_\tau g \in \mathbb{Z}$ where:

- τ is the direct unit tangent vector of Γ ($\tau = \nu^\perp$ where ν is the outward normal unit vector of $\text{int}(\Gamma)$, the bounded open set whose boundary is Γ),
- $\partial_\tau := \tau \cdot \nabla$ is the tangential derivative on Γ . For further use we denote $\partial_\nu = \nu \cdot \nabla$ the normal derivative on Γ .

For simplicity of the presentation, when there is no ambiguity, we may omit the dependance on the Jordan curve in the notation of the degree. For example:

- if Γ is a Jordan curve and if $h \in H^{1/2}(\Gamma, \mathbb{S}^1)$, then we may write $\text{deg}(h)$ instead of $\text{deg}_\Gamma(h)$.
- If $\mathcal{D} = \Omega \setminus \bar{\omega}$ is an annular type domain and $u \in H^1(\mathcal{D}, \mathbb{S}^1)$, then $\text{deg}_{\partial\Omega}(u) = \text{deg}_{\partial\omega}(u)$. Consequently, without ambiguity, we may write $\text{deg}(u)$ instead of $\text{deg}_{\partial\Omega}(u)$ or $\text{deg}_{\partial\omega}(u)$.

If \mathcal{D} is a perforated domain and if $u \in H^1(\mathcal{D}, \mathbb{S}^1)$ then we write

$$\text{deg}(u) := (\text{deg}_{\partial\omega_1}(u), \dots, \text{deg}_{\partial\omega_P}(u)) \in \mathbb{Z}^P.$$

Note that for $\mathbf{d} \in \mathbb{Z}^P$ we have $\mathcal{E}_{\mathbf{d}}(\mathcal{D}) := \{u \in H^1(\mathcal{D}, \mathbb{S}^1) \mid \text{deg}(u) = \mathbf{d}\} \neq \emptyset$.

2.3 Data of the problem

In this article we consider:

- $\omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ be a smooth bounded simply connected open set s.t. $0 \in \omega$,
- $N \in \mathbb{N}^*$, $\mathbf{d} = (d_1, \dots, d_N) \in \mathbb{Z}^N$ and we let $d := \sum_{i=1}^N d_i \in \mathbb{Z}$,
- $\mathbf{z} \in (\omega^N)^* := \{(z_1, \dots, z_N) \in \omega^N \mid z_i \neq z_j \text{ for } i \neq j\}$,
- $B \in (0; 1)$, $b \in [B; B^{-1}]$ and $\alpha \in L^\infty(\mathbb{R}^2, [B^2; B^{-2}])$ s.t. $\alpha \equiv b^2$ in ω .

We define

$$R_0 := \max\{1; 10^2 \cdot \text{diameter}(\omega)\} \text{ and } \rho_0 := 10^{-2} \cdot \min \left\{ 1, \min_{i \neq j} |z_i - z_j|, \min_i \text{dist}(z_i, \partial\omega) \right\}.$$

For $R > R_0$ and $\rho_0 > \rho > 0$, we denote

- $\mathcal{D}_{R, \mathbf{z}} := B_R \setminus \cup_{i=1}^N \overline{B(z_i, \rho)}$,
- $\Omega_R := B_R \setminus \overline{\omega}$,
- $\Omega_{\rho, \mathbf{z}} := \omega \setminus \cup_{i=1}^N \overline{B(z_i, \rho)}$.

The main purpose of this article is the following minimization problem:

$$I(R, \rho, \mathbf{z}, \mathbf{d}) := \inf_{u \in \mathcal{E}_{\mathbf{d}}(\mathcal{D}_{R, \mathbf{z}})} \frac{1}{2} \int_{\mathcal{D}_{R, \mathbf{z}}} \alpha |\nabla u|^2. \quad (2)$$

Namely, we are interested in the asymptotic behavior of $I(R, \rho, \mathbf{z}, \mathbf{d})$ when $R \rightarrow \infty$ and $\rho \rightarrow 0$.

Without loss of generality and for simplicity of the presentation, $R > R_0$ is considered as the major parameter writing $\rho = \rho(R)$.

In order to study the minimization problem (2) we will define other similar minimization problems. In particular we handle minimization problems of the following form:

$$\inf_{u \in \mathcal{E}_{\mathbf{d}'}(\mathcal{D})} \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla u|^2 \quad (3)$$

where

- $\mathcal{D} := \Omega \setminus \cup_{i=1}^P \overline{\omega_i}$ is a perforated domain as in Definition 5,
- $\mathbf{d}' \in \mathbb{Z}^P$,
- $\alpha' \in L^\infty(\mathcal{D}; [B^2; B^{-2}])$, $B \in (0; 1)$.

We have the following classical proposition [whose proof is left to the reader]:

Proposition 6. *Minimization problem (3) admits solutions. Moreover if u is a solution of (3) then v is a solution of (3) if and only if there exists $\lambda \in \mathbb{S}^1$ s.t. $v = \lambda u$.*

Moreover a minimizer $u_{\mathbf{d}}$ solves

$$\begin{cases} -\text{div}(\alpha' \nabla u_{\mathbf{d}}) = \alpha' u_{\mathbf{d}} |\nabla u_{\mathbf{d}}|^2 & \text{in } \mathcal{D} \\ \partial_\nu u_{\mathbf{d}} = 0 & \text{on } \partial\mathcal{D} \end{cases} \quad (4)$$

And there exists $\psi_{\mathbf{d}}$ which is locally defined in \mathcal{D} and whose gradient is in $L^2(\mathcal{D}, \mathbb{R}^2)$ s.t. $u_{\mathbf{d}} = e^{i\psi_{\mathbf{d}}}$ and

$$\begin{cases} -\text{div}(\alpha' \nabla \psi_{\mathbf{d}}) = 0 & \text{in } \mathcal{D} \\ \partial_\nu \psi_{\mathbf{d}} = 0 & \text{on } \partial\mathcal{D} \end{cases} \quad (5)$$

3 First step in the proof of Theorem 1: splitting of the domain

The first step in the proof of Theorem 1 consists in a strategy which was already used in [Dos15]. It is a splitting of the integral over $\mathcal{D}_{R,\mathbf{z}}$ [in (2)] in two parts: the integral over Ω_R and the one over $\Omega_{\rho,\mathbf{z}}$.

For each integral we consider a mixed minimization problem by adding an arbitrary Dirichlet boundary condition on $\partial\omega$: $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d = \sum d_i$.

We then claim that these mixed minimization problems admit "unique" solutions.

In the next steps we will solve these problems, we will minimize among $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$ and finally we will decouple the minimal energy according to the different data.

The splitting consists in the following obvious equality:

$$I(R, \rho, \mathbf{z}, \mathbf{d}) = \inf_{\substack{h \in H^{1/2}(\partial\omega, \mathbb{S}^1) \\ \text{s.t. } \deg(h)=d}} \left\{ \inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \text{tr}_{\partial\omega}(v)=h}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2 + \inf_{\substack{w \in \mathcal{E}_{\mathbf{d}}(\Omega_{\rho,\mathbf{z}}) \\ \text{tr}_{\partial\omega}(w)=h}} \frac{b^2}{2} \int_{\Omega_{\rho,\mathbf{z}}} |\nabla w|^2 \right\}. \quad (6)$$

The three previous minimization problems admit "unique" solutions. Indeed we have the following proposition [whose proof is left to the reader].

Proposition 7. 1. Both minimization problems in (6) having a [partial] Dirichlet boundary condition $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ in (6) admit each a unique solution.

2. The minimization problem in (6) among $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$ admits solutions. Moreover if h_0 is a solution, then \tilde{h}_0 is a minimizer if and only if there exists $\lambda \in \mathbb{S}^1$ s.t. $\tilde{h}_0 = \lambda h_0$.

4 Second step in the proof of Theorem 1: the key ingredient

The key ingredient in this article is the use of *special solutions*. It is expressed in the following proposition.

Proposition 8. Let $\mathcal{D} = \Omega \setminus \cup_{i=1}^P \bar{\omega}_i$ be a perforated domain, $B \in (0, 1)$, $\alpha' \in L^\infty(\mathcal{D}; [B^2; B^{-2}])$ and $\mathbf{d}' \in \mathbb{Z}^P$. We let $u_{\mathbf{d}'}$ be a minimizer of (3). Then for $\varphi \in H^1(\mathcal{D}, \mathbb{R})$ we have

$$\frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla(u_{\mathbf{d}'} e^{i\varphi})|^2 = \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla u_{\mathbf{d}'}|^2 + \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla \varphi|^2.$$

Proof. We fix $\mathcal{D}, B, \alpha', \mathbf{d}'$ be as in the proposition. First note that, from Proposition 6, we get the existence of $u_{\mathbf{d}'}$. Moreover $u_{\mathbf{d}'}$ is a solution of (4). We may thus write $u_{\mathbf{d}'} = e^{i\psi_{\mathbf{d}'}}$ where $\psi_{\mathbf{d}'}$ is locally defined in \mathcal{D} and $\nabla \psi_{\mathbf{d}'} \in L^2(\mathcal{D}, \mathbb{R}^2)$. Thus $\psi_{\mathbf{d}'}$ solves (5). Let $\varphi \in H^1(\mathcal{D}, \mathbb{R})$. We have

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla(u_{\mathbf{d}'} e^{i\varphi})|^2 &= \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla(\psi_{\mathbf{d}'} + \varphi)|^2 \\ &= \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla \psi_{\mathbf{d}'}|^2 + \int_{\mathcal{D}} \alpha' \nabla \psi_{\mathbf{d}'} \cdot \nabla \varphi + \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla \varphi|^2. \end{aligned}$$

From (5) and an integration by parts we get $\int_{\mathcal{D}} \alpha' \nabla \psi_{\mathbf{d}'} \cdot \nabla \varphi = 0$ and this equality ends the proof of the proposition since $\frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla \psi_{\mathbf{d}'}|^2 = \frac{1}{2} \int_{\mathcal{D}} \alpha' |\nabla u_{\mathbf{d}'}|^2$. □

Remark 9. It is easy to check that Proposition 8 allows to prove in a "different" way the uniqueness, up to a constant rotation, of a minimizer of (3).

Because minimizers of (3) are not unique, in order to fix such a minimizer we add an extra condition. This choice leads to the crucial notion of *special solution*.

In both next sections we define the special solutions in $\Omega_R = B_R \setminus \bar{\omega}$ [Section 4.1] and in $\Omega_{\rho,\mathbf{z}} = \omega \setminus \cup \bar{B}(z_i, \rho)$ [Section 4.2].

4.1 The special solution in Ω_R

In this section we focus on the annular type domain Ω_R . We first treat the case $d = 1$ by considering:

$$\inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \deg(v)=1}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2. \quad (7)$$

With the help of Proposition 6, we may fix a map $v_R \in H^1(\Omega_R, \mathbb{S}^1)$ s.t. $\deg(v_R) = 1$ which is a solution of (7). We freeze the non-uniqueness of v_R by letting v_R be in the form

$$v_R = \frac{x}{|x|} e^{i\gamma_R} \text{ with } \gamma_R \in H^1(\Omega_R, \mathbb{R}) \text{ s.t. } \int_{\partial\omega} \gamma_R = 0. \quad (8)$$

It is clear that such map v_R is uniquely and well defined. Moreover, for $d \in \mathbb{Z}$, we have v_R^d which is a solution of the minimization problem:

$$\inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \deg(v)=d}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2. \quad (9)$$

It is direct to check that v_R^d is the unique solution of the minimization problem (9) of the form $\left(\frac{x}{|x|}\right)^d e^{i\tilde{\gamma}}$ with $\tilde{\gamma} \in H^1(\Omega_R, \mathbb{R})$ s.t. $\int_{\partial\omega} \tilde{\gamma} = 0$.

The special solution v_R is fundamental in the analysis since it allows to get a decoupling of the weighted Dirichlet energy. Namely, from Proposition 8 we have:

Lemma 10. *For $d \in \mathbb{Z}$ and $\varphi \in H^1(\Omega_R, \mathbb{R})$ we have:*

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla (v_R^d e^{i\varphi})|^2 = \frac{d^2}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2 + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi|^2.$$

The above lemma allows to get a crucial information on the asymptotic behavior of $(\gamma_R)_R$:

Proposition 11. *There exists $\gamma_\infty \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega, \mathbb{R})$ s.t. when $R \rightarrow \infty$ we have $\gamma_R \rightarrow \gamma_\infty$ in $H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega)$.*

Proof. Let $R' > R > R_0$ and $\varphi_R = \gamma_{R'} - \gamma_R$ in order to have $v_{R'} = v_R e^{i\varphi_R}$ in Ω_R .

From Lemma 10 we have

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla v_{R'}|^2 = \frac{1}{2} \int_{\Omega_R} \alpha |\nabla (v_R e^{i\varphi_R})|^2 = \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2 + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R|^2. \quad (10)$$

We need the following lemma:

Lemma 12. *There exists a constant $C_{B,\omega} > 0$ depending only on B and ω s.t.*

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R|^2 \leq C_{B,\omega}.$$

For the convenience of the reader the proof of this lemma is postponed in Appendix [see Appendix A].

From Lemma 12 we have $\frac{1}{2} \int_{B_{\sqrt{R}} \setminus \overline{B_{R^{1/4}}}} \alpha |\nabla \varphi_R|^2 \leq C_{B,\omega}$.

Notation 13. In the rest of this proof, C_0 stands for a constant depending only on ω and B derived from $C_{B,\omega}$ and with universal multiplicative constants. Its values may change from line to line.

Therefore, with the help of a mean value argument, we have the existence of $r \in (R^{1/4}, \sqrt{R})$ and of a constant C_0 depending only on B and ω s.t.:

$$\int_0^{2\pi} |\partial_\theta \varphi_R(re^{i\theta})|^2 d\theta \leq \frac{C_0}{\ln R}.$$

We denote $m_R := \int_0^{2\pi} \varphi_R(re^{i\theta}) d\theta$. From the above estimate and with the help of a Poincaré-Wirtinger inequality, we have

$$\int_0^{2\pi} [\varphi_R(re^{i\theta}) - m_R]^2 d\theta \leq \frac{C_0}{\ln R}.$$

We now define $\tilde{\varphi}_R \in H^1(B_R, \mathbb{R})$:

$$\tilde{\varphi}_R(se^{i\theta}) = \begin{cases} m_R & \text{for } s \in [0, r/2] \\ \frac{s-r/2}{r/2} \varphi_R(re^{i\theta}) + \frac{r-s}{r/2} m_R & \text{for } s \in (r/2, r) \\ \varphi_R(se^{i\theta}) & \text{for } s \in [r, R] \end{cases}.$$

It is easy to check that $\tilde{\varphi}_R \in H^1(B_R, \mathbb{R})$ and with direct calculations we obtain:

$$\int_{B_r} |\nabla \tilde{\varphi}_R|^2 = \int_{B_r \setminus B_{r/2}} |\nabla \tilde{\varphi}_R|^2 \leq \frac{C_0}{\ln R}. \quad (11)$$

By noting that $\text{tr}_{\partial B_R}(v_{R_e^{i\tilde{\varphi}_R}}) = \text{tr}_{\partial B_R}(v_{R_e^{i\varphi_R}}) = \text{tr}_{\partial B_R}(v_{R'})$, with the help of $\tilde{\varphi}_R$ we construct $\tilde{v}_R \in H^1(\Omega_{R'}, \mathbb{S}^1)$:

$$\tilde{v}_R = \begin{cases} v_{R'} & \text{in } B_{R'} \setminus \overline{B_R} \\ v_{R_e^{i\tilde{\varphi}_R}} & \text{in } \Omega_R \end{cases}.$$

From the minimality of $v_{R'}$ and Lemma 10 we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla v_{R'}|^2 &\leq \frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla \tilde{v}_R|^2 \\ &= \frac{1}{2} \int_{\Omega_{R'} \setminus \overline{\Omega_R}} \alpha |\nabla v_{R'}|^2 + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2 + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \tilde{\varphi}_R|^2. \end{aligned} \quad (12)$$

Estimate (12) implies: $\frac{1}{2} \int_{\Omega_R} \alpha |\nabla v_{R'}|^2 \leq \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2 + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \tilde{\varphi}_R|^2$. This inequality coupled with (10) gives $\frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R|^2 \leq \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \tilde{\varphi}_R|^2$.

On the other hand, from the definition of $\tilde{\varphi}_R$ we have $\tilde{\varphi}_R = \varphi_R$ in $B_R \setminus \overline{B_r}$. Consequently we deduce $\frac{1}{2} \int_{\Omega_r} \alpha |\nabla \varphi_R|^2 \leq \frac{1}{2} \int_{\Omega_r} \alpha |\nabla \tilde{\varphi}_R|^2$. With (11) and since $r \in (R^{1/4}, \sqrt{R})$ we may conclude

$$\frac{1}{2} \int_{\Omega_{R^{1/4}}} \alpha |\nabla \varphi_R|^2 \leq \frac{C_0}{\ln R}.$$

In particular, for a compact set $K \subset \mathbb{R}^2 \setminus \omega$ s.t. $\partial\omega \subset \partial K$ we have for sufficiently large R

$$\frac{1}{2} \int_K \alpha |\nabla \varphi_R|^2 \leq \frac{C_0}{\ln R}.$$

Since $\int_{\partial\omega} \varphi_R = 0$, we may use a Poincaré type inequality to get $\|\varphi_R\|_{H^1(K)} \rightarrow 0$ when $R \rightarrow \infty$ independently of $R' > R$.

It suffices to note that $\varphi_R = \gamma_{R'} - \gamma_R$ in order to conclude that $(\gamma_R)_R$ is a Cauchy family in $H^1(K)$. Then $(\gamma_R)_R$ is a Cauchy family in $H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega)$. The completeness of $H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega, \mathbb{R})$ allows to get the existence of $\gamma_\infty \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega, \mathbb{R})$ s.t. $\gamma_R \rightarrow \gamma_\infty$ in $H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega)$. \square

Corollary 14. *We have two direct consequences of Proposition 11 :*

1. $\text{tr}_{\partial\omega}(\gamma_R) \rightarrow \text{tr}_{\partial\omega}(\gamma_\infty)$ in $H^{1/2}(\partial\omega)$,
2. $v_R = \frac{x}{|x|}e^{v\gamma_R} \rightarrow v_\infty := \frac{x}{|x|}e^{v\gamma_\infty}$ in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \omega)$.

4.2 The special solution in $\Omega_{\rho, \mathbf{z}}$

As for the special solution in Ω_R , we first consider the minimization problem:

$$\inf_{w \in \mathcal{E}_{\mathbf{d}}(\Omega_{\rho, \mathbf{z}})} \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w|^2. \quad (13)$$

From Proposition 6, we may fix $w_{\rho, \mathbf{z}, \mathbf{d}}$, a unique solution of (13), by imposing

$$w_{\rho, \mathbf{z}, \mathbf{d}} = \prod_{i=1}^N \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} e^{v\gamma_{\rho, \mathbf{z}, \mathbf{d}}} \text{ with } \int_{\partial\omega} \gamma_{\rho, \mathbf{z}, \mathbf{d}} = 0. \quad (14)$$

For $i \in \{1, \dots, N\}$, we may locally define θ_i in $\mathbb{R}^2 \setminus \{z_i\}$ as a lifting of $\frac{x - z_i}{|x - z_i|}$, i.e., $e^{i\theta_i} = \frac{x - z_i}{|x - z_i|}$. Moreover $\nabla\theta_i$ is globally defined. We denote $\Theta := d_1\theta_1 + \dots + d_N\theta_N$ which is locally defined in $\mathbb{R}^2 \setminus \{z_1, \dots, z_N\}$ and whose gradient is globally defined in $\mathbb{R}^2 \setminus \{z_1, \dots, z_N\}$. We then may write $w_{\rho, \mathbf{z}, \mathbf{d}} = e^{i(\Theta + \gamma_{\rho, \mathbf{z}, \mathbf{d}})}$.

In contrast with the previous section, the asymptotic behavior of $w_{\rho, \mathbf{z}, \mathbf{d}}$ is well known when $\rho \rightarrow 0$. For example Lefter and Rădulescu proved the following theorem.

Theorem 2. [Theorem 1 [LR96]] *For $\rho_0 > \rho > 0$ we let w_ρ be a minimizer of (13) and we consider a sequence $\rho_n \downarrow 0$. Up to pass to a subsequence, there exists $w_0 \in C^\infty(\bar{\omega} \setminus \{z_1, \dots, z_N\}, \mathbb{S}^1)$ s.t. $w_{\rho_n} \rightarrow w_0$ dans $C^k_{\text{loc}}(\bar{\omega} \setminus \{z_1, \dots, z_N\})$ for all $k \geq 0$.*

Moreover the limits w_0 are unique up to the multiplication by a constant in \mathbb{S}^1 .

From Theorem 2, we get that the possible limits w_0 's are unique up to a constant rotation. Thus there exists a unique limit $w_{0, \mathbf{z}, \mathbf{d}}$ [given by Theorem 2] which may be written:

$$w_{0, \mathbf{z}, \mathbf{d}} = \prod_{i=1}^N \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} e^{i\gamma_{0, \mathbf{z}, \mathbf{d}}} \text{ with } \int_{\partial\omega} \gamma_{0, \mathbf{z}, \mathbf{d}} = 0. \quad (15)$$

We thus have the following corollary:

Corollary 15. *Let $\gamma_{0, \mathbf{z}, \mathbf{d}} \in H^1_{\text{loc}}(\bar{\omega} \setminus \{z_1, \dots, z_N\}, \mathbb{R})$ be defined by (15). When $\rho \rightarrow 0$ we have $\gamma_{\rho, \mathbf{z}, \mathbf{d}} \rightarrow \gamma_{0, \mathbf{z}, \mathbf{d}}$ in $H^1_{\text{loc}}(\bar{\omega} \setminus \{z_1, \dots, z_N\})$. Thus we also get $\text{tr}_{\partial\omega}(\gamma_{\rho, \mathbf{z}, \mathbf{d}}) \rightarrow \text{tr}_{\partial\omega}(\gamma_{0, \mathbf{z}, \mathbf{d}})$ in $H^{1/2}(\partial\omega)$.*

Proof. Let $K \subset \bar{\omega} \setminus \{z_1, \dots, z_N\}$ be a connected compact set s.t. $\partial\omega \subset \partial K$ and let $\rho_n \downarrow 0$ be s.t. $w_{\rho_n, \mathbf{z}, \mathbf{d}} = e^{i(\Theta + \gamma_{\rho_n, \mathbf{z}, \mathbf{d}})} \rightarrow w_0 = e^{i(\Theta + \gamma_0)}$ in $C^1(K)$ for some $\gamma_0 \in C^1(K)$. It suffices to prove that we may choose $\gamma_0 = \gamma_{0, \mathbf{z}, \mathbf{d}}$ defined by (15).

On the one hand, we have $\nabla\gamma_{\rho_n, \mathbf{z}, \mathbf{d}} = w_{\rho_n, \mathbf{z}, \mathbf{d}} \wedge \nabla w_{\rho_n, \mathbf{z}, \mathbf{d}} - \nabla\Theta \rightarrow w_0 \wedge \nabla w_0 - \nabla\Theta = \nabla\gamma_0$ in $L^2(K)$. Then $\gamma_0 = \gamma_{0, \mathbf{z}, \mathbf{d}} + \lambda$ for some $\lambda \in \mathbb{R}$.

On the other hand $(\gamma_{\rho_n, \mathbf{z}, \mathbf{d}})_n$ is bounded in $H^1(K)$, consequently, up to pass to a subsequence, we have $\gamma_{\rho_n, \mathbf{z}, \mathbf{d}} \rightharpoonup \gamma_0$ in $H^1(K)$. With the help of the previous paragraph, we get that the convergence is in fact strong. Thus $\text{tr}_{\partial\omega}(\gamma_{\rho_n, \mathbf{z}, \mathbf{d}}) \rightarrow \text{tr}_{\partial\omega}(\gamma_0)$ in $L^2(\partial\omega)$.

In conclusion

$$0 = \int_{\partial\omega} \gamma_{\rho_n, \mathbf{z}, \mathbf{d}} \rightarrow \int_{\partial\omega} \gamma_0 = \lambda + \int_{\partial\omega} \gamma_{0, \mathbf{z}, \mathbf{d}}.$$

This means $\lambda = 0$ and thus $\gamma_0 = \gamma_{0, \mathbf{z}, \mathbf{d}}$. □

About the asymptotic energetic expanding, Lefter and Rădulescu proved the following result:

Theorem 3. [Theorem 2 [LR96]] *For $N \in \mathbb{N}^*$, there exists a map $W : (\omega^N)^* \times \mathbb{Z}^N \rightarrow \mathbb{R}$ s.t. for $\mathbf{d} \in \mathbb{Z}^N$ and $\mathbf{z} \in (\omega^N)^*$ when $\rho \rightarrow 0$ we have:*

$$\inf_{w \in \mathcal{E}_{\mathbf{d}}(\Omega_{\rho, \mathbf{z}})} \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w|^2 = \pi \sum_{i=1}^N d_i^2 |\ln \rho| + W(\mathbf{z}, \mathbf{d}) + o_\rho(1).$$

5 Upper Bound

We are now in position to start the proof of Theorem 1. To this end, the goal of this section is to identify a map $\mathcal{K} : \{h \in H^{1/2}(\partial\omega, \mathbb{S}^1) \mid \deg(h) = d\} \rightarrow \mathbb{R}$ s.t. for a fixed $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ with $\deg(h) = d$, when $R \rightarrow \infty$ we have

$$\begin{aligned} & \inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \text{tr}_{\partial\omega}(v) = h}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2 + \inf_{\substack{w \in \mathcal{E}_d(\Omega_{\rho, \mathbf{z}}) \\ \text{tr}_{\partial\omega}(w) = h}} \frac{b^2}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w|^2 \\ &= \mathcal{K}(h) + d^2 f(R) + b^2 \left[\pi \left(\sum_{i=1}^N d_i^2 \right) |\ln \rho| + W(\mathbf{z}, \mathbf{d}) \right] + o(1). \end{aligned} \quad (16)$$

In the above estimate we have:

- \mathcal{K} is independent of R, ρ ;
- f is defined by Remark 1.2 and is independent of $h, \rho, \mathbf{z}, \mathbf{d}$ and $d = \sum d_i$;
- W is defined in Theorem 3 and is independent of b, B, h, ρ and R .

Note that from Corollaries 14 and 15, we have the existence of

- $\gamma_\infty \in H^{1/2}(\partial\omega)$ s.t. $\gamma_R \rightarrow \gamma_\infty$ in $H^{1/2}(\partial\omega)$,
- $\gamma_{0, \mathbf{z}, \mathbf{d}} \in H^{1/2}(\partial\omega)$ s.t. $\gamma_{\rho, \mathbf{z}, \mathbf{d}} \rightarrow \gamma_{0, \mathbf{z}, \mathbf{d}}$ in $H^{1/2}(\partial\omega)$.

It is important to claim that since $\int_{\partial\omega} \gamma_R = 0$ and $\int_{\partial\omega} \gamma_{\rho, \mathbf{z}, \mathbf{d}} = 0$, we have $\int_{\partial\omega} \gamma_\infty = 0$ and $\int_{\partial\omega} \gamma_{0, \mathbf{z}, \mathbf{d}} = 0$.

5.1 Study in the domain Ω_R

For $R \in (R_0, \infty)$ and $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$ we consider

$$I_R(h) := \inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \text{tr}_{\partial\omega}(v) = h}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2. \quad (17)$$

Let $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$. In this section we want to estimate $I_R(h)$ when $R \rightarrow \infty$.

We let $g_h := h \left(\frac{|x|}{x} \right)^d \in H^{1/2}(\partial\omega)$. It is clear that $\deg(g_h) = 0$ and then we may fix a unique

$\phi_h \in H^{1/2}(\partial\omega, \mathbb{R})$ s.t. $g_h = e^{i\phi_h}$ and $\int_{\partial\omega} \phi_h \in (-\pi, \pi]$.

Remark 16. Since our goal is to estimate $I_R(h)$ and since for $\beta \in \mathbb{R}$ we have $I_R(h) = I_R(e^{i\beta} h)$, up to replace h by $e^{i\beta} h$ with $\beta = - \int_{\partial\omega} \phi_h$, we may assume that $\int_{\partial\omega} \phi_h = 0$.

Recall that for $R \in (R_0, \infty]$ we have $v_R = \frac{x}{|x|} e^{i\gamma_R}$ [see (8) and Corollary 14]. For $R \in (R_0, \infty]$ we let $\phi_R^h := \phi_h - d \text{tr}_{\partial\omega}(\gamma_R) \in H^{1/2}(\partial\omega, \mathbb{R})$ and so we get $h = \text{tr}_{\partial\omega}(v_R^d) e^{i\phi_R^h}$ and $\int_{\partial\omega} \phi_R^h = 0$.

From Corollary 14 we immediately obtain:

Corollary 17. $\phi_R^h \xrightarrow{R \rightarrow \infty} \phi_\infty^h$ in $H^{1/2}(\partial\omega)$.

For $R \in (R_0, \infty)$ and $v \in H^1(\Omega_R, \mathbb{S}^1)$ it is clear that

$$\text{tr}_{\partial\omega}(v) = h \iff v = v_R^d e^{i\varphi} \text{ with } \left| \begin{array}{l} \varphi \in H^1(\Omega_R, \mathbb{R}) \\ \text{tr}_{\partial\omega}(\varphi) = \phi_R^h \end{array} \right. .$$

On the other hand, for $v = v_R^d e^{i\varphi} \in H^1(\Omega_R, \mathbb{S}^1)$, from Lemma 10 we have

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2 = \frac{d^2}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2 + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi|^2. \quad (18)$$

Therefore, one may obtain that $v = v_R^d e^{i\varphi}$ with $\text{tr}_{\partial\omega}(\varphi) = \phi_R^h$ is a solution of the minimization problem (17) if and only if $\varphi \in H^1(\Omega_R, \mathbb{R})$ is a solution of the minimization problem

$$\inf_{\substack{\varphi \in H^1(\Omega_R, \mathbb{R}) \\ \text{tr}_{\partial\omega}(\varphi) = \phi_R^h}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi|^2. \quad (19)$$

It is classic to prove that Problem (19) admits a unique solution denoted by φ_R^h . Moreover this minimizer is the unique solution of

$$\begin{cases} -\text{div}(\alpha \nabla \varphi_R^h) = 0 & \text{in } \Omega_R \\ \text{tr}_{\partial\omega}(\varphi_R^h) = \phi_R^h \\ \partial_\nu \varphi_R^h = 0 & \text{on } \partial B_R \end{cases}.$$

We denote $\Omega_\infty := \mathbb{R}^2 \setminus \bar{\omega}$ and for $\phi \in H^{1/2}(\partial\omega, \mathbb{R})$ we let

$$\mathcal{H}_\phi := \{\varphi \in H_{\text{loc}}^1(\bar{\Omega}_\infty, \mathbb{R}) \mid \nabla \varphi \in L^2(\Omega_\infty) \text{ and } \text{tr}_{\partial\omega}(\varphi) = \phi\}. \quad (20)$$

We are now interested in the minimization problem:

$$\inf_{\varphi \in \mathcal{H}_{\phi_R^h}} \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi|^2. \quad (21)$$

By direct minimization, first order variations and from the strict convexity of the energy we get:

Proposition 18. *Problem (21) admits a unique solution denoted by φ_∞^h . Moreover φ_∞^h is a solution of*

$$-\text{div}(\alpha \nabla \varphi_\infty^h) = 0 \text{ in } \Omega_\infty. \quad (22)$$

We are now able to prove the main result of this section:

Proposition 19. *When $R \rightarrow \infty$, we have $\varphi_R^h \rightarrow \varphi_\infty^h$ in $H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega)$ and $\nabla \varphi_R^h \mathbb{I}_{\Omega_R} \rightarrow \nabla \varphi_\infty^h$ in $L^2(\mathbb{R}^2 \setminus \bar{\omega})$, with $\mathbb{I}_{\Omega_R}(x) = \begin{cases} 1 & \text{if } x \in \Omega_R \\ 0 & \text{if } x \notin \Omega_R \end{cases}$.*

And consequently: $\int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2 = \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty^h|^2 + o_R(1)$.

Proof. From Corollary 17 we have $\phi_R^h - \phi_\infty^h \rightarrow 0$ in $H^{1/2}(\partial\omega)$. Consequently, there exists $\xi_R \in H^1(\Omega_\infty, \mathbb{R})$ s.t. $\text{tr}_{\partial\omega}(\xi_R) = \phi_R^h - \phi_\infty^h$ and $\|\xi_R\|_{H^1(\Omega_\infty)} \rightarrow 0$.

The test function $\varphi_\infty^h + \xi_R$ satisfies the boundary condition of Problem (19), therefore:

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2 \leq \frac{1}{2} \int_{\Omega_R} \alpha |\nabla(\varphi_\infty^h + \xi_R)|^2 = \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_\infty^h|^2 + o(1). \quad (23)$$

Note we used $\int_{\Omega_R} \alpha |\nabla \varphi_\infty^h|^2 \leq C_0 := \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty^h|^2 < \infty$. From (23), we obtain

$$\limsup_{R \rightarrow \infty} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2 \leq \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty^h|^2. \quad (24)$$

We now prove the "lim inf"-lower bound:

$$\liminf_{R \rightarrow \infty} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2 \geq \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty^h|^2. \quad (25)$$

On the one hand, from (23), for $R \in (R_0, \infty)$ sufficiently large, we have $\int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2 \leq C_0 + 1$ and thus, up to pass to a subsequence, we get that $\nabla \varphi_R^h \mathbb{1}_{\Omega_R}$ weakly converges in $L^2(\mathbb{R}^2 \setminus \bar{\omega}, \mathbb{R}^2)$.

On the other hand, for a connected compact set $K \subset \mathbb{R}^2 \setminus \omega$ s.t. $\partial\omega \subset \partial K$, the test function $\varphi_\infty^h + \xi_R$ is bounded in $H^1(K)$. We let $\chi_R := \varphi_R - (\varphi_\infty^h + \xi_R) \in H^1(K)$ and then, for sufficiently large R , we have $\|\nabla \chi_R\|_{L^2(\Omega_R)} \leq 2C_0 + 2$. It is easy to check that $\text{tr}_{\partial\omega}(\chi_R) = 0$. Consequently, from a Poincaré type inequality, there exists a constant $C_K > 1$ s.t. $\|\chi_R\|_{L^2(K)} \leq C_K \|\nabla \chi_R\|_{L^2(K)} \leq C_K \times (2C_0 + 2)$. Thus there exists a constant C'_K s.t., for sufficiently large R , $\|\varphi_R^h\|_{L^2(K)} \leq C'_K$.

Consequently, with the help of an exhaustion by compact sets and a diagonal extraction process, we have the existence of a sequence $R_k \uparrow \infty$ and $\tilde{\varphi}_\infty \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega, \mathbb{R})$ s.t.

$$\begin{cases} \varphi_{R_k}^h \rightharpoonup \tilde{\varphi}_\infty \text{ in } H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega) \text{ and } \nabla \varphi_{R_k}^h \mathbb{1}_{\Omega_{R_k}} \rightharpoonup \nabla \tilde{\varphi}_\infty \text{ in } L^2(\mathbb{R}^2 \setminus \bar{\omega}). \\ \liminf_{R \rightarrow \infty} \int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2 = \lim_{R_k \rightarrow \infty} \int_{\Omega_{R_k}} \alpha |\nabla \varphi_{R_k}^h|^2 \end{cases} \quad (26)$$

We thus get $\nabla \tilde{\varphi}_\infty \in L^2(\mathbb{R}^2 \setminus \bar{\omega})$ and $\text{tr}_{\partial\omega}(\tilde{\varphi}_\infty) = \phi_\infty^h$, i.e., $\tilde{\varphi}_\infty \in \mathcal{H}_{\phi_\infty^h}$.

From the definition of φ_∞^h [Proposition 18] we have with (26)

$$\frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty^h|^2 \leq \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \tilde{\varphi}_\infty|^2 \leq \liminf_{R \rightarrow \infty} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2.$$

We thus obtained (25). Therefore by combining (24) and (25) we get:

$$\int_{\Omega_R} \alpha |\nabla \varphi_R^h|^2 = \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty^h|^2 + o_R(1). \quad (27)$$

The above estimate implies that a limiting map $\tilde{\varphi}_\infty \in \mathcal{H}_{\phi_\infty^h}$ as previously obtained satisfies:

$$\frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \tilde{\varphi}_\infty|^2 \leq \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty^h|^2.$$

On the other hand φ_∞^h is the unique solution of Problem (21). Therefore $\tilde{\varphi}_\infty = \varphi_\infty^h$. Consequently, the convergences in (26) hold for $R \rightarrow \infty$ and from (27), these convergences are strong. \square

5.2 Study in the domain $\Omega_{\rho, \mathbf{z}}$

Recall that we fixed a map $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$. We are interested in getting an asymptotic estimate for the minimal energy

$$I_{\rho, \mathbf{z}, d}(h) = \inf_{\substack{w \in \mathcal{E}_d(\Omega_{\rho, \mathbf{z}}) \\ \text{tr}_{\partial\omega}(w) = h}} \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w|^2. \quad (28)$$

First note that letting $g_{\mathbf{z}, d}^h := h \prod_{i=1}^N \left(\frac{|x - z_i|}{x - z_i} \right)^{d_i} \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ we have $\deg(g_{\mathbf{z}, d}^h) = 0$. Thus, from

a standard lifting result, we may fix $\phi_{\mathbf{z}, d}^h \in H^{1/2}(\partial\omega, \mathbb{R})$ s.t. $g_{\mathbf{z}, d}^h = e^{i\phi_{\mathbf{z}, d}^h}$ and $\int_{\partial\omega} \phi_{\mathbf{z}, d}^h \in (-\pi, \pi]$.

It is clear that $\phi_{\mathbf{z}, d}^h$ is uniquely defined.

Remark 20. As in the previous section [see Remark 16], for $\beta \in \mathbb{R}$ we have $I_{\rho, \mathbf{z}, d}(h) = I_{\rho, \mathbf{z}, d}(he^{i\beta})$. Thus up to replace h by $he^{i\beta}$, with $\beta = -\int_{\partial\omega} \phi_{\mathbf{z}, d}^h$, in order to estimate $I_{\rho, \mathbf{z}, d}(h)$, we may assume

that $\int_{\partial\omega} \phi_{\mathbf{z}, d}^h = 0$.

For $\rho \in [0, \rho_0)$ we let $\phi_{\rho, \mathbf{z}, d}^h := \phi_{\mathbf{z}, d}^h - \gamma_{\rho, \mathbf{z}, d} \in H^{1/2}(\partial\omega, \mathbb{R})$ [$\gamma_{\rho, \mathbf{z}, d}$ is defined in (14) and (15)] in order to have $h = \text{tr}_{\partial\omega}(w_{\rho, \mathbf{z}, d}) e^{i\phi_{\rho, \mathbf{z}, d}^h}$. Moreover it is clear that $\int_{\partial\omega} \phi_{\rho, \mathbf{z}, d}^h = 0$.

Notation 21. For simplicity of the presentation, until the end of this section, we omit the subscripts \mathbf{z}, \mathbf{d} e.g. writing, for $\rho \in [0, \rho_0)$, ϕ_ρ^h instead of $\phi_{\rho, \mathbf{z}, \mathbf{d}}^h$.

From Corollary 15 we get:

Corollary 22. $\phi_\rho^h \xrightarrow{\rho \rightarrow 0} \phi_0^h$ in $H^{1/2}(\partial\omega)$.

For $\rho \in (0, \rho_0)$ and $w \in H^1(\Omega_{\rho, \mathbf{z}}, \mathbb{S}^1)$, we have

$$\mathrm{tr}_{\partial\omega}(w) = h \iff w = w_\rho e^{i\varphi} \text{ with } \begin{cases} \varphi \in H^1(\Omega_{\rho, \mathbf{z}}, \mathbb{R}) \\ \mathrm{tr}_{\partial\omega}(\varphi) = \phi_\rho^h \end{cases}.$$

We follow the same strategy than in the previous section. For $\varphi \in H^1(\Omega_{\rho, \mathbf{z}}, \mathbb{R})$, from Proposition 8 we have for $w = w_\rho e^{i\varphi}$

$$\frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w|^2 = \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w_\rho|^2 + \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi|^2. \quad (29)$$

Consequently a test function $w = w_\rho e^{i\varphi}$ with $\mathrm{tr}_{\partial\omega}(\varphi) = \phi_\rho^h$ is a solution of the minimizing problem (28) if and only if $\varphi \in H^1(\Omega_{\rho, \mathbf{z}}, \mathbb{R})$ is a solution of the minimizing problem

$$\inf_{\substack{\varphi \in H^1(\Omega_{\rho, \mathbf{z}}, \mathbb{R}) \\ \mathrm{tr}_{\partial\omega}(\varphi) = \phi_\rho^h}} \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi|^2. \quad (30)$$

And then for $\rho \in (0, \rho_0)$, the minimizing Problem (30) admits a unique solution denoted by φ_ρ^h . About the asymptotic behavior of φ_ρ^h we have the following result:

Proposition 23. *When $\rho \rightarrow 0$, we have*

$$\frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi_\rho^h|^2 = \frac{1}{2} \int_{\omega} |\nabla \tilde{\phi}_0^h|^2 + o_\rho(1)$$

where $\tilde{\phi}_0^h$ is the harmonic extension of ϕ_0^h in ω .

Proof. Let ξ_ρ be the harmonic extension of $\phi_0^h - \phi_\rho^h$ in ω . Since $\|\phi_0^h - \phi_\rho^h\|_{H^{1/2}(\partial\omega)} \rightarrow 0$, we have $\xi_\rho \rightarrow 0$ in $H^1(\omega)$.

We now prove the proposition. On the one hand, by minimality of φ_ρ^h and since $\mathrm{tr}_{\partial\omega}(\tilde{\phi}_0^h - \xi_\rho) = \phi_\rho^h$ we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi_\rho^h|^2 &\leq \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla(\tilde{\phi}_0^h - \xi_\rho)|^2 \\ &\leq \frac{1}{2} \int_{\omega} |\nabla \tilde{\phi}_0^h|^2 + o_\rho(1). \end{aligned} \quad (31)$$

On the other hand, from the Estimate (31), denoting $C_0 := \int_{\omega} |\nabla \tilde{\phi}_0^h|^2 + 1$, for sufficiently small ρ we get

$$\sum_{i=1}^N \frac{1}{2} \int_{B(z_i, \sqrt{\rho}) \setminus B(z_i, \rho)} |\nabla \varphi_\rho^h|^2 < C_0. \quad (32)$$

Thus for small ρ , we get the existence of $\rho' \in (\rho, \sqrt{\rho})$ s.t.:

$$\sum_{i=1}^N \frac{1}{2} \int_0^{2\pi} |\partial_\theta \varphi_\rho^h(z_i + \rho' e^{i\theta})|^2 \leq \frac{2C_0}{|\ln \rho|}.$$

For $i \in \{1, \dots, N\}$ we let $m_{i,\rho} := \int_0^{2\pi} \varphi_\rho^h(z_i + \rho' e^{i\theta})$. We now define $\tilde{\varphi} \in H^1(\omega)$ by $\tilde{\varphi} = \varphi_\rho^h$ in $\omega \setminus \cup_i \overline{B(z_i, \rho')}$ and for $x = z_i + se^{i\theta} \in B(z_i, \rho')$ [with $i \in \{1, \dots, N\}$]

$$\tilde{\varphi}(z_i + se^{i\theta}) = \begin{cases} \frac{2s - \rho'}{\rho'} \varphi_\rho^h(z_i + \rho' e^{i\theta}) + \frac{2(\rho' - s)}{\rho'} m_{i,\rho} & \text{if } s \in (\frac{\rho'}{2}, \rho') \\ m_{i,\rho} & \text{if } s \leq \frac{\rho'}{2} \end{cases}.$$

A direct calculation gives for $z \in \{z_1, \dots, z_N\}$

$$\int_{B(z, \rho')} |\nabla \tilde{\varphi}|^2 = \mathcal{O} \left[\int_0^{2\pi} |\partial_\theta \varphi_\rho^h(z + \rho' e^{i\theta})|^2 \right] = o_\rho(1).$$

Therefore we obtain

$$\frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi_\rho^h|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\varphi}|^2 + o_\rho(1).$$

But $\text{tr}_{\partial\omega}(\tilde{\varphi} + \xi_\rho) = \phi_0^h$ and consequently, from the Dirichlet principle, we have:

$$\frac{1}{2} \int_\omega |\nabla(\tilde{\varphi} + \xi_\rho)|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_0^h|^2$$

and thus with (31)

$$\frac{1}{2} \int_\omega |\nabla \tilde{\varphi}|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_0^h|^2 + o_\rho(1).$$

On the other hand, since $\tilde{\varphi} = \varphi_\rho^h$ in $\omega \setminus \cup_i \overline{B(z_i, \rho')} \subset \Omega_{\rho, \mathbf{z}}$ and $\frac{1}{2} \int_{\cup_i B(z_i, \rho')} |\nabla \tilde{\varphi}|^2 = o_\rho(1)$ we obtain:

$$\frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi_\rho^h|^2 \geq \frac{1}{2} \int_{\omega \setminus \cup_i \overline{B(z_i, \rho')}} |\nabla \varphi_\rho^h|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_0^h|^2 + o_\rho(1).$$

Finally, using (31), by matching upper bound and lower bound we conclude:

$$\frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi_\rho^h|^2 = \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_0^h|^2 + o_\rho(1).$$

The last estimates ends the proof of the proposition. \square

5.3 Conclusion

For $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$ we have from (18) and Proposition 19:

$$\inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \text{tr}_{\partial\omega}(v) = h}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2 = \frac{d^2}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2 + \inf_{\varphi \in \mathcal{H}_{\phi_\infty}^h} \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi|^2 + o_R(1). \quad (33)$$

Using Theorem 3, (29) and Proposition 23 we have

$$\inf_{\substack{w \in H^1(\Omega_{\rho, \mathbf{z}}, \mathbb{S}^1) \\ \text{tr}_{\partial\omega}(w) = h}} \frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w|^2 = \left(\sum_i d_i^2 \right) \pi |\ln \rho| + W(\mathbf{z}, \mathbf{d}) + \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_0^h|^2 + o_\rho(1). \quad (34)$$

Letting $\mathcal{K} : \{h \in H^{1/2}(\partial\omega, \mathbb{S}^1) \mid \deg(h) = d\} \rightarrow \mathbb{R}^+$ defined by

$$\mathcal{K}(h) := \inf_{\varphi \in \mathcal{H}_{\phi_\infty}^h} \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi|^2 + \frac{b^2}{2} \int_\omega |\nabla \tilde{\phi}_0^h|^2 \quad (35)$$

we get (16). Recall that, without loss of generality, the parameter "R" is considered as the major parameter writing $\rho = \rho(R)$. From (16), we get for $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$:

$$\limsup_{R \rightarrow \infty} \left\{ I(R, \rho, \mathbf{z}, \mathbf{d}) - \left[d^2 f(R) + b^2 \left(\sum_i d_i^2 \pi |\ln \rho| + W(\mathbf{z}, \mathbf{d}) \right) \right] \right\} \leq \mathcal{K}(h). \quad (36)$$

6 Lower bound

In this section we prove the existence of a map $h_\infty \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h_\infty) = d$ and

$$\liminf_{R \rightarrow \infty} \left\{ I(R, \rho, \mathbf{z}, \mathbf{d}) - \left[d^2 f(R) + b^2 \left(\sum_i d_i^2 \pi |\ln \rho| + W(\mathbf{z}, \mathbf{d}) \right) \right] \right\} \geq \mathcal{K}(h_\infty). \quad (37)$$

Clearly a such map h_∞ should minimize $\mathcal{K} : \{h \in H^{1/2}(\partial\omega, \mathbb{S}^1) \mid \deg(h) = d\} \rightarrow \mathbb{R}^+$. But in order to get an explicit expression for h_∞ we do not define h_∞ in this way.

We let $R_n \uparrow \infty$ be a sequence which realizes the "lim inf" in the left hand side of (37).

In order to keep notation simple, we drop the subscript n writing $R = R_n$ when it will not be necessary to specify the dependance on n .

Let u_R be a minimizer of (2) [Proposition 6]. In Ω_R , we may decompose u_R under the form $u_R = v_R^d e^{i\varphi_R}$ where $\varphi_R \in H^1(\Omega_R, \mathbb{R})$ and v_R is defined in (8).

Since u_R is unique up to a multiplicative constant [Proposition 6], we may freeze the non uniqueness by imposing $\int_{\partial\omega} \varphi_R = 0$.

Notation 24. For sake of simplicity of the presentation we use the shorthands:

- " $R \in (R_0, \infty)$ " to consider an arbitrary term of the sequence $(R_n)_n$;
- " $R \in (R_0, \infty]$ " to consider either an arbitrary term of the sequence $(R_n)_n$ or the limiting case $R = \infty$.

We denote:

- for $R \in (R_0, \infty)$, $h_R := \text{tr}_{\partial\omega} u_R$, and thus we have $h_R = \text{tr}_{\partial\omega} \left[\left(\frac{x}{|x|} \right)^d e^{i(d\gamma_R + \varphi_R)} \right]$;
- $g_{\mathbf{z}, \mathbf{d}} := \text{tr}_{\partial\omega} \left[\left(\frac{|x|}{x} \right)^d \prod_{i=1}^N \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} \right]$.

Since $g_{\mathbf{z}, \mathbf{d}} \in C^\infty(\partial\omega, \mathbb{S}^1)$ and $\deg_{\partial\omega}(g_{\mathbf{z}, \mathbf{d}}) = 0$ we may fix $\xi_{\mathbf{z}, \mathbf{d}} \in C^\infty(\partial\omega, \mathbb{R})$ s.t. $e^{i\xi_{\mathbf{z}, \mathbf{d}}} = g_{\mathbf{z}, \mathbf{d}}$ and $\int_{\partial\omega} \xi_{\mathbf{z}, \mathbf{d}} \in [-\pi, \pi]$.

6.1 Compatibility conditions

From the minimality of u_R , it is obvious that the restriction of u_R to Ω_R [resp. $\Omega_{\rho, \mathbf{z}}$] is a solution of the problem (17) [resp. (28)] with $h = h_R$.

It is easy to check that we may write for $R \in (R_0, \infty)$

$$h_R = \text{tr}_{\partial\omega} [v_R^d e^{i\varphi_R}] = \text{tr}_{\partial\omega} [w_{\rho, \mathbf{z}, \mathbf{d}} e^{i\varphi_{\rho, \mathbf{z}, \mathbf{d}}}]$$

where, omitting the superscript h_R , we have:

- v_R is the special solution in Ω_R defined in (8).
- $\varphi_R = \varphi_R^{h_R} \in H^1(\Omega_R, \mathbb{R})$ is the unique solution of Problem (19) [for the Dirichlet data h_R on $\partial\omega$] s.t. $u_R = v_R^d e^{i\varphi_R}$ in Ω_R and $\int_{\partial\omega} \varphi_R = 0$ [φ_R is defined above].
- $w_{\rho, \mathbf{z}, \mathbf{d}} = \prod_{i=1}^N \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} e^{i\gamma_{\rho, \mathbf{z}, \mathbf{d}}}$ is defined in (14);
- $\varphi_{\rho, \mathbf{z}, \mathbf{d}} = \varphi_{\rho, \mathbf{z}, \mathbf{d}}^{h_R} \in H^1(\Omega_{\rho, \mathbf{z}}, \mathbb{R})$ is the unique solution of (30) [for the Dirichlet data h_R on $\partial\omega$] s.t. $u_R = w_{\rho, \mathbf{z}, \mathbf{d}} e^{i\varphi_{\rho, \mathbf{z}, \mathbf{d}}}$ in $\Omega_{\rho, \mathbf{z}}$ and $\int_{\partial\omega} \varphi_{\rho, \mathbf{z}, \mathbf{d}} \in (-\pi, \pi]$.

By using Corollaries 14 and 15, we have the existence of $\gamma_\infty, \gamma_{0,\mathbf{z},\mathbf{d}} \in H^{1/2}(\partial\omega, \mathbb{R})$ s.t. $\gamma_R \rightarrow \gamma_\infty$ and $\gamma_{\rho,\mathbf{z},\mathbf{d}} \rightarrow \gamma_{0,\mathbf{z},\mathbf{d}}$ in $H^{1/2}(\partial\omega)$. It is fundamental to note that

- γ_∞ and $\gamma_{0,\mathbf{z},\mathbf{d}}$ are independent of the sequence $(R_n)_n$;
- $\int_{\partial\omega} \gamma_R = \int_{\partial\omega} \gamma_\infty = \int_{\partial\omega} \gamma_{0,\mathbf{z},\mathbf{d}} = \int_{\partial\omega} \gamma_{\rho,\mathbf{z},\mathbf{d}} = 0$.

We have the following equivalences:

$$\begin{aligned} e^{i[\operatorname{tr}_{\partial\omega}(\varphi_R) - \operatorname{tr}_{\partial\omega}(\varphi_{\rho,\mathbf{z},\mathbf{d}})]} &= \operatorname{tr}_{\partial\omega}(w_{\rho,\mathbf{z},\mathbf{d}}) \times \operatorname{tr}_{\partial\omega}(\overline{v_R^d}) \\ \Leftrightarrow e^{i[\operatorname{tr}_{\partial\omega}(\varphi_R) - \operatorname{tr}_{\partial\omega}(\varphi_{\rho,\mathbf{z},\mathbf{d}})]} &= e^{i[\xi_{\mathbf{z},\mathbf{d}} + \operatorname{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}}) - d\operatorname{tr}_{\partial\omega}(\gamma_R)]} \\ \Leftrightarrow \operatorname{tr}_{\partial\omega}(\varphi_R) - \operatorname{tr}_{\partial\omega}(\varphi_{\rho,\mathbf{z},\mathbf{d}}) &= \xi_{\mathbf{z},\mathbf{d}} + \operatorname{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}}) - d\operatorname{tr}_{\partial\omega}(\gamma_R) + 2k_0\pi \quad \text{with } k_0 \in \mathbb{Z}. \end{aligned} \quad (38)$$

We thus have

$$-\int_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} = \int_{\partial\omega} \varphi_R - \int_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} = \int_{\partial\omega} [\xi_{\mathbf{z},\mathbf{d}} + \operatorname{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}}) - d\operatorname{tr}_{\partial\omega}(\gamma_R) + 2k_0\pi] = 2k_0\pi + \int_{\partial\omega} \xi_{\mathbf{z},\mathbf{d}}.$$

Since $\int_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} \in (-\pi, \pi]$ and $\int_{\partial\omega} \xi_{\mathbf{z},\mathbf{d}} \in [-\pi, \pi)$, the above equalities imply that $k_0 = 0$ in (38).

Consequently we get:

$$\operatorname{tr}_{\partial\omega}(\varphi_R) - \operatorname{tr}_{\partial\omega}(\varphi_{\rho,\mathbf{z},\mathbf{d}}) = \xi_{\mathbf{z},\mathbf{d}} + \operatorname{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}}) - d\operatorname{tr}_{\partial\omega}(\gamma_R). \quad (39)$$

6.2 Asymptotic estimate of the energy

By using (18) and (29), we have the following decoupling:

$$\begin{aligned} I(R, \rho, \mathbf{z}, \mathbf{d}) &= \frac{1}{2} \int_{\mathcal{D}_{R,\mathbf{z}}} \alpha |\nabla u_R|^2 \\ &= \frac{1}{2} \int_{\Omega_R} \alpha |\nabla(v_R^d e^{i\varphi_R})|^2 + \frac{b^2}{2} \int_{\Omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}} e^{i\varphi_{\rho,\mathbf{z},\mathbf{d}}}|^2 \\ &= d^2 f(R) + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R|^2 + \frac{b^2}{2} \int_{\Omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2 + \frac{b^2}{2} \int_{\Omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2. \end{aligned} \quad (40)$$

From the minimality of u_R and by using (36), letting $C_0 := \mathcal{K} \left(\frac{x^d}{|x|^d} \right) + 1$, for sufficiently large R , we have:

$$I(R, \rho, \mathbf{z}, \mathbf{d}) - \left[d^2 f(R) + \frac{b^2}{2} \int_{\Omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2 \right] = \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R|^2 + \frac{b^2}{2} \int_{\Omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \leq C_0. \quad (41)$$

Since $\int_{\partial\omega} \varphi_R = 0$ [resp. $\int_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} \in (-\pi, \pi]$] for K_1 a connected compact set of $\mathbb{R}^2 \setminus \omega$ [resp. K_2 a connected compact set of $\overline{\omega} \setminus \{z_1, \dots, z_N\}$] s.t. $\partial\omega \subset \partial K_1$ [resp. $\partial\omega \subset \partial K_2$], there exists $C_1 > 0$ [resp. $C_2 > 0$] s.t. for large R we have $\int_{K_1} |\varphi_R|^2 \leq C_1$ and $\int_{K_2} |\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \leq C_2$.

Consequently :

- $(\varphi_R)_R$ is bounded in $H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega)$. Thus there exists $\varphi_\infty \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega)$ s.t., up to pass to a subsequence, we have

$$\varphi_R \rightharpoonup \varphi_\infty \text{ in } H_{\text{loc}}^1(\mathbb{R}^2 \setminus \omega). \quad (42)$$

- $(\varphi_{\rho,\mathbf{z},\mathbf{d}})_R$ is bounded in $H_{\text{loc}}^1(\overline{\omega} \setminus \{z_1, \dots, z_N\})$. Thus there exists $\varphi_{0,\mathbf{z},\mathbf{d}} \in H_{\text{loc}}^1(\overline{\omega} \setminus \{z_1, \dots, z_N\})$ s.t., up to pass to a subsequence, we have

$$\varphi_{\rho,\mathbf{z},\mathbf{d}} \rightharpoonup \varphi_{0,\mathbf{z},\mathbf{d}} \text{ in } H_{\text{loc}}^1(\overline{\omega} \setminus \{z_1, \dots, z_N\}). \quad (43)$$

From (39), we have $\text{tr}_{\partial\omega}(\varphi_R) - \text{tr}_{\partial\omega}(\varphi_{\rho,\mathbf{z},\mathbf{d}}) = \xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}}) - d\text{tr}_{\partial\omega}(\gamma_R)$. On the other hand, with Corollaries 14&15, we get that $\xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}}) - d\text{tr}_{\partial\omega}(\gamma_R)$ strongly converges in $H^{1/2}(\partial\omega)$ to $\xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}}) - d\text{tr}_{\partial\omega}(\gamma_\infty)$. Consequently $\text{tr}_{\partial\omega}(\varphi_R) - \text{tr}_{\partial\omega}(\varphi_{\rho,\mathbf{z},\mathbf{d}})$ strongly converges in $H^{1/2}(\partial\omega)$ to $\text{tr}_{\partial\omega}(\varphi_\infty) - \text{tr}_{\partial\omega}(\varphi_{0,\mathbf{z},\mathbf{d}}) = \xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}}) - d\text{tr}_{\partial\omega}(\gamma_\infty)$.

We thus may deduce $e^{i[\text{tr}_{\partial\omega}(\varphi_\infty) - \text{tr}_{\partial\omega}(\varphi_{0,\mathbf{z},\mathbf{d}})]} = e^{i[\xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}}) - d\text{tr}_{\partial\omega}(\gamma_\infty)]}$, *i.e.*,

$$\left(\frac{x}{|x|}\right)^d e^{i\text{tr}_{\partial\omega}(d\gamma_\infty + \varphi_\infty)} = \prod_{i=1}^N \left(\frac{x - z_i}{|x - z_i|}\right)^{d_i} e^{i\text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}} + \varphi_{0,\mathbf{z},\mathbf{d}})}. \quad (44)$$

We now define:

$$h_\infty := \text{tr}_{\partial\omega} \left[\left(\frac{x}{|x|}\right)^d e^{i(d\gamma_\infty + \varphi_\infty)} \right] \in H^{1/2}(\partial\omega, \mathbb{S}^1). \quad (45)$$

It is clear that $\deg(h_\infty) = d$. We prove in the three next subsections [Sections 6.3&6.4&6.5] that h_∞ satisfies (37).

6.3 Calculations in $\mathbb{R}^2 \setminus \bar{\omega}$

From (41), we get that $\nabla\varphi_R \mathbb{1}_{\Omega_R}$ is bounded in $L^2(\mathbb{R}^2 \setminus \bar{\omega})$ and thus, up to pass to a subsequence, $\nabla\varphi_R \mathbb{1}_{\Omega_R}$ weakly converges in $L^2(\mathbb{R}^2 \setminus \bar{\omega})$. Consequently, we may improve the convergence in (42), up to pass to a subsequence, we obtain that $\nabla\varphi_R \mathbb{1}_{\Omega_R} \rightharpoonup \nabla\varphi_\infty$ in $L^2(\mathbb{R}^2 \setminus \bar{\omega})$. In particular we obtain $\nabla\varphi_\infty \in L^2(\mathbb{R}^2 \setminus \bar{\omega})$.

Consequently, denoting $\phi_\infty := \text{tr}_{\partial\omega}(\varphi_\infty)$ we obtain $\varphi_\infty \in \mathcal{H}_{\phi_\infty}$. Therefore, with $\Omega_\infty = \mathbb{R}^2 \setminus \bar{\omega}$, we have:

$$\begin{aligned} \liminf_{R_n \rightarrow \infty} \left\{ \frac{1}{2} \int_{\Omega_{R_n}} \alpha |\nabla u_{R_n}|^2 - \frac{d^2}{2} \int_{\Omega_{R_n}} \alpha |\nabla v_{R_n}|^2 \right\} &= \liminf_{R_n \rightarrow \infty} \frac{1}{2} \int_{\Omega_{R_n}} \alpha |\nabla \varphi_{R_n}|^2 \\ &\geq \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi_\infty|^2 \\ &\geq \inf_{\varphi \in \mathcal{H}_{\phi_\infty}} \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi|^2. \end{aligned} \quad (46)$$

6.4 Calculations on ω

We continue the calculations by proving:

$$\frac{1}{2} \int_{\Omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \geq \frac{1}{2} \int_{\omega} |\nabla \tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1) \quad (47)$$

where $\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}$ is the harmonic extension of $\phi_{0,\mathbf{z},\mathbf{d}} := \text{tr}_{\partial\omega} \varphi_{0,\mathbf{z},\mathbf{d}}$ in ω , $\varphi_{0,\mathbf{z},\mathbf{d}}$ is defined in (43).

In order to get (47), we adapt the argument done to prove Proposition 23. From (41), we have

$$\sum_{i=1}^N \frac{1}{2} \int_{B(z_i, \sqrt{\rho}) \setminus \overline{B(z_i, \rho)}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \leq C_0.$$

Thus, with a mean value argument, there exists $\rho' \in (\rho, \sqrt{\rho})$ s.t.

$$\sum_{i=1}^N \frac{1}{2} \int_0^{2\pi} |\partial_\theta \varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i + \rho' e^{i\theta})|^2 d\theta \leq \frac{2C_0}{|\ln \rho|}.$$

We now define $\tilde{\varphi}_\rho \in H^1(\omega)$ by $\tilde{\varphi}_\rho = \varphi_{\rho,\mathbf{z},\mathbf{d}}$ in $\omega \setminus \cup_i \overline{B(z_i, \rho')}$ and for $i \in \{1, \dots, N\}$ & $x = z_i + se^{i\theta} \in B(z_i, \rho')$ we let

$$\tilde{\varphi}_\rho(z_i + se^{i\theta}) = \begin{cases} 2 \frac{s - \rho'/2}{\rho'} \varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i + \rho' e^{i\theta}) + \frac{\rho' - s}{\pi \rho'} \int_0^{2\pi} \varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i + \rho' e^{i\theta}) d\theta & \text{if } s \in (\frac{\rho'}{2}, \rho') \\ \frac{1}{2\pi} \int_0^{2\pi} \varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i + \rho' e^{i\theta}) d\theta & \text{if } s \leq \frac{\rho'}{2} \end{cases}.$$

A direct calculation gives:

$$\sum_{i=1}^N \int_{B(z_i, \rho')} |\nabla \tilde{\varphi}_\rho|^2 = \mathcal{O} \left[\sum_{i=1}^N \int_0^{2\pi} |\partial_\theta \varphi_{\rho, \mathbf{z}, \mathbf{d}}(z_i + \rho' e^{i\theta})|^2 \right] = o_\rho(1). \quad (48)$$

Thus, letting $\Omega_{\rho', \mathbf{z}} = \omega \setminus \cup_{i=1}^N \overline{B(z_i, \rho')}$ and $\tilde{\mathcal{D}}_{\rho'} = \cup_{i=1}^N B(z_i, \rho') \setminus \overline{B(z_i, \rho)}$, we obtain:

$$\begin{aligned} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla \varphi_{\rho, \mathbf{z}, \mathbf{d}}|^2 &= \int_{\Omega_{\rho', \mathbf{z}}} |\nabla \tilde{\varphi}_\rho|^2 + \int_{\tilde{\mathcal{D}}_{\rho'}} |\nabla \varphi_{\rho, \mathbf{z}, \mathbf{d}}|^2 \\ &\geq \int_{\Omega_{\rho', \mathbf{z}}} |\nabla \tilde{\varphi}_\rho|^2 \\ &\stackrel{(48)}{=} \int_{\omega} |\nabla \tilde{\varphi}_\rho|^2 + o_\rho(1). \end{aligned} \quad (49)$$

Since $\tilde{\varphi}_\rho$ is bounded in $H^1(\omega)$, up to pass to a subsequence, we may assume the existence of $\tilde{\varphi}_0 \in H^1(\omega)$ s.t. $\tilde{\varphi}_\rho \rightharpoonup \tilde{\varphi}_0$ in $H^1(\omega)$.

On the other hand, it is clear that $\text{tr}_{\partial\omega} \tilde{\varphi}_0 = \text{tr}_{\partial\omega} \varphi_{0, \mathbf{z}, \mathbf{d}} = \phi_{0, \mathbf{z}, \mathbf{d}}$. Consequently from the Dirichlet principle we get [denoting $\rho_n = \rho(R_n)$]

$$\liminf_{\rho_n \rightarrow 0} \int_{\omega} |\nabla \tilde{\varphi}_\rho|^2 \geq \int_{\omega} |\nabla \tilde{\varphi}_0|^2 \geq \int_{\omega} |\nabla \tilde{\phi}_{0, \mathbf{z}, \mathbf{d}}|^2. \quad (50)$$

By combining (49) and (50) we obtain (47). From (40) and (47) we may write

$$\begin{aligned} \liminf_{\rho_n \rightarrow 0} \left\{ \frac{1}{2} \int_{\Omega_{\rho_n, \mathbf{z}}} |\nabla u_{R_n}|^2 - \frac{1}{2} \int_{\Omega_{\rho_n, \mathbf{z}}} |\nabla w_{\rho_n, \mathbf{z}, \mathbf{d}}|^2 \right\} &= \liminf_{\rho_n \rightarrow 0} \frac{1}{2} \int_{\Omega_{\rho_n, \mathbf{z}}} |\nabla \varphi_{\rho_n, \mathbf{z}, \mathbf{d}}|^2 \\ &\geq \frac{1}{2} \int_{\omega} |\nabla \tilde{\phi}_{0, \mathbf{z}, \mathbf{d}}|^2. \end{aligned} \quad (51)$$

6.5 Conclusion

Using (46), (51) and the definition of the sequence $(R_n)_n$ we get

$$\begin{aligned} &\liminf_{R \rightarrow \infty} \left\{ I(R, \rho, \mathbf{z}, \mathbf{d}) - \left(d^2 f(R) + \frac{b^2}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w_{\rho, \mathbf{z}, \mathbf{d}}|^2 \right) \right\} \\ &= \lim_{R_n \rightarrow \infty} \left\{ \frac{1}{2} \int_{\mathcal{D}_{R_n, \mathbf{z}}} \alpha |\nabla u_{R_n}|^2 - \left(d^2 f(R_n) + \frac{b^2}{2} \int_{\Omega_{\rho_n, \mathbf{z}}} |\nabla w_{\rho_n, \mathbf{z}, \mathbf{d}}|^2 \right) \right\} \\ &\geq \liminf_{R_n \rightarrow \infty} \left\{ \frac{1}{2} \int_{\Omega_{R_n}} \alpha |\nabla u_{R_n}|^2 - d^2 f(R_n) \right\} + b^2 \liminf_{\rho_n \rightarrow 0} \left\{ \frac{1}{2} \int_{\Omega_{\rho_n, \mathbf{z}}} |\nabla u_{R_n}|^2 - \frac{1}{2} \int_{\Omega_{\rho_n, \mathbf{z}}} |\nabla w_{\rho_n, \mathbf{z}, \mathbf{d}}|^2 \right\} \\ &\geq \inf_{\varphi \in \mathcal{H}_{\phi_\infty}} \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi|^2 + \frac{b^2}{2} \int_{\omega} |\nabla \tilde{\phi}_{0, \mathbf{z}, \mathbf{d}}|^2. \end{aligned} \quad (52)$$

Recall that $h_\infty = \left(\frac{x}{|x|} \right)^d e^{i(d\gamma_\infty + \phi_\infty)} \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ [see (45)]. Therefore from (35) and (44) we may write

$$\mathcal{K}(h_\infty) = \inf_{\varphi \in \mathcal{H}_{\phi_\infty}} \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi|^2 + \frac{b^2}{2} \int_{\omega} |\nabla \tilde{\phi}_{0, \mathbf{z}, \mathbf{d}}|^2.$$

Consequently (52) becomes

$$\liminf_{R \rightarrow \infty} \left\{ I(R, \rho, \mathbf{z}, \mathbf{d}) - \left(d^2 f(R) + \frac{b^2}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w_{\rho, \mathbf{z}, \mathbf{d}}|^2 \right) \right\} \geq \mathcal{K}(h_\infty). \quad (53)$$

It suffices now to see that, from Theorem 3 we have

$$\frac{1}{2} \int_{\Omega_{\rho, \mathbf{z}}} |\nabla w_{\rho, \mathbf{z}, \mathbf{d}}|^2 = \sum_i d_i^2 \pi |\ln \rho| + W(\mathbf{z}, \mathbf{d}) + o_\rho(1),$$

this combined with (36) gives

$$\lim_{R \rightarrow \infty} \left\{ I(R, \rho, \mathbf{z}, \mathbf{d}) - \left[f(R) + b^2 \left(\pi \sum_{i=1}^N d_i^2 |\ln \rho| + W(\mathbf{z}, \mathbf{d}) \right) \right] \right\} \text{ exists}$$

and

$$\lim_{R \rightarrow \infty} \left\{ I(R, \rho, \mathbf{z}, \mathbf{d}) - \left[f(R) + b^2 \left(\pi \sum_{i=1}^N d_i^2 |\ln \rho| + W(\mathbf{z}, \mathbf{d}) \right) \right] \right\} = \mathcal{K}(h_\infty).$$

We now define:

$$W^{\text{micro}}(\mathbf{z}, \mathbf{d}) := b^2 W(\mathbf{z}, \mathbf{d}) + \min_{\substack{h \in H^{1/2}(\partial\omega, \mathbb{S}^1) \\ \deg(h) = d}} \mathcal{K}(h) \quad (54)$$

in order to write

$$I(R, \rho, \mathbf{z}, \mathbf{d}) = d^2 f(R) + b^2 \pi \sum_{i=1}^N d_i^2 |\ln \rho| + W^{\text{micro}}(\mathbf{z}, \mathbf{d}) + o_\rho(1).$$

The last equality ends the proof of Theorem 1.

7 The case of the radially symmetric diluted impurity: $\omega = \mathbb{D}$

In this section we focus on the circular case with $\omega = \mathbb{D}$ is the unit disc and for $b \in (0, \infty)$ we let

$$\alpha : \mathbb{R}^2 \rightarrow \begin{cases} \{b^2; 1\} \\ x \mapsto \begin{cases} b^2 & \text{if } x \in \mathbb{D} \\ 1 & \text{if } x \in \mathbb{R}^2 \setminus \mathbb{D} \end{cases} \end{cases}.$$

We fix

- $N \in \mathbb{N}^*$, $\mathbf{d} = (d_1, \dots, d_N) \in \mathbb{Z}^N$ and we let $d := \sum_{i=1}^N d_i \in \mathbb{Z}$;
- $\mathbf{z} \in (\mathbb{D}^N)^* := \{(z_1, \dots, z_N) \in \mathbb{D}^N \mid z_i \neq z_j \text{ for } i \neq j\}$.

7.1 Explicit expression of the special solutions

We use the same notation as in Section 4.

Notation 25. In this section and in the next sections, in order to keep notation simple, we use the shorthand "x" to stand the identity map. Namely we use the abuse of notation $\text{Id} = x$ where $\text{Id} : U \rightarrow U$, $x \mapsto \text{Id}(x) = x$ and $U \subset \mathbb{R}^2 \simeq \mathbb{C}$ is an arbitrary set .

We let v_∞ be the limiting function obtained in Corollary 14. It is easy to prove that $v_\infty(x) = \frac{x}{|x|}$, i.e. $\gamma_\infty \equiv 0$.

We let $w_{0, \mathbf{z}, \mathbf{d}} = \prod_{i=1}^N \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} e^{i\gamma_{0, \mathbf{z}, \mathbf{d}}}$ be the function defined in (15). This function is the canonical harmonic map in \mathbb{D} associated to the singularities (\mathbf{z}, \mathbf{d}) .

On the unit circle \mathbb{S}^1 we have $\text{tr}_{\mathbb{S}^1}(w_{0,\mathbf{z},\mathbf{d}}) = e^{i\psi_{0,\mathbf{z},\mathbf{d}}}$ with

$$\partial_\tau \psi_{0,\mathbf{z},\mathbf{d}} = \partial_\nu \left[\sum_{j=1}^N d_j (\ln |x - z_j| - \ln |1 - \bar{z}_j x|) \right].$$

This result comes from [LM14] Eq. (2.25) and (4.1). From Identity (4.14) in [LM14] we get

$$\partial_\tau \psi_{0,\mathbf{z},\mathbf{d}} = \sum_{j=1}^N d_j [2\partial_\nu (\ln |x - z_j|) - 1].$$

Thus

$$\partial_\tau \psi_{0,\mathbf{z},\mathbf{d}} = \sum_{j=1}^N d_j [2\partial_\tau (\arg(x - z_j)) - 1] \text{ with } \frac{x - z_j}{|x - z_j|} = e^{i\arg(x - z_j)}.$$

Consequently we get

$$\text{tr}_{\mathbb{S}^1}(w_{0,\mathbf{z},\mathbf{d}}) = e^{i\psi_{0,\mathbf{z},\mathbf{d}}} = \text{Cst} \times x^{-d} \prod_{j=1}^N \left(\frac{x - z_j}{|x - z_j|} \right)^{2d_j} \quad (55)$$

where $\text{Cst} \in \mathbb{S}^1$ is a constant.

7.2 Use of Fourier decompositions

In order to get an explicit expression of $W^{\text{micro}}(\mathbf{z}, \mathbf{d})$ it seems natural to work on \mathcal{K} . For $h \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ we have [see (20) and (35)]

$$\mathcal{K}(h) = \inf_{\varphi \in \mathcal{H}_{\phi_\infty^h}} \frac{1}{2} \int_{\Omega_\infty} |\nabla \varphi|^2 + \frac{b^2}{2} \int_{\omega} |\nabla \tilde{\phi}_0^h|^2,$$

where:

- on the unit circle we have

$$h = x^d e^{i\phi_\infty^h} = w_{0,\mathbf{z},\mathbf{d}} e^{i\phi_0^h} \text{ with } \int_{\partial\omega} \phi_\infty^h, \int_{\partial\omega} \phi_0^h \in (-\pi, \pi]; \quad (56)$$

- $\tilde{\phi}_0^h$ is the harmonic extension of ϕ_0^h in \mathbb{D} .

Condition (56) is a compatibility condition between the function ϕ_∞^h and ϕ_0^h . Since our goal is to estimate $\mathcal{K}(h)$, it is clear that we may slightly modify Condition (56) by imposing

$$\left(\frac{x}{|x|} \right)^d e^{i\phi_\infty^h} = \text{Cst} \times w_{0,\mathbf{z},\mathbf{d}} e^{i\phi_0^h} \text{ with } \text{Cst} \in \mathbb{S}^1. \quad (57)$$

We may easily prove that

$$\inf_{\varphi \in \mathcal{H}_{\phi_\infty^h}} \frac{1}{2} \int_{\Omega_\infty} |\nabla \varphi|^2 = \frac{1}{2} \int_{\Omega_\infty} |\nabla \hat{\phi}_\infty^h|^2$$

where for $\phi \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$, $\hat{\phi} \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \overline{\mathbb{D}})$ is the unique solution of

$$\begin{cases} -\Delta \varphi = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\mathbb{D}} \\ \text{tr}_{\mathbb{S}^1}(\varphi) = \phi, \quad \nabla \varphi \in L^2(\mathbb{R}^2 \setminus \overline{\mathbb{D}}) \end{cases}.$$

[See Proposition 26 for more details about $\hat{\phi}$]

From (55), an equivalent reformulation of (57) is

$$\text{Cst} \prod_{j=1}^N \left(\frac{x - z_j}{|x - z_j| \times x} \right)^{2d_j} = e^{i(\phi_\infty^h - \phi_0^h)} \text{ with } \text{Cst} \in \mathbb{S}^1.$$

The above condition is equivalent to the compatibility condition:

$$\phi_\infty^h - \phi_0^h = \Psi_{\mathbf{z}, \mathbf{d}} + \text{Cst} \text{ where } \text{Cst} \in \mathbb{R} \quad (58)$$

and $\Psi_{\mathbf{z}, \mathbf{d}} \in C^\infty(\mathbb{S}^1, \mathbb{R})$ is a lifting of $\prod_{j=1}^N \left(\frac{x - z_j}{|x - z_j| \times x} \right)^{2d_j}$.

With a direct calculation, for $z_0 \in \mathbb{D}$ and $x \in \mathbb{S}^1$, we have

$$\left(\frac{x - z_0}{|x - z_0| x} \right)^2 = \frac{x - z_0}{x - z_0 \times x^2} = \frac{x - z_0}{1 - \bar{z}_0 x} \times \frac{1}{x} = M_{z_0}(x) \times \frac{1}{x}$$

where $M_{z_0} : \mathbb{D} \rightarrow \mathbb{D}$ is the Moebius function defined by $M_{z_0}(x) = \frac{x - z_0}{1 - \bar{z}_0 x}$.

In [Dos15], it is proved [Section 7] that if $z_0 \in \mathbb{D} \cap \mathbb{R}^+$ then for $e^{i\theta} \in \mathbb{S}^1$

$$M_{z_0}(e^{i\theta})e^{-i\theta} = e^{\Psi_{z_0,1}(e^{i\theta})} \text{ where } \Psi_{z_0,1}(e^{i\theta}) = \sum_{n \in \mathbb{Z}^*} \frac{z_0^{|n|}}{in} e^{in\theta} + \text{Cst}, \text{ Cst} \in \mathbb{R}.$$

In the general case $z_0 = te^{i\gamma} \in \mathbb{D}$ [with $t \geq 0, \gamma \in \mathbb{R}$] we easily deduce from the previous equality:

$$M_{z_0}(e^{i\theta})e^{-i\theta} = M_t[e^{i(\theta-\gamma)}]e^{-i(\theta-\gamma)}.$$

Then

$$\begin{aligned} \Psi_{z_0,1}(e^{i\theta}) &= \Psi_{t,1}(e^{i(\theta-\gamma)}) + \text{Cst} \\ &= \sum_{n \in \mathbb{Z}^*} \frac{t^{|n|}}{in} e^{in(\theta-\gamma)} + \text{Cst} \\ &= \sum_{n \in \mathbb{N}^*} \left[\frac{\bar{z}_0^n}{in} e^{in\theta} - \frac{z_0^n}{in} e^{-in\theta} \right] + \text{Cst}, \text{ Cst} \in \mathbb{R}. \end{aligned}$$

It is easy to prove that we have $\Psi_{\mathbf{z}, \mathbf{d}} = \sum_{j=1}^N d_j \Psi_{z_j,1} + \text{Cst}$ [Cst $\in \mathbb{R}$] and then

$$\Psi_{\mathbf{z}, \mathbf{d}}(e^{i\theta}) = \text{Cst} + \sum_{n \in \mathbb{N}^*} \sum_{j=1}^N d_j \left[\frac{\bar{z}_j^n}{in} e^{in\theta} - \frac{z_j^n}{in} e^{-in\theta} \right]. \quad (59)$$

We now go back to the previously fixed function $h \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$. We are in position to reformulate the compatibility condition (58) in term of Fourier series.

Let $\phi_0^h, \phi_\infty^h \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ [defined in (56)], consider their Fourier decompositions [we drop the superscript h for the coefficients]:

$$\phi_0^h(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_{0,n} e^{in\theta} \text{ and } \phi_\infty^h(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_{\infty,n} e^{in\theta}. \quad (60)$$

The compatibility condition (57) is equivalent to (58). From (59), the condition (58) reads with Fourier decompositions:

$$\forall n \in \mathbb{Z}^*, c_{\infty,n} - c_{0,n} = \begin{cases} \sum_{j=1}^N d_j \frac{\bar{z}_j^n}{in} & \text{if } n > 0 \\ -\sum_{j=1}^N d_j \frac{z_j^n}{in} & \text{if } n < 0 \end{cases}.$$

7.3 Explicit expression of the minimal value of \mathcal{K}

Before going further we recall some basic facts.

Proposition 26. *Let $\phi \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ and consider $\phi(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$ be its Fourier decomposition.*

Then we have

1. For all $n \in \mathbb{N}$, $c_n = \overline{c_{-n}}$.
2. $\sum_{n \in \mathbb{Z}} |n| |c_n|^2 < \infty$.
3. The map $\tilde{\phi} : \mathbb{D} \rightarrow \mathbb{R}$, $re^{i\theta} \mapsto \sum_{n \in \mathbb{Z}} c_n r^{|n|} e^{in\theta}$ is the harmonic extension of ϕ . Moreover

$$\frac{1}{2} \int_{\mathbb{D}} |\nabla \tilde{\phi}|^2 = \pi \sum_{n \in \mathbb{Z}} |n| |c_n|^2.$$
4. The map $\hat{\phi} : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \rightarrow \mathbb{R}$, $re^{i\theta} \mapsto \sum_{n \in \mathbb{Z}} c_n r^{-|n|} e^{in\theta}$ is an exterior harmonic extension of ϕ .
 Moreover $\frac{1}{2} \int_{\mathbb{R}^2 \setminus \overline{\mathbb{D}}} |\nabla \hat{\phi}|^2 = \pi \sum_{n \in \mathbb{Z}} |n| |c_n|^2.$

5. $\hat{\phi}$ is the unique solution of

$$\begin{cases} -\Delta \varphi = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\mathbb{D}}, \\ \varphi \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \mathbb{D}, \mathbb{R}) \\ \text{tr}_{\mathbb{S}^1}(\varphi) = \phi, \nabla \varphi \in L^2(\mathbb{R}^2 \setminus \overline{\mathbb{D}}, \mathbb{R}^2) \end{cases}. \quad (61)$$

Therefore it is also the unique solution of the problem

$$\inf_{\varphi \in \mathcal{H}_\phi} \frac{1}{2} \int_{\mathbb{R}^2 \setminus \overline{\mathbb{D}}} |\nabla \varphi|^2. \quad (62)$$

Proof. Assertions 1 and 2 are quite standard. Assertions 3 and 4 follow from standard calculations.

We now prove Assertion 5. Let $\phi \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ and let $\hat{\phi}$ be defined by Assertion 4. It is clear that $\hat{\phi}$ solves (61). Assume that φ_0 is a solution of (61) and let $\eta := \hat{\phi} - \varphi_0$. Then η satisfies:

$$\begin{cases} -\Delta \eta = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\mathbb{D}}, \\ \eta \in H_{\text{loc}}^1(\mathbb{R}^2 \setminus \mathbb{D}, \mathbb{R}) \\ \text{tr}_{\mathbb{S}^1}(\eta) = 0, \nabla \eta \in L^2(\mathbb{R}^2 \setminus \overline{\mathbb{D}}, \mathbb{R}^2) \end{cases}.$$

From [SS96] [Theorem II.6.2.ii] we get $\eta = 0$. This clearly gives the uniqueness of the solution of (61).

On the one hand, by direct minimization we know that Problem (62) admits solution(s). It is standard to check that a minimizer for (62) solves (61). Consequently $\hat{\phi}$ is the unique solution of Problem (62). \square

Notation 27. From now on, for $\phi \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ with Fourier decomposition $\phi(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$, we let the semi-norm $|\phi|_{H^{1/2}} := \sqrt{\pi \sum_{n \in \mathbb{Z}} |n| |c_n|^2}$.

For $n \in \mathbb{N}^*$, letting $\gamma_n = \sum_{j=1}^N d_j \frac{\bar{z}_j^n}{in}$, i.e. $\Psi_{\mathbf{z}, \mathbf{d}}(e^{i\theta}) = \text{Cst} + \sum_{n \in \mathbb{Z}^*} \gamma_n e^{in\theta}$ [see (59)], we get

$$\begin{aligned}
\inf_{\substack{h \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1) \\ \deg(h)=d}} \mathcal{K}(h) &= \inf_{\substack{\phi_0, \phi_\infty \in H^{1/2}(\mathbb{S}^1, \mathbb{R}) \\ x^d e^{i\phi_\infty} = \text{Cst} \times w_0, \mathbf{z}, \mathbf{d} e^{i\phi_0}}} \left(\frac{1}{2} \int_{\Omega_\infty} |\nabla \hat{\phi}_\infty|^2 + \frac{b^2}{2} \int_w |\nabla \tilde{\phi}_0|^2 \right) \\
&= 2\pi \inf_{\substack{(c_0, n)_{n \in \mathbb{N}^*}, (c_\infty, n) \in \ell^2(\mathbb{N}^*) \\ c_\infty, n - c_0, n = \gamma_n \forall n \in \mathbb{N}^*}} \left(\sum_{n \in \mathbb{N}} n |c_{0, n}|^2 + b^2 \sum_{n \in \mathbb{N}} n |c_{\infty, n}|^2 \right) \\
&= 2\pi \sum_{n \in \mathbb{N}^*} \left[n \times \inf_{\substack{c_0, n, c_\infty, n \in \mathbb{C} \\ c_\infty, n - c_0, n = \gamma_n}} (|c_{0, n}|^2 + b^2 |c_{\infty, n}|^2) \right] \\
&= 2\pi \sum_{n \in \mathbb{N}^*} \left[n \times \inf_{c_0, n \in \mathbb{C}} (|c_{0, n}|^2 + b^2 |c_0, n + \gamma_n|^2) \right] \\
&= 2\pi \sum_{n \in \mathbb{N}^*} \left[n \times \left(\left| \frac{-b^2}{1+b^2} \gamma_n \right|^2 + b^2 \left| \frac{-b^2}{1+b^2} \gamma_n + \gamma_n \right|^2 \right) \right] \\
&= \frac{b^2}{1+b^2} 2\pi \sum_{n \in \mathbb{N}^*} n |\gamma_n|^2 = \frac{b^2}{1+b^2} |\Psi_{\mathbf{z}, \mathbf{d}}|_{H^{1/2}}^2. \tag{63}
\end{aligned}$$

7.4 Explicit expression of W^{micro} : Proof of Proposition 2

We first recall the expression of $W(\mathbf{z}, \mathbf{d})$ [see Proposition 1 in [LR96]]:

$$W(\mathbf{z}, \mathbf{d}) = -\pi \sum_{i \neq j} d_i d_j \ln |z_i - z_j| + \pi \sum_{i=1}^N d_i^2 \ln(1 - |z_i|^2) + \pi \sum_{i \neq j} d_i d_j \ln |1 - z_i \bar{z}_j|.$$

From (54) we have $W^{\text{micro}}(\mathbf{z}, \mathbf{d}) = b^2 W(\mathbf{z}, \mathbf{d}) + \min_{\substack{h \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1) \\ \deg(h)=d}} \mathcal{K}(h)$.

By combining (59) and (63) we may write

$$\min_{\substack{h \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1) \\ \deg(h)=d}} \mathcal{K}(h) = \frac{2b^2}{1+b^2} \pi \sum_{n \in \mathbb{N}^*} n \left| \sum_{j=1}^N d_j \frac{\bar{z}_j^n}{in} \right|^2 = \frac{2b^2}{1+b^2} \pi \sum_{n \in \mathbb{N}^*} \frac{1}{n} \left| \sum_{j=1}^N d_j z_j^n \right|^2.$$

For $n \in \mathbb{N}^*$ we have the following expanding

$$\left| \sum_{j=1}^N d_j z_j^n \right|^2 = \sum_{j=1}^N d_j^2 |z_j|^{2n} + 2 \text{Re} \left[\sum_{i < j} d_i d_j (z_i \bar{z}_j)^n \right].$$

Therefore we obtain

$$\begin{aligned}
\sum_{n \in \mathbb{N}^*} \frac{1}{n} \left| \sum_{j=1}^N d_j z_j^n \right|^2 &= \sum_{j=1}^N d_j^2 \left(\sum_{n \in \mathbb{N}^*} \frac{1}{n} |z_j|^{2n} \right) + 2 \sum_{i < j} d_i d_j \text{Re} \left[\sum_{n \in \mathbb{N}^*} \frac{1}{n} (z_i \bar{z}_j)^n \right] \\
&= -\sum_{j=1}^N d_j^2 \ln(1 - |z_j|^2) - 2 \sum_{i < j} d_i d_j \text{Re} [\ln(1 - z_i \bar{z}_j)] \\
&= -\sum_{j=1}^N d_j^2 \ln(1 - |z_j|^2) - \sum_{i \neq j} d_i d_j \ln |1 - z_i \bar{z}_j|.
\end{aligned}$$

We may thus conclude:

$$\begin{aligned}
W^{\text{micro}}(\mathbf{z}, \mathbf{d}) &= b^2\pi \left[-\sum_{i \neq j} d_i d_j \ln |z_i - z_j| + \sum_{i=1}^N d_i^2 \ln(1 - |z_i|^2) + \sum_{i \neq j} d_i d_j \ln |1 - z_i \overline{z_j}| - \right. \\
&\quad \left. -\frac{2}{1+b^2} \left(\sum_{j=1}^N d_j^2 \ln(1 - |z_j|^2) + \sum_{i \neq j} d_i d_j \ln |1 - z_i \overline{z_j}| \right) \right] \\
&= -b^2\pi \left[\sum_{i \neq j} d_i d_j \ln |z_i - z_j| + \frac{1-b^2}{1+b^2} \sum_{j=1}^N d_j^2 \ln(1 - |z_j|^2) + \frac{1-b^2}{1+b^2} \sum_{i \neq j} d_i d_j \ln |1 - z_i \overline{z_j}| \right].
\end{aligned}$$

These calculations end the proof of Proposition 2.

7.5 Minimization of W^{micro} in some particular cases

We first claim that if $\mathbf{d} = \mathbf{0}_{\mathbb{Z}^N}$ then $W^{\text{micro}}(\cdot, \mathbf{d}) \equiv 0$. In the following we consider $\mathbf{d} \in \mathbb{Z}^N \setminus \{\mathbf{0}_{\mathbb{Z}^N}\}$.

7.5.1 The case $N = 1$ and the case $N \geq 2 \& \exists! k_0 \in \{1, \dots, N\}$ s.t. $d_{k_0} \neq 0$

We first treat the case $N = 1$. In this situation, we have for $z \in \mathbb{D}$ and $d \in \mathbb{Z}^*$:

$$W^{\text{micro}}(z, d) = -\frac{b^2(1-b^2)}{1+b^2} \pi d^2 \ln(1 - |z|^2)$$

Therefore, if $b < 1$ then $z = 0$ is the unique minimizer of W^{micro} .

Remark 28. This simple fact is the main result of [Dos15] [where the explicit expression of W^{micro} was unknown].

If $b = 1$ then $W^{\text{micro}}(\cdot, d) \equiv 0$.

If $b > 1$ then $W^{\text{micro}}(z, d) \rightarrow -\infty$ when $|z| \rightarrow 1$. This implies that $W^{\text{micro}}(\cdot, d)$ does not admit minimizers.

Remark 29. We may conclude that the condition $b < 1$ creates a confinement effect for the points of minimum of $W^{\text{micro}}(\cdot, d)$. This confinement effect does not hold for $b \geq 1$.

We now consider the case $N \geq 2$. We assume that $d_1 \neq 0$ and $d_l = 0$ for $l \neq 1$.

This situation is similar to the above one since for $\mathbf{z} = (z_1, \dots, z_N) \in (\omega^N)^*$ we have $W^{\text{micro}}(\mathbf{z}, \mathbf{d}) = W^{\text{micro}}(z_1, d_1)$. Consequently as previously we have:

- If $b < 1$ then the set of global minimizers of W^{micro} is $\{\mathbf{z} \in (\omega^N)^* \mid z_1 = 0\}$.
- If $b = 1$ then $W^{\text{micro}}(\cdot, \mathbf{d}) \equiv 0$.
- If $b > 1$ then $W^{\text{micro}}(\mathbf{z}, \mathbf{d}) \rightarrow -\infty$ when $|z_1| \rightarrow 1$.

7.5.2 The case $N \geq 2$ and there exist k, l s.t. $d_k d_l < 0$

Let $\mathbf{d} \in \mathbb{Z}^N$ s.t. there exist $k \neq l$ satisfying $d_k d_l < 0$. In this situation we have

$$\inf_{\mathbf{z} \in (\omega^N)^*} W^{\text{micro}}(\mathbf{z}, \mathbf{d}) = -\infty.$$

Indeed, without loss of generality, we may assume that $d_1 d_2 < 0$. For $n \in \mathbb{N}^*$, we consider $z_1^{(n)} := -1/n$, $z_2^{(n)} := 1/n$ and for $k \in \{1, \dots, N\} \setminus \{1, 2\}$, $z_k := e^{i2k\pi/N}/2$.

With direct calculations, we obtain $\lim_n W(\mathbf{z}_n, \mathbf{d}) = -\infty$.

Remark 30. This fact underline that if we impose $d_1 d_2 < 0$ then the main part of the *optimal* energy $I(R, \rho, \mathbf{z}, \mathbf{d})$ is not

$$\left(\sum_{i=1}^N d_i \right)^2 f(R) + b^2 \pi \sum_{i=1}^N d_i^2 |\ln \rho|.$$

Indeed when we consider *very near* singularities z_1 & z_2 we may optimize the divergent term $b^2 \left(\sum_{i=1}^N d_i^2 \right) |\ln \rho|$. The key argument is that with degrees having different signs (*e.g* $d_1 d_2 < 0$) we have

$$\sum_{i=1}^N d_i^2 > (d_1 + d_2)^2 + \sum_{i=3}^N d_i^2.$$

This is an example of the standard attractive effect of singularities having degrees with different signs.

7.5.3 The case $b = 1$, $N \geq 2$, $d_k d_l \geq 0 \forall k, l$ and there exist k_0, l_0 s.t. $d_{k_0} d_{l_0} > 0$

When $b = 1$, for $(\mathbf{z}, \mathbf{d}) \in (\omega^N)^* \times \mathbb{Z}^N$ we have $W^{\text{micro}}(\mathbf{z}, \mathbf{d}) = -\pi \sum_{i \neq j} d_i d_j \ln |z_i - z_j|$. Thus

$$\inf_{\mathbf{z} \in (\omega^N)^*} W^{\text{micro}}(\mathbf{z}, \mathbf{d}) > -\infty$$

but the lower bound is not attained. Indeed, it is easy to check that for $\mathbf{z} \in (\omega^N)^*$

$$\inf_{\mathbf{z} \in (\omega^N)^*} W^{\text{micro}}(\mathbf{z}, \mathbf{d}) > -\pi \sum_{i \neq j} d_i d_j \ln 2.$$

Consequently $W^{\text{micro}}(\cdot, \mathbf{d})$ is bounded from below. We now prove that the lower bound is not reached. Let $\mathbf{z} \in (\omega^N)^*$, and consider $\tilde{\mathbf{z}} \in (\omega^N)^*$ be s.t. $\tilde{z}_k = \lambda z_k$ with $\lambda := \frac{2}{1 + \max_l |z_l|}$. It is easy to check that $\tilde{\mathbf{z}} \in (\omega^N)^*$. Moreover we get $W^{\text{micro}}(\tilde{\mathbf{z}}, \mathbf{d}) = W^{\text{micro}}(\mathbf{z}, \mathbf{d}) - \pi \ln \lambda \sum_{i \neq j} d_i d_j$.

Since $\lambda > 1$, we have $W^{\text{micro}}(\tilde{\mathbf{z}}, \mathbf{d}) < W^{\text{micro}}(\mathbf{z}, \mathbf{d})$. This fact implies that the lower bound is not reached.

Remark 31. When $b = 1$, the impurity $\omega = \mathbb{D}$ does not play any role. Then, due to the standard repulsion effect between vortices, the more the vortices are distant the smaller the energy. Consequently, for fixed degrees having all the same sign, minimal sequences of singularities go to the boundary of the impurity which is not an admissible configuration in this framework.

7.5.4 The case $b > 1$ and $N \geq 2$

If $b > 1$ then taking, for $n \in \mathbb{N}^*$ and $k \in \{1, \dots, N\}$, $z_k^{(n)} := (1 - 1/n)e^{i2\pi k/N}$ we have

$$W^{\text{micro}}(\mathbf{z}_n, \mathbf{d}) = \mathcal{O}(1) + \frac{b^2 - 1}{1 + b^2} \sum_{j=1}^N d_j^2 \ln(1 - |z_j^{(n)}|^2) \rightarrow -\infty \text{ when } n \rightarrow \infty.$$

Remark 32. The case $b > 1$ corresponds to an impurity $\omega = \mathbb{D}$ which have a repulsive effect on the singularities.

7.5.5 The case $0 < b < 1$, $N = 2$ and $\mathbf{d} \in (\mathbb{N}^*)^2$

This situation is the most challenging. Note that with the help of [DM11] we may obtain the existence of minimizers for $W^{\text{micro}}(\cdot, \mathbf{d})$ with $d_i = 1$ for $i \in \{1, \dots, N\}$, $N \in \mathbb{N}^*$. But [DM11] does not give any information on the location of minimizers and for other configurations of degrees.

From technical issues, we restrict the study to $N = 2$ and $p, q \in \mathbb{N}^*$. Note that the case $p, q < 0$ is obviously symmetric.

We are going to prove that there exist minimizers and they are unique up to a rotation [see (69)&(70)].

We may assume $p \leq q$. For $z_1, z_2 \in \mathbb{D}$ we have, writing $(\mathbf{z}, \mathbf{d}) = ((z_1, p), (z_2, q))$

$$\frac{W^{\text{micro}}(\mathbf{z}, \mathbf{d})}{-b^2\pi} = 2pq \ln |z_1 - z_2| + \frac{1-b^2}{1+b^2} [p^2 \ln(1-|z_1|^2) + q^2 \ln(1-|z_2|^2) + 2pq \ln |1 - z_1 \bar{z}_2|].$$

We let:

- $\mathcal{B} := \frac{1-b^2}{1+b^2}$ and $\mathcal{A} := \frac{p}{q} \leq 1$;
- $f(z_1, z_2) = 2 \ln |z_1 - z_2| + \mathcal{B} [\mathcal{A} \ln(1-|z_1|^2) + \mathcal{A}^{-1} \ln(1-|z_2|^2) + 2 \ln |1 - z_1 \bar{z}_2|]$.

Note that $W^{\text{micro}}[(z_1, z_2), (p, q)] = -b^2 pq \pi f(z_1, z_2)$. Consequently, in order to study minimizing points of $W^{\text{micro}}[\cdot, (p, q)]$, we have to maximize $f(\cdot)$. We first claim that if either $|z_1| \rightarrow 1$ or $|z_2| \rightarrow 1$ or $|z_1 - z_2| \rightarrow 0$, then $f(z_1, z_2) \rightarrow -\infty$. Consequently, from the continuity of f , f admits maximum points in $(\mathbb{D}^2)^*$.

Since $z_1 \neq z_2$ and since for $t \in \mathbb{R}$ we have $f(z_1, z_2) = f(z_1 e^{it}, z_2 e^{it})$, we may assume that $z_1 = s \geq 0$. We thus have for $z_2 = \rho e^{i\theta}$ [$0 \leq \rho < 1, \theta \in \mathbb{R}$]

$$f(s, \rho e^{i\theta}) = \ln [s^2 + \rho^2 - 2s\rho \cos \theta] + \mathcal{B} [\mathcal{A} \ln(1-s^2) + \mathcal{A}^{-1} \ln(1-\rho^2) + \ln(1+s^2\rho^2 - 2s\rho \cos \theta)].$$

We first claim that if $s = 0$ then $\rho > 0$ and for $\varepsilon > 0$ we have

$$f(\varepsilon, -\rho) = f(0, \rho e^{i\theta}) + \varepsilon(\rho^{-1} + 2\beta\rho) + \mathcal{O}(\varepsilon^2).$$

Consequently, for $\varepsilon > 0$ sufficiently small we have $f(\varepsilon, -\rho) > f(0, \rho e^{i\theta})$. Therefore, if $(s, \rho e^{i\theta})$ maximizes f , then $s \in (0; 1)$. Using a similar argument, we may prove that for $s > 0$, if $(s, \rho e^{i\theta})$ maximizes f , then $\rho \in (0; 1)$.

On the other hand, from direct checking, for $s, \rho > 0$, the map $\theta \in [0, 2\pi] \mapsto f(s, \rho e^{i\theta})$ is maximal if and only if $\theta = \pi$.

Consequently, we focus on the map

$$g: (0; 1)^2 \rightarrow \mathbb{R} \\ (s, t) \mapsto f(s, -t) = 2 \ln (s+t) + \mathcal{B} [\mathcal{A} \ln(1-s^2) + \mathcal{A}^{-1} \ln(1-t^2) + 2 \ln(1+st)].$$

We first look for critical points of g :

$$\begin{aligned} \nabla g(s, t) = \mathbf{0} &\Leftrightarrow \begin{cases} \frac{1}{s+t} + \mathcal{B} \left(\frac{-\mathcal{A}s}{1-s^2} + \frac{t}{1+st} \right) = 0 \\ \frac{1}{s+t} + \mathcal{B} \left(\frac{-\mathcal{A}^{-1}t}{1-t^2} + \frac{s}{1+st} \right) = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} (1-s^2)(1+st) + \mathcal{B} [-\mathcal{A}s(1+st)(s+t) + t(1-s^2)(s+t)] = 0 \\ (1-t^2)(1+st) + \mathcal{B} [-\mathcal{A}^{-1}t(1+st)(s+t) + s(1-t^2)(s+t)] = 0 \end{cases} \end{aligned} \quad (64)$$

By considering the difference of both lines in (64) we get:

$$\begin{aligned} &(t^2 - s^2)(1+st) + \mathcal{B} [(\mathcal{A}^{-1}t - \mathcal{A}s)(1+st)(s+t) + (t - s^2t - s + st^2)(s+t)] = 0 \\ \Leftrightarrow &(1+st)(s+t) [t - s + \mathcal{B}((\mathcal{A}^{-1} + 1)t - (\mathcal{A} + 1)s)] = 0 \\ \stackrel{[s, t > 0]}{\Leftrightarrow} &[1 + \mathcal{B}(\mathcal{A}^{-1} + 1)]t - [1 + \mathcal{B}(\mathcal{A} + 1)]s = 0 \\ \Leftrightarrow &t = \lambda s \text{ with } \lambda := \frac{1 + \mathcal{B}(\mathcal{A} + 1)}{1 + \mathcal{B}(\mathcal{A}^{-1} + 1)}. \end{aligned} \quad (65)$$

Remark 33. It is important to note that $0 < \lambda \leq 1$. Moreover $\lambda = 1$ if and only if $p = q$.

Using (65) in the first line of (64) we have

$$(1 - s^2)(1 + \lambda s^2) + \mathcal{B} [-\mathcal{A}s^2(1 + \lambda s^2)(1 + \lambda) + \lambda s^2(1 - s^2)(1 + \lambda)] = 0. \quad (66)$$

Thus, letting $\sigma = s^2$, we get the following equation:

$$[\lambda + (\mathcal{A} + 1)\mathcal{B}\lambda(1 + \lambda)]\sigma^2 + [1 - \lambda + (\mathcal{A} - \lambda)\mathcal{B}(1 + \lambda)]\sigma - 1 = 0. \quad (67)$$

We let $\Delta := [1 - \lambda + (\mathcal{A} - \lambda)\mathcal{B}(1 + \lambda)]^2 + 4[\lambda + (\mathcal{A} + 1)\mathcal{B}\lambda(1 + \lambda)]$. Note that $\Delta > 0$ and $\sqrt{\Delta} > 1 - \lambda + (\mathcal{A} - \lambda)\mathcal{B}(1 + \lambda)$.

We obtain immediately that

$$\sigma_0 = \frac{-[1 - \lambda + (\mathcal{A} - \lambda)\mathcal{B}(1 + \lambda)] + \sqrt{\Delta}}{2[\lambda + (\mathcal{A} + 1)\mathcal{B}\lambda(1 + \lambda)]} \quad (68)$$

is the unique positive solution of (67).

Consequently

$$s_0 = \sqrt{\frac{-[1 - \lambda + (\mathcal{A} - \lambda)\mathcal{B}(1 + \lambda)] + \sqrt{\Delta}}{2[\lambda + (\mathcal{A} + 1)\mathcal{B}\lambda(1 + \lambda)]}} \quad (69)$$

is the unique positive solution of (66).

In conclusion, the set of global minimizers of $W^{\text{micro}}[\cdot, (p, q)]$ is

$$\{(s_0 e^{i\theta}; -\lambda s_0 e^{i\theta}) \in (\mathbb{D}^2)^* \mid \theta \in \mathbb{R}\} \quad (70)$$

where s_0 is given by (69) and λ by (65).

Remark 34. It is interesting to note that if $((z_1, z_2), (p, q)) \in (\mathbb{D}^2)^* \times (\mathbb{N}^*)^2$ is a minimizers for W^{micro} , then we have:

$$|z_1| \leq |z_2| \iff p \geq q$$

and

$$|z_1| = |z_2| \iff p = q.$$

A Proof of Lemma 12

The key ingredient to get Lemma 12 is Proposition C.4 in [Dos13] previously proved for $W^{2,\infty}$ weights by Sauvageot in [Sau11] [in fact Sauvageot's article treats the anisotropic case which is more general than Proposition 35 below].

For the convenience of the reader we state this proposition:

Proposition 35. [Proposition C.4 in [Dos13]]

Let $\alpha \in L^\infty(\mathbb{R}^2, [B^2; B^{-2}])$ and $R > r > 0$ we denote:

- $\mu^{\text{Dir}}(B_R \setminus \overline{B_r}) := \inf \left\{ \frac{1}{2} \int_{B_R \setminus \overline{B_r}} \alpha |\nabla w|^2 \mid \begin{array}{l} w \in H^1(B_R \setminus \overline{B_r}, \mathbb{S}^1) \text{ s.t. } w(re^{i\theta}) = e^{i\theta}, \\ w(Re^{i\theta}) = e^{i(\theta + \theta_0)}, \theta_0 \in \mathbb{R} \end{array} \right\},$
- $\mu(B_R \setminus \overline{B_r}) := \inf \left\{ \frac{1}{2} \int_{B_R \setminus \overline{B_r}} \alpha |\nabla w|^2 \mid \begin{array}{l} w \in H^1(B_R \setminus \overline{B_r}, \mathbb{S}^1) \\ \text{s.t. } \deg(w) = 1 \end{array} \right\}.$

There exists a constant C_B depending only on B s.t.

$$\mu(B_R \setminus \overline{B_r}) \leq \mu^{\text{Dir}}(B_R \setminus \overline{B_r}) \leq \mu(B_R \setminus \overline{B_r}) + C_B.$$

Remark 36. In [Dos13], Proposition C.4, was initially stated for $\tilde{\alpha} \in L^\infty(\mathbb{R}^2, [b^2; 1])$ and $b \in (0; 1)$. Some obvious modifications allow to get the aforementioned formulation.

Lemma 12 is equivalent to

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_{R'}|^2 - \frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_R|^2 \leq C_{B,\omega}. \quad (71)$$

Recall that $R_0 := \max\{1; 10^2 \cdot \text{diam}(\omega)\}$, thus $\bar{\omega} \subset B_{R_0}$.

We let

$$C_\omega := \frac{1}{2} \int_{B_{R_0} \setminus \bar{\omega}} \left| \nabla \left(\frac{x}{|x|} \right) \right|^2. \quad (72)$$

It is easy to check, *e.g.* using the direct method of minimization, that the minima $\mu^{\text{Dir}}(B_{R'} \setminus \overline{B_R})$ and $\mu^{\text{Dir}}(B_R \setminus \overline{B_{R_0}})$ are reached. Let u_1 [resp. u_2] be a minimizer of $\mu^{\text{Dir}}(B_{R'} \setminus \overline{B_R})$ [resp. $\mu^{\text{Dir}}(B_R \setminus \overline{B_{R_0}})$]. Up to multiply u_1 by a constant rotation we may assume $\text{tr}_{\partial B_{R'}}(u_1) = \text{tr}_{\partial B_R}(u_2)$. We are now in position to define

$$u = \begin{cases} u_1 & \text{in } B_{R'} \setminus \overline{B_R} \\ u_2 & \text{in } B_R \setminus \overline{B_{R_0}} \\ \frac{x}{|x|} & \text{in } B_{R_0} \setminus \bar{\omega} \end{cases}.$$

It is clear that $u \in H^1(\Omega_{R'}, \mathbb{S}^1)$ and $\deg(u) = 1$. Consequently

$$\begin{aligned} \frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla u_{R'}|^2 &\leq \frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla u|^2 \\ &= \mu^{\text{Dir}}(B_{R'} \setminus \overline{B_R}) + \mu^{\text{Dir}}(B_R \setminus \overline{B_{R_0}}) + \frac{1}{2} \int_{B_{R_0} \setminus \bar{\omega}} \alpha \left| \nabla \left(\frac{x}{|x|} \right) \right|^2 \\ [\text{Prop. 35 \& Eq. (72)}] &\leq \mu(B_{R'} \setminus \overline{B_R}) + \mu(B_R \setminus \overline{B_{R_0}}) + 2C_B + B^{-2}C_\omega. \end{aligned}$$

Since $\mu(B_{R'} \setminus \overline{B_R}) \leq \frac{1}{2} \int_{B_{R'} \setminus \overline{B_R}} \alpha |\nabla u_{R'}|^2$ and $\mu(B_R \setminus \overline{B_{R_0}}) \leq \frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_R|^2$ we obtain:

$$\frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla u_{R'}|^2 \leq \frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_R|^2 + 2C_B + B^{-2}C_\omega.$$

Letting $C_{B,\omega} := 2C_B + B^{-2}C_\omega$ the above inequality is exactly (71).

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