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Explicit expression of the microscopic renormalized energy for a pinned Ginzburg-Landau functional

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Abstract

We get a new expression of the microscopic renormalized energy for a pinned Ginzburg-Landau type energy modeling small impurities. This is done by obtaining a sharp decomposition for the minimal energy of a Dirichlet type functional with an L^∞-weight.

In particular we get an explicit expression of the microscopic renormalized energy for a circular impurity. We proceed also to the minimization of this renormalized energy in some cases.

Keywords. Ginzburg-Landau type energy, pinning, renormalized energy
Mathematics Subject Classification (2000). 49K20 35J66 35J50

1 Introduction

1.1 Main results

The goal of this article is to give an explicit formula for a *microscopic renormalized energy* in the context of the study of a *pinned Ginzburg-Landau type energy.*

This renormalized energy allows to know the location of vorticity defects inside small impurities in an heterogenous superconductor. The microscopic renormalized energy may be defined *via* an auxiliary minimization problem involving unimodular maps.

The study of this auxiliary problem is the heart of this work. The main result of this article is the following theorem:

Theorem 1. *Let*

- $\omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ *be a smooth bounded simply connected open set s.t.* $0 \in \omega$,
- $N \in \mathbb{N}^*$ *and* $(\omega^N)^* := \{(z_1, ..., z_N) \in \omega^N \,|\, z_i \neq z_j \text{ for } i \neq j\}$,
- $B \in (0;1)$, $b \in [B; B^{-1}]$ *and* $\alpha \in L^\infty(\mathbb{R}^2, [B^2; B^{-2}])$ *be s.t.* $\alpha \equiv b^2$ *in* ω.

Then there exist

- $f :]R_0, \infty[\to \mathbb{R}^+$ *which satisfies* $B^2\pi\ln(R) \leq f(R) \leq B^{-2}\pi\ln(R)$ *[with* R_0 *sufficiently large],*
- $$W^{\text{micro}} : \begin{array}{ccc} (\omega^N)^* \times \mathbb{Z}^N & \to & \mathbb{R} \\ (\mathbf{z}, \mathbf{d}) & \mapsto & W^{\text{micro}}(\mathbf{z}, \mathbf{d}) \end{array}$$

s.t. for $\mathbf{d} = (d_1, ..., d_N) \in \mathbb{Z}^N$ *and* $\mathbf{z} = (z_1, ..., z_N) \in (\omega^N)^*$, *when* $R \to \infty$ *and* $\rho \to 0^+$, *we have*

$$\inf_{\substack{u \in H^1(B(0,R)\setminus \cup_i \overline{B(z_i,\rho)}, \mathbb{S}^1) \\ \deg_{\partial B(z_i,\rho)}(u) = d_i,\, i=1,\dots,N}} \left\{ \frac{1}{2}\int_{B(0,R)\setminus\overline{\omega}} \alpha|\nabla u|^2 + \frac{b^2}{2}\int_{\omega\setminus\cup_i \overline{B(z_i,\rho)}} |\nabla u|^2 \right\}$$
$$= \left(\sum_{i=1}^N d_i\right)^2 f(R) + b^2\left(\sum_{i=1}^N d_i^2\right)|\ln\rho| + W^{\text{micro}}(\mathbf{z}, \mathbf{d}) + o(1).$$

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[Note that the *degree* of a function is defined in Section 2.3].

Remark 1. 1. The expression of W^{micro} is given in (63). The map $W^{\text{micro}} : (\omega^N)^* \times \mathbb{Z}^N \to \mathbb{R}$ depends only on α, ω and N.

2. The function $f(\cdot)$ is defined by $f(R) := \inf_{\substack{v \in H^1(B_R \setminus \overline{\omega}, \mathbb{S}^1) \\ \deg(v)=1}} \frac{1}{2} \int_{B_R \setminus \overline{\omega}} \alpha |\nabla v|^2$.

In the circular case, *i.e.*, the set ω is the unit disk $\mathbb{D}$, we may obtain an explicit expression for W^{micro}.

Proposition 2. *If ω is the unit disk $\mathbb{D}$, then the microscopic renormalized energy with N vortices $(\mathbf{z}, \mathbf{d}) = \{(z_1, d_1), ..., (z_N, d_N)\}$ is*

$$W^{\text{micro}}(\mathbf{z}, \mathbf{d}) = -b^2 \pi \left[\sum_{i \neq j} d_i d_j \ln |z_i - z_j| + \frac{1-b^2}{1+b^2} \sum_{j=1}^{N} d_j^2 \ln(1 - |z_j|^2) + \frac{1-b^2}{1+b^2} \sum_{i \neq j} d_i d_j \ln |1 - z_i \overline{z_j}| \right].$$

Remark 3. Section 7 is dedicated to the case $\omega = \mathbb{D}$. Proposition 2 is proved Section 7.4. The minimization of the renormalized energy W^{micro} when ω is the unit disk is presented in some particular cases Section 7.5.

Theorem 1 may have several applications. For us, the main motivation appears in the study of a pinned Ginzburg-Landau type energy modeling a superconductor with impurities.

1.2 Motivations

Vorticity defects

The superconductivity phenomenon is an impressive property that appears on some materials called *superconductors*. When a superconductor is cooled below a critical temperature, it carries electric currents without dissipation [no electrical resistance] and expels magnetic fields from its body [Meissner effect].

But if the conditions imposed on the material are too strong [*e.g.* a strong magnetic field] then the superconductivity properties may be destroyed: the material has a classical behavior. According to the response of the superconductor to intense magnetic fields, essentially two kinds of superconductors are distinguished. The *type I* superconductors are those which pass abruptly to the superconducting state everywhere to the normal state everywhere. The *type II* superconductors admits an intermediate state called *mixed state*. Namely, for a type II superconductor, there exists intermediate critical fields $0 < H_{c_1} < H_{c_2}$ s.t. if the intensity of the applied field H is less than H_{c_1} then the superconductor is everywhere in the superconducting state. While if $H > H_{c_2}$, then the superconductor is everywhere in the normal state. For the intermediate regime $[H_{c_1} < H < H_{c_2}]$ there are "small" areas where the superconductivity is destroyed. While the rest of the sample is in a superconducting state. See [SS07] for a rigorous and quite complete presentation of these facts.

The areas where the superconductivity is destroyed are called *vorticity defects*. In an *homogeneous* superconductor, the vorticity defects arrange themselves into triangular Abrikosov lattice. In the presence of current, vorticity defects may move, generating dissipation, and destroying zero-resistance state. A way to prevent this motion is to trap the vorticity defects in small areas called *pinning sites*. In practice, pinning sites are often impurities which are present in a non perfect sample or intentionally introduced by irradiation, doping of impurities.

In order to prevent displacements in the superconductor, the key idea is to consider very small impurities. The heart of this article is to answer to the following question: *Once the vorticity defects are trapped by small impurities, what is their locations inside the impurities [microscopic location] ?*

The simplified Ginzburg-Landau functional

The mathematical theory of the superconductivity knew a increasing popularity with the pioneering work of Bethuel, Brezis and Hélein [BBH94]&[BBH93]. They studied the minimizers of the simplified Ginzburg-Landau energy

$$\begin{array}{rccc} E_\varepsilon : & H^1(\Omega,\mathbb{C}) & \to & \mathbb{R}^+ \\ & u & \mapsto & \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \end{array}$$

submitted to a Dirichlet boundary condition in the asymptotic $\varepsilon \to 0$. In their works Ω is a simply connected domain which is a cross section of an homogenous superconducting cylinder $\Omega \times \mathbb{R}$. The number $\varepsilon > 0$ is a characteristic parameter of the superconductor; the case $\varepsilon \to 0$ consists in considering *extrem type II superconductor*.

In this simplified model, a map u which minimizes E_ε [under boundary conditions] models the state of the superconductor in the mixed state. The superconducting area is the set $\{|u| \simeq 1\}$ and the vorticity defects are the connected components of $\{|u| \simeq 0\}$. One may mention that a quantization of the vorticity defects may be done by observing the *degree* of a minimizers around the connected components of $\{|u| \simeq 0\}$. In this context we say that z is a *vortex* of u when it is an isolated zero of u with a non zero degree. Namely, a vorticity defect may be seen as a small disc [with radius of order of ε] centered at a vortex. A Dirichlet boundary condition [with a non zero degree] mimics the application of a magnetic field by forcing the presence of vorticity defects.

A part of the main results of [BBH94] concerns *quantization* & *location* of the vorticity defects and an asymptotic estimate of the energy of a minimizer. All these results are related with the crucial notion of *renormalized energy*.

Theorem 2. *[Bethuel-Brezis-Hélein] Let Ω be a smooth and bounded simply connected open set and let $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ with degree $d > 0$.*

For $\varepsilon > 0$ we let u_ε be a minimizer of E_ε in H^1_g. Then:

1. *There exist $\varepsilon_0 > 0$ and $C > 1$ s.t. for $0 < \varepsilon < \varepsilon_0$:*
   - *u_ε has exactly d zeros $x_1^\varepsilon, ..., x_d^\varepsilon$ and $\{x \in \Omega \,|\, |u_\varepsilon(x)| \le 1/2\} \subset \cup_i B(x_i^\varepsilon, C\varepsilon)$; [Here $B(z,r) \subset \mathbb{R}^2$ is the open ball with center z and radius r]*
   - *Each zero is of degree 1: $\deg(u_\varepsilon, x_i^\varepsilon) = 1$ for all $i = 1, ..., d$;*
   - *As $\varepsilon \to 0$, up to extraction of a subsequence, there exist d distinct points $a_1, ..., a_d \in \Omega$ s.t. (up to relabeling of the points x_i^ε) we have $x_i^\varepsilon \to a_i$.*
2. *There exists a smooth map $W_g : \{(x_1, ..., x_d) \in \Omega^d \,|\, x_i \neq x_j \text{ for } i \neq j\} \to \mathbb{R}$, called renormalized energy, s.t.*
   - *$E_\varepsilon(u_\varepsilon) = \pi d |\ln \varepsilon| + W(a_1, ..., a_d) + d\gamma + o_\varepsilon(1)$ where $\gamma \in \mathbb{R}$ is a universal constant;*
   - *the set $(a_1, ..., a_d)$ minimizes the renormalized energy W_g.*

Remark 4. 1. In the work [BBH94], the renormalized energy W_g plays an important role. It is defined *via* auxiliary minimization problems involving $\mathbb{S}^1$-valued maps: for $N \in \mathbb{N}^*$ and $\mathbf{z} = (z_1, ...z_N) \in \Omega^N$ s.t. $z_i \neq z_j$ for $i \neq j$, $(d_1, ..., d_N) \in \mathbb{Z}^N$ s.t. $\sum d_i = d$ and $\rho \to 0^+$ we have:

$$\pi \sum_{i=1}^N d_i^2 |\ln \rho| + W_g(\mathbf{z}, \mathbf{d}) = \inf_{\substack{u \in H^1(\Omega \setminus \cup_i \overline{B(z_i,\rho)}, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\Omega}(u) = g \\ \deg_{\partial B(z_i,\rho)}(u) = d_i,\, i=1,...,N}} \frac{1}{2}\int_{\Omega \setminus \cup_i \overline{B(z_i,\rho)}} |\nabla u|^2 + o(1) \quad (1)$$

$$= \inf_{\substack{u \in H^1(\Omega \setminus \cup_i \overline{B(z_i,\rho)}, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\Omega}(u) = g \\ u(z_i + \rho e^{\imath\theta}) = \alpha_i e^{\imath d_i \theta},\, \alpha_i \in \mathbb{S}^1, \\ i=1,...,N}} \frac{1}{2}\int_{\Omega \setminus \cup_i \overline{B(z_i,\rho)}} |\nabla u|^2 + o(1). \quad (2)$$

In Theorem 2, we have $N = d$ and $d_i = 1$ for all i and we wrote $W_g(z_1, ..., z_d) := W_g((z_1, 1), ..., (z_d, 1))$.

2. The minimization of E_ε with a Dirichlet boundary condition is not relevant from the physical point of view since the Dirichlet boundary condition is not gauge invariant. In particular, the renormalized energy W_g is not physically relevant. But, in their work, Bethuel-Brezis-Hélein introduced systematic tools and asymptotic estimates to study vorticity defects.

The simplified Ginzburg-Landau functional with a pinning term

One may modify the above model in order to consider a superconducting cylinder with impurities. This is done with the help of a *pinning term* $a : \Omega \to \mathbb{R}^+$ by considering the functional

$$\begin{array}{rccc} E_\varepsilon^{\text{pinned}} : & H^1(\Omega, \mathbb{C}) & \to & \mathbb{R}^+ \\ & u & \mapsto & \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a^2 - |u|^2)^2 \end{array}.$$

There are a lot of works which deal with a such energy. Some variants are studied in the literature with the function a which is "smooth" or piecewise constant; independent of ε or depending on ε... See the Introduction of [Dos15] for a more complete presentation of this models.

In order to present the interpretation of the pinning term, we focus on the case of a pinning term $a : \Omega \to \mathbb{R}$ piecewise constant. Say, for some $b \in (0; 1)$ we have $a(\Omega) = \{1; b\}$ and $\overline{a^{-1}(\{b\})}$ is a smooth compact subset of Ω whose connected components represent the impurities.

A possible interpretation of a such pinning term is an heterogeneity in temperature. Letting T_c be the critical temperature of the superconductor, if $T_1 < T_c$ is the temperature in $a^{-1}(\{1\})$, then $T_b = (1 - b^2)T_c + b^2 T_1$ is the temperature in $a^{-1}(\{b\})$. Here the impurities are "heat" areas [note that $T_1 < T_2 < T_c$]. See Section 2.2 of the Introduction of [Dos10].

In order to consider "small" impurities we need to use an ε-dependent pinning term [$a_\varepsilon : \Omega \to \{b; 1\}$ with b independent of ε]. Then we may model *shrinking* impurities: the diameter of the connected components of $\overline{a^{-1}(\{b\})}$ tend to 0.

Essentially three kinds of pinning term may be used.

First kind of pinning term. The first kind of pinning term are those having a **fixed number of impurities** $P \in \mathbb{N}^*$:

- the impurities have the same form given by a smooth simply connected open set $\omega \subset \mathbb{R}^2$, $0 \in \omega$;
- the impurities are "centered" at some distinct points $y_1, ..., y_P \in \Omega$;
- the impurities have size $\lambda = \lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$.

This kind of pinning term is represented in Figure 1 and it is studied in [DM11].

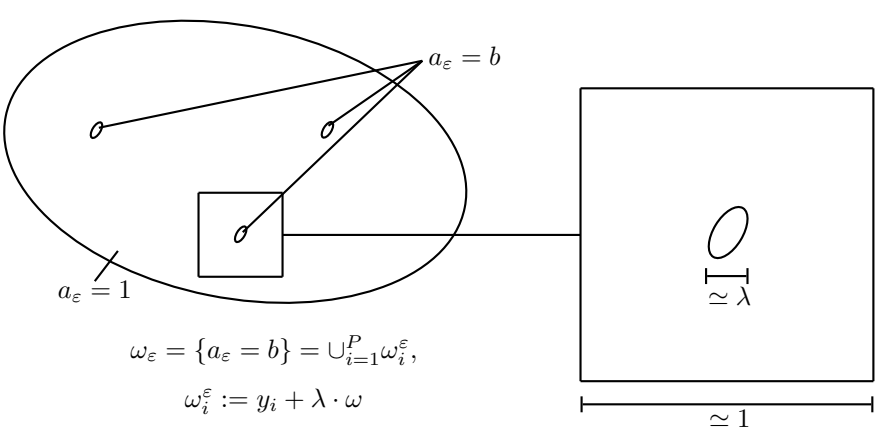
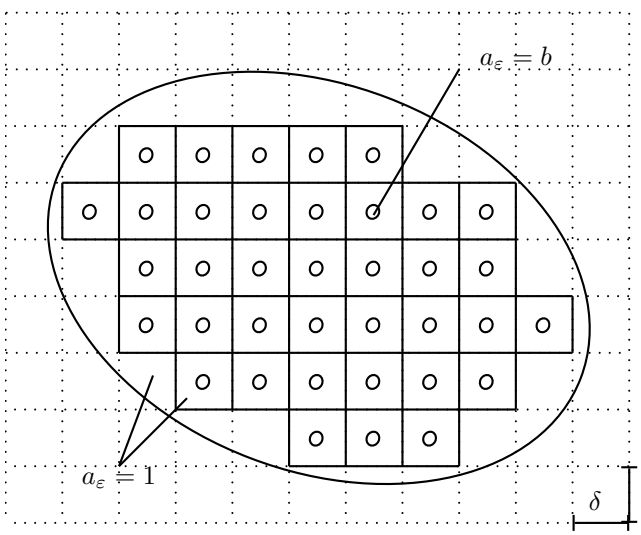
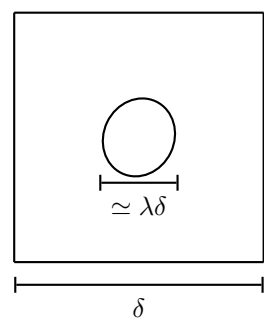


Figure 1: A pinning term modeling $P = 3$ small impurities ($\lambda = \lambda(\varepsilon) \underset{\varepsilon \to 0}{\to} 0^+$)

Second kind of pinning term. The second kind of pinning term correspond to the **periodic case**. This case is studied in [Dos13]. For $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ we consider a $\delta \times \delta$ squared grid covering $\mathbb{R}^2$. In the center of each cells entirely contained in Ω we insert an impurity with size $\lambda\delta$. Here λ may be equal to 1 or $\lambda \to 0^+$ as $\varepsilon \to 0$; it is a factor of dilution when $\lambda \to 0$. [See Figure 2]



(a) A $\delta \times \delta$-periodic pinning term



(b) λ controls the size of an impurity inside a cell

Figure 2: A periodic [rapidly oscillating] pinning term ($\lambda = \lambda(\varepsilon), \delta = \delta(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$)

This periodic pinning term illustrate the fundamental notion of *dilution* when $\lambda \to 0$. The *diluted impurities* are small impurities with the inter-distance between two impurities which is very larger than their diameters. Note that for the periodic pinning term and $\lambda \equiv 1$, the size of the impurities is of order δ and two neighboring impurities have their inter-distance of order δ. Consequently, despite the impurities are small, when $\lambda \equiv 1$, the impurities are not diluted.

Third kind of pinning term. The notion of diluted impurities leads to the third kind of pinning term: the **general diluted pinning term** [See Figure 3]. This case is studied in [Dos13]. The general diluted pinning term correspond to the presence of diluted impurities possibly having different sizes: $\lambda\delta, \ldots, \lambda\delta^P$ for some $P \in \mathbb{N}^*$.

For all these pinning terms, from [DM11] and [Dos13] we may state the following theorem:

Theorem 3. *Let Ω be a smooth and bounded simply connected open set and let $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ with degree $d > 0$. Let $b \in (0;1)$ and $a_\varepsilon : \Omega \to \{b;1\}$ be as in Figure 1 or Figure 2 or Figure 3.*

Assume that $[\ln(\lambda)]^3/\ln(\varepsilon) \to 0$ for the first kind of pinning term and $[\ln(\lambda\delta)]^3/\ln(\varepsilon) \to 0$ for both other cases.

For $\varepsilon > 0$ we let u_ε be a minimizer of E_ε in H^1_g. Then there exist $\varepsilon_0 > 0$ and $C > 1$ s.t. for $0 < \varepsilon < \varepsilon_0$:

1. *Quantization informations*
   - *u_ε has exactly d zeros $x_1^\varepsilon, ..., x_d^\varepsilon$ and $\{x \in \Omega \,|\, |u_\varepsilon(x)| \le b/2\} \subset \cup_i B(x_i^\varepsilon, C\varepsilon)$;*
   - *Each zero is of degree 1: $\deg(u_\varepsilon, x_i^\varepsilon) = 1$ for all $i = 1, ..., d$;*
2. *Pinning*

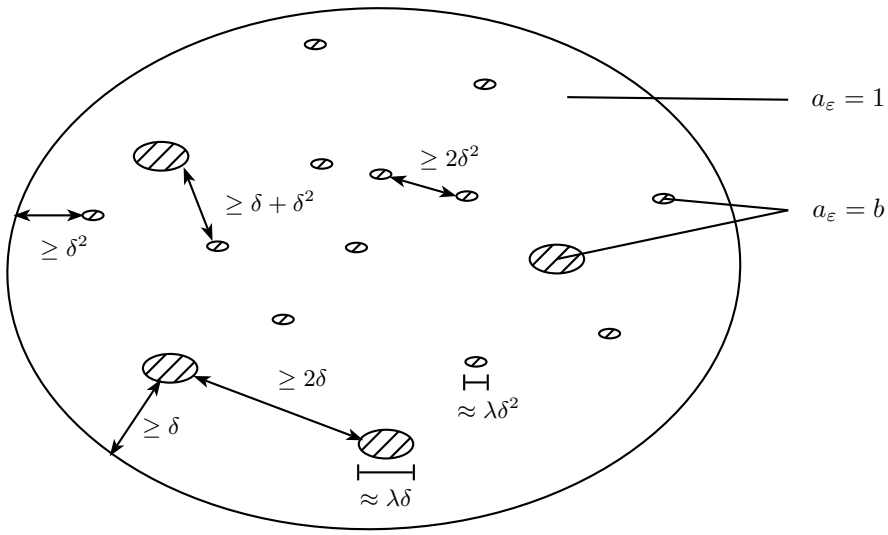


Figure 3: Representation of the general diluted pinning term with $P = 2$

- *For the first kind of pinning term:* $\cup_i B(x_i^\varepsilon, \lambda/C) \subset \{a_\varepsilon = b\}$
- *For the second kind of pinning term:* $\cup_i B(x_i^\varepsilon, \lambda\delta/C) \subset \{a_\varepsilon = b\}$
- *For the third kind of pinning term, in order to make a simple presentation of the situation, we assume that there is* $\eta_0 > 0$ *[independent of* ε*] s.t. for small* ε
  - *there are at least* d *impurities with size* $\lambda\delta$*:* $\omega_1^\varepsilon, ..., \omega_d^\varepsilon$*;*
  - $\mathrm{dist}(\omega_i^\varepsilon, \omega_j^\varepsilon), \mathrm{dist}(\omega_i^\varepsilon, \partial\Omega) > \eta_0$ *for all* $i, j \in \{1, ..., d\}$*,* $i \neq j$*.*

  Under these extra assumptions we have $\cup_i B(x_i^\varepsilon, \lambda\delta/C) \subset \{a_\varepsilon = b\}$*. In particular the vorticity defects are trapped by the largest impurities.*

3. *Macroscopic location*

   - *For the first kind of pinning term, the vortices are distributed in the impurities according to the most uniform way. Namely each impurities contain* $[d/P]$ *or* $[d/P] + 1$ *vortices. Here for* $x \in \mathbb{R}$*,* $[x]$ *is the integer part of* x*.*

     The choice between $[d/P]$ *or* $[d/P] + 1$ *is done via the minimization of* W_g*.*
   - *For the second kind of pinning term, each impurities contain at most one vortex. Moreover if* $\lambda \to 0$*, then* $(x_1^\varepsilon, ..., x_d^\varepsilon)$ *tends to minimize* W_g *with all the degree equal to* 1*. If* $\lambda \equiv 1$*, then there is no sharp macroscopic information except some classical vortex/vortex Columbian repulsion and confinent effect for the vortices [repulsion effect of* $\partial\Omega$*].*
   - *For the third kind of pinning term and under the above assumption on the largest impurities, the vortices are trapped by the largest impurities and each impurities contain at most one vortex. Moreover, the choice of the impurities containing a vortex is related with the minimization of the renormalized energy* W_g *with all the degree equal to* 1*.*

4. *Microscopic location*

   When $\lambda \to 0$*, for the three kinds of pinning term, the asymptotic location of a vortices inside an impurities tends to be independent on the Dirichlet boundary condition* g*. The microscopic location of the vortices trapped by an impurity tends to minimize a microscopic renormalized energy* W^{micro} *which depends only on* b*, the form of the impurity and the number of vortices trapped by the impurity.*

In the non diluted case [a periodic pinning term with $\lambda \equiv 1$], there is no sharp microscopic informations except some classical vortex/vortex Columbian repulsion and confinent effect for the vortices.

In [Dos15] [Section 2] it is explained in detailed the link between the minimization problem considered in Theorem 1 and the microscopic location of vortices in a diluted case.

Remark 5.

1. In [DM11], the existence and the role of W^{micro} was established. But its expression was not really explicit.
2. In particular, in the easiest case, the case of an impurity which is a disk containing a unique vortex, we expect that the limiting location is the center of the disc. The expression of W^{micro} obtained in [DM11] does not allow to get this result easily. This result was obtained from scratch in [Dos15].
3. Theorem 1 has a more general scope than needed. Indeed:
   i. In Theorem 1, the points z_i's corresponds to the location of the vortices inside an impurity. The weight α is a_ε^2 rescaled at the size of the impurity.
   Essentially, in the diluted case, we have to consider $\alpha = \begin{cases} 1 & \text{outside } \omega \\ b^2 & \text{in } \omega \end{cases}$ where ω is the form of the impurity.
   ii. With the help of the above theorem, in order to study W^{micro} in the context of a pinned Ginzburg-Landau type function, we may focus on the case $d_i = 1$ for $i \in \{1, ..., N\}$. But, since the minimization problem considered in Theorem 1 is of its self-interest we treat the case of general degrees.
4. In contrast with the renormalized energy W_g, we believe that the microscopic renormalized energy W^{micro} may play a role in a more physical problem.
5. If
   - $\omega \subset Y := (-1/2; 1/2] \times (-1/2; 1/2]$ is as in Theorem 1,
   - $\alpha = \begin{cases} 1 & \text{in } Y \setminus \omega \\ b^2 & \text{in } \omega \end{cases}$,
   - α is 1-periodic,

   then W^{micro} [given in Theorem 1] should govern the limiting location of vortices inside an impurity for the periodic non diluted case. But, there is no result which asserts that in the non diluted case the microscopic location of the vortices may be studied with this minimization problem. [Despite we believe that, in the non diluted periodic case, microscopic location of vortices should be given by minimal configuration of W^{micro} with degrees 1]

   Note that in [Dos15] [Section 2] the key use of the dilution property is clearly mentioned.

2 Notations and basic properties

2.1 General notations

2.1.1 Set and number

- For $z \in \mathbb{C}$, $|z|$ is the modulus of z, $\text{Re}(z) \in \mathbb{R}$ is the real part of z, $\text{Im}(z) \in \mathbb{R}$ is the imaginary part of z, $\overline{z}$ is the conjugate of z.
- "$\wedge$" stands for the vectorial product in $\mathbb{C}$, *i.e.* $z_1 \wedge z_2 = \text{Im}(\overline{z_1} z_2)$, $z_1, z_2 \in \mathbb{C}$.
- For $z \in \mathbb{C}$ and $r > 0$, $B(z, r) = \{\tilde{z} \in \mathbb{C} \,|\, |z - \tilde{z}| < r\}$. When $z = 0$ we write $B_r := B(0, r)$.

- For a set $A \subset \mathbb{R}^2 \simeq \mathbb{C}$, we let $\overline{A}$ be the adherence of A and ∂A be the boundary of A [with respect to the usual Euclidean distance in $\mathbb{R}^2$].
- We denote by $\mathbb{D} = B(0,1)$ the unit open disk and $\mathbb{S}^1 = \partial\mathbb{D}$ the unit circle.

2.1.2 Asymptotic

- In this article $R > 1$ is a "large" number and $\rho \in (0;1)$ is a small number. We are essentially interested in the asymptotic $R \to \infty$ and $\rho \to 0^+$.
- The notation $o_R(1)$ [resp. $o_\rho(1)$] means a quantity depending on R [resp. ρ] which tends to 0 when $R \to +\infty$ [resp. $\rho \to 0^+$]. When there is no ambiguity we just write $o(1)$.
- The notation $o[f(R)]$ [resp. $o[f(\rho)]$] means a quantity $g(R)$ [resp. $g(\rho)$] s.t. $\dfrac{g(R)}{f(R)} \to 0$ when $R \to +\infty$ [resp. $\dfrac{g(\rho)}{f(\rho)} \to 0$ when $\rho \to 0$]. When there is no ambiguity we just write $o(f)$.
- The notation $\mathcal{O}[f(R)]$ [resp. $\mathcal{O}[f(\rho)]$] means a quantity $g(R)$ [resp. $g(\rho)$] s.t. $\dfrac{g(R)}{f(R)}$ [resp. $\dfrac{g(\rho)}{f(\rho)}$] is bounded (independently of the variable) when R is large [resp. $\rho > 0$ is small]. When there is no ambiguity we just write $\mathcal{O}(f)$.

2.2 Data of the problem

Along this article we fix:

- $\omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ be a smooth bounded simply connected open set s.t. $0 \in \omega$,
- $N \in \mathbb{N}^*$, $\mathbf{d} = (d_1, ..., d_N) \in \mathbb{Z}^N$ and we let $d := \displaystyle\sum_{i=1}^N d_i \in \mathbb{Z}$,
- $\mathbf{z} \in (\omega^N)^* := \{(z_1, ..., z_N) \in \omega^N \mid z_i \neq z_j \text{ for } i \neq j\}$,
- $B \in (0;1)$, $b \in [B; B^{-1}]$ and $\alpha \in L^\infty(\mathbb{R}^2, [B^2; B^{-2}])$ s.t. $\alpha \equiv b^2$ in ω.

We define

$$R_0 := \max\{1; 10^2 \cdot \text{diameter}(\omega)\} \text{ and } \rho_0 := 10^{-2} \cdot \min\left\{1, \min_{i \neq j} |z_i - z_j|, \min_i \text{dist}(z_i, \partial\omega)\right\}.$$

For $R > R_0$ and $\rho_0 > \rho > 0$, we denote $\mathcal{D}_{R,\mathbf{z}} = B_R \setminus \cup_{i=1}^N \overline{B(z_i, \rho)}$.
The main purpose of this article is the following minimization problem :

$$I(R, \rho, \mathbf{z}, \mathbf{d}) := \inf_{u \in \mathcal{I}(R,\rho,\mathbf{z},\mathbf{d})} \frac{1}{2} \int_{\mathcal{D}_{R,\mathbf{z}}} \alpha |\nabla u|^2 \tag{3}$$

where

$$\mathcal{I}(R, \rho, \mathbf{z}, \mathbf{d}) := \{u \in H^1(\mathcal{D}_{R,\mathbf{z}}, \mathbb{S}^1) \mid \deg_{\partial B(z_i,\rho)}(u) = d_i,\ i = 1, ..., N\}.$$

Namely, we are interested in the asymptotic behavior of $I(R, \rho, \mathbf{z}, \mathbf{d})$ when $R \to \infty$ and $\rho \to 0$.
Without loss of generality and for simplicity of the presentation, $R > R_0$ is considered as the major parameter writing $\rho = \rho(R)$.
Before going further we recall some basic facts related with this minimization problem.

2.3 Test functions and degree

The functions we consider are essentially defined on *perforated domains*:

Definition 6. We say that $\mathcal{D} \subset \mathbb{R}^2$ is a *perforated domain* when $\mathcal{D} = \Omega \setminus \cup_{i=1}^{P} \overline{\omega_i}$ where $P \in \mathbb{N}^*$ and $\Omega, \omega_1, ..., \omega_P$ are smooth simply connected bounded open sets s.t. for $i \in \{1, ..., P\}$ we have $\overline{\omega_i} \subset \Omega$ and, for $i \neq j$, $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$.

If $P = 1$ we say that $\mathcal{D}$ is an annular type domain.

In this article the test functions stand in the standard Sobolev space of order 1 with complex values modeled on L^2, $H^1(\Omega, \mathbb{C})$, where Ω is a smooth open set.

We use the standard norm on $H^1(\Omega, \mathbb{C})$:

$$\|u\|_{H^1} = \left(\int_\Omega |u|^2 + |\nabla u|^2 \right)^{1/2}.$$

Our main interest is based on unimodular map, *i.e*, the test functions are $\mathbb{S}^1$-valued. Thus we focus on maps lying in

$$H^1(\Omega, \mathbb{S}^1) := \{u \in H^1(\Omega, \mathbb{C}) \,|\, |u| = 1 \text{ a.e in } \Omega\}$$

where Ω is a smooth open set.

For $\Omega \subset \mathbb{R}^2$ a smooth open set, we let $\mathrm{tr}_{\partial\Omega} : H^1(\Omega, \mathbb{C}) \to H^{1/2}(\partial\Omega, \mathbb{C})$ be the *trace* operator. Here $H^{1/2}(\partial\Omega, \mathbb{C})$ is the *trace space*

Note if $u \in H^1(\Omega, \mathbb{S}^1)$ then $\mathrm{tr}_{\partial\Omega}(u) \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$.

Recall that for $\Gamma \subset \mathbb{R}^2$ a Jordan curve and $g \in H^{1/2}(\Gamma, \mathbb{S}^1)$, the degree (winding number) of g is defined as

$$\deg_\Gamma(g) := \frac{1}{2\pi} \int_\Gamma g \wedge \partial_\tau g \in \mathbb{Z}. \tag{4}$$

Here

- τ is the direct unit tangent vector of Γ ($\tau = \nu^\perp$ where ν is the outward normal unit vector of $\mathrm{int}(\Gamma)$, the bounded open set whose boundary is Γ),
- $\partial_\tau := \tau \cdot \nabla$ is the tangential derivative on Γ. For further use we denote $\partial_\nu = \nu \cdot \nabla$ the normal derivative on Γ.

Remark 7.

1. Note that (4) may be understood *via* $H^{1/2} - H^{-1/2}$ duality. Another way to define the degree of an $H^{1/2}(\Gamma)$-map consists in using a density argument [see Appendix in [BGP91]].
2. Let $\mathcal{D} = \Omega \setminus \cup_{i=1}^{P} \overline{\omega_i}$ be a perforated domain [see Definition 6]. The orientation with respect to which we calculate degrees is counter-clockwise on $\partial\omega_i$ and clockwise on $\partial\Omega$.
3. If $\mathcal{D}$ is a perforated domain and if $u \in H^1(\mathcal{D}, \mathbb{S}^1)$ then we write

$$\deg(u) := (\deg_{\partial\omega_1}(u), ..., \deg_{\partial\omega_P}(u)) \in \mathbb{Z}^P.$$

For the convenience of the reader we recall some basic properties related with the degree.

Proposition 8 ([Bre01])**.** *Let $\Gamma \subset \mathbb{R}^2$ be a Jordan curve and let $\mathcal{D} := \Omega \setminus \cup_{i=1}^{P} \overline{\omega_i}$ be a perforated domain.*

1. *For $\mathbf{d} \in \mathbb{Z}^P$ we have*

$$\mathcal{E}_{\mathbf{d}} := \left\{ u \in H^1(\mathcal{D}, \mathbb{S}^1) \,|\, \deg(u) = \mathbf{d} \right\} \neq \emptyset.$$

2. *If $u \in H^{1/2}(\Gamma, \mathbb{S}^1)$ then we have*

$$\exists \phi \in H^{1/2}(\Gamma, \mathbb{R}) \text{ s.t. } u = \mathrm{e}^{\imath\phi} \Longleftrightarrow \deg_\Gamma(u) = 0.$$

Moreover, if for $\phi_1, \phi_2 \in H^{1/2}(\Gamma, \mathbb{R})$ we have $\mathrm{e}^{\imath\phi_1} = \mathrm{e}^{\imath\phi_2}$ then $\phi_1 - \phi_2 = \lambda \in 2\pi\mathbb{Z}$.

3. *If* $u, v \in H^{1/2}(\Gamma, \mathbb{S}^1)$*, then we have*

$$\deg_\Gamma(uv) = \deg_\Gamma(u) + \deg_\Gamma(v) \text{ and } \deg_\Gamma(1/u) = -\deg_\Gamma(u).$$

4. *If* $u \in H^1(\mathcal{D}, \mathbb{S}^1)$ *then* $\deg_{\partial\Omega}(u) = \sum_{i=1}^{P} \deg_{\partial\omega_i}(u)$.

5. *If* $u \in H^1(\mathcal{D}, \mathbb{S}^1)$ *then there exists* $\varphi \in H^1(\mathcal{D}, \mathbb{R})$ *s.t.* $u = \mathrm{e}^{\imath\varphi}$ *if and only if* $\deg_{\partial\omega_i}(u) = 0$ *for* $i \in \{1, ..., P\}$.

   - *In particular for* $u_0 \in H^1(\mathcal{D}, \mathbb{S}^1)$ *we have*

   $$\left\{u \in H^1(\mathcal{D}, \mathbb{S}^1) \,|\, \deg(u) = \deg(u_0)\right\} = \left\{u_0 \mathrm{e}^{\imath\varphi} \,|\, \varphi \in H^1(\mathcal{D}, \mathbb{R})\right\}.$$

   - *Moreover, if for* $\varphi_1, \varphi_2 \in H^1(\mathcal{D}, \mathbb{R})$ *we have* $\mathrm{e}^{\imath\varphi_1} = \mathrm{e}^{\imath\varphi_2}$ *then* $\varphi_1 - \varphi_2 = \lambda \in 2\pi\mathbb{Z}$.

6. *For* $\mathbf{d} = (d_1, ..., d_P) \in \mathbb{Z}^P$ *and* $(z_1, ..., z_P) \in \omega_1 \times \cdots \times \omega_P$ *we have*

   $$\prod_{i=1}^{P} \left(\frac{z - z_i}{|z - z_i|}\right)^{d_i} \in \mathcal{E}_{\mathbf{d}}.$$

   Locally, one may define θ_i*, a determination of the argument of* $z - z_i$ *in* $\mathbb{R}^2 \setminus \{z_i\}$*. Note that* $\nabla\theta_i$ *is globally defined in* $\mathcal{D}$ *and* $\sum_{i=1}^{P} d_i \nabla\theta_i \in L^2(\mathcal{D}, \mathbb{R}^2)$.

   Therefore, letting $\Theta := \sum_{i=1}^{P} d_i\theta_i$*, we have, for* $u \in \mathcal{E}_{\mathbf{d}}$*, the existence of* $\varphi \in H^1(\mathcal{D}, \mathbb{R})$ *s.t.* $u = \mathrm{e}^{\imath(\Theta+\varphi)}$.

   In other words, for $u \in H^1(\mathcal{D}, \mathbb{S}^1)$ *there exists* ψ *which is locally defined in* $\mathcal{D}$ *and whose gradient is in* $L^2(\mathcal{D}, \mathbb{R}^2)$ *s.t.* $u = \mathrm{e}^{\imath\psi}$.

Notation 9. 1. It is important to note that for $u \in H^1(\mathcal{D}, \mathbb{S}^1)$, the function ψ given by Proposition 8.6 is locally defined modulo 2π while $\nabla\psi$ is globally well defined. Namely $\nabla\psi = u \wedge \nabla u$.

2. For simplicity of the presentation, when there is no ambiguity, we may omit the dependance on the Jordan curve in the notation of the degree. For example:

   - if Γ is a Jordan curve and if $h \in H^{1/2}(\Gamma, \mathbb{S}^1)$, then we may write $\deg(h)$ instead of $\deg_\Gamma(h)$.
   - If $\mathcal{D} = \Omega \setminus \overline{\omega}$ is an annular type domain and $u \in H^1(\mathcal{D}, \mathbb{S}^1)$, then $\deg_{\partial\Omega}(u) = \deg_{\partial\omega}(u)$. Consequently, without ambiguity, we may write $\deg(u)$ instead of $\deg_{\partial\Omega}(u)$ or $\deg_{\partial\omega}(u)$.

2.4 Minimization problems

One of the main issue in this article is the study of minimization problems of weighted Dirichlet functionals with prescribed degrees :

$$\inf_{u \in \mathcal{E}_{\mathbf{d}}} \frac{1}{2} \int_{\mathcal{D}} \alpha |\nabla u|^2 \tag{5}$$

where

- $\mathcal{D} := \Omega \setminus \cup_{i=1}^{P} \overline{\omega_i}$ is a perforated domain as in Definition 6,
- $\mathbf{d} = (d_1, ..., d_P) \in \mathbb{Z}^P$,
- $\mathcal{E}_{\mathbf{d}} := \{u \in H^1(\mathcal{D}, \mathbb{S}^1) \,|\, \deg_{\partial\omega_i}(u) = d_i \text{ for } i \in \{1, ..., P\}\}$,

- $\alpha \in L^\infty(\mathcal{D}; [B^2; B^{-2}])$, $B \in (0;1)$.

Problem (5) admits solutions which are unique up to a constant rotation. Namely we have the following proposition:

Proposition 10. *Minimisation problem (5) admits solutions. Moreover if u is a solution of (5) then v is a solution of (5) if and only if there exists $\lambda \in \mathbb{S}^1$ s.t. $v = \lambda u$.*

Moreover a minimizer $u_{\mathbf{d}}$ solves

$$\begin{cases} -\mathrm{div}(\alpha \nabla u_{\mathbf{d}}) = \alpha u_{\mathbf{d}} |\nabla u_{\mathbf{d}}|^2 & \text{in } \mathcal{D} \\ \partial_\nu u_{\mathbf{d}} = 0 & \text{on } \partial\mathcal{D} \end{cases}. \tag{6}$$

Proof. Since from Proposition 8.5, the set $\mathcal{E}_{\mathbf{d}}$ is closed under the weak-H^1 convergence, the existence of solution of (5) is obtained by direct minimization.

If $u_{\mathbf{d}}$ is a solution of (5), then from standard computations of directional derivatives we get that (6) is satisfied [see *e.g.* Appendix A in [Dos13]].

Let $u_{\mathbf{d}}$ be a solution of (5). From Proposition 8.6, there exists $\psi_{\mathbf{d}}$ which is locally defined in $\mathcal{D}$ and whose gradient is in $L^2(\mathcal{D}, \mathbb{R}^2)$ s.t. $u_{\mathbf{d}} = \mathrm{e}^{\imath \psi_{\mathbf{d}}}$. In terms of $\psi_{\mathbf{d}}$, Equations (6) reads :

$$\begin{cases} -\mathrm{div}(\alpha \nabla \psi_{\mathbf{d}}) = 0 & \text{in } \mathcal{D} \\ \partial_\nu \psi_{\mathbf{d}} = 0 & \text{on } \partial\mathcal{D} \end{cases}. \tag{7}$$

Thus, if $v_{\mathbf{d}}$ is a minimizers, then, with the help of Proposition 8.5, there exists $\varphi \in H^1(\mathcal{D}, \mathbb{R})$ s.t. $v_{\mathbf{d}} = \mathrm{e}^{\imath(\psi_{\mathbf{d}} + \varphi)}$. Then, using the minimality of $v_{\mathbf{d}}$ we get

$$\begin{cases} -\mathrm{div}[\alpha \nabla (\psi_{\mathbf{d}} + \varphi)] = 0 & \text{in } \mathcal{D} \\ \partial_\nu (\psi_{\mathbf{d}} + \varphi) = 0 & \text{on } \partial\mathcal{D} \end{cases}.$$

Consequently, using (7) we obtain

$$\begin{cases} -\mathrm{div}(\alpha \nabla \varphi) = 0 & \text{in } \mathcal{D} \\ \partial_\nu \varphi = 0 & \text{on } \partial\mathcal{D} \end{cases}. \tag{8}$$

With the help an integration by parts, we easily get that $\varphi \in H^1(\mathcal{D}, \mathbb{R})$ solves (8) if and only if φ is a constant. This argument yields the uniqueness of the solution up to a constant rotation.

□

3 First step in the proof of Theorem 1: splitting of the domain

The first step in the proof of Theorem 1 is standard. The strategy employed was already used in [Dos15]. It consists in splitting the integral over $\mathcal{D}_{R,\mathbf{z}}$ [in (3)] in two parts: the integral over $\Omega_R := B_R \setminus \overline{\omega}$ and the one over $\omega_{\rho,\mathbf{z}} := \omega \setminus \cup_{i=1}^N \overline{B(z_i, \rho)}$ [as presented in Theorem 1].

For each integrals we consider a mixed minimization problem by adding an arbitrary Dirichlet boundary condition on $\partial\omega$: $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d = \sum d_i$.

We then claim that these mixed minimization problems admit "unique" solutions.

In the next steps we will solve these problems, we will minimize among $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$ and finally we will decouple the minimal energy according to the different data.

The splitting consists in the following obvious equality:

$$I(R, \rho, \mathbf{z}, \mathbf{d}) = \inf_{\substack{h \in H^{1/2}(\partial\omega, \mathbb{S}^1) \\ \text{s.t. } \deg(h) = d}} \left\{ \inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(v) = h}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2 + \inf_{\substack{w \in H^1(\omega_{\rho,\mathbf{z}}, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(w) = h \\ \deg_{\partial B(z_i,\rho)}(w) = d_i \ \forall i}} \frac{b^2}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla w|^2 \right\}. \tag{9}$$

The three previous minimization problems admit "unique" solutions:

Proposition 11. *1. Both minimization problems in (9) having a [partial] Dirichlet boundary condition $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ in (9) admit each a unique solution. For $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ let $v_{R,h}$ be the solution of*

$$\inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(v) = h}} \int_{\Omega_R} \alpha |\nabla v|^2 \tag{10}$$

and $w_{\rho,h}$ be the one of

$$\inf_{\substack{w \in H^1(\omega_{\rho,\mathbf{z}}, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(w) = h \\ \deg_{\partial B(z_i,\rho)}(w) = d_i \, \forall i}} \int_{\omega_{\rho,\mathbf{z}}} |\nabla w|^2. \tag{11}$$

Then $v_{R,h}$ is the unique solution of

$$\begin{cases} -\mathrm{div}(\alpha \nabla v_{R,h}) = \alpha v_{R,h} |\nabla v_{R,h}|^2 & \text{in } \Omega_R \\ v_{R,h} = h & \text{on } \partial\omega \\ \partial_\nu v_{R,h} = 0 & \text{on } \partial B_R \end{cases} \tag{12}$$

and $w_{\rho,h}$ is the unique solution of

$$\begin{cases} -\Delta w_{\rho,h} = w_{\rho,h} |\nabla w_{\rho,h}|^2 & \text{in } \omega_{\rho,\mathbf{z}} \\ w_{\rho,h} = h & \text{on } \partial\omega \\ \partial_\nu w_{\rho,h} = 0 & \text{on } \partial B(z_i, \rho),\ i \in \{1, ..., N\} \\ \deg_{\partial B(z_i,\rho)}(w_{\rho,h}) = d_i & i \in \{1, ..., N\} \end{cases}. \tag{13}$$

2. The minimization problem in (9) among $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$ admits solutions. Moreover if h_0 is a solution, then $\tilde{h}_0$ is a minimizer if and only if there exists $\lambda \in \mathbb{S}^1$ s.t. $\tilde{h}_0 = \lambda h_0$.

Proof. It is clear [by considering a minimizing sequence] that (10) and (11) admit solutions. Moreover these minimizers solve the corresponding equations (12) and (13).

We now prove that (12) admits a unique solution. The argument is similar to prove that the solution of (13) is unique. Let v_0 be a solution of (12) and $v \in H^1(\Omega_R, \mathbb{S}^1)$ s.t. $\mathrm{tr}_{\partial\omega}(v) = h$. On the one hand, writing $v_0 = \mathrm{e}^{\imath\psi_0}$ where ψ_0 is locally defined in Ω_R and $\nabla\psi_0 \in L^2(\Omega_R)$ is globally defined [Proposition 8.6], it standard to get that

$$\begin{cases} v \in H^1(\Omega_R, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(v) = h \end{cases} \iff \begin{cases} v = \mathrm{e}^{\imath(\psi_0 + \varphi)} \\ \varphi \in H^1(\Omega_R, \mathbb{R}) \text{ s.t. } \mathrm{tr}_{\partial\omega}(\varphi) = 0 \end{cases}.$$

On the other hand, from direct calculations, we have, for $v = \mathrm{e}^{\imath(\psi_0+\varphi)}$ s.t. $\varphi \in H^1(\Omega_R, \mathbb{R})\&\mathrm{tr}_{\partial\omega}(\varphi) = 0$, the following equivalence

$$\begin{cases} -\mathrm{div}(\alpha\nabla v) = \alpha v |\nabla v|^2 \text{ in } \Omega_R \\ \partial_\nu v = 0 \text{ on } \partial B_R \end{cases} \iff \begin{cases} -\mathrm{div}[\alpha\nabla(\psi_0 + \varphi)] = 0 \text{ in } \Omega_R \\ \partial_\nu(\psi_0 + \varphi) = 0 \text{ on } \partial B_R \end{cases}.$$

Thus $-\mathrm{div}(\alpha\nabla\psi_0) = 0$ in Ω_R and $\partial_\nu\psi_0 = 0$ on ∂B_R. Consequently, if $v = \mathrm{e}^{\imath(\psi_0+\varphi)}$ is a solution of (12), then φ solves

$$\begin{cases} -\mathrm{div}(\alpha\nabla\varphi) = 0 \text{ in } \Omega_R \\ \partial_\nu\varphi = 0 \text{ on } \partial B_R \end{cases}.$$

Noting that $\mathrm{tr}_{\partial\omega}(\varphi) = 0$ we immediatly obtained that $\varphi = 0$, *i.e.*, $v = v_0$.

The second part of the proposition is a direct consequence of Proposition 10 and of the first part of Proposition 11. □

4 Second step in the proof of Theorem 1: the key ingredient

The key ingredient in this article is the use of *special solutions*. In order to motivate their use, we focus on the fully radial homogeneous case: $\omega = \mathbb{D}, \alpha \equiv 1$, $N = 1$, $z = 0$.

It is easy to check that, letting $\mathscr{R}(R, \rho, 0) := B_R \setminus \overline{B_\rho}$ with $R > \rho > 0$, for $d \in \mathbb{Z}$, the map

$$\begin{array}{rccc} u_d : & \mathscr{R}(R,\rho,0) & \to & \mathbb{S}^1 \\ & x & \mapsto & \left(\dfrac{x}{|x|}\right)^d \end{array}$$

is a global minimizer of the Dirichlet functional $\dfrac{1}{2}\displaystyle\int_{\mathscr{R}(R,\rho,0)} |\nabla \cdot|^2$ in the space

$$\mathcal{E}_d := \{u \in H^1[\mathscr{R}(R,\rho,0), \mathbb{S}^1] \,|\, \deg(u) = d\}.$$

Letting $\theta(x)$ be a determination of the argument of $x \in \mathbb{C}\setminus\{0\}$ which is locally defined in $\mathscr{R}(R,\rho,0)$ and whose gradient is globally defined, we have $u_d = \mathrm{e}^{\imath d\theta}$.

Let $u \in \mathcal{E}_d$ and $\varphi \in H^1[\mathscr{R}(R,\rho,0), \mathbb{R}]$ be s.t. $u = u_d \mathrm{e}^{\imath\varphi} = \mathrm{e}^{\imath(d\theta+\varphi)}$ [Proposition 8.5]. Since θ solves $-\Delta\theta = 0$ in $\mathscr{R}(R,\rho,0)$ and $\partial_\nu\theta = 0$ on $\partial\mathscr{R}(R,\rho,0)$ with the help of an integration by parts we obtain

$$\begin{aligned} \frac{1}{2}\int_{\mathscr{R}(R,\rho,0)} |\nabla u|^2 &= \frac{1}{2}\int_{\mathscr{R}(R,\rho,0)} |\nabla(d\theta+\varphi)|^2 \\ &= \frac{d^2}{2}\int_{\mathscr{R}(R,\rho,0)} |\nabla\theta|^2 + \frac{1}{2}\int_{\mathscr{R}(R,\rho,0)} |\nabla\varphi|^2 \\ &= \frac{1}{2}\int_{\mathscr{R}(R,\rho,0)} |\nabla u_d|^2 + \frac{1}{2}\int_{\mathscr{R}(R,\rho,0)} |\nabla\varphi|^2. \end{aligned}$$

These calculations are standard and give an easy decoupling for the energy of $u = u_d \mathrm{e}^{\imath\varphi}$ as the energy of u_d pulse those of the dephasing φ.

The main argument of this article consists in the fact that this argument is not restricted to the fully radial homogeneous case. Indeed we have the following proposition:

Proposition 12. *Let $\mathcal{D}$ be a perforated domain, $B \in (0;1)$, $\alpha \in L^\infty(\mathcal{D}; [B^2; B^{-2}])$ and $\mathbf{d} \in \mathbb{Z}^N$. We let $u_{\mathbf{d}}$ be a minimizer of (5). Then for $\varphi \in H^1(\mathcal{D}, \mathbb{R})$ we have*

$$\frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla(u_{\mathbf{d}}\mathrm{e}^{\imath\varphi})|^2 = \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla u_{\mathbf{d}}|^2 + \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla\varphi|^2.$$

Proof. We fix $\mathcal{D}, B, \alpha, \mathbf{d}$ be as in the proposition. First note that, from Proposition 10, we get the existence of $u_{\mathbf{d}}$. Moreover $u_{\mathbf{d}}$ is a solution of

$$\begin{cases} -\mathrm{div}(\alpha\nabla u_{\mathbf{d}}) = \alpha u_{\mathbf{d}}|\nabla u_{\mathbf{d}}|^2 & \text{in } \mathcal{D} \\ \partial_\nu u_{\mathbf{d}} = 0 & \text{on } \partial\mathcal{D} \end{cases}.$$

We may write $u_{\mathbf{d}} = \mathrm{e}^{\imath\psi}$ where ψ is locally defined in $\mathcal{D}$ and $\nabla\psi \in L^2(\mathcal{D}, \mathbb{R}^2)$ [Proposition 8.6].

Thus ψ solves

$$\begin{cases} -\mathrm{div}(\alpha\nabla\psi) = 0 & \text{in } \mathcal{D} \\ \partial_\nu\psi = 0 & \text{on } \partial\mathcal{D} \end{cases}. \tag{14}$$

Let $\varphi \in H^1(\mathcal{D}, \mathbb{R})$. We have

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla(u_{\mathbf{d}}\mathrm{e}^{\imath\varphi})|^2 &= \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla(\mathrm{e}^{\imath(\psi+\varphi)})|^2 \\ &= \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla(\psi+\varphi)|^2 \\ &= \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla\psi|^2 + \int_{\mathcal{D}} \alpha\nabla\psi\cdot\nabla\varphi + \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla\varphi|^2. \end{aligned}$$

From (14) and an integration by parts we get $\int_{\mathcal{D}} \alpha \nabla\psi \cdot \nabla\varphi = 0$ and this equality ends the proof of the proposition since

$$\frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla\psi|^2 = \frac{1}{2}\int_{\mathcal{D}} \alpha|\nabla u_{\mathbf{d}}|^2.$$

□

Remark 13. It is easy to check that Proposition 12 allows to prove in a "different" way the uniqueness, up to a constant rotation, of a minimizer of (5).

Because minimizers of (5) are not unique, in order to fix such a minimizer we add an extra condition. This choice leads to the crucial notion of *special solution*.

In both next sections we define the special solutions in $\Omega_R = B_R \setminus \overline{\omega}$ [Section 4.1] and in $\omega_{\rho,\mathbf{z}} = \omega \setminus \cup\overline{B(z_i,\rho)}$ [Section 4.2].

4.1 The special solution in Ω_R

In this section we focus on the annular type domain $\Omega_R = B_R \setminus \overline{\omega}$. We first treat the case $d = 1$ by considering:

$$\inf_{\substack{v\in H^1(\Omega_R,\mathbb{S}^1)\\ \deg(v)=1}} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v|^2. \tag{15}$$

With the help of Proposition 10, we may fix a map $v_R \in H^1(\Omega_R,\mathbb{S}^1)$ s.t. $\deg(v_R) = 1$ which is a solution of (15). We freeze the non-uniqueness of v_R by letting v_R be in the form

$$v_R = \frac{x}{|x|}\mathrm{e}^{\imath\gamma_R} \text{ with } \gamma_R \in H^1(\Omega_R,\mathbb{R}) \text{ s.t. } \int_{\partial\omega} \gamma_R = 0. \tag{16}$$

It is clear that such map v_R is uniquely and well defined.

It is easy to check that, for $d \in \mathbb{Z}$, we have v_R^d which is a solution of the minimization problem:

$$\inf_{\substack{v\in H^1(\Omega_R,\mathbb{S}^1)\\ \deg(v)=d}} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v|^2. \tag{17}$$

Moreover it is the unique solution of the form $v_R^d = \left(\frac{x}{|x|}\right)^d \mathrm{e}^{\imath\tilde{\gamma}}$ with $\tilde{\gamma} \in H^1(\Omega_R,\mathbb{R})$ s.t. $\int_{\partial\omega}\tilde{\gamma} = 0$.

We have the following proposition:

Proposition 14. *For $x = |x|\mathrm{e}^{\imath\theta} \in \Omega_R$ we have $v_R(x) = \mathrm{e}^{\imath(\theta+\gamma_R(x))}$ with $\gamma_R \in H^1(\Omega_R)$ which is a solution of*

$$\begin{cases} -\mathrm{div}\,[\alpha\nabla(\theta+\gamma_R)] = 0 \text{ in } \Omega_R \\ \nabla(\theta+\gamma_R)\cdot\nu = 0 \text{ on } \partial\Omega_R \\ \int_{\partial\omega}\gamma_R = 0 \end{cases}. \tag{18}$$

The special solution v_R is fundamental in the analysis since it allows to get a decoupling of weighted Dirichlet energy. Namely, from Proposition 12 we have:

Lemma 15. *For $d \in \mathbb{Z}$ and $\varphi \in H^1(\Omega_R,\mathbb{R})$ we have:*

$$\frac{1}{2}\int_{\Omega_R} \alpha|\nabla(v_R^d\mathrm{e}^{\imath\varphi})|^2 = \frac{d^2}{2}\int_{\Omega_R} \alpha|\nabla v_R|^2 + \frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi|^2.$$

The above lemma allows to get a crucial information on the asymptotic behavior of $(\gamma_R)_R$:

Proposition 16. *There exists $\gamma_\infty \in H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\omega,\mathbb{R})$ s.t. when $R \to \infty$ we have $\gamma_R \to \gamma_\infty$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\omega)$.*

Proof. Let $R' > R > R_0$ and $\varphi_R = \gamma_{R'} - \gamma_R$ in order to have $v_{R'} = v_R \mathrm{e}^{\imath\varphi_R}$ in Ω_R.

From Lemma 15 we have

$$\frac{1}{2}\int_{\Omega_R} \alpha|\nabla v_{R'}|^2 = \frac{1}{2}\int_{\Omega_R} \alpha|\nabla(v_R \mathrm{e}^{\imath\varphi_R})|^2 = \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v_R|^2 + \frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi_R|^2. \tag{19}$$

We need the following lemma:

Lemma 17. *There exists a constant $C_{B,\omega} > 0$ depending only on B and ω s.t.*

$$\frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi_R|^2 \leq C_{B,\omega}.$$

For the convenience of the reader the proof of this lemma in postponed in Appendix [see Appendix A].

From Lemma 17 we have

$$\frac{1}{2}\int_{B_{\sqrt{R}}\setminus\overline{B_{R^{1/4}}}} \alpha|\nabla\varphi_R|^2 \leq C_{B,\omega}.$$

Notation 18. In the rest of this proof, C_0 stands for a constant depending only on ω and B derived from $C_{B,\omega}$ and with universal multiplicative constants. Its values may change from line to line.

Therefore, with the help of a mean value argument, we have the existence of $r \in (R^{1/4}, \sqrt{R})$ and of a constant C_0 depending only on B and ω s.t.:

$$\int_0^{2\pi} |\partial_\theta\varphi_R(r\mathrm{e}^{\imath\theta})|^2 \,\mathrm{d}\theta \leq \frac{C_0}{\ln R}.$$

We denote $m_R := \fint_0^{2\pi} \varphi_R(r\mathrm{e}^{\imath\theta})\,\mathrm{d}\theta$.

From the above estimate and with the help of a Poincaré-Wirtinger inequality, we have

$$\int_0^{2\pi} \left(\varphi_R(r\mathrm{e}^{\imath\theta}) - m_R\right)^2 \,\mathrm{d}\theta \leq \frac{C_0}{\ln R}.$$

We now define $\tilde{\varphi}_R \in H^1(B_R, \mathbb{R})$ using polar coordinates:

$$\tilde{\varphi}_R(s,\theta) = \begin{cases} m_R & \text{for } s \in [0, r/2] \\ \dfrac{s - r/2}{r/2}\varphi_R(r,\theta) + \dfrac{r-s}{r/2} m_R & \text{for } s \in]r/2, r[\\ \varphi_R(s,\theta) & \text{for } s \in [r, R[\end{cases}.$$

It is easy to check that $\tilde{\varphi}_R \in H^1(B_R, \mathbb{R})$ and with direct calculations we obtain:

$$\begin{aligned} \int_{B_r} |\nabla\tilde{\varphi}|^2 &= \int_{B_r\setminus B_{r/2}} |\nabla\tilde{\varphi}|^2 \\ &\leq \int_{r/2}^r s\,\mathrm{d}s \int_0^{2\pi} \left\{ \frac{4}{r^2}\left(\varphi_R(r,\theta) - m_R\right)^2 + \frac{1}{s^2}\left[\frac{2(s-r)}{r}\right]^2 |\partial_\theta\varphi_R(r,\theta)|^2 \right\} \mathrm{d}\theta \\ &\leq \frac{C_0}{\ln R}. \end{aligned} \tag{20}$$

By noting that $\mathrm{tr}_{\partial B_R}(v_R \mathrm{e}^{\imath\tilde{\varphi}_R}) = \mathrm{tr}_{\partial B_R}(v_R \mathrm{e}^{\imath\varphi_R}) = \mathrm{tr}_{\partial B_R}(v_{R'})$, with the help of $\tilde{\varphi}_R$ we construct $\tilde{v}_R \in H^1(\Omega_{R'}, \mathbb{S}^1)$:

$$\tilde{v}_R = \begin{cases} v_{R'} & \text{in } B_{R'} \setminus \overline{B_R} \\ v_R \mathrm{e}^{\imath\tilde{\varphi}_R} & \text{in } \Omega_R \end{cases}.$$

From the minimality of $v_{R'}$ and Lemma 15 we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega_{R'}}\alpha|\nabla v_{R'}|^2 &\leq \frac{1}{2}\int_{\Omega_{R'}}\alpha|\nabla\tilde{v}_R|^2\\ &= \frac{1}{2}\int_{\Omega_{R'}\setminus\overline{\Omega_R}}\alpha|\nabla\tilde{v}_R|^2+\frac{1}{2}\int_{\Omega_R}\alpha|\nabla\tilde{v}_R|^2\\ &= \frac{1}{2}\int_{\Omega_{R'}\setminus\overline{\Omega_R}}\alpha|\nabla v_{R'}|^2+\frac{1}{2}\int_{\Omega_R}\alpha|\nabla v_R|^2+\int_{\Omega_R}\alpha|\nabla\tilde{\varphi}_R|^2. \qquad (21)\end{aligned}$$

Estimate (21) implies:

$$\frac{1}{2}\int_{\Omega_R}\alpha|\nabla v_{R'}|^2\leq\frac{1}{2}\int_{\Omega_R}\alpha|\nabla v_R|^2+\frac{1}{2}\int_{\Omega_R}\alpha|\nabla\tilde{\varphi}_R|^2.$$

The above inequality coupled with (19) gives:

$$\frac{1}{2}\int_{\Omega_R}\alpha|\nabla\varphi_R|^2\leq\frac{1}{2}\int_{\Omega_R}\alpha|\nabla\tilde{\varphi}_R|^2.$$

On the other hand, from the definition of $\tilde{\varphi}_R$ we have $\tilde{\varphi}_R=\varphi_R$ in $B_R\setminus\overline{B_r}$. Consequently we deduce:

$$\frac{1}{2}\int_{\Omega_r}\alpha|\nabla\varphi_R|^2\leq\frac{1}{2}\int_{\Omega_r}\alpha|\nabla\tilde{\varphi}_R|^2.$$

With (20) and since $r\in(R^{1/4},\sqrt{R})$ we may conclude

$$\frac{1}{2}\int_{\Omega_{R^{1/4}}}\alpha|\nabla\varphi_R|^2\leq\frac{C_0}{\ln R}.$$

In particular, for a compact set $K\subset\mathbb{R}^2\setminus\omega$ s.t. $\partial\omega\subset\partial K$ we have for sufficiently large R

$$\frac{1}{2}\int_K\alpha|\nabla\varphi_R|^2\leq\frac{C_0}{\ln R}.$$

Since $\fint_{\partial\omega}\varphi_R=0$, we may use a Poincaré type inequality to get:

$$\|\varphi_R\|_{H^1(K)}\to 0 \text{ when } R\to\infty \text{ independently of } R'>R.$$

It suffices to note that $\varphi_R=\gamma_{R'}-\gamma_R$ in order to conclude that $(\gamma_R)_R$ is a Cauchy family in $H^1(K)$. Then $(\gamma_R)_R$ is a Cauchy family in $H^1_{\text{loc}}(\mathbb{R}^2\setminus\omega)$. The completeness of $H^1_{\text{loc}}(\mathbb{R}^2\setminus\omega,\mathbb{R})$ allows to get the existence of $\gamma_\infty\in H^1_{\text{loc}}(\mathbb{R}^2\setminus\omega,\mathbb{R})$ s.t. $\gamma_R\to\gamma_\infty$ in $H^1_{\text{loc}}(\mathbb{R}^2\setminus\omega)$. □

Corollary 19. *We have two direct consequences of Proposition 16 :*

1. $\text{tr}_{\partial\omega}(\gamma_R)\to\text{tr}_{\partial\omega}(\gamma_\infty)$ *in* $H^{1/2}(\partial\omega)$,
2. $v_R=\dfrac{x}{|x|}\mathrm{e}^{\imath\gamma_R}\to v_\infty:=\dfrac{x}{|x|}\mathrm{e}^{\imath\gamma_\infty}$ *in* $H^1_{\text{loc}}(\mathbb{R}^2\setminus\omega)$.

4.2 The special solution in $\omega_{\rho,\mathbf{z}}$

As for the special solution in Ω_R, we first consider the minimization problem:

$$\inf_{\substack{w\in H^1(\omega_{\rho,\mathbf{z}},\mathbb{S}^1)\\ \deg_{\partial B(z_i,\rho)}(w)=d_i\ \forall i}}\frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}}|\nabla w|^2. \qquad (22)$$

From Proposition 10, we may fix $w_{\rho,\mathbf{z},\mathbf{d}}$, a unique solution of (22), by imposing

$$w_{\rho,\mathbf{z},\mathbf{d}} = \prod_{i=1}^{N} \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} \mathrm{e}^{\imath \gamma_{\rho,\mathbf{z},\mathbf{d}}} \text{ with } \int_{\partial\omega} \gamma_{\rho,\mathbf{z},\mathbf{d}} = 0. \tag{23}$$

For $i \in \{1, ..., N\}$, we may locally define θ_i in $\mathbb{R}^2 \setminus \{z_i\}$ as a lifting of $\dfrac{x - z_i}{|x - z_i|}$, *i.e.*, $\mathrm{e}^{\imath\theta_i} = \dfrac{x - z_i}{|x - z_i|}$. Moreover $\nabla\theta_i$ is globally defined.

We denote $\Theta := d_1\theta_1 + ... + d_N\theta_N$ which is locally defined in $\mathbb{R}^2 \setminus \{z_1, ..., z_N\}$ and whose gradient is globally defined in $\mathbb{R}^2 \setminus \{z_1, ..., z_N\}$. It is clear that

$$\prod_{i=1}^{N} \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} = \mathrm{e}^{\imath\Theta} \text{ in } \mathbb{R}^2 \setminus \{z_1, ..., z_N\}.$$

From the definition of $w_{\rho,\mathbf{z},\mathbf{d}}$ we have the following proposition.

Proposition 20. *$w_{\rho,\mathbf{z},\mathbf{d}} = \mathrm{e}^{\imath(\Theta + \gamma_{\rho,\mathbf{z},\mathbf{d}})}$ with $\gamma_{\rho,\mathbf{z},\mathbf{d}} \in H^1(\omega_{\rho,\mathbf{z}})$ which is a solution of*

$$\begin{cases} -\Delta\gamma_{\rho,\mathbf{z},\mathbf{d}} = 0 \text{ in } \omega_{\rho,\mathbf{z}} \\ \nabla(\Theta + \gamma_{\rho,\mathbf{z},\mathbf{d}}) \cdot \nu = 0 \text{ on } \partial\omega_{\rho,\mathbf{z}} \\ \int_{\partial\omega} \gamma_{\rho,\mathbf{z},\mathbf{d}} = 0 \end{cases}. \tag{24}$$

In contrast with the previous section, the asymptotic behavior of $w_{\rho,\mathbf{z},\mathbf{d}}$ is well known when $\rho \to 0$. For example Lefter and Rădulescu proved the following theorem.

Theorem 4. *[Theorem 1 [LR96]] For $\rho_0 > \rho > 0$ we let w_ρ be a minimizer of (22) and we consider a sequence $\rho_n \downarrow 0$. Up to pass to a subsequence, there exists $w_0 \in C^\infty(\overline{\omega} \setminus \{z_1, ..., z_N\}, \mathbb{S}^1)$ s.t. $w_{\rho_n} \to w_0$ dans $C^k_{\mathrm{loc}}(\overline{\omega} \setminus \{z_1, ..., z_N\})$ for all $k \geq 0$.*

Moreover the limits w_0 are unique up to the multiplication by a constant in $\mathbb{S}^1$.

From Theorem 4, we get that the possible limits w_0's are unique up to a constant rotation. Thus there exists a unique limit $w_{0,\mathbf{z},\mathbf{d}}$ [given by Theorem 4] which may be written:

$$w_{0,\mathbf{z},\mathbf{d}} = \prod_{i=1}^{N} \left(\frac{x - z_i}{|x - z_i|} \right)^{d_i} \mathrm{e}^{\imath \gamma_{0,\mathbf{z},\mathbf{d}}} \text{ with } \int_{\partial\omega} \gamma_{0,\mathbf{z},\mathbf{d}} = 0. \tag{25}$$

On the other hand, it is easy to check that for $\rho = \rho_n \downarrow 0$, if $w_{\rho,\mathbf{z},\mathbf{d}} = \mathrm{e}^{\imath(\Theta + \gamma_{\rho,\mathbf{z},\mathbf{d}})} \to w_0 = \mathrm{e}^{\imath(\Theta + \gamma_0)}$ in $C^1(\partial\omega)$ then $\mathrm{e}^{\imath\gamma_{\rho,\mathbf{z},\mathbf{d}}} \to \mathrm{e}^{\imath\gamma_0}$ in $C^1(\partial\omega)$. Moreover if we impose $\int_{\partial\omega} \gamma_0 \in [0, 2\pi[$ then we immediately get $\int_{\partial\omega} \gamma_0 = 0$.

We thus have the following corollary:

Corollary 21. *Let $\gamma_{0,\mathbf{z},\mathbf{d}} \in H^1_{\mathrm{loc}}(\overline{\omega} \setminus \{z_1, ..., z_N\}, \mathbb{R})$ be defined by (25). When $\rho \to 0$ we have $\gamma_{\rho,\mathbf{z},\mathbf{d}} \to \gamma_{0,\mathbf{z},\mathbf{d}}$ in $H^1_{\mathrm{loc}}(\overline{\omega} \setminus \{z_1, ..., z_N\})$.*

Thus we also get $\mathrm{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}}) \to \mathrm{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}})$ in $H^{1/2}(\partial\omega)$.

Proof. Let $K \subset \overline{\omega} \setminus \{z_1, ..., z_N\}$ be a connected compact set s.t. $\partial\omega \subset \partial K$ and let $\rho_n \downarrow 0$ be s.t. $w_{\rho_n,\mathbf{z},\mathbf{d}} = \mathrm{e}^{\imath(\Theta + \gamma_{\rho_n,\mathbf{z},\mathbf{d}})} \to w_0 = \mathrm{e}^{\imath(\Theta + \gamma_0)}$ in $C^1(K)$ for some $\gamma_0 \in C^1(K)$. It suffices to prove that we may choose $\gamma_0 = \gamma_{0,\mathbf{z},\mathbf{d}}$ defined by (25).

On the one hand, we have $\nabla\gamma_{\rho_n,\mathbf{z},\mathbf{d}} = w_{\rho_n,\mathbf{z},\mathbf{d}} \wedge \nabla w_{\rho_n,\mathbf{z},\mathbf{d}} - \nabla\Theta \to w_0 \wedge \nabla w_0 - \nabla\Theta = \nabla\gamma_0$ in $L^2(K)$. Then $\gamma_0 = \gamma_{0,\mathbf{z},\mathbf{d}} + \lambda$ for some $\lambda \in \mathbb{R}$.

On the other hand $(\gamma_{\rho_n,\mathbf{z},\mathbf{d}})_n$ is bounded in $H^1(K)$, consequently, up to pass to a subsequence, we have $\gamma_{\rho_n,\mathbf{z},\mathbf{d}} \rightharpoonup \gamma_0$ in $H^1(K)$. We the help of the previous paragraph, we get that the convergence is in fact strong. Thus $\mathrm{tr}_{\partial\omega}(\gamma_{\rho_n,\mathbf{z},\mathbf{d}}) \to \mathrm{tr}_{\partial\omega}(\gamma_0)$ in $L^2(\partial\omega)$.

In conclusion

$$0 = \fint_{\partial\omega} \gamma_{\rho_n,\mathbf{z},\mathbf{d}} \to \fint_{\partial\omega} \gamma_0 = \lambda + \fint_{\partial\omega} \gamma_{0,\mathbf{z},\mathbf{d}} = \lambda = 0.$$

This means $\gamma_0 = \gamma_{0,\mathbf{z},\mathbf{d}}$. ∎

About the asymptotic energetic expanding, Lefter and Rădulescu proved the following result:

Theorem 5. *[Theorem 2 [LR96]] For $N \in \mathbb{N}^*$, there exists a map $W : (\omega^N)^* \times \mathbb{Z}^N \to \mathbb{R}$ s.t. for $\mathbf{d} \in \mathbb{Z}^N$ and $\mathbf{z} \in (\omega^N)^*$ when $\rho \to 0$ we have:*

$$\inf_{\substack{w \in H^1(\omega_{\rho,\mathbf{z}},\mathbb{S}^1) \\ \deg_{\partial B(z_i,\rho)}(w)=d_i \,\forall i}} \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w|^2 = \left(\sum_{i=1}^N d_i^2\right)\pi|\ln\rho| + W(\mathbf{z},\mathbf{d}) + o_\rho(1).$$

5 Upper Bound

We are now in position to start the proof of Theorem 1. To this end, the goal of this section is to identify a map

$$\mathcal{K} : \{h \in H^{1/2}(\partial\omega,\mathbb{S}^1) \,|\, \deg(h) = d\} \to \mathbb{R}$$

s.t. for a fixed $h \in H^{1/2}(\partial\omega,\mathbb{S}^1)$ with $\deg(h) = d$, when $R \to \infty$ we have

$$\begin{aligned}\inf_{\substack{v \in H^1(\Omega_R,\mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(v)=h}} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v|^2 \quad + \quad & \inf_{\substack{w \in H^1(\omega_{\rho,\mathbf{z}},\mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(w)=h \\ \deg_{\partial B(z_i,\rho)}(w)=d_i \,\forall\, i}} \frac{b^2}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w|^2 \\ & = \mathcal{K}(h) + d^2 f(R) + b^2\left[\left(\sum_{i=1}^N d_i^2\right)|\ln\rho| + W(\mathbf{z},\mathbf{d})\right] + o(1). \quad (26)\end{aligned}$$

In the above estimate we have:

- $\mathcal{K}$ is independent of R, ρ ;
- f is defined by Remark 1.2 and is independent of $h, \rho, \mathbf{z}, \mathbf{d}$ and $d = \sum d_i$;
- W is independent of b, B, h, ρ and R.

For this purpose we fix $h \in H^{1/2}(\partial\omega,\mathbb{S}^1)$ s.t. $\deg(h) = d$. In this section we identify $\mathcal{K}(h)$ in order to have a such expanding.

Note that from Corollaries 19 and 21, we have the existence of

- $\gamma_\infty \in H^{1/2}(\partial\omega)$ s.t. $\gamma_R \to \gamma_\infty$ in $H^{1/2}(\partial\omega)$,
- $\gamma_{0,\mathbf{z},\mathbf{d}} \in H^{1/2}(\partial\omega)$ s.t. $\gamma_{\rho,\mathbf{z},\mathbf{d}} \to \gamma_{0,\mathbf{z},\mathbf{d}}$ in $H^{1/2}(\partial\omega)$.

It is important to claim that since $\int_{\partial\omega}\gamma_R = 0$ and $\int_{\partial\omega}\gamma_{\rho,\mathbf{z},\mathbf{d}} = 0$, we have $\int_{\partial\omega}\gamma_\infty = 0$ and $\int_{\partial\omega}\gamma_{0,\mathbf{z},\mathbf{d}} = 0$.

5.1 Study in the domain Ω_R

For $R \in [R_0,\infty[$ and $h \in H^{1/2}(\partial\omega,\mathbb{S}^1)$ s.t. $\deg(h) = d$ we consider

$$I_R(h) := \inf_{\substack{v \in H^1(\Omega_R,\mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(v)=h}} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v|^2. \quad (27)$$

Our goal is to estimate $I_R(h)$ when $R \to \infty$.

We let

$$g_0 := h\left(\frac{\overline{x}}{|x|}\right)^d \in H^{1/2}(\partial\omega).$$

From Proposition 8.3, we have $\deg(g_0) = 0$. Consequently, using Proposition 8.2 we may fix a unique $\phi_0 \in H^{1/2}(\partial\omega,\mathbb{R})$ s.t.

$$g_0 = \mathrm{e}^{\imath\phi_0} \text{ and } \fint_{\partial\omega} \phi_0 \in [0, 2\pi[.$$

Remark 22. It is clear that for $\beta \in \mathbb{R}$ we have $I_R(h) = I_R(\mathrm{e}^{\imath\beta}h)$. Thus, up to replace h by $\mathrm{e}^{\imath\beta}h$ with $\beta = -\fint_{\partial\omega} \phi_0$, we may assume that $\fint_{\partial\omega} \phi_0 = 0$.

For $R \in [R_0, \infty]$ we let

$$g_R := h\mathrm{tr}_{\partial\omega}(\overline{v_R^d}) = g_0 \mathrm{e}^{-\imath d \mathrm{tr}_{\partial\omega}(\gamma_R)},$$

in order to have $h = g_R \mathrm{tr}_{\partial\omega}(v_R^d)$. [Note that v_∞ is defined in Corollary 19]

Consequently we have $g_R = \mathrm{e}^{\imath(\phi_0 - d\mathrm{tr}_{\partial\omega}(\gamma_R))}$. Finally we let $\phi_R := \phi_0 - d\mathrm{tr}_{\partial\omega}(\gamma_R) \in H^{1/2}(\partial\omega, \mathbb{R})$ and so we get $h = \mathrm{tr}_{\partial\omega}(v_R^d)\mathrm{e}^{\imath\phi_R}$ and $\int_{\partial\omega} \phi_R = 0$.

From Corollary 19 we immediately obtain:

Corollary 23. *$\phi_R \underset{R\to\infty}{\longrightarrow} \phi_\infty$ in $H^{1/2}(\partial\omega)$.*

For $R \in [R_0, +\infty[$ and $v \in H^1(\Omega_R, \mathbb{S}^1)$, from Proposition 8, we may prove the following equivalence:

$$\mathrm{tr}_{\partial\omega}(v) = h \Longleftrightarrow v = v_R^d \mathrm{e}^{\imath\varphi} \text{ with } \left| \begin{array}{c} \varphi \in H^1(\Omega_R, \mathbb{R}) \\ \mathrm{tr}_{\partial\omega}(\varphi) = \phi_R \end{array} \right. .$$

On the other hand, for $v = v_R^d \mathrm{e}^{\imath\varphi} \in H^1(\Omega_R, \mathbb{S}^1)$, from Lemma 15 we have

$$\frac{1}{2}\int_{\Omega_R} \alpha|\nabla v|^2 \quad = \quad \frac{d^2}{2}\int_{\Omega_R} \alpha|\nabla v_R|^2 + \frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi|^2. \tag{28}$$

Therefore, one may obtain that $v = v_R^d \mathrm{e}^{\imath\varphi}$ with $\mathrm{tr}_{\partial\omega}(\varphi) = \phi_R$ is a solution of the minimization problem

$$\inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(v) = h}} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v|^2$$

if and only if $\varphi \in H^1(\Omega_R, \mathbb{R})$ is a solution of the minimization problem

$$\inf_{\substack{\varphi \in H^1(\Omega_R, \mathbb{R}) \\ \mathrm{tr}_{\partial\omega}(\varphi) = \phi_R}} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi|^2. \tag{29}$$

It is easy to get:

Proposition 24. *For $R \in [R_0, \infty[$, Problem (29) admits a unique solution denoted by φ_R. Moreover this minimizer is the unique solution of*

$$\begin{cases} -\mathrm{div}(\alpha\nabla\varphi_R) = 0 \text{ in } \Omega_R \\ \mathrm{tr}_{\partial\omega}(\varphi_R) = \phi_R \\ \partial_\nu \varphi_R = 0 \text{ on } \partial B_R \end{cases} .$$

For $R = \infty$, we denote $\Omega_\infty := \mathbb{R}^2 \setminus \overline{\omega}$ and

$$\mathscr{H}_{\phi_\infty} := \{\varphi \in H^1_{\mathrm{loc}}(\overline{\Omega_\infty}, \mathbb{R}) \,|\, \nabla\varphi \in L^2(\Omega_\infty) \text{ and } \mathrm{tr}_{\partial\omega}(\varphi) = \phi_\infty\}. \tag{30}$$

We are now interested in the minimization problem:

$$\inf_{\varphi \in \mathscr{H}_{\phi_\infty}} \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi|^2. \tag{31}$$

By direct minimization we get:

Proposition 25. *Problem (31) admits a unique solution denoted by φ_∞. Moreover φ_∞ is a solution of*

$$-\mathrm{div}(\alpha\nabla\varphi_\infty) = 0 \text{ dans } \Omega_\infty. \tag{32}$$

We are now able to prove the main result of this section:

Proposition 26. *We have:*

$$\varphi_R \to \varphi_\infty \text{ in } H^1_{\text{loc}}(\mathbb{R}^2 \setminus \omega)$$

and

$$\nabla\varphi_R \mathbb{1}_{\Omega_R} \to \nabla\varphi_\infty \text{ in } L^2(\mathbb{R}^2 \setminus \overline{\omega}), \text{ with } \mathbb{1}_{\Omega_R}(x) = \begin{cases} 1 & \text{if } x \in \Omega_R \\ 0 & \text{if } x \notin \Omega_R \end{cases}.$$

And consequently:

$$\int_{\Omega_R} \alpha|\nabla\varphi_R|^2 = \int_{\Omega_\infty} \alpha|\nabla\varphi_\infty|^2 + o_R(1).$$

Proof. From Corollary 23 we have $\phi_R - \phi_\infty \to 0$ in $H^{1/2}(\partial\omega)$. Consequently, there exists $\xi_R \in H^1(\Omega_\infty, \mathbb{R})$ s.t.

$$\text{tr}_{\partial\omega}(\xi_R) = \phi_R - \phi_\infty \text{ and } \|\xi_R\|_{H^1(\Omega_\infty)} \to 0.$$

The test function $\varphi_\infty + \xi_R$ satisfies the boundary condition of Problem (29), therefore:

$$\frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi_R|^2 \leq \frac{1}{2}\int_{\Omega_R} \alpha|\nabla(\varphi_\infty + \xi_R)|^2 = \frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi_\infty|^2 + o(1). \tag{33}$$

Note we used $\displaystyle\int_{\Omega_R} \alpha|\nabla\varphi_\infty|^2 \leq C_0 := \int_{\Omega_\infty} \alpha|\nabla\varphi_\infty|^2 < \infty$. From (33), we obtain

$$\limsup_{R\to\infty} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi_R|^2 \leq \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi_\infty|^2. \tag{34}$$

We now prove *the* "lim inf"-lower bound:

$$\liminf_{R\to\infty} \frac{1}{2}\int_{\Omega_R} \alpha|\nabla\varphi_R|^2 \geq \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi_\infty|^2. \tag{35}$$

On the one hand, for $R \in [R_0, \infty[$, sufficiently large we have $\displaystyle\int_{\Omega_R} \alpha|\nabla\varphi_R|^2 \leq C_0 + 1$ and thus, up to pass to a subsequence, we have $\nabla\varphi_R \mathbb{1}_{\Omega_R}$ which weakly converges in $L^2(\mathbb{R}^2 \setminus \overline{\omega}, \mathbb{R}^2)$.

On the other hand, for a connected compact set $K \subset \mathbb{R}^2 \setminus \omega$ s.t. $\partial\omega \subset \partial K$, the test function $\varphi_\infty + \xi_R$ is bounded in $H^1(K)$.

We let $\chi_R := \varphi_R - (\varphi_\infty + \xi_R) \in H^1(K)$. It is easy to check that $\text{tr}_{\partial\omega}(\chi_R) = 0$. Then, from a Poincaré type inequality, there exists a constant $C_K > 1$ s.t.

$$\|\chi_R\|_{L^2(K)} \leq C_K \|\nabla\chi_R\|_{L^2(K)}.$$

Thus

$$\begin{aligned} \|\varphi_R\|_{L^2(K)} &\leq C_K(\|\nabla\varphi_R\|_{L^2(K)} + \|\nabla\varphi_\infty\|_{L^2(K)} + \|\nabla\xi_R\|_{L^2(K)}) + \|\varphi_\infty\|_{L^2(K)} + \|\xi_R\|_{L^2(K)} \\ &\leq \tilde{C}_K. \end{aligned}$$

Consequently, with the help of an exhaustion by compacts set and a diagonal extraction process, we have the existence of a sequence $R_k \uparrow \infty$ and $\tilde{\varphi}_\infty \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \omega, \mathbb{R})$ s.t.

$$\begin{cases} \varphi_{R_k} \rightharpoonup \tilde{\varphi}_\infty \text{ in } H^1_{\text{loc}}(\mathbb{R}^2 \setminus \omega) \text{ and } \nabla\varphi_{R_k}\mathbb{1}_{\Omega_{R_k}} \rightharpoonup \nabla\tilde{\varphi}_\infty \text{ in } L^2(\mathbb{R}^2 \setminus \overline{\omega}). \\ \displaystyle\liminf_{R\to\infty} \int_{\Omega_R} \alpha|\nabla\varphi_R|^2 = \lim_{R_k\to 0} \int_{\Omega_{R_k}} \alpha|\nabla\varphi_{R_k}|^2 \end{cases}. \tag{36}$$

We thus get $\nabla\tilde{\varphi}_\infty \in L^2(\mathbb{R}^2 \setminus \overline{\omega})$ and $\text{tr}_{\partial\omega}(\tilde{\varphi}_\infty) = \phi_\infty$, *i.e.*, $\tilde{\varphi}_\infty \in \mathscr{H}_{\phi_\infty}$.

From the definition of φ_∞ [Proposition 25] we have

$$\frac{1}{2}\int_{\Omega_\infty}\alpha|\nabla\varphi_\infty|^2 \leq \frac{1}{2}\int_{\Omega_\infty}\alpha|\nabla\tilde{\varphi}_\infty|^2 \leq \liminf_{R\to\infty}\frac{1}{2}\int_{\Omega_R}\alpha|\nabla\varphi_R|^2.$$

We thus obtained (35). Therefore by combining (34) and (35) we have:

$$\int_{\Omega_R}\alpha|\nabla\varphi_R|^2 = \int_{\Omega_\infty}\alpha|\nabla\varphi_\infty|^2 + o_R(1). \tag{37}$$

The above estimate implies that a limiting map $\tilde{\varphi}_\infty \in \mathscr{H}_{\phi_\infty}$ as previously obtained satisfies:

$$\frac{1}{2}\int_{\Omega_\infty}\alpha|\nabla\tilde{\varphi}_\infty|^2 \leq \frac{1}{2}\int_{\Omega_\infty}\alpha|\nabla\varphi_\infty|^2.$$

On the other hand φ_∞ is the unique solution of Problem (31). Therefore $\tilde{\varphi}_\infty = \varphi_\infty$. Consequently, the convergences in (36) hold for $R\to\infty$ and from (37), these convergences are strong. This ends the proof of the proposition. □

5.2 Study in the domain $\omega_{\rho,\mathbf{z}}$

Recall that we fixed a map $h \in H^{1/2}(\partial\omega,\mathbb{S}^1)$ s.t. $\deg(h) = d$. We are interested in the minimizing problem

$$I_{\rho,\mathbf{z},\mathbf{d}}(h) = \inf_{\substack{w\in H^1(\omega_{\rho,\mathbf{z}},\mathbb{S}^1)\\ \mathrm{tr}_{\partial\omega}(w)=h\\ \deg_{\partial B(z_i,\rho)}(w)=d_i\ \forall\, i}} \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}}|\nabla w|^2. \tag{38}$$

First note that letting

$$g_{\mathbf{z},\mathbf{d}} = h\prod_{i=1}^{N}\left(\frac{|x-z_i|}{x-z_i}\right)^{d_i}$$

we have $\deg(g_{\mathbf{z},\mathbf{d}}) = \mathbf{0}_{\mathbb{Z}^N}$. Thus, from Proposition 8.2, we may fix $\phi^h_{\mathbf{z},\mathbf{d}} \in H^{1/2}(\partial\omega,\mathbb{R})$ s.t. $g_{\mathbf{z},\mathbf{d}} = \mathrm{e}^{\imath\phi^h_{\mathbf{z},\mathbf{d}}}$ and $\fint_{\partial\omega}\phi^h_{\mathbf{z},\mathbf{d}} \in [0,2\pi[$. It is clear that $\phi^h_{\mathbf{z},\mathbf{d}}$ is uniquely defined.

Remark 27. As in the previous section [see Remark 22], for $\beta\in\mathbb{R}$ we have $I_{\rho,\mathbf{z},\mathbf{d}}(h) = I_{\rho,\mathbf{z},\mathbf{d}}(h\mathrm{e}^{\imath\beta})$. Thus up to replace h by $h\mathrm{e}^{\imath\beta}$, with $\beta = -\fint_{\partial\omega}\phi^h_{\mathbf{z},\mathbf{d}}$, in order to estimate $I_{\rho,\mathbf{z},\mathbf{d}}(h)$, we may assume that $\fint_{\partial\omega}\phi^h_{\mathbf{z},\mathbf{d}} = 0$.

For $\rho\in[0,\rho_0]$ we let

$$g_{\rho,\mathbf{z},\mathbf{d}} := h\mathrm{tr}_{\partial\omega}(\overline{w}_{\rho,\mathbf{z},\mathbf{d}}) = g_{\mathbf{z},\mathbf{d}}\mathrm{e}^{-\imath\gamma_{\rho,\mathbf{z},\mathbf{d}}}$$

in order to have $h = g_{\rho,\mathbf{z},\mathbf{d}}\mathrm{tr}_{\partial\omega}(w_{\rho,\mathbf{z},\mathbf{d}})$. Note that, for $\rho\in[0,\rho_0]$, $w_{\rho,\mathbf{z},\mathbf{d}}$ is defined in (23) and (25).

Thus letting $\phi_{\rho,\mathbf{z},\mathbf{d}} := \phi^h_{\mathbf{z},\mathbf{d}} - \gamma_{\rho,\mathbf{z},\mathbf{d}} \in H^{1/2}(\partial\omega,\mathbb{R})$ we have $g_{\rho,\mathbf{z},\mathbf{d}} = \mathrm{e}^{\imath\phi_{\rho,\mathbf{z},\mathbf{d}}}$ and $\fint_{\partial\omega}\phi_{\rho,\mathbf{z},\mathbf{d}} = 0$

As in the previous section [Corollary 23], we easily get the following convergence result:

Corollary 28. *$\phi_{\rho,\mathbf{z},\mathbf{d}} \underset{\rho\to 0}{\longrightarrow} \phi_{0,\mathbf{z},\mathbf{d}}$ in $H^{1/2}(\partial\omega)$.*

For $\rho\in]0,\rho_0]$ and $w\in H^1(\omega_{\rho,\mathbf{z}},\mathbb{S}^1)$, we have

$$\mathrm{tr}_{\partial\omega}(w) = h \Longleftrightarrow w = w_{\rho,\mathbf{z},\mathbf{d}}\mathrm{e}^{\imath\varphi} \text{ with } \left|\begin{array}{l}\varphi\in H^1(\omega_{\rho,\mathbf{z}},\mathbb{R})\\ \mathrm{tr}_{\partial\omega}(\varphi) = \phi_{\rho,\mathbf{z},\mathbf{d}}\end{array}\right. .$$

From Proposition 8.6, we have the existence of a map $\psi_{\rho,\mathbf{z},\mathbf{d}}$ locally defined in $\omega_{\rho,\mathbf{z}}$ [whose gradient is in $L^2(\omega_{\rho,\mathbf{z}},\mathbb{R}^2)$] s.t. $w_{\rho,\mathbf{z},\mathbf{d}} = \mathrm{e}^{\imath\psi_{\rho,\mathbf{z},\mathbf{d}}}$.

For $w = w_{\rho,\mathbf{z},\mathbf{d}}\mathrm{e}^{\imath\varphi} \in H^1(\omega_{\rho,\mathbf{z}}, \mathbb{S}^1)$ we have

$$\begin{aligned}
\frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w|^2 &= \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla \psi_{\rho,\mathbf{z},\mathbf{d}} + \nabla\varphi|^2 \\
&= \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla \psi_{\rho,\mathbf{z},\mathbf{d}}|^2 + \int_{\omega_{\rho,\mathbf{z}}} \nabla\psi_{\rho,\mathbf{z},\mathbf{d}}\cdot\nabla\varphi + \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla\varphi|^2 \\
[\text{with } (24)] &= \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2 + \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla\varphi|^2. \qquad (39)
\end{aligned}$$

Consequently a test function $w = w_{\rho,\mathbf{z},\mathbf{d}}\mathrm{e}^{\imath\varphi}$ with $\mathrm{tr}_{\partial\omega}(\varphi) = \phi_{\rho,\mathbf{z},\mathbf{d}}$ is a solution of the minimizing problem (38) if and only if $\varphi \in H^1(\omega_{\rho,\mathbf{z}}, \mathbb{R})$ is a solution of the minimizing problem

$$\inf_{\substack{\varphi\in H^1(\omega_{\rho,\mathbf{z}},\mathbb{R}) \\ \mathrm{tr}_{\partial\omega}(\varphi)=\phi_{\rho,\mathbf{z},\mathbf{d}}}} \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla\varphi|^2. \qquad (40)$$

It is easy to get the following proposition:

Proposition 29. *For $\rho \in]0, \rho_0]$, the minimizing Problem (40) admits a unique solution denoted by $\varphi_{\rho,\mathbf{z},\mathbf{d}}$. Moreover this solution satisfies:*

$$\begin{cases} -\Delta\varphi_{\rho,\mathbf{z},\mathbf{d}} = 0 \text{ in } \omega_{\rho,\mathbf{z}} \\ \mathrm{tr}_{\partial\omega}(\varphi_{\rho,\mathbf{z},\mathbf{d}}) = \phi_{\rho,\mathbf{z},\mathbf{d}} \\ \partial_\nu\varphi_{\rho,\mathbf{z},\mathbf{d}} = -\partial_\nu\Theta \text{ on } \partial B(z_i,\rho),\ i = 1, ..., N \end{cases}$$

where Θ is defined by Proposition 8.6 s.t.

$$\prod_{i=1}^{N}\left(\frac{x - z_i}{|x - z_i|}\right)^{d_i} = \mathrm{e}^{\imath\Theta} \text{ in } \mathbb{R}^2 \setminus \{z_1, ..., z_N\}.$$

About the asymptotic behavior of $\varphi_{\rho,\mathbf{z},\mathbf{d}}$ we have the following result:

Proposition 30. *When $\rho \to 0$, we have*

$$\frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 = \frac{1}{2}\int_{\omega} |\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1)$$

where $\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}$ is the harmonic extension of $\phi_{0,\mathbf{z},\mathbf{d}}$ in ω.

Proof. Let ξ_ρ be the harmonic extension of $\phi_{0,\mathbf{z},\mathbf{d}} - \phi_{\rho,\mathbf{z},\mathbf{d}}$ in ω. Since $\|\phi_{0,\mathbf{z},\mathbf{d}} - \phi_{\rho,\mathbf{z},\mathbf{d}}\|_{H^{1/2}(\partial\omega)} \to 0$, we have $\xi_\rho \to 0$ in $H^1(\omega)$.

We now prove the proposition. On the one hand, by minimality of $\varphi_{\rho,\mathbf{z},\mathbf{d}}$ and since $\mathrm{tr}_{\partial\omega}(\tilde{\phi}_{0,\mathbf{z},\mathbf{d}} - \xi_\rho) = \phi_{\rho,\mathbf{z},\mathbf{d}}$ we get

$$\begin{aligned}
\frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 &\le \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla(\tilde{\phi}_{0,\mathbf{z},\mathbf{d}} - \xi_\rho)|^2 \\
&\le \frac{1}{2}\int_{\omega} |\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1). \qquad (41)
\end{aligned}$$

On the other hand, from the Estimate (41), denoting $C_0 := \int_\omega |\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + 1$, for sufficiently small ρ we get

$$\sum_{i=1}^{N}\frac{1}{2}\int_{B(z_i,\sqrt{\rho})\setminus\overline{B(z_i,\rho)}} |\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 < C_0. \qquad (42)$$

Thus for small ρ, we get the existence of $\rho' \in]\rho, \sqrt{\rho}[$ s.t.:

$$\sum_{i=1}^{N} \frac{1}{2} \int_0^{2\pi} |\partial_\theta \varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i + \rho' \mathrm{e}^{\imath\theta})|^2 \leq \frac{2C_0}{|\ln \rho|}.$$

For $i \in \{1, ..., N\}$ we let

$$m_{i,\rho} := \frac{1}{2\pi} \int_0^{2\pi} \varphi(z_i + \rho' \mathrm{e}^{\imath\theta}).$$

We now define $\tilde{\varphi} \in H^1(\omega)$ by $\tilde{\varphi} = \varphi_{\rho,\mathbf{z},\mathbf{d}}$ in $\omega \setminus \cup_i \overline{B(z_i, \rho')}$ and for $x = z_i + s\mathrm{e}^{\imath\theta} \in B(z_i, \rho')$ [with $i \in \{1, ..., N\}$]

$$\tilde{\varphi}(z_i + s\mathrm{e}^{\imath\theta}) = \left| \begin{array}{ll} \dfrac{2s - \rho'}{\rho'} \varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i + \rho' \mathrm{e}^{\imath\theta}) + \dfrac{2(\rho' - s)}{\rho'} m_{i,\rho} & \text{if } s \in]\dfrac{\rho'}{2}, \rho'[\\ m_{i,\rho} & \text{if } s \leq \dfrac{\rho'}{2} \end{array} \right. .$$

A direct calculation [similar to (20)] gives for $z \in \{z_1, ..., z_N\}$

$$\int_{B(z,\rho')} |\nabla \tilde{\varphi}|^2 = \mathcal{O}\left[\int_0^{2\pi} |\partial_\theta \varphi_{\rho,\mathbf{z},\mathbf{d}}(z + \rho' \mathrm{e}^{\imath\theta})|^2 \right] = o_\rho(1).$$

Therefore we obtain

$$\frac{1}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\varphi}|^2 + o_\rho(1).$$

But $\mathrm{tr}_{\partial\omega}(\tilde{\varphi} + \xi_\rho) = \phi_{0,\mathbf{z},\mathbf{d}}$ and consequently, from the Dirichlet principle, we have:

$$\frac{1}{2} \int_\omega |\nabla(\tilde{\varphi} + \xi_\rho)|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2$$

and thus:

$$\frac{1}{2} \int_\omega |\nabla \tilde{\varphi}|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1).$$

On the other hand, since $\tilde{\varphi} = \varphi_{\rho,\mathbf{z},\mathbf{d}}$ in $\omega \setminus \cup_i \overline{B(z_i, \rho')} \subset \omega_{\rho,\mathbf{z}}$ and $\frac{1}{2} \int_{\cup_i B(z_i,\rho')} |\nabla \tilde{\varphi}|^2 = o_\rho(1)$ we obtain:

$$\frac{1}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \geq \frac{1}{2} \int_{\omega \setminus \cup_i \overline{B(z_i,\rho')}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \geq \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1).$$

Finally, using (41), by matching upper bound and lower bound we conclude:

$$\frac{1}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 = \frac{1}{2} \int_\omega |\nabla \tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1).$$

The last estimates ends the proof of the proposition. □

5.3 Conclusion

For $h \in H^{1/2}(\partial\omega, \mathbb{S}^1)$ s.t. $\deg(h) = d$ we have from (28) and Proposition 26:

$$\inf_{\substack{v \in H^1(\Omega_R, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\omega}(v) = h}} \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v|^2 = \frac{d^2}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2 + \inf_{\varphi \in \mathscr{H}_{\phi_\infty}} \frac{1}{2} \int_{\Omega_\infty} \alpha |\nabla \varphi|^2 + o_R(1). \tag{43}$$

Recall that ϕ_∞ is defined in Corollary 23 and $\mathscr{H}_{\phi_\infty}$ in (30).

Using Theorem 5, (39) and Proposition 30, letting $\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}$ be the harmonic extension of $\phi_{0,\mathbf{z},\mathbf{d}}$ in ω [recall that $\phi_{0,\mathbf{z},\mathbf{d}}$ is defined in Corollary 28], we have

$$\inf_{\substack{w\in H^1(\omega_{\rho,\mathbf{z}},\mathbb{S}^1)\\ \mathrm{tr}_{\partial\omega}(w)=h}} \frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w|^2 = \left(\sum_i d_i^2\right)\pi|\ln\rho| + W(\mathbf{z},\mathbf{d}) + \frac{1}{2}\int_\omega |\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1). \tag{44}$$

We let $\mathcal{K} : \left\{h \in H^{1/2}(\partial\omega,\mathbb{S}^1)\,|\,\deg(h)=d\right\} \to \mathbb{R}^+$ be defined by:

$$\mathcal{K}(h) := \inf_{\varphi\in\mathscr{H}_{\phi_\infty}} \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi|^2 + \frac{b^2}{2}\int_\omega |\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 \tag{45}$$

and

$$f(R) := \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v_R|^2$$

which gives (26).

Recall that, without loss of generality, the parameter "R" is considered as the major parameter writing $\rho=\rho(R)$. From (26), we get for $h\in H^{1/2}(\partial\omega,\mathbb{S}^1)$ s.t. $\deg(h)=d$:

$$\limsup_{R\to\infty}\left\{I(R,\rho,\mathbf{z},\mathbf{d}) - \left[d^2 f(R) + b^2\left(\sum_i d_i^2\pi|\ln\rho| + W(\mathbf{z},\mathbf{d})\right)\right]\right\} \le \mathcal{K}(h). \tag{46}$$

6 Lower bound

In this section we prove the existence of a map $h_\infty\in H^{1/2}(\partial\omega,\mathbb{S}^1)$ s.t. $\deg(h_\infty)=d$ and

$$\liminf_{R\to\infty}\left\{I(R,\rho,\mathbf{z},\mathbf{d}) - \left[d^2 f(R) + b^2\left(\sum_i d_i^2\pi|\ln\rho| + W(\mathbf{z},\mathbf{d})\right)\right]\right\} \quad\ge\quad \mathcal{K}(h_\infty). \tag{47}$$

We let $R_n\uparrow\infty$ be a sequence which realizes the "lim inf" in the left hand side of (47).

In order to keep notations simple, we drop the subscript n writing $R=R_n$ when it will not be necessary to specify the dependance on n.

Let u_R be a minimizer of (3) [Proposition 10]. From Proposition 8.5 we may decompose u_R under the form $u_R = v_R^d \mathrm{e}^{\imath\varphi_R}$ where $\varphi_R\in H^1(\Omega_R,\mathbb{R})$ and v_R is defined in (16).

Since u_R is unique up to a multiplicative constant [Proposition 10], we may freeze the non uniqueness by imposing $\int_{\partial\omega}\varphi_R=0$.

Notation 31. For sake of simplicity of the presentation we use the shorthands:

- "$R\in[R_0,\infty[$" to consider an arbitrary term of the sequence $(R_n)_n$;
- "$R\in[R_0,\infty]$" to consider an arbitrary term of the sequence $(R_n)_n$ or the limiting case $R=\infty$.

We denote:

- $h_R := \mathrm{tr}_{\partial\omega}u_R$, we thus have $h_R = \mathrm{tr}_{\partial\omega}\left[\left(\frac{x}{|x|}\right)^d \mathrm{e}^{\imath(d\gamma_R+\varphi_R)}\right]$ where $\int_{\partial\omega}\varphi_R=0$;
- $g_{\mathbf{z},\mathbf{d}} := \mathrm{tr}_{\partial\omega}\left[\left(\frac{|x|}{x}\right)^d \prod_{i=1}^N \left(\frac{x-z_i}{|x-z_i|}\right)^{d_i}\right]$.

Since $g_{\mathbf{z},\mathbf{d}}\in C^\infty(\partial\omega,\mathbb{S}^1)$ and $\deg_{\partial\omega}(g_{\mathbf{z},\mathbf{d}})=0$, from Proposition 8.2, we may fix $\xi_{\mathbf{z},\mathbf{d}}\in C^\infty(\partial\omega,\mathbb{R})$ s.t.

$$\mathrm{e}^{\imath\xi_{\mathbf{z},\mathbf{d}}} = g_{\mathbf{z},\mathbf{d}} \text{ and } \fint_{\partial\omega}\xi_{\mathbf{z},\mathbf{d}} \in]-2\pi,0].$$

6.1 Compatibility conditions

We write for $R \in [R_0, \infty[$

$$h_R := \mathrm{tr}_{\partial\omega} u_R = \mathrm{tr}_{\partial\omega}[v_R^d \mathrm{e}^{\imath\varphi_R}] = \mathrm{tr}_{\partial\omega}[w_{\rho,\mathbf{z},\mathbf{d}} \mathrm{e}^{\imath\varphi_{\rho,\mathbf{z},\mathbf{d}}}]$$

where

- $w_{\rho,\mathbf{z},\mathbf{d}} = \prod_{i=1}^{N} \left(\frac{x - z_i}{|x - z_i|}\right)^{d_i} \mathrm{e}^{\imath\gamma_{\rho,\mathbf{z},\mathbf{d}}}$ is defined in (23);
- $\varphi_{\rho,\mathbf{z},\mathbf{d}} \in H^1(\omega_{\rho,\mathbf{z}}, \mathbb{R})$ is defined by Proposition 8.5 s.t. $u_R = w_{\rho,\mathbf{z},\mathbf{d}} \mathrm{e}^{\imath\varphi_{\rho,\mathbf{z},\mathbf{d}}}$ in $\omega_{\rho,\mathbf{z}}$ and $\fint_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} \in [0, 2\pi[$.

By using Corollaries 19 and 21, we have the existence of $\gamma_\infty, \gamma_{0,\mathbf{z},\mathbf{d}} \in H^{1/2}(\partial\omega, \mathbb{R})$ s.t. $\gamma_R \to \gamma_\infty$ and $\gamma_{\rho,\mathbf{z},\mathbf{d}} \to \gamma_{0,\mathbf{z},\mathbf{d}}$ dans $H^{1/2}(\partial\omega)$. It is fundamental to note that

- γ_∞ and $\gamma_{0,\mathbf{z},\mathbf{d}}$ are independent of the sequence $(R_n)_n$;
- $\int_{\partial\omega} \gamma_R = \int_{\partial\omega} \gamma_\infty = \int_{\partial\omega} \gamma_{0,\mathbf{z},\mathbf{d}} = \int_{\partial\omega} \gamma_{\rho,\mathbf{z},\mathbf{d}} = 0.$

We have the following equivalences:

$$\begin{aligned} & \mathrm{e}^{\imath \mathrm{tr}_{\partial\omega}(\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}})} = \mathrm{tr}_{\partial\omega}[w_{\rho,\mathbf{z},\mathbf{d}} \overline{v_R^d}] \\ \Leftrightarrow \quad & \mathrm{e}^{\imath \mathrm{tr}_{\partial\omega}(\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}})} = \mathrm{e}^{\imath[\xi_{\mathbf{z},\mathbf{d}} + \mathrm{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}} - d\gamma_R)]} \\ \Leftrightarrow \quad & \mathrm{tr}_{\partial\omega}(\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}}) = \xi_{\mathbf{z},\mathbf{d}} + \mathrm{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}} - d\gamma_R) + 2k_0\pi \quad \text{with } k_0 \in \mathbb{Z}. \end{aligned} \tag{48}$$

We thus have

$$-\fint_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} = \fint_{\partial\omega} (\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}}) = \fint_{\partial\omega} [\xi_{\mathbf{z},\mathbf{d}} + \mathrm{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}} - d\gamma_R) + 2k_0\pi] = 2k_0\pi + \fint_{\partial\omega} \xi_{\mathbf{z},\mathbf{d}}.$$

Since $\fint_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} \in [0, 2\pi[$ and $\fint_{\partial\omega} \xi_{\mathbf{z},\mathbf{d}} \in]-2\pi, 0]$, the above equalities imply that $k_0 = 0$ in (48).

Consequently we get:

$$\mathrm{tr}_{\partial\omega}(\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}}) = \xi_{\mathbf{z},\mathbf{d}} + \mathrm{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}} - d\gamma_R). \tag{49}$$

6.2 Asymptotic estimate of the energy

By using (28) and (39), we have the following decoupling:

$$\begin{aligned} I(R, \rho, \mathbf{z}, \mathbf{d}) &= \frac{1}{2} \int_{\mathcal{D}_{R,\mathbf{z}}} \alpha |\nabla u_R| \\ &= \frac{1}{2} \int_{\Omega_R} \alpha |\nabla (v_R^d \mathrm{e}^{\imath\varphi_R})|^2 + \frac{b^2}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}} \mathrm{e}^{\imath\varphi_{\rho,\mathbf{z},\mathbf{d}}}|^2 \\ &= d^2 f(R) + \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R|^2 + \frac{b^2}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2 + \frac{b^2}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2. \end{aligned} \tag{50}$$

Recall that we denoted: $f(R) = \frac{1}{2} \int_{\Omega_R} \alpha |\nabla v_R|^2$.

From the minimality of u_R and by using (46), letting $C_0 := \mathcal{K}\left(\frac{x^d}{|x|^d}\right) + 1$, for sufficiently large R, we have:

$$I(R, \rho, \mathbf{z}, \mathbf{d}) - \left[d^2 f(R) + \frac{b^2}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2\right] = \frac{1}{2} \int_{\Omega_R} \alpha |\nabla \varphi_R|^2 + \frac{b^2}{2} \int_{\omega_{\rho,\mathbf{z}}} |\nabla \varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \leq C_0. \tag{51}$$

Since $\fint_{\partial\omega} \varphi_R = 0$ [resp. $\fint_{\partial\omega} \varphi_{\rho,\mathbf{z},\mathbf{d}} \in [0, 2\pi[$] for K_1 a connected compact set of $\mathbb{R}^2 \setminus \omega$ [resp. K_2 a connected compact set of $\overline{\omega} \setminus \{z_1, ..., z_N\}$] s.t. $\partial\omega \subset \partial K_1$ [resp. $\partial\omega \subset \partial K_2$], there exists $C_1 > 0$ [resp. $C_2 > 0$] s.t. for large R we have

$$\int_{K_1} |\varphi_R|^2 \leq C_1 \int_{K_1} |\nabla\varphi_R|^2 \left[\text{resp. } \int_{K_2} |\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \leq C_2 \int_{K_2} |\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2\right].$$

Consequently :

- $(\varphi_R)_R$ is bounded in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \omega)$. Thus there exists $\varphi_\infty \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \omega)$ s.t., up to pass to a subsequence, we have
$$\varphi_R \rightharpoonup \varphi_\infty \text{ in } H^1_{\text{loc}}(\mathbb{R}^2 \setminus \omega). \tag{52}$$
- $(\varphi_{\rho,\mathbf{z},\mathbf{d}})_R$ is bounded in $H^1_{\text{loc}}(\overline{\omega} \setminus \{z_1, ..., z_N\}$. Thus there exists $\varphi_{0,\mathbf{z},\mathbf{d}} \in H^1_{\text{loc}}(\overline{\omega} \setminus \{z_1, ..., z_N\})$ s.t., up to pass to a subsequence, we have
$$\varphi_{\rho,\mathbf{z},\mathbf{d}} \rightharpoonup \varphi_{0,\mathbf{z},\mathbf{d}} \text{ in } H^1_{\text{loc}}(\overline{\omega} \setminus \{z_1, ..., z_N\}). \tag{53}$$

From (49), we have

$$\text{tr}_{\partial\omega}(\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}}) = \xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}} - d\gamma_R)$$

where $\xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{\rho,\mathbf{z},\mathbf{d}} - d\gamma_R)$ is strongly converging to $\xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}} - d\gamma_\infty)$ in $H^{1/2}(\partial\omega)$ [Corollaries 19&21]. Consequently we get the same for $\text{tr}_{\partial\omega}(\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}})$, namely $\text{tr}_{\partial\omega}(\varphi_R - \varphi_{\rho,\mathbf{z},\mathbf{d}})$ is strongly convergent in $H^{1/2}(\partial\omega)$ to

$$\text{tr}_{\partial\omega}(\varphi_\infty - \varphi_{0,\mathbf{z},\mathbf{d}}) = \xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}} - d\gamma_\infty).$$

We thus may deduce:

$$\mathrm{e}^{\imath \text{tr}_{\partial\omega}(\varphi_\infty - \varphi_{0,\mathbf{z},\mathbf{d}})} = \mathrm{e}^{\imath[\xi_{\mathbf{z},\mathbf{d}} + \text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}} - d\gamma_\infty)]}$$

i.e.

$$\left(\frac{x}{|x|}\right)^d \mathrm{e}^{\imath \text{tr}_{\partial\omega}(d\gamma_\infty + \varphi_\infty)} = \prod_{i=1}^N \left(\frac{x - z_i}{|x - z_i|}\right)^{d_i} \mathrm{e}^{\imath \text{tr}_{\partial\omega}(\gamma_{0,\mathbf{z},\mathbf{d}} + \varphi_{0,\mathbf{z},\mathbf{d}})}. \tag{54}$$

We now define:

$$h_\infty := \text{tr}_{\partial\omega}\left[\left(\frac{x}{|x|}\right)^d \mathrm{e}^{\imath(d\gamma_\infty + \varphi_\infty)}\right] \in H^{1/2}(\partial\omega, \mathbb{S}^1).$$

It is clear that $\deg(h_\infty) = d$.

We prove in the three next subsections [Sections 6.3&6.4&6.5] that h_∞ satisfies (47).

6.3 Calculations in $\mathbb{R}^2 \setminus \overline{\omega}$

From (51), we get that $\nabla\varphi_R \mathbb{1}_{\Omega_R}$ is bounded in $L^2(\mathbb{R}^2 \setminus \overline{\omega})$ and thus, up to pass to a subsequence, $\nabla\varphi_R \mathbb{1}_{\Omega_R}$ weakly converge in $L^2(\mathbb{R}^2 \setminus \overline{\omega})$. Consequently, we may improve the convergence in (52), up to pass to a subsequence, we obtain that $\nabla\varphi_R \mathbb{1}_{\Omega_R} \rightharpoonup \nabla\varphi_\infty$ in $L^2(\mathbb{R}^2 \setminus \overline{\omega})$. In particular we obtain $\nabla\varphi_\infty \in L^2(\mathbb{R}^2 \setminus \overline{\omega})$.

Consequently, denoting $\phi_\infty := \text{tr}_{\partial\omega}(\varphi_\infty)$ we obtain $\varphi_\infty \in \mathscr{H}_{\phi_\infty}$ [see (30) for the definition of $\mathscr{H}_{\phi_\infty}$]. Therefore, letting $\Omega_\infty = \mathbb{R}^2 \setminus \overline{\omega}$, we have:

$$\begin{aligned}
\liminf_{R_n \to \infty} \left\{\frac{1}{2}\int_{\Omega_{R_n}} \alpha|\nabla u_{R_n}|^2 - \frac{d^2}{2}\int_{\Omega_{R_n}} \alpha|\nabla v_{R_n}|^2\right\} &= \liminf_{R_n \to \infty} \frac{1}{2}\int_{\Omega_{R_n}} \alpha|\nabla\varphi_{R_n}|^2 \\
&\geq \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi_\infty|^2 \\
&\geq \inf_{\varphi \in \mathscr{H}_{\phi_\infty}} \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi|^2.
\end{aligned} \tag{55}$$

6.4 Calculations on ω

We continue the calculations by proving:

$$\frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}}|\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \geq \frac{1}{2}\int_{\omega}|\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2 + o_\rho(1) \tag{56}$$

where $\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}$ is the harmonique extension of $\phi_{0,\mathbf{z},\mathbf{d}} := \mathrm{tr}_{\partial\omega}\varphi_{0,\mathbf{z},\mathbf{d}}$ in ω, $\varphi_{0,\mathbf{z},\mathbf{d}}$ is defined in (53).

In order to get (56), we adapt the argument done to prove Proposition 30.

From (51), we have

$$\sum_{i=1}^{N}\frac{1}{2}\int_{B(z_i,\sqrt{\rho})\setminus\overline{B(z_i,\rho)}}|\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \leq C_0.$$

Thus, from a mean value argument, there exists $\rho' \in]\rho,\sqrt{\rho}[$ s.t.

$$\sum_{i=1}^{N}\frac{1}{2}\int_0^{2\pi}|\partial_\theta\varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i+\rho'\mathrm{e}^{\imath\theta})|^2\mathrm{d}\theta \leq \frac{2C_0}{|\ln\rho|}.$$

We now define $\tilde{\varphi}_\rho \in H^1(\omega)$ by $\tilde{\varphi}_\rho = \varphi_{\rho,\mathbf{z},\mathbf{d}}$ in $\omega\setminus\cup_i\overline{B(z_i,\rho')}$ and for $i\in\{1,...,N\}$ & $x = z_i + s\mathrm{e}^{\imath\theta} \in B(z_i,\rho')$ we let

$$\tilde{\varphi}_\rho(z_i+s\mathrm{e}^{\imath\theta}) = \left|\begin{array}{ll} 2\dfrac{s-\rho'/2}{\rho'}\varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i+\rho'\mathrm{e}^{\imath\theta}) + \dfrac{\rho'-s}{\pi\rho'}\displaystyle\int_0^{2\pi}\varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i+\rho'\mathrm{e}^{\imath\theta})\mathrm{d}\theta & \text{if } s\in]\dfrac{\rho'}{2},\rho'[\\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i+\rho'\mathrm{e}^{\imath\theta})\mathrm{d}\theta & \text{if } s\leq\dfrac{\rho'}{2}\end{array}\right. .$$

A direct calculation gives:

$$\sum_{i=1}^{N}\int_{B(z_i,\rho')}|\nabla\tilde{\varphi}_\rho|^2 = \mathcal{O}\left[\sum_{i=1}^{N}\int_0^{2\pi}|\partial_\theta\varphi_{\rho,\mathbf{z},\mathbf{d}}(z_i+\rho'\mathrm{e}^{\imath\theta})|^2\right] = o_\rho(1). \tag{57}$$

Thus, letting $\omega_{\rho',\mathbf{z}} = \omega\setminus\cup_{i=1}^N\overline{B(z_i,\rho')}$ and $\tilde{\mathcal{D}}_{\rho'} = \cup_{i=1}^N B(z_i,\rho')\setminus\overline{B(z_i,\rho)}$, we obtain:

$$\begin{aligned}\int_{\omega_{\rho,\mathbf{z}}}|\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 &= \int_{\omega_{\rho',\mathbf{z}}}|\nabla\tilde{\varphi}_\rho|^2 + \int_{\tilde{\mathcal{D}}_{\rho'}}|\nabla\varphi_{\rho,\mathbf{z},\mathbf{d}}|^2 \\ &\geq \int_{\omega_{\rho',\mathbf{z}}}|\nabla\tilde{\varphi}_\rho|^2 \\ &\overset{(57)}{=} \int_{\omega}|\nabla\tilde{\varphi}_\rho|^2 + o_\rho(1).\end{aligned} \tag{58}$$

Since $\tilde{\varphi}_\rho$ is bounded in $H^1(\omega)$, up to pass to a subsequence, we may assume the existence of $\tilde{\varphi}_0 \in H^1(\omega)$ s.t. $\tilde{\varphi}_\rho \rightharpoonup \tilde{\varphi}_0$ in $H^1(\omega)$.

On the other hand, it is clear that $\mathrm{tr}_{\partial\omega}\tilde{\varphi}_0 = \mathrm{tr}_{\partial\omega}\varphi_{0,\mathbf{z},\mathbf{d}} = \phi_{0,\mathbf{z},\mathbf{d}}$. Consequently from the Dirichlet principle we get

$$\liminf_{\rho=\rho(R_n)\to 0}\int_\omega|\nabla\tilde{\varphi}_\rho|^2 \geq \int_\omega|\nabla\tilde{\varphi}_0|^2 \geq \int_\omega|\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2. \tag{59}$$

By combining (58) and (59) we obtain (56).

From (50) and (56) we may write [denoting $\rho_n = \rho(R_n)$]

$$\begin{aligned}\liminf_{\rho_n\to 0}\left\{\frac{1}{2}\int_{\omega_{\rho_n,\mathbf{z}}}|\nabla u_{R_n}|^2 - \frac{1}{2}\int_{\omega_{\rho_n,\mathbf{z}}}|\nabla w_{\rho_n,\mathbf{z},\mathbf{d}}|^2\right\} &= \liminf_{\rho_n\to 0}\frac{1}{2}\int_{\omega_{\rho_n,\mathbf{z}}}|\nabla\varphi_{\rho_n,\mathbf{z},\mathbf{d}}|^2 \\ &\geq \frac{1}{2}\int_\omega|\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2.\end{aligned} \tag{60}$$

6.5 Conclusion

Using (55), (60), the definition of the sequence $(R_n)_n$ and letting $f(R) = \frac{1}{2}\int_{\Omega_R} \alpha|\nabla v_R|^2$ we get

$$\begin{aligned}
&\liminf_{R\to\infty}\left\{I(R,\rho,\mathbf{z},\mathbf{d}) - \left(d^2 f(R) + \frac{b^2}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2\right)\right\} \\
= \quad & \lim_{R_n\to\infty}\left\{\frac{1}{2}\int_{\mathcal{D}_{R_n,\mathbf{z}}} \alpha|\nabla u_{R_n}|^2 - \left(d^2 f(R_n) + \frac{b^2}{2}\int_{\omega_{\rho_n,\mathbf{z}}} |\nabla w_{\rho_n,\mathbf{z},\mathbf{d}}|^2\right)\right\} \\
\geq \quad & \liminf_{R_n\to\infty}\left\{\frac{1}{2}\int_{\Omega_{R_n}} \alpha|\nabla u_{R_n}|^2 - d^2 f(R_n)\right\} + b^2 \liminf_{\rho_n\to 0}\left\{\frac{1}{2}\int_{\omega_{\rho_n,\mathbf{z}}} |\nabla u_{R_n}|^2 - \frac{1}{2}\int_{\omega_{\rho_n,\mathbf{z}}} |\nabla w_{\rho_n,\mathbf{z},\mathbf{d}}|^2\right\} \\
\geq \quad & \inf_{\varphi\in\mathscr{H}_{\phi_\infty}} \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi|^2 + \frac{b^2}{2}\int_{\omega} |\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2. \qquad (61)
\end{aligned}$$

Recall that

$$h_\infty := \left(\frac{x}{|x|}\right)^d \mathrm{e}^{\imath(d\gamma_\infty+\phi_\infty)} \in H^{1/2}(\partial\omega,\mathbb{S}^1).$$

Therefore from (45) and (54) we may write

$$\mathcal{K}(h_\infty) = \inf_{\varphi\in\mathscr{H}_{\phi_\infty}} \frac{1}{2}\int_{\Omega_\infty} \alpha|\nabla\varphi|^2 + \frac{b^2}{2}\int_{\omega} |\nabla\tilde{\phi}_{0,\mathbf{z},\mathbf{d}}|^2.$$

Consequently (61) becomes

$$\liminf_{R\to\infty}\left\{I(R,\rho,\mathbf{z},\mathbf{d}) - \left(d^2 f(R) + \frac{b^2}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2\right)\right\} \geq \mathcal{K}(h_\infty). \qquad (62)$$

It suffices now to see that, from Theorem 5 we have

$$\frac{1}{2}\int_{\omega_{\rho,\mathbf{z}}} |\nabla w_{\rho,\mathbf{z},\mathbf{d}}|^2 = \sum_i d_i^2\pi|\ln\rho| + W(\mathbf{z},\mathbf{d}) + o_\rho(1),$$

this combined with (46) gives

$$\lim_{R\to\infty}\left\{I(R,\rho,\mathbf{z},\mathbf{d}) - \left[f(R) + b^2\left(\pi\left(\sum_{i=1}^N d_i^2\right)|\ln\rho| + W(\mathbf{z},\mathbf{d})\right)\right]\right\} \text{ exists}$$

and

$$\lim_{R\to\infty}\left\{I(R,\rho,\mathbf{z},\mathbf{d}) - \left[f(R) + b^2\left(\pi\left(\sum_{i=1}^N d_i^2\right)|\ln\rho| + W(\mathbf{z},\mathbf{d})\right)\right]\right\} = \mathcal{K}(h_\infty).$$

Remark 32. It is direct to get that h_∞ is a minimizer of $\mathcal{K} : \{h \in H^{1/2}(\partial\omega,\mathbb{S}^1)\,|\,\deg(h) = d\} \to \mathbb{R}$.

We now define:

$$W^{\text{micro}}(\mathbf{z},\mathbf{d}) := b^2 W(\mathbf{z},\mathbf{d}) + \min_{\substack{h\in H^{1/2}(\partial\omega,\mathbb{S}^1)\\ \deg(h)=d}} \mathcal{K}(h) \qquad (63)$$

in order to write

$$I(R,\rho,\mathbf{z},\mathbf{d}) = d^2 f(R) + b^2\pi\left(\sum_{i=1}^N d_i^2\right)|\ln\rho| + W^{\text{micro}}(\mathbf{z},\mathbf{d}) + o_\rho(1).$$

The last equality ends the proof of Theorem 1.

7 The case of the radially symmetric impurity: $\omega = \mathbb{D}$

In this section we focus on the circular case with $\omega = \mathbb{D} = B(0,1)$ is the unit disc and for $b \in (0,\infty)$ we let

$$\begin{array}{rccc} \alpha: & \mathbb{R}^2 & \to & \{b^2;1\} \\ & x & \mapsto & \begin{cases} b^2 & \text{if } x \in \mathbb{D} \\ 1 & \text{if } x \in \mathbb{R}^2 \setminus \mathbb{D} \end{cases} \end{array}.$$

We fix

- $N \in \mathbb{N}^*$, $\mathbf{d} = (d_1, ..., d_N) \in \mathbb{Z}^N$ and we let $d := \sum_{i=1}^N d_i \in \mathbb{Z}$;
- $\mathbf{z} \in (\mathbb{D}^N)^* := \{(z_1, ..., z_N) \in \mathbb{D}^N \mid z_i \neq z_j \text{ for } i \neq j\}$.

7.1 Explicit expression of the special solutions

We use the same notations as in Section 4.

Notation 33. In this section and in the next sections, in order to keep notations simple, we use the shorthand "x" to stand the identity map. Namely we use the abuse of notation $\mathrm{Id} = x$ where $\mathrm{Id} : U \to U$, $x \mapsto \mathrm{Id}(x) = x$ and $U \subset \mathbb{R}^2 \simeq \mathbb{C}$ is an arbitrary set .

We let v_∞ be the limiting function obtained in Corollary 19. It is easy to prove that $v_\infty(x) = \dfrac{x}{|x|}$, *i.e.* $\gamma_\infty \equiv 0$.

We let $w_{0,\mathbf{z},\mathbf{d}} = \prod_{i=1}^N \left(\dfrac{x - z_i}{|x - z_i|}\right)^{d_i} \mathrm{e}^{\imath\gamma_{0,\mathbf{z},\mathbf{d}}}$ be the function defined in (25). This function is the canonical harmonique map in $\mathbb{D}$ associated to the singularities $(\mathbf{z}, \mathbf{d})$.

On the unit circle $\mathbb{S}^1$ we have $\mathrm{tr}_{\mathbb{S}^1}(w_{0,\mathbf{z},\mathbf{d}}) = \mathrm{e}^{\imath\psi_{0,\mathbf{z},\mathbf{d}}}$ with

$$\partial_\tau \psi_{0,\mathbf{z},\mathbf{d}} = \partial_\nu \left[\sum_{j=1}^N d_j \left(\ln|x - z_j| - \ln|1 - \overline{z_j}x|\right)\right].$$

This result comes from [LM14] Eq. (2.25) et (4.1).

From (4.14) in [LM14] we have

$$\partial_\tau \psi_{0,\mathbf{z},\mathbf{d}} = \sum_{j=1}^N d_j \left[2\partial_\nu \left(\ln|x - z_j|\right) - 1\right].$$

Thus

$$\partial_\tau \psi_{0,\mathbf{z},\mathbf{d}} = \sum_{j=1}^N d_j \left[2\partial_\tau \left(\arg(x - z_j)\right) - 1\right]$$

with $\dfrac{x - z_j}{|x - z_j|} = \mathrm{e}^{\imath\arg(x - z_j)}$.

Consequently we get

$$\mathrm{tr}_{\mathbb{S}^1}(w_{0,\mathbf{z},\mathbf{d}}) = \mathrm{e}^{\imath\psi_{0,\mathbf{z},\mathbf{d}}} = \mathrm{Cst} \times x^{-d} \prod_{j=1}^N \left(\frac{x - z_j}{|x - z_j|}\right)^{2d_j} \tag{64}$$

where $\mathrm{Cst} \in \mathbb{S}^1$ is a constant.

7.2 Expression of the dephasing

For $h_\infty \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ we have [see (30) and (45)]

$$\mathcal{K}(h_\infty) = \inf_{\varphi \in \mathscr{H}_{\phi_\infty}} \frac{1}{2} \int_{\Omega_\infty} |\nabla \varphi|^2 + \frac{b^2}{2} \int_\omega |\nabla \tilde{\phi}_0|^2,$$

where:

- on the unit circle we have
$$h_\infty = x^d \mathrm{e}^{\imath \phi_\infty} = w_{0,\mathbf{z},\mathbf{d}} \mathrm{e}^{\imath \phi_0}, \tag{65}$$
- $\tilde{\phi}_0$ is the harmonic extension of ϕ_0 in $\mathbb{D}$.

Condition (65) is a compatibility condition between the fonctions ϕ_∞ et ϕ_0. It is clear that from the definition of $\mathcal{K}$ we may slightly modify Condition (65) by imposing

$$\left(\frac{x}{|x|}\right)^d \mathrm{e}^{\imath \phi_\infty} = \mathrm{Cst} \times w_{0,\mathbf{z},\mathbf{d}} \mathrm{e}^{\imath \phi_0} \text{ with } \mathrm{Cst} \in \mathbb{S}^1. \tag{66}$$

We may easily prove that

$$\inf_{\varphi \in \mathscr{H}_{\phi_\infty}} \frac{1}{2} \int_{\Omega_\infty} |\nabla \varphi|^2 = \frac{1}{2} \int_{\Omega_\infty} |\nabla \hat{\phi}_\infty|^2$$

where for $\phi \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$, $\hat{\phi} \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{\mathbb{D}})$ is the unique solution of

$$\begin{cases} -\Delta \varphi = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\mathbb{D}} \\ \mathrm{tr}_{\mathbb{S}^1}(\varphi) = \phi, & \nabla \varphi \in L^2(\mathbb{R}^2 \setminus \overline{\mathbb{D}}) \end{cases}.$$

[See Proposition 34 for more details about $\hat{\phi}$]

From (64), an equivalent reformulation of (66) is

$$\mathrm{Cst} \prod_{j=1}^N \left(\frac{x - z_j}{|x - z_j| \times x}\right)^{2d_j} = \mathrm{e}^{\imath(\phi_\infty - \phi_0)} \text{ with } \mathrm{Cst} \in \mathbb{S}^1.$$

The above condition is equivalent to the compatibility condition:

$$\phi_\infty - \phi_0 = \Psi_{\mathbf{z},\mathbf{d}} + \mathrm{Cst} \text{ where } \mathrm{Cst} \in \mathbb{R} \tag{67}$$

with $\Psi_{\mathbf{z},\mathbf{d}} \in C^\infty(\mathbb{S}^1, \mathbb{R})$ which is a lifting of

$$\prod_{j=1}^N \left(\frac{x - z_j}{|x - z_j| \times x}\right)^{2d_j}.$$

Here we used Proposition 8.2 and the smoothness of $\mathrm{e}^{\imath \Psi_{\mathbf{z},\mathbf{d}}}$.

With a direct calculation, for $z_0 \in \mathbb{D}$ and $x \in \mathbb{S}^1$, we have

$$\left(\frac{x - z_0}{|x - z_0| x}\right)^2 = \frac{x - z_0}{\overline{x - z_0} \times x^2} = \frac{x - z_0}{1 - \overline{z_0} x} \times \frac{1}{x} = M_{z_0}(x) \times \frac{1}{x}$$

where $M_{z_0} : \mathbb{D} \to \mathbb{D}$ is the Moebius function defined by $M_{z_0}(x) = \dfrac{x - z_0}{1 - \overline{z_0} x}$.

In [Dos15], it is proved [Section 7] that if $z_0 \in \mathbb{D} \cap \mathbb{R}^+$ then for $\mathrm{e}^{\imath\theta} \in \mathbb{S}^1$

$$M_{z_0}(\mathrm{e}^{\imath\theta}) \mathrm{e}^{-\imath\theta} = \mathrm{e}^{\Psi_{z_0,1}(\mathrm{e}^{\imath\theta})} \text{ where } \Psi_{z_0,1}(\mathrm{e}^{\imath\theta}) = \sum_{n \in \mathbb{Z}^*} \frac{z_0^{|n|}}{\imath n} \mathrm{e}^{\imath n \theta} + \mathrm{Cst}, \ \mathrm{Cst} \in \mathbb{R}.$$

In the general case $z_0 = t\mathrm{e}^{\imath\gamma} \in \mathbb{D}$ [with $t \geq 0, \gamma \in \mathbb{R}$] we easily deduce from the previous equality:

$$\begin{aligned} M_{z_0}(\mathrm{e}^{\imath\theta})\mathrm{e}^{-\imath\theta} &= \frac{\mathrm{e}^{\imath\theta} - t\mathrm{e}^{\imath\gamma}}{(1 - t\mathrm{e}^{-\imath\gamma}\mathrm{e}^{\imath\theta})\mathrm{e}^{\imath\theta}} \\ &= \frac{\mathrm{e}^{\imath(\theta-\gamma)} - t}{(1 - t\mathrm{e}^{\imath(\theta-\gamma)})\mathrm{e}^{\imath(\theta-\gamma)}} \\ &= M_t[\mathrm{e}^{\imath(\theta-\gamma)}]\mathrm{e}^{-\imath(\theta-\gamma)}. \end{aligned}$$

Then

$$\begin{aligned} \Psi_{z_0,1}(\mathrm{e}^{\imath\theta}) &= \Psi_{t,1}(\mathrm{e}^{\imath(\theta-\gamma)}) + \mathrm{Cst} \\ &= \sum_{n\in\mathbb{Z}^*} \frac{t^{|n|}}{\imath n}\mathrm{e}^{\imath n(\theta-\gamma)} + \mathrm{Cst} \\ &= \sum_{n\in\mathbb{N}^*} \left[\frac{\overline{z_0}^n}{\imath n}\mathrm{e}^{\imath n\theta} - \frac{{z_0}^n}{\imath n}\mathrm{e}^{-\imath n\theta}\right] + \mathrm{Cst},\ \mathrm{Cst} \in \mathbb{R}. \end{aligned}$$

It is easy to prove that we have $\Psi_{\mathbf{z},\mathbf{d}} = \sum_{j=1}^N d_j \Psi_{z_0,1} + \mathrm{Cst}$ [$\mathrm{Cst} \in \mathbb{R}$] and then

$$\Psi_{\mathbf{z},\mathbf{d}}(\mathrm{e}^{\imath\theta}) = \mathrm{Cst} + \sum_{n\in\mathbb{N}^*}\sum_{j=1}^N d_j \left[\frac{\overline{z_j}^n}{\imath n}\mathrm{e}^{\imath n\theta} - \frac{{z_j}^n}{\imath n}\mathrm{e}^{-\imath n\theta}\right]. \tag{68}$$

We are now in position to reformulate the compatibility condition (67).

Let $\phi_0, \phi_\infty \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, consider their Fourier decompositions:

$$\phi_0(\mathrm{e}^{\imath\theta}) = \sum_{n\in\mathbb{Z}} c_{0,n}\mathrm{e}^{\imath n\theta} \text{ and } \phi_\infty(\mathrm{e}^{\imath\theta}) = \sum_{n\in\mathbb{Z}} c_{\infty,n}\mathrm{e}^{\imath n\theta}. \tag{69}$$

We have

$$\left(\frac{x}{|x|}\right)^d \mathrm{e}^{\imath\phi_\infty} = \mathrm{Cst} \times w_{0,\mathbf{z},\mathbf{d}}\mathrm{e}^{\imath\phi_0},\ \mathrm{Cst} \in \mathbb{S}^1 \tag{66}$$

$$\Leftrightarrow \qquad \phi_\infty - \phi_0 = \Psi_{\mathbf{z},\mathbf{d}} + \mathrm{Cst},\ \mathrm{Cst} \in \mathbb{R} \tag{67}$$

$$\Leftrightarrow \qquad \forall\, n \in \mathbb{Z}^*,\ c_{\infty,n} - c_{0,n} = \begin{cases} \displaystyle\sum_{j=1}^N d_j \frac{\overline{z_j}^n}{\imath n} & \text{if } n > 0 \\ \displaystyle -\sum_{j=1}^N d_j \frac{{z_j}^n}{\imath n} & \text{if } n < 0 \end{cases}. \tag{70}$$

7.3 Explicit expression of the minimal value of $\mathcal{K}$

For $\phi_\infty, \phi_0 \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ we use Notation (69) for their Fourier coefficients:

- the Fourier coefficients of ϕ_∞ are denoted by $(c_{\infty,n})_{n\in\mathbb{Z}}$,
- the Fourier coefficients of ϕ_0 are denoted by $(c_{0,n})_{n\in\mathbb{Z}}$.

Before going further we recall some basic facts.

Proposition 34. *Let $\phi \in H^{1/2}(\mathbb{S}^1, \mathbb{R})$ and consider $\phi(\mathrm{e}^{\imath\theta}) = \sum_{n\in\mathbb{Z}} c_n\mathrm{e}^{\imath n\theta}$ be its Fourier decomposition.*

Then we have

1. $c_n = \overline{c_{-n}}$

2. $\sum_{n\in\mathbb{Z}} |n||c_n|^2 < \infty$ *and we may choose the quantity* $\sqrt{\pi \sum_{n\in\mathbb{Z}} |n||c_n|^2}$ *as a semi-norm in* $H^{1/2}(\mathbb{S}^1,\mathbb{R})$.

3. *The map*

$$\begin{array}{rccc} \tilde{\phi}: & \mathbb{D} & \to & \mathbb{R} \\ & re^{\imath\theta} & \mapsto & \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{\imath n\theta} \end{array}$$

is the harmonic extension of ϕ. *Moreover*

$$\frac{1}{2}\int_{\mathbb{D}} |\nabla\tilde{\phi}|^2 = \pi \sum_{n\in\mathbb{Z}} |n||c_n|^2.$$

4. *The map*

$$\begin{array}{rccc} \hat{\phi}: & \mathbb{R}^2\setminus\overline{\mathbb{D}} & \to & \mathbb{R} \\ & re^{\imath\theta} & \mapsto & \sum_{n\in\mathbb{Z}} c_n r^{-|n|} e^{\imath n\theta} \end{array}$$

is an exterior harmonic extension of ϕ. *Moreover*

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\overline{\mathbb{D}}} |\nabla\hat{\phi}|^2 = \pi \sum_{n\in\mathbb{Z}} |n||c_n|^2.$$

5. $\hat{\phi}$ *is the unique solution of*

$$\begin{cases} -\Delta\varphi = 0 \text{ in } \mathbb{R}^2\setminus\overline{\mathbb{D}}, \\ \varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\mathbb{D},\mathbb{R}) \\ \mathrm{tr}_{\mathbb{S}^1}(\varphi) = \phi,\ \nabla\varphi \in L^2(\mathbb{R}^2\setminus\overline{\mathbb{D}},\mathbb{R}^2) \end{cases}. \tag{71}$$

Therefore it is also the unique solution of the problem

$$\inf_{\substack{\varphi\in H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\mathbb{D},\mathbb{R}) \\ \mathrm{tr}_{\mathbb{S}^1}(\varphi)=\phi,\ \nabla\varphi\in L^2(\mathbb{R}^2\setminus\overline{\mathbb{D}},\mathbb{R}^2)}} \frac{1}{2}\int_{\mathbb{R}^2\setminus\overline{\mathbb{D}}} |\nabla\varphi|^2.$$

Proof. Assertions 1 and 2 are quite standard. Assertions 3 and 4 follow from standard calculations.

We now prove Assertion 5. Let $\phi \in H^{1/2}(\mathbb{S}^1,\mathbb{R})$ and let $\hat{\phi}$ be defined by Assertion 4. It is clear that $\hat{\phi}$ solves (71). Assume that φ_0 is a solution of (71) and let $\eta := \hat{\phi} - \varphi_0$. Then η satisfies:

$$\begin{cases} -\Delta\eta = 0 \text{ in } \mathbb{R}^2\setminus\overline{\mathbb{D}}, \\ \eta \in H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\mathbb{D},\mathbb{R}) \\ \mathrm{tr}_{\mathbb{S}^1}(\eta) = 0,\ \nabla\eta \in L^2(\mathbb{R}^2\setminus\overline{\mathbb{D}},\mathbb{R}^2) \end{cases}.$$

From [SS96] [Theorem II.6.2.ii] we get $\eta = 0$. This clearly gives the uniqueness of the solution of (71).

On the one hand, by direct minimization we know that Problem 72 admits solution(s). It is standard to check that a minimizer for (72) solves (71). Consequently $\hat{\phi}$ is the unique solution of (72). ☐

Notation 35. From now on, for $\phi \in H^{1/2}(\mathbb{S}^1,\mathbb{R})$ with Fourier decomposition $\phi(e^{\imath\theta}) = \sum_{n\in\mathbb{Z}} c_n e^{\imath n\theta}$, we let

$$|\phi|_{H^{1/2}} := \sqrt{\pi \sum_{n\in\mathbb{Z}} |n||c_n|^2} = \sqrt{2\pi \sum_{n\in\mathbb{N}^*} |n||c_n|^2} = \sqrt{\frac{1}{2}\int_{\mathbb{R}^2\setminus\overline{\mathbb{D}}} |\nabla\hat{\phi}|^2} = \sqrt{\frac{1}{2}\int_{\mathbb{D}} |\nabla\tilde{\phi}|^2}.$$

For $n \in \mathbb{N}^*$, letting $\gamma_n = \sum_{j=1}^N d_j \frac{\overline{z_j}^n}{\imath n}$, *i.e.* $\Psi_{\mathbf{z},\mathbf{d}}(\mathrm{e}^{\imath\theta}) = \mathrm{Cst} + \sum_{n\in\mathbb{Z}^*} \gamma_n \mathrm{e}^{\imath n\theta}$ [see (68)], we get

$$\begin{aligned}
\inf_{\substack{h\in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)\\ \deg(h)=d}} \mathcal{K}(h) &= \inf_{\substack{\phi_0,\phi_\infty\in H^{1/2}(\mathbb{S}^1,\mathbb{R})\\ x^d\mathrm{e}^{\imath\phi_\infty}=\mathrm{Cst}\times w_{0,\mathbf{z},\mathbf{d}}\mathrm{e}^{\imath\phi_0}}} \left(\frac{1}{2}\int_{\Omega_\infty}|\nabla\hat{\phi}_\infty|^2 + \frac{b^2}{2}\int_\omega |\nabla\tilde{\phi}_0|^2\right)\\
&= 2\pi \inf_{\substack{(c_{0,n})_{n\in\mathbb{N}^*},(c_{\infty,n})\in\ell^2(\mathbb{N}^*)\\ c_{\infty,n}-c_{0,n}=\gamma_n\ \forall\, n\in\mathbb{N}^*}} \left(\sum_{n\in\mathbb{N}} n|c_{0,n}|^2 + b^2\sum_{n\in\mathbb{N}} n|c_{\infty,n}|^2\right)\\
&= 2\pi\sum_{n\in\mathbb{N}^*}\left[n\times \inf_{\substack{c_{0,n},c_{\infty,n}\in\mathbb{C}\\ c_{\infty,n}-c_{0,n}=\gamma_n\ \forall\, n\in\mathbb{N}^*}} \left(|c_{0,n}|^2 + b^2|c_{\infty,n}|^2\right)\right]\\
&= 2\pi\sum_{n\in\mathbb{N}^*}\left[n\times \inf_{c_{0,n}\in\mathbb{C}} \left(|c_{0,n}|^2 + b^2|c_{0,n}+\gamma_n|^2\right)\right]\\
&= 2\pi\sum_{n\in\mathbb{N}^*}\left[n\times \left(\left|\frac{-b^2}{1+b^2}\gamma_n\right|^2 + b^2\left|\frac{-b^2}{1+b^2}\gamma_n+\gamma_n\right|^2\right)\right]\\
&= \frac{b^2}{1+b^2}2\pi\sum_{n\in\mathbb{N}^*} n|\gamma_n|^2 = \frac{b^2}{1+b^2}|\Psi_{\mathbf{z},\mathbf{d}}|^2_{H^{1/2}}. \qquad (72)
\end{aligned}$$

7.4 Explicit expression of W^{micro}: Proof of Proposition 2

We first recall the expression of $W(\mathbf{z},\mathbf{d})$ [see Proposition 1 in [LR96]]:

$$W(\mathbf{z},\mathbf{d}) = -\pi\sum_{i\neq j} d_i d_j \ln|z_i - z_j| + \pi\sum_{i=1}^N d_i^2 \ln(1-|z_i|^2) + \pi\sum_{i\neq j} d_i d_j \ln|1 - z_i\overline{z_j}|.$$

From (63) we have

$$W^{\mathrm{micro}}(\mathbf{z},\mathbf{d}) = b^2 W(\mathbf{z},\mathbf{d}) + \min_{\substack{h\in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)\\ \deg(h)=d}} \mathcal{K}(h).$$

By combining (68) and (72) we may write

$$\min_{\substack{h\in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)\\ \deg(h)=d}} \mathcal{K}(h) = \frac{2b^2}{1+b^2}\pi\sum_{n\in\mathbb{N}^*} n\left|\sum_{j=1}^N d_j\frac{\overline{z_j}^n}{\imath n}\right|^2 = \frac{2b^2}{1+b^2}\pi\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left|\sum_{j=1}^N d_j z_j^n\right|^2.$$

For $n\in\mathbb{N}^*$ we have the following expanding

$$\left|\sum_{j=1}^N d_j z_j^n\right|^2 = \sum_{j=1}^N d_j^2|z_j|^{2n} + \sum_{i\neq j} d_i d_j (z_i\overline{z_j})^n = \sum_{j=1}^N d_j^2|z_j|^{2n} + 2\mathrm{Re}\left[\sum_{i<j} d_i d_j (z_i\overline{z_j})^n\right].$$

Therefore we obtain

$$\begin{aligned}
\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left|\sum_{j=1}^N d_j z_j^n\right|^2 &= \sum_{j=1}^N d_j^2\left(\sum_{n\in\mathbb{N}^*}\frac{1}{n}|z_j|^{2n}\right) + 2\sum_{i<j} d_i d_j \mathrm{Re}\left[\sum_{n\in\mathbb{N}^*}\frac{1}{n}(z_i\overline{z_j})^n\right]\\
&= -\sum_{j=1}^N d_j^2\ln(1-|z_j|^2) - 2\sum_{i<j} d_i d_j \mathrm{Re}\left[\ln(1-z_i\overline{z_j})\right]\\
&= -\sum_{j=1}^N d_j^2\ln(1-|z_j|^2) - \sum_{i\neq j} d_i d_j \ln|1-z_i\overline{z_j}|.
\end{aligned}$$

We may thus conclude:

$$\begin{aligned} W^{\text{micro}}(\mathbf{z},\mathbf{d}) &= b^2\pi\left[-\sum_{i\neq j} d_i d_j \ln|z_i - z_j| + \sum_{i=1}^{N} d_i^2 \ln(1-|z_i|^2) + \sum_{i\neq j} d_i d_j \ln|1 - z_i\overline{z_j}| - \right. \\ &\qquad \left. -\frac{2}{1+b^2}\left(\sum_{j=1}^{N} d_j^2 \ln(1-|z_j|^2) + \sum_{i\neq j} d_i d_j \ln|1-z_i\overline{z_j}|\right)\right] \\ &= -b^2\pi\left[\sum_{i\neq j} d_i d_j \ln|z_i - z_j| + \frac{1-b^2}{1+b^2}\sum_{j=1}^{N} d_j^2 \ln(1-|z_j|^2) + \frac{1-b^2}{1+b^2}\sum_{i\neq j} d_i d_j \ln|1-z_i\overline{z_j}|\right]. \end{aligned}$$

These calculations end the proof of Proposition 2.

7.5 Minimization of W^{micro} in some particular cases

We first claim that if $\mathbf{d} = \mathbf{0}_{\mathbb{Z}^N}$ then $W^{\text{micro}}(\cdot,\mathbf{d}) \equiv 0$. In the following we consider $\mathbf{d} \in \mathbb{Z}^N \setminus \{\mathbf{0}_{\mathbb{Z}^N}\}$.

7.5.1 The case $N = 1$ and the case $N \geq 2 \& \exists! k_0 \in \{1, ..., N\}$ s.t. $d_{k_0} \neq 0$

We first treat the case $N = 1$. In this situation, we have for $z \in \mathbb{D}$ and $d \in \mathbb{Z}$:

$$W^{\text{micro}}(z,d) = -\frac{b^2(1-b^2)}{1+b^2}\pi d^2 \ln(1-|z|^2)$$

Therefore, if $b < 1$ then $z = 0$ is the unique minimizer of W^{micro}.

Remark 36. This simple fact is the main result of [Dos15] [where the explicit expression of W^{micro} was unknown].

If $b = 1$ then $W^{\text{micro}}(\cdot,d) \equiv 0$.

If $b > 1$ then

$$W^{\text{micro}}(z,d) = \frac{b^2(b^2-1)}{1+b^2}\pi d^2 \ln(1-|z|^2).$$

Consequently $W^{\text{micro}}(z,d) \to -\infty$ when $|z| \to 1$. This implies that $W^{\text{micro}}(\cdot,d)$ does not admit minimizers.

Remark 37. We may conclude that the condition $b < 1$ creates a confinement effect for the points of minimum of $W^{\text{micro}}(\cdot,d)$. This confinement effect does not hold for $b \geq 1$.

We now consider the case $N \geq 2$. We assume that $d_1 \neq 0$ and $d_l = 0$ for $l \neq 1$.

This situation is similar to the above one since for $\mathbf{z} = (z_1, ..., z_N) \in (\omega^N)^*$ we have $W^{\text{micro}}(\mathbf{z},\mathbf{d}) = W^{\text{micro}}(z_1, d_1)$. Consequently as previously we have:

- If $b < 1$ then the set of global minimizers of W^{micro} is $\{\mathbf{z} \in (\omega^N)^* \mid z_1 = 0\}$.
- If $b = 1$ then $W^{\text{micro}}(\cdot,\mathbf{d}) \equiv 0$.
- If $b > 1$ then $W^{\text{micro}}(\mathbf{z},\mathbf{d}) \to -\infty$ when $|z_1| \to 1$.

7.5.2 The case $N \geq 2$ and there exist k, l s.t. $d_k d_l < 0$

Let $\mathbf{d} \in \mathbb{Z}^N$ s.t. there exist $k \neq l$ satisfying $d_k d_l < 0$.

In this situation we have

$$\inf_{\mathbf{z}\in(\omega^N)^*} W^{\text{micro}}(\mathbf{z},\mathbf{d}) = -\infty.$$

Indeed, without loss of generality, we may assume that $d_1 d_2 < 0$. We thus consider $z_1^{(n)} = -1/n$, $z_2^{(n)} = 1/n$ and for $k \in \{1, ..., N\} \setminus \{1, 2\}$, $z_k = \mathrm{e}^{\imath 2k\pi/N}/2$.

With direct calculations, we obtain $\lim_n W(\mathbf{z}_n,\mathbf{d}) = -\infty$.

Remark 38. This fact underline that if we impose $d_1 d_2 < 0$ then the main part of the *optimal* energy $I(R, \rho, \mathbf{z}, \mathbf{d})$ is not

$$\left(\sum_{i=1}^{N} d_i\right)^2 f(R) + b^2 \left(\sum_{i=1}^{N} d_i^2\right) |\ln \rho|.$$

Indeed when we consider *very near* singularities $z_1 \& z_2$ we may optimize the divergent term $b^2 \left(\sum_{i=1}^{N} d_i^2\right) |\ln \rho|$. The key argument is that with degrees having different signs (*e.g* $d_1 d_2 < 0$) we have

$$\sum_{i=1}^{N} d_i^2 > (d_1 + d_2)^2 + \sum_{i=3}^{N} d_i^2.$$

This is an example of the standard attractive effect of singularities having degrees with different signs.

7.5.3 The case $b = 1$, $N \geq 2$, $d_k d_l \geq 0$ $\forall k, l$ and there exist k_0, l_0 s.t. $d_{k_0} d_{l_0} > 0$

When $b = 1$, for $(\mathbf{z}, \mathbf{d}) \in (\omega^N)^* \times \mathbb{Z}^N$ we have

$$W^{\text{micro}}(\mathbf{z}, \mathbf{d}) \quad = \quad -\pi \sum_{i \neq j} d_i d_j \ln |z_i - z_j|.$$

Thus

$$\inf_{\mathbf{z} \in (\omega^N)^*} W^{\text{micro}}(\mathbf{z}, \mathbf{d}) > -\infty$$

but the lower bound is not attained.

Indeed, it is easy to check for $\mathbf{z} \in (\omega^N)^*$

$$0 > \inf_{\mathbf{z} \in (\omega^N)^*} W^{\text{micro}}(\mathbf{z}, \mathbf{d}) > -\pi \sum_{i \neq j} d_i d_j \ln 2.$$

Consequently $W^{\text{micro}}(\cdot, \mathbf{d})$ is bounded from below.

We now prove that the lower bound is not reached. Let $\mathbf{z} \in (\omega^N)^*$, and consider $\tilde{\mathbf{z}} \in (\omega^N)^*$ be s.t. $\tilde{z}_k = \lambda z_k$ with $\lambda := \dfrac{2}{1 + \max\{|z_l|, l \in \{1, ..., N\}\}}$. It is easy to check that $\tilde{\mathbf{z}} \in (\omega^N)^*$.

We get

$$W^{\text{micro}}(\tilde{\mathbf{z}}, \mathbf{d}) = W^{\text{micro}}(\mathbf{z}, \mathbf{d}) - \pi \ln \lambda \sum_{i \neq j} d_i d_j.$$

Since $\lambda > 1$, we have $W^{\text{micro}}(\tilde{\mathbf{z}}, \mathbf{d}) < W^{\text{micro}}(\mathbf{z}, \mathbf{d})$. This fact implies that the lower bound is not reached.

Remark 39. When $b = 1$, the impurity $\omega = \mathbb{D}$ does not play any role. Then, due to the standard repulsion effect between vortices, the more the vortices are distant the smaller the energy. Consequently, for fixed degrees having all the same sign, minimal sequences of singularities go to the boundary of the impurity which is not an admissible configuration in this framework.

7.5.4 The case $b > 1$ and $N \geq 2$

If $b > 1$ then for $(\mathbf{z}, \mathbf{d}) \in (\omega^N)^* \times \mathbb{Z}^N$ we have

$$W^{\text{micro}}(\mathbf{z}, \mathbf{d}) \quad = \quad b^2 \pi \left[-\sum_{i \neq j} d_i d_j \ln |z_i - z_j| + \frac{b^2 - 1}{1 + b^2} \sum_{j=1}^{N} d_j^2 \ln(1 - |z_j|^2) + \frac{b^2 - 1}{1 + b^2} \sum_{i \neq j} d_i d_j \ln |1 - z_i \overline{z_j}| \right]$$

Taking, for $k \in \{1, ..., N\}$, $z_k^{(n)} = (1 - 1/n) \mathrm{e}^{\imath 2\pi k/N}$ we have

$$W^{\text{micro}}(\mathbf{z}, \mathbf{d}) = \mathcal{O}(1) + \frac{b^2 - 1}{1 + b^2} \sum_{j=1}^{N} d_j^2 \ln(1 - |z_j|^2) \to -\infty \text{ when } n \to \infty.$$

Remark 40. The case $b > 1$ corresponds to an impurity $\omega = \mathbb{D}$ which have a repulsive effect on the singularities.

7.5.5 The case $0 < b < 1$, $N \geq 2$ and $\mathbf{d} \in \mathbb{N}^N$

This situation is the most challenging.

Note that with the help of [DM11] we may obtain the existence of minimizers for $W^{\text{micro}}(\cdot, \mathbf{d})$ with $d_i = 1$ for $i \in \{1, ..., N\}$, $N \in \mathbb{N}^*$. But [DM11] does not give any information on the location of minimizers and for other configurations of degrees.

From technical issues, we restrict the study to $N = 2$ and $p, q \in \mathbb{N}^*$. Note that the case $p, q < 0$ is obviously symmetric.

We are going to prove that there exist minimizers and there are unique up to a rotation [see (78)&(79)].

We may assume $p \leq q$. For $z_1, z_2 \in \mathbb{D}$ we have, writing $(\mathbf{z}, \mathbf{d}) = ((z_1, p), (z_2, q))$

$$\frac{W^{\text{micro}}(\mathbf{z}, \mathbf{d})}{-b^2\pi} = 2pq \ln|z_1 - z_2| + \frac{1-b^2}{1+b^2}\left[p^2 \ln(1-|z_1|^2) + q^2 \ln(1-|z_2|^2) + 2pq \ln|1 - z_1\overline{z_2}|\right].$$

We let:

- $\mathcal{B} := \dfrac{1-b^2}{1+b^2}$ and $\mathcal{A} := \dfrac{p}{q} \leq 1$;
- $f(z_1, z_2) = 2\ln|z_1 - z_2| + \mathcal{B}\left[\mathcal{A}\ln(1-|z_1|^2) + \mathcal{A}^{-1}\ln(1-|z_2|^2) + 2\ln|1 - z_1\overline{z_2}|\right]$.

Note that $W^{\text{micro}}[(z_1, z_2), (p, q)] = -b^2 pq\pi f(z_1, z_2)$. Consequently, in order to study minimizing points of $W^{\text{micro}}[\cdot, (p, q)]$, we have to maximize $f(\cdot)$.

Since $z_1 \neq z_2$ and since for $t \in \mathbb{R}$ we have $f(z_1, z_2) = f(z_1 \mathrm{e}^{\imath t}, z_2 \mathrm{e}^{\imath t})$, we may assume that $z_1 = s \geq 0$. We thus have for $z_2 = \rho \mathrm{e}^{\imath\theta}$ $[0 \leq \rho < 1, \theta \in \mathbb{R}]$

$$f(s, \rho\mathrm{e}^{\imath\theta}) = \ln\left[s^2 + \rho^2 - 2s\rho\cos\theta\right] + \mathcal{B}\left[\mathcal{A}\ln(1-s^2) + \mathcal{A}^{-1}\ln(1-\rho^2) + \ln(1 + s^2\rho^2 - 2s\rho\cos\theta)\right].$$

We first claim that if $s = 0$ then $\rho > 0$ and for $\varepsilon > 0$ we have

$$f(\varepsilon, -\rho) = f(0, \rho\mathrm{e}^{\imath\theta}) + \varepsilon(\rho^{-1} + 2\beta\rho) + \mathcal{O}(\varepsilon^2).$$

Consequently, for $\varepsilon > 0$ sufficiently small we have $f(\varepsilon, -\rho) > f(0, \rho\mathrm{e}^{\imath\theta})$. Therefore, if $(s, \rho\mathrm{e}^{\imath\theta})$ maximizes f, then $s \in (0; 1)$. Using a similar argument, we may prove that for $s > 0$, if $(s, \rho\mathrm{e}^{\imath\theta})$ maximizes f, then $\rho \in (0; 1)$.

On the other hand, from direct checking, for $s, \rho > 0$, the map $\theta \in [0, 2\pi] \mapsto f(s, \rho\mathrm{e}^{\imath\theta})$ is maximal if and only if $\theta = \pi$.

Consequently, we focus on the map

$$g: \begin{array}{ccc} (0;1)^2 & \to & \mathbb{R} \\ (s,t) & \mapsto & f(s,-t) = 2\ln(s+t) + \mathcal{B}\left[\mathcal{A}\ln(1-s^2) + \mathcal{A}^{-1}\ln(1-t^2) + 2\ln(1+st)\right] \end{array}.$$

We first look for critical points of g:

$$\begin{aligned} \nabla g(s,t) = \mathbf{0} \quad &\Leftrightarrow \quad \begin{cases} \dfrac{1}{s+t} + \mathcal{B}\left(\dfrac{-\mathcal{A}s}{1-s^2} + \dfrac{t}{1+st}\right) = 0 \\ \dfrac{1}{s+t} + \mathcal{B}\left(\dfrac{-\mathcal{A}^{-1}t}{1-t^2} + \dfrac{s}{1+st}\right) = 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} (1-s^2)(1+st) + \mathcal{B}\left[-\mathcal{A}s(1+st)(s+t) + t(1-s^2)(s+t)\right] = 0 \\ (1-t^2)(1+st) + \mathcal{B}\left[-\mathcal{A}^{-1}t(1+st)(s+t) + s(1-t^2)(s+t)\right] = 0 \end{cases} \end{aligned} \tag{73}$$

By considering the difference of both lines in (73) we get:

$$\begin{aligned} & (t^2 - s^2)(1+st) + \mathcal{B}\left[(\mathcal{A}^{-1}t - \mathcal{A}s)(1+st)(s+t) + (t - s^2t - s + st^2)(s+t)\right] = 0 \\ \Longleftrightarrow \quad & (1+st)(s+t)\left[t - s + \mathcal{B}((\mathcal{A}^{-1}+1)t - (\mathcal{A}+1)s)\right] = 0 \\ \overset{[s,t>0]}{\Longleftrightarrow} \quad & [1 + \mathcal{B}(\mathcal{A}^{-1}+1)]t - [1 + \mathcal{B}(\mathcal{A}+1)]s = 0 \\ \Longleftrightarrow \quad & t = \lambda s \text{ with } \lambda := \frac{1 + \mathcal{B}(\mathcal{A}+1)}{1 + \mathcal{B}(\mathcal{A}^{-1}+1)}. \end{aligned} \tag{74}$$

Remark 41. It is important to note that $0 < \lambda \leq 1$. Moreover $\lambda = 1$ if and only if $p = q$.

Using (74) in the first line of (73) we have

$$(1-s^2)(1+\lambda s^2) + \mathcal{B}\left[-\mathcal{A}s^2(1+\lambda s^2)(1+\lambda) + \lambda s^2(1-s^2)(1+\lambda)\right] = 0. \tag{75}$$

Thus, letting $\sigma = s^2$, we get the following equation:

$$[\lambda + (\mathcal{A}+1)\mathcal{B}\lambda(1+\lambda)]\sigma^2 + [1-\lambda+(\mathcal{A}-\lambda)\mathcal{B}(1+\lambda)]\sigma - 1 = 0 \tag{76}$$

We let

$$\Delta := [1-\lambda+(\mathcal{A}-\lambda)\mathcal{B}(1+\lambda)]^2 + 4[\lambda+(\mathcal{A}+1)\mathcal{B}\lambda(1+\lambda)].$$

Note that $\Delta > 0$ and $\sqrt{\Delta} > 1-\lambda+(\mathcal{A}-\lambda)\mathcal{B}(1+\lambda)$.

We obtain immediately that

$$\sigma_0 = \frac{-[1-\lambda+(\mathcal{A}-\lambda)\mathcal{B}(1+\lambda)] + \sqrt{\Delta}}{2[\lambda+(\mathcal{A}+1)\mathcal{B}\lambda(1+\lambda)]} \tag{77}$$

is the unique positive solution of (76).

Consequently

$$s_0 = \sqrt{\frac{-[1-\lambda+(\mathcal{A}-\lambda)\mathcal{B}(1+\lambda)] + \sqrt{\Delta}}{2[\lambda+(\mathcal{A}+1)\mathcal{B}\lambda(1+\lambda)]}} \tag{78}$$

is the unique positive solution of (75).

In order to prove that $(s_0, -\lambda s_0) \in (\mathbb{D}^2)^*$, since $0 < \lambda \leq 1$ and $s_0 = \sqrt{\sigma_0}$, it suffices to check that the positive roots σ_0 given in (77) satisfies $\sigma_0 < 1$. To this end we let P be the quadratic polynomial function expresses in the LHS of (76) with variable σ. With direct computations we get $P(0) = -1 < 0$ and $P(1) = \mathcal{B}(1+\lambda)^2\mathcal{A} > 0$. Therefore the equation (76) admits at least a solution $\tilde{\sigma} \in (0;1)$. Since σ_0 given in (77) is the unique positive solution of (76) we get $\sigma_0 \in (0;1)$.

In conclusion, the set of global minimizers of $W^{\text{micro}}[\cdot,(p,q)]$ is

$$\left\{\left(s_0 \mathrm{e}^{\imath\theta}; -\lambda s_0 \mathrm{e}^{\imath\theta}\right) \in (\mathbb{D}^2)^* \,|\, \theta \in \mathbb{R}\right\} \tag{79}$$

where s_0 is given by (78) and λ by (74).

In particular, if $p = q$ then $\mathcal{A} = \lambda = 1$ and in this case the set of minimizers of $W^{\text{micro}}[\cdot,(p,p)]$ is

$$\left\{\left(\left(1+4\frac{1-b^2}{1+b^2}\right)^{-1/4} \mathrm{e}^{\imath\theta}; -\left(1+4\frac{1-b^2}{1+b^2}\right)^{-1/4} \mathrm{e}^{\imath\theta}\right) \in (\mathbb{D}^2)^* \,|\, \theta \in \mathbb{R}\right\}.$$

Remark 42. It is interesting to note that if $((z_1,z_2),(p,q)) \in (\mathbb{D}^2)^* \times (\mathbb{N}^*)^2$ is a minimizers for W^{micro}, then we have:

$$|z_1| \leq |z_2| \Longleftrightarrow p \geq q$$

and

$$|z_1| = |z_2| \Longleftrightarrow p = q.$$

A Proof of Lemma 17

The key ingredient to get Lemma 17 is Proposition C.4 in [Dos13]. For the convenience of the reader we state this proposition:

Proposition 43. *[Proposition C.4 in [Dos13]]*
Let $\alpha \in L^\infty(\mathbb{R}^2, [B^2; B^{-2}])$ and $R > r > 0$ we denote:

- $\mathscr{R} := B_R \setminus \overline{B_r}$,
- $\mu^{\text{Dir}}(\mathscr{R}) := \inf\left\{\frac{1}{2}\int_{\mathscr{R}} \alpha|\nabla w|^2 \;\middle|\; \begin{array}{c} w \in H^1(\mathscr{R},\mathbb{S}^1) \text{ s.t. }, w(r\mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath\theta}, \\ w(R\mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath(\theta+\theta_0)}, \theta_0 \in \mathbb{R} \end{array}\right\}$,

- $\mu(\mathscr{R}) := \inf \left\{ \frac{1}{2} \int_{\mathscr{R}} \alpha |\nabla w|^2 \;\middle|\; \begin{array}{c} w \in H^1(\mathscr{R}, \mathbb{S}^1) \\ s.t.\ \deg(w) = 1 \end{array} \right\}.$

There exists a constant C_B depending only on B s.t.

$$\mu(\mathscr{R}) \leq \mu^{\mathrm{Dir}}(\mathscr{R}) \leq \mu(\mathscr{R}) + C_B.$$

Remark 44. In [Dos13], Proposition C.4, was initially stated for $\tilde{\alpha} \in L^\infty(\mathbb{R}^2, [b^2; 1])$ and $b \in (0; 1)$. Some obvious modifications allow to get the aforementioned formulation.

Lemma 17 is equivalent to

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_{R'}|^2 - \frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_R|^2 \leq C_{B,\omega}. \tag{80}$$

Recall that $R_0 := \max\{1; 10^2 \cdot \mathrm{diam}(\omega)\}$, thus $\overline{\omega} \subset B_{R_0}$.

We let

$$C_\omega := \frac{1}{2} \int_{B_{R_0} \setminus \overline{\omega}} \left| \nabla \left(\frac{x}{|x|} \right) \right|^2. \tag{81}$$

It is obvious that we have:

$$\frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla u_{R'}|^2 \geq \mu(B_{R'} \setminus \overline{B_R}) + \mu(B_R \setminus \overline{B_{R_0}}).$$

Using Proposition 43 we have:

$$\frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla u_{R'}|^2 \geq \mu^{\mathrm{Dir}}(B_{R'} \setminus \overline{B_R}) + \mu^{\mathrm{Dir}}(B_R \setminus \overline{B_{R_0}}) - 2C_B.$$

It is easy to check, *e.g.* using the direct method of minimization, that the minima $\mu^{\mathrm{Dir}}(B_{R'} \setminus \overline{B_R})$ and $\mu^{\mathrm{Dir}}(B_R \setminus \overline{B_{R_0}})$ are reached. Let u_1 [resp. u_2] be a minimizer of $\mu^{\mathrm{Dir}}(B_{R'} \setminus \overline{B_R})$ [resp. $\mu^{\mathrm{Dir}}(B_R \setminus \overline{B_{R_0}})$].

Up to multiply u_1 by a constant rotation we may assume $\mathrm{tr}_{\partial B_R} u_1 = \mathrm{tr}_{\partial B_R} u_2$.

We are now in position to define

$$u = \begin{cases} u_1 & \text{in } B_{R'} \setminus \overline{B_R} \\ u_2 & \text{in } B_R \setminus \overline{B_{R_0}} \\ \dfrac{x}{|x|} & \text{in } B_{R_0} \setminus \overline{\omega} \end{cases}.$$

It is clear that $u \in H^1(\Omega_{R'}, \mathbb{S}^1)$ and $\deg(u) = 1$. Consequently

$$\begin{aligned} \frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla u_{R'}|^2 &\leq \frac{1}{2} \int_{\Omega_{R'}} \alpha |\nabla u|^2 \\ &= \mu^{\mathrm{Dir}}(B_{R'} \setminus \overline{B_R}) + \mu^{\mathrm{Dir}}(B_R \setminus \overline{B_{R_0}}) + \frac{1}{2} \int_{B_{R_0} \setminus \overline{\omega}} \alpha \left| \nabla \left(\frac{x}{|x|} \right) \right|^2 \\ [\text{Prop. } 43 \& \text{ Eq. } (81)] \quad &\leq \mu(B_{R'} \setminus \overline{B_R}) + \mu(B_R \setminus \overline{B_{R_0}}) + 2C_B + B^{-2} C_\omega. \end{aligned}$$

Since $\mu(B_{R'} \setminus \overline{B_R}) \leq \frac{1}{2} \int_{B_{R'} \setminus \overline{B_R}} \alpha |\nabla u_{R'}|^2$ and $\mu(B_R \setminus \overline{B_{R_0}}) \leq \frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_R|^2$ we obtain:

$$\frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_{R'}|^2 \leq \frac{1}{2} \int_{\Omega_R} \alpha |\nabla u_R|^2 + 2C_B + B^{-2} C_\omega.$$

Letting $C_{B,\omega} := 2C_B + B^{-2} C_\omega$ the above inequality is exactly (80).

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References

[BBH93] F. Bethuel, H. Brezis, and F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. Var. Partial Differential Equations **1** (1993), no. 2, 123–148.

[BBH94] _____, *Ginzburg-Landau Vortices*, Progress in Nonlinear Differential Equations and their Applications, 13, Birkhäuser Boston Inc., Boston, MA, 1994.

[BGP91] A. Boutet de Monvel-Berthier, V. Georgescu, and R. Purice, *A boundary value problem related to the Ginzburg-Landau model*, Comm. Math. Phys. **142** (1991), no. 1, 1–23.

[Bre01] H. Brezis, *Équations de Ginzburg-Landau et singularités. Lecture notes of the graduate course written by Vicenţiu Rădulescu*, http://inf.ucv.ro/ radulescu/articles/coursHB.pdf (2001).

[DM11] M. Dos Santos and O. Misiats, *Ginzburg-Landau model with small pinning domains*, Netw. Heterog. Media **6** (2011), no. 4, 715–753.

[Dos10] M. Dos Santos, *Défauts de vorticité dans un supraconducteur en présence d'impuretés*, Ph.D. thesis, Univ. Lyon 1, 2010.

[Dos13] _____, *The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part II: the non-zero degree case*, Indiana Univ. Math. J. **62** (2013), no. 2.

[Dos15] _____, *Microscopic renormalized energy for a pinned Ginzburg–Landau functional*, Calc. Var. Partial Differential Equations **53** (2015), no. 1-2, 65–89.

[LM14] X. Lamy and P. Mironescu, *Existence of critical points with semi-stiff boundary conditions for singular perturbation problems in simply connected planar domains*, J. Math. Pures Appl. **102** (2014), no. 2, 385–418.

[LR96] C. Lefter and V. Rădulescu, *Minimization problems and corresponding renormalized energies*, Diff. Int. Eq **9** (1996), no. 5, 903–917.

[SS96] C.G. Simader and H. Sohr, *The Dirichlet problem for the Laplacian in bounded and unbounded domains*, vol. 360, CRC Press, 1996.

[SS07] E. Sandier and S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Birkhäuser Boston Inc., Boston, MA, 2007.