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ON STRONG L^2 CONVERGENCE OF NUMERICAL SCHEMES FOR THE STOCHASTIC 2D NAVIER-STOKES EQUATIONS

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ABSTRACT. We prove that some discretization schemes for the 2D Navier-Stokes equations subject to a random perturbation converge in $L^2(\Omega)$. This refines previous results which only established the convergence in probability of these numerical approximations. Using exponential moment estimates of the solution of the stochastic NS equations and convergence of a localized scheme, we can prove strong convergence of fully implicit and semi-implicit time Euler discretizations, and of a splitting scheme. The speed of the $L^2(\Omega)$-convergence depends on the diffusion coefficient and on the viscosity parameter.

1. INTRODUCTION

An incompressible fluid flow dynamic is described by the so-called incompressible Navier-Stokes equations. The fluid flow is defined by a velocity field and a pressure term that evolve in a very particular way. These equations are parametrized by the viscosity coefficient $\nu > 0$. Their quantitative and qualitative properties depend on the dimensional setting. For example, while the well posedness of global weak solutions of the 2D Navier-Stokes is well known and established, the uniqueness of global weak solutions for the 3D case is completely open. In this paper, we will focus on the 2D incompressible Navier-Stokes equations in a bounded domain $D = [0, L]^2$, subject to an external forcing defined as:

$$\partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla\pi = G(u)dW \quad \text{in} \quad D_T, \tag{1.1}$$

$$\text{div } u = 0 \quad \text{in} \quad D_T, \tag{1.2}$$

where $T > 0$ and $D_T = (0, T) \times D$. The process $u : \Omega \times D_T \to \mathbb{R}^2$ is the velocity field with initial condition u_0 in D and periodic boundary conditions $u(t, x + Lv_i) = u(t, x)$ on $(0, T) \times \partial D$, where v_i, $i = 1, 2$ denotes the canonical basis of $\mathbb{R}^2$, and $\pi : \Omega \times D_T \to \mathbb{R}$ is the pressure.

The external force is described by a stochastic perturbation and will be defined in detail later. Here G is a diffusion coefficient with global Lipschitz conditions. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ denote a filtered probability space and W be a Wiener process to be precisely defined later. Given $T > 0$, let $\Omega_T = \Omega \times [0, T]$.

There is an extensive literature concerning the deterministic models and we refer to the books of Temam; see [25, 26] for known results. The stochastic case has also been widely investigated, see [14] for some very general results and the references therein. For the 2D case, unique global weak and strong solutions (in the PDE sense) are constructed for both additive and multiplicative noise, and without being exhaustive, we refer to [9, 12].

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Numerical schemes and algorithms have been introduced to best approximate and construct solutions for PDEs. A similar approach has started to emerge for stochastic models and in particular SPDEs and has known a strong interested by the probability community. Many algorithms based on either finite difference, finite element or spectral Galerkin methods (for the space discretization), and on either Euler schemes, Crank-Nicolson or Runge-Kutta schemes (for the temporal discretization) have been introduced for both the linear and nonlinear cases. Their rate of convergence have been widely investigated. The literature on numerical analysis for SPDEs is now very extensive. When the models are either linear, have global Lipschitz properties or more generally some monotonicity property, then there is extensive literature, see [2, 3]. Moreover, in this case the convergence is proven to be in mean square. When nonlinearities are involved that are not of Lipschitz or monotone type, then a rate of convergence in mean square is difficult to obtain. Indeed, because of the stochastic perturbation, there is no way of using the Gronwall lemma after taking the expectation of the error estimate because it involves a nonlinear term that is usually in a quadratic form. One way of getting around it is to localize the nonlinear term in order to get a linear inequality and then use the Gronwall lemma. This gives rise to a rate of convergence in probability, that was first introduced by J. Printems [24].

The stochastic Navier-Stokes equations with a multiplicative noise (1.1) have been investigated by Z. Brzezniak, E. Carelli and A. Prohl in [10]. There, space discretization based on finite elements and Euler schemes for the time discretization have been implemented. The numerical scheme was proven to converge in probability with a particular rate. A similar problem has been investigated by E. Carelli and A. Prohl in [11], with more focus on various Euler schemes including semi-implicit and fully implicit ones. This gave rise to a slightly different rate of convergence, although still in probability. Again, the main tool used is the localization of the nonlinear term over a probability space of "large" probability. In [4], the authors used a splitting method, based on the Lie-Trotter formula, proving again some rate of convergence in probability of the numerical scheme. In [15], P. Dörsek studied a semigroup splitting and used cubature approximations, obtaining interesting results for an additive noise. When the noise is additive, a pathwise argument was used by H. Breckner in [9]; convergence almost sure and in mean was obtained, although no rate of convergence was explicitly given. To the best of our knowledge there is no result about a strong speed of convergence for the stochastic Navier-Stokes equations in the current literature.

Numerical schemes for stochastic nonlinear models with local Lipschitz nonlinearities have been studied by many authors. In the setting of SDEs, let us mention the memoir by M. Hutzenthaler and A. Jentzen [17] and the references therein. Note that this strong convergence need not always be true as proved by M. Hutzenthaler, A. Jentzen and P. Kloeden in [18]. In the case of stochastic partial differential equations (SPDEs), there are several attempts to obtain a speed of convergence for non linear equations, such as the non linear Schrödinger equation proved by A. de Bouard and A. Debussche [13], where a strong and weak speed of convergence were obtained. In the parabolic setting, in [8] D. Blömker and A. Jentzen proved a speed of convergence in probability of Galerkin approximations of the stochastic Burgers equation. In [1], the strong speed of convergence of schemes is proved in a general setting; the authors deduce a strong speed of convergence for the stochastic Allen-Cahn equation. Let us also mention [21] where a strong speed of convergence is proved for non linear reaction diffusion equations. For the Burgers and Cahn-Hilliard equations, as well as more general non linear SPDEs subject to space-time white noise

driven perturbation, A. Jentzen, D. Salinova and T. Welti proved in [23] the strong convergence of the scheme but did not give a rate of convergence. Some attempts have also been done to prove exponential moments of tamed or stopped convergence schemes for non linear SPDEs, such as the Burgers and Navier Stokes equations in [22].

In this paper, we focus on the stochastic 2D Navier Stokes equations and would like to go one step further, that is, obtain a strong speed of convergence in mean square instead of the convergence in probability. In fact, the main goal is twofold. On one hand, we will improve the convergence from convergence in probability to $L^2(\Omega)$- convergence, the so-called strong convergence in mean square. On the other hand, we will also improve the rate of convergence from logarithmic to almost polynomial.

To explain the method, the paper will deal with two different algorithms: the splitting scheme used in [4] and the implicit Euler schemes used in [11]. In the case of a diffusion coefficient G with linear growth conditions, which may depend on the solution and its gradient for the Euler schemes, we prove that the speed of convergence of both schemes is any negative power of the logarithm of the time mesh $\frac{T}{N}$ when the initial condition belongs to $\mathbb{W}^{1,2}$ and is divergence free. In the case of an additive noise - or under a slight generalization of such a noise - we prove that the strong $L^2(\Omega)$ speed of convergence of the fully or semi implicit Euler schemes introduced by Carelli and Prohl in [11] is polynomial in the time mesh. This speed depends on the viscosity coefficient ν and on the length of the time interval T. When T is small, or when ν is large, this speed is close to the best one which can be achieved in time, that is almost $\frac{1}{4}$. This is consistent with the time regularity of the strong solution to the stochastic Navier-Stokes equations, due to the scaling between the time and space variables in the heat kernel, and to the stochastic integral.

Let us try to explain the steps of our method here before going into more details later on in the paper. As we explained earlier, the main difficulty that prevents getting the strong convergence in mean square is due to the nonlinear term $(u \cdot \nabla)u$. Indeed, in order to estimate the error $e_k := u(t_k) - u_N(t_k)$ over the grid points $t_k,\ k = 1, \dots, N$, in some implicit Euler method, one has to estimate

$$\mathbb{E}\|(u(t_k) \cdot \nabla)u(t_k) - (u_N(t_k) \cdot \nabla)u_N(t_k)\|_{V'}.$$

To close the estimates and use some Gronwall lemma, the tool used in [11] (as well as in [4]) is to localize on a subspace of Ω. However, in both previous results, the localization set was depending on the discretization. In this work, we make slightly different computations, based on the antisymmetry of the bilinear term, and localize on sets which only depend on the solution to the stochastic Navier Stokes equations (1.1), such as Ω_N^M defined by (4.6) for the Euler schemes. Hence, one obtains for example

$$\mathbb{E}\Big(1_{\Omega_N^M} \max_{1\leq k\leq N} |e_k|^2_{\mathbb{L}^2}\Big) \leq C\ \exp\big[C_1(M)T\big]\ \Big(\frac{T}{N}\Big)^{\eta},$$

where $\eta < \frac{1}{2}$ and $C_1(M)$ is a constant depending on the bound M of the $\mathbb{L}^2$-norm of ∇u imposed on the localization set Ω_N^M. The exponent η is natural and related to the time regularity of the solution u when the initial condition u_0 belongs to $\mathbb{W}^{1,2}$.

In order to prove the strong speed of convergence, we will use the partition of Ω into Ω_N^M and its complement for some threshold M depending on N. More precisely, we have to balance the upper estimate of the moments localized on the set $\Omega_N^{M(N)}$ for some well chosen sequence $M(N)$, going to infinity as N does, and a similar upper estimate of the $L^2(\Omega)$ moment of the error localized on the complement of the set $\Omega_N^{M(N)}$. This is performed

by upper estimating moments of $u(t_k)$ and of $u_N(t_k)$ uniformly in N and k, estimating $\mathbb{P}\big((\Omega_N^{M(N)})^c\big)$, and using Hölder's inequality

$$\mathbb{E}\Big(1_{(\Omega_N^{M(N)})^c}\max_{1\leq k\leq N}|e_k|^2_{\mathbb{L}^2}\Big)\leq\Big(\mathbb{P}\big((\Omega_N^{M(N)})^c\big)\Big)^{\frac{1}{p}}\Big[\mathbb{E}\Big(\sup_{0\leq s\leq T}|u(s)|^{2q}_{\mathbb{L}^2}+\max_{0\leq k\leq N}|u_N(t_k)|^{2q}_{\mathbb{L}^2}\Big)\Big]^{\frac{1}{q}},$$

where p and q are conjugate exponents. A similar estimate was already used in[17] and [19]. The upper estimate of the probability of the "bad" set depends on the assumptions on the diffusion coefficient. Note that since we are localizing on a set which does not depend on the discretization scheme, only moments of the solution to the stochastic Navier Stokes equation (1.1) have to be dealt with.

In the case of globally Lipschitz coefficient G, we use bounds of various moments of u in $\mathbb{W}^{1,2}$. For both schemes, the strong speed of convergence is again in the logarithmic scale; when the initial condition is deterministic, it is any negative power of $\ln(N)$.

In the case of an additive noise, we use a slight extension of exponential moments of the solution of (1.1) in vorticity formulation proved previously by M. Hairer and J. Mattingly in [16], given by $\mathbb{E}\big(\sup_{t\in[0,T]}\exp(\alpha_0|\nabla u(t)|^2_{\mathbb{L}^2})\big)<\infty$ for some $\alpha_0>0$. This yields a better speed of convergence, due to that fact that the polynomial Markov inequality is replaced by an exponential one.

For the implicit Euler scheme the strong speed of convergence is polynomial with exponent $\gamma<\frac{1}{2}$ that depends on the viscosity ν. Note that for large ν, γ approaches $\frac{1}{2}$. For the splitting scheme, the strong speed of convergence we obtain in this paper is better than that of the convergence in probability proven in [4], although not polynomial; it is of the form $c\exp(-C\sqrt{N})$.

To simplify the presentation of our results, we only deal with time discretization, unlike in [11] where a space-time discretization is studied. Furthermore, in order to keep the paper in a reasonable size and present simple arguments to follow by the reader, we assume that G does not depend on time. We add relevant comments and remarks on the assumptions to be added in case of time dependent coefficients for the implicit Euler schemes (see section 4.5).

The paper is organized as follows. Section 2 recalls basic properties of the 2D Navier Stokes equations, functional spaces and strong solutions. We formulate the assumptions on the noise, which may be of gradient type are stated, provided that the stochastic parabolicity condition holds. The splitting scheme is described in Section 3 and various moments estimates previously used in [4] are recalled. The strategy for proving the strong speed of convergence is described and explained in details. The same strategy is used for the Euler schemes in Section 4, which is devoted to the fully implicit and semi implicit Euler schemes studied previously in [11]. Their strong speed of convergence is proved with a rate of convergence that is polynomial when the exponential moment is used. Finally, Section 5 provides on one hand an improved moment estimate for an auxiliary process used in the splitting scheme, and on the other hand the proof for the exponential moment estimates of the gradient of the solution to (1.1) .

As usual, except if specified otherwise, C denotes a positive constant that may change throughout the paper, and $C(a)$ denotes a positive constant depending on the parameter a.

2. Notations and preliminary results

Let $\mathbb{L}^p := L^p(D)^2$ (resp. $\mathbb{W}^{k,p} := W^{k,p}(D)^2$) denote the usual Lebesgue and Sobolev spaces of vector-valued functions endowed with the norms $|\cdot|_{\mathbb{L}^2}$ (resp. $\|\cdot\|_{\mathbb{W}^{k,p}}$). In what follows, we will consider velocity fields that have mean zero over $[0,L]^2$. Let $\mathbb{L}^2_{per}$ denote the subset of $\mathbb{L}^2$ periodic functions with mean zero over $[0,L]^2$, and let

$$H :=\{u \in \mathbb{L}^2_{per} \ : \ \mathrm{div} u = 0 \quad \text{weakly in} \ \ D\}, \qquad V := H \cap \mathbb{W}^{1,2}$$

be separable Hilbert spaces. The space H inherits its inner product denoted by $(\cdot,\cdot)$ and its norm from $\mathbb{L}^2$. The norm in V, inherited from $\mathbb{W}^{1,2}$, is denoted by $\|\cdot\|_V$. Moreover, let V' be the dual space of V, $\langle\cdot,\cdot\rangle$ denotes the duality between V' and V. Let $A = -\Delta$ with its domain $D(A) = \mathbb{W}^{2,2} \cap H$.

Let $b : V^3 \to \mathbb{R}$ denote the trilinear map defined by

$$b(u_1,u_2,u_3) := \int_D (u_1 \cdot \nabla u_2) \cdot u_3,$$

which by the incompressibility condition satisfies $b(u_1,u_2,u_3) = -b(u_1,u_3,u_2)$ for $u_i \in V$, $i = 1,2,3$. There exists a continuous bilinear map $B : V \times V \mapsto V'$ such that

$$\langle B(u_1,u_2),u_3\rangle = b(u_1,u_2,u_3), \quad \text{for all} \ \ u_i \in V, \ i = 1,2,3.$$

The map B satisfies the following antisymmetry relations:

$$\langle B(u_1,u_2),u_3\rangle = -\langle B(u_1,u_3),u_2\rangle, \quad \langle B(u_1,u_2),u_2\rangle = 0 \qquad \text{for all} \quad u_i \in V. \tag{2.1}$$

Furthermore, the Gagliardo-Nirenberg inequality implies that for $X := H \cap \mathbb{L}^4(D)$ we have

$$\|u\|_X^2 \le \bar{C} \ |u|_{\mathbb{L}^2} \, |\nabla u|_{\mathbb{L}^2} \le \frac{\bar{C}}{2} \|u\|_V^2 \tag{2.2}$$

for some positive constant $\bar{C}$. For $u \in V$ set $B(u) := B(u,u)$ and recall some well-known properties of B, which easily follow from Hölder's and Young's inequalities: given any $\beta > 0$ we have

$$|\langle B(u_1,u_2),u_3\rangle| \le \beta \|u_3\|_V^2 + \frac{1}{4\beta} \|u_1\|_X \, \|u_2\|_X, \tag{2.3}$$

$$|\langle B(u_1) - B(u_2) \, , u_1 - u_2\rangle| \le \beta \|u_1 - u_2\|_V^2 + C_\beta |u_1 - u_2|_{\mathbb{L}^2}^2 \|u_1\|_X^4, \tag{2.4}$$

for $u_i \in V$, $i = 1,2,3$, where

$$C_\beta = \frac{\bar{C}^2 3^3}{4^4 \beta^3}, \tag{2.5}$$

and $\bar{C}$ is defined by (2.2).

Let K be a separable Hilbert space and $(W(t), t \in [0,T])$ be a K-cylindrical Wiener process defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For technical reasons, we assume that the initial condition u_0 belongs to $L^p(\Omega; V)$ for some $p \in [2,\infty]$, and only consider *strong solutions* in the PDE sense. Given two Hilbert spaces H_1 and H_2, let $\mathcal{L}_2(H_1,H_2)$ denote the set of Hilbert-Schmidt operators from H_1 to H_2. The diffusion coefficient G satisfies the following assumption:

Condition (G1) Assume that $G : V \to \mathcal{L}_2(K,H)$ is continuous and there exist positive constants K_i, $i = 0,1,2$, L_1 and L_2 such that for $u, v \in V$:

$$\|G(u)\|_{\mathcal{L}_2(K,H)}^2 \le K_0 + K_1 |u|_{\mathbb{L}^2}^2 + K_2 \|u\|_V^2, \tag{2.6}$$

$$\|G(u) - G(v)\|_{\mathcal{L}_2(K,H)}^2 \le L_1 |u - v|_{\mathbb{L}^2}^2 + L_2 \|u - v\|_V^2. \tag{2.7}$$

Finally, note that the following identity involving the Stokes operator A and the bilinear term holds (see e.g. [25] Lemma 3.1):

$$\langle B(u), Au\rangle = 0, \quad u \in \text{Dom } (A). \tag{2.8}$$

We also suppose that G satisfies the following assumptions:
Condition (G2) The coefficient $G : [0,T] \times \text{Dom}(A) \to \mathcal{L}_2(K,V)$ and there exist positive constants K_i, $i = 0, 1, 2$, L_1 and L_2 such that for every $t \in [0,T]$, $u, v \in \text{Dom } (A)$:

$$\|G(u)\|^2_{\mathcal{L}_2(K,V)} \leq K_0 + K_1\|u\|^2_V + K_2|Au|^2_{\mathbb{L}^2}, \tag{2.9}$$

$$\|G(u) - G(v)|^2_{\mathcal{L}_2(K,V)} \leq L_1\|u - v\|^2_V + L_2|A(u - v)|^2_{\mathbb{L}^2}. \tag{2.10}$$

We define a strong solution of (1.1) as usual:

Definition 2.1. *We say that equation* (1.1) *has a unique strong solution if:*

- u *is an adapted* V*-valued process,*
- $\mathbb{P}$ *a.s. we have* $u(.,.,\omega) \in C([0,T];V) \cap L^2(0,T;\text{Dom } (A))$*,*
- $\mathbb{P}$ a.s.

$$\big(u(t),\phi\big)+\nu\int_0^t \big(\nabla u(s), \nabla\phi\big)ds + \int_0^t \big\langle[u(s)\cdot\nabla]u(s),\phi\big\rangle ds$$
$$= \big(u_0,\phi\big) + \int_0^t \big(\phi, G(u(s))dW(s)\big)$$

for every $t \in [0,T]$ *and every* $\phi \in V$*.*

As usual, by projecting (1.1) on divergence free fields, the pressure term is implicitly in the space V and can be recovered afterwards. Proposition 2.2 in [5] (see also [4], Theorem 4.1) shows the following:

Theorem 2.2. *Assume that* u_0 *is a* V*-valued,* $\mathcal{F}_0$*-measurable random variable such that* $\mathbb{E}\big(\|u_0\|^{2p}_V\big) < \infty$ *for some real number* $p \in [2,\infty)$*. Assume that the conditions* **(G1)** *and* **(G2)** *are satisfied with* $K_2 < \frac{2\nu}{2p-1}$ *and* $L_2 < 2\nu$*. Then there exists a unique strong solution* u *to equation* (1.1)*. Furthermore, for some positive constant* C *we have*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|u(t)\|^{2p}_V + \int_0^T |Au(s)|^2_{\mathbb{L}^2}\big(1+\|u(s)\|^{2(p-1)}_V\big)ds\Big) \leq C\big[1+\mathbb{E}(\|u_0\|^{2p}_V)\big]. \tag{2.11}$$

3. Time splitting scheme

In this section, we prove the strong $L^2(\Omega)$ convergence of the splitting scheme introduced in [4].

3.1. **Description of the splitting scheme.** Let $N > 1$, $h = \frac{T}{N}$ denote the time mesh, and $t_i = \frac{iT}{N}$, $i = 0, \cdots, N$ denote a partition of the time interval $[0,T]$. Let $F : V \to V'$ be defined by

$$F(u) = \nu Au + B(u,u). \tag{3.1}$$

In this section, we will require the coefficient G to satisfy the following stronger conditions, which are reformulations of **(G1)** and **(G2)** with $K_2 = L_2 = 0$. There exist non negative constants K_0, K_1 and L_1 such that for every $t \in [0,T]$ and $u, v \in H$ (resp. $u, v \in V$)

(G1Bis) $\|G(u)\|^2_{\mathcal{L}_2(K,H)} \leq K_0 + K_1|u|^2_{\mathbb{L}^2}$, $\|G(u) - G(v)\|^2_{\mathcal{L}_2(K,H)} \leq L_1|u - v|^2_{\mathbb{L}^2}$,

(G2Bis) $\|G(u)\|^2_{\mathcal{L}_2(K,V)} \leq K_0 + K_1\|u\|^2_V$, $\|G(u) - G(v)\|^2_{\mathcal{L}_2(K,V)} \leq L_1\|u - v\|^2_V$.

Note that the formulation of **(G2)** is slighty different from that used in [4]. They are equivalent due to the inequality $|\nabla u|_{\mathbb{L}^2} \leq C|\text{curl } u|_{\mathbb{L}^2}$ for $u \in V$, where curl $u = \partial_{x_1} u_2 - \partial_{x_2} u_1$.

Set $t_{-1} = -\frac{T}{N}$. For $t \in [t_{-1}, 0)$ set $y^N(t) = u^N(t) = u_0$ and $\mathcal{F}_t = \mathcal{F}_0$. The approximation. (y^N, u^N) is defined by induction as follows. Suppose that the processes $u^N(t)$ and $y^N(t)$ are defined for $t \in [t_{i-1}, t_i)$ and that $y^N(t_i^-)$ is H-valued and $\mathcal{F}_{t_i}$-measurable. Then for $t \in [t_i, t_{i+1})$, $u^N(t)$ with initial condition $y^N(t_i^-)$ at time t_i, is the unique solution of equation:

$$\frac{d}{dt}u^N(t) + F(u^N(t)) = 0, \quad t \in [t_i, t_{i+1}), \quad u^N(t_i) = u^N(t_i^+) = y^N(t_i^-). \tag{3.2}$$

Then $u^N(t_{i+1}^-)$ is well-defined, H-valued and $\mathcal{F}_{t_i}$-measurable. Then set $y^N(t_i) = u^N(t_{i+1}^-)$, and for $t \in [t_i, t_{i+1})$ define $y^N(t)$ as the unique solution of equation

$$dy^N(t) = G(y^N(t))dW(t), \quad t \in [t_i, t_{i+1}), \quad y^N(t_i) = y^N(t_i^+) = u^N(t_{i+1}^-). \tag{3.3}$$

Finally, set $u^N(T) = y^N(T) = y^N(T^-)$. The processes u^N and y^N are well-defined and have finite moments as proved in [4], Lemma 4.2.

Theorem 3.1. *Let u_0 be a V-valued, $\mathcal{F}_0$ random variable such that $\mathbb{E}(\|u_0\|_V^{2p}) < \infty$ for some real number $p \geq 2$. Suppose that G satisfies conditions* **(G1Bis)** *and* **(G2Bis)**. *Then there exists a positive constant C such that for every integer $N \geq 1$*

$$\sup_{t\in[0,T]} \mathbb{E}\big(\|u^N(t)\|_V^{2p} + \|y^N(t)\|_V^{2p}\big) + \mathbb{E}\int_0^T \big(1 + \|u^N(t)\|_V^{2(p-1)}\big)|Au^N(t)|_{\mathbb{L}^2}^2 dt \leq C. \tag{3.4}$$

The following result gives a bound of the difference between u^N and y^N (see [4], Proposition 4.3).

Proposition 3.2. *Let u_0 be $\mathcal{F}_0$-measurable such that $\mathbb{E}(\|u_0\|_V^4) < \infty$ and G satisfies conditions* **(G1Bis)** *and* **(G2Bis)**. *Then there exists a positive constant $C := C(T)$ such that for every integer $N \geq 1$*

$$\mathbb{E}\int_0^T \|y^N(t) - u^N(t)\|_V^2 dt \leq \frac{C}{N}. \tag{3.5}$$

For technical reasons, let us consider the process $(z^N(t), t \in [0, T])$ which mixes u^N and y^N, and is defined by

$$z^N(t) = u_0 - \int_0^t F(u^N(s))ds + \int_0^t G(y^N(s))dW(s). \tag{3.6}$$

The process z^N is a.s. continuous on $[0, T]$. Note that $z^N(t_k) = y^N(t_k^-) = u^N(t_k)$ for $k = 0, 1, \cdots, N$. The following result gives bounds of the V-norm of $z^N - u^N$ and $z^N - y^N$. It is an extension of Lemma 4.4 in [4].

Proposition 3.3. *Let u_0 be V-valued, $\mathcal{F}_0$-measurable such that $\mathbb{E}(\|u_0\|_V^{2p}) < \infty$ for some integer $p \geq 2$ and let G satisfy conditions* **(G1Bis)** *and* **(G2Bis)**. *Then there exists a constant $C := C(p, T)$ such that for every integer $N \geq 1$*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|z^N(t) - u^N(t)\|_V^{2p}\Big) \leq \frac{C}{N^{p-1}} \text{ and } \sup_{t\in[0,T]} \mathbb{E}\big(\|z^N(t) - u^N(t)\|_V^{2p}\big) \leq \frac{C}{N^p}. \tag{3.7}$$

Note that combining (3.5) and (3.7), we deduce that if $\mathbb{E}(\|u_0\|_V^4) < \infty$,

$$\mathbb{E}\int_0^T \|z^N(t) - y^N(t)\|_V^2 dt \leq \frac{C}{N} \tag{3.8}$$

for some constant $C := C(T)$ independent of N.

Proof. For $t \in [t_k, t_{k+1})$, $k = 0, \cdots, N-1$ we have

$$z^N(t) - u^N(t) = \int_{t_k}^t G(y^N(s))dW(s).$$

Since $z^N(T) = u^N(T)$, for any $p \geq 2$ the Burkholder-Davies-Gundy inequality implies

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[0,T]} \|z^N(t) - u^N(t)\|_V^{2p}\Big) &= \mathbb{E}\Big(\sup_{0\leq k<N}\sup_{t\in[t_k,t_{k+1})} \|z^N(t) - u^N(t)\|_V^{2p}\Big) \\
&\leq C_p \sum_{k=0}^{N-1} \mathbb{E}\Big(\Big|\int_{t_k}^{t_{k+1}} \|G\big(s, y^N(s)\big)\|_{\mathcal{L}_2(K,V)}^2 ds\Big|^p\Big) \\
&\leq C_p \sum_{k=0}^{N-1} \Big(\frac{T}{N}\Big)^p \sup_{t\in[0,T]} \big(K_0 + K_1\mathbb{E}(\|y^N(t)\|_V^{2p})\big).
\end{aligned}$$

Inequality (3.4) concludes the proof of the first part of (3.7). Furthermore,

$$\sup_{t\in[0,T]} \mathbb{E}\big(\|z^N(t) - u^N(t)\|_V^{2p}\big) = \sup_{0\leq k<N}\sup_{t\in[t_k,t_{k+1})} \mathbb{E}\big(\|z^N(t) - u^N(t)\|_V^{2p}\big).$$

A similar argument concludes the proof. □

3.2. **A localized $L^2(\Omega)$ convergence.** Recall that $X = \mathbb{L}^4(D) \cap H$ is an interpolation space between H and V such that (2.2) holds. For every $M > 0$, set

$$\tilde{\Omega}_M(t) := \Big\{ \sup_{s\in[0,t]} \|u(s)\|_X^4 \leq M\Big\}. \tag{3.9}$$

Note that once more, and unlike [4], this set only depends on the solution u of (1.1) and does not depend on the scheme. Let $\tau_M =: \inf\{t \geq 0 : \|u(t)\|_X^4 \geq M\} \wedge T$; then $\tau_M = T$ on $\Omega_M(T)$. The following result improves Proposition 5.1 in [4]. Note that in our case, both processes u and z^N have a.s. continuous trajectories.

Proposition 3.4. *Let u_0 be V-valued, $\mathcal{F}_0$-measurable such that $\mathbb{E}(\|u_0\|_V^8) < \infty$ and suppose that G satisfies the conditions* **(G1Bis)** *and* **(G2Bis)**. *Then, there exist constants C and $\widetilde{C(M)}$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,\tau_M]} |z^N(t) - u(t)|_{\mathbb{L}^2}^2 + \int_0^{\tau_M} \big[\|u^N(t) - u(t)\|_V^2 + \|y^N(t) - u(t)\|_V^2\big]dt\Big) \leq \frac{C}{N} e^{T\widetilde{C(M)}}, \tag{3.10}$$

where

$$\widetilde{C(M)} := \frac{3^3\bar{C}^2}{2^5\beta^3\nu^3} M + C(\nu, L_1, \beta, \epsilon), \tag{3.11}$$

and $\bar{C}$ is the constant defined in (2.2) *and $\beta < 1$.*

Proof. Let us apply the Itô formula to $|z^N(t\wedge\tau_M)-u(t\wedge\tau_M)|^2_{\mathbb{L}^2}$. This is possible even if z^N and u are not regular enough. Indeed, we can apply the Itô formula to their Galerkin approximations and then pass to the limit using some Yoshida approximation. This implies

$$|z^N(t\wedge\tau_M)-u(t\wedge\tau_M)|^2_{\mathbb{L}^2}=\sum_{i=1}^{2}T_i(t\wedge\tau_M)+I(t\wedge\tau_M), \tag{3.12}$$

where

$$\begin{aligned}
T_1(t\wedge\tau_M)=&-2\int_0^{t\wedge\tau_M}\big\langle F(u^N(s))-F(u(s))\,,\,z^N(s)-u(s)\big\rangle ds,\\
T_2(t\wedge\tau_M)=&\int_0^{t\wedge\tau_M}\|G(y^N(s))-G(u(s))\|^2_{\mathcal{L}_2(K,H)}ds,\\
I(t\wedge\tau_M)=&2\int_0^{t\wedge\tau_M}\big(\big[G(y^N(s))-G(u(s))\big]dW(s)\,,\,z^N(s)-u(s)\big).
\end{aligned}$$

The Lipschitz property **(G1Bis)** implies

$$T_2(t\wedge\tau_M)\leq 2L_1\Big[\int_0^{t\wedge\tau_M}|z^N(s)-u(s)|^2_{\mathbb{L}^2}ds+\int_0^{t\wedge\tau_M}|z^N(s)-y^N(s)|^2_{\mathbb{L}^2}ds\Big]. \tag{3.13}$$

Furthermore, $T_1(t\wedge\tau_M)\leq\sum_{i=1}^4 T_{1,i}(t\wedge\tau_M)$, where

$$\begin{aligned}
T_{1,1}(t\wedge\tau_M)=&-2\nu\int_0^{t\wedge\tau_M}\langle Au^N(s)-Au(s))\,,\,u^N(s)-u(s)\rangle ds,\\
T_{1,2}(t\wedge\tau_M)=&-2\nu\int_0^{t\wedge\tau_M}\langle Au^N(s)-Au(s))\,,\,z^N(s)-u^N(s)\rangle ds,\\
T_{1,3}(t\wedge\tau_M)=&-2\int_0^{t\wedge\tau_M}\langle B(u^N(s),u^N(s))-B(u(s),u(s))\,,\,u^N(s)-u(s)\rangle ds,\\
T_{1,4}(t\wedge\tau_M)=&-2\int_0^{t\wedge\tau_M}\langle B(u^N(s),u^N(s))-B(u(s),u(s))\,,\,z^N(s)-u^N(s)\rangle ds.
\end{aligned}$$

The definition of A implies that

$$T_{1,1}(t\wedge\tau_M)=-2\nu\int_0^{t\wedge\tau_M}|\nabla u^N(s)-\nabla u(s)|^2_{\mathbb{L}^2}ds. \tag{3.14}$$

The Cauchy Schwarz and Young inequalities imply that for any $\beta_2>0$,

$$T_{1,2}(t\wedge\tau_M)\leq\nu\beta_2\int_0^{t\wedge\tau_M}|\nabla u^N(s)-\nabla u(s)|^2_{\mathbb{L}^2}ds+\frac{\nu}{\beta_2}\int_0^t\|z^N(s)-u^N(s)\|^2_V ds. \tag{3.15}$$

Using the inequality (2.4), we deduce that for any $\beta_3>0$ and $\epsilon>0$,

$$\begin{aligned}
T_{1,3}(t\wedge\tau_M)\leq&\,2\nu\beta_3\int_0^{t\wedge\tau_M}\big(|\nabla u^N(s)-\nabla u(s)|^2_{\mathbb{L}^2}+|u^N(s)-u(s)|^2_{\mathbb{L}^2}\big)ds\\
&+2C_{\nu\beta_3}\int_0^{t\wedge\tau_M}\|u(s)\|^4_X\,|u^N(s)-u(s)|^2_{\mathbb{L}^2}ds\\
\leq&2\nu\beta_3\int_0^{t\wedge\tau_M}|\nabla[u^N(s)-u(s)]|^2_{\mathbb{L}^2}ds+2(\nu\beta_3+C_{\nu\beta_3}M)(1+\epsilon)\int_0^{t\wedge\tau_M}|z^N(t)-u(t)|^2_{\mathbb{L}^2}ds\\
&+C(\nu,\beta_3,\epsilon)\int_0^{t\wedge\tau_M}\big(1+\|u(s)\|^4_X\big)|z^N(t)-u^N(t)|^2_{\mathbb{L}^2}ds.
\end{aligned} \tag{3.16}$$

Finally, since B is bilinear, Hölder's inequality implies

$$T_{1,4}(t\wedge\tau_M) = 2\int_0^{t\wedge\tau_N} \Big[\big\langle B(u^N(s)-u(s), z^N(s)-u^N(s))\,,\, u^N(s)\big\rangle + \langle B(u(s), z^N(s)-u^N(s))\,,\, u^N(s)-u(s)\rangle\Big]ds$$
$$\leq 2\int_0^{t\wedge\tau_M} \|u^N(s)-u(s)\|_X\big[\|u^N(s)\|_X+\|u(s)\|_X\big]\|z^N(s)-u^N(s)\|_V ds.$$

Using the interpolation inequality (2.2) and the Young inequality, we deduce that for any $\beta_4>0$, there exists a positive constant $C(\nu,\beta_4)$ such that

$$\sqrt{C}\big|\nabla\big(u^N(s)-u(s)\big)\big|_{\mathbb{L}^2}^{\frac12}|u^N(s)-u(s)|_{\mathbb{L}^2}^{\frac12}\|z^N(s)-u^N(s)\|_V\big[\|u^N(s)\|_X+\|u(s)\|_X\big]$$
$$\leq \nu\beta_4\big|\nabla\big(u^N(s)-u(s)\big)\big|_{\mathbb{L}^2}^2 + C|z^N(s)-u(s)|_{\mathbb{L}^2}^2 + C|z^N(s)-u^N(s)|_{\mathbb{L}^2}^2$$
$$+ C(\nu,\beta_4)\big[\|u^N(s)\|_X^2+\|u(s)\|_X^2\big]\|z^N(s)-u^N(s)\|_V^2.$$

This implies

$$T_{1,4}(t\wedge\tau_M) \leq \nu\beta_4\int_0^{t\wedge\tau_M}\big|\nabla\big[u^N(s)-u(s)\big]\big|_{\mathbb{L}^2}^2 ds + C\int_0^{t\wedge\tau_M}|z^N(s)-u(s)|_{\mathbb{L}^2}^2 ds$$
$$+ C(\nu,\beta_4)\int_0^{t\wedge\tau_M}\Big(|u^N(s)-z^N(s)|_{\mathbb{L}^2}^2 + \big[\|u^N(s)\|_X^2+\|u(s)\|_X^2\big]\|z^N(s)-u^N(s)\|_V^2\Big)ds. \tag{3.17}$$

Collecting the upper estimates (3.13)–(3.17), we deduce that for $\beta_2+2\beta_3+\beta_4<2$ and $t\in[0,T]$,

$$\sup_{0\leq s\leq t}|z^N(s\wedge\tau_N)-u(s\wedge\tau_N)|_{\mathbb{L}^2}^2 + \nu(2-\beta_2-2\beta_3-\beta_4)\int_0^{t\wedge\tau_N}|\nabla[u^N(s)-u(s)]|_{\mathbb{L}^2}^2 ds$$
$$\leq R(t) + \sup_{s\in[0,t]} I(s\wedge\tau_N) + \Big[2(1+\epsilon)C_{\nu\beta_3}M + 2L_1 + C(\nu,\beta_3,\epsilon)\Big]\int_0^{t\wedge\tau_M}|z^N(s)-u(s)|_{\mathbb{L}^2}^2 ds, \tag{3.18}$$

where, gathering all error terms, we let

$$R(t) = \int_0^{t\wedge\tau_M}\Big(2L_1|z^N(s)-y^N(s)|_{\mathbb{L}^2}^2 + C(\nu,\beta_2,\beta_3,\beta_4)\|z^N(s)-u^N(s)\|_V^2\Big)ds$$
$$+ C(\nu,\beta_3,\epsilon)\Big\{\int_0^{t\wedge\tau_M}\|u(s)\|_X^8 ds\Big\}^{\frac12}\Big\{\int_0^{t\wedge\tau_M}|z^N(s)-u^N(s)|_{\mathbb{L}^2}^4 ds\Big\}^{\frac12}$$
$$+ C(\nu,\beta_4)\int_0^{t\wedge\tau_M}\big[\|u^N(s)\|_X^2+\|u(s)\|_X^2\big]\|z^N(s)-u^N(s)\|_V^2 ds.$$

The Cauchy-Schwarz inequality, the integrability property $\mathbb{E}(\|u_0\|_V^8)<\infty$, and the upper estimates (2.11), (3.4), (3.7) and (3.8) imply the existence of some positive constant C (depending on the parameter β_i and ϵ such that for every N, M,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} R(t)\Big)\leq\frac{C}{N}. \tag{3.19}$$

Using the Burkholder-Davis-Gundy inequality and then Young's inequality and the Lipschitz condition **(G1Bis)** on G, we deduce that for any $\delta > 0$,

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[s\le t]} I(t)\Big) &\le 6\mathbb{E}\Big(\Big\{ \int_0^{t\wedge\tau_M} \|G(s,y^N(s)) - G(s,u(s))\|^2_{\mathcal{L}_2(K,H)} |u^N(s)-u(s)|^2_{\mathbb{L}^2} ds\Big\}^{\frac12}\Big)\\
&\le 6\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_M]} |z^N(s)-u(s)|_{\mathbb{L}^2}\Big\{ \int_0^{t\wedge\tau_M} L_1 |y^N(s)-u(s)|^2_{\mathbb{L}^2} ds\Big\}^{\frac12}\Big).\\
&\le \delta\mathbb{E}\Big(\sup_{s\in[0,t]} |z^N(s\wedge\tau_M)-u(s\wedge\tau_M)|^2_{\mathbb{L}^2}\Big) + C(\delta)\mathbb{E}\int_0^{t\wedge\tau_M} |z^N(s)-u(s)|^2_{\mathbb{L}^2} ds\\
&\quad + C(\delta)\mathbb{E}\int_0^{t\wedge\tau_M} |y^N(s)-z^N(s)|^2_{\mathbb{L}^2} ds\\
&\le \delta\mathbb{E}\Big(\sup_{s\in[0,t]} |z^N(s\wedge\tau_M)-u(s\wedge\tau_M)|^2_{\mathbb{L}^2}\Big) + C(\delta)\mathbb{E}\int_0^{t\wedge\tau_M} |z^N(s)-u(s)|^2_{\mathbb{L}^2} ds + \frac{C(\delta)}{N},
\end{aligned} \tag{3.20}$$

where the last inequality is a consequence of (3.8). The upper estimates (3.18)–(3.20) imply that if $\delta < 1$ and $\bar{C}(\beta) := 2 - \beta_2 + 2\beta_3 + \beta_4 > 0$,

$$\begin{aligned}
&(1-\delta)\mathbb{E}\Big(\sup_{s\in[0,t]} |z^N(s\wedge\tau_M)-u(s\wedge\tau_M)|^2_{\mathbb{L}^2}\Big) + \nu\bar{C}(\beta)\mathbb{E}\int_0^{t\wedge\tau_M} |\nabla[u^N(s)-u(s)||^2_{\mathbb{L}^2} ds\\
&\le \frac{C}{N} + \Big[2(1+\epsilon)C_{\nu\beta_3}M + C(L_1,\nu,\beta_3,\epsilon,\delta)\Big]\mathbb{E}\int_0^t |z^N(s\wedge\tau_M)-u(s\wedge\tau_M)|^2_{\mathbb{L}^2} ds.
\end{aligned} \tag{3.21}$$

The Gronwall inequality (disregarding the second term in the left hand side of (3.21)) proves that

$$\mathbb{E}\Big(\sup_{s\in[0,t]} |z^N(s\wedge\tau_M)-u(s\wedge\tau_M)|^2_{\mathbb{L}^2}\Big) \le \frac{C}{N}\exp\big(T\,\widetilde{C(M)}\big),$$

where $\widetilde{C(M)}$ is defined by (3.11). Indeed, the term $\beta_3 < 1$ has to be chosen first since the other positive constants β_2 and β_4, which have to satisfy $\beta_2 + \beta_4 < 2(1-\beta_3)$, only appear in the "error" term $R(t)$. Using this inequality in (3.21), the definition of $C_{\nu\beta_3}$ given in (2.5), (3.7) and (3.8), and changing the ratio $\frac{1+\epsilon}{(1-\delta)\beta_3^3}$ into $\frac{1}{\beta^3}$ for some $\beta \in (0,1)$, we conclude the proof of (3.10). □

3.3. **Strong $L^2(\Omega)$ speed of convergence of the splitting scheme.** Since $\tau_M = T$ on $\tilde{\Omega}_M$, we can rewrite (3.10) as

$$\mathbb{E}\Big(1_{\tilde{\Omega}_M}\Big\{ \sup_{t\in[0,T]} |z^N(t)-u(t)|^2_{\mathbb{L}^2} + \int_0^T \big[\|u^N(t)-u(t)\|^2_V + \|y^N(t)-u(t)\|^2_V\big] dt\Big\}\Big) \le \frac{C}{N} e^{T\tilde{C}(M)}.$$

The previous section provides an upper estimate of the $L^2(\Omega)$ - norm of the maximal error on each time step localized on the set Ω_M. In order to deduce the strong speed of convergence, we next have to analyze this error on the complement of this localization set.

The first step is described in the following simple upper estimates, which follow from Hölder's inequality. Hence, we suppose that $\mathbb{E}(\|u_0\|^{2q}_V) < \infty$ and that p and q are conjugate

exponents; then

$$\mathbb{E}\Big(1_{\tilde{\Omega}_M^c}\sup_{t\in[0,T]}|z^N(t)-u(t)|_{\mathbb{L}^2}^2\Big)\leq C\Big[\mathbb{P}(\tilde{\Omega}_M^c)\Big]^{\frac{1}{p}}\Big[\mathbb{E}\Big(\sup_{t\in[0,T]}\big(|z^N(t)|_{\mathbb{L}^2}^{2q}+|u(t)|_{\mathbb{L}^2}^{2q}\big)\Big)\Big]^{\frac{1}{q}}. \quad (3.22)$$

Note that we have the similar upper estimate

$$\mathbb{E}\Big(1_{\tilde{\Omega}_M^c}\int_0^T\|z^N(t)-u(t)\|_V^2dt\Big)$$
$$\leq C\Big[\mathbb{P}(\tilde{\Omega}_M^c)\Big]^{\frac{1}{p}}\Big[\mathbb{E}\int_0^T\big(\|z^N(t)-u^N(t)\|_V^{2q}+\|u^N(t)\|_V^{2q}+\|u(t)\|_V^{2q}\big)dt\Big]^{\frac{1}{q}}\leq C\Big[\mathbb{P}(\tilde{\Omega}_M^c)\Big]^{\frac{1}{p}},$$

where the last upper estimates is deduced from (2.11), (3.4) and (3.7). Theorem 2.2 shows that $\mathbb{E}(\sup_{t\in[0,T]}\|u(t)\|_V^2)<\infty$. We next prove that $\mathbb{E}(\sup_{t\in[0,T]}|z^N(t)|_{\mathbb{L}^2}^{2q})$ is bounded by a constant independent of N.

Lemma 3.5. *Let $p\geq 2$ and u_0 be V-valued, $\mathcal{F}_0$-measurable such that $\mathbb{E}(\|u_0\|_V^{2p+1})<\infty$. Suppose that G satisfies the conditons* **(G1Bis)** *and* **(G2Bis)**. *Then there exists a positive constant C_p such that for every integer $N\geq 1$,*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|z^N(t)|_{\mathbb{L}^2}^{2p}+\nu\int_0^T\|z^N(t)\|_V^{2p}dt\Big)\leq C_p.$$

The proof of this lemma is given in the Appendix, Section 5.1.

We next have to make sure that the threshold $M(N)$ is chosen to balance the upper estimates (3.10) and (3.22).

Case 1: Linear growth diffusion coefficient. Suppose that $\mathbb{E}(\|u_0\|_V^{2q})<\infty$. Then using the Gagliardo-Nirenberg inequality (2.2), we deduce

$$\mathbb{P}\big(\tilde{\Omega}_{M(N)}^c\big)=\mathbb{P}\Big(\sup_{t\in[0,T]}\|u(t)\|_X^4\geq M(N)\Big)=\mathbb{P}\Big(\sup_{t\in[0,T]}\|u(t)\|_V^4\geq\frac{4M(N)}{\bar{C}^2}\Big)$$
$$\leq\frac{\bar{C}^q}{2^qM(N)^{\frac{q}{2}}}\mathbb{E}\Big(\sup_{t\in[0,T]}\|u(t)\|_V^{2q}\Big)\leq\frac{C}{M(N)^{\frac{q}{2}}},$$

where the last upper estimates is a consequence of (2.11). To balance the right hand sides of (3.10) and (3.22), we choose $M(N)\to\infty$ as $N\to\infty$ such that

$$\frac{1}{N}\exp(TC\widetilde{(M(N))})\asymp C(q)\frac{1}{M(N)^{\frac{q-1}{2}}}. \quad (3.23)$$

Taking logarithms and using (3.11), this comes down to

$$-\ln(N)+2(1+\epsilon)C_{\nu\beta}M(N)T\asymp-\frac{q-1}{2}\ln(M(N)),$$

for some $\beta\in(0,1)$ and $\epsilon\in(0,1)$, where $C_{\nu\beta}$ is defined in (2.5). Let

$$M(N):=\frac{1}{2(1+\epsilon)C_{\nu\beta}T}\Big[\ln(N)-\frac{q-1}{2}\ln\big(\ln(N)\big)\Big]\asymp C(\nu,\beta,\epsilon,T)\ln(N). \quad (3.24)$$

Then $M(N)\to\infty$ as $N\to\infty$ and (3.23) is satisfied. This yields the following result, where the first upper estimate follows from (3.23) and (3.7), while the second one is deduced from the fact that $y^N(t_k^+)=u^N(t_{k+1}^-)$.

Theorem 3.6. *Suppose that u_0 is V-valued such that $\mathbb{E}(\|u_0\|_V^{2q+1}) < \infty$ for some $q \geq 4$ and that G satisfies conditions* **(G1Bis)** *and* **(G2Bis)**. *Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \big[|z^N(t) - u(t)|_{\mathbb{L}^2}^2 + |u^N(t) - u(t)|_{\mathbb{L}^2}^2\big] + \int_0^T \big[\|z^N(s) - u(s)\|_V^2 + \|u^N(s) - u(s)\|_V^2 + \|y^N(s) - u(s)\|_V^2\big]ds\Big) \leq \frac{C}{\ln(N)^{\frac{q-1}{2}}}, \tag{3.25}$$

$$\mathbb{E}\Big(\sup_{k=1,\cdots,N} \big[|u^N(t_k^+) - u(t_k)|_{\mathbb{L}^2}^2 + |y^N(t_k^+) - u(t_k)|_{\mathbb{L}^2}^2\big]\Big) \leq \frac{C}{\ln(N)^{\frac{q-1}{2}}}. \tag{3.26}$$

Remark 3.7. *Note that if u_0 is a deterministic element of V, then the speed of convergence of the current splitting scheme is any negative power of* $\ln(N)$.

Case 2: Additive noise. Suppose that $G(u) := G \in \mathcal{L}_2(K, V)$, that is the noise is additive, or more generally that G satisfies the conditions **(G1Bis)** and **(G2Bis)** with $K_1 = 0$, that is $\|G(u)\|_{\mathcal{L}_2(K,V)}^2 \leq K_0$. Then for any constant $\alpha > 0$ using an exponential Markov inequality and the Gagliardo-Niremberg inequality (2.2), we deduce

$$\mathbb{P}(\tilde{\Omega}_M^c) \leq \mathbb{P}\Big(\sup_{t\in[0,T]} \|u(t)\|_X^2 \geq \sqrt{M}\Big) \leq \mathbb{P}\Big(\sup_{t\in[0,T]} \|u(t)\|_V^2 \geq \frac{2\sqrt{M}}{\bar{C}}\Big). \tag{3.27}$$

We next prove that for an additive noise, or in a slightly more general setting, for $\alpha > 0$ small enough, $\mathbb{E}[\exp(\alpha \sup_{t\in[0,T]} \|u(t)\|_V^2)] < \infty$. In the case of an additive noise, this result is a particular case of [16], Lemma A.1. In this reference, the periodic Navier Stokes equation is written in vorticity formulation $\xi(t) = \partial_1 u_2(t) - \partial_2 u_1(t)$. The velocity u projected on divergence free fields can be deduced from ξ by the Biot-Savart kernel, and $|\nabla u(t)|_{\mathbb{L}^2} \leq C|\xi(t)|_{\mathbb{L}^2}$. We extend this result to the case of a more general diffusion coefficient whose Hilbert-Schmidt norm is bounded. Recall that the Poincaré inequality implies the existence of a constant $\tilde{C} > 0$ such that if we set $|||u|||^2 := |\nabla u|_{\mathbb{L}^2}^2 + |Au|_{\mathbb{L}^2}^2$ for $u \in Dom(A)$, then we have

$$\|u\|_V^2 = |u|_{\mathbb{L}^2}^2 + |\nabla u|_{\mathbb{L}^2}^2 \leq \tilde{C}|||u|||^2. \tag{3.28}$$

Lemma 3.8. *Let $u_0 \in V$ and G satisfy conditions* **(C1Bis)** *and* **(C2Bis)** *with $K_1 = 0$, that is $\|G(t,u)\|_{\mathcal{L}_2(K,V)}^2 \leq K_0$. Then the solution u to* (1.1) *satisfies*

$$\mathbb{E}\Big\{ \exp\Big[\alpha\Big(\sup_{0\leq s\leq T} \|u(s)\|_V^2\Big) + \nu \int_0^T |Au(s)|_{\mathbb{L}^2}^2 ds\Big)\Big]\Big\} \leq 3\exp\big(\alpha[\|u_0\|_V^2 + TK_0]\big) \tag{3.29}$$

for $\alpha \in (0, \alpha_0]$ and $\alpha_0 = \frac{\nu}{4K_0\tilde{C}}$, where $\tilde{C}$ is defined in (3.28).

In order to make this paper as self-contained as possible, we prove this lemma in the appendix, section 5.2.

Let $u_0 \in V$; then for $\alpha_0 = \frac{\nu}{4K_0\tilde{C}}$, the inequalities (3.27) and (3.29) imply

$$\mathbb{P}(\tilde{\Omega}_M^c) \leq 3e^{\alpha_0 K_0 T} \exp\Big(- 2\alpha_0 \frac{\sqrt{M}}{\bar{C}}\Big),$$

where $\bar{C}$ is defined by (2.2). We then have to choose $M(N) \to \infty$ as $N \to \infty$ to balance the right hand sides of (3.10) and (3.22), that is such that for some $p > 1$,

$$\exp\Big(- \frac{2\alpha_0 \sqrt{M(N)}}{p\bar{C}}\, T\Big) \asymp c_2 \frac{T}{N} \exp\big(\widetilde{C(M(N))T}\big)$$

for some positive constant c_2. Taking logarithms, we look for $M(N)$ such that

$$-\frac{2\alpha_0}{p\bar{C}} \sqrt{M(N)} \asymp -\ln(N) + 2(1+\epsilon) C_{\nu\beta} M(N) T,$$

where $\beta \in (0,1)$, $\epsilon > 0$ and $C_{\nu\beta}$ is defined by (2.5). Set $X = \sqrt{M(N)}$, $a_2 = 2(1+\epsilon)C_{\nu\beta}T$ and $a_1 = \frac{2\alpha_0}{p\bar{C}}$. We have to solve the equation $a_2 X^2 + a_1 X - \ln(N) = 0$. The positive root of this polynomial is equal to $\sqrt{\frac{\ln(N)}{a_2}} + O(1)$ as $N \to \infty$ and $\exp(-\frac{2\alpha_0\sqrt{M(N)}}{p\bar{C}}) \asymp C\exp(-2\alpha_0 \frac{\sqrt{\ln(N)}}{\sqrt{a_2} p\bar{C}})$. Thus, we deduce the following rate of convergence of the splitting scheme.

Theorem 3.9. *Let $u_0 \in V$ and G satisfy the conditions* **(G1Bis)** *and* **(G2Bis)** *with $K_1 = 0$, that is $\|G(s,u)\|^2_{\mathcal{L}(K,V)} \le K_0$. Then*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \big[|z^N(t) - u(t)|^2_{\mathbb{L}^2} + |u^N(t) - u(t)|^2_{\mathbb{L}^2}\big] \tag{3.30}$$
$$+ \int_0^T \big[\|z^N(s) - u(s)\|^2_V + \|u^N(s) - u(s)\|^2_V + \|y^N(s) - u(s)\|^2_V\big] ds\Big) \le C e^{-\gamma\sqrt{\ln(N)}},$$

$$\mathbb{E}\Big(\sup_{k=1,\cdots,N} \big[|u^N(t_k^+) - u(t_k)|^2_{\mathbb{L}^2} + |y^N(t_k^+) - u(t_k)|^2_{\mathbb{L}^2}\big]\Big) \le C e^{-\gamma\sqrt{\ln(N)}}, \tag{3.31}$$

where

$$\gamma < \frac{\alpha_0}{\bar{C}^2} \sqrt{\frac{2^9 \nu^3}{3^3 T}}.$$

Note that when ν increases, the upper bound of the exponent γ increases.

Remark 3.10. *Note that the statements of Theorems 3.6 and 3.9 are valid if the diffusion coefficient G depends on the time parameter $t \in [0,T]$ and satisfies the global growth and Lipschitz versions of* **(G1Bis)** *and* **(G2Bis)**. *We have removed the time dependence of G to focus on the main arguments used to obtain strong convergence results.*

4. Euler time schemes

4.1. **Description of the fully implicit scheme and first results.** Let us first recall the fully implicit time discretization scheme of the stochastic 2D Navier-Stokes introduced by E. Carelli and A. Prohl in [11]. As in the previous section, let $t_k = \frac{kT}{N}$, $k = 0, \cdots, N$, denote the time grid. Since we are not using finite elements for the space discretization, we may disregard the pressure and define the following time discretization scheme:
Fully implicit Euler scheme *Let u_0 be a V-valued, $\mathcal{F}_0$-measurable random variable and set $u_N(t_0) = u_0$. For $k = 1, \cdots, N$, find $u_N(t_k) \in V$ such that $\mathbb{P}$ a.s. for $k = 1, \cdots, N$ and for all $\phi \in V$,*

$$\big(u_N(t_k) - u_N(t_{k-1}), \phi\big) + \frac{T}{N}\Big[\nu\big(\nabla u_N(t_k), \nabla\phi\big) + \big\langle B(u_N(t_k), u_N(t_k)), \phi\big\rangle\Big]$$
$$= \big(G(u_N(t_{k-1}))\, \Delta_k W, \phi\big), \tag{4.1}$$

where $\Delta_k W = W(t_k) - W(t_{k-1})$.

In the study of the Euler discretization schemes, we will need some Hölder regularity of the solution. This is proved by means of semigroup theory; see [24], Proposition 3.4 and [11], Lemma 2.3. An easy modification of these proofs in our slightly different functional setting for the stochastic integral yields the following.

Proposition 4.1. *Let* u_0 *be* $\mathcal{F}_0$*-measurable such that* $\mathbb{E}(\|u_0\|_V^{2p}) < \infty$ *for some* $p \in [2,4]$. *Let* G *satisfy conditions* **(C1)** *and* **(C2)** *with the constraints of Theorem 2.2. Then for* $\eta \in (0, \frac{1}{2})$, *we have*

$$\mathbb{E}\big(\|u(t) - u(s)\|_{\mathbb{L}^4}^p\big) \le C\,|t-s|^{\eta p}, \tag{4.2}$$

$$\mathbb{E}\big(\|u(t) - u(s)\|_V^p\big) \le C\,|t-s|^{\frac{\eta p}{2}}. \tag{4.3}$$

Let us recall Lemma 3.1 in [11], which proves moment estimates of the solution to (4.1). Despite the slightly different assumptions on the diffusion coefficient G, its proof clearly extends as well as the induction argument of Lemma 3.1 in reference [10], which is used to prove (i) below for any exponent q.

Lemma 4.2. *Let* u_0 *be* $\mathcal{F}_0$*-measurable such that* $\mathbb{E}\big(\|u_0\|_V^{2q}\big) < \infty$ *for some integer* $q \in [2,\infty)$. *Assume that* G *satisfies the conditions* **(G1)** *and* **(G2)** *and that* K_2 *satisfies the same stochastic parabolicity constraint as in Theorem 2.2, i.e.,* $K_2 < \frac{2\nu}{2^q-1}$, *and that* $L_2 < 2\nu$. *Then there exists a* $\mathbb{P}$ *a.s. unique sequence* $\{u_N(t_k)\}_{k=1}^N$ *solution of* (4.1), *such that each random variable* $u_N(t_k)$ *is* $\mathcal{F}_{t_k}$*-measurable and satisfies:*

$$\sup_{N\ge 1} \mathbb{E}\Big(\max_{1\le k\le N} \|u_N(t_k)\|_V^{2q} + \nu \frac{T}{N} \sum_{k=1}^N \|u_N(t_k)\|_V^{2q-2} |Au_N(t_k)|_{\mathbb{L}^2}^2 \Big) \le C(T,q), \tag{4.4}$$

where $C(T,q)$ *is a constant which depends on* T, *the constants* K_i, $i = 0,1,2$ *in conditions* **(G1)** *and* **(G2)**, *and also depends on* $\mathbb{E}(\|u_0\|_V^{2q})$.

For $k = 0, \cdots, N$, let $e_k := u(t_k) - u_N(t_k)$ denote the error of this scheme (note that $e_0 = 0$). Then for any $\phi \in V$ and $j = 1, \cdots, N$, we have

$$\begin{aligned}(e_j - e_{j-1}, \phi) + \int_{t_{j-1}}^{t_j} \Big[\nu\big(\nabla u(s) - \nabla u_N(t_j)\,,\, \nabla\phi\big) + \big\langle B(u(s)) - B(u_N(t_j))\,,\, \phi\big\rangle \Big] ds \\ = \Big(\int_{t_{j-1}}^{t_j} \big[G(u(s)) - G(u_N(t_{j-1}))\big] dW(s)\,,\, \phi \Big).\end{aligned} \tag{4.5}$$

4.2. **A localized convergence result.** The first result states a localized upper estimate of the error terms. This is due to the non linear term, but unlike [11], it depends on u and not on u_N. Given $M > 0$ and $k = 1, \cdots, N$, set

$$\Omega_k^M := \Big\{ \omega \in \Omega \,:\, \max_{1\le j\le k} |\nabla u(t_j)|_{\mathbb{L}^2}^2 \le M \Big\} \in \mathcal{F}_{t_k}. \tag{4.6}$$

The following proposition is one of the main results of this section. The modification with respect to Theorem 3.1 in [11] is the localization set which does not depend on the approximation. This will be crucial to obtain a speed of $L^2(\Omega)$- strong convergence, and not only that the scheme converges in probability.

Proposition 4.3. *Let* G *satisfy the growth and Lipschitz conditions* **(C1)**, **(C2)** *with* $K_2 < \frac{2\nu}{7}$, $L_2 < \frac{4\nu}{19}$. *Let* u_0 *be such that* $\mathbb{E}(\|u_0\|_V^8) < \infty$. *Then for* Ω_k^M *be defined by* (4.6)

and N large enough, we have for every $k = 1, \cdots, N$:

$$\mathbb{E}\Big(1_{\Omega_{k-1}^M} \max_{1\leq j\leq k}\Big[|e_k|_{\mathbb{L}^2}^2 + \nu\frac{T}{N}\sum_{j=1}^{k}|\nabla e_k|_{\mathbb{L}^2}^2\Big]\Big) \leq C\ \exp\big[C_1(M)T\big]\ \Big(\frac{T}{N}\Big)^{\eta}, \tag{4.7}$$

for some constant $C > 0$, $\eta \in (0, \frac{1}{2})$, and

$$C_1(M) = \frac{\bar{C}^2}{\delta(2-4\tilde{\delta})\nu}\,M + C(\tilde{\delta},\bar{\epsilon})\,L_1, \tag{4.8}$$

where $\bar{C}$ is defined in (2.2)*, $\delta \in (0,1)$ and $\tilde{\delta} \in (0, \frac{1}{2})$ are chosen such that*

$$\nu(1-\delta) - \frac{1}{4}\big(1 + \frac{9}{\tilde{\delta}}\big)L_2 > 0.$$

Proof. We follow the scheme of the arguments in [11], pages 2480-2484, but the upper estimate of the duality involving the difference of the bilinear terms is dealt with differently, which leads to a different localization set. Furthermore, in order to describe the strong speed of convergence of the scheme, we need a more precise control of various constants appearing in some upper estimates. Hence we give a detailed proof below.

Step 1: Upper estimates for the bilinear term

Let us consider the duality between the difference of bilinear terms and e_j, that is upper estimate $\int_{t_{j-1}}^{t_j}\langle B(u(s)) - B(u_N(t_j))\,,\,e_j\rangle ds$. For every $s \in (t_{j-1}, t_j]$, using the bilinearity of B and the antisymmetry property (2.1), we deduce

$$\big\langle B\big(u(s),u(s)\big) - B\big(u_N(t_j),u_N(t_j)\big)\,,\,e_j\big\rangle = \sum_{i=1}^{3} T_i(s), \tag{4.9}$$

where, since $\big\langle B\big(v,u_N(t_j)\big)\,,\,e_j\big\rangle = \big\langle B\big(v,u(t_j)\big)\,,\,e_j\big\rangle$ for every $v \in V$,

$$\begin{aligned}
T_1(s) :=& \big\langle B\big(e_j,u_N(t_j)\big)\,,\,e_j\big\rangle = \big\langle B\big(e_j,u(t_j)\big)\,,\,e_j\big\rangle,\\
T_2(s) :=& \big\langle B\big(u(s)-u(t_j),u(t_j)\big)\,,\,e_j\big\rangle,\\
T_3(s) :=& \big\langle B\big(u(s),u(s)-u(t_j)\big)\,,\,e_j\big\rangle = -\big\langle B\big(u(s),e_j\big)\,,\,u(s)-u(t_j)\big\rangle.
\end{aligned}$$

Note that, unlike the first formulation of $T_1(s)$, the second one only depends on the error and on the solution to (1.1), and not on the approximation scheme. The Hölder inequality and (2.2) yield for every $\delta_1 > 0$ and $\bar{C}$ defined in the interpolation inequality (2.2)

$$\begin{aligned}
\int_{t_{j-1}}^{t_j}|T_1(s)|\,ds &\leq \bar{C}\frac{T}{N}\,|e_j|_{\mathbb{L}^2}\,|\nabla e_j|_{\mathbb{L}^2}\,|\nabla u(t_j)|_{\mathbb{L}^2}\\
&\leq \delta_1\nu\frac{T}{N}\,|\nabla e_j|_{\mathbb{L}^2}^2 + \frac{\bar{C}^2}{4\delta_1\nu}\,\frac{T}{N}\,|e_j|_{\mathbb{L}^2}^2\,|\nabla u(t_j)|_{\mathbb{L}^2}^2,
\end{aligned}$$

where the last upper estimate follows from Young's inequality (with conjugate exponents 2 and 2). A similar argument using Hölder's and Young's inequalities with exponents 4,4 and 2 imply that for any $\delta_2 > 0$ and $\gamma_2 > 0$,

$$|T_2(s)| \leq \delta_2\nu|\nabla e_j|_{\mathbb{L}^2}^2 + \gamma_2|e_j|_{\mathbb{L}^2}^2 + C(\nu,\delta_2,\gamma_2)\|u(t_j)-u(s)\|_{\mathbb{L}^4}^2|\nabla u(t_j)|_{\mathbb{L}^2}^2.$$

Using the Cauchy-Schwarz inequality we deduce

$$\int_{t_{j-1}}^{t_j}|T_2(s)|\,ds \leq \delta_2\nu\frac{T}{N}|\nabla e_j|_{\mathbb{L}^2}^2 + \gamma_2\frac{T}{N}|e_j|_{\mathbb{L}^2}^2$$

$$+ C(\nu,\delta_2,\gamma_2)|\nabla u(t_j)|^2_{\mathbb{L}^2}\int_{t_{j-1}}^{t_j}\|u(t_j)-u(s)\|^2_{\mathbb{L}^4}ds.$$

Similar computations using the Hölder and Young inequalities imply

$$\int_{t_{j-1}}^{t_j}|T_3(s)|\,ds \le \delta_3\nu\frac{T}{N}|\nabla e_j|^2_{\mathbb{L}^2}+\frac{1}{4\nu\delta_3}\int_{t_{j-1}}^{t_j}\|u(s)\|^2_{\mathbb{L}^4}\,\|u(t_j)-u(s)\|^2_{\mathbb{L}^4}ds \tag{4.10}$$

for any $\delta_3>0$. Note that

$$\nu\int_{t_{j-1}}^{t_j}\big(\nabla(u(s)-u_N(t_j))\,,\,\nabla e_j\big)ds=\nu\frac{T}{N}|\nabla e_j|^2_{\mathbb{L}^2}+\nu\int_{t_{j-1}}^{t_j}\big(\nabla(u(s)-u(t_j)),\nabla e_j\big)ds.$$

Using the Cauchy-Schwarz and Young inequalities, we deduce

$$\nu\int_{t_{j-1}}^{t_j}\big|\big(\nabla(u(s)-u(t_j)),\nabla e_j\big)\big|ds\le\delta_0\nu\frac{T}{N}|\nabla e_j|^2_{\mathbb{L}^2}+\frac{\nu}{4\delta_0}\int_{t_{j-1}}^{t_j}\big|\nabla\big(u(s)-u(t_j)\big)\big|^2_{\mathbb{L}^2}ds$$

for any $\delta_0>0$. Hence using the above upper estimates in (4.5) with $\phi=e_j$, we deduce

$$\begin{aligned}\big(e_j-e_{j-1}\,,e_j\big)+\nu\frac{T}{N}|\nabla e_j|^2_{\mathbb{L}^2}&\le\nu\sum_{r=0}^{3}\delta_r\frac{T}{N}|\nabla e_j|^2_{\mathbb{L}^2}+\Big(\gamma_2+\frac{\bar C^2}{4\delta_1\nu}|\nabla u(t_j)|^2_{\mathbb{L}^2}\Big)\frac{T}{N}\,|e_j|^2_{\mathbb{L}^2}\\&+\sum_{l=1}^{3}\tilde T_j(l)+\Big(\int_{t_{j-1}}^{t_j}\big[G(u(s))-G(u_N(t_{j-1}))\big]dW(s)\,,\,e_j\Big),\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}\tilde T_j(1)=&\frac{\nu}{4\delta_0}\int_{t_{j-1}}^{t_j}\big|\nabla\big(u(s)-u(t_j)\big)\big|^2_{\mathbb{L}^2}ds,\\ \tilde T_j(2)=&C(\nu,\delta_2,\gamma_2)\,|\nabla u(t_j)|^2_{\mathbb{L}^2}\int_{t_{j-1}}^{t_j}\|u(s)-u(t_j)\|^2_{\mathbb{L}^4}ds,\\ \tilde T_j(3)=&\frac{1}{4\nu\delta_3}\int_{t_{j-1}}^{t_j}\|u(s)\|^2_{\mathbb{L}^4}\|u(t_j)-u(s)\|^2_{\mathbb{L}^4}ds.\end{aligned}$$

Using the time regularity (4.3) with $p=2$, we deduce

$$\mathbb{E}\big(\tilde T_j(1)\big)\le C\frac{\nu}{4\delta_0}\Big(\frac{T}{N}\Big)^{1+\eta}.\tag{4.12}$$

The Cauchy-Schwarz inequality, (2.11) with $p=2$ and (4.2) imply

$$\mathbb{E}\big(\tilde T_j(2)\big)\le\,C(\nu,\delta_2,\gamma_2)\Big(\frac{T}{N}\Big)^{1+2\eta},\tag{4.13}$$

$$\mathbb{E}\big(\tilde T_j(3)\big)\le C\,\frac{1}{4\nu\delta_3}\,\Big(\frac{T}{N}\Big)^{1+2\eta}.\tag{4.14}$$

Step 2: Localization

In order to use a discrete version of the Gronwall lemma to upper estimate $|e_j|^2_{\mathbb{L}^2}$, due to the factor $|\nabla u(t_j)|^2_{\mathbb{L}^2}$ in the RHS of (4.11), we have to localize on the random set Ω^M_{j-1} defined in (4.6). The shift of index is due to the fact that, in order to deal with the stochastic integral, we have to make sure that the localization set is $\mathcal{F}_{t_{j-1}}$-measurable. This set depends on j, but we will need to add the localized inequalities (4.11) and take expected values.

Note that for $1 \leq j \leq k$, $\Omega_k^M \subset \Omega_j^M$. Hence, since $e_0 = 0$, as proved in [11], estimate (3.25), we have

$$\max_{1\leq j\leq k} \sum_{l=1}^{j} 1_{\Omega_{l-1}^M} \Big(|e_l|_{\mathbb{L}^2}^2 - |e_{l-1}|_{\mathbb{L}^2}^2 \Big) = \max_{1\leq j\leq k} \Big(1_{\Omega_{j-1}^M} |e_j|_{\mathbb{L}^2}^2 + \sum_{l=2}^{j} \big(1_{\Omega_{l-2}^M} - 1_{\Omega_{l-1}^M} \big) |e_{l-1}|_{\mathbb{L}^2}^2 \Big)$$
$$\geq \max_{1\leq j\leq k} 1_{\Omega_{j-1}^M} |e_j|_{\mathbb{L}^2}^2. \tag{4.15}$$

Thus, we will localize $|e_j|_{\mathbb{L}^2}^2$ on the set Ω_{j-1}^M and - shifting the index by one - control some "error term" $|e_j - e_{j-1}|_{\mathbb{L}^2}^2$ localized on the same set. Note that this localization set only depends on the projection of the solution u of equation (1.1) on divergence free fields, and not on its approximation.

Adding the inequalities (4.11) with $\phi = e_j$ localized on the set Ω_{j-1}^M, using $e_0 = 0$ and the identity $(a, a-b) = \frac{1}{2}\big[|a|_{\mathbb{L}^2}^2 - |b|_{\mathbb{L}^2}^2 + |a-b|_{\mathbb{L}^2}^2\big]$, we deduce for $k = 1, \cdots, N$

$$\max_{1\leq j\leq k} \Big(\frac{1}{2} 1_{\Omega_{j-1}^M} |e_j|_{\mathbb{L}^2}^2 + \frac{1}{2} \sum_{l=1}^{j} 1_{\Omega_{l-1}^M} |e_l - e_{l-1}|_{\mathbb{L}^2}^2 \Big)$$
$$\leq \frac{1}{2} \Big(\max_{1\leq j\leq k} \sum_{l=1}^{j} 1_{\Omega_{l-1}^M} \Big(|e_l|_{\mathbb{L}^2}^2 - |e_{l-1}|_{\mathbb{L}^2}^2 \Big) + \sum_{l=1}^{j} 1_{\Omega_{l-1}^M} |e_l - e_{l-1}|_{\mathbb{L}^2}^2 \Big)$$
$$\leq \max_{1\leq j\leq k} \sum_{1\leq l\leq j} 1_{\Omega_{l-1}^M} \big(e_l - e_{l-1}, e_l \big).$$

The upper estimates (4.11) for $j = 1, \cdots, k$ imply for any $\epsilon > 0$

$$\max_{1\leq j\leq k} \Big[\frac{1}{2} 1_{\Omega_{j-1}^M} |e_j|_{\mathbb{L}^2}^2 + \sum_{l=1}^{j} 1_{\Omega_{l-1}^M} |e_l - e_{l-1}|_{\mathbb{L}^2}^2 + \nu \big(1 - \sum_{r=0}^{3} \delta_r \big) \frac{T}{N} \sum_{l=1}^{j} 1_{\Omega_{l-1}^M} |\nabla e_l|_{\mathbb{L}^2}^2 \Big]$$
$$\leq \Big[\gamma_2 + (1+\epsilon) \frac{\bar{C}^2 M}{4\delta_1 \nu} \Big] \frac{T}{N} \sum_{j=1}^{k} 1_{\Omega_{j-1}^M} |e_j|_{\mathbb{L}^2}^2 + \sum_{i=1}^{3} \sum_{j=1}^{k} \tilde{T}_j(i)$$
$$+ C(\nu, \delta_1, \epsilon) \frac{T}{N} \sum_{j=1}^{k} 1_{\Omega_{j-1}^M} |e_j|_{\mathbb{L}^2}^2 |\nabla \big[u(t_j) - u(t_{j-1}) \big]|_{\mathbb{L}^2}^2 + M_k(1) + M_k(2), \tag{4.16}$$

where

$$M_k(1) = \sum_{j=1}^{k} 1_{\Omega_{j-1}^M} \Big(\int_{t_{j-1}}^{t_j} \big[G(u(s)) - G(u_N(t_{j-1})) \big] dW(s) \, , \, e_{j-1} \Big),$$
$$M_k(2) = \sum_{j=1}^{k} 1_{\Omega_{j-1}^M} \Big(\int_{t_{j-1}}^{t_j} \big[G(u(s)) - G(u_N(t_{j-1})) \big] dW(s) \, , \, e_j - e_{j-1} \Big).$$

The inequalities (4.12)– (4.14) imply the existence of a constant C depending on T, ν, δ_i, $i = 0, \cdots 3$ and γ_2 such that

$$\sum_{i=1}^{3} \sum_{j=1}^{N} \mathbb{E}\big(\tilde{T}_j(i) \big) \leq C \left(\frac{T}{N} \right)^{\eta}. \tag{4.17}$$

The Cauchy-Schwarz inequality, (2.11) and (4.4) for $p = q = 2$, and the time regularity (4.3) for $p = 4$ imply the existence of a constant C such that

$$\frac{T}{N}\sum_{j=1}^{N}\mathbb{E}\Big(|e_j|^2_{\mathbb{L}^2}|\nabla\big[u(t_j)) - u(t_{j-1})\big]|^2_{\mathbb{L}^2}\Big) \leq C\Big(\frac{T}{N}\Big)^{\eta}. \tag{4.18}$$

We next upper estimate $\mathbb{E}\big(\max_{1\leq k\leq N} M_k(2)\big)$. The Cauchy-Schwarz and Young inequalities imply that for any $\tilde{\delta}_2 > 0$

$$\begin{aligned}
\mathbb{E}\Big(\max_{1\leq j\leq k} M_k(2)\Big) &\leq \sum_{j=1}^{k}\Big\{\mathbb{E}\big(1_{\Omega^M_{j-1}}|e_j - e_{j-1}|^2_{\mathbb{L}^2}\big)\Big\}^{\frac{1}{2}} \\
&\qquad\times\Big\{\mathbb{E}\Big(1_{\Omega^M_{j-1}}\int_{t_{j-1}}^{t_j}\|G(u(s)) - G(u_N(t_{j-1})\|^2_{\mathcal{L}_2(K,H)}ds\Big)\Big\}^{\frac{1}{2}} \\
&\leq \tilde{\delta}_2\sum_{j=1}^{k}\mathbb{E}\big(1_{\Omega^M_{j-1}}|e_j - e_{j-1}|^2_{\mathbb{L}^2}\big) \\
&\qquad + \frac{1}{4\tilde{\delta}_2}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\mathbb{E}\Big(1_{\Omega^M_{j-1}}\|G(u(s)) - G(u_N(t_{j-1}))\|^2_{\mathcal{L}_2(K,H)}\Big)ds.
\end{aligned}$$

The growth condition **(G2)** and (4.3) imply for any $\epsilon > 0$ and $s \in [t_{l-1}, t_l]$,

$$\begin{aligned}
&\mathbb{E}\big(1_{\Omega^M_{l-1}}\|G(u(s)) - G(u_N(t_{l-1}))\|^2_{\mathcal{L}_2(K,H)}\big) \\
&\quad\leq (1+\epsilon)\mathbb{E}\big(1_{\Omega^M_{l-1}}\|G(u(t_{l-1})) - G(u_N(t_{l-1}))\|^2_{\mathcal{L}_2(K,H)}\big) \\
&\qquad + \big(1+\frac{1}{\epsilon}\big)\mathbb{E}\big(1_{\Omega^M_{l-1}}\|G(u(s)) - G(u(t_{l-1}))\|^2_{\mathcal{L}_2(K,H)}\big) \\
&\quad\leq L_1\,(1+\epsilon)\,\mathbb{E}\Big(1_{\Omega^N_{l-1}}|e_{l-1}|^2_{\mathbb{L}^2}\Big) + L_2\,(1+\epsilon)\,\mathbb{E}\Big(1_{\Omega^N_{l-1}}|\nabla e_{l-1}|^2_{\mathbb{L}^2}\Big) + C(\epsilon)\Big(\frac{T}{N}\Big)^{\eta}.
\end{aligned} \tag{4.19}$$

Since $\Omega^M_j \subset \Omega^M_{j-1}$ and $e_0 = 0$, we deduce for any $k = 2, \cdots, N$,

$$\begin{aligned}
\mathbb{E}\Big(\max_{1\leq j\leq k} M_j(2)\Big) \leq& \tilde{\delta}_2\frac{T}{N}\sum_{j=1}^{k}\mathbb{E}\big(1_{\Omega^M_{j-1}}|e_j - e_{j-1}|^2_{\mathbb{L}^2}\big) + \frac{1+\epsilon}{4\tilde{\delta}_2}L_1\,\frac{T}{N}\sum_{j=1}^{k-1}\mathbb{E}\Big(1_{\Omega^N_{j-1}}|e_j|^2_{\mathbb{L}^2}\Big) \\
&+ \frac{1+\epsilon}{4\tilde{\delta}_2}L_2\,\frac{T}{N}\sum_{j=1}^{k-1}\mathbb{E}\Big(1_{\Omega^N_{j-1}}|\nabla e_j|^2_{\mathbb{L}^2}\Big) + C(\epsilon,\tilde{\delta}_2)T\Big(\frac{T}{N}\Big)^{\eta}.
\end{aligned} \tag{4.20}$$

Since $1_{\Omega^M_{l-1}}$ and e_{l-1} are $\mathcal{F}_{t_{l-1}}$-measurable, using the Burkholder-Davies-Gundy inequality, the Young inequality, (4.19), and using once more the inclusion $\Omega^M_j \subset \Omega^M_{j-1}$, we deduce that for any $\tilde{\delta}_1 > 0$,

$$\begin{aligned}
\mathbb{E}\Big(\max_{1\leq j\leq k} M_j(1)\Big) &\leq 3\sum_{l=1}^{k}\mathbb{E}\Big[\Big\{1_{\Omega^M_{l-1}}\int_{t_{l-1}}^{t_l}\|G(u(s)) - G(u_N(t_{l-1}))\|^2_{\mathcal{L}_2(K,H)}\,|e_{l-1}|^2_{\mathbb{L}^2}\Big\}^{\frac{1}{2}}\Big] \\
&\leq 3\mathbb{E}\Big[\Big(\max_{1\leq l\leq k} 1_{\Omega^M_{l-1}}|e_{l-1}|_{\mathbb{L}^2}\Big)\Big\{\sum_{l=1}^{k}1_{\Omega^M_{l-1}}\int_{t_{l-1}}^{t_l}\|G(u(s)) - G(u_N(t_{l-1}))\|^2_{\mathcal{L}_2(K,H)}ds\Big\}^{\frac{1}{2}}\Big]
\end{aligned}$$

$$\leq \tilde{\delta}_1 \mathbb{E}\Big(\max_{1\leq l\leq k} 1_{\Omega^M_{l-1}} |e_{l-1}|^2_{\mathbb{L}^2}\Big) + \frac{9(1+\epsilon)}{4\tilde{\delta}_1} L_1 \, \frac{T}{N} \sum_{j=1}^{k-1} \mathbb{E}\Big(1_{\Omega^M_{j-1}} |e_j|^2_{\mathbb{L}^2}\Big).$$

$$+ \frac{9(1+\epsilon)}{4\tilde{\delta}_1} L_2 \, \frac{T}{N} \sum_{j=1}^{k-1} \mathbb{E}\Big(1_{\Omega^M_{j-1}} |\nabla e_j|^2_{\mathbb{L}^2}\Big) + C(\epsilon, \tilde{\delta}_1)\Big(\frac{T}{N}\Big)^{\eta}, \tag{4.21}$$

Collecting the upper estimates (4.16)–(4.21) and taking $\tilde{\delta}_2 = 1$, we deduce for $k = 2, \cdots, N$

$$\mathbb{E}\Big(\max_{1\leq j\leq k} \Big\{ \frac{1}{2} 1_{\Omega^M_{j-1}} |e_j|^2_{\mathbb{L}^2}\Big) + \nu\Big(1 - \sum_{i=0}^{3} \delta_i\Big) \frac{T}{N} \sum_{l=1}^{j} 1_{\Omega^M_{l-1}} |\nabla e_l|^2_{\mathbb{L}^2} \Big\}\Big)$$

$$\leq \Big[\tilde{\delta}_1 + \gamma_2 + (1+\epsilon)\frac{\bar{C}^2 M}{4\delta_1 \nu} \frac{T}{N}\Big] \mathbb{E}\Big(\max_{1\leq j\leq k} 1_{\Omega^M_{j-1}} |e_j|^2_{\mathbb{L}^2}\Big)$$

$$+ \Big[(1+\epsilon)\frac{\bar{C}^2 M}{4\delta_1 \nu} + \gamma_2 + \frac{1+\epsilon}{4}\Big(\frac{1}{\tilde{\delta}_2} + \frac{9}{\tilde{\delta}_1}\Big) L_1\Big] \frac{T}{N} \sum_{j=1}^{k-1} \mathbb{E}\big(1_{\Omega^M_{j-1}} |e_j|^2_{\mathbb{L}^2}\big)$$

$$+ \frac{1+\epsilon}{4}\Big(\frac{1}{\tilde{\delta}_2} + \frac{9}{\tilde{\delta}_1}\Big) L_2 \frac{T}{N} \sum_{j=1}^{k} \mathbb{E}\big(1_{\Omega^M_{j-1}} |\nabla e_j|^2_{\mathbb{L}^2}\big) + C\Big(\frac{T}{N}\Big)^{\eta}. \tag{4.22}$$

Step 3: Discrete Gronwall lemma

Since $\frac{19}{4} L_2 < \nu$, we may choose $\tilde{\delta_1} < \frac{1}{2}$, $\tilde{\delta}_2 < 1$, $\epsilon > 0$ and $\delta_1 < 1$ such that

$$\nu(1-\delta_1) - \frac{1+\epsilon}{4}\Big[\frac{1}{\tilde{\delta}_2} + \frac{9}{\tilde{\delta}_1}\Big] L_2 \geq \alpha\nu$$

for some $\alpha > 0$. Fix $\tilde{\epsilon} \in (0,1)$ and $\gamma_2 \in \big(0, \frac{\tilde{\epsilon}}{2}(\frac{1}{2} - \tilde{\delta}_1)\big)$; suppose that N is large enough to imply

$$\frac{1}{2} - \tilde{\delta}_1 - \Big(\gamma_2 + (1+\epsilon)\frac{\bar{C}^2 M}{4\delta_1 \nu}\Big) \frac{T}{N} \geq (1-\tilde{\epsilon})\big(\frac{1}{2} - \tilde{\delta}_1\big),$$

and choose δ_i, $i = 0, 2, 3$ such that $\delta_0 + \delta_2 + \delta_3 < \frac{\alpha}{2}\nu$. Then for N large enough,

$$(1-\tilde{\epsilon})\big(\frac{1}{2} - \tilde{\delta}_1\big) \mathbb{E}\Big(\max_{1\leq j\leq k} 1_{\Omega^M_{j-1}} |e_j|^2_{\mathbb{L}^2}\Big) + \frac{\alpha\nu}{2} \frac{T}{N} \sum_{j=1}^{k} \mathbb{E}\big(1_{\Omega^M_{j-1}} |\nabla e_j|^2_{\mathbb{L}^2}\big)$$

$$\leq \Big[(1+\epsilon)\frac{\bar{C}^2 M}{4\delta_1 \nu} + C(\gamma_2, \tilde{\delta}_1, \tilde{\delta}_2, \epsilon) L_1\Big] \frac{T}{N} \sum_{j=1}^{k-1} \mathbb{E}\big(1_{\Omega^M_{j-1}} |e_j|^2_{\mathbb{L}^2}\big) + C\Big(\frac{T}{N}\Big)^{\eta}.$$

Set

$$C_1(M) := \frac{\frac{(1+\epsilon)\bar{C}^2}{4\delta_1\nu} M + (1+\epsilon) C(\gamma_2, \tilde{\delta}_1, \tilde{\delta}_2) L_1}{(1-\tilde{\epsilon})\big(\frac{1}{2} - \tilde{\delta}_1\big)}. \tag{4.23}$$

Neglecting the second term in the right hand side and using a discrete version of the Gronwall lemma, we deduce that

$$\mathbb{E}\Big(\max_{1\leq j\leq k} 1_{\Omega^M_{j-1}} |e_j|^2_{\mathbb{L}^2}\Big) \leq C\, e^{C_1(M)\, T} \Big(\frac{T}{N}\Big)^{\eta}.$$

Note that the choices of the "main" coefficients has to made such that

$$\nu(1-\delta_1) - \frac{1}{4}\Big(1 + \frac{9}{\tilde{\delta}_1}\Big) L_2 > 0 \quad \text{for } \delta_1 \in (0,1),\ \tilde{\delta}_1 \in (0, \frac{1}{2}).$$

Once these inequalities hold, choosing γ_2, ϵ and $\tilde{\epsilon}$ small enough, one can let

$$C_1(M) = \frac{(1+\bar{\epsilon})\bar{C}^2}{\delta_1(2-4\tilde{\delta}_1)\nu} M + C(\tilde{\delta}_1, \bar{\epsilon}) L_1,$$

for $\bar{\epsilon} > 0$ arbitrary close to 0. Plugging this in the previous upper estimate and using the inclusions $\Omega_k^M \subset \Omega_j^M$ for $j = 1, \cdots, k$, we deduce (4.7) and (4.8). Indeed, we change $\frac{1+\bar{\epsilon}}{\delta}$ into δ for another constant $\delta \in (0,1)$. □

Remark 4.4. *Note that if $L_2 = 0$, the constant $C_1(M)$ in the statement of Proposition 4.3 has a simpler expression. Choosing $\tilde{\delta}_1 \sim 0$, $\gamma_2 \sim 0$, $\delta_1 \sim 1$, $\delta_i \sim 0$ for $i = 0, 2, 3$, we may take $C_1(M)$ arbitrary such that*

$$C_1(M) = \frac{(1+\bar{\epsilon})\,\bar{C}^2 M}{2\nu} + C(\bar{\epsilon}) L_1, \tag{4.24}$$

where $\bar{C}$ is the constant defined in (2.2) and $\bar{\epsilon} > 0$ is arbitrary close to 0.

4.3. **Strong speed of convergence of the implicit Euler scheme.** As in section 3.3, let us use Hölder's inequality with conjugate exponents p and 2^{q-1}. We obtain

$$\begin{aligned}\mathbb{E}\Big(1_{(\Omega_N^M)^c} \max_{1\le k\le N} |e_k|_{\mathbb{L}^2}^2\Big) \le & C\left[\mathbb{P}\big((\Omega_N^M)^c\big)\right]^{\frac{1}{p}} \\ & \times \left[\mathbb{E}\Big(\sup_{0\le s\le T} |u(s)|_{\mathbb{L}^2}^{2^q} + \max_{0\le k\le N} |u_N(t_k)|_{\mathbb{L}^2}^{2^q}\Big)\right]^{\frac{1}{q}}, \end{aligned} \tag{4.25}$$

$$\begin{aligned}\mathbb{E}\Big(1_{(\Omega_N^M)^c} \frac{T}{N}\sum_{k=1}^{N} |\nabla e_k|_{\mathbb{L}^2}^2\Big) \le & C\Big[\mathbb{P}\big((\Omega_N^M)^c\big)\Big]^{\frac{1}{p}} \\ & \times \left[\mathbb{E}\Big(\sup_{0\le s\le T} |\nabla u(s)|_{\mathbb{L}^2}^{2^q} + \max_{0\le k\le N} |\nabla u_N(t_k)|_{\mathbb{L}^2}^{2^q}\Big)\right]^{\frac{1}{q}}. \end{aligned} \tag{4.26}$$

The inequalities (2.11) and (4.4) applied with $K_2 < \frac{2\nu}{7}$ and $L_2 < \frac{4\nu}{19}$ prove that if $\mathbb{E}(\|u_0\|_V^{2^q}) < \infty$, the second factors in the right hand sides of (4.25) and (4.26) are bounded by a constant independent of N.

We now upper estimate the probability of the complement of the localization set and to balance the upper estimates of the L^2 moments localized on the set Ω_N^M and its complement. To obtain a strong speed of convergence will require the threshold M to depend on N. Two cases are studied.

Case 1: Linear growth diffusion coefficient

Suppose that G satisfies conditions **(C1)** and **(C2)** with $L_2 < \frac{19\nu}{4}$ and that $\mathbb{E}(\|u_0\|^{2^q}) < \infty$. Then (2.11) implies

$$\begin{aligned}\mathbb{P}\Big(\big(\Omega_N^{M(N)}\big)^c\Big) \; &\le \mathbb{P}\Big(\sup_{0\le s\le T} |\nabla u(s)|_{\mathbb{L}^2}^2 > M(n)\Big) \\ &\le \Big(\frac{1}{M(N)}\Big)^{2^{q-1}} \mathbb{E}\Big(\sup_{0\le s\le T} \|u(s)\|_V^{2^q}\Big) \le C_q\, M(N)^{-2^{q-1}}. \end{aligned} \tag{4.27}$$

If we suppose that $\mathbb{E}(\|u_0\|_V^{2^q}) < \infty$, in order to balance the upper estimates (4.25), (4.26) with (4.27) and (4.7), we have to choose $M(N) \to \infty$ as $N \to \infty$, such that as $N \to \infty$,

$$\Big(\frac{T}{N}\Big)^{\eta} \exp[C_1\big(M(N)\big)\, T] \;\asymp\; C(q) M(N)^{-2^{q-1}+1}.$$

where $C_1(M(N))$ is defined in (4.8). For $\delta \in (0,1)$, $\tilde{\delta} \in (0, \frac{1}{2})$ such that $\nu(1-\delta) - \frac{1}{4}\big(1 + \frac{9}{\tilde{\delta}}\big)L_2 > 0$. Taking logarithms and neglecting constants leads to

$$-\eta \ln(N) + \frac{\bar{C}^2 M(N) T}{\delta(2-4\tilde{\delta})\nu} \asymp -\big(2^{q-1}-1\big)\ln(M(N)) + O(1) \quad \text{as } N \to \infty.$$

Let

$$M(N) := \frac{\delta(2-4\tilde{\delta})\nu}{\bar{C}^2 T}\Big\{\eta \ln(N) - \big(2^{q-1}-1\big)\ln\big(\ln(N)\big)\Big\} \asymp \frac{\eta\delta(2-4\tilde{\delta})\nu \ln(N)}{\bar{C}^2 T} \tag{4.28}$$

for $\bar{\epsilon} > 0$. Then, for this choice of $M(N)$, we have

$$-\eta \ln(N) + C_1\big(M(N)\big)T = -\ln\big[\big(2^{q-1}-1\big)\ln(N)\big] + O(1),$$

which implies $\big(\frac{T}{N}\big)^{\eta}\exp[C_1(M(N))T] \asymp C\big(\ln(N)\big)^{-2^{q-1}+1}$ for some positive constant C. Furthermore, $M(N)^{-2^{q-1}+1} \asymp C\big(\ln(N)\big)^{-2^{q-1}+1}$ for some positive constant C. Similar computations with sum of the V norms of the error on the time grid yield

$$\mathbb{E}\Big(\max_{1\le k\le N}|e_k|^2_{\mathbb{L}^2} + \nu\frac{T}{N}\sum_{k=1}^{N}|\nabla e_k|^2_{\mathbb{L}^2}\Big) \le C\big(\ln(N)\big)^{-(2^{q-1}-1)},$$

for some constant C depending on T, q and the coefficients K_i, $i = 0,1,2$. This completes the proof of the following

Theorem 4.5. *Let u_0 be such that $\mathbb{E}(\|u_0\|_V^{2q}) < \infty$ for some $q \ge 3$, G satisfy assumptions* **(G1)**-**(G3)**, *and suppose that $K_2 < \frac{2\nu}{2^q-1}$ and $L_2 < \frac{4\nu}{19}$. Then the fully implicit scheme u_N solution of* (4.1) *converges in $L^2(\Omega)$ to the solution u of* (1.1). *More precisely, for N large enough we have*

$$\mathbb{E}\Big(\max_{1\le k\le N}|u(t_k)-u_N(t_k)|^2_{\mathbb{L}^2} + \frac{T}{N}\sum_{k=1}^{N}\big|\nabla\big[u(t_k)-u_N(t_k)\big]\big|^2_{\mathbb{L}^2}\Big) \le C\big[\ln(N)\big]^{-(2^{q-1}-1)}. \tag{4.29}$$

Remark 4.6. *Note that, as for the splitting scheme, if u_0 is a deterministic element of V and G satisfies the assumptions of Theorem 4.5, we have*

$$\mathbb{E}\Big(\max_{1\le k\le N}|u(t_k)-u_N(t_k)|^2_{\mathbb{L}^2} + \frac{T}{N}\sum_{k=1}^{N}\big|\nabla\big[u(t_k)-u_N(t_k)\big]\big|^2_{\mathbb{L}^2}\Big) \le C\big[\ln(N)\big]^{-\gamma}$$

for any $\gamma > 0$.

Case 2: Additive noise Suppose that $G(u) := G \in \mathcal{L}_2(K,V)$, that is the noise is additive, or more generally that the conditions **(G1)** and **(G2)** are satisfied with $K_i = 0$ for $i = 1,2$ and $L_2 = 0$. Using an exponential Markov inequality, we deduce that for any constant $\alpha > 0$

$$\mathbb{P}\Big(\big(\Omega_N^{M(N)}\big)^c\Big) \le \exp\big(-\alpha M(N)\big)\,\mathbb{E}\Big[\exp\Big(\alpha\sup_{0\le t\le T}|\nabla u(t)|^2_{\mathbb{L}^2}\Big)\Big]. \tag{4.30}$$

Recall that Lemma 3.8 implies that for $\alpha \in (0,\alpha_0]$, where $\alpha_0 = \frac{\nu}{4K_0\tilde{C}}$ and $\tilde{C}$ is defined in (3.28), we have $\mathbb{E}\big[\sup_{t\in[0,T]}\exp(\alpha\|u(t)\|_V^2)\big] < \infty$. Using (4.7) with (4.24), (4.30) and (3.29), for $\tilde{\delta}_1 \sim 0$ and $\delta_1 \sim 1$, we choose $M(N)$ such that

$$\Big(\frac{T}{N}\Big)^{\eta}\exp\Big(\frac{(1+\bar{\epsilon})\,\bar{C}^2 M(N) T}{2}\nu\Big) = c_2\exp\Big(-\frac{\nu M(N)}{p4K_0\tilde{C}}\Big), \tag{4.31}$$

for some $p > 1$, $\bar{\epsilon} > 0$, and some positive constant c_2, where $\bar{C}$ (resp. $\tilde{C}$) is defined by (2.2) (resp. (3.28)). For any $p, \bar{\epsilon} \in (1, \infty)$ since u_0 is deterministic, $\mathbb{E}(\|u_0\|_V^q) < \infty$ for conjugate exponents p and q. Set

$$M(N) := \frac{\eta \ln(N)}{\frac{\nu}{p4K_0\tilde{C}} + \frac{(1+\bar{\epsilon})\bar{C}^2T}{2\nu}}$$

for some $\bar{\epsilon} > 0$. Then $M(N) \to \infty$ as $n \to \infty$, and both hand sides of (4.31) are equal to some constant multiple of $N^{-\beta\eta}$, where, choosing p close enough to 1, we have $\beta < \frac{\frac{\nu}{4K_0\tilde{C}}}{\frac{\nu}{4K_0\tilde{C}} + \frac{\bar{C}^2T}{2\nu}}$. Since $\eta < \frac{1}{2}$ can be chosen as close to $\frac{1}{2}$ as wanted, this yields the following rate of convergence.

Theorem 4.7. *Let $u_0 \in V$, G satisfy assumptions* **(G1)** *and* **(G2)** *with $K_i = 0$, $i = 1, 2$, and $L_2 = 0$. Let u denote the solution of* (1.1) *and u_N be the fully implicit scheme solution of* (4.1)*. Then for N large enough, $\bar{C}$ (resp. $\tilde{C}$) defined by* (2.2) *(resp.* (3.28)*),*

$$\mathbb{E}\Big(\max_{1\le k\le N} |u(t_k) - u_N(t_k)|_{\mathbb{L}^2}^2 + \frac{T}{N}\sum_{k=1}^{N} \big|\nabla\big[u(t_k) - u_N(t_k)\big]\big|_{\mathbb{L}^2}^2 \Big) \le C\Big(\frac{T}{N}\Big)^{\gamma}, \tag{4.32}$$

where $\gamma < \frac{1}{2}\left(\frac{\frac{\nu}{4K_0\tilde{C}}}{\frac{\nu}{4K_0\tilde{C}} + \frac{\bar{C}^2T}{2\nu}}\right)$.

Note that if ν is large, the speed of convergence of the H and V norms in Theorem 4.7 is "close" to $C(T)N^{-\frac{1}{2}}$. Intuitively, it cannot be better because of the stochastic integral and the scaling between the time and space parameters in the heat kernel, which is behind the time regularity of the solution stated in (4.3).

4.4. **Semi-implicit Euler scheme.** In this section, we prove the strong $L^2(\Omega)$ convergence of a discretization scheme with a linearized drift. Let v_N be defined on the time grid $(t_k, k = 0, \cdots, N)$ as follows.

Semi implicit Euler scheme Let u_0 be a V-valued $\mathcal{F}_0$-measurable random variable and set $v_N(0) = u_0$. For $k = 1, \cdots, N$, let $v_N(t) \in V$ be such that $\mathbb{P}$ a.s. for all $\phi \in V$,

$$\begin{aligned}\big(v_N(t_k) - v_N(t_{k-1})\,,\,\phi\big) + \frac{T}{N}\Big[\nu\big(\nabla v_N(t_k)\,,\nabla\phi\big) + \big\langle B(v_N(t_{k-1}), v_N(t_k))\,,\,\phi\big\rangle \\ = \big(G(v_N(t_{k-1}))\,\Delta_k W\,,\,\phi\big),\end{aligned} \tag{4.33}$$

where $\Delta_k W = W(t_k) - W(t_{k-1})$.

Note that since in general $\langle B(u, v)\,,\,Av\rangle \neq 0$ for $u, v \in Dom(A)$, the moments of v_N are bounded in a weaker norm than that of the fully implicit scheme u_N.

Lemma 4.8. *Let $u_0 \in L^{2^q}(\Omega, V)$ for some integer $q \ge 2$ be $\mathcal{F}_0$-measurable and let G satisfy the condition* **(G1)** *with $K_2 < \frac{2\nu}{2q-1}$ and $L_2 < 2\nu$. Then each random variable $v_N(t_k)$, $k = 0, \cdots, N$ is $\mathcal{F}_{t_k}$-measurable such that*

$$\sup_N \mathbb{E}\Big(\max_{1\le k\le N} |v_N(t_k)|_{\mathbb{L}^2}^{2^q} + \nu\frac{T}{N}\sum_{k=1}^{N} |v_N(t_k)|_{\mathbb{L}^2}^{2^q-1} \|v_N(t_k)\|_V^2 \Big) \le C(T, q). \tag{4.34}$$

For $k = 0, \cdots, N$, set $\bar{e}_k = u(t_k) - v_N(t_k)$. Unlike [11], we will not compare the schemes u_N and v_N since the norm of the difference would require a localization in terms of the gradient of u_N. Instead of that, we prove the following analog of Proposition 4.3.

Proposition 4.9. *Let G satisfy the growth and Lipschitz conditions* **(G1)** *and* **(G2)** *with $K_2 < \frac{2\nu}{7}$ and $L_2 < \frac{4\nu}{19}$, and u_0 be $\mathcal{F}_0$-measurable such that $\mathbb{E}(\|u_0\|_V^8) < \infty$. Then for Ω_k^M defined by* (4.6) *and N large enough, we have for $k = 1, \cdots, N$ and $\eta < \frac{1}{2}$*

$$\mathbb{E}\Big(1_{\Omega_{k-1}^M} \max_{1\leq j\leq k}\Big[|\bar{e}_k|_{\mathbb{L}^2}^2 + \nu\frac{T}{N}\sum_{j=1}^{k}|\nabla\bar{e}_k|_{\mathbb{L}^2}^2\Big]\Big) \leq C\Big(\frac{T}{N}\Big)^{\eta}\exp\big[C_1(M)T\big], \tag{4.35}$$

where $C > 0$ is some constant and $C_1(M)$ is defined by (4.8) *for $\delta \in (0,1)$ and $\tilde{\delta} \in (0, \frac{1}{2})$ such that $\nu(1-\delta) - \frac{1}{4}\big(1 + \frac{9}{\tilde{\delta}}\big)L_2 > 0$.*

Proof. Many parts of the argument are similar to the corresponding ones in the proof of Proposition 4.3; we only focus on the differences.

We first consider the duality between the difference of bilinear terms and $\bar{e}_j$, that is upper estimate $\int_{t_{j-1}}^{t_j}\langle B(u(s),u(s)) - B(v_N(t_{j-1}),v_N(t_j))\,,\,\bar{e}_j\rangle ds$. For every $s \in [t_{j-1},t_j]$, using the bilinearity and antisymetry of B we deduce

$$\langle B(u(s),u(s)) - B(v_N(t_{j-1}),v_N(t_j))\,,\,\bar{e}_j\rangle = \sum_{i=1}^{3}\bar{T}_i(s),$$

where

$$\begin{aligned}
\bar{T}_1(s) :=& \big\langle B\big(\bar{e}_{j-1},v_N(t_j)\big)\,,\,\bar{e}_j\big\rangle = \big\langle B\big(\bar{e}_{j-1},u(t_j)\big)\,,\,\bar{e}_j\big\rangle,\\
\bar{T}_2(s) :=& \big\langle B\big(u(s)-u(t_{j-1}),u(t_j)\big)\,,\,\bar{e}_j\big\rangle,\\
\bar{T}_3(s) :=& \big\langle B\big(u(s),u(s)-u(t_j)\big)\,,\,\bar{e}_j\big\rangle = -\big\langle B(u(s),\bar{e}_j)\,,\,u(s)-u(t_j)\big\rangle.
\end{aligned}$$

Using Hölder's inequality, (2.2) and Young's inequality, we deduce that for every $\delta_1 > 0$,

$$\begin{aligned}
\int_{t_{j-1}}^{t_j}|\bar{T}_1(s)|ds \leq& \bar{C}\frac{T}{N}|\bar{e}_{j-1}|_{\mathbb{L}^2}^{\frac{1}{2}}|\bar{e}_j|_{\mathbb{L}^2}^{\frac{1}{2}}|\nabla\bar{e}_{j-1}|_{\mathbb{L}^2}^{\frac{1}{2}}|\nabla\bar{e}_j|_{\mathbb{L}^2}^{\frac{1}{2}}|\nabla u(t_j)|_{\mathbb{L}^2}.\\
\leq& \frac{\delta_1}{2}\nu\frac{T}{N}|\nabla\bar{e}_{j-1}|_{\mathbb{L}^2}^2 + \frac{\delta_1}{2}\nu\frac{T}{N}|\nabla\bar{e}_j|_{\mathbb{L}^2}^2\\
&+ \frac{\delta_1\bar{C}^2}{8\delta_1\nu}\frac{T}{N}|\bar{e}_{j-1}|_{\mathbb{L}^2}^2|\nabla u(t_j)|_{\mathbb{L}^2}^2 + + \frac{\delta_1\bar{C}^2}{8\delta_1\nu}\frac{T}{N}|\bar{e}_j|_{\mathbb{L}^2}^2|\nabla u(t_j)|_{\mathbb{L}^2}^2.
\end{aligned} \tag{4.36}$$

The upper estimates of $\int_{t_{j-1}}^{t_j}\bar{T}_i(s)ds$, $i = 2,3$ are similar to the corresponding ones in the first step of the proof of Theorem 4.3. This yields the following analog of (4.11) with the same upper estimates (4.12) – (4.14) of the terms $\tilde{T}_j(i)$, $i = 1,2,3$

$$\begin{aligned}
(\bar{e}_j - \bar{e}_{j-1}\,,\,\bar{e}_j) + \nu|\nabla\bar{e}_j|_{\mathbb{L}^2}^2 \leq& \nu(\delta_0 + \frac{1}{2}\delta_1 + \delta_2 + \delta_3)\frac{T}{N}|\nabla\bar{e}_j|_{\mathbb{L}^2}^2 + \frac{1}{2}\delta_1\nu\frac{T}{N}|\nabla\bar{e}_{j-1}|_{\mathbb{L}^2}^2\\
&+ \Big(\gamma_2 + \frac{\bar{C}^2}{8\delta_1\nu}|\nabla u(t_j)|_{\mathbb{L}^2}^2\Big)\frac{T}{N}|\bar{e}_j|_{\mathbb{L}^2}^2 + \frac{\bar{C}^2}{8\delta_1\nu}|\nabla u(t_j)|_{\mathbb{L}^2}^2|\bar{e}_{j-1}|_{\mathbb{L}^2}^2\\
&+ \sum_{i=1}^{3}\tilde{T}_j(i) + \Big(\int_{t_{j-1}}^{t_j}\big[G(u(s)) - G(v_N(t_{j-1}))\big]\,dW(s)\,,\,\bar{e}_j\Big).
\end{aligned} \tag{4.37}$$

Once adding these estimates localized on the set $\Omega_{t_{j-1}}^M$, we deduce an upper estimate similar to (4.16) where e_j is replaced by $\bar{e}_j$. Following the same steps as in the proof of Proposition 4.3, we conclure the proof. □

The arguments in section 4.3 prove that the statements of Theorems 4.5 and 4.7 remain valid if we replace the solution $u_N(t_k)$ of the fully implicit Euler scheme by the solution $v_N(t_k)$ of the semi implicit one.

4.5. **Time dependent coefficients.** For the sake of simplicity, we have supposed that the diffusion coefficient G does not depend on time. An easy modification of the proofs of this section shows that the statements of Theorems 4.5, 4.7 for the fully or semi implicit Euler schemes remain true if we suppose that $G : [0,T] \times V \to \mathcal{L}_2(K,H)$ (resp. $G : [0,T] \times \mathrm{Dom(A)} \to \mathcal{L}_2(K,V)$) satisfy the global linear growth and Lipschitz versions of **(G1)** and **(G2)**

$$\|G(t,u)\|^2_{\mathcal{L}_2(K,H)} \leq K_0 + K_1|u|^2_{\mathbb{L}^2} + K_2\|u\|^2_V, \quad \|G(t,u)\|^2_{\mathcal{L}_2(K,V)} \leq K_0 + K_1\|u\|^2_V + K_2|Au|^2_{\mathbb{L}^2},$$
$$\|G(t,u) - G(t,v)\|^2_{\mathcal{L}_2(K,H)^2} \leq L_1|u-v|^2_{\mathbb{L}^2} + L_2\|u-v\|^2_V,$$
$$\|G(t,u) - G(t,v)\|^2_{\mathcal{L}_2(K,V)^2} \leq L_1\|u-v\|^2_V + L_2|A(u-v)|^2_{\mathbb{L}^2},$$

for $u, v \in V$ (resp. $u, v \in Dom(A)$) with the same constraints on K_2 and L_2, and if G also satisfies the following time regularity condition:
(G3) There exits a constant $C > 0$ such that for any $u, v \in V$ and $s, t \in [0,T]$:

$$\|G(t,u) - G(s,u)\|^2_{\mathcal{L}_2(K,H)} \leq C|t-s|^{\frac{1}{2}}\,(1 + \|u\|^2_V).$$

In that case, the fully implicit scheme u^N (resp. semi implicit scheme v_N) is defined replacing $G(u_N(t_{k-1}))$ by $G(t_{k-1}, u_N(t_{k-1}))$ (resp. by $G(t_{k-1}, v_N(t_{k-1}))$). in the right hand side of (4.1).

5. Appendix

In this section we prove two technical lemmas used to obtain the strong convergence results.

5.1. **Proof of Lemma 3.5.** First note that using (3.4) and (3.7), we deduce that if $\mathbb{E}(\|u_0\|^{2p}_V) < \infty$ then

$$\sup_{N\geq 1} \mathbb{E}\int_0^T \|z^N(s)\|^{2p}_V ds \leq C(p).$$

Thus only moments of $\sup_{t\in[0,T]} |z^N(t)|^{2p}_{\mathbb{L}^2}$ have to be dealt with.

The process z^N defined by (3.6) is not regular enough to apply directly Itô's formula to $|z^N(t)|^2_{\mathbb{L}^2}$. However, we may use Itô's formula with the Galerkin approximations of the processes u^N, y^N and z^N and then pass to the limit. This yields for every $t \in [0,T]$

$$|z^N(t)|^2_{\mathbb{L}^2} = |u_0|^2_{\mathbb{L}^2} - 2\int_0^t \langle F(u^N(s))\,,\, z^N(s)\rangle ds + \int_0^t \|G(y^N(s)\|^2_{\mathcal{L}_2(K,H)} ds$$
$$+ 2\int_0^t \big(G(y^N(s))dW(s)\,,\, z^N(s)\big).$$

Using once more the Itô formula, we deduce

$$|z^N(t)|^{2p}_{\mathbb{L}^2} = |u_0|^{2p}_{\mathbb{L}^2} + I(t) + \sum_{i=1}^3 J_i(t), \tag{5.1}$$

where

$$
\begin{aligned}
I(t) =& 2p\int_0^t \big(G(y^N(s))dW(s)\,,\, z^N(s)\big)|z^N(s)|_{\mathbb{L}^2}^{2(p-1)},\\
J_1(t) =& -2p\nu\int_0^t |z^N(s)|_{\mathbb{L}^2}^{2(p-1)}\big(\nabla u^N(s)\,,\nabla z^N(s)\big)ds\\
J_2(t) =& -2p\int_0^t |z^N(s)|_{\mathbb{L}^2}^{2(p-1)}\big\langle B(u^N(s),u^N(s))\,,\, z^N(s)\big\rangle ds\\
J_3(t) =& +p\int_0^t |z^N(s)|_{\mathbb{L}^2}^{2(p-1)}\|G(y^N(s)\|_{\mathcal{L}_2(K,H)}^2 ds\\
&+2p(p-1)\int_0^t \|G^*(y^N(s))z^N(s)\|_K^2|z^N(s)|_{\mathbb{L}^2}^{2(p-2)}ds.
\end{aligned}
$$

Hölder's and Young's inequalities imply

$$
|J_1(t)| \leq 2(p-1)\nu\int_0^t |z^N(s)|_{\mathbb{L}^2}^{2p}ds+\nu\int_0^t \big[|\nabla z^N(s)|_{\mathbb{L}^2}^{2p}+|\nabla u^N(s)|_{\mathbb{L}^2}^{2p}\big]ds
$$

Using again the Hölder and Young inequalities with exponents $\frac{2p+1}{2p-1}$ and $\frac{2p+1}{2}$, we deduce

$$
\begin{aligned}
|J_2(t)| \leq& 2p\int_0^t |z^N(s)|_{\mathbb{L}^2}^{2(p-1)}|\nabla z^N(s)|_{\mathbb{L}^2}\,\|u^N(s)\|_X^2 ds\\
\leq& \frac{(2p-1)2p}{2p+1}\int_0^t \|z^N(s)\|_V^{2p+1}ds+\frac{4p}{2p+1}\Big(\frac{\bar{C}}{2}\Big)^{\frac{2p+1}{2}}\int_0^t \|u^N(s)\|_V^{2p+1}ds,
\end{aligned}
$$

where $\bar{C}$ is the constant defined in (2.2). Finally, using the growth condition **(G1Bis)**, we deduce

$$
\begin{aligned}
|J_3(t)| \leq& (2p^2-p)\int_0^t |z^N(s)|_{\mathbb{L}^2}^{2(p-1)}\big[K_0+K_1|y^N(s)|_{\mathbb{L}^2}^2\big]ds\\
\leq& (2p-1)(p-1)\int_0^t |z^N(s)|_{\mathbb{L}^2}^{2p}ds+C(p)K_1^p\int_0^t \|y^N(s)\|_{\mathbb{L}^2}^{2p}ds+C(p)K_0T.
\end{aligned}
$$

where $C(p)$ is a constant depending on p. The inequalities (3.4) and (3.7), and the above estimates of $J_i(t)$ for $i=1,2,3$, imply the existence of a positive constant $C(p)$ depending on p such that for every integer. $N\geq 1$,

$$
\sum_{i=1}^3 \mathbb{E}\Big(\sup_{t\in[0,T]}|J_i(t)|\Big)\leq C(p). \tag{5.2}
$$

Furthermore, the Burkholder-Davies-Gundy inequality, the growth condition in **(G1Bis)** and the Young inequality imply

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[0,T]}|I(s)|\Big) \leq& 6p\mathbb{E}\Big(\Big\{\int_0^T |z^N(s)|_{\mathbb{L}^2}^{4p-2}\|G(y^N(s))\|_{\mathcal{L}_2(K,H)}^2 ds\Big\}^{\frac{1}{2}}\Big)\\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}|z^N(t)|_{\mathbb{L}^2}^{2p}\Big)+C(p)\mathbb{E}\int_0^T \big[K_0^p+K_1^p|y^N(s)|_{\mathbb{L}^2}^{2p}\big]ds\\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}|z^N(t)|_{\mathbb{L}^2}^{2p}\Big)+C(p),
\end{aligned}\tag{5.3}
$$

where the last inequality is deduced from (3.4).

The upper estimates (5.1)–(5.3), (2.11), (3.4) and (3.7) conclude the proof. □

5.2. **Proof of Lemma 3.8.** We prove the existence of exponential moments for the square of the V norm of the solution u of (1.1).

Proof of Lemma 3.8. Let $\alpha > 0$; even if the solution u to (1.1) is not regular enough, we may apply the Itô formula to the square of its V norm. Indeed, we can apply Itô's formula to the more regular Galerkin approximation of u and then pass to the limit using for example a Yoshida approximation or computations similar to those in [12], step 4 of the Appendix on page 416. This yields

$$\alpha\|u(t)|_V^2 + \alpha\nu\int_0^t |||u(s)|||^2 ds = \alpha\|u_0\|_V^2 + \alpha\int_0^t \|G(s,u(s))\|^2_{\mathcal{L}_2(K,V)} ds + 2\alpha\int_0^t \big(u(s)\,,\, G(s,u(s))dW(s)\big)_V - \alpha\nu\int_0^t |||u(s)|||^2 ds, \tag{5.4}$$

where for $u, v \in V$ we set $(u,v)_V = (u,v) + (\nabla u, \nabla v)$ and recall that $|||u|||^2 := |\nabla u|^2_{\mathbb{L}^2} + |Au|^2_{\mathbb{L}^2}$.

Let $M(t) := 2\alpha\int_0^t \big(u(s)\,,\, G(s,u(s))dW(s)\big)_V$; then M is a martingale with quadratic variation

$$\begin{aligned}\langle M\rangle_t &\le 4\alpha^2\int_0^t \|u(s)\|_V^2\|G(s,u(s))\|^2_{\mathcal{L}_2(K,V)} ds \le 4\alpha^2 K_0\int_0^t \|u(s)\|_V^2 ds,\\ &\le 4\alpha^2 K_0\tilde{C}\int_0^t |||u(s)|||^2 ds,\end{aligned}$$

where the last inequality follows from (2.9) and the definition of $\tilde{C}$ in (3.28).

Let $\alpha_0 := \frac{\nu}{4K_0\tilde{C}}$; then we deduce

$$M_t - \alpha\nu\int_0^t |||u(s)|||^2 ds \le M_t - \frac{\alpha_0}{\alpha}\langle M\rangle_t.$$

Therefore, using the previous inequality and classical exponential martingale arguments, we deduce

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\le t\le T}\Big(M_t - \alpha\nu\int_0^t |||u(s)|||^2 ds\Big) \ge K\Big] &\le \mathbb{P}\Big[\sup_{0\le t\le T}\big(M_t - \frac{\alpha_0}{\alpha}\langle M\rangle_t\big) \ge K\Big]\\ &\le \mathbb{P}\Big[\sup_{0\le t\le T}\exp\Big(\frac{2\alpha_0}{\alpha}M_t - \frac{1}{2}\langle\frac{2\alpha_0}{\alpha}M\rangle_t\Big) \ge \exp\Big(\frac{2\alpha_0}{\alpha}K\Big)\Big]\\ &\le \exp\Big(-\frac{2\alpha_0}{\alpha}K\Big)\,\mathbb{E}\Big[\exp\Big(\frac{2\alpha_0}{\alpha}M_T - \frac{1}{2}\langle\frac{\beta}{\alpha}M\rangle_T\Big)\Big] \le \exp\Big(-\frac{2\alpha_0}{\alpha}K\Big)\end{aligned}$$

for any $K > 0$. Set

$$X = \exp\Big(\alpha\Big\{\sup_{t\in[0,T]} 2\int_0^t \big(u(s)\,,\, G(s,u(s))\big)_V - \nu\int_0^t |||u(s)|||^2 ds\Big\}\Big);$$

then for $\alpha \in (0,\alpha_0]$, we deduce that $\mathbb{P}(X \ge e^K) \le \exp(-K\frac{2\alpha_0}{\alpha}) \le e^{-2K} = (e^{-K})^2$ for any $K > 0$.

Using this inequality for any $C = e^K > 1$ with $K > 0$, we deduce $\mathbb{E}(X) \le 2 + \int_0^\infty \mathbb{P}(X \ge C)dC \le 3$. Since (5.4) implies

$$\alpha\Big(\sup_{t\in[0,T]}\Big[\|u(t)\|_V^2 + \nu\int_0^t |||u(s)|||^2 ds\Big]\Big) \le \alpha\|u_0\|_V^2 + \alpha K_0 T$$

$$+\sup_{t\in[0,T]}\Big(\alpha\Big\{2\int_0^t\big(u(s)\,,\,G(s,u(s))\big)_V-\nu\int_0^t|||u(s)|||^2ds\Big\}\Big),$$

we conclude the proof of (3.29). □

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