Asymptotic behavior for the Vlasov-Poisson equations with strong external magnetic field. Straight magnetic field lines

Mihai Bostan

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Mihaï BOSTAN *

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Abstract

The subject matter of this paper concerns the Vlasov-Poisson equations in the framework of magnetic confinement . We study the behavior of the Vlasov-Poisson system with strong external magnetic field, when neglecting the curvature of the magnetic lines. The arguments rely on averaging techniques. We intend to determine second order approximations and to retrieve the usual electric cross field drift, the magnetic gradient drift, etc.

Keywords: Vlasov-Poisson system, two-scale analysis, averaging, homogenization.

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1 Introduction

Let $f = f(t,x,v)$ be the presence density of a population of charged particles of mass m, charge q, depending on time t, position x and velocity v. Motivated by the magnetic confinement, we study the Vlasov-Poisson equations, with a strong external non vanishing magnetic field. Neglecting the curvature of the magnetic lines, we assume that the external magnetic field is orthogonal to Ox_1, Ox_2. In the two dimensional setting $x = (x_1, x_2), v = (v_1, v_2)$, the Vlasov equation writes

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m} \left\{ E[f^\varepsilon(t)](x) + B^\varepsilon(x)\, {}^\perp v \right\} \cdot \nabla_v f^\varepsilon = 0, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2. \quad (1)$$

Here the notation ${}^\perp(\cdot)$ stands for the rotation of angle $-\pi/2$, *i.e.*, ${}^\perp v = \mathcal{R}(-\pi/2)v = (v_2, -v_1)$, $v = (v_1, v_2) \in \mathbb{R}^2$ and the magnetic field writes $\mathbf{B}^\varepsilon(x) = (0, 0, B^\varepsilon(x)) = (0, 0, B(x)/\varepsilon)$, where $B(x)$ is a given function and $\varepsilon > 0$ is a small parameter. The electric field $E[f^\varepsilon(t)] = -\nabla_x \Phi[f^\varepsilon(t)]$ derives from the potential

$$\Phi[f^\varepsilon(t)](x) = -\frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x - x'| f^\varepsilon(t, x', v')\, \mathrm{d}v' \mathrm{d}x' \quad (2)$$

*Aix Marseille Université, CNRS, Centrale Marseille, I2M, Marseille France, Centre de Mathématiques et Informatique, UMR 7373, 39 rue Frédéric Joliot Curie, 13453 Marseille Cedex 13 France. E-mail : mihai.bostan@univ-amu.fr.

satisfying the Poisson equation

$$-\varepsilon_0 \Delta_x \Phi[f^\varepsilon(t)] = q \int_{\mathbb{R}^2} f^\varepsilon(t,x,v)\, \mathrm{d}v, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

whose fundamental solution is $z \to -\frac{1}{2\pi} \ln|z|, z \in \mathbb{R}^2 \setminus \{0\}$. Here ε_0 represents the electric permittivity. For any presence density $f = f(x,v)$, the notation $E[f]$ stands for the Poisson electric field

$$E[f](x) = \frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x',v') \frac{x-x'}{|x-x'|^2}\, \mathrm{d}v'\mathrm{d}x' \tag{3}$$

and $\rho[f], j[f]$ are the charge and current densities respectively

$$\rho[f] = q \int_{\mathbb{R}^2} f(\cdot,v)\, \mathrm{d}v, \quad j[f] = q \int_{\mathbb{R}^2} f(\cdot,v) v\, \mathrm{d}v.$$

We complete the above system by the initial condition

$$f^\varepsilon(0,x,v) = f_{\mathrm{in}}(x,v), \quad (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{4}$$

We intend to investigate the asymptotic behavior of the problem (1), (3), (4) when ε goes to 0. We refer to [17, 18, 19, 24, 25, 11, 13, 15, 22, 3, 5, 6, 7] for previous results on this topic where, most of the time, the authors studied the case of uniform magnetic fields (see also [20] for results with magnetic field of constant direction but variable strenght and with magnetic field of constant strenght but variable direction).

Solving numerically (1), (2), when ε becomes small, requires a huge amount of computations. For example, when explicit numerical methods are used, CFL stability conditions apply, leading to a small time step of order ε. One alternative is to construct suitable numerical schemes, preserving the asymptotic cf. [14, 16, 12]. Here we have in mind another possibility. Instead of solving the problem (1), (2), which appears in a singular form, due to the large magnetic field, we are looking for a regular reformulation of it, whose numerical resolution is not penalized anymore by the smallness of the parameter ε. Certainly, the new problem will not be equivalent to the original one, but up to a second order term with respect to ε, the solutions will coincide. Therefore, when ε becomes small we can obtain very good approximations for the Vlasov-Poisson system with large external magnetic field, with a numerical cost not depending on ε.

The derivation of the regular formulation follows by averaging with respect to the fast cyclotronic motion, whose invariants are the Larmor center and radius. Its well posedness is established : a classical solution exists globally in time and it is unique. For any $k \in \mathbb{N}$, the notation C_b^k stands for k times continuously differentiable functions, whose all partial derivatives, up to order k, are bounded.

Theorem 1.1
Consider a non negative, smooth, compactly supported initial presence density $\tilde{f}_{\mathrm{in}} \in C_c^1(\mathbb{R}^2 \times \mathbb{R}^2)$ and a smooth magnetic field $B^\varepsilon = \frac{B}{\varepsilon} \in C_b^2(\mathbb{R}^2)$ such that $\inf_{x\in\mathbb{R}^2} |B^\varepsilon(x)| = B_0^\varepsilon > 0$ (that is $B_0^\varepsilon = \frac{B_0}{\varepsilon}$, $\inf_{x\in\mathbb{R}^2} |B(x)| = B_0 > 0$). There is a unique presence density $\tilde{f} \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ whose restriction on $[0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ is compactly supported for any $T \in \mathbb{R}_+$, whose Poisson electric field belongs to $C^1(\mathbb{R}_+ \times \mathbb{R}^2)$

$$E[\tilde{f}(t)](x) = \frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{f}(t,x',\tilde{v}') \frac{x-x'}{|x-x'|^2}\, \mathrm{d}\tilde{v}'\mathrm{d}x', \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \tag{5}$$

satisfying

$$\partial_t \tilde{f} + \left(\frac{{}^{\perp}E[\tilde{f}(t)]}{B^\varepsilon} - \frac{m|\tilde{v}|^2}{2qB^\varepsilon} \frac{{}^{\perp}\nabla B^\varepsilon}{B^\varepsilon} \right) \cdot \nabla_x \tilde{f} + \frac{1}{2} \left(\frac{{}^{\perp}E[\tilde{f}(t)]}{B^\varepsilon} \cdot \frac{\nabla_x B^\varepsilon}{B^\varepsilon} \right) \tilde{v} \cdot \nabla_{\tilde{v}} \tilde{f} = 0 \quad (6)$$

$$\tilde{f}(0, x, \tilde{v}) = \tilde{f}_{\text{in}}(x, \tilde{v}), \ \ (x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Moreover, if for some integer $k \geq 2$ we have $\tilde{f}_{\text{in}} \in C_c^k(\mathbb{R}^2 \times \mathbb{R}^2), B^\varepsilon \in C^{k+1}(\mathbb{R}^2)$, then $\tilde{f} \in C^k(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ and $E[\tilde{f}] \in C^k(\mathbb{R}_+ \times \mathbb{R}^2)$.

We prove that the solution of the previous model approximates the solution of the Vlasov-Poisson system (1), (2) up to a second order term with respect to ε.

Theorem 1.2
Let $B \in C_b^3(\mathbb{R}^2)$ be a smooth magnetic field, such that $\inf_{x \in \mathbb{R}^2} |B(x)| = B_0 > 0$. Consider a family of non negative, smooth, uniformly compactly supported presence densities $(g^\varepsilon)_{\varepsilon > 0} \subset C_c^2(\mathbb{R}^2 \times \mathbb{R}^2)$

$$\exists\, R_{\tilde{x}}, R_{\tilde{v}} > 0 \ : \ \text{supp}\, g^\varepsilon \subset \{(\tilde{x}, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ |\tilde{x}| \leq R_{\tilde{x}} \ \text{ and } |\tilde{v}| \leq R_{\tilde{v}}\}, \ \sup_{\varepsilon > 0} \|g^\varepsilon\|_{C^2} < +\infty.$$

We assume that the presence densities are well prepared i.e.,

$$\sup_{\varepsilon > 0} \frac{\|{}^{\perp}\tilde{v} \cdot \nabla_{\tilde{v}} g^\varepsilon\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}}{\varepsilon^2} < +\infty.$$

We denote by $(f^\varepsilon)_{\varepsilon > 0}$ the solutions of the Vlasov-Poisson equations with external magnetic field (1), (2) corresponding to the initial conditions

$$f^\varepsilon(0, x, v) = g^\varepsilon \left(x + \varepsilon \frac{{}^{\perp}v}{\omega_c}, v - \varepsilon \frac{{}^{\perp}E[g^\varepsilon]}{B} \right), \ \ (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2, \ \ \varepsilon > 0.$$

Then for any $T \in \mathbb{R}_+$, there is $\varepsilon_T > 0$ and $C_T > 0$ such that for any $0 < \varepsilon \leq \varepsilon_T$

$$\sup_{t \in [0,T]} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[f^\varepsilon(t, x, v) - \tilde{f} \left(t, x + \varepsilon \frac{{}^{\perp}v}{\omega_c}, v - \varepsilon \frac{{}^{\perp}E[\tilde{f}(t)]}{B} \right) \right]^2 \, \mathrm{d}v\mathrm{d}x \right\}^{1/2} \leq C_T \varepsilon^2$$

where $\tilde{f}$ is the solution of (6), (5) corresponding to the initial condition $\tilde{f}(0) = \langle g^\varepsilon \rangle$ (here the notation $\langle \cdot \rangle$ stands for the average along the characteristic flow of the vector field $\omega_c(x)\, {}^{\perp}v \cdot \nabla_v$, see Proposition 3.1).

When the magnetic field is uniform, we obtain second order approximations independently of the initial conditions being well prepared or not, see also [25].

Theorem 1.3 *Let $f_{\text{in}} \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2)$ be a non negative, smooth, compactly supported presence density. We denote by $(f^\varepsilon)_{\varepsilon > 0}$ the solutions of the Vlasov-Poisson equations (1), (2) with uniform external magnetic field $\frac{B}{\varepsilon} \neq 0$, corresponding to the initial condition f_{in}. Then for any $T \in \mathbb{R}_+$, there is a constant $C_T > 0$ such that for any $\varepsilon > 0$*

$$\sup_{t \in [0,T]} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[f^\varepsilon(t, x, v) - \tilde{F} \left(t, x + \varepsilon \frac{{}^{\perp}v}{\omega_c}, \mathcal{R}(\omega_c \frac{t}{\varepsilon}) \left(v - \varepsilon \frac{{}^{\perp}E[\tilde{F}(t)]}{B} \right) \right) \right]^2 \mathrm{d}v\mathrm{d}x \right\}^{1/2} \leq C_T \varepsilon^2$$

where $\tilde{F}$ *is the solution of*

$$\partial_t \tilde{F} + \varepsilon \frac{{}^\perp E[\tilde{F}(t)]}{B} \cdot \nabla_X \tilde{F} + \varepsilon \frac{{}^\perp (j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V})}{2\varepsilon_0 B} \cdot \nabla_{\tilde{V}} \tilde{F} = 0, \quad (t, X, \tilde{V}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

corresponding to the initial condition

$$\tilde{F}(0, X, \tilde{V}) = f_{\text{in}} \left(X - \varepsilon \frac{{}^\perp \tilde{V}}{\omega_c}, \tilde{V} + \varepsilon \frac{{}^\perp E[f_{\text{in}}](X)}{B} \right), \quad (X, \tilde{V}) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

The above results extend to the three dimensional Vlasov-Poisson system, when the curvature of the magnetic lines is not neglected anymore. The regular reformulations of this problem will be the topic of a future work [10].

Our paper is organized as follows. In Section 2 we discuss the well posedness of the Vlasov-Poisson problem with external magnetic field. The regular reformulation of the Vlasov-Poisson problem is derived by formal computations in Section 3. Its well posedness is established in Section 4. The error estimate, when the initial conditions are well prepared, is shown in Section 5 and relies on the construction of a corrector term. More general results, for initial conditions not necessarily well prepared, are discussed in the last section, for uniform magnetic fields.

2 Strong solutions for the Vlasov-Poisson problem with external magnetic field

The well posedness of the Vlasov-Poisson problem is well known. We refer to [1] for weak solutions, and to [26, 21, 23] for strong solutions. Using essentially the same arguments, leads to global existence and uniqueness for the strong solution of the Vlasov-Poisson problem with external magnetic field. More exactly we prove the following result, see Appendix A for the main lines of the proof.

Theorem 2.1
Consider a non negative, smooth, compactly supported initial presence density $f_{\text{in}} \in C_c^1(\mathbb{R}^2 \times \mathbb{R}^2)$ *and a smooth magnetic field* $B \in C_b^1(\mathbb{R}^2)$*. There is a unique presence density* $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$*, whose restriction on* $[0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$ *is compactly supported for any* $T \in \mathbb{R}_+$*, whose Poisson electric field is smooth* $E[f] \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$*, satisfying*

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m} \left(E[f(t)] + B\, {}^\perp v \right) \cdot \nabla_v f = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \tag{7}$$

$$E[f(t)](x) = \frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(t, x', v') \frac{x - x'}{|x - x'|^2} \, \mathrm{d}v' \mathrm{d}x', \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \tag{8}$$

$$f(0, x, v) = f_{\text{in}}(x, v), \quad (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{9}$$

Moreover, if for some integer $k \geq 2$ *we have* $f_{\text{in}} \in C_c^k(\mathbb{R}^2 \times \mathbb{R}^2), B \in C_b^k(\mathbb{R}^2)$*, then* $f \in C^k(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)$ *and* $E[f] \in C^k(\mathbb{R}_+ \times \mathbb{R}^2)$*.*

Remark 2.1
1. The solution constructed in Theorem 2.1 satisfies the conservation of the particle number and total energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,v)\,\mathrm{d}v\mathrm{d}x = 0, \quad t\in\mathbb{R}_+$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{m|v|^2}{2}f(t,x,v)\,\mathrm{d}v\mathrm{d}x - \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho[f(t)](x)\rho[f(t)](x')\ln|x-x'|\,\mathrm{d}x'\mathrm{d}x\right\} = 0.$$

2. Notice also that we have the following balance for the total momentum

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,v)mv\,\mathrm{d}v\mathrm{d}x - q\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,v)B(x)\,{}^{\perp}v\,\mathrm{d}v\mathrm{d}x &= \int_{\mathbb{R}^2}\rho[f(t)]E[f(t)]\,\mathrm{d}x\\ &= \varepsilon_0\int_{\mathbb{R}^2}\mathbf{1}_{\mathrm{supp}\,\rho[f(t)]}\,\mathrm{div}_x E[f(t)]\;E[f(t)]\,\mathrm{d}x\\ &= \varepsilon_0\int_{\mathbb{R}^2}\mathbf{1}_{\mathrm{supp}\,\rho[f(t)]}\,\mathrm{div}_x\left(E[f(t)]\otimes E[f(t)] - \frac{|E[f(t)]|^2}{2}I_2\right)\,\mathrm{d}x = 0.\end{aligned}$$

When the magnetic field is uniform, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,v)mv\,\mathrm{d}v\mathrm{d}x = \frac{qB}{m}\;{}^{\perp}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,v)mv\,\mathrm{d}v\mathrm{d}x\right)$$

saying that the total momentum rotates at the cyclotronic frequency $\omega_c = \frac{qB}{m}$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,v)mv\,\mathrm{d}v\mathrm{d}x = \mathcal{R}(-\omega_c t)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_{\mathrm{in}}(x,v)mv\,\mathrm{d}v\mathrm{d}x, \quad t\in\mathbb{R}_+.$$

3. By direct computation, when the magnetic field is uniform, we obtain

$$\left[v\cdot\nabla_x + \frac{q}{m}(E[f] + B\,{}^{\perp}v)\cdot\nabla_v\right]\left(\frac{1}{2}\left|x + \frac{{}^{\perp}v}{\omega_c}\right|^2 - \frac{1}{2}\frac{|v|^2}{\omega_c^2}\right) = -\frac{E[f]}{B}\cdot\,{}^{\perp}x.$$

Therefore, after integration by parts, we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,v)\left(\frac{1}{2}\left|x + \frac{{}^{\perp}v}{\omega_c}\right|^2 - \frac{1}{2}\frac{|v|^2}{\omega_c^2}\right)\,\mathrm{d}v\mathrm{d}x &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\frac{E[f]}{B}\cdot\,{}^{\perp}x\,\mathrm{d}v\mathrm{d}x\\ &= -\frac{\varepsilon_0}{Bq}\int_{\mathbb{R}^2}\mathbf{1}_{\mathrm{supp}\,\rho[f(t)]}\,\mathrm{div}_x E[f(t)]\;E[f(t)]\cdot\,{}^{\perp}x\,\mathrm{d}x\\ &= -\frac{\varepsilon_0}{Bq}\int_{\mathbb{R}^2}\mathbf{1}_{\mathrm{supp}\,\rho[f(t)]}\,\mathrm{div}_x\left(E[f(t)]\otimes E[f(t)] - \frac{|E[f(t)]|^2}{2}I_2\right)\cdot\,{}^{\perp}x\,\mathrm{d}x\\ &= \frac{\varepsilon_0}{Bq}\int_{\mathbb{R}^2}\mathbf{1}_{\mathrm{supp}\,\rho[f(t)]}\left(E[f(t)]\otimes E[f(t)] - \frac{|E[f(t)]|^2}{2}I_2\right) : \partial_x^{\perp}x\,\mathrm{d}x = 0.\end{aligned}$$

3 Asymptotic analysis by formal arguments

We are interested on the asymptotic behavior of the presence densities $(f^\varepsilon)_{\varepsilon>0}$ satisfying (1), (2), (4) when $\varepsilon > 0$ becomes small. We assume that the initial presence density and the external magnetic field $B^\varepsilon(x) = B(x)/\varepsilon$ are smooth

$$f_{\mathrm{in}} \geq 0, \quad f_{\mathrm{in}}\in C_c^1(\mathbb{R}^2\times\mathbb{R}^2), \quad B\in C_b^1(\mathbb{R}^2). \tag{10}$$

Under the above assumptions, we know by Theorem 2.1 that for every $\varepsilon > 0$, there is a unique strong solution $f^\varepsilon \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ (whose restriction on $[0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ is compactly supported for any $T \in \mathbb{R}_+$), $E^\varepsilon := E[f^\varepsilon] \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$ for the Vlasov-Poisson problem with external magnetic field $B^\varepsilon = B/\varepsilon$. By the arguments in the proof of Theorem 2.1 we also have uniform estimates with respect to $\varepsilon > 0$ for the L^∞ norm of the electric field E^ε and the size of the support of the presence density f^ε. Let us denote by $(X^\varepsilon, V^\varepsilon)(t; t_0, x, v)$ the characteristics associated to (1)

$$\frac{\mathrm{d}X^\varepsilon}{\mathrm{d}t} = V^\varepsilon(t; t_0, x, v), \ \frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t} = \frac{q}{m}\left[E^\varepsilon(t, X^\varepsilon(t; t_0, x, v)) + B^\varepsilon(X^\varepsilon(t; t_0, x, v))\, {}^\perp V^\varepsilon(t; t_0, x, v)\right]$$

$$X^\varepsilon(t; t_0, x, v) = x, \ \ V^\varepsilon(t; t_0, x, v) = v.$$

Clearly, the strong external magnetic field induces a large cyclotronic frequency $\omega_c^\varepsilon = qB^\varepsilon/m = \omega_c/\varepsilon, \omega_c = qB/m$, and therefore a fast dynamics. The key point is to find out quantities which are left invariant with respect to the fast motion. For example, $X^\varepsilon(t) + \varepsilon\, {}^\perp V^\varepsilon(t)/\omega_c(X^\varepsilon(t))$ has small variations in time

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[X^\varepsilon(t) + \varepsilon\frac{{}^\perp V^\varepsilon(t)}{\omega_c(X^\varepsilon(t))}\right] = \varepsilon\left[\frac{{}^\perp E^\varepsilon(t, X^\varepsilon(t))}{B(X^\varepsilon(t))} - \frac{{}^\perp V^\varepsilon(t) \otimes V^\varepsilon(t)}{\omega_c(X^\varepsilon(t))^2}\nabla_x \omega_c(X^\varepsilon(t))\right].$$

Another quantity having small variations is $\mathcal{R}\left(\int_0^t \frac{\omega_c(X^\varepsilon(\sigma))}{\varepsilon}\mathrm{d}\sigma\right)[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B(X^\varepsilon(t))}]$. Indeed, notice that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t, X^\varepsilon(t))}{B(X^\varepsilon(t))}\right] &= \frac{\omega_c(X^\varepsilon(t))}{\varepsilon}\, {}^\perp\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t, X^\varepsilon(t))}{B(X^\varepsilon(t))}\right] \\ &\quad - \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{{}^\perp E^\varepsilon(t, X^\varepsilon(t))}{B(X^\varepsilon(t))}\right]\end{aligned}$$

and therefore

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathcal{R}\left(\int_0^t \frac{\omega_c(X^\varepsilon(\sigma))}{\varepsilon}\mathrm{d}\sigma\right)\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t, X^\varepsilon(t))}{B(X^\varepsilon(t))}\right]\right\} \\ &\quad = -\varepsilon\mathcal{R}\left(\int_0^t \frac{\omega_c(X^\varepsilon(\sigma))}{\varepsilon}\mathrm{d}\sigma\right)\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{{}^\perp E^\varepsilon(t, X^\varepsilon(t))}{B(X^\varepsilon(t))}\right].\end{aligned} \tag{11}$$

Certainly, the above quantities are not exactly left invariant. The idea will be to split the advection field appearing in the Vlasov equation (1) into a fast and slow dynamics in such a way that the previous quantities become invariant. First of all, in order to simplify our computations, we perform a change of coordinates. Motivated by the calculation in (11), we introduce the relative velocity with respect to the electric cross field drift

$$\tilde{v} = v - \varepsilon\frac{{}^\perp E^\varepsilon(t, x)}{B(x)}. \tag{12}$$

Accordingly, at any time $t \in \mathbb{R}_+$, we consider the new presence density

$$\tilde{f}^\varepsilon(t, x, \tilde{v}) = f^\varepsilon\left(t, x, \tilde{v} + \varepsilon\frac{{}^\perp E[f^\varepsilon(t)](x)}{B(x)}\right), \ \ (x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{13}$$

Notice that this change of coordinates, which depends on the presence density through the electric field, does not change the charge density

$$\rho[\tilde{f}^\varepsilon(t)] = q\int_{\mathbb{R}^2} \tilde{f}^\varepsilon(t,\cdot,\tilde{v})\,\mathrm{d}\tilde{v} = q\int_{\mathbb{R}^2} f^\varepsilon(t,\cdot,v)\,\mathrm{d}v = \rho[f^\varepsilon(t)],\ \ t\in\mathbb{R}_+.$$

Therefore the Poisson electric fields corresponding to the presence densities $f^\varepsilon, \tilde{f}^\varepsilon$ coincide

$$E[f^\varepsilon(t)] = E[\tilde{f}^\varepsilon(t)],\ \ t\in\mathbb{R}_+$$

and we can use the same notation $E^\varepsilon(t)$ for denoting them. We add to (10) the hypothesis

$$B_0 := \inf_{x\in\mathbb{R}^2} |B(x)| > 0 \ \text{ or equivalently } \omega_0 := \inf_{x\in\mathbb{R}^2} |\omega_c(x)| > 0 \tag{14}$$

such that (12), (13) are well defined. Observe that the new presence densities $(\tilde{f}^\varepsilon)_{\varepsilon>0}$ are smooth, $\tilde{f}^\varepsilon \in C^1(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2)$ and that the restrictions to $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$ are compactly supported, uniformly with respect to $\varepsilon\in]0,1]$, for any $T\in\mathbb{R}_+$. The last statement comes from the similar property of the presence densities $(f^\varepsilon)_{\varepsilon>0}$, together with the uniform bound for the electric fields $(E^\varepsilon)_{\varepsilon>0}$ and the hypothesis (14). A straightforward computation leads to the following problem in the new coordinates $(x,\tilde{v})$

$$\begin{aligned}\partial_t\tilde{f}^\varepsilon + \left(\tilde{v}+\varepsilon\frac{{}^\perp E^\varepsilon}{B}\right)\cdot\nabla_x\tilde{f}^\varepsilon - \varepsilon\left[\partial_t\left(\frac{{}^\perp E^\varepsilon}{B}\right)+\partial_x\left(\frac{{}^\perp E^\varepsilon}{B}\right)\left(\tilde{v}+\varepsilon\frac{{}^\perp E^\varepsilon}{B}\right)\right]\cdot\nabla_{\tilde{v}}\tilde{f}^\varepsilon \\ + \frac{\omega_c(x)}{\varepsilon}\,{}^\perp\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}^\varepsilon = 0,\ \ (t,x,\tilde{v})\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2\end{aligned} \tag{15}$$

$$\tilde{f}^\varepsilon(0,x,\tilde{v}) = f_{\text{in}}\left(x,\tilde{v}+\varepsilon\frac{{}^\perp E[f_{\text{in}}](x)}{B(x)}\right),\ \ (x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2.$$

Notice that the time derivative of the electric field E^ε can be written in terms of the presence density f^ε (or $\tilde{f}^\varepsilon$). Indeed, thanks to the continuity equation

$$\partial_t\rho[f^\varepsilon] + \mathrm{div}_x j[f^\varepsilon] = 0$$

we have

$$\begin{aligned}\partial_t E[f^\varepsilon] &= \frac{1}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\partial_t\rho[f^\varepsilon(t)](x-x')\frac{x'}{|x'|^2}\,\mathrm{d}x' \\ &= -\frac{1}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\mathrm{div}_x j[f^\varepsilon](x-x')\frac{x'}{|x'|^2}\,\mathrm{d}x' \\ &= -\frac{1}{2\pi\varepsilon_0}\mathrm{div}_x\int_{\mathbb{R}^2}\frac{x'}{|x'|^2}\otimes j[f^\varepsilon(t)](x-x')\,\mathrm{d}x' \\ &= -\frac{1}{2\pi\varepsilon_0}\mathrm{div}_x\int_{\mathbb{R}^2}\frac{x-x'}{|x-x'|^2}\otimes\left(j[\tilde{f}^\varepsilon(t)](x') + \varepsilon\frac{{}^\perp E^\varepsilon(t,x')}{B(x')}\rho[\tilde{f}^\varepsilon(t)](x')\right)\,\mathrm{d}x'.\end{aligned}$$

Finally the Vlasov equation (15) writes

$$\partial_t\tilde{f}^\varepsilon + \varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon + \frac{b^\varepsilon(x,\tilde{v})}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon = 0,\ \ (t,x,\tilde{v})\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2 \tag{16}$$

where $b^\varepsilon \cdot \nabla_{x,\tilde v} = (\varepsilon \tilde v + \varepsilon^2 A_x^\varepsilon(x,\tilde v)) \cdot \nabla_x + \omega_c(x)\, {}^\perp\tilde v \cdot \nabla_{\tilde v}$ and for any presence density $\tilde f$, $a^\varepsilon[\tilde f] \cdot \nabla_{x,\tilde v}$ stands for the vector field

$$a^\varepsilon[\tilde f] \cdot \nabla_{x,\tilde v} = \left(\frac{{}^\perp E[\tilde f]}{B} - A_x^\varepsilon \right) \cdot \nabla_x + \left[-\partial_x \left(\frac{{}^\perp E[\tilde f]}{B} \right) \left(\tilde v + \varepsilon \frac{{}^\perp E[\tilde f]}{B} \right) \right.$$
$$\left. + \frac{1}{2\pi\varepsilon_0 B} \mathrm{div}_x \int_{\mathbb{R}^2} \frac{{}^\perp(x-x')}{|x-x'|^2} \otimes \left(j[\tilde f] + \varepsilon \frac{{}^\perp E[\tilde f]}{B} \rho[\tilde f] \right)(x')\, \mathrm{d}x' \right] \cdot \nabla_{\tilde v}.$$

Here $A_x^\varepsilon(x,\tilde v) \cdot \nabla_x$ is a vector field, to be determined later on, not depending on the presence density $\tilde f$. We will distinguish between the fast dynamics along the vector field $\frac{b^\varepsilon}{\varepsilon} \cdot \nabla_{x,\tilde v}$ and the slow dynamics along the vector field $\varepsilon a^\varepsilon \cdot \nabla_{x,\tilde v}$. We pick up the vector field $A_x^\varepsilon(x,\tilde v) \cdot \nabla_x$ entering the corrections in $a^\varepsilon \cdot \nabla_{x,\tilde v}$ and $b^\varepsilon \cdot \nabla_{x,\tilde v}$ such that $x + \varepsilon \frac{{}^\perp \tilde v}{\omega_c(x)}$ (which is not exactly the Larmor center, but a perturbation of it, up to a term in ε^2) is left invariant by the fast dynamics

$$b^\varepsilon \cdot \nabla_{x,\tilde v} \left(x + \varepsilon \frac{{}^\perp \tilde v}{\omega_c(x)} \right) = 0$$

that is

$$\left(I_2 - \varepsilon \frac{{}^\perp \tilde v \otimes \nabla \omega_c}{\omega_c^2(x)} \right) A_x^\varepsilon(x,\tilde v) = \frac{{}^\perp \tilde v \otimes \tilde v}{\omega_c^2(x)} \nabla_x \omega_c. \tag{17}$$

Obviously, when the magnetic field is uniform, *i.e.*, $\nabla_x \omega_c = 0$, there is no correction, $A_x^\varepsilon = 0$. Notice that A_x^ε is well defined for any $(x,\tilde v)$ such that $\varepsilon \frac{{}^\perp \tilde v \cdot \nabla \omega_c}{\omega_c^2(x)} \neq 1$, that is, for almost all $(x,\tilde v) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Remark 3.1 *The vector field in the Vlasov equation* (16) *is divergence free*

$$\mathrm{div}_{x,\tilde v} \left(\varepsilon a^\varepsilon[\tilde f] + \frac{b^\varepsilon}{\varepsilon} \right) = \mathrm{div}_x \left(\frac{{}^\perp E[\tilde f]}{B} \right) - \mathrm{div}_{\tilde v} \left[\partial_x \left(\frac{{}^\perp E[\tilde f]}{B} \right) \tilde v \right] = 0.$$

Studying the asymptotic behavior of (16), when ε goes to 0 reduces to averaging with respect to the flow of the fast dynamics generated by the advection field $\frac{b^\varepsilon(x,\tilde v)}{\varepsilon} \cdot \nabla_{x,\tilde v}$ cf. [3, 5, 9, 6, 7]. Actually, the characteristic flow associated to $b^\varepsilon(x,\tilde v) \cdot \nabla_{x,\tilde v}$ is periodic, which simplifies a lot the asymptotic analysis.

Proposition 3.1 *We denote by* $(\mathcal{X}^\varepsilon(s;x,\tilde v), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v))$ *the characteristic flow of the autonomous vector field* $b^\varepsilon(x,\tilde v) \cdot \nabla_{x,\tilde v}$

$$\frac{\mathrm{d}\mathcal{X}^\varepsilon}{\mathrm{d}s} = \varepsilon \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v) + \varepsilon^2 A_x^\varepsilon(\mathcal{X}^\varepsilon(s;x,\tilde v), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v)), \quad \frac{\mathrm{d}\tilde{\mathcal{V}}^\varepsilon}{\mathrm{d}s} = \omega_c(\mathcal{X}^\varepsilon(s;x,\tilde v))\, {}^\perp\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v)$$

$$\mathcal{X}^\varepsilon(0;x,\tilde v) = x, \quad \tilde{\mathcal{V}}^\varepsilon(0;x,\tilde v) = \tilde v$$

and by $(\mathcal{X}(s;x,\tilde v), \tilde{\mathcal{V}}(s;x,\tilde v))$ *the characteristic flow of the autonomous vector field* $b(x,\tilde v) \cdot \nabla_{x,\tilde v} = \omega_c(x)\, {}^\perp\tilde v \cdot \nabla_{\tilde v}$

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}s} = 0, \quad \frac{\mathrm{d}\tilde{\mathcal{V}}}{\mathrm{d}s} = \omega_c(\mathcal{X}(s;x,\tilde v))\, {}^\perp\tilde{\mathcal{V}}(s;x,\tilde v), \quad \mathcal{X}(0;x,\tilde v) = x, \quad \tilde{\mathcal{V}}(0;x,\tilde v) = \tilde v.$$

1. *For $\varepsilon > 0$ small enough, the flow $(\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)$ is periodic. More exactly for any $(x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\varepsilon > 0$ such that $\varepsilon \frac{|\tilde{v}| \, \|\nabla \omega_c\|_{L^\infty}}{\omega_0^2} < 1$, the characteristic $s \to (\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(s; x, \tilde{v})$ is periodic, with smallest period $S^\varepsilon(x, \tilde{v}) > 0$.*

2. *For any $(x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\varepsilon > 0$ such that $\varepsilon \frac{|\tilde{v}| \, \|\nabla \omega_c\|_{L^\infty}}{\omega_0^2} \leq \frac{1}{2}$ we have*

$$|\mathcal{X}^\varepsilon(s; x, \tilde{v}) - \mathcal{X}(s; x, \tilde{v})| = |\mathcal{X}^\varepsilon(s; x, \tilde{v}) - x| \leq \varepsilon \frac{4\pi}{\omega_0} |\tilde{v}|, \quad s \in \mathbb{R}$$

$$S^\varepsilon(x, \tilde{v}) \leq \frac{2\pi}{\omega_0}, \quad |S^\varepsilon(x, \tilde{v}) - S(x, \tilde{v})| \leq \varepsilon \|\nabla \omega_c\|_{L^\infty} \frac{8\pi^2 |\tilde{v}|}{\omega_0^3}, \quad S(x, \tilde{v}) = \frac{2\pi}{|\omega_c(x)|} \leq \frac{2\pi}{\omega_0}$$

and

$$|\tilde{\mathcal{V}}^\varepsilon(s; x, \tilde{v}) - \tilde{\mathcal{V}}(s; x, \tilde{v})| = |\tilde{\mathcal{V}}^\varepsilon(s; x, \tilde{v}) - \mathcal{R}(-s\omega_c(x))\tilde{v}| \leq \varepsilon \|\nabla \omega_c\|_{L^\infty} \frac{8\pi^2}{\omega_0^2} |\tilde{v}|^2, \quad s \in \left[0, \frac{2\pi}{\omega_0}\right].$$

3. *For any continuous function $u \in C(\mathbb{R}^2 \times \mathbb{R}^2)$ we define the averages along the flows of $b \cdot \nabla_{x,\tilde{v}}$, $b^\varepsilon \cdot \nabla_{x,\tilde{v}}$*

$$\langle u \rangle (x, \tilde{v}) = \frac{1}{S(x, \tilde{v})} \int_0^{S(x,\tilde{v})} u(\mathcal{X}(s; x, \tilde{v}), \tilde{\mathcal{V}}(s; x, \tilde{v})) \, \mathrm{d}s, \quad (x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$$

$$\langle u \rangle_\varepsilon (x, \tilde{v}) = \frac{1}{S^\varepsilon(x, \tilde{v})} \int_0^{S^\varepsilon(x,\tilde{v})} u(\mathcal{X}^\varepsilon(s; x, \tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s; x, \tilde{v})) \, \mathrm{d}s, \quad (x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

For any $R_x, R_{\tilde{v}} \in \mathbb{R}_+$ we have

$$\| \langle u \rangle \|_{L^\infty(B(R_x) \times B(R_{\tilde{v}}))} \leq \|u\|_{L^\infty(B(R_x) \times B(R_{\tilde{v}}))}$$

$$\| \langle u \rangle_\varepsilon \|_{L^\infty(B(R_x) \times B(R_{\tilde{v}}))} \leq \|u\|_{L^\infty(B(R_x^\varepsilon) \times B(R_{\tilde{v}}))}, \quad R_x^\varepsilon = R_x + 2\varepsilon R_{\tilde{v}}/\omega_0$$

where $B(R)$ stands for the closed ball of radius R in $\mathbb{R}^2$.

4. *If u is Lipschitz continuous, then for any $(x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\varepsilon > 0$ such that $\varepsilon \frac{|\tilde{v}| \, \|\nabla \omega_c\|_{L^\infty}}{\omega_0^2} \leq \frac{1}{2}$ we have*

$$\begin{aligned} \frac{|\langle u \rangle_\varepsilon (x, \tilde{v}) - \langle u \rangle (x, \tilde{v})|}{\varepsilon} &\leq \mathrm{Lip}(u) \frac{4\pi}{\omega_0} |\tilde{v}| \left[1 + \|\nabla \omega_c\|_{L^\infty} \frac{2\pi}{\omega_0} |\tilde{v}|\right] \\ &\quad + \sup_{|\tilde{v}'| = |\tilde{v}|} |u(x, \tilde{v}')| \|\nabla \omega_c\|_{L^\infty} \frac{8\pi}{\omega_0^2} |\tilde{v}|. \end{aligned}$$

5. *For any function $u \in C_c^1(\mathbb{R}^2 \times \mathbb{R}^2)$ we have the inequality*

$$\|u - \langle u \rangle\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \leq \frac{2\pi}{\omega_0} \|b \cdot \nabla_{x,\tilde{v}} u\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}.$$

6. *For any function $u \in C^1(\mathbb{R}^2 \times \mathbb{R}^2)$, we have $\langle u \rangle \in C^1(\mathbb{R}^2 \times \mathbb{R}^2)$ and*

$$\langle \nabla_x u \rangle = \nabla_x \langle u \rangle, \quad \tilde{v} \cdot \nabla_{\tilde{v}} \langle u \rangle = \langle \tilde{v} \cdot \nabla_{\tilde{v}} u \rangle, \quad {}^\perp \tilde{v} \cdot \nabla_{\tilde{v}} \langle u \rangle = 0.$$

Proof.
1. For any $s \in \mathbb{R}$ we have

$$\tilde{\mathcal{V}}^\varepsilon(s) = \mathcal{R}\left(-\int_0^s \omega_c(\mathcal{X}^\varepsilon(\sigma))\,\mathrm{d}\sigma\right)\tilde{v}$$

and therefore $|\tilde{\mathcal{V}}^\varepsilon(s)| = |\tilde{v}|$. By hypothesis (14) we know that ω_c has constant sign, and there is a unique $S^\varepsilon(x,\tilde{v}) > 0$ such that

$$\operatorname{sgn}\omega_c \int_0^{S^\varepsilon(x,\tilde{v})} \omega_c(\mathcal{X}^\varepsilon(\sigma;x,\tilde{v}))\,\mathrm{d}\sigma = \int_0^{S^\varepsilon(x,\tilde{v})} |\omega_c(\mathcal{X}^\varepsilon(\sigma;x,\tilde{v}))|\,\mathrm{d}\sigma = 2\pi.$$

Clearly we have $\tilde{\mathcal{V}}^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}) = \tilde{v}$ and we claim that, for small ε, we also have $\mathcal{X}^\varepsilon(S^\varepsilon(x,\tilde{v});x,\tilde{v}) = x$. Indeed, by using the invariance

$$\mathcal{X}^\varepsilon(S^\varepsilon) + \varepsilon\frac{{}^\perp\tilde{\mathcal{V}}^\varepsilon(S^\varepsilon)}{\omega_c(\mathcal{X}^\varepsilon(S^\varepsilon))} = x + \varepsilon\frac{{}^\perp\tilde{v}}{\omega_c(x)}$$

we deduce

$$|\mathcal{X}^\varepsilon(S^\varepsilon) - x| \le \varepsilon\frac{|\tilde{v}|}{\omega_0^2}|\omega_c(\mathcal{X}^\varepsilon(S^\varepsilon)) - \omega_c(x)| \le \frac{\varepsilon|\tilde{v}|\ \|\nabla\omega_c\|_{L^\infty}}{\omega_0^2}|\mathcal{X}^\varepsilon(S^\varepsilon) - x|.$$

Therefore, if $\frac{\varepsilon|\tilde{v}|\ \|\nabla\omega_c\|_{L^\infty}}{\omega_0^2} < 1$, we obtain $\mathcal{X}^\varepsilon(S^\varepsilon(x,\tilde{v})) = x$.
2. Notice that for any $s \in \mathbb{R}$

$$\begin{aligned}\det\left(I_2 - \varepsilon\frac{{}^\perp\tilde{\mathcal{V}}^\varepsilon(s)\otimes\nabla\omega_c(\mathcal{X}^\varepsilon(s))}{\omega_c^2(\mathcal{X}^\varepsilon(s))}\right) &= 1 - \varepsilon\frac{{}^\perp\tilde{\mathcal{V}}^\varepsilon(s)\cdot\nabla\omega_c(\mathcal{X}^\varepsilon(s))}{\omega_c^2(\mathcal{X}^\varepsilon(s))}\\ &\ge 1 - \varepsilon\frac{|\tilde{\mathcal{V}}^\varepsilon(s)|\ |\nabla\omega_c(\mathcal{X}^\varepsilon(s))|}{\omega_c^2(\mathcal{X}^\varepsilon(s))}\\ &\ge 1 - \frac{\varepsilon|\tilde{v}|\ \|\nabla\omega_c\|_{L^\infty}}{\omega_0^2} \ge \frac{1}{2}\end{aligned}$$

and therefore $A_x^\varepsilon(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))$ is well defined for any $s \in \mathbb{R}$. Moreover (17) implies

$$\begin{aligned}|A_x^\varepsilon(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))| &\le \frac{|\tilde{\mathcal{V}}^\varepsilon(s)|^2}{\omega_c^2(\mathcal{X}^\varepsilon(s))}|\nabla\omega_c(\mathcal{X}^\varepsilon(s))| + \varepsilon\frac{|\tilde{\mathcal{V}}^\varepsilon(s)|\ |\nabla\omega_c(\mathcal{X}^\varepsilon(s))|}{\omega_c^2(\mathcal{X}^\varepsilon(s))}|A_x^\varepsilon(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))|\\ &\le \frac{|\tilde{v}|^2}{\omega_0^2}\|\nabla\omega_c\|_{L^\infty} + \frac{1}{2}|A_x^\varepsilon(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))|\end{aligned}$$

and thus

$$\varepsilon|A_x^\varepsilon(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))| \le 2\frac{\varepsilon|\tilde{v}|^2\ \|\nabla\omega_c\|_{L^\infty}}{\omega_0^2} \le |\tilde{v}|.$$

By the previous point we know that $s \to \mathcal{X}^\varepsilon(s;x,\tilde{v})$ is $S^\varepsilon(x,\tilde{v})$ periodic, where

$$S^\varepsilon(x,\tilde{v}) \le \frac{1}{\omega_0}\int_0^{S^\varepsilon(x,\tilde{v})} |\omega_c(\mathcal{X}^\varepsilon(\sigma))|\,\mathrm{d}\sigma = \frac{2\pi}{\omega_0}.$$

Finally we write for any $s \in [0, S^\varepsilon(x,\tilde v)] \subset [0, 2\pi/\omega_0]$

$$|\mathcal{X}^\varepsilon(s;x,\tilde v) - x| = \left| \int_0^s \{\varepsilon \tilde{\mathcal{V}}^\varepsilon(\sigma) + \varepsilon^2 A_x^\varepsilon(\mathcal{X}^\varepsilon(\sigma), \tilde{\mathcal{V}}^\varepsilon(\sigma))\}\, \mathrm{d}\sigma \right| \\ \leq 2\varepsilon S^\varepsilon(x,\tilde v)|\tilde v| \leq \varepsilon \frac{4\pi}{\omega_0}|\tilde v|.$$

By periodicity we have

$$|\mathcal{X}^\varepsilon(s;x,\tilde v) - \mathcal{X}(s;x,\tilde v)| = |\mathcal{X}^\varepsilon(s;x,\tilde v) - x| \leq \varepsilon \frac{4\pi}{\omega_0}|\tilde v|, \quad s \in \mathbb{R}.$$

Let us compare now $S^\varepsilon(x,\tilde v)$ to $S(x,\tilde v) = 2\pi/|\omega_c(x)|$, which is the smallest period of the flow $(\mathcal{X}, \tilde{\mathcal{V}})$. Integrating over $[0, S^\varepsilon(x,\tilde v)]$ the inequality

$$|\omega_c(x)| - \|\nabla\omega_c\|_{L^\infty} \frac{\varepsilon 4\pi|\tilde v|}{\omega_0} \leq |\omega_c(\mathcal{X}^\varepsilon(\sigma)| \leq |\omega_c(x)| + \|\nabla\omega_c\|_{L^\infty} \frac{\varepsilon 4\pi|\tilde v|}{\omega_0}$$

one gets

$$S^\varepsilon(x,\tilde v)\left(|\omega_c(x)| - \|\nabla\omega_c\|_{L^\infty} \frac{\varepsilon 4\pi|\tilde v|}{\omega_0}\right) \leq 2\pi \leq S^\varepsilon(x,\tilde v)\left(|\omega_c(x)| + \|\nabla\omega_c\|_{L^\infty} \frac{\varepsilon 4\pi|\tilde v|}{\omega_0}\right)$$

implying that

$$\left| \frac{1}{S^\varepsilon(x,\tilde v)} - \frac{1}{S(x,\tilde v)} \right| \leq \varepsilon \|\nabla\omega_c\|_{L^\infty} \frac{2|\tilde v|}{\omega_0}$$

and

$$|S^\varepsilon(x,\tilde v) - S(x,\tilde v)| = S^\varepsilon(x,\tilde v)S(x,\tilde v)\left| \frac{1}{S^\varepsilon(x,\tilde v)} - \frac{1}{S(x,\tilde v)} \right| \leq \varepsilon \|\nabla\omega_c\|_{L^\infty} \frac{8\pi^2|\tilde v|}{\omega_0^3}.$$

Thanks to the inequality $\|\mathcal{R}(\theta) - \mathcal{R}(\theta')\| \leq |\theta - \theta'|$, $\theta, \theta' \in \mathbb{R}$, we obtain for any $s \in [0, 2\pi/\omega_0]$

$$|\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v) - \tilde{\mathcal{V}}(s;x,\tilde v)| = \left| \mathcal{R}\left(-\int_0^s \omega_c(\mathcal{X}^\varepsilon(\sigma))\, \mathrm{d}\sigma\right)\tilde v - \mathcal{R}\left(-\int_0^s \omega_c(\mathcal{X}(\sigma))\, \mathrm{d}\sigma\right)\tilde v \right| \\ \leq \int_0^s |\omega_c(\mathcal{X}^\varepsilon(\sigma)) - \omega_c(\mathcal{X}(\sigma))|\, \mathrm{d}\sigma\, |\tilde v| \\ \leq \varepsilon \|\nabla\omega_c\|_{L^\infty} \frac{8\pi^2}{\omega_0^2}|\tilde v|^2.$$

3. The first inequality is obvious, sice $\mathcal{X}(s;x,\tilde v) = x$ and $|\tilde{\mathcal{V}}(s;x,\tilde v)| = |\tilde v|$, for any $(s,x,\tilde v) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$. For the second one use $|\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v)| = |\tilde v|$ and notice that by the invariance of the center

$$\mathcal{X}^\varepsilon(s;x,\tilde v) + \varepsilon \frac{{}^\perp\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v)}{\omega_c(\mathcal{X}^\varepsilon(s;x,\tilde v))} = x + \varepsilon \frac{{}^\perp\tilde v}{\omega_c(x)}$$

we have

$$|\mathcal{X}^\varepsilon(s;x,\tilde v)| \leq |x| + |\mathcal{X}^\varepsilon(s;x,\tilde v) - x| \leq |x| + 2\varepsilon \frac{|\tilde v|}{\omega_0}, \quad (s,x,\tilde v) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

4. It is a direct consequence of the previous statements. We have

$$
\begin{aligned}
&|\langle u\rangle_\varepsilon(x,\tilde v)-\langle u\rangle(x,\tilde v)|\le\frac{1}{S^\varepsilon(x,\tilde v)}\int_0^{S^\varepsilon(x,\tilde v)}|u(\mathcal{X}^\varepsilon(s),\tilde{\mathcal{V}}^\varepsilon(s))-u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s\\
&+\left|\frac{1}{S^\varepsilon(x,\tilde v)}-\frac{1}{S(x,\tilde v)}\right|\int_0^{S^\varepsilon(x,\tilde v)}|u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s+\frac{1}{S(x,\tilde v)}\left|\int_{S(x,\tilde v)}^{S^\varepsilon(x,\tilde v)}|u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))|\,\mathrm{d}s\right|\\
&\le\mathrm{Lip}(u)\sup_{0\le s\le 2\pi/\omega_0}\Big(|\mathcal{X}^\varepsilon(s)-\mathcal{X}(s)|+|\tilde{\mathcal{V}}^\varepsilon(s)-\tilde{\mathcal{V}}(s)|\Big)\\
&+\left[\left|\frac{1}{S^\varepsilon(x,\tilde v)}-\frac{1}{S(x,\tilde v)}\right|S^\varepsilon(x,\tilde v)+\frac{|S^\varepsilon(x,\tilde v)-S(x,\tilde v)|}{S(x,\tilde v)}\right]\sup_{|\tilde v'|=|\tilde v|}|u(x,\tilde v')|\\
&\le\varepsilon\,\mathrm{Lip}(u)\frac{4\pi}{\omega_0}|\tilde v|\left[1+\|\nabla\omega_c\|_{L^\infty}\frac{2\pi}{\omega_0}|\tilde v|\right]+\varepsilon\sup_{|\tilde v'|=|\tilde v|}|u(x,\tilde v')|\|\nabla\omega_c\|_{L^\infty}\frac{8\pi}{\omega_0^2}|\tilde v|.
\end{aligned}
$$

5. We can write for any $(x,\tilde v)\in\mathbb{R}^2\times\mathbb{R}^2$

$$
\begin{aligned}
\langle u\rangle(x,\tilde v)-u(x,\tilde v)&=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}[u(\mathcal{X}(s;x,\tilde v),\tilde{\mathcal{V}}(s;x,\tilde v))-u(x,\tilde v)]\,\mathrm{d}s\\
&=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}\int_0^s\frac{\mathrm{d}}{\mathrm{d}\sigma}u(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma\mathrm{d}s\\
&=\frac{1}{S(x,\tilde v)}\int_0^{S(x,\tilde v)}\int_0^s(b\cdot\nabla u)(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma\mathrm{d}s\\
&=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\theta}{\omega_c(x)}}(b\cdot\nabla u)(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))\,\mathrm{d}\sigma\mathrm{d}\theta.
\end{aligned}
$$

As u has compact support, its average $\langle u\rangle$ is also compactly supported and therefore $u-\langle u\rangle$ belongs to $L^2(\mathbb{R}^2\times\mathbb{R}^2)$. By the previous computation we have

$$
|\langle u\rangle(x,\tilde v)-u(x,\tilde v)|\le\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{2\pi}{\omega_0}}|(b\cdot\nabla u)(\mathcal{X}(\sigma;x,\tilde v),\tilde{\mathcal{V}}(\sigma;x,\tilde v))|\,\mathrm{d}\sigma\mathrm{d}\theta
$$

and therefore

$$
\|\langle u\rangle-u\|_{L^2}\le\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{2\pi}{\omega_0}}\|(b\cdot\nabla u)(\mathcal{X}(\sigma;\cdot,\cdot),\tilde{\mathcal{V}}(\sigma;\cdot,\cdot))\|_{L^2}\,\mathrm{d}\sigma\mathrm{d}\theta=\frac{2\pi}{\omega_0}\|b\cdot\nabla u\|_{L^2}.
$$

The above Poincaré inequality still holds true for any $u\in L^2(\mathbb{R}^2\times\mathbb{R}^2)$ such that $b\cdot\nabla u\in L^2(\mathbb{R}^2\times\mathbb{R}^2)$, see [3] for details about the average of L^p functions, $1\le p\le\infty$.
6. Observe that for any $(x,\tilde v)\in\mathbb{R}^2\times\mathbb{R}^2$ we have

$$
\langle u\rangle(x,\tilde v)=\frac{1}{2\pi}\int_0^{2\pi}u(x,\mathcal{R}(-\theta)\tilde v)\,\mathrm{d}\theta
$$

and therefore $\langle u\rangle\in C^1(\mathbb{R}^2\times\mathbb{R}^2)$. Taking the derivatives with respect to $x,\tilde v$ under the integral sign, one gets

$$
(\nabla_x\langle u\rangle)(x,\tilde v)=\frac{1}{2\pi}\int_0^{2\pi}(\nabla_x u)(x,\mathcal{R}(-\theta)\tilde v)\,\mathrm{d}\theta=\langle\nabla_x u\rangle(x,\tilde v)
$$

and

$$\tilde{v}\cdot(\nabla_{\tilde{v}}\langle u\rangle)(x,\tilde{v})=\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(-\theta)\tilde{v}\cdot(\nabla_{\tilde{v}}u)(x,\mathcal{R}(-\theta)\tilde{v})\,\mathrm{d}\theta=\langle\tilde{v}\cdot\nabla_{\tilde{v}}u\rangle\,(x,\tilde{v}).$$

By the definition, the average $\langle u\rangle$ is constant along the flow of $b\cdot\nabla_{x,\tilde{v}}$, and therefore belongs to the kernel of $b\cdot\nabla_{x,\tilde{v}}$, that is ${}^{\perp}\tilde{v}\cdot\nabla_{\tilde{v}}\langle u\rangle=0$. □

It is easily seen that the periods S, S^{ε} are invariant along the flows of $b\cdot\nabla_{x,\tilde{v}}$, $b^{\varepsilon}\cdot\nabla_{x,\tilde{v}}$ as well as the averages $\langle u\rangle, \langle u\rangle_{\varepsilon}$. If u is a C^1 function, we have by periodicity

$$\langle b\cdot\nabla_{x,\tilde{v}}u\rangle\,(x,\tilde{v})=\frac{1}{S(x,\tilde{v})}\int_0^{S(x,\tilde{v})}\frac{\mathrm{d}}{\mathrm{d}s}u(\mathcal{X}(s),\tilde{\mathcal{V}}(s))\,\mathrm{d}s=0,\quad(x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2$$

and similarly $\langle b^{\varepsilon}\cdot\nabla_{x,\tilde{v}}u\rangle_{\varepsilon}=0$.

In the sequel we derive formally the limit model of (16), as ε goes to 0. The rigorous arguments will be discussed in Section 5. As we are interested by a second order model, we keep all the first order corrections. When ε becomes small, we expect that $b^{\varepsilon}(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}\tilde{f}^{\varepsilon}$ will vanish, saying that $\tilde{f}^{\varepsilon}$ behaves like its average $\left\langle\tilde{f}^{\varepsilon}\right\rangle_{\varepsilon}$ along the flow $(\mathcal{X}^{\varepsilon},\tilde{\mathcal{V}}^{\varepsilon})$. Taking the average of (16) along the flow $(\mathcal{X}^{\varepsilon},\tilde{\mathcal{V}}^{\varepsilon})$ we obtain

$$\partial_t\left\langle\tilde{f}^{\varepsilon}(t)\right\rangle_{\varepsilon}+\varepsilon\left\langle a^{\varepsilon}[\tilde{f}^{\varepsilon}(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}^{\varepsilon}(t)\right\rangle_{\varepsilon}=0,\quad(t,x,\tilde{v})\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2.$$

As the difference between the averages along $b^{\varepsilon}(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}$ and $b(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}$ is of order ε cf. Proposition 3.1, up to a second order term, we can replace $\varepsilon\left\langle a^{\varepsilon}[\tilde{f}^{\varepsilon}(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}^{\varepsilon}(t)\right\rangle_{\varepsilon}$ by $\varepsilon\left\langle a[\tilde{f}(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}(t)\right\rangle$ where

$$\begin{aligned}a[\tilde{f}]\cdot\nabla_{x,\tilde{v}}=&\left(\frac{{}^{\perp}E[\tilde{f}]}{B}-A_x\right)\cdot\nabla_x\\&+\left[\frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\frac{{}^{\perp}(x-x')}{|x-x'|^2}\otimes j[\tilde{f}](x')\,\mathrm{d}x'-\partial_x\left(\frac{{}^{\perp}E[\tilde{f}]}{B}\right)\tilde{v}\right]\cdot\nabla_{\tilde{v}}\end{aligned}$$

and $A_x(x,\tilde{v})=\frac{{}^{\perp}\tilde{v}\otimes\tilde{v}}{\omega_c^2(x)}\nabla\omega_c$. More exactly, we expect that solving

$$\partial_t\tilde{f}+\varepsilon\left\langle a[\tilde{f}(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}(t)\right\rangle=0,\quad b\cdot\nabla_{x,\tilde{v}}\tilde{f}=0,\quad(t,x,\tilde{v})\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2$$

for a suitable (well prepared) initial condition (see Section 5), will provide a second order approximation for (1), (2). Although the above solution depends on ε, we use the notation $\tilde{f}$, for saying that it is an approximation of $\tilde{f}^{\varepsilon}$, when ε becomes small. It remains to compute the average of $a[\tilde{f}(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}(t)$, where the presence density $\tilde{f}(t)$ satisfies the constraint $b\cdot\nabla_{x,\tilde{v}}\tilde{f}(t)=0$, that is, $\tilde{f}(t)$ depends only on x and $|\tilde{v}|$. By the definition of the average operator along $b\cdot\nabla_{x,\tilde{v}}$, it is easily seen that $\left\langle{}^{\perp}\tilde{v}\otimes\tilde{v}\right\rangle=\frac{|\tilde{v}|^2}{2}\mathcal{R}(-\pi/2)$ and therefore

$$\begin{aligned}\varepsilon\left\langle\left(\frac{{}^{\perp}E[\tilde{f}]}{B}-A_x\right)\cdot\nabla_x\tilde{f}\right\rangle&=\varepsilon\frac{{}^{\perp}E[\tilde{f}]}{B(x)}\cdot\nabla_x\tilde{f}-\varepsilon\,\langle A_x\rangle\cdot\nabla_x\tilde{f}\\&=\varepsilon\left(\frac{{}^{\perp}E[\tilde{f}]}{B(x)}-\frac{|\tilde{v}|^2}{2\omega_c^2(x)}\,{}^{\perp}\nabla\omega_c\right)\cdot\nabla_x\tilde{f}.\end{aligned}$$

We recognize here the electric cross field drift $\varepsilon\frac{{}^{\perp}E[\tilde{f}]}{B} = \frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon}$ and the magnetic gradient drift

$$-\varepsilon\frac{|\tilde{v}|^2}{2\omega_c^2(x)}\,{}^{\perp}\nabla\omega_c = -\frac{m|\tilde{v}|^2}{2qB^\varepsilon}\frac{{}^{\perp}\nabla B^\varepsilon}{B^\varepsilon}.$$

As $\left\langle \nabla_{\tilde{v}}\tilde{f}\right\rangle = 0$, we have

$$\left\langle \frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x \int_{\mathbb{R}^2}\frac{{}^{\perp}(x-x')}{|x-x'|^2}\otimes j[\tilde{f}](x')\,\mathrm{d}x'\cdot\nabla_{\tilde{v}}\tilde{f}\right\rangle = 0.$$

It remains to compute the average $-\varepsilon\left\langle \partial_x\left(\frac{{}^{\perp}E[\tilde{f}]}{B}\right)\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}\right\rangle$. As $\tilde{f}$ satisfies the constraint $b\cdot\nabla_{x,\tilde{v}}\tilde{f}=0$, we have $\nabla_{\tilde{v}}\tilde{f} = \frac{\nabla_{\tilde{v}}\tilde{f}\cdot\tilde{v}}{|\tilde{v}|^2}\tilde{v}$ and therefore

$$\begin{aligned}
-\varepsilon\left\langle \partial_x\left(\frac{{}^{\perp}E[\tilde{f}]}{B}\right)\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}\right\rangle &= \varepsilon\left\langle \partial_x\left(\frac{E[\tilde{f}]}{B}\right)\tilde{v}\cdot\,{}^{\perp}\tilde{v}\right\rangle\frac{\nabla_{\tilde{v}}\tilde{f}\cdot\tilde{v}}{|\tilde{v}|^2}\\
&= \varepsilon\partial_x\left(\frac{E[\tilde{f}]}{B}\right) : \left\langle {}^{\perp}\tilde{v}\otimes\tilde{v}\right\rangle\frac{\nabla_{\tilde{v}}\tilde{f}\cdot\tilde{v}}{|\tilde{v}|^2}\\
&= \frac{\varepsilon}{2}\left(\frac{{}^{\perp}E[\tilde{f}]}{B}\cdot\frac{\nabla B}{B}\right)\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}.
\end{aligned}$$

Finally, the presence density $\tilde{f}$ satisfies

$$\partial_t\tilde{f} + \left(\frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon(x)} - \frac{m|\tilde{v}|^2}{2qB^\varepsilon}\frac{{}^{\perp}\nabla B^\varepsilon}{B^\varepsilon(x)}\right)\cdot\nabla_x\tilde{f} + \frac{1}{2}\left(\frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon(x)}\cdot\frac{\nabla B^\varepsilon}{B^\varepsilon(x)}\right)\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f} = 0.$$

4 Well posedness of the limit model

In this section we focus on the existence, uniqueness and other properties of the limit model

$$\partial_t\tilde{f} + \left(\frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon(x)} - \frac{m|\tilde{v}|^2}{2qB^\varepsilon}\frac{{}^{\perp}\nabla B^\varepsilon}{B^\varepsilon(x)}\right)\cdot\nabla_x\tilde{f} + \frac{1}{2}\left(\frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon(x)}\cdot\frac{\nabla B^\varepsilon}{B^\varepsilon(x)}\right)\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f} = 0 \tag{18}$$

where $E[\tilde{f}]$ stands for the Poisson electric field

$$E[\tilde{f}(t)](x) = \frac{q}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{f}(t,x',v')\frac{x-x'}{|x-x'|^2}\,\mathrm{d}v'\mathrm{d}x',\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2. \tag{19}$$

We supplement our model by the initial condition

$$\tilde{f}(0,x,\tilde{v}) = \tilde{f}_{\mathrm{in}}(x,\tilde{v}),\quad (x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2 \tag{20}$$

where $\tilde{f}_{\mathrm{in}}$ is a smooth compactly supported presence density. The parameter $\varepsilon>0$ is fixed and B^ε is a smooth external magnetic field.

Proof. (of Theorem 1.1)
We follow the same arguments as in the well posedness proof for the Vlasov-Poisson problem with external magnetic field. It is enough to establish the existence and uniqueness on any time interval $[0,T], T \in \mathbb{R}_+$. We construct a map acting on electric fields. Given a C^1 electric field E on $[0,T]\times\mathbb{R}^2$ such that $E, \partial_x E$ are bounded, $\mathrm{div}_x {}^\perp E = 0$, we consider the solution by characteristics of the Vlasov problem (18), (19), (20) on $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$, corresponding to the electric field E

$$\tilde{f}^\varepsilon(t,x,\tilde{v}) = \tilde{f}_{\mathrm{in}}(X^E(0;t,x,\tilde{v}), \tilde{V}^E(0;t,x,\tilde{v})), \quad (t,x,\tilde{v}) \in [0,T]\times\mathbb{R}^2\times\mathbb{R}^2$$

where this time the characteristics $(X^E(s;t,x,\tilde{v}), V^E(s;t,x,\tilde{v}))$ are given by

$$\frac{\mathrm{d}X^E}{\mathrm{d}s} = \frac{{}^\perp E(s,X^E(s))}{B^\varepsilon(X^E(s))} - \frac{m|\tilde{V}^E(s)|^2}{2qB^\varepsilon(X^E(s))}\frac{{}^\perp\nabla B^\varepsilon(X^E(s))}{B^\varepsilon(X^E(s))} \tag{21}$$

$$\frac{\mathrm{d}\tilde{V}^E}{\mathrm{d}s} = \frac{1}{2}\left(\frac{{}^\perp E(s,X^E(s))}{B^\varepsilon(X^E(s))}\cdot\frac{\nabla B^\varepsilon(X^E(s))}{B^\varepsilon(X^E(s))}\right)\tilde{V}^E(s) \tag{22}$$

$$X^E(s;t,x,\tilde{v}) = x, \quad V^E(s;t,x,\tilde{v}) = \tilde{v}, \quad (s,t,x,\tilde{v}) \in [0,T]\times[0,T]\times\mathbb{R}^2\times\mathbb{R}^2.$$

Notice that the magnetic moment $\mu^\varepsilon(x,\tilde{v}) = \frac{m|\tilde{v}|^2}{2qB^\varepsilon(x)}$ is left invariant along these characteristics :

$$\frac{\mathrm{d}\mu^\varepsilon}{\mathrm{d}s} = -\frac{m|\tilde{V}^E(s)|^2}{2qB^\varepsilon(X^E(s))}\frac{\nabla B^\varepsilon(X^E(s))}{B^\varepsilon(X^E(s))}\cdot\left(\frac{{}^\perp E(s,X^E(s))}{B^\varepsilon(X^E(s))} - \frac{m|\tilde{V}^E(s)|^2}{2qB^\varepsilon(X^E(s))}\frac{{}^\perp\nabla B^\varepsilon(X^E(s))}{B^\varepsilon(X^E(s))}\right)$$
$$+\frac{m|\tilde{V}^E(s)|^2}{2qB^\varepsilon(X^E(s))}\left(\frac{{}^\perp E(s,X^E(s))}{B^\varepsilon(X^E(s))}\cdot\frac{\nabla B^\varepsilon(X^E(s))}{B^\varepsilon(X^E(s))}\right) = 0$$

and therefore the velocity remains bounded, independently with respect to the electric field

$$|\tilde{V}^E(s)| \le \left(\frac{\|B^\varepsilon\|_{L^\infty}}{B_0^\varepsilon}\right)^{1/2}|\tilde{v}| = \left(\frac{\|B\|_{L^\infty}}{B_0}\right)^{1/2}|\tilde{v}|, \quad s\in\mathbb{R}.$$

We deduce that (21) also writes

$$\frac{\mathrm{d}X^E}{\mathrm{d}s} = \frac{{}^\perp E(s,X^E(s))}{B^\varepsilon(X^E(s))} - \frac{m|\tilde{v}|^2}{2qB^\varepsilon(x)}\frac{{}^\perp\nabla B^\varepsilon(X^E(s))}{B^\varepsilon(X^E(s))} \tag{23}$$

whose right hand side remains bounded for $s\in[0,T]$ and thus X^E exists globally on $[0,T]$ and has C^1 regularity in all the arguments. Once that X^E is determined, $\tilde{V}^E$ comes easily by solving (22), which is linear with respect to $\tilde{V}^E$, and therefore $\tilde{V}^E$ exists also globally on $[0,T]$, and has C^1 regularity in all the arguments. As before, we consider the map $\tilde{\mathcal{F}}$, whose fixed point gives the solution of the limit model (18), (19), (20)

$$E \to \tilde{\mathcal{F}}(E) := \frac{q}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{f}^E(t,x',\tilde{v}')\frac{x-x'}{|x-x'|^2}\,\mathrm{d}v'\mathrm{d}x'.$$

We are looking for a set $\tilde{X}_T$ of smooth electric fields, which is left invariant by the map $\tilde{\mathcal{F}}$ and for an estimate like

$$\|\tilde{\mathcal{F}}E(t) - \tilde{\mathcal{F}}E'(t)\|_{L^\infty} \le \varepsilon\tilde{C}_T\int_0^t\|E(s)-E'(s)\|_{L^\infty}\,\mathrm{d}s, \quad E,E'\in\tilde{X}_T,\ t\in[0,T], \varepsilon\in]0,1] \tag{24}$$

for some constant $\tilde{C}_T$, not depending on E, E'. Assume that

$$\mathrm{supp}\tilde{f}_{\mathrm{in}} \subset \{(x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ |x| \leq \tilde{R}_x^{\mathrm{in}} \text{ and } |\tilde{v}| \leq \tilde{R}_{\tilde{v}}^{\mathrm{in}}\}.$$

By the conservation of the magnetic moment and of the particle number (thanks to the condition $\mathrm{div}_x {}^{\perp}E = 0$), it is easily seen that at any time $t \in [0,T]$ we have

$$\mathrm{supp}\tilde{f}^E(t) \subset \{(x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ |\tilde{v}| \leq \tilde{R}_{\tilde{v}} := \tilde{R}_{\tilde{v}}^{\mathrm{in}}(\|B\|_{L^\infty}/B_0)^{1/2}\}$$

implying that

$$\|\rho[\tilde{f}(t)]\|_{L^\infty} \leq |q| \ \|\tilde{f}_{\mathrm{in}}\|_{L^\infty} \pi (\tilde{R}_{\tilde{v}}^{\mathrm{in}})^2 \frac{\|B\|_{L^\infty}}{B_0}.$$

Therefore, as in (57) we obtain the L^∞ bound for the electric field

$$\begin{aligned}\|E[\tilde{f}(t)]\|_{L^\infty(\mathbb{R}^2)} &\leq \frac{1}{\varepsilon_0}\|\rho[\tilde{f}(t)]\|_{L^\infty(\mathbb{R}^2)}^{1/2}\left(\frac{|q|}{2\pi}\|\tilde{f}_{\mathrm{in}}\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\right)^{1/2} \\ &\leq \alpha := \frac{|q|}{\sqrt{2}\varepsilon_0}\|\tilde{f}_{\mathrm{in}}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{1/2} R_{\tilde{v}}^{\mathrm{in}}\left(\frac{\|B\|_{L^\infty}}{B_0}\right)^{1/2}\|\tilde{f}_{\mathrm{in}}\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}^{1/2}, \quad t \in [0,T].\end{aligned}$$

Integrating in time (23) leads immediately to

$$\mathrm{supp}\tilde{f}(t) \subset \{(x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ |x| \leq \tilde{R}_x(t) \ \text{ and } |\tilde{v}| \leq \tilde{R}_{\tilde{v}}\}$$

where

$$\tilde{R}_x(t) = \tilde{R}_x^{\mathrm{in}} + \varepsilon t\left[\frac{\alpha}{B_0} + \frac{m(\tilde{R}_{\tilde{v}}^{\mathrm{in}})^2}{2|q|B_0}\frac{\|B\|_{L^\infty}}{B_0}\frac{\|\nabla B\|_{L^\infty}}{B_0}\right], \quad t \in [0,T].$$

Taking the derivative with respect to x in (23), (22) we deduce that there is a constant $C_5(m,\varepsilon_0,q,T,\|B\|_{W^{2,\infty}(\mathbb{R}^2)},B_0)$ such that for any $(x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2, |\tilde{v}| \leq \tilde{R}_{\tilde{v}}$

$$|\partial_x X^E(0;t,x,\tilde{v})| + |\partial_x \tilde{V}^E(0;t,x,\tilde{v})| \leq C_5 \exp\left(\varepsilon\int_0^t \|\partial_x E(s)\|_{L^\infty}\,\mathrm{d}s\right), \quad t \in [0,T]. \tag{25}$$

As before we write

$$\begin{aligned}\nabla_x\rho[\tilde{f}(t)] &= q\nabla_x\int_{\mathbb{R}^2}\tilde{f}_{\mathrm{in}}(X^E(0;t,x,\tilde{v}),\tilde{V}^E(0;t,x,\tilde{v}))\,\mathrm{d}\tilde{v} \\ &= q\int_{\mathbb{R}^2}\mathbf{1}_{\{|\tilde{v}|\leq\tilde{R}_{\tilde{v}}\}}{}^t\partial_x X^E(0;t,x,\tilde{v})(\nabla_X\tilde{f}_{\mathrm{in}})(X^E(0;t,x,\tilde{v}),\tilde{V}^E(0;t,x,\tilde{v}))\,\mathrm{d}\tilde{v} \\ &\quad + q\int_{\mathbb{R}^2}\mathbf{1}_{\{|\tilde{v}|\leq\tilde{R}_{\tilde{v}}\}}{}^t\partial_x V^E(0;t,x,\tilde{v})(\nabla_V\tilde{f}_{\mathrm{in}})(X^E(0;t,x,\tilde{v}),\tilde{V}^E(0;t,x,\tilde{v}))\,\mathrm{d}\tilde{v}\end{aligned}$$

and by coming back to (25), we obtain

$$\|\nabla_x\rho[\tilde{f}(t)]\|_{L^\infty} \leq C_6\exp\left(\varepsilon\int_0^t\|\partial_x E(s)\|_{L^\infty}\,\mathrm{d}s\right), \quad t \in [0,T]$$

for some constant $C_6(m,\varepsilon_0,q,T,\|B\|_{W^{2,\infty}(\mathbb{R}^2)},B_0) \geq 1$ and thus

$$\ln^+\|\nabla_x\rho[\tilde{f}(t)]\|_{L^\infty} \leq \ln C_6 + \varepsilon\int_0^t\|\partial_x E(s)\|_{L^\infty}\,\mathrm{d}s, \quad t \in [0,T], \ \varepsilon \in]0,1].$$

Combining with (62) yields

$$\|\partial_x E[\tilde{f}(t)]\|_{L^\infty} \leq C_7 \left(1 + \varepsilon \int_0^t \|\partial_x E(s)\|_{L^\infty} \, \mathrm{d}s\right), \;\; t \in [0,T], \; \varepsilon \in]0,1]$$

for some constant $C_7(m, \varepsilon_0, q, T, \|B\|_{W^{2,\infty}(\mathbb{R}^2)}, B_0)$. The previous computations allow us to consider the set

$$\begin{aligned}\tilde{X}_T = \{&E \in C^1([0,T] \times \mathbb{R}^2) \; : \\ &\mathrm{div}_x{}^\perp E = 0, \; \|E(t)\|_{L^\infty} \leq \alpha, \; \|\partial_x E(t)\|_{L^\infty} \leq C_7 \exp(\varepsilon t C_7), \; t \in [0,T]\}.\end{aligned}$$

It remains to establish (24). Let us consider $E, E' \in \tilde{X}_T$ and denote by $\tilde{f}^E, \tilde{f}^{E'}$ the solutions by characteristics of (18), (20) corresponding to the electric fields E, E' respectively. Recall that at any time $t \in [0,T]$, the densities $\tilde{f}^E(t), \tilde{f}^{E'}(t)$ are supported in $\{(x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 \; : \; |x| \leq \tilde{R}_x(t) \text{ and } |\tilde{v}| \leq \tilde{R}_{\tilde{v}}\}$ and therefore at any time $t \in [0,T]$, the charge densities $\rho[\tilde{f}^E(t)], \rho[\tilde{f}^{E'}(t)]$ are supported in $\{x \in \mathbb{R}^2 : |x| \leq \tilde{R}_x(t)\}$. It is easily seen by (57) that

$$\begin{aligned}\|\tilde{\mathcal{F}}(E)(t) - \tilde{\mathcal{F}}(E')(t)\|_{L^\infty(\mathbb{R}^2)} &= \|E[\tilde{f}^E(t)] - E[\tilde{f}^{E'}(t)]\|_{L^\infty(\mathbb{R}^2)} \\ &\leq \frac{1}{\varepsilon_0} \|\rho[\tilde{f}^E(t)] - \rho[\tilde{f}^{E'}(t)]\|_{L^\infty(\mathbb{R}^2)}^{1/2} \left(\frac{1}{2\pi} \|\rho[\tilde{f}^E(t)] - \rho[\tilde{f}^{E'}(t)]\|_{L^1(\mathbb{R}^2)}\right)^{1/2} \\ &\leq \frac{1}{\varepsilon_0} \frac{|q|}{\sqrt{2}} \tilde{R}_x(t) \left\| \mathbf{1}_{\{|x| \leq \tilde{R}_x(t)\}} \int_{\mathbb{R}^2} \mathbf{1}_{\{|\tilde{v}| \leq \tilde{R}_{\tilde{v}}\}} (\tilde{f}^E - \tilde{f}^{E'})(t, \cdot, \tilde{v}) \, \mathrm{d}\tilde{v} \right\|_{L^\infty(\mathbb{R}^2)}.\end{aligned}$$

By standard computations, one gets for any $(t,x,\tilde{v}) \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ such that $|x| \leq \tilde{R}_x(t)$ and $|\tilde{v}| \leq \tilde{R}_{\tilde{v}}$

$$\begin{aligned}|(\tilde{f}^E - \tilde{f}^{E'})(t,x,\tilde{v})| &= \left|\tilde{f}_{\mathrm{in}}(X^E(0;t,x,\tilde{v}), \tilde{V}^E(0;t,x,\tilde{v})) - \tilde{f}_{\mathrm{in}}(X^{E'}(0;t,x,\tilde{v}), \tilde{V}^{E'}(0;t,x,\tilde{v}))\right| \\ &\leq \mathrm{Lip}(f_{\mathrm{in}})[|X^E - X^{E'}|(0;t,x,\tilde{v}) + |\tilde{V}^E - \tilde{V}^{E'}|(0;t,x,\tilde{v})] \\ &\leq \varepsilon \, \mathrm{Lip}(f_{\mathrm{in}}) C_8 \int_0^t \|E(s) - E'(s)\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{d}s \; \exp(\varepsilon t C_8)\end{aligned}$$

for some constant $C_8(m, \varepsilon_0, q, T, \|B\|_{W^{2,\infty}(\mathbb{R}^2)}, B_0)$ and (24) follows. □

Remark 4.1

1. *The total number of particles is conserved. Indeed, as* $\mathrm{div}_x^\perp E = 0$*, we have*

$$\mathrm{div}_x \left(\frac{{}^\perp E}{B^\varepsilon} - \frac{m|\tilde{v}|^2}{2qB^\varepsilon} \frac{{}^\perp \nabla B^\varepsilon}{B^\varepsilon}\right) + \mathrm{div}_{\tilde{v}} \left[\frac{1}{2} \left(\frac{{}^\perp E}{B^\varepsilon} \cdot \frac{\nabla B^\varepsilon}{B^\varepsilon}\right) \tilde{v}\right] = 0$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{f}(t,x,\tilde{v}) \, \mathrm{d}\tilde{v} \mathrm{d}x = 0, \;\; t \in \mathbb{R}_+.$$

2. *The continuity equation writes*

$$\partial_t \int_{\mathbb{R}^2} \tilde{f}(t,x,\tilde{v}) \, \mathrm{d}\tilde{v} + \mathrm{div}_x \int_{\mathbb{R}^2} \left(\frac{{}^\perp E}{B^\varepsilon} - \frac{m|\tilde{v}|^2}{2qB^\varepsilon} \frac{{}^\perp \nabla B^\varepsilon}{B^\varepsilon}\right) \tilde{f} \, \mathrm{d}\tilde{v} = 0$$

and, when the magnetic field is uniform, becomes

$$\partial_t \rho[\tilde{f}(t)] + \frac{{}^{\perp}E[\tilde{f}(t)]}{B^\varepsilon} \cdot \nabla_x \rho[\tilde{f}(t)] = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

that is, the vorticity formulation of the two dimensional Euler equations, with velocity ${}^{\perp}E/B^\varepsilon$ *and vorticity* $\mathrm{div}_x\, {}^{\perp}(\frac{{}^{\perp}E}{B^\varepsilon}) = -\frac{\rho[\tilde{f}]}{\varepsilon_0 B^\varepsilon}$ *see also [11, 4, 22].*

3. *The balance of the total momentum is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{f}(t,x,\tilde{v}) m\tilde{v}\, \mathrm{d}\tilde{v}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} m\, \mathrm{div}_x \left(\frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon(x)}\right) \tilde{f}(t,x,\tilde{v})\tilde{v}\, \mathrm{d}\tilde{v}\mathrm{d}x = 0.$$

In particular, the total momentum is conserved when the magnetic field is uniform.

4. *As the magnetic moment* μ^ε *is left invariant along the characteristics of* (18)

$$\left(\frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon} - \mu^\varepsilon \frac{{}^{\perp}\nabla B^\varepsilon}{B^\varepsilon}\right) \cdot \nabla_x \mu^\varepsilon + \frac{1}{2}\left(\frac{{}^{\perp}E[\tilde{f}]}{B^\varepsilon} \cdot \frac{\nabla B^\varepsilon}{B^\varepsilon}\right) \tilde{v} \cdot \nabla_{\tilde{v}} \mu^\varepsilon = 0$$

we deduce the conservation of the total magnetic moment

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{m|\tilde{v}|^2}{2qB^\varepsilon} \tilde{f}(t,x,\tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}x = 0, \quad t \in \mathbb{R}_+.$$

5. *The total, i.e., kinetic and electric energy is conserved. Indeed, by using the fundamental solution of the Poisson equation in* $\mathbb{R}^2$*, we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(x-x')\rho[\tilde{f}(t)](x)\rho[\tilde{f}(t)](x')\, \mathrm{d}x'\mathrm{d}x \\
&\quad = \frac{1}{\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(x-x')\rho[\tilde{f}(t)](x')\partial_t\rho[\tilde{f}(t)](x)\, \mathrm{d}x'\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^2} \Phi[\tilde{f}(t)](x)\partial_t\rho[\tilde{f}(t)](x)\, \mathrm{d}x \\
&\quad = -\int_{\mathbb{R}^2} \Phi[\tilde{f}(t)](x)\mathrm{div}_x \int_{\mathbb{R}^2} q\tilde{f}(t,x,\tilde{v}) \left(\frac{{}^{\perp}E[\tilde{f}(t)]}{B^\varepsilon} - \frac{m|\tilde{v}|^2}{2qB^\varepsilon}\frac{{}^{\perp}\nabla B^\varepsilon}{B^\varepsilon}\right) \mathrm{d}\tilde{v}\, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(\frac{E[\tilde{f}(t)]}{B^\varepsilon} \cdot \frac{{}^{\perp}\nabla B^\varepsilon}{B^\varepsilon}\right) \frac{m|\tilde{v}|^2}{2} \tilde{f}\, \mathrm{d}\tilde{v}\mathrm{d}x \\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{m|\tilde{v}|^2}{2} \tilde{f}(t,x,\tilde{v})\, \mathrm{d}\tilde{v}\mathrm{d}x
\end{aligned}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{m|\tilde{v}|^2}{2}\tilde{f}\, \mathrm{d}\tilde{v}\mathrm{d}x - \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho[\tilde{f}](x)\rho[\tilde{f}](x')\ln|x-x'|\, \mathrm{d}x'\mathrm{d}x\right] = 0, \quad t \in \mathbb{R}_+.$$

6. *The model* (18), (19), (20) *is invariant under rotation in the velocity space. More exactly, if* $\tilde{f}$ *solves* (18), (19), (20), *then* $\tilde{f}_\theta(t,x,\tilde{V}) = \tilde{f}(t,x,\tilde{v} = \mathcal{R}(\theta)\tilde{V})$ *solves* (18), (19) *as well, together with the initial condition* $\tilde{f}_\theta(0,x,\tilde{V}) = f_{\rm in}(x,\tilde{v} = \mathcal{R}(\theta)\tilde{V})$. *Indeed, we have* $\tilde{V}\cdot\nabla_{\tilde{V}}\tilde{f}_\theta = \tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}|_{\tilde{v}=\mathcal{R}(\theta)\tilde{V}}$ *and* $E[\tilde{f}_\theta(t)] = E[\tilde{f}(t)]$, *implying that*

$$\partial_t\tilde{f}_\theta + \left(\frac{{}^\perp E[\tilde{f}_\theta]}{B^\varepsilon} - \frac{m|\tilde{V}|^2}{2qB^\varepsilon}\frac{{}^\perp\nabla B^\varepsilon}{B^\varepsilon}\right)\cdot\nabla_x\tilde{f}_\theta + \frac{1}{2}\left(\frac{{}^\perp E[\tilde{f}_\theta(t)]}{B^\varepsilon}\cdot\frac{\nabla B^\varepsilon}{B^\varepsilon}\right)(\tilde{V}\cdot\nabla_{\tilde{V}}\tilde{f}_\theta) = 0.$$

In particular, if the initial presence density depends only on x *and* $|\tilde{v}|$ *i.e.,* $b\cdot\nabla_{x,\tilde{v}}\tilde{f}_{\rm in} = 0$, *then at any time* $t\in\mathbb{R}_+$ *the presence density* $\tilde{f}(t)$ *depends only on* x *and* $|\tilde{v}|$, *that is* $b\cdot\nabla_{x,\tilde{v}}\tilde{f}(t) = 0$. *Indeed, for any* $\theta\in\mathbb{R}$ *the presence densities* $\tilde{f}, \tilde{f}_\theta$ *satisfy the problem* (18), (19), (20) *and thanks to the uniqueness, we obtain* $\tilde{f} = \tilde{f}_\theta$, *saying that at any time* $t\in\mathbb{R}_+$, $\tilde{f}(t)$ *depends only on* x *and* $|\tilde{v}|$.

5 Error analysis

In this section we study rigorously the error when replacing the Vlasov-Poisson problem with strong external magnetic field (1), (2) by the model (18), (19). We expect an error of order ε^2, when the initial presence density $f_{\rm in}$ is well prepared. As usual when performing the error analysis in homogenization theory, we need to construct a corrector. We will use the following easy lemma.

Lemma 5.1 *Let* M *be a square matrix of size* 2. *Then we have the equalities*

$$\langle M\tilde{v}\cdot\tilde{v}\rangle = \mathrm{trace}(M)\frac{|\tilde{v}|^2}{2}$$

$$M\tilde{v}\cdot\tilde{v} - \langle M\tilde{v}\cdot\tilde{v}\rangle + b\cdot\nabla_{x,\tilde{v}}\left(\frac{M\mathcal{R}(-\pi/2)+\mathcal{R}(\pi/2)M}{4\omega_c(x)}\tilde{v}\cdot\tilde{v}\right)$$
$$\left\langle\frac{M\mathcal{R}(-\pi/2)+\mathcal{R}(\pi/2)M}{4\omega_c(x)}\tilde{v}\cdot\tilde{v}\right\rangle = 0.$$

Proof. By direct computations we check that $\langle\tilde{v}_1^2\rangle = \langle\tilde{v}_2^2\rangle = |\tilde{v}|^2/2$, $\langle\tilde{v}_1\tilde{v}_2\rangle = 0$,

$$b\cdot\nabla_{x,\tilde{v}}(\tilde{v}_1^2-\tilde{v}_2^2) = 4\omega_c(x)\tilde{v}_2\tilde{v}_1,\quad b\cdot\nabla_{x,\tilde{v}}(\tilde{v}_2\tilde{v}_1) = \omega_c(x)(\tilde{v}_2^2-\tilde{v}_1^2)$$

and the first and second statements follow easily. The last statement is a consequence of the first one

$$\left\langle\frac{M\mathcal{R}(-\pi/2)+\mathcal{R}(\pi/2)M}{4\omega_c(x)}\tilde{v}\cdot\tilde{v}\right\rangle = \mathrm{trace}\left(M\mathcal{R}\left(-\frac{\pi}{2}\right)+\mathcal{R}\left(\frac{\pi}{2}\right)M\right)\frac{|\tilde{v}|^2}{8\omega_c(x)} = 0.$$

□

Based on the previous formulae, we are able to construct a corrector $\tilde{f}^2$ which will allow us to estimate $\tilde{f}^\varepsilon - \tilde{f} - \varepsilon^2\tilde{f}^2$.

Proposition 5.1 *Assume that* $\tilde{f} = \tilde{f}(x,\tilde{v}) \in C^1_c(\mathbb{R}^2 \times \mathbb{R}^2), E = E(x) \in C^1(\mathbb{R}^2), B = B(x) \in C^1(\mathbb{R}^2)$ *such that* $b \cdot \nabla_{x,\tilde{v}} \tilde{f} = 0, \mathrm{div}_x \, {}^{\perp}E = 0, \inf_{x\in\mathbb{R}^2} |B(x)| > 0$. *Then we have the equality*

$$\left(\frac{{}^{\perp}E}{B} - A_x\right) \cdot \nabla_x \tilde{f} + \left[\frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x \int_{\mathbb{R}^2} \frac{{}^{\perp}(x-x')}{|x-x'|^2} \otimes j[\tilde{f}](x') \, \mathrm{d}x' - \partial_x\left(\frac{{}^{\perp}E}{B}\right)\tilde{v}\right] \cdot \nabla_{\tilde{v}} \tilde{f}$$
$$- \left(\frac{{}^{\perp}E}{B} - \frac{m|\tilde{v}|^2}{2qB}\frac{{}^{\perp}\nabla B}{B}\right) \cdot \nabla_x \tilde{f} - \frac{1}{2}\left(\frac{{}^{\perp}E}{B} \cdot \frac{\nabla B}{B}\right)\tilde{v} \cdot \nabla_{\tilde{v}} \tilde{f} + b \cdot \nabla_{x,\tilde{v}} \tilde{f}^2 = 0$$

where $A_x(x,\tilde{v}) = \frac{{}^{\perp}\tilde{v}\otimes\tilde{v}}{\omega_c^2}\nabla\omega_c$ *and*

$$\begin{aligned} \tilde{f}^2(x,\tilde{v}) = {} & \frac{\nabla\omega_c \otimes \nabla_x \tilde{f} - {}^{\perp}\nabla\omega_c \otimes {}^{\perp}\nabla_x \tilde{f}}{4\omega_c^3(x)}\tilde{v} \cdot \tilde{v} \\ & + \frac{1}{2\pi\varepsilon_0 B}\left(\mathrm{div}_x \int_{\mathbb{R}^2} \frac{{}^{\perp}(x-x')}{|x-x'|^2} \otimes j[\tilde{f}](x') \, \mathrm{d}x' \cdot \frac{{}^{\perp}\tilde{v}}{\omega_c(x)}\right)\frac{\tilde{v} \cdot \nabla_{\tilde{v}} \tilde{f}}{|\tilde{v}|^2} \\ & - \left(\frac{\partial_x\left(\frac{{}^{\perp}E}{B}\right)\mathcal{R}\left(-\frac{\pi}{2}\right) + \mathcal{R}\left(\frac{\pi}{2}\right)\partial_x\left(\frac{{}^{\perp}E}{B}\right)}{4\omega_c(x)}\tilde{v} \cdot \tilde{v}\right)\frac{\tilde{v} \cdot \nabla_{\tilde{v}} \tilde{f}}{|\tilde{v}|^2}. \end{aligned} \tag{26}$$

Proof. Notice that

$$-A_x(x,\tilde{v}) \cdot \nabla_x \tilde{f} = -\frac{({}^{\perp}\tilde{v} \cdot \nabla_x \tilde{f})(\tilde{v} \cdot \nabla\omega_c)}{\omega_c^2(x)} = \frac{(\tilde{v} \cdot {}^{\perp}\nabla_x \tilde{f})(\tilde{v} \cdot \nabla\omega_c)}{\omega_c^2(x)} = \frac{\nabla\omega_c \otimes {}^{\perp}\nabla_x \tilde{f}}{\omega_c^2(x)}\tilde{v} \cdot \tilde{v}$$

and therefore, thanks to Lemma 5.1, we have

$$\begin{aligned} -A_x(x,\tilde{v}) \cdot \nabla_x \tilde{f} + \frac{m|\tilde{v}|^2}{2qB(x)}\frac{{}^{\perp}\nabla B}{B(x)} \cdot \nabla_x \tilde{f} &= \frac{\nabla\omega_c \otimes {}^{\perp}\nabla_x \tilde{f}}{\omega_c^2(x)}\tilde{v} \cdot \tilde{v} - \left\langle \frac{\nabla\omega_c \otimes {}^{\perp}\nabla_x \tilde{f}}{\omega_c^2(x)}\tilde{v} \cdot \tilde{v} \right\rangle \\ &= -b \cdot \nabla_{x,\tilde{v}} \tilde{f}_I^2 \end{aligned}$$

with

$$\begin{aligned} \tilde{f}_I^2 &:= \left(\frac{\nabla\omega_c \otimes {}^{\perp}\nabla_x \tilde{f}}{4\omega_c^3(x)}\mathcal{R}\left(-\frac{\pi}{2}\right) + \mathcal{R}\left(\frac{\pi}{2}\right)\frac{\nabla\omega_c \otimes {}^{\perp}\nabla_x \tilde{f}}{4\omega_c^3(x)}\right)\tilde{v} \cdot \tilde{v} \\ &= \frac{\nabla\omega_c \otimes \nabla_x \tilde{f} - {}^{\perp}\nabla\omega_c \otimes {}^{\perp}\nabla_x \tilde{f}}{4\omega_c^3(x)}\tilde{v} \cdot \tilde{v}. \end{aligned}$$

Observing that $b \cdot \nabla_{x,\tilde{v}}(-{}^{\perp}\tilde{v}/\omega_c) = \tilde{v}$ and that $\nabla_{\tilde{v}} \tilde{f} = \frac{\nabla_{\tilde{v}} \tilde{f} \cdot \tilde{v}}{|\tilde{v}|^2}\tilde{v}$, it is easily seen that

$$\frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x \int_{\mathbb{R}^2} \frac{{}^{\perp}(x-x')}{|x-x'|^2} \otimes j[\tilde{f}](x') \, \mathrm{d}x' \cdot \nabla_{\tilde{v}} \tilde{f} = -b \cdot \nabla_{x,\tilde{v}} \tilde{f}_{II}^2$$

with

$$\tilde{f}_{II}^2 := \frac{1}{2\pi\varepsilon_0 B}\left(\mathrm{div}_x \int_{\mathbb{R}^2} \frac{{}^{\perp}(x-x')}{|x-x'|^2} \otimes j[\tilde{f}](x') \, \mathrm{d}x' \cdot \frac{{}^{\perp}\tilde{v}}{\omega_c(x)}\right)\frac{\tilde{v} \cdot \nabla_{\tilde{v}} \tilde{f}}{|\tilde{v}|^2}.$$

Using one more time Lemma 5.1, yields

$$-\partial_x\left(\frac{^\perp E}{B}\right)\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}-\frac{1}{2}\left(\frac{^\perp E}{B}\cdot\frac{\nabla B}{B}\right)(\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f})$$
$$=-\frac{\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}}{|\tilde{v}|^2}\left[\partial_x\left(\frac{^\perp E}{B}\right)\tilde{v}\cdot\tilde{v}-\left\langle\partial_x\left(\frac{^\perp E}{B}\right)\tilde{v}\cdot\tilde{v}\right\rangle\right]$$
$$=-b\cdot\nabla_{x,\tilde{v}}\tilde{f}^2_{III}$$

with

$$\tilde{f}^2_{III}=-\frac{\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}}{|\tilde{v}|^2}\,\frac{\partial_x\left(\frac{^\perp E}{B}\right)\mathcal{R}\left(-\frac{\pi}{2}\right)+\mathcal{R}\left(\frac{\pi}{2}\right)\partial_x\left(\frac{^\perp E}{B}\right)}{4\omega_c(x)}\tilde{v}\cdot\tilde{v}.$$

Our conclusion follows by taking $\tilde{f}^2=\tilde{f}^2_I+\tilde{f}^2_{II}+\tilde{f}^2_{III}$. □

Remark 5.1 *Notice that $\tilde{f}^2\in C_c(\mathbb{R}^2\times\mathbb{R}^2)$ and if there is $k\geq 2$ such that $\tilde{f}\in C^k_c(\mathbb{R}^2\times\mathbb{R}^2), E\in C^k(\mathbb{R}^2), B\in C^k(\mathbb{R}^2)$, then $\tilde{f}^2\in C^{k-1}_c(\mathbb{R}^2\times\mathbb{R}^2)$.*

In the sequel we will use the map $T^\varepsilon:\mathbb{R}^2\times\mathbb{R}^2$, given by

$$T^\varepsilon(x,\tilde{v})=\left(x+\varepsilon\frac{^\perp\tilde{v}}{\omega_c(x)},\tilde{v}\right).$$

Notice that for any application $\varphi(x,v)$ in the kernel of $b\cdot\nabla_{x,\tilde{v}}$ *i.e.*, $\varphi(x,v)=\psi(x,|\tilde{v}|)$, the composition product $\varphi\circ T^\varepsilon$ depends only on $x+\varepsilon\frac{^\perp\tilde{v}}{\omega_c(x)},|\tilde{v}|$

$$(\varphi\circ T^\varepsilon)(x,\tilde{v})=\varphi\left(x+\varepsilon\frac{^\perp\tilde{v}}{\omega_c(x)},\tilde{v}\right)=\psi\left(x+\varepsilon\frac{^\perp\tilde{v}}{\omega_c(x)},|\tilde{v}|\right)$$

and therefore belongs to the kernel of $b^\varepsilon\cdot\nabla_{x,\tilde{v}}$. Actually, by direct computations we check that

$$\partial_{x,\tilde{v}}T^\varepsilon b^\varepsilon=\lambda^\varepsilon b\circ T^\varepsilon,\quad\lambda^\varepsilon(x,\tilde{v})=\frac{\omega_c(x)}{\omega_c(\tilde{x})},\quad\tilde{x}=x+\varepsilon\frac{^\perp\tilde{v}}{\omega_c(x)}.\tag{27}$$

We are ready to prove our main result.

Proof. (of Theorem 1.2)
As g^ε belongs to $C^2_c(\mathbb{R}^2\times\mathbb{R}^2)$, it is easily seen that $\langle g^\varepsilon\rangle\in C^2_c(\mathbb{R}^2\times\mathbb{R}^2)$. We deduce by Theorem 1.1 that the restriction on $[0,T]$ of the solution $\tilde{f}$ for (18), (19) corresponding to the initial condition $\langle g^\varepsilon\rangle$ belongs to $C^2_c([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$. The restriction on $[0,T]$ of the Poisson electric field $E[\tilde{f}]$ belongs to $C^2([0,T]\times\mathbb{R}^2)$. As $b\cdot\nabla\langle g^\varepsilon\rangle=0$, we deduce by the last statement in Remark 4.1 that $b\cdot\nabla\tilde{f}(t)=0, t\in\mathbb{R}_+$. Applying Proposition 5.1 with the presence density $\tilde{f}(t)\in C^2_c(\mathbb{R}^2\times\mathbb{R}^2)$, the electric field $E[\tilde{f}(t)]\in C^2(\mathbb{R}^2)$, which verify $b\cdot\nabla_{x,\tilde{v}}\tilde{f}(t)=0,\mathrm{div}_x\,{}^\perp E[\tilde{f}(t)]=0$, one gets for any $t\in\mathbb{R}_+$

$$\left[\frac{^\perp E[\tilde{f}(t)]}{B}-A_x\right]\cdot\nabla_x\tilde{f}+\left[\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\frac{^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde{f}](x')\,\mathrm{d}x'-\partial_x\left(\frac{^\perp E[\tilde{f}(t)]}{B}\right)\tilde{v}\right]\cdot\nabla_{\tilde{v}}\tilde{f}\tag{28}$$
$$-\left(\frac{^\perp E[\tilde{f}(t)]}{B}-\frac{m|\tilde{v}|^2}{2qB}\frac{^\perp\nabla B}{B}\right)\cdot\nabla_x\tilde{f}-\frac{1}{2}\left(\frac{^\perp E[\tilde{f}(t)]}{B}\cdot\frac{\nabla B}{B}\right)\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}+b\cdot\nabla_{x,\tilde{v}}\tilde{f}^2(t)=0$$

where $\tilde{f}^2(t) \in C_c^1(\mathbb{R}^2 \times \mathbb{R}^2)$ is given by the formula (26) which corresponds to the presence density $\tilde{f}(t)$ and the electric field $E[\tilde{f}(t)]$. Multiplying (28) by ε and adding to (18) yield

$$\partial_t \tilde{f} + \varepsilon a[\tilde{f}(t)] \cdot \nabla \tilde{f}(t) + \varepsilon b \cdot \nabla \tilde{f}^2(t) = 0. \tag{29}$$

Notice that thanks to (27) we can write

$$\lambda^\varepsilon (b \cdot \nabla \tilde{f}^2) \circ T^\varepsilon = \partial T^\varepsilon b^\varepsilon \cdot (\nabla \tilde{f}^2) \circ T^\varepsilon = b^\varepsilon \cdot \nabla (\tilde{f}^2 \circ T^\varepsilon)$$

and therefore, after composition with T^ε and multiplication by λ^ε, the equation (29) becomes

$$\lambda^\varepsilon (\partial_t \tilde{f}) \circ T^\varepsilon + \varepsilon \lambda^\varepsilon (a[\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon + \varepsilon b^\varepsilon \cdot \nabla (\tilde{f}^2 \circ T^\varepsilon) = 0$$

or equivalently

$$\partial_t (\langle \lambda^\varepsilon \rangle \, \tilde{f} \circ T^\varepsilon) + \varepsilon \lambda^\varepsilon (a[\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon + \varepsilon b^\varepsilon \cdot \nabla (\tilde{f}^2 \circ T^\varepsilon) = -\varepsilon (\langle \lambda^\varepsilon \rangle - \lambda^\varepsilon)(\langle a \rangle \, [\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon \tag{30}$$

where we have used the notation

$$\langle a \rangle \, [\tilde{f}] := \left(\frac{{}^\perp E[\tilde{f}]}{B} - \frac{m|\tilde{v}|^2}{2qB} \frac{{}^\perp \nabla B}{B}, \frac{1}{2} \left(\frac{{}^\perp E[\tilde{f}]}{B} \cdot \frac{\nabla B}{B} \right) \tilde{v} \right).$$

Notice that, as $\tilde{f}(t)$ belongs to the kernel of $b \cdot \nabla$, then $\tilde{f} \circ T^\varepsilon$ belongs to the kernel of $b^\varepsilon \cdot \nabla$, as well as $\langle \lambda^\varepsilon \rangle \, \tilde{f} \circ T^\varepsilon$. Combining (16), (30) we obtain

$$\begin{aligned}
&\partial_t \{ \tilde{f}^\varepsilon - \langle \lambda^\varepsilon \rangle \, \tilde{f} \circ T^\varepsilon - \varepsilon^2 \tilde{f}^2 \circ T^\varepsilon \} + \varepsilon a^\varepsilon [\tilde{f}^\varepsilon(t)] \cdot \nabla (\langle \lambda^\varepsilon \rangle \, \tilde{f} \circ T^\varepsilon) - \varepsilon \lambda^\varepsilon (a[\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon \\
&+ \left(\varepsilon a^\varepsilon [\tilde{f}^\varepsilon(t)] + \frac{b^\varepsilon}{\varepsilon} \right) \cdot \nabla \{ \tilde{f}^\varepsilon - \langle \lambda^\varepsilon \rangle \, \tilde{f} \circ T^\varepsilon - \varepsilon^2 \tilde{f}^2 \circ T^\varepsilon \} = -\varepsilon^2 (\partial_t \tilde{f}^2) \circ T^\varepsilon \\
&- \varepsilon^3 a^\varepsilon [\tilde{f}^\varepsilon(t)] \cdot \nabla (\tilde{f}^2 \circ T^\varepsilon) - \varepsilon (\lambda^\varepsilon - \langle \lambda^\varepsilon \rangle)(\langle a \rangle \, [\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon.
\end{aligned} \tag{31}$$

By Remark 3.1 we know that $\operatorname{div}(\varepsilon a^\varepsilon [\tilde{f}^\varepsilon(t)] + b^\varepsilon/\varepsilon) = 0$ and therefore, multiplying (31) by $r^\varepsilon := \tilde{f}^\varepsilon - \langle \lambda^\varepsilon \rangle \, \tilde{f} \circ T^\varepsilon - \varepsilon^2 \tilde{f}^2 \circ T^\varepsilon$ yields

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|r^\varepsilon\|_{L^2}^2 \leq \varepsilon \|r^\varepsilon(t)\|_{L^2} \|a^\varepsilon [\tilde{f}^\varepsilon(t)] \cdot \nabla (\langle \lambda^\varepsilon \rangle \, \tilde{f} \circ T^\varepsilon) - \lambda^\varepsilon (a[\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon\|_{L^2} + \|r^\varepsilon(t)\|_{L^2} \\
&\times \|\varepsilon^2 (\partial_t \tilde{f}^2) \circ T^\varepsilon + \varepsilon^3 a^\varepsilon [\tilde{f}^\varepsilon(t)] \cdot \nabla (\tilde{f}^2 \circ T^\varepsilon) + \varepsilon (\lambda^\varepsilon - \langle \lambda^\varepsilon \rangle)(\langle a \rangle \, [\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon\|_{L^2}.
\end{aligned}$$

By Bellman lemma one gets

$$\begin{aligned}
\|r^\varepsilon(t)\|_{L^2} \leq \|r^\varepsilon(0)\|_{L^2} &+ \varepsilon \int_0^t \|a^\varepsilon [\tilde{f}^\varepsilon(s)] \cdot \nabla (\langle \lambda^\varepsilon \rangle \, \tilde{f}(s) \circ T^\varepsilon) - \lambda^\varepsilon (a[\tilde{f}(s)] \cdot \nabla \tilde{f}(s)) \circ T^\varepsilon\|_{L^2} \, \mathrm{d}s \\
&+ \varepsilon^2 \int_0^t \|(\partial_t \tilde{f}^2(s)) \circ T^\varepsilon\|_{L^2} \, \mathrm{d}s + \varepsilon^3 \int_0^t \|a^\varepsilon [\tilde{f}^\varepsilon(s)] \cdot \nabla (\tilde{f}^2(s) \circ T^\varepsilon)\|_{L^2} \, \mathrm{d}s \\
&+ \varepsilon \int_0^t \|(\lambda^\varepsilon - \langle \lambda^\varepsilon \rangle)(\langle a \rangle \, [\tilde{f}(s)] \cdot \nabla \tilde{f}(s)) \circ T^\varepsilon\|_{L^2} \, \mathrm{d}s.
\end{aligned} \tag{32}$$

We need to analyze one by one all the terms in the right hand side of the previous inequality, for any $t \in [0, T], T \in \mathbb{R}_+$. We will denote by C any constant depending on

$m, \varepsilon_0, q, T, B$ and the uniform bounds satisfied by the presence densities $(g^\varepsilon)_{\varepsilon>0}$, but not on ε. As $(g^\varepsilon)_{\varepsilon>0}$ are uniformly compactly supported in $\mathbb{R}^2\times\mathbb{R}^2$ and uniformly bounded in $C^2(\mathbb{R}^2\times\mathbb{R}^2)$, it is easily seen that so are $(\langle g^\varepsilon\rangle)_{\varepsilon>0}$. By Theorem 1.1 the solutions $\tilde f$ for (18), (19) associated to the initial conditions $(\langle g^\varepsilon\rangle)_{\varepsilon>0}$, remain uniformly compactly supported in $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2, T\in\mathbb{R}_+$, and uniformly bounded in $C^2([0,T]\times\mathbb{R}^2), T\in\mathbb{R}_+$. Clearly, at least for ε small enough $0<\varepsilon\le\varepsilon_T$ we have

$$\varepsilon^2\|\tilde f^2(t)\circ T^\varepsilon\|_{L^2}\le C\varepsilon^2,\ \ t\in[0,T].$$

As $B\in C^2(\mathbb{R}^2), \inf_{x\in\mathbb{R}^2}|B(x)|>0$, for any compact set $B(R_x)\times B(R_{\tilde v})$ there is a constant $C(R_x,R_{\tilde v})$ such that

$$\left|\frac{\omega_c(x)}{\omega_c(x+\varepsilon^\perp\tilde v/\omega_c(x))}-1+\varepsilon\frac{\nabla\omega_c(x)\cdot^\perp\tilde v}{\omega_c^2(x)}\right|\le C(R_x,R_{\tilde v})\varepsilon^2,\ \ (x,\tilde v)\in B(R_x)\times B(R_{\tilde v}) \tag{33}$$

and by the third statement in Proposition 3.1, we deduce

$$|\langle\lambda^\varepsilon\rangle-1|=\left|\left\langle\frac{\omega_c(x)}{\omega_c(\tilde x)}-1+\varepsilon\frac{\nabla\omega_c\cdot{}^\perp\tilde v}{\omega_c^2}\right\rangle\right|\le C(R_x,R_{\tilde v})\varepsilon^2,\ \ (x,\tilde v)\in B\left(R_x-\varepsilon\frac{R_{\tilde v}}{\omega_0}\right)\times B(R_{\tilde v}). \tag{34}$$

Notice that $\|f^\varepsilon(0)-g^\varepsilon\|_{L^2}$, and therefore $\|\rho[f^\varepsilon(0)]-\rho[g^\varepsilon]\|_{L^2}$ are of order ε, implying that

$$\|\,E[f^\varepsilon(0)]-E[g^\varepsilon]\,\|_{L^2(\mathbb{R}^2)}\le C\varepsilon.$$

We intend to compare $\tilde f^\varepsilon(0)$ with respect to $\tilde f(0)\circ T^\varepsilon$. We have

$$\begin{aligned}\tilde f^\varepsilon(0,x,\tilde v)&=f^\varepsilon\left(0,x,\tilde v+\varepsilon\frac{{}^\perp E[f^\varepsilon(0)]}{B(x)}\right)\\&=g^\varepsilon\left(x+\frac{\varepsilon}{\omega_c(x)}{}^\perp\left(\tilde v+\varepsilon\frac{{}^\perp E[f^\varepsilon(0)]}{B(x)}\right),\tilde v+\varepsilon\frac{{}^\perp E[f^\varepsilon(0)]}{B(x)}-\varepsilon\frac{{}^\perp E[g^\varepsilon]}{B(x)}\right)\end{aligned}$$

and therefore $\|\tilde f^\varepsilon(0)-g^\varepsilon\circ T^\varepsilon\|_{L^2}\le C\varepsilon^2$. Combining the previous arguments, together with the inequality

$$\|g^\varepsilon-\langle g^\varepsilon\rangle\,\|_{L^2}\le\frac{2\pi}{\omega_0}\|b\cdot\nabla g^\varepsilon\|_{L^2}\le\varepsilon^2\frac{2\pi}{\omega_0}\sup_{\eta>0}\frac{\|b\cdot\nabla g^\eta\|_{L^2}}{\eta^2}$$

cf. statement 5 in Proposition 3.1, we obtain

$$\begin{aligned}\|r^\varepsilon(0)\|_{L^2}&\le\|\tilde f^\varepsilon(0)-\langle\lambda^\varepsilon\rangle\,\tilde f(0)\circ T^\varepsilon\|_{L^2}+\varepsilon^2\|\tilde f^2(0)\circ T^\varepsilon\|_{L^2}\\&\le\|\tilde f^\varepsilon(0)-g^\varepsilon\circ T^\varepsilon\|_{L^2}+\|g^\varepsilon\circ T^\varepsilon-\langle g^\varepsilon\rangle\circ T^\varepsilon\|_{L^2}\\&\quad+\|(1-\langle\lambda^\varepsilon\rangle)\,\langle g^\varepsilon\rangle\circ T^\varepsilon\|_{L^2}+\varepsilon^2\|\tilde f^2(0)\circ T^\varepsilon\|_{L^2}\le C\varepsilon^2.\end{aligned}$$

Notice that (33), (34) also imply that $\left(\frac{\lambda^\varepsilon-\langle\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}$ is bounded on compact sets of $\mathbb{R}^2\times\mathbb{R}^2$, and thus

$$\varepsilon\int_0^t\|(\lambda^\varepsilon-\langle\lambda^\varepsilon\rangle)(\langle a\rangle\,[\tilde f(s)]\cdot\nabla\tilde f(s))\circ T^\varepsilon\|_{L^2}\,\mathrm{d}s\le C\varepsilon^2.$$

Using the C^1_c regularity of $\tilde{f}^2$, and therefore the C^2_c regularity of $\tilde{f}$, it is straightforward that

$$\varepsilon^2 \int_0^t \|(\partial_t \tilde{f}^2(s)) \circ T^\varepsilon\|_{L^2}\, \mathrm{d}s + \varepsilon^3 \int_0^t \|a^\varepsilon[\tilde{f}^\varepsilon(s)] \cdot \nabla(\tilde{f}^2(s) \circ T^\varepsilon)\|_{L^2}\, \mathrm{d}s \leq C\varepsilon^2, \;\; t \in [0,T].$$

We claim that

$$\varepsilon \int_0^t \|a^\varepsilon[\tilde{f}^\varepsilon(s)] \cdot \nabla(\langle\lambda^\varepsilon\rangle\, \tilde{f}(s) \circ T^\varepsilon) - \lambda^\varepsilon(a[\tilde{f}(s)] \cdot \nabla \tilde{f}(s)) \circ T^\varepsilon\|_{L^2} \leq C\varepsilon^2 + C\varepsilon \int_0^t \|\tilde{f}^\varepsilon(s) - \tilde{f}(s) \circ T^\varepsilon\|_{L^2}\, \mathrm{d}s, \;\; t \in [0,T], \;\; 0 < \varepsilon \leq \varepsilon_T.$$

Indeed, recall that for any $(x, \tilde{v}) \in \mathbb{R}^2 \times B(R)$ and any $\varepsilon > 0$ such that $\frac{\varepsilon R \|\nabla \omega_c\|_{L^\infty}}{\omega_0^2} \leq \frac{1}{2}$, we have

$$|A^\varepsilon_x(x,\tilde{v})| \leq 2|A_x(x,\tilde{v})|, \;\; |A^\varepsilon_x(x,\tilde{v}) - A_x(x,\tilde{v})| \leq \frac{\varepsilon|\tilde{v}|\, \|\nabla\omega_c\|_{L^\infty}}{\omega_0^2}|A^\varepsilon_x(x,\tilde{v})| \leq \frac{|A^\varepsilon_x(x,\tilde{v})|}{2}$$

implying that

$$|A^\varepsilon_x(x,\tilde{v})| \leq 2|A_x(x,\tilde{v})| \leq \frac{2R^2\|\nabla\omega_c\|_{L^\infty}}{\omega_0^2}, \;\; |A^\varepsilon_x(x,\tilde{v}) - A_x(x,\tilde{v})| \leq 2\varepsilon\frac{R^3\|\nabla\omega_c\|^2_{L^\infty}}{\omega_0^4}.$$

As $\{\tilde{f}^\varepsilon(t), t \in [0,T], 0 < \varepsilon \leq 1\}$ are uniformly compactly supported in $\mathbb{R}^2 \times \mathbb{R}^2$, and thanks to the uniform bounds

$$\sup_{\varepsilon>0, t\in[0,T]} \{\|\tilde{f}^\varepsilon(t)\|_{C^1(\mathbb{R}^2\times\mathbb{R}^2)} + \|E[\tilde{f}^\varepsilon(t)]\|_{C^1(\mathbb{R}^2)}\} < +\infty$$

we have

$$\|(a^\varepsilon[\tilde{f}^\varepsilon(t)] - a[\tilde{f}^\varepsilon(t)]) \cdot \nabla(\langle\lambda^\varepsilon\rangle\, \tilde{f}(t) \circ T^\varepsilon)\|_{L^2} \leq C\varepsilon, \;\; t \in [0,T].$$

Using elliptic regularity results, the quantity

$$\|(a[\tilde{f}^\varepsilon(t)] - a[\tilde{f}(t) \circ T^\varepsilon]) \cdot \nabla(\langle\lambda^\varepsilon\rangle\, \tilde{f}(t) \circ T^\varepsilon)\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}$$

is bounded by the L^2 norms of the charge and current densities

$$\|\rho[\tilde{f}^\varepsilon(t)] - \rho[\tilde{f}(t) \circ T^\varepsilon]\|_{L^2(\mathbb{R}^2)} + \|j[\tilde{f}^\varepsilon(t)] - j[\tilde{f}(t) \circ T^\varepsilon]\|_{L^2(\mathbb{R}^2)}$$

and thus by the L^2 norm of the presence densities $\|\tilde{f}^\varepsilon(t) - \tilde{f}(t) \circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}$. Therefore we have the inequality

$$\|(a[\tilde{f}^\varepsilon(t)] - a[\tilde{f}(t) \circ T^\varepsilon]) \cdot \nabla(\langle\lambda^\varepsilon\rangle\, \tilde{f}(t) \circ T^\varepsilon)\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} \leq C\|\tilde{f}^\varepsilon(t) - \tilde{f}(t) \circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}, t \in [0,T].$$

We are done if we prove that

$$\|a[\tilde{f}(t) \circ T^\varepsilon] \cdot \nabla(\langle\lambda^\varepsilon\rangle\, \tilde{f}(t) \circ T^\varepsilon) - \lambda^\varepsilon(a[\tilde{f}(t)] \cdot \nabla \tilde{f}(t)) \circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} \leq C\varepsilon, \;\; t \in [0,T], \;\; \varepsilon > 0$$

which comes easily by the C^2 regularity of $\tilde{f}$ and the compactness of its support, and the boundedness of $\left(\frac{\lambda^\varepsilon - \langle\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}, \left(\frac{\nabla\langle\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}$ on compact sets of $\mathbb{R}^2 \times \mathbb{R}^2$. We only

have to check the boundedness of $\left(\frac{\nabla\langle\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}$, because that of $\left(\frac{\lambda^\varepsilon-\langle\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}$ has been established before. By the statement 6 in Proposition 3.1 we know that

$$\nabla_x\langle\lambda^\varepsilon\rangle=\langle\nabla_x\lambda^\varepsilon\rangle\,,\quad \nabla_{\tilde v}\langle\lambda^\varepsilon\rangle=\frac{\tilde v\otimes\tilde v}{|\tilde v|^2}\nabla_{\tilde v}\langle\lambda^\varepsilon\rangle=\left\langle\frac{\tilde v\cdot\nabla_{\tilde v}\lambda^\varepsilon}{|\tilde v|}\right\rangle\frac{\tilde v}{|\tilde v|}.$$

Notice that

$$\nabla_x\lambda^\varepsilon(x,\tilde v)=\frac{\nabla\omega_c(x)}{\omega_c(\tilde x)}-\frac{\omega_c(x)}{\omega_c^2(\tilde x)}\,{}^t\left(I_2-\varepsilon\frac{{}^\perp\tilde v\otimes\nabla\omega_c(x)}{\omega_c^2(x)}\right)\nabla\omega_c(\tilde x)$$

and $\nabla_x\lambda^\varepsilon(x,\tilde v)|_{\varepsilon=0}=0$, implying that $\left(\frac{\nabla_x\langle\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}$, and therefore $\left(\frac{\langle\nabla_x\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}=\left(\frac{\nabla_x\langle\lambda^\varepsilon\rangle}{\varepsilon}\right)_{\varepsilon>0}$ is bounded on compact sets of $\mathbb{R}^2\times\mathbb{R}^2$. Similarly we have

$$\tilde v\cdot\nabla_{\tilde v}\lambda^\varepsilon=-\varepsilon\,{}^\perp\tilde v\cdot\frac{\nabla_x\omega_c(\tilde x)}{\omega_c^2(\tilde x)}$$

and therefore

$$\left|\frac{\nabla_{\tilde v}\langle\lambda^\varepsilon\rangle}{\varepsilon}\right|=\left|\frac{\tilde v\cdot\nabla_{\tilde v}\langle\lambda^\varepsilon\rangle}{\varepsilon}\frac{\tilde v}{|\tilde v|^2}\right|=\left|\frac{\langle\tilde v\cdot\nabla_{\tilde v}\lambda^\varepsilon\rangle}{\varepsilon}\frac{\tilde v}{|\tilde v|^2}\right|\le\frac{\|\nabla\omega_c\|_{L^\infty}}{\omega_0^2},\quad\varepsilon>0.$$

Collecting all the previous estimates into (32) leads to the inequality

$$\begin{aligned}\|\tilde f^\varepsilon(t)-\tilde f(t)\circ T^\varepsilon\|_{L^2}&\le\|r^\varepsilon(t)\|_{L^2}+\|(\langle\lambda^\varepsilon\rangle-1)\tilde f(t)\circ T^\varepsilon\|_{L^2}+\varepsilon^2\|\tilde f^2(t)\circ T^\varepsilon\|_{L^2}\\&\le C\varepsilon^2+C\varepsilon\int_0^t\|\tilde f^\varepsilon(s)-\tilde f(s)\circ T^\varepsilon\|_{L^2}\,\mathrm{d}s,\quad t\in[0,T],\quad 0<\varepsilon\le\varepsilon_T\end{aligned}$$

and by Gronwall lemma we deduce that

$$\|\tilde f^\varepsilon(t)-\tilde f(t)\circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}\le C\varepsilon^2\exp(C\varepsilon t),\quad t\in[0,T],\quad 0<\varepsilon\le\varepsilon_T.$$

In particular, as the L^2 norm of the Poisson electric field is controlled by the L^2 norm of the charge density, and therefore by the L^2 norm of the presence density (because of the compactness of its support), we have

$$\|E[\tilde f^\varepsilon(t)]-E[\tilde f(t)\circ T^\varepsilon]\|_{L^2(\mathbb{R}^2)}\le C\varepsilon^2,\quad t\in[0,T],\quad 0<\varepsilon\le\varepsilon_T.$$

It is easily seen that, thanks to the inequality $\|\tilde f(t)\circ T^\varepsilon-\tilde f(t)\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}\le C\varepsilon$ that

$$\|E[\tilde f(t)\circ T^\varepsilon]-E[\tilde f(t)]\|_{L^2(\mathbb{R}^2)}\le C\varepsilon,\quad t\in[0,T]$$

and therefore

$$\|E[f^\varepsilon(t)]-E[\tilde f(t)]\|_{L^2(\mathbb{R}^2)}=\|E[\tilde f^\varepsilon(t)]-E[\tilde f(t)]\|_{L^2(\mathbb{R}^2)}\le C\varepsilon,\quad t\in[0,T],\quad 0<\varepsilon\le\varepsilon_T.$$

Finally we obtain for any $t \in [0,T], 0 < \varepsilon \leq \varepsilon_T$

$$\left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[f^\varepsilon(t,x,v) - \tilde{f}\left(t, x + \varepsilon \frac{{}^\perp v}{\omega_c(x)}, v - \varepsilon \frac{{}^\perp E[\tilde{f}(t)]}{B(x)} \right) \right]^2 \mathrm{d}v\mathrm{d}x \right\}^{1/2} = \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big[\tilde{f}^\varepsilon(t,x,\tilde{v}) \right.$$

$$\left. - \tilde{f}\left(t, x + \frac{\varepsilon}{\omega_c(x)} {}^\perp\left(\tilde{v} + \varepsilon \frac{{}^\perp E[f^\varepsilon(t)]}{B(x)} \right), \tilde{v} + \varepsilon \frac{{}^\perp E[f^\varepsilon(t)] - {}^\perp E[\tilde{f}(t)]}{B(x)} \right) \right]^2 \mathrm{d}\tilde{v}\mathrm{d}x \right\}^{1/2}$$

$$\leq \| \tilde{f}^\varepsilon(t) - \tilde{f}(t) \circ T^\varepsilon \|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} + \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[\tilde{f}\left(t, x + \varepsilon \frac{{}^\perp \tilde{v}}{\omega_c(x)}, \tilde{v} \right) \right.\right.$$

$$\left.\left. - \tilde{f}\left(t, x + \varepsilon \frac{{}^\perp \tilde{v}}{\omega_c(x)} - \frac{\varepsilon^2}{\omega_c(x)} \frac{E[f^\varepsilon(t)]}{B(x)}, \tilde{v} + \varepsilon \frac{{}^\perp E[f^\varepsilon(t)] - {}^\perp E[\tilde{f}(t)]}{B(x)} \right) \right]^2 \mathrm{d}\tilde{v}\mathrm{d}x \right\}^{1/2}$$

$$\leq C\varepsilon^2 + C\varepsilon \| E[f^\varepsilon(t)] - E[\tilde{f}(t)] \|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^2.$$

□

6 Asymptotic analysis for uniform magnetic fields

When the magnetic field is uniform, it is possible to go further in our analysis, by considering smooth initial presence densities, not necessarily well prepared. For any non negative initial presence density $f_{\rm in} \in C^2_c(\mathbb{R}^2 \times \mathbb{R}^2)$, let us denote by $f^\varepsilon \in C^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ the solution of (1), (2), with $E[f^\varepsilon] \in C^2(\mathbb{R}_+ \times \mathbb{R}^2)$, where the magnetic field writes $B^\varepsilon = \frac{B}{\varepsilon}$, for some constant $B \neq 0$. By Theorem 2.1 we know that, for any $T \in \mathbb{R}_+$, we have uniform estimates with respect to $\varepsilon > 0$ for the L^∞ norm of $E[f^\varepsilon]|_{[0,T]\times\mathbb{R}^2}$ and the size of the support of $f^\varepsilon|_{[0,T]\times\mathbb{R}^2\times\mathbb{R}^2}$. The change of coordinates (12), (13) leads to the problem

$$\partial_t \tilde{f}^\varepsilon + \varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)] \cdot \nabla_{x,\tilde{v}} \tilde{f}^\varepsilon + \frac{b^\varepsilon(x,\tilde{v})}{\varepsilon} \cdot \nabla_{x,\tilde{v}} \tilde{f}^\varepsilon = 0, \quad (t,x,\tilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \tag{35}$$

$$\tilde{f}^\varepsilon(0,x,\tilde{v}) = f_{\rm in}\left(x, \tilde{v} + \varepsilon \frac{{}^\perp E[f_{\rm in}](x)}{B} \right), \quad (x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$$

where $b^\varepsilon \cdot \nabla_{x,\tilde{v}} = \varepsilon \tilde{v} \cdot \nabla_x + \omega_c \, {}^\perp\tilde{v} \cdot \nabla_{\tilde{v}}$ and for any presence density $\tilde{f}$, $a^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$ stands for the vector field

$$a^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}} = \frac{{}^\perp E[\tilde{f}]}{B} \cdot \nabla_x + \frac{1}{2\pi\varepsilon_0 B} \mathrm{div}_x \int_{\mathbb{R}^2} \frac{{}^\perp(x-x')}{|x-x'|^2} \otimes \left(j[\tilde{f}] + \varepsilon \frac{{}^\perp E[\tilde{f}]}{B} \rho[\tilde{f}] \right)(x') \, \mathrm{d}x' \cdot \nabla_{\tilde{v}}$$

$$- \partial_x \left(\frac{{}^\perp E[\tilde{f}]}{B} \right) \left(\tilde{v} + \varepsilon \frac{{}^\perp E[\tilde{f}]}{B} \right) \cdot \nabla_{\tilde{v}}.$$

Notice that there is no correction A^ε_x when the magnetic field is uniform, since in that case the solution of (17) is $A^\varepsilon_x = 0$. As before we have $E[f^\varepsilon(t)] = E[\tilde{f}^\varepsilon(t)]$ and the new presence densities $\tilde{f}^\varepsilon$ are smooth, $\tilde{f}^\varepsilon \in C^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$, and uniformly compactly

supported in $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$, with respect to $\varepsilon>0$, for any $T\in\mathbb{R}_+$. The characteristic flow of the vector field $b^\varepsilon\cdot\nabla_{x,\tilde v}$

$$\frac{\mathrm{d}\mathcal{X}^\varepsilon}{\mathrm{d}s}=\varepsilon\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v),\quad \frac{\mathrm{d}\tilde{\mathcal{V}}^\varepsilon}{\mathrm{d}s}=\omega_c\,{}^\perp\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v),\quad \mathcal{X}^\varepsilon(0;x,\tilde v)=x,\quad \tilde{\mathcal{V}}^\varepsilon(0;x,\tilde v)=\tilde v$$

is given by

$$\mathcal{X}^\varepsilon(s;x,\tilde v)=x+\varepsilon[I_2-\mathcal{R}(-\omega_c s)]\frac{{}^\perp v}{\omega_c},\quad \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde v)=\mathcal{R}(-\omega_c s)\tilde v,\quad \omega_c=\frac{qB}{m}. \tag{36}$$

It is periodic, and has the same period as the characteristic flow of the vector field $b(x,\tilde v)\cdot\nabla_{x,\tilde v}=\omega_c\,{}^\perp\tilde v\cdot\nabla_{\tilde v}$

$$S^\varepsilon(x,\tilde v)=S(x,\tilde v)=\frac{2\pi}{|\omega_c|},\quad (x,\tilde v)\in\mathbb{R}^2\times\mathbb{R}^2,\quad \varepsilon>0.$$

In order to filter out the fast oscillations corresponding to the vector field $\frac{b^\varepsilon(x,\tilde v)}{\varepsilon}\cdot\nabla_{x,\tilde v}$, we perform one more change of coordinates

$$\tilde f^\varepsilon(t,x,\tilde v)=\tilde F^\varepsilon(t,X,\tilde V),\quad (X,\tilde V)=(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-t/\varepsilon;x,\tilde v). \tag{37}$$

By applying the chain rule, we obtain

$$\partial_t\tilde F^\varepsilon+\varepsilon\partial_{x,\tilde v}(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-t/\varepsilon;(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(t/\varepsilon;X,\tilde V))a^\varepsilon[\tilde F^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon}\cdot\nabla_{X,\tilde V}\tilde F^\varepsilon=0 \tag{38}$$

where we have used the notations

$$\tilde F^\varepsilon(t)_{-t/\varepsilon}=\tilde F^\varepsilon(t)\circ(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-\frac{t}{\varepsilon}),\quad a^\varepsilon[\tilde F^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon}=a^\varepsilon[\tilde F^\varepsilon(t)\circ(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-\frac{t}{\varepsilon})]\circ(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\frac{t}{\varepsilon}).$$

As the characteristic flow $(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)$ in (36) is linear, the jacobian matrix simply writes for any $(x,\tilde v)\in\mathbb{R}^2\times\mathbb{R}^2$

$$\partial_{x,\tilde v}(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-t/\varepsilon;x,\tilde v)=\begin{pmatrix} I_2 & \frac{\varepsilon}{\omega_c}\mathcal{R}(-\pi/2)[I_2-\mathcal{R}(\omega_c t/\varepsilon)]\\ O_2 & \mathcal{R}(\omega_c t/\varepsilon)\end{pmatrix}$$

and therefore (38) becomes

$$\begin{aligned}\partial_t\tilde F^\varepsilon+\varepsilon\left\{a^\varepsilon_x[\tilde F^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon}+\frac{\varepsilon}{\omega_c}\mathcal{R}(-\pi/2)[I_2-\mathcal{R}(\omega_c t/\varepsilon)]a^\varepsilon_{\tilde v}[\tilde F^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon}\right\}\cdot\nabla_X\tilde F^\varepsilon\\ +\varepsilon\mathcal{R}(\omega_c t/\varepsilon)a^\varepsilon_{\tilde v}[\tilde F^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon}\cdot\nabla_{\tilde V}\tilde F^\varepsilon=0,\quad (t,X,\tilde V)\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2.\end{aligned}$$

We have obtained a two scale problem and we expect that the asymptotic behavior when ε becomes small will follow by averaging with respect to the fast time variable $s=t/\varepsilon$. As we are looking for second order approximations, we only need to average, with respect to s, when ε is small

$$a^\varepsilon_x[\tilde F^\varepsilon(t)_{-s}]_s\cdot\nabla_X\tilde F^\varepsilon\quad\text{and}\quad\mathcal{R}(\omega_c s)a^\varepsilon_{\tilde v}[\tilde F^\varepsilon(t)_{-s}]_s\cdot\nabla_{\tilde V}\tilde F^\varepsilon$$

up to terms of order ε. By the second statement in Proposition 3.1, observe that

$$\begin{aligned}
E[\tilde{F}^\varepsilon(t)_{-s}]_s(X,\tilde{V}) &= \frac{q}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\tilde{F}^\varepsilon(t,(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-s;x',\tilde{v}'))\frac{\mathcal{X}^\varepsilon(s;X,\tilde{V})-x'}{|\mathcal{X}^\varepsilon(s;X,\tilde{V})-x'|^2}\,\mathrm{d}\tilde{v}'\mathrm{d}x' \\
&= \frac{q}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\tilde{F}^\varepsilon(t,X',\tilde{V}')\frac{\mathcal{X}^\varepsilon(s;X,\tilde{V})-\mathcal{X}^\varepsilon(s;X',\tilde{V}')}{|\mathcal{X}^\varepsilon(s;X,\tilde{V})-\mathcal{X}^\varepsilon(s;X',\tilde{V}')|^2}\,\mathrm{d}\tilde{V}'\mathrm{d}X' \\
&= E[\tilde{F}^\varepsilon(t)](X)+\mathcal{O}(\varepsilon) \qquad (39)
\end{aligned}$$

and therefore we deduce that

$$\frac{\omega_c}{2\pi}\int_0^{2\pi/\omega_c} a_x^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s\cdot\nabla_X\tilde{F}^\varepsilon(t)\,\mathrm{d}s = \frac{{}^\perp E[\tilde{F}^\varepsilon(t)](X)}{B}\cdot\nabla_X\tilde{F}^\varepsilon+\mathcal{O}(\varepsilon). \qquad (40)$$

We concentrate now on the average of $\mathcal{R}(\omega_c s)a_{\tilde{v}}^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s\cdot\nabla_{\tilde{V}}\tilde{F}^\varepsilon$. We only need to consider the contributions of order 1 in $a_{\tilde{v}}^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s$, that is, those of

$$\begin{aligned}
&\frac{q}{2\pi\varepsilon_0 B}\left(\mathrm{div}_x\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\frac{{}^\perp(\cdot\,-x')}{|\,\cdot\,-x'|^2}\otimes\tilde{v}'\tilde{F}^\varepsilon(t,(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-s;x',\tilde{v}'))\,\mathrm{d}\tilde{v}'\mathrm{d}x'\right)(\mathcal{X}^\varepsilon(s;X,\tilde{V})) \\
&= \frac{q}{2\pi\varepsilon_0 B}\mathrm{div}_X\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\frac{{}^\perp(\mathcal{X}^\varepsilon(s;X,\tilde{V})-\mathcal{X}^\varepsilon(s;X',\tilde{V}'))}{|\mathcal{X}^\varepsilon(s;X,\tilde{V})-\mathcal{X}^\varepsilon(s;X',\tilde{V}')|}\otimes\mathcal{R}(-\omega_c s)\tilde{V}'\tilde{F}^\varepsilon(t,X',\tilde{V}')\,\mathrm{d}\tilde{V}'\mathrm{d}X'
\end{aligned}$$

and

$$-\left(\partial_x\left(\frac{{}^\perp E[\tilde{F}^\varepsilon(t)_{-s}]}{B}\right)\tilde{v}\right)_s(X,\tilde{V}) = \frac{\mathcal{R}(\pi/2)}{B}\partial_X E[\tilde{F}^\varepsilon(t)_{-s}]_s\mathcal{R}(-\omega_c s)\tilde{V}.$$

Notice that, up to terms of order ε, the average of the first contribution writes

$$\begin{aligned}
&\frac{\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}}\mathcal{R}(\omega_c s)\frac{q}{2\pi\varepsilon_0 B}\mathrm{div}_X\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\frac{{}^\perp(X-X')}{|X-X'|^2}\otimes\mathcal{R}(-\omega_c s)\tilde{V}'\tilde{F}^\varepsilon(t,X',\tilde{V}')\,\mathrm{d}\tilde{V}'\mathrm{d}X'\mathrm{d}s \\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\mathrm{div}_X\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\frac{\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}}\frac{\mathcal{R}(\omega_c s)(X-X')}{|X-X'|^2}\otimes\mathcal{R}(-\omega_c s)\tilde{V}'\mathrm{d}s\tilde{F}^\varepsilon(t,X',\tilde{V}')\,\mathrm{d}\tilde{V}'\mathrm{d}X' \\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\mathrm{div}_X\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\frac{(X-X')\otimes\tilde{V}'-{}^\perp(X-X')\otimes{}^\perp\tilde{V}'}{2|X-X'|^2}\tilde{F}^\varepsilon(t,X',\tilde{V}')\,\mathrm{d}\tilde{V}'\mathrm{d}X' \\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}\frac{(X-X')\otimes\tilde{V}'-{}^\perp(X-X')\otimes{}^\perp\tilde{V}'}{2|X-X'|^2}\nabla_{X'}\tilde{F}^\varepsilon(t,X',\tilde{V}')\,\mathrm{d}\tilde{V}'\mathrm{d}X'.
\end{aligned}$$

It is easily seen that

$$\mathrm{div}_{X'}\frac{(X-X')\otimes\tilde{V}'-{}^\perp(X-X')\otimes{}^\perp\tilde{V}'}{2|X-X'|^2}=0,\quad X\neq X'$$

and thus the above computations lead to

$$\frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\lim_{r\searrow 0}\int_{\mathbb{R}^2}\int_{|X-X'|>r}\frac{(X-X')\otimes\tilde{V}'-{}^{\perp}(X-X')\otimes{}^{\perp}\tilde{V}'}{2|X-X'|^2}\nabla_{X'}\tilde{F}^{\varepsilon}(t,X',\tilde{V}')\,\mathrm{d}X'\,\mathrm{d}\tilde{V}'$$
$$=\frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\lim_{r\searrow 0}\int_{\mathbb{R}^2}\int_{|X-X'|=r}\tilde{F}^{\varepsilon}\frac{(X-X')\otimes\tilde{V}'-{}^{\perp}(X-X')\otimes{}^{\perp}\tilde{V}'}{2|X-X'|^2}\frac{X-X'}{|X-X'|}\,\mathrm{d}\sigma(X')\,\mathrm{d}\tilde{V}'$$
$$=\frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\lim_{r\searrow 0}\frac{1}{2r}\int_{|X-X'|=r}\int_{\mathbb{R}^2}\tilde{F}^{\varepsilon}(t,X',\tilde{V}')\tilde{V}'\,\mathrm{d}\tilde{V}'\mathrm{d}\sigma(X')$$
$$=\frac{{}^{\perp}j[\tilde{F}^{\varepsilon}(t)](X)}{2\varepsilon_0 B}. \tag{41}$$

Similarly, up to terms of order ε, the second contribution is, thanks to (39)

$$\frac{\omega_c}{2\pi B}\int_0^{2\pi/\omega_c}\mathcal{R}(\pi/2)\mathcal{R}(\omega_c s)\partial_X E[\tilde{F}^{\varepsilon}(t)]\mathcal{R}(-\omega_c s)\tilde{V}\mathrm{d}s=-\frac{\mathrm{trace}(\partial_X E[\tilde{F}^{\varepsilon}(t)])}{2B}{}^{\perp}\tilde{V}$$
$$=-\frac{\mathrm{div}_X E[\tilde{F}^{\varepsilon}(t)]}{2B}{}^{\perp}\tilde{V}=-\frac{\rho[\tilde{F}^{\varepsilon}(t)]^{\perp}\tilde{V}}{2\varepsilon_0 B}. \tag{42}$$

Combining (41), (42) we obtain

$$\frac{\omega_c}{2\pi}\int_0^{2\pi/\omega_c}\mathcal{R}(\omega_c s)a^{\varepsilon}_{\tilde{v}}[\tilde{F}^{\varepsilon}(t)_{-s}]_s\cdot\nabla_{\tilde{V}}\tilde{F}^{\varepsilon}\mathrm{d}s=\frac{{}^{\perp}\{j[\tilde{F}^{\varepsilon}(t)]-\rho[\tilde{F}^{\varepsilon}(t)]\tilde{V}\}}{2\varepsilon_0 B}\cdot\nabla_{\tilde{V}}\tilde{F}^{\varepsilon}+\mathcal{O}(\varepsilon). \tag{43}$$

Thanks to (40), (43) we are led to the model

$$\partial_t\tilde{F}+\varepsilon\frac{{}^{\perp}E[\tilde{F}(t)]}{B}\cdot\nabla_X\tilde{F}+\varepsilon\frac{{}^{\perp}\{j[\tilde{F}(t)]-\rho[\tilde{F}(t)]\tilde{V}\}}{2\varepsilon_0 B}\cdot\nabla_{\tilde{V}}\tilde{F}=0,\ (t,X,\tilde{V})\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2 \tag{44}$$

which is supplemented by the initial condition

$$\tilde{F}(0,X,\tilde{V})=f_{\mathrm{in}}\left(X-\varepsilon\frac{{}^{\perp}\tilde{V}}{\omega_c},\tilde{V}+\varepsilon\frac{{}^{\perp}E[f_{\mathrm{in}}](X)}{B}\right),\ \ (X,\tilde{V})\in\mathbb{R}^2\times\mathbb{R}^2. \tag{45}$$

More exactly, we expect that solving (44) together with the initial condition (45), will provide a second order approximation for (1), (2). Although the above solution depends on ε, we use the notation $\tilde{F}$, saying that it is an approximation, when ε becomes small. The well posedness of the limit model (44), (45) is a direct consequence of the well posedness of the vorticity formulation for the $2D$ incompressible Euler equations. Indeed, integrating (44) with respect to the velocity leads to the Euler equations

$$\partial_t\rho[\tilde{F}]+\frac{{}^{\perp}E}{B^{\varepsilon}}\cdot\nabla_X\rho[\tilde{F}]=0, E(t,X)=\frac{1}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\rho[\tilde{F}(t)](X')\frac{X-X'}{|X-X'|^2}\,\mathrm{d}X', (t,X)\in\mathbb{R}_+\times\mathbb{R}^2$$

which allows us to determine ρ and E. Multiplying (44) by $\tilde{V}$ and integrating with respect to the velocity give a transport equation for the current density as well

$$\partial_t j[\tilde{F}]+\left(\frac{{}^{\perp}E}{B^{\varepsilon}}\cdot\nabla_X\right)j[\tilde{F}]=0$$

and finally the presence density $\tilde{F}$ comes by solving the linear transport equation (44), with smooth advection field

$$\frac{^\perp E}{B^\varepsilon} \cdot \nabla_X + \frac{1}{2\varepsilon_0 B^\varepsilon} {}^\perp(j - \rho\tilde{V}) \cdot \nabla_{\tilde{V}}.$$

The proof details are left to the reader, see also Lemma 3.3 [25].

Theorem 6.1 *Consider a non negative, smooth, compactly supported initial presence density $\tilde{F}_{\rm in} \in C^1_c(\mathbb{R}^2 \times \mathbb{R}^2)$ and a uniform magnetic field $B^\varepsilon = \frac{B}{\varepsilon} \neq 0$. There is a unique presence density $\tilde{F} \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ whose restriction on $[0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ is compactly supported for any $T \in \mathbb{R}_+$, whose Poisson electric field is smooth $E[\tilde{F}] \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$, satisfying* (44), (45). *Moreover, if for some integer $k \geq 2$ we have $\tilde{F}_{\rm in} \in C^k_c(\mathbb{R}^2 \times \mathbb{R}^2)$, then $\tilde{F} \in C^k(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ and $E[\tilde{F}] \in C^k(\mathbb{R}_+ \times \mathbb{R}^2)$.*

Remark 6.1

1. *The total kinetic energy is conserved. Indeed, multiplying* (44) *by $m|\tilde{V}|^2/2$ and integrating with respect to velocity yields*

$$\partial_t \int_{\mathbb{R}^2} m\frac{|\tilde{V}|^2}{2}\tilde{F}\,\mathrm{d}\tilde{V} + \frac{^\perp E[\tilde{F}(t)]}{B^\varepsilon} \cdot \nabla_X \int_{\mathbb{R}^2} m\frac{|\tilde{V}|^2}{2}\tilde{F}\,\mathrm{d}\tilde{V}$$
$$= \frac{m}{2\varepsilon_0 B^\varepsilon} \int_{\mathbb{R}^2} {}^\perp(j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V}) \cdot \tilde{V}\tilde{F}\,\mathrm{d}\tilde{V} = 0$$

   and our conclusion follows by integrating also with respect to X.

2. *The total electric energy is conserved. As before, using the fundamental solution of the Poisson equation in $\mathbb{R}^2$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(X-X')\rho[\tilde{F}(t)](X)\rho[\tilde{F}(t)](X')\,\mathrm{d}X'\mathrm{d}X$$
$$= \frac{1}{\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(X-X')\rho[\tilde{F}(t)](X')\partial_t\rho[\tilde{F}(t)](X)\,\mathrm{d}X'\mathrm{d}X$$
$$= \int_{\mathbb{R}^2} \Phi[\tilde{F}(t)](X)\partial_t\rho[\tilde{F}(t)](X)\,\mathrm{d}X - \int_{\mathbb{R}^2} \Phi[\tilde{F}(t)](X)\mathrm{div}_X\left(\rho[\tilde{F}(t)]\frac{^\perp E[\tilde{F}(t)]}{B^\varepsilon}\right)\,\mathrm{d}X$$
$$= \int_{\mathbb{R}^2} \nabla_X\Phi[\tilde{F}(t)] \cdot \frac{^\perp E[\tilde{F}(t)]}{B^\varepsilon}\rho[\tilde{F}(t)](X)\,\mathrm{d}X = 0.$$

3. *The function $(X,\tilde{V}) \to \frac{1}{2}|j[\tilde{F}(t)](X) - \rho[\tilde{F}(t)](X)\tilde{V}|^2$ is an invariant of the transport operator in* (44). *Indeed, thanks to the mass and momentum balances, we deduce that*

$$\left(\partial_t + \frac{^\perp E[\tilde{F}(t)]}{B^\varepsilon} \cdot \nabla_X + \frac{^\perp(j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V})}{2\varepsilon_0 B^\varepsilon} \cdot \nabla_{\tilde{V}}\right)\frac{|j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V}|^2}{2}$$
$$= -\rho[\tilde{F}(t)]\frac{^\perp(j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V})}{2\varepsilon_0 B^\varepsilon} \cdot (j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V}) = 0.$$

4. *The model* (44), (45) *is invariant under rotation in the velocity space. More exactly, if* $\tilde{F}$ *solves* (44), (45), *then* $\tilde{F}_\theta(t,X,\tilde{V}) = \tilde{F}(t,X,\mathcal{R}(\theta)\tilde{V})$ *solves* (44) *together with the initial condition*

$$\tilde{F}_\theta(0,X,\tilde{V}) = \tilde{F}_{\rm in}(X,\mathcal{R}(\theta)\tilde{V}),\ \ (X,\tilde{V}) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

In particular, if the initial presence density satisfies $b \cdot \nabla_{X,\tilde{V}} \tilde{F}_{\rm in} = 0$, *then, thanks to the uniqueness, we have* $b \cdot \nabla_{X,\tilde{V}} \tilde{F}(t) = 0$ *at any time* $t \in \mathbb{R}_+$.

5. *When the initial presence density is well prepared i.e.,* $b \cdot \nabla_{X,\tilde{V}} \tilde{F}_{\rm in} = 0$, *then* ${}^\perp(j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V}) \cdot \nabla_{\tilde{V}} \tilde{F} = 0, t \in \mathbb{R}_+$ *and therefore* (44) *reduces to* (18), *with uniform magnetic field.*

The solution of (44), (45) will allow us to describe the asymptotic behavior of the family $(f^\varepsilon)_{\varepsilon>0}$ corresponding to the initial condition $f_{\rm in}$, when ε becomes small. Let us introduce the presence density

$$\tilde{f}(t,s,x,\tilde{v}) = \tilde{F}(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})),\ \ (t,s,x,\tilde{v}) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$$

where $\tilde{F}$ solves (44), (45) and $(\mathcal{X},\tilde{\mathcal{V}})$ is the characteristic flow associated to the vector field $b(x,\tilde{v})\cdot\nabla_{x,\tilde{v}} = \omega_c\, {}^\perp\tilde{v}\cdot\nabla_{\tilde{v}}$, see Proposition 3.1. The idea is to compare $\tilde{f}^\varepsilon(t,x,\tilde{v})$ with respect to $\overline{f}(t,x+\varepsilon^\perp\tilde{v}/\omega_c,\tilde{v})$, where $\overline{f}(t,x,\tilde{v}) = \tilde{f}(t,t/\varepsilon,x,\tilde{v})$. By direct computation we check that

$$\partial_t \overline{f} + \varepsilon \left(\frac{{}^\perp E[\overline{f}(t)]}{B} \cdot \nabla_x \overline{f} + \frac{{}^\perp(j[\overline{f}(t)] - \rho[\overline{f}(t)]\tilde{v})}{2\varepsilon_0 B} \cdot \nabla_{\tilde{v}} \overline{f} \right) + \frac{b(x,\tilde{v})}{\varepsilon} \cdot \nabla_{x,\tilde{v}} \overline{f} = 0. \quad (46)$$

Notice that $\overline{f}(0) \circ T^\varepsilon$ is a second order approximation of $\tilde{f}^\varepsilon(0)$ in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$

$$\begin{aligned}
&\left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[\tilde{f}^\varepsilon(0,x,\tilde{v}) - \overline{f}\left(0, x + \varepsilon \frac{{}^\perp\tilde{v}}{\omega_c}, \tilde{v}\right) \right]^2 \,{\rm d}\tilde{v}{\rm d}x \right\}^{1/2} \qquad (47)\\
&= \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[f_{\rm in}\left(x, \tilde{v} + \varepsilon \frac{{}^\perp E[f_{\rm in}]}{B}(x)\right) - \tilde{F}\left(0, x + \varepsilon \frac{{}^\perp\tilde{v}}{\omega_c}, \tilde{v}\right) \right]^2 \,{\rm d}\tilde{v}{\rm d}x \right\}^{1/2}\\
&= \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[f_{\rm in}\left(x, \tilde{v} + \varepsilon \frac{{}^\perp E[f_{\rm in}]}{B}(x)\right) - f_{\rm in}\left(x, \tilde{v} + \varepsilon \frac{{}^\perp E[f_{\rm in}]}{B}(x + \varepsilon\, {}^\perp\tilde{v}/\omega_c)\right) \right]^2 \,{\rm d}\tilde{v}{\rm d}x \right\}^{1/2}\\
&\leq C\varepsilon^2.
\end{aligned}$$

By introducing a corrector term, we will prove that $\overline{f} \circ T^\varepsilon$ is a second order approximation of $\tilde{f}^\varepsilon$ in $L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$. We mention that the asymptotic behavior for a very similar problem has been investigated in Theorem 1.2 [25], but without indicating the convergence rate. Our goal in this section is to complete the asymptotic analysis by justifying the second order approximation. For any smooth presence density $\tilde{f} \in C^1_c(\mathbb{R}^2 \times \mathbb{R}^2)$ we use the notations

$$a[\tilde{f}] \cdot \nabla_{x,\tilde{v}} = \frac{{}^\perp E[\tilde{f}]}{B} \cdot \nabla_x + \left[\frac{{\rm div}_x}{2\pi\varepsilon_0 B} \int_{\mathbb{R}^2} \frac{{}^\perp(x-x')}{|x-x'|^2} \otimes j[\tilde{f}](x')\, {\rm d}x' - \partial_x \left(\frac{{}^\perp E[\tilde{f}]}{B} \right) \tilde{v} \right] \cdot \nabla_{\tilde{v}}$$

and

$$\langle a\rangle\,[\tilde f]\cdot\nabla_{x,\tilde v}=\frac{{}^\perp E[\tilde f]}{B}\cdot\nabla_x+\frac{{}^\perp(j[\tilde f]-\rho[\tilde f]\tilde v)}{2\varepsilon_0 B}\cdot\nabla_{\tilde v}.$$

For constructing the corrector, we need essentially to invert the transport operator $\partial_s+b\cdot\nabla_{x,\tilde v}$ on the subspace of functions with zero average with respect to the characteristic flow of the vector field $\partial_s+b\cdot\nabla_{x,\tilde v}$, see [8, 9]. As in Proposition 5.1, the expression of the corrector is explicit and follows by direct computations. We consider presence densities depending also on the fast time variable and therefore we work in the phase space $(s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$.

Proposition 6.1 *Assume that $\tilde f=\tilde f(s,x,\tilde v)\in C^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$ is $S=\frac{2\pi}{|\omega_c|}$ periodic and uniformly compactly supported in $(x,\tilde v)$ with respect to $s\in\mathbb{R}$, such that*

$$\partial_s\tilde f+b(x,\tilde v)\cdot\nabla_{x,\tilde v}\tilde f=0,\quad (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Then we have the equality

$$a[\tilde f(s)]\cdot\nabla_{x,\tilde v}\tilde f-\langle a\rangle\,[\tilde f(s)]\cdot\nabla_{x,\tilde v}\tilde f+(\partial_s+b(x,\tilde v)\cdot\nabla_{x,\tilde v})\tilde f^2=0$$

where

$$\begin{aligned}\tilde f^2=&\frac{\cos(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes{}^\perp j[\tilde F]+\frac{x-x'}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v)\\&+\frac{\sin(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{(x-x')}{|x-x'|^2}\otimes{}^\perp j[\tilde F]-\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v)\\&+\frac{\cos(2\omega_c s)}{4\omega_c}\left[\mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde F]}{B}\right)+\partial_x\left(\frac{E[\tilde F]}{B}\right)\mathcal{R}(-\pi/2)\right]:{}^\perp\nabla_{\tilde V}\tilde F\otimes\mathcal{R}(\omega_c s)\tilde v\\&-\frac{\sin(2\omega_c s)}{4\omega_c}\left[\partial_x\left(\frac{E[\tilde F]}{B}\right)-\mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde F]}{B}\right)\mathcal{R}(-\pi/2)\right]:{}^\perp\nabla_{\tilde V}\tilde F\otimes\mathcal{R}(\omega_c s)\tilde v.\end{aligned}$$

Proof. The presence density $\tilde f$ satisfies the constraint $(\partial_s+b\cdot\nabla_{x,\tilde v})\tilde f=0$ and therefore we have

$$\tilde f(s,x,\tilde v)=\tilde f(0,(\mathfrak{X},\tilde{\mathfrak{V}})(-s;x,\tilde v)),\quad (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Therefore there is a function $\tilde F\in C^1_c(\mathbb{R}^2\times\mathbb{R}^2)$ such that

$$\tilde f(s,x,\tilde v)=\tilde F(x,\mathcal{R}(\omega_c s)\tilde v),\quad (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Observe that $\rho[\tilde f(s)]=\rho[\tilde F]$, $j[\tilde f(s)]=\mathcal{R}(-\omega_c s)j[\tilde F]$ and

$$\nabla_{\tilde v}\tilde f(s,x,\tilde v)=\mathcal{R}(-\omega_c s)\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v),\quad (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Notice that

$$\begin{aligned}&\frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde f(s)](x')\,\mathrm{d}x'\cdot\nabla_{\tilde v}\tilde f(s)-\frac{{}^\perp j[\tilde f(s)]}{2\varepsilon_0 B}\cdot\nabla_{\tilde v}\tilde f(s)\qquad(48)\\&\quad=\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\mathcal{R}(\omega_c s)\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes\mathcal{R}(-\omega_c s)j[\tilde F](x')\,\mathrm{d}x'\cdot(\nabla_{\tilde V}\tilde F)(x,\mathcal{R}(\omega_c s)\tilde v)\\&\quad\quad-\frac{{}^\perp j[\tilde F](x)}{2\varepsilon_0 B}\cdot(\nabla_{\tilde V}\tilde F)(x,\mathcal{R}(\omega_c s)\tilde v).\end{aligned}$$

As already observed in (41), the average with respect to s of

$$\frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x \int_{\mathbb{R}^2} \mathcal{R}(\omega_c s)\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes \mathcal{R}(-\omega_c s) j[\tilde F](x')\,\mathrm{d}x'$$

coincides with ${}^\perp j[\tilde F]/(2\varepsilon_0 B)$, and therefore we have

$$\begin{aligned}
&\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2} \mathcal{R}(\omega_c s)\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes \mathcal{R}(-\omega_c s) j[\tilde F](x')\,\mathrm{d}x' - \frac{{}^\perp j[\tilde F](x)}{2\varepsilon_0 B}\\
&= \frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\left[\cos^2(\omega_c s)-\frac12\right]\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde F] + \left[\sin^2(\omega_c s)-\frac12\right]\frac{x-x'}{|x-x'|^2}\otimes {}^\perp j[\tilde F]\,\mathrm{d}x'\\
&+ \frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\cos(\omega_c s)\sin(\omega_c s)\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde F] + \frac{(x-x')}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'
\end{aligned}$$

whose zero average primitive with respect to s is

$$\begin{aligned}
&\frac{\sin(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde F] - \frac{x-x'}{|x-x'|^2}\otimes {}^\perp j[\tilde F]\right]\,\mathrm{d}x'\\
&-\frac{\cos(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde F] + \frac{x-x'}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'.
\end{aligned}$$

Coming back to (48), and taking into account that $\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v)$ belongs to the kernel of $\partial_s + b\cdot\nabla_{x,\tilde v}$, since it depends only on the invariants $x, \mathcal{R}(\omega_c s)\tilde v$ of $\partial_s + b\cdot\nabla_{x,\tilde v}$, it is easily seen that

$$\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde f(s)]\,\mathrm{d}x'\cdot\nabla_{\tilde v}\tilde f(s) - \frac{{}^\perp j[\tilde f(s)]}{2\varepsilon_0 B}\cdot\nabla_{\tilde v}\tilde f(s) + (\partial_s + b\cdot\nabla_{x,\tilde v})\tilde f_I^2 = 0$$

where

$$\begin{aligned}
\tilde f_I^2 &= \frac{\cos(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde F] + \frac{x-x'}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v)\\
&+ \frac{\sin(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde F] - \frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v).
\end{aligned}\tag{49}$$

Similarly we obtain

$$-\partial_x\left(\frac{{}^\perp E[\tilde f(s)]}{B}\right)\tilde v\cdot\nabla_{\tilde v}\tilde f + \frac{\rho[\tilde f(s)]}{2\varepsilon_0 B}\,{}^\perp\tilde v\cdot\nabla_{\tilde v}\tilde f + (\partial_s + b\cdot\nabla_{x,\tilde v})\tilde f_{II}^2 = 0$$

where

$$\begin{aligned}
\tilde f_{II}^2 &= \frac{\cos(2\omega_c s)}{4\omega_c}\left[\mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde F]}{B}\right) + \partial_x\left(\frac{E[\tilde F]}{B}\right)\mathcal{R}(-\pi/2)\right] : {}^\perp\nabla_{\tilde V}\tilde F\otimes\mathcal{R}(\omega_c s)\tilde v\\
&- \frac{\sin(2\omega_c s)}{4\omega_c}\left[\partial_x\left(\frac{E[\tilde F]}{B}\right) - \mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde F]}{B}\right)\mathcal{R}(-\pi/2)\right] : {}^\perp\nabla_{\tilde V}\tilde F\otimes\mathcal{R}(\omega_c s)\tilde v.
\end{aligned}\tag{50}$$

Our conclusion follows by combining (49), (50). □

Remark 6.2 *Notice that $\tilde{f}^2 = \tilde{f}^2(s,x,\tilde{v})$ is continuous, $S = \frac{2\pi}{|\omega_c|}$ periodic and uniformly compactly supported in $(x,\tilde{v})$ with respect to s. Moreover, if $\tilde{f} \in C^k(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$ for some integer $k \geq 2$, then $\tilde{f}^2 \in C^{k-1}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$.*

We appeal one more time to the application $T^\varepsilon(x,\tilde{v}) = (x + \varepsilon^\perp\tilde{v}/\omega_c, \tilde{v})$. Since the magnetic field is uniform, the formula (27) simply writes

$$\partial T^\varepsilon b^\varepsilon = b\circ T^\varepsilon, \quad b\cdot\nabla_{x,\tilde{v}} = \omega_c\,{}^\perp\tilde{v}\cdot\nabla_{\tilde{v}}, \quad b^\varepsilon\cdot\nabla_{x,\tilde{v}} = \varepsilon\tilde{v}\cdot\nabla_x + \omega_c\,{}^\perp\tilde{v}\cdot\nabla_{\tilde{v}}. \tag{51}$$

Finally we are ready to prove that $\overline{f}(t)\circ T^\varepsilon$, where $\overline{f}(t,x,\tilde{v}) = \tilde{F}(t,x,\mathcal{R}(\omega_c t/\varepsilon)\tilde{v})$, is a second order approximation of $\tilde{f}^\varepsilon(t,x,\tilde{v})$.

Proof. (of Theorem 1.3)
For any $t\in\mathbb{R}_+$, the presence density $(s,x,\tilde{v}) \to \tilde{f}(t,s,x,\tilde{v}) = \tilde{F}(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))$ belongs to $C^2(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$, is $S = \frac{2\pi}{|\omega_c|}$ periodic with respect to s and uniformly compactly supported in $(x,\tilde{v})$, with respect to $s\in\mathbb{R}$ and $t\in[0,T], T\in\mathbb{R}_+$. By definition, for any $t\in\mathbb{R}_+$, the presence density $\tilde{f}(t)$ satisfies the constraint $(\partial_s + b\cdot\nabla_{x,\tilde{v}})\tilde{f}(t) = 0$. Thanks to Proposition 6.1, there is $\tilde{f}^2(t,s,x,\tilde{v})$ such that

$$a[\tilde{f}(t,s)]\cdot\nabla_{x,\tilde{v}}\tilde{f} - \langle a\rangle\,[\tilde{f}(t,s)]\cdot\nabla_{x,\tilde{v}}\tilde{f} + (\partial_s + b\cdot\nabla_{x,\tilde{v}})\tilde{f}^2 = 0. \tag{52}$$

It is easily seen, by the explicit formula of $\tilde{f}^2$, that the corrector $\tilde{f}^2$ belongs to $C^1(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$, is $S = \frac{2\pi}{|\omega_c|}$ periodic with respect to s, and uniformly compactly supported in $(x,\tilde{v})$ with respect to $s\in\mathbb{R}$ and $t\in[0,T], T\in\mathbb{R}_+$. Taking $s = t/\varepsilon$ in (52), multiplying by ε and combining with (46) yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\overline{f}(t) + \varepsilon^2\tilde{f}^2(t,t/\varepsilon)\right\} + \varepsilon a[\overline{f}(t)]\cdot\nabla_{x,\tilde{v}}\overline{f}(t) + \frac{b}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\left\{\overline{f}(t) + \varepsilon^2\tilde{f}^2(t,t/\varepsilon)\right\}\\ = \varepsilon^2\partial_t\tilde{f}^2(t,t/\varepsilon,x,\tilde{v}).\end{aligned}$$

After composition with T^ε, the above equation becomes, thanks to (51)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\overline{f}(t)\circ T^\varepsilon + \varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\right\} + \varepsilon a[\overline{f}(t)]\circ T^\varepsilon\cdot(\nabla\overline{f}(t))\circ T^\varepsilon\\ &\quad + \frac{b^\varepsilon}{\varepsilon}\cdot\nabla\left\{\overline{f}(t)\circ T^\varepsilon + \varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\right\} = \varepsilon^2\partial_t\{\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\}.\end{aligned}$$

Combining with (35) and using the notation $r^\varepsilon(t) = \tilde{f}^\varepsilon(t) - \overline{f}(t)\circ T^\varepsilon - \varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon$, lead to

$$\partial_t r^\varepsilon + \varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon - \varepsilon(a[\overline{f}(t)]\cdot\nabla\overline{f}(t))\circ T^\varepsilon + \frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}}r^\varepsilon = -\varepsilon^2\partial_t\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon. \tag{53}$$

As the magnetic field is uniform, the vector fields $a^\varepsilon[\tilde{f}^\varepsilon], b^\varepsilon$ are divergence free and therefore

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon r^\varepsilon\,\mathrm{d}\tilde{v}\mathrm{d}x = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla(\overline{f}(t)\circ T^\varepsilon + \varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon)r^\varepsilon\,\mathrm{d}\tilde{v}\mathrm{d}x$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}}r^\varepsilon\,r^\varepsilon\,\mathrm{d}\tilde{v}\mathrm{d}x = 0.$$

Multiplying (53) by $r^\varepsilon(t,x,\tilde v)$ and integrating with respect to $(x,\tilde v)$ imply

$$\begin{aligned}
&\frac12\frac{\mathrm d}{\mathrm dt}\|r^\varepsilon(t)\|_{L^2}^2+\varepsilon\int_{\mathbb R^2}\int_{\mathbb R^2}a^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla(\overline f(t)\circ T^\varepsilon)r^\varepsilon\,\mathrm d\tilde v\mathrm dx\\
&\quad-\varepsilon\int_{\mathbb R^2}\int_{\mathbb R^2}(a[\overline f(t)]\cdot\nabla\overline f(t))\circ T^\varepsilon r^\varepsilon\,\mathrm d\tilde v\mathrm dx\\
&\quad=-\varepsilon^2\int_{\mathbb R^2}\int_{\mathbb R^2}\partial_t\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon r^\varepsilon\,\mathrm d\tilde v\mathrm dx-\varepsilon^3\int_{\mathbb R^2}\int_{\mathbb R^2}a^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla(\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon)r^\varepsilon\,\mathrm d\tilde v\mathrm dx
\end{aligned}$$

and by Bellman lemma one gets

$$\begin{aligned}
\|r^\varepsilon(t)\|_{L^2}\le\|r^\varepsilon(0)\|_{L^2}+\varepsilon\int_0^t\|a^\varepsilon[\tilde f^\varepsilon(t')]\cdot\nabla(\overline f(t')\circ T^\varepsilon)-(a[\overline f(t')]\cdot\nabla\overline f(t'))\circ T^\varepsilon\|_{L^2}\,\mathrm dt'\\
+\varepsilon^2\int_0^t\|\partial_t\tilde f^2(t',t'/\varepsilon)\circ T^\varepsilon\|_{L^2}\,\mathrm dt'+\varepsilon^3\int_0^t\|a^\varepsilon[\tilde f^\varepsilon(t')]\cdot\nabla(\tilde f^2(t',t'/\varepsilon)\circ T^\varepsilon)\|_{L^2}\,\mathrm dt'.
\end{aligned}\tag{54}$$

As before, we are working for $t\in[0,T], T\in\mathbb R_+$, and we denote by C any constant depending on m,ε_0,q,T,B and the initial presence density f_{in}, but not on ε. Thanks to (47) we have

$$\|r^\varepsilon(0)\|_{L^2}\le\|\tilde f^\varepsilon(0)-\overline f(0)\circ T^\varepsilon\|_{L^2}+\varepsilon^2\|\tilde f^2(0,0)\circ T^\varepsilon\|_{L^2}$$

and thus clearly $\|r^\varepsilon(0)\|_{L^2}\le C\varepsilon^2$. Using the C^1_c regularity of $\tilde f^2$ which comes from the C^2_c regularity of $\tilde F$, it is straightforward that

$$\varepsilon^2\int_0^T\|\partial_t\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon\|_{L^2}\,\mathrm dt'+\varepsilon^3\int_0^T\|a^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla(\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon)\|_{L^2}\,\mathrm dt\le C\varepsilon^2.$$

We claim that

$$\begin{aligned}
\varepsilon\int_0^t\|a^\varepsilon[\tilde f^\varepsilon(t')]\cdot\nabla(\overline f(t')\circ T^\varepsilon)-(a[\overline f(t')]\cdot\nabla\overline f(t'))\circ T^\varepsilon\|_{L^2}\,\mathrm dt'\le C\varepsilon^2\\
+C\varepsilon\int_0^t\|\tilde f^\varepsilon(t')-\overline f(t')\circ T^\varepsilon\|_{L^2},\quad t\in[0,T].
\end{aligned}\tag{55}$$

Thanks to the uniform bounds

$$\sup_{\varepsilon>0,t\in[0,T]}\{\|\tilde f^\varepsilon(t)\|_{C^1(\mathbb R^2\times R^2)}+\|E[\tilde f^\varepsilon(t)]\|_{C^1(\mathbb R^2)}\}<+\infty$$

it is easily seen that

$$\|(a^\varepsilon[\tilde f^\varepsilon(t)]-a[\tilde f^\varepsilon(t)])\cdot\nabla(\overline f(t)\circ T^\varepsilon)\|_{L^2}\le C\varepsilon,\quad t\in[0,T],\quad\varepsilon>0.$$

Thanks to elliptic regularity results, the quantity

$$\|(\,a[\tilde f^\varepsilon(t)]-a[\overline f(t)\circ T^\varepsilon]\,)\cdot\nabla(\overline f(t)\circ T^\varepsilon)\|_{L^2}$$

is bounded by the L^2 norms of the charge and current densities

$$\|\rho[\tilde f^\varepsilon(t)]-\rho[\overline f(t)\circ T^\varepsilon]\|_{L^2(\mathbb R^2)}+\|j[\tilde f^\varepsilon(t)]-j[\overline f(t)\circ T^\varepsilon]\|_{L^2(\mathbb R^2)}$$

and thus by the L^2 norm of the presence densities $\|\tilde{f}^\varepsilon(t) - \overline{f}(t) \circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}$. We obtain the inequality

$$\|(a[\tilde{f}^\varepsilon(t)]-a[\overline{f}(t)\circ T^\varepsilon])\cdot\nabla(\overline{f}(t)\circ T^\varepsilon)\|_{L^2} \leq C\|\tilde{f}^\varepsilon(t)-\overline{f}(t)\circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}, \quad t\in[0,T], \quad \varepsilon>0.$$

The inequality (55) follows immediately, noticing that

$$\|a[\overline{f}(t)\circ T^\varepsilon]\cdot\nabla(\overline{f}(t)\circ T^\varepsilon) - (a[\overline{f}(t)]\cdot\nabla\overline{f}(t))\circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} \leq C\varepsilon, \quad t\in[0,T], \quad \varepsilon>0.$$

Finally (54) writes

$$\|r^\varepsilon(t)\|_{L^2} \leq C\varepsilon^2 + C\varepsilon\int_0^t \|\tilde{f}^\varepsilon(t') - \overline{f}(t')\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t'$$

implying that

$$\begin{aligned}\|\tilde{f}^\varepsilon(t) - \overline{f}(t)\circ T^\varepsilon\|_{L^2} &\leq \|r^\varepsilon(t)\|_{L^2} + \varepsilon^2\|\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\|_{L^2}\\ &\leq C\varepsilon^2 + C\varepsilon\int_0^t \|\tilde{f}^\varepsilon(t') - \overline{f}(t')\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t', \quad t\in[0,T], \quad \varepsilon>0.\end{aligned}$$

By Gronwall lemma we deduce

$$\|\tilde{f}^\varepsilon(t) - \overline{f}(t)\circ T^\varepsilon\|_{L^2} \leq C\varepsilon^2\exp(C\varepsilon t), \quad t\in[0,T], \quad \varepsilon>0.$$

Clearly we have

$$\begin{aligned}\|E[f^\varepsilon(t)] - E[\tilde{F}(t)]\|_{L^2} = \|E[\tilde{f}^\varepsilon(t)] - E[\overline{f}(t)]\|_{L^2} &\leq \|E[\tilde{f}^\varepsilon(t)] - E[\overline{f}(t)\circ T^\varepsilon]\|_{L^2}\\ &+ \|E[\overline{f}(t)\circ T^\varepsilon] - E[\overline{f}(t)]\|_{L^2} \leq C\varepsilon\end{aligned}$$

and therefore

$$\begin{aligned}&\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\tilde{F}\left(t, x+\varepsilon\frac{{}^\perp v}{\omega_c}, \mathcal{R}(\omega_c t/\varepsilon)\left(v-\varepsilon\frac{{}^\perp E[\tilde{F}(t)]}{B}\right)\right) - f^\varepsilon(t,x,v)\right]^2 \mathrm{d}v\mathrm{d}x\right\}^{1/2}\\
&= \left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\tilde{F}\left(t, x+\frac{\varepsilon}{\omega_c}\left({}^\perp\tilde{v}-\varepsilon\frac{E[f^\varepsilon(t)]}{B}\right), \mathcal{R}(\omega_c t/\varepsilon)\left(\tilde{v}+\varepsilon\frac{{}^\perp E[f^\varepsilon(t)] - {}^\perp E[\tilde{F}(t)]}{B}\right)\right)\right.\right.\\
&\left.\left. -\tilde{f}^\varepsilon(t,x,\tilde{v})\right]^2 \mathrm{d}\tilde{v}\mathrm{d}x\right\}^{1/2}\\
&\leq \|\tilde{f}^\varepsilon(t) - \overline{f}(t)\circ T^\varepsilon\|_{L^2} + \left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\tilde{F}\left(t, x+\varepsilon\frac{{}^\perp\tilde{v}}{\omega_c}, \mathcal{R}(\omega_c t/\varepsilon)\tilde{v}\right) -\right.\right.\\
&\left.\left.\tilde{F}\left(t, x+\varepsilon\frac{{}^\perp\tilde{v}}{\omega_c} - \frac{\varepsilon^2}{\omega_c}\frac{E[f^\varepsilon(t)]}{B}, \mathcal{R}(\omega_c t/\varepsilon)\tilde{v} + \varepsilon\mathcal{R}(\omega_c t/\varepsilon)\frac{{}^\perp E[f^\varepsilon(t)] - {}^\perp E[\tilde{F}(t)]}{B}\right)\right]^2 \mathrm{d}\tilde{v}\mathrm{d}x\right\}^{1/2}\\
&\leq C\varepsilon^2 + C\varepsilon\|E[f^\varepsilon(t)] - E[\tilde{F}(t)]\|_{L^2} \leq C\varepsilon^2, \quad t\in[0,T], \quad \varepsilon>0.\end{aligned}$$

□

A Well posedness of the Vlasov-Poisson problem with external magnetic field

Proof. (of Theorem 2.1)
We are done if we prove the well posedness on any time interval $[0,T], T \in \mathbb{R}_+$. We construct a map acting on electric fields. Given a C^1 electric field E on $[0,T]\times\mathbb{R}^2$ such that $E, \partial_x E$ are bounded, we consider the solution by characteristics of the Vlasov problem (7), (9) on $[0,T]\times\mathbb{R}^2$, that is

$$f^E(t,x,v) = f_{\rm in}(X^E(0;t,x,v), V^E(0;t,x,v)), \quad (t,x,v)\in[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$$

where the characteristics $(X^E, V^E)(s;t,x,v)$ are given by

$$\frac{\mathrm{d}X^E}{\mathrm{d}s} = V^E(s), \quad \frac{\mathrm{d}V^E}{\mathrm{d}s} = \frac{q}{m}\left(E(s,X^E(s)) + B(X^E(s))\,{}^\perp V^E(s)\right)$$

$$(X^E, V^E)(s=t;t,x,v) = (x,v), \quad (s,t,x,v)\in[0,T]\times[0,T]\times\mathbb{R}^2\times\mathbb{R}^2.$$

Notice that the right hand side terms in the characteristic equations have at most linear growth at infinity (*i.e.*, when $s,t\in[0,T], |x|+|v|\to+\infty$), and therefore the characteristics exist globally on $[0,T]$ and have C^1 regularity in all the arguments. Considering now the Poisson electric field corresponding to the presence density f, leads to a map $\mathcal{F}$, whose fixed point gives the solution of the Vlasov-Poisson problem (7), (8), (9) on $[0,T]$

$$E \to \mathcal{F}(E) = \frac{q}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^E(t,x',v')\frac{x-x'}{|x-x'|^2}\,\mathrm{d}v'\mathrm{d}x'.$$

We need to find a set X_T of smooth electric fields, which is left invariant by the map $\mathcal{F}$, and to establish an estimate like

$$\|\mathcal{F}E(t) - \mathcal{F}\tilde{E}(t)\|_{L^\infty(\mathbb{R}^2)} \le C_T\int_0^t \|E(s)-\tilde{E}(s)\|_{L^\infty(\mathbb{R}^2)}\,\mathrm{d}s, \quad E,\tilde{E}\in X_T, t\in[0,T] \quad (56)$$

for some constant C_T, not depending on $E, \tilde{E}$. After that, the well posedness of the Vlasov-Poisson system follows immediately, by iterating the map $\mathcal{F}$. We are not indicating all the details of this construction, but only the a priori estimates, for smooth solutions of (7), (8), (9). Let f be a smooth solution corresponding to the non negative initial presence density $f_{\rm in}\in C^1_c(\mathbb{R}^2\times\mathbb{R}^2)$. We are looking for estimating $E, \partial_x E$ in $C([0,T]\times\mathbb{R}^2)$. Notice that for any $R>0$ we have

$$\begin{aligned}|\varepsilon_0 E[f(t)](x)| &\le \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathbf{1}_{\{|x-x'|<R\}}\frac{\rho[f(t)](x')}{|x-x'|}\,\mathrm{d}x' + \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathbf{1}_{\{|x-x'|\ge R\}}\frac{\rho[f(t)](x')}{|x-x'|}\,\mathrm{d}x'\\ &\le R\|\rho[f(t)]\|_{L^\infty(\mathbb{R}^2)} + \frac{1}{2\pi R}\|\rho[f(t)]\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

As the total charge is conserved, after minimization with respect to R, that is by taking $R = (\|\rho[f_{\rm in}]\|_{L^1(\mathbb{R}^2)}/2\pi\|\rho[f(t)]\|_{L^\infty(\mathbb{R}^2)})^{1/2}$, we obtain

$$\varepsilon_0\|E[f(t)]\|_{L^\infty(\mathbb{R}^2)} \le \|\,\rho[f(t)]\,\|^{1/2}_{L^\infty(\mathbb{R}^2)}\left(\frac{|q|}{2\pi}\|f_{\rm in}\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\right)^{1/2}. \quad (57)$$

For estimating the L^∞ norm of the charge density, we analyze the support of the presence density f. By the characteristic equations we have for any $0 \le s \le t \le T$

$$|X(s;t,x,v) - x| \le \int_s^t |V(\sigma;t,x,v)| \, \mathrm{d}\sigma \tag{58}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|V(s;t,x,v)|^2 = \frac{q}{m} E[f(s)](X(s;t,x,v)) \cdot V(s;t,x,v) \tag{59}$$

implying that

$$\big|\, |V(s;t,x,v)| - |v| \,\big| \le \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{d}\sigma.$$

Assume that $\operatorname{supp} f_{\mathrm{in}} \subset \{(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ |x| \le R_x^{\mathrm{in}} \text{ and } |v| \le R_v^{\mathrm{in}}\}$. Clearly, for any $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $|v| > R_v(t) := R_v^{\mathrm{in}} + \frac{|q|}{m}\int_0^t \|E[f(s)]\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{d}s$ we have

$$|V(0;t,x,v)| \ge |v| - \frac{|q|}{m} \int_0^t \|E[f(s)]\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{d}s > R_v^{\mathrm{in}}$$

and therefore

$$f(t,x,v) = f_{\mathrm{in}}(X(0;t,x,v), V(0;t,x,v)) = 0.$$

Consider now $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that

$$|x| > R_x(t) := R_x^{\mathrm{in}} + tR_v(t) + \int_0^t \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{d}\sigma \, \mathrm{d}s.$$

If $|v| > R_v(t)$ we already know that $f(t,x,v) = 0$. If $|v| \le R_v(t)$, we have by (58), (59)

$$\begin{aligned}
|X(0;t,x,v)| &\ge |x| - \int_0^t |V(s;t,x,v)| \, \mathrm{d}s \\
&\ge |x| - \int_0^t \left[|v| + \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{d}\sigma \right] \mathrm{d}s \\
&\ge |x| - tR_v(t) - \int_0^t \frac{|q|}{m} \int_s^t \|E[f(\sigma)]\|_{L^\infty(\mathbb{R}^2)} \, \mathrm{d}\sigma \, \mathrm{d}s > R_x^{\mathrm{in}}
\end{aligned}$$

implying that $f(t,x,v) = f_{\mathrm{in}}(X(0;t,x,v), V(0;t,x,v)) = 0$. We have proved that f is compactly supported

$$\operatorname{supp} f(t) \subset \{(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ |x| \le R_x(t) \ \text{ and } |v| \le R_v(t)\}, \ \ t \in [0,T].$$

Notice that the above computations are not depending on the magnetic field B; more exactly the equations (59) will be the same when considering an external magnetic field or not. This is why the arguments used for the Vlasov-Poisson problem also apply for the Vlasov-Poisson problem with external magnetic field. The charge density is bounded by

$$|\rho[f(t)](x)| \ = |q| \textstyle\int_{\mathbb{R}^2} f_{\mathrm{in}}(X(0;t,x,v), V(0;t,x,v)) \, \mathrm{d}v \le |q| \|f_{\mathrm{in}}\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \pi R_v^2(t). \tag{60}$$

The a priori bound for the L^∞ norm of the electric field comes by Gronwall lemma. Actually, combining (57), (60) yields

$$\begin{aligned}\varepsilon_0\|E[f(t)]\|_{L^\infty(\mathbb{R}^2)} &\leq \left(\frac{|q|}{2\pi}\|f_{\rm in}\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\right)^{1/2}\left(|q|\|f_{\rm in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\pi\right)^{1/2} R_v(t)\\ &= \frac{|q|}{\sqrt{2}}\|f_{\rm in}\|^{1/2}_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\|f_{\rm in}\|^{1/2}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\left(R_v^{\rm in}+\frac{|q|}{m}\int_0^t\|E[f(s)]\|_{L^\infty(\mathbb{R}^2)}\,{\rm d}s\right).\end{aligned}$$

Notice that once that we have obtained a L^∞ bound for the electric field, we also get a bound for the support size of the presence density f. The derivation of the L^∞ a priori estimate for $\partial_x E$ is more elaborated. It was shown in [2] that there is a constant $\tilde{C}$ (depending only on m,ε_0) such that

$$\|\partial_x E[f(t)]\|_{L^\infty}\leq \tilde{C}\left[(1+\|\rho[f(t)]\|_{L^\infty})(1+\ln^+\|\nabla_x\rho[f(t)]\|_{L^\infty})+\|\rho[f(t)]\|_{L^1}\right]. \quad (61)$$

Here the notation $\ln^+$ stands for the positive part of ln. We already have a priori bounds for the L^∞ norm of $\rho[f(t)]$ (use the estimate for the size of the support of f) and for the L^1 norm of $\rho[f(t)]$ (use the conservation of the total charge), and therefore (61) becomes

$$\|\partial_x E[f(t)]\|_{L^\infty}\leq C_1(1+\ln^+\|\nabla_x\rho[f(t)]\|_{L^\infty}),\ \ t\in[0,T] \quad (62)$$

for some constant C_1 depending on m,ε_0,q,T. We need to estimate the L^∞ norm of $\nabla_x\rho[f(t)]$ which writes

$$\begin{aligned}\nabla_x\rho[f(t)] &= q\nabla_x\int_{\mathbb{R}^2} f_{\rm in}(X(0;t,x,v),V(0;t,x,v))\,{\rm d}v \quad (63)\\ &= q\int_{\mathbb{R}^2}\mathbf{1}_{\{|v|\leq R_v(t)\}}\,{}^t\partial_x X(0;t,x,v)(\nabla_X f_{\rm in})(X(0;t,x,v),V(0;t,x,v))\,{\rm d}v\\ &+ q\int_{\mathbb{R}^2}\mathbf{1}_{\{|v|\leq R_v(t)\}}\,{}^t\partial_x V(0;t,x,v)(\nabla_V f_{\rm in})(X(0;t,x,v),V(0;t,x,v))\,{\rm d}v.\end{aligned}$$

A straightforward computation on the characteristic equations shows that there is a constant $C_2(m,q,T,\|B\|_{W^{1,\infty}})\geq 1$ such that for any $(x,v)\in\mathbb{R}^2\times\mathbb{R}^2, |v|\leq R_v(t)$, we have

$$|\partial_x X(0;t,x,v)|+|\partial_x V(0;t,x,v)|\leq C_2\exp\left(\int_0^t\|\partial_x E[f(s)]\|_{L^\infty}\,{\rm d}s\right),\ \ t\in[0,T].$$

Coming back to (63) we obtain

$$\|\nabla_x\rho[f(t)]\|_{L^\infty}\leq C_3\exp\left(\int_0^t\|\partial_x E[f(s)]\|_{L^\infty}\,{\rm d}s\right),\ \ t\in[0,T]$$

for some constant $C_3(m,q,T,\|B\|_{W^{1,\infty}})\geq 1$, and therefore

$$\ln^+\|\nabla_x\rho[f(t)]\|_{L^\infty}\leq \ln C_3+\int_0^t\|\partial_x E[f(s)]\|_{L^\infty}\,{\rm d}s,\ \ t\in[0,T].$$

Finally, combining with (62) yields

$$\|\partial_x E[f(t)]\|_{L^\infty}\leq C_4\left(1+\int_0^t\|\partial_x E[f(s)]\|_{L^\infty}\,{\rm d}s\right),\ \ t\in[0,T]$$

and the a priori estimate for the L^∞ norm of $\partial_x E[f]$ follows by Gronwall lemma. Based on the L^∞ estimates for the electric field together with its spatial derivatives, and also on the compactness of the presence density support, the inequality (56) comes immediately. □

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