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► To cite this version:

Jean Néraud, Carla Selmi. Embedding a θ -invariant code into a complete one. Theoretical Computer Science, In press, 10.1016/j.tcs.2018.08.022 . hal-01683320v3

HAL Id: hal-01683320

<https://hal.science/hal-01683320v3>

Submitted on 31 Aug 2018

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Embedding a θ -invariant code into a complete one

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Abstract

Let A be an arbitrary alphabet and let θ be an (anti-)automorphism of A^* (by definition, such a correspondence is determined by a permutation of the alphabet). This paper deals with sets which are invariant under θ (θ -invariant for short) that is, languages L satisfying $\theta(L) \subseteq L$. We establish an extension of the famous defect theorem. With regard to the so-called notion of completeness, we provide a series of examples of finite complete θ -invariant codes. Moreover, we establish a formula which allows to embed any non-complete θ -invariant code into a complete one. As a consequence, in the family of the so-called thin θ -invariant codes, maximality and completeness are two equivalent notions.

Keywords: antimorphism, anti-automorphism, automorphism, (anti-)automorphism, Bernoulli distribution, bifix, code, complete, context-free, defect, equation, finite, invariant, involutive, label, maximal, morphism, order, overlap, overlapping-free, prefix, regular, suffix, thin, tree, θ -invariant, θ -code, uniform, variable-length code, word

1. Introduction

In the free monoid theory, during the last decade, research involving one-to-one *morphic* or *antimorphic* substitutions has played a particularly important part: this is due to the powerful applications of these objects, in particular in the framework of DNA-computing. In the case of automorphisms or anti-automorphisms -for short we write *(anti-)automorphisms*- given an arbitrary *alphabet*, say A , any such mapping is completely determined by extending a unique permutation of A to A^* , the *free monoid* that is generated by A .

In the special case of *involutive* (anti-)automorphisms, lots of successful investigations have been done for extending most of the now classical combinatorial properties on words. The topics of the so-called *pseudo-palindromes* [9], that of θ -*episturmian* words [3], and the one of *pseudo-repetitions* [7, 14] have been particularly involved. The framework of some peculiar families of *variable-length codes* [15] and that of *equations in words* [5, 8, 16, 21] have been concerned. Generalizations of the famous theorem of Fine and Wilf ([13],[18, Proposition 1.3.5]) were also established [6, 20].

Equations in words are also the starting point of the study in the present paper, which consists in some full version of [22]. Let A be an arbitrary alphabet and let θ be an (anti-)automorphism of A^* ; we adopt the point of view from [18, Ch. 9], by considering a finite

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Preprint submitted to Elsevier

August 31, 2018

collection of unknown words, say Z . We assume that a (minimum) positive integer k (i.e. the so-called *order* of θ) exists such that $\theta^k = id_{A^*}$. This condition is particularly satisfied by every (anti-)automorphism whenever A is finite. In view of making the present forward more easily readable, in the first instance let us take θ as an involutive (anti-)automorphism (that is, $\theta^2 = id_{A^*}$). We assign that the words in Z and their images by θ to satisfy a given equation, and we ask for the computation of a finite set of words, say Y , such that all the words of Z can be expressed as a concatenation of words in Y . Actually, such a question appears more complex than in the classical configuration, where θ does not interfere: in this classical case, according to the famous defect theorem [18, Theorem 1.2.5], it is well known that at most $|Z| - 1$ words allow to compute the words in Z . At the contrary, due to the interference of (anti-)automorphisms, in [16], examples where $|Y| = |Z|$ are provided by the authors.

Along the way, for solving our problem, applying the defect theorem to the set $X = Z \cup \theta(Z)$ might appear natural. Such a methodology guarantees the existence of a set Y , with $|Y| \leq |X| - 1$ and whose elements allow by concatenation to rebuild all the words in X . It is also well known that Y can be chosen in such a way that only trivial equations may hold among its elements: with the terminology of [1, 18, 19], Y is a *code*, or equivalently Y^* , the submonoid that it generates, is *free*. Unfortunately, since both the words in Z and $\theta(Z)$ are expressed as concatenations of words in Y , among the words of $Y \cup \theta(Y)$ non-trivial equations can still hold. In other words, by applying that methodology, the initial problem would be transferred among the words in $Y \cup \theta(Y)$.

An alternative methodology will consist in asking for codes Y which are invariant under θ (θ -invariant for short) that is, satisfying $\theta(Y) = Y$. Returning to the general case, where θ is an arbitrary (anti-)automorphism, this is equivalent to say that the union of the sets $\theta^i(Y)$, for all $i \in \mathbb{Z}$, is θ -invariant. By the way, it is straightforward to show that the intersection of an arbitrary family of free θ -invariant submonoids is itself a free θ -invariant submonoid. In the present paper we prove the following result:

Theorem 1. *Let θ be an (anti-)automorphism of A^* and let X be a finite θ -invariant set. If X is not a code, then the smallest θ -invariant free submonoid of A^* containing X is generated by a θ -invariant code Y , which furthermore satisfies $|Y| \leq |X| - 1$.*

For illustrating this result in terms of equation, we refer to [5, 21], where the authors considered generalizations of the famous three unknown variables equation of Lyndon-Shützenberger [18, § 9.2]. They proved that, an involutive (anti-)automorphism θ being fixed, given such an equation with sufficiently long members, a word t exists such that any 3-uple of “solutions” can be expressed as a concatenation of words in $\{t\} \cup \{\theta(t)\}$. With the notation of Theorem 1, the elements of the θ -invariant set X are $x, y, z, \theta(x), \theta(y), \theta(z)$ and those of Y are t and $\theta(t)$: we verify that, in every case Y is a θ -invariant code, furthermore we have $|Y| \leq |X| - 1$.

With regard to the theory of codes, completeness is one of the most challenging notions: a subset X of the free monoid A^* is *complete* if any word is a factor of some word in X^* . Maximality is another important notion: a code is *maximal* if it cannot be strictly included in some other code of A^* . Actually, according to Zorn’s Lemma, any code is included in a maximal one moreover, a famous result due to Schützenberger states that, for the family of the so-called *thin* codes (which contains the regular codes), maximality and completeness are two equivalent notions [1, Theorem 2.5.16]. From this point of view, in the second part of our study we are interested in *complete* θ -invariant codes. It is natural to prealably examine the case of finite codes. Clearly, the well-known complete *uniform codes* that is, the codes A^n (with $n \geq 1$), are invariant under every (anti-)automorphism. Beside that, non-trivial

finite complete θ -invariant codes exist: for instance, take for A the binary alphabet $\{a, b\}$, choose for θ the anti-automorphism that swaps the letters a and b , and consider the complete code which was introduced in [4]:

$$X = \{a^3, ab, a^2ba, a^2b^2, ba^2, baba, bab^2, b^2a, b^3\}.$$

It is straightforward to verify that X is θ -invariant. In our paper, we provide some other examples: each of the classes of bifix codes, prefix codes, and non-prefix non-suffix codes is concerned.

Despite that, the question of describing a general structure for finite complete θ -invariant codes remains largely open: this is not surprising since, with the exception of certain special families (e.g. [11, 12, 24]), no general structure that could embrace finite complete codes is described in the literature.

Another issue could consist in developing methods for embedding a code into a complete one. However, in [23], the author presents a class of codes that cannot be embedded into any finite complete one. With regard to θ -invariance, as far as we know, the question of embedding finite codes into complete ones remains open.

Actually, in [23], the question whether any finite code can be embedded into a regular one was implicitly asked: a positive answer was brought in [10], where the authors provided a now classical formula for embedding any regular code into a complete one. In the present paper, we put a corresponding problem in the framework of θ -invariant codes. Actually, by establishing the following result, we bring a positive answer:

Theorem 2. *Any non-complete θ -invariant code $X \subseteq A^*$, can be embedded into a complete one. Moreover, if A is finite and X regular, then X can be embedded into a regular complete θ -invariant code.*

As a consequence, we obtain the following result: it states that, in the framework of θ -invariant codes, a property similar to a famous one due to Schützenberger [1, Theorem 2.5.16] holds:

Theorem 3. *Given a thin θ -invariant code $X \subseteq A^*$, the five following conditions are equivalent:*

- (i) *X is complete.*
- (ii) *X is a maximal code.*
- (iii) *X is maximal in the family of the θ -invariant codes.*
- (iv) *A positive Bernoulli distribution π exists such that $\pi(X) = 1$.*
- (v) *For any positive Bernoulli distribution π , we have $\pi(X) = 1$.*

We now describe the contents of our paper. Section 2 contains the preliminaries: the terminology of the free monoid is settled, and we recall some classical notions and results concerning the codes. The preceding Theorem 1 is established in Section 3, where an original example of equation is studied. In Section 4, we present several examples of finite complete θ -invariant codes. The problem of embedding a finite θ -invariant code into a complete one is also discussed: this ensures a transition to the question of embedding a regular θ -invariant code into a complete one. This last question is studied in Section 5, where the preceding Theorem 2 and Theorem 3 are established.

2. Preliminaries

2.1. Words and free monoid

We adopt the notation of the free monoid theory. In the whole paper, we consider an alphabet A , and we denote by A^* the free monoid that it generates. Given a word $w \in A^*$,

we denote by $|w|$ its length, the empty word, which we denote by ε , being the word with length 0. Given a subset X of A^* , we denote by X^* the submonoid of A^* that is generated by X , moreover we set $X^+ = X^* \setminus \{\varepsilon\}$.

Let $x \in A^*$ and $w \in A^+$. We say that x is a *prefix* (*suffix*) of w if a word u exists such that $w = xu$ ($w = ux$). Similarly, x is a *factor* of w if two words u, v exist such that $w = uxv$. Given a non-empty set $X \subseteq A^*$, we denote by $P(X)$ ($S(X)$, $F(X)$) the set of the words that are prefix (suffix, factor) of some word in X . Clearly, we have $X \subseteq P(X) \subseteq F(X)$ ($X \subseteq S(X) \subseteq F(X)$). Given a pair of non-empty words w, w' , we say that it *overlaps* if words u, v exist such that $uw' = wv$ or $w'u = vw$, with $1 \leq |u| \leq |w| - 1$ and $1 \leq |v| \leq |w'| - 1$; otherwise, the pair is *overlapping-free* (in such a case, if $w = w'$, we simply say that w is overlapping-free).

2.2. Variable length codes

It is assumed that the reader has a fundamental understanding with the main concepts of the theory of variable-length codes: we only recall some of the main definitions and we suggest, if necessary, that he (she) report to [1]. A subset X of A^* is a *variable-length code* (a *code* for short) if any equation among the words of X is trivial that is, for any pair of sequences of words in X , say $(x_i)_{1 \leq i \leq n}$, $(y_j)_{1 \leq j \leq p}$, the equation $x_1 \cdots x_n = y_1 \cdots y_p$ implies $n = p$ and $x_i = y_i$, for each integer $i \in [1, n]$. By definition X^* is a *free* submonoid of A^* .

In the present paper the so-called *prefix*, *suffix* and *bifix* codes play an noticeable part: a code $X \subseteq A^*$ is prefix (suffix) if $X \cap XA^+ = \emptyset$ ($X \cap A^+X = \emptyset$). A code is bifix if it is both prefix and suffix.

A code $X \subseteq A^*$ is *maximal* if it is not strictly included in some other code of A^* . Given a set $X \subseteq A^*$, it is *complete* if $A^* = F(X^*)$; X is *thin* if $A^* \neq F(X)$. Regular codes are well known examples of thin codes [1, Proposition 2.5.20].

A *positive Bernoulli distribution* is a morphism π from the free monoid A^* onto the multiplicative monoid $[0, 1]$, such that we have $\pi(a) > 0$ for every $a \in A$, and such that $\sum_{a \in A} \pi(a) = 1$. The *uniform* distribution corresponds to $\pi(a) = 1/|A|$, for every letter a . For any subset X of A^* , we set $\pi(X) = \sum_{x \in X} \pi(x)$. Clearly, the last sum may be finite or not, however if X is a thin subset we have $\pi(X) < \infty$ [1, Proposition 2.5.12]; moreover for every code $X \subseteq A^*$, we have $\pi(X) \leq 1$. From this point of view, the following result was established by Shützenberger (e.g. [1, Theorem 2.5.16]):

Theorem 2.1. *Given a thin code $X \subseteq A^*$, the four following conditions are equivalent:*

- (i) X is complete.
- (ii) X is a maximal code.
- (iii) A positive Bernoulli distribution π exists such that $\pi(X) = 1$.
- (iv) For any positive Bernoulli distribution π , we have $\pi(X) = 1$.

2.3. (Anti-)automorphisms

In the whole paper, we fix an alphabet A and a mapping θ onto A^* which is either an *automorphism* or an *anti-automorphism*: it is an anti-automorphism if it is one-to-one, with $\theta(\varepsilon) = \varepsilon$ and $\theta(xy) = \theta(y)\theta(x)$, for any pair of words x, y . For short in any case we write that θ is an *(anti-)automorphism*.

We say that the (anti-)automorphism θ is of *finite order* if some positive integer k exist such that $\theta^k = id_{A^*}$, the smallest one being the so-called *order* of θ (trivially id_{A^*} is of order 1). It is well known that such a condition is satisfied whenever A is a finite set; in particular, over a two letter alphabet, any non-trivial (anti-)automorphism is of order 2 that is, it is *involution*.

In the whole paper, we are interested in the family of sets $X \subseteq A^*$ that are invariant under θ (θ -invariant for short) that is, which satisfy $\theta(X) \subseteq X$; the mapping θ being one-to-one, this is equivalent to $\theta(X) = X$.

Example 1. Let $A = \{a, b, c, d\}$. Consider the (unique) anti-automorphism θ that is defined by $\theta(a) = a, \theta(b) = b, \theta(c) = d, \theta(d) = c$. It is straightforward to verify that the mapping θ is involutive, moreover the sets $\{cd\}$ and $\{abcd, cdba\}$ are θ -invariant.

Remark 1. In the spirit of the families of codes that were introduced in [15], given an (anti-)automorphism θ , define a θ -code as a set X such that $\bigcup_{i \in \mathbb{Z}} \theta^i(X)$ is a code. Clearly, with this definition any θ -code is a code; the converse is false, as attested below by Example 2.

Actually, any θ -code that is a maximal code, is necessarily θ -invariant. Indeed, assuming X not θ -invariant, we have $X \subsetneq X \cup \theta(X)$, thus X is strictly included in the code $\bigcup_{i \in \mathbb{Z}} \theta^i(X)$.

A similar argument proves that if X is maximal as a θ -code, then it is θ -invariant (indeed, $\bigcup_{i \in \mathbb{Z}} \theta^i(X)$ itself is a θ -code).

Taking account of the fundamental importance of the concept of maximality in the theory of codes, such properties reinforces the relevance of the notion of θ -invariant code.

Example 2. Let $A = \{a, b\}$ and θ be the so-called *mirror antimorphism*: $\theta(a) = a, \theta(b) = b$. Take for X the finite (prefix) code $\{a, ba\}$. We have $X \cup \theta(X) = \{a, ab, ba\}$, which is not a code ($ab \cdot a = a \cdot ba$).

3. A defect effect for invariant sets

We start with some considerations about θ -invariant submonoids of A^* . Clearly the intersection of a non-empty family of θ -invariant free submonoids of A^* is itself a θ -invariant free submonoid. Given a submonoid M of A^* , recall that its *minimal generating set* is $(M \setminus \{\varepsilon\}) \setminus (M \setminus \{\varepsilon\})^2$. The following property holds:

Proposition 3.1. *Given an alphabet A and given an (anti-)automorphism θ of A^* , let M be a submonoid of A^* and let $S \subseteq A^*$ such that $M = S^*$. Then the two following properties hold:*

- (i) *If S is θ -invariant then the same holds for M .*
- (ii) *If S is the minimal generating set of M and if M is θ -invariant then S is θ -invariant.*

Proof. (i) Assume that the set S is θ -invariant, and let $w \in M$. Since $M = S^*$, a finite sequence of words in S , namely $(s_i)_{1 \leq i \leq n}$, exists such that $w = s_1 \cdots s_n$. Since θ is an (anti-)automorphism, in every case $\theta(w)$ is some concatenation of the words $\theta(s_i)$ ($1 \leq i \leq n$), therefore we have $\theta(w) \in S^* = M$. Consequently M is θ -invariant.

(ii) Assume that M is θ -invariant and let $s \in S$. It follows from $S \subseteq M$ that we have $\theta(s) \in \theta(M) = M$ therefore, a sequence of words in S , namely $(s_i)_{1 \leq i \leq n}$, exists such that $\theta(s) = s_1 \cdots s_n$. Since θ is an (anti-)automorphism, s is in fact some concatenation of the words $\theta^{-1}(s_1), \dots, \theta^{-1}(s_n) \in M$. Moreover, for each integer $i \in [1, n]$, we have $\theta^{-1}(s_i) = s_i^1 \cdots s_i^{n_i}$, with $s_i^j \in S$ ($1 \leq j \leq n_i$). It follows from the definition of S that we have $n = 1$ and $s = s_1^1 = \theta^{-1}(s_1)$, thus $\theta(s) = s_1 \in S$. As a consequence, S itself is θ -invariant. $\square$

Informally, the famous defect theorem says that if some words in a set X satisfy a non-trivial equation, then these words can be written upon an alphabet of smaller size. In this section, we will examine whether a corresponding result may be stated in the framework of θ -invariant sets.

Theorem 3.2. *Given an alphabet A and given an (anti-)automorphism θ of A^* , let $X \subseteq A^*$ be a θ -invariant set. Let Y be the minimal generating set of the smallest θ -invariant free submonoid of A^* that contains X . If X is not a code, then we have $|Y| \leq |X| - 1$.*

With the notation of Theorem 3.2, since Y is a code, each word $x \in X$ has a unique factorization upon the words of Y , namely $x = y_1 \cdots y_n$, with $y_i \in Y$ ($1 \leq i \leq n$). In a classical way, we say that y_1 (y_n) is the *initial* (*terminal*) factor of x (with respect to such a factorization). From this point of view, before to prove Theorem 3.2, we need to establish the following statement:

Lemma 3.3. *With the preceding notation, each word in Y is the initial (terminal) factor of some word in X .*

Proof. By contradiction, assume that a word $y \in Y$ that is never initial of any word in X exists. Set $Z_0 = (Y \setminus \{y\})\{y\}^*$ and $Z_i = \theta^i(Z_0)$, for each integer $i \in \mathbb{Z}$. In a classical way (see e.g. [18, p. 7]), since Y is a code, Z_0 itself is a code.

For each integer $i \in \mathbb{Z}$, since θ^i is itself an (anti-)automorphism, Z_i is a code that is, Z_i^* is a free submonoid of A^* . Consequently, the intersection, say M , of the family $(Z_i^*)_{i \in \mathbb{Z}}$ is itself a free submonoid of A^* .

Let $w \in M$. For each integer $i \in \mathbb{Z}$, we have $w \in Z_i^*$, thus $\theta(w) \in \theta(Z_i^*) \subseteq (\theta(Z_i))^* = (\theta^{i+1}(Z_0))^* = Z_{i+1}^*$. Consequently we have $\theta(w) \in \bigcap_{i \in \mathbb{Z}} Z_{i+1}^* = \bigcap_{i \in \mathbb{Z}} Z_i^* = M$, whence we have $\theta(M) \subseteq M$ therefore, since θ is onto, we obtain $\theta(M) = M$.

Let x be an arbitrary word in X . Since $X \subseteq Y^*$, and according to the definition of y , we have $x = (z_1 y^{k_1})(z_2 y^{k_2}) \cdots (z_n y^{k_n})$, with $n \geq 1$, $z_1, \dots, z_n \in Y \setminus \{y\}$ and $k_1, \dots, k_n \geq 0$. Consequently x belongs to Z_0^* , therefore we have $X \subseteq Z_0^*$. Since X is θ -invariant, this implies $X = \theta^i(X) \subseteq \theta^i(Z_0^*) \subseteq Z_i^*$, for each $i \in \mathbb{Z}$, thus $X \subseteq M$.

But the word y belongs to Y^* and does not belong to Z_0^* thus, it doesn't belong to M . This implies $X \subseteq M \subsetneq Y^*$: a contradiction with the minimality of Y^* . Clearly, similar arguments may be applied to words $y \in Y$ that are never terminal of any word in X : this completes the proof. $\square$

Proof of Theorem 3.2. Let α be the mapping from X onto Y which, with every word $x \in X$, associates the initial factor of x in its (unique) factorization over Y^* . According to Lemma 3.3, α is onto. We will prove that it is not one-to-one. Classically, since X is not a code, a non-trivial equation may be written among its words, say: $x_1 \cdots x_n = x'_1 \cdots x'_p$, with $x_i, x'_j \in X$ $x_1 \neq x'_1$ ($1 \leq i \leq n, 1 \leq j \leq p$). Since Y is a code, a unique sequence of words in Y , namely $y_1, \dots, y_m$ ($m \geq 1$) exists such that: $x_1 \cdots x_n = x'_1 \cdots x'_p = y_1 \cdots y_m$. This implies $y_1 = \alpha(x_1) = \alpha(x'_1)$ and completes the proof. $\square$

In what follows we discuss some interpretation of Theorem 3.2 with regard to equations in words. For this purpose, we assume that A is finite, θ being of order k , and we consider a finite set of words, say Z . Let X be the union of the sets $\theta^i(Z)$, for $i \in [0, k-1]$, and assume that a non-trivial equation holds among the words of X , namely $x_1 \cdots x_m = y_1 \cdots y_p$. By construction X is θ -invariant therefore, according to Theorem 3.2, a θ -invariant code Y exists such that $X \subseteq Y^*$, with $|Y| \leq |X| - 1$. This means that each of the words in X can be expressed by making use of at most $|X| - 1$ words of type $\theta^i(u)$, with $u \in Y$ and $0 \leq i \leq k-1$. It will be easily verified that the examples from [5, 16, 21] corroborate this fact; moreover, below we mention an original one:

Example 3. Let θ be an anti-automorphism of order 3. Consider two different words x, y , with $|x| > |y| > 0$, satisfying the equation: $x\theta(y) = \theta^2(y)\theta(x)$. With this condition, a pair of words u, v exists such that $x = uv$, $\theta(x) = v\theta(y)$, $\theta^2(y) = u$, thus $y = \theta(u)$. It follows

from $x = uv$ that $v\theta(y) = \theta(x) = \theta(v)\theta(u)$, thus $v = \theta(v)$ and $\theta(y) = \theta(u)$. This implies $y = u = \theta(u) = \theta^2(u)$ and $v = \theta(v) = \theta^2(v)$. Moreover, we have $\theta(x) = vu$, $\theta^2(x) = uv$: we obtain $x = \theta^2(x)$ thus, $x = \theta(x) = \theta^2(x)$; hence we have $uv = vu$. Consequently, a non-empty word t and integers i, j exist such that $u = t^i$, $v = t^j$. With the preceding notation, we have $Z = \{x, y\}$, $X = Z \cup \theta(Z) \cup \theta^2(Z) = \{x, y\}$, $Y = \{t\}$. We verify that $|Y| \leq |X| - 1$.

4. Finite complete θ -invariant codes

In this section we are interested in finite complete θ -invariant codes over an alphabet A . Given an arbitrary letter $a \in A$, since for every non-negative integer n , we have $a^n \in F(X^*)$, necessarily a (unique) positive integer p exists such that $a^p \in X$; therefore, A is necessarily finite. Several examples of finite complete θ -invariant codes will be presented. We start with prefix codes, which certainly constitute the best-known class of them.

4.1. Finite complete prefix θ -invariant codes

Actually finite complete prefix codes play a peculiar part in the framework of codes. A famous result due to Schützenberger [25] (cf. also [2]) states that any finite complete code with a *finite deciphering delay* (e.g. [1, Ch. 5]) is necessarily a prefix code. In particular, over A^* only one finite complete *circular* code (or, equivalently, finite complete *uniformly synchronized code*) can exist, namely the alphabet A itself (cf. [1, Ch. 7, Ch. 10], [17]).

It is well-known that each prefix set, say X , can be represented by a tree, say $\mathcal{T}(X)$, of arity $|A|$: in this representation, each node (i.e. vertice) is a prefix of some word in X (i.e. an element of $P(X)$), the root being ε , the empty word. Moreover, given two nodes u, v and a letter $a \in A$, an edge with label a exists from u to v in $\mathcal{T}(X)$ if, and only if, we have $v = ua$: we denote such a labeled edge by (u, a, v) and we say that v is a successor of u . In that representation, complete prefix codes correspond to *complete trees*, in the sense where each interior node has exactly $|A|$ successors.

We start with the case where θ is an automorphism of A^* . Given a prefix set $X \subseteq A^*$, we say that the corresponding tree $\mathcal{T}(X)$ is *invariant* under θ whenever (u, a, v) is an edge of $\mathcal{T}(X)$ if, and only if, $(\theta(u), \theta(a), \theta(v))$ is an edge of $\mathcal{T}(X)$. With this notion, a characterization of θ -invariant prefix codes may be stated:

Claim 1. *Let A be a finite alphabet, let θ be an automorphism of A^* and let X be a prefix code. Then X is θ -invariant if, and only if, the tree $\mathcal{T}(X)$ itself is invariant under θ .*

Proof. Assume that X is θ -invariant, and let (u, a, ua) an arbitrary edge in $\mathcal{T}(X)$. By construction a word $s \in S(X)$ exists such that $uas \in X$. Since X is a θ -invariant set, this implies $\theta(u)\theta(as) = \theta(ua)\theta(s) \in X$, thus $\theta(u)$ and $\theta(ua) \in P(X)$. Consequently, $(\theta(u), \theta(a), \theta(u)\theta(a))$ is an edge of $\mathcal{T}(X)$, therefore $\mathcal{T}(X)$ is invariant under θ .

Conversely, assume that $\mathcal{T}(X)$ is invariant under θ . Let $w = w_1 \cdots w_n \in X$, with $w_i \in A$ ($1 \leq i \leq n$). By construction, the following sequence of edges exists in $\mathcal{T}(X)$ (for $i = 0$, we set $w_1 \cdots w_i = \varepsilon$):

$$(w_1 \cdots w_i, w_{i+1}, w_1 \cdots w_{i+1}) \quad (0 \leq i \leq n-1),$$

moreover the node $w = w_1 \cdots w_n$ has no successor. Since $\mathcal{T}(X)$ is invariant under θ , a corresponding sequence of edges exists in $\mathcal{T}(X)$, namely:

$$(\theta(w_1) \cdots \theta(w_i), \theta(w_{i+1}), \theta(w_1) \cdots \theta(w_{i+1})) \quad (0 \leq i \leq n-1).$$

Since the node $w_1 \cdots w_n$ has no successor, the same holds for the corresponding node $\theta(w) = \theta(w_1 \cdots w_n)$: this implies $\theta(w) \in X$. $\square$

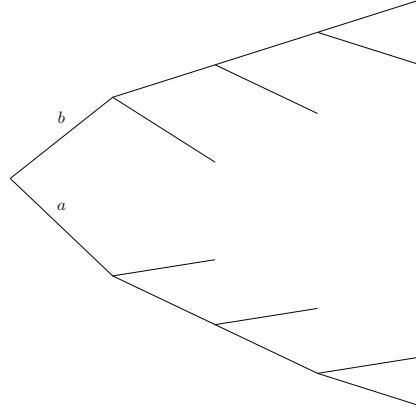


Figure 1: Example 4 with $n = 4$. In the tree $\mathcal{T}(X)$, each bottom-up (top-down) branch represents an edge with label b (a).

Example 4. Let $A = \{a, b\}$, and θ be the automorphism defined by $\theta(a) = b$, $\theta(b) = a$. Given an arbitrary integer $n \geq 3$, consider the following set:

$$X = \bigcup_{1 \leq i \leq n-1} \{a^i b, b^i a\} \cup \{a^n, b^n\}.$$

By construction, X is a prefix code. Moreover, X is complete: this can be directly verified by examining $\mathcal{T}(X)$ (an alternative method consists in applying Theorem 2.1 (iii), with π the uniform Bernoulli distribution). It is also straightforward to verify that X is θ -invariant.

Note that X is not bifix: indeed, for each integer $i \in [2, n-1]$, the word $ab \in X$ is a suffix of $a^i b \in X$. Figure 1 illustrates the corresponding tree $\mathcal{T}(X)$ for $n = 4$.

In the case where θ is an anti-automorphism, the following property is noticeable:

Claim 2. *Let θ be an anti-automorphism onto A^* and let $X \subseteq A^*$ be a finite θ -invariant code. If X is prefix, then it is necessarily bifix.*

Proof. By contradiction, assume X not bifix, thus not suffix: words $p \in A^*$, $s \in A^+$ exist

such that $s, ps \in X$. Since X is θ -invariant, we have $\theta(s), \theta(ps) \in X$, thus $\theta(s), \theta(s)\theta(p) \in X$: this contradicts the fact that X is a prefix code. $\square$

The result of Claim 2 directly leads to examine the behavior of finite complete bifix codes with regard to (anti-)automorphisms.

4.2. Finite complete bifix θ -invariant codes

At first, it is worth mentioning a well-known class of finite bifix codes:

Example 5. A set X is *uniform* if a positive integer n exists such that $X \subseteq A^n$. Trivially, such a set is a bifix code moreover, it is complete if, and only if, we have $X = A^n$. It is straightforward to verify that X is invariant under every (anti-)automorphism of A^* : indeed, the restriction of such a mapping on words of length n induces a permutation of A^n .

It is a natural question to ask whether non-uniform finite complete bifix θ -invariant codes exist. By exhibiting infinite classes of convenient codes, the three following examples allow to bring a positive answer. Actually, the two first families of codes have been constructed by applying a famous internal transformation to some uniform code [4] (cf. also [1, § 6.2])

Example 6. Let $A = \{a, b\}$ and θ be the anti-automorphism of A^* that is defined by $\theta(a) = b, \theta(b) = a$.

Let $n = 2k + 1$, with $k \geq 1$. Consider the following set:

$$X = (A^n \setminus (Aa^kb^k \cup a^kb^kA)) \cup \{a^kb^k\} \cup Aa^kb^kA.$$

The set Aa^kb^kA is a (uniform) bifix code. Since the condition $a^kb^k \in P(X)$ ($a^kb^k \in S(X)$) necessarily implies $a^kb^k \in P(a^kb^kA)$ ($a^kb^k \in S(Aa^kb^kA)$), X is a finite (non-uniform) bifix code. The code X is complete: indeed, we have $Aa^kb^k \cap a^kb^kA = \emptyset$ therefore, given an arbitrary positive Bernoulli distribution π over A^* , we have:

$$\pi(X) = \pi(A^n \setminus (Aa^kb^k \cup a^kb^kA)) + \pi(a^kb^k) + \pi(Aa^kb^kA) = 1 - 2\pi(a^kb^k) + 2\pi(a^kb^k) = 1.$$

Furthermore, since we have $\theta(A) = A$ and $\theta(a^kb^k) = a^kb^k$, X is θ -invariant.

For $n = 3$, the preceding construction leads to the following finite complete bifix θ -invariant code [4, (1)]:

$$X = \{a^3, ba^2, b^2a, b^3, ab, a^2ba, a^2b^2, baba, bab^2\}.$$

Example 7. Let $A = \{a, b\}$ and θ be the so-called mirror-image, which is in fact the anti-automorphism defined by $\theta(a) = a, \theta(b) = b$.

Take $n = 3k + 1$, with $k \geq 1$. We have $a^kb^ka^k \notin P(Aa^kb^ka^k) \cup S(a^kb^ka^kA)$; therefore an examination similar to the one we applied at Example 6 leads to verify that the following set is a finite complete bifix θ -invariant code:

$$X = (A^n \setminus (Aa^kb^ka^k \cup a^kb^ka^kA)) \cup \{a^kb^ka^k\} \cup Aa^kb^ka^kA.$$

For $n = 4$ (i.e. $k = 1$) the corresponding binary tree $\mathcal{T}(X)$ is represented in Figure 2.

We observe that, in view of constructing arbitrarily large non-uniform finite bifix θ -invariant codes over arbitrarily large finite alphabets, the two last constructions can be generalized, as illustrated by the following example:

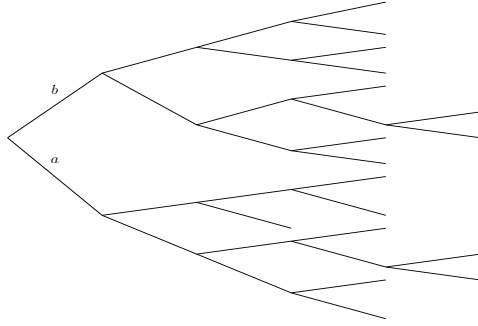


Figure 2: Example 7: the case where $n = 4$, thus $k = 1$.

Example 8. 1) Let $A = \{a, b, c\}$ and θ be the anti-automorphism defined by $\theta(a) = b$, $\theta(b) = c$, $\theta(c) = a$.

Take $n = 2k + 1$, with $k \geq 1$, and

$$W = \bigcup_{i \in \mathbb{Z}} \{\theta^i(a^k b^k)\} = \{a^k b^k, c^k b^k, c^k a^k, b^k a^k, b^k c^k, a^k c^k\}.$$

By construction we have $W \cap (P(AW) \cup S(WA)) = \emptyset$ therefore, the following set is a θ -invariant finite bifix code:

$$X = (A^n \setminus (AW \cup WA)) \cup W \cup AW A. \quad (1)$$

Moreover, X is complete: indeed, by construction we have $AW \cap WA = \emptyset$ therefore, for any positive Bernoulli distribution over A^* , we have $\pi(X) = 1 - 2\pi(W) + 2\pi(W) = 1$.

2) Similarly, over $A = \{a, b, c\}$ take for θ the anti-automorphism onto A^* defined by $\theta(a) = b$, $\theta(b) = c$, $\theta(c) = a$. Set $n = 3k + 1$ and

$$W = \bigcup_{i \in \mathbb{Z}} \{\theta^i(a^k b^k a^k)\} = \{a^k b^k a^k, b^k c^k b^k, c^k a^k c^k\}.$$

Applying the construction from (1) also leads to obtain a finite complete bifix θ -invariant code.

4.3. Non-prefix non-suffix finite complete θ -invariant codes

In the most general case, given an (anti-)automorphism θ of A^* , we are looking to finite codes $X \subseteq A^*$ which are both complete and θ -invariant. With regard to the last condition, the following statement brings some characterization:

Claim 3. *Let θ be an (anti-)automorphism onto A^* and let X be a finite subset of A^* . Then X is θ -invariant if, and only if, it is the disjoint union of a finite family of uniform θ -invariant codes.*

Proof. Let $\ell_1 < \dots < \ell_n$ be the unique increasing finite sequence of the lengths of the words in X . For each $i \in [1, n]$, set $X_i = X \cap A^{\ell_i}$. By construction, each set X_i is a uniform code, moreover we have:

$$X = \bigcup_{1 \leq i \leq n} X_i.$$

Clearly, the set X is θ -invariant if, and only if, for each integer $i \in [1, n]$, θ induces a permutation of X_i itself. $\square$

When X is required to be a code, Claim 3 only leads to some necessary condition. For instance, the set $\{a, ab, b\} = \{a, b\} \cup \{ab\}$, which satisfies the condition of the claim, is θ -invariant, but clearly it is not a code. Actually, despite that in any case θ -invariance is preserved with respect to the union of sets, the main obstacle is that, given two (disjoint) codes, there is no characterization that can guarantee that their union remains a code.

Of course, one can wonder about the impact of θ -invariance itself on the structure of a finite complete code. Indeed, in view of the above, such an influence is very strong with regard to two special families of codes: the uniform ones and, with respect to automorphisms, the family of prefix non-suffix codes. However, the part of θ -invariance appeared in fact of lesser importance in the construction of our families of bifix codes, where it essentially involved the structure of a few convenient words (e.g. the elements of W).

Things become even more complex when attempting to construct finite complete θ -invariant codes that are neither prefix nor suffix. Indeed, with regard to finite complete codes, although that some famous families have been exhibited (e.g. [11, 12]), no general structure is known. However, finite complete θ -invariant codes that are neither prefix nor suffix exist as attested by the following example:

Example 9. With the anti-automorphism θ that was introduced in Example 6 (which swaps the letters a and b), consider the classical finite complete code $X = \{a^2, ab, a^2b, ab^2, b^2\}$ [24, Example 2], which is neither prefix, nor suffix. It is straightforward to verify that it is θ -invariant (we have $\theta(ab) = ab$).

4.4. Toward the construction of regular complete θ -invariant codes

In [23], by making use of factorizations of the so-called cyclotomic polynomials, the author provided a family of non-finitely completable codes. It is therefore a natural question to ask whether corresponding objects exist in the framework of θ -invariant codes.

Let A be a finite alphabet, and let θ be an (anti-)automorphism of A^* . Given a finite code X , if X is embeddable into a complete θ -invariant code, say Y , then, with the terminology of Remark 1, it has to be a θ -code. Indeed the set $\bigcup_{i \in \mathbb{Z}} \theta^i(X)$ is necessarily a θ -invariant code that is included in Y . Therefore, our problem comes down to wonder whether a given finite θ -invariant code can be embedded into a complete one.

We begin by strictly restraining the problem to the framework of prefix codes. Given a (non-trivial) automorphism θ , according to the preceding Claim 1 any θ -invariant prefix

code can be embedded into a θ -invariant complete one. Informally, it suffices to complete the corresponding tree with convenient ones of arity $|A|$ that are invariant under θ .

In the case of anti-automorphisms, according to Claim 2, for being embeddable into a complete one, a finite prefix θ -invariant code has to be bifix. However, the converse is false; indeed there are finite bifix θ -invariant codes that cannot be included into any complete one, as attested by the following example:

Example 10. 1) Let $A = \{a, b\}$, and let θ be the mirror anti-automorphism of Example 7. At first, we observe that the finite θ -invariant bifix code $X = \{aa, b\}$ cannot embed into any finite complete bifix (not necessarily θ -invariant) code. Indeed, assume that such a complete code, say Z , exists: necessarily Z is prefix and complete, hence for any positive integer p , we have $ab^p \in P(Z^*)$. Therefore a positive integer n exists such that ab^n belongs to Z ; since b belongs to Z this contradicts the fact that Z is bifix.

As a consequence X cannot be included in any finite complete prefix θ -invariant code. Indeed, according to Claim 2, such a code should be bifix.

2) Note that the infinite (regular) set $Z = \{b\} \cup \{ab^n a : n \in \mathbb{N}\}$ is a θ -invariant bifix code which contains X . Moreover, taking for π the uniform Bernoulli distribution, it is straightforward to verify that we have $\pi(Z) = 1/2 + 1/4 \sum_{n \in \mathbb{N}} (1/2^n) = 1$, thus Z is complete.

We do not know whether there are finite θ -invariant complete codes that contain the code X of Example 10. Actually, as far as we know, the question of embedding a finite θ -invariant code into a complete one remains open.

From another angle, the study in [23] led its author to conclude that the study of all finite codes requires also investigations on the infinite ones. From that, the question of embedding a finite code into a regular one was open. A positive answer was given in [10], where a now famous method for embedding a regular code into a complete one was published.

From this last point of view, in the next section, we will interest in the problem of embedding a regular θ -invariant code into a regular complete one.

5. Embedding a regular θ -invariant code into a complete one

5.1. Some notation

In this section we consider an (anti-)automorphism θ of A^* , and a non-complete θ -invariant code $X \subseteq A^*$. We ask for a complete regular θ -invariant code Y such that $X \subseteq Y$. We will bring a positive answer: let's begin by describing our construction.

Let X be a non-complete θ -invariant code, and let $y \notin F(X^*)$. Necessarily, we have $|A| \geq 2$ (otherwise, X should be complete). Without loss of generality, we may assume that the initial and the terminal letters of y are different (otherwise, substitute to y the word $ay\bar{a}$, with $a, \bar{a} \in A$ and $a \neq \bar{a}$): in particular, we have $|y| \geq 2$. Set:

$$z = \bar{a}^{|y|} y a^{|y|} \quad (\text{with } y \in aA^*\bar{a}). \quad (2)$$

Since θ is an (anti-)automorphism, for each integer $i \in \mathbb{Z}$, two different letters $b, \bar{b}$ exist such that the following property holds:

$$\theta^i(z) = \bar{b}^{|y|} \theta^i(y) b^{|y|} \quad (\text{with } \theta^i(y) \in bA^*\bar{b}). \quad (3)$$

Finally, we introduce the three following sets:

$$Z = \bigcup_{i \in \mathbb{Z}} \{\theta^i(z)\}, \quad (4)$$

$$W = ZA^* \cap A^*Z, \quad (5)$$

$$T = W \setminus (W \cup X)(W \cup X)^+. \quad (6)$$

By construction, the following inclusion holds:

$$W \subseteq (X \cup T)^+. \quad (7)$$

5.2. Basic properties of Z

By construction, each element of the preceding set Z has length $3|y|$. Given two (not necessarily different) integers $i, j \in \mathbb{Z}$, we will accurately study how the two words $\theta^i(z), \theta^j(z)$ may overlap.

Lemma 5.1. *With the notation in (3), let $u, v \in A^+$, $i, j \in \mathbb{Z}$ such that $|u| \leq |z| - 1$ and $\theta^i(z)v = u\theta^j(z)$. Then we have $|u| = |v| \geq 2|y|$, moreover a letter b and a unique positive integer k (depending of $|u|$) exist such that we have $\theta^i(z) = ub^k$, $\theta^j(z) = b^kv$, with $k \leq |y|$.*

Proof. According to (3), we set $\theta^i(z) = \bar{b}^{|y|}bx'\bar{b}b^{|y|}$ and $\theta^j(z) = \bar{c}^{|y|}cx''\bar{c}c^{|y|}$, with $b, \bar{b}, c, \bar{c} \in A$, $b \neq \bar{b}, c \neq \bar{c}$ and $|x'| = |x''| = |y| - 2$. Since θ is an (anti-)automorphism, we have $|\theta^i(z)| = |\theta^j(z)|$, thus $|u| = |v|$; since we have $1 \leq |u| \leq 3|y| - 1$, exactly one of the following cases occurs:

Case 1: $1 \leq |u| \leq |y| - 1$. With this condition, we have $(\theta^i(z))_{|u|+1} = \bar{b} = \bar{c} = (u\theta^j(z))_{|u|+1}$ and $(\theta^i(z))_{|y|+1} = b = \bar{c} = (u\theta^j(z))_{|y|+1}$, which contradicts $b \neq \bar{b}$.

Case 2: $|u| = |y|$. This condition implies $(\theta^i(z))_{|u|+1} = b = \bar{c} = (u\theta^j(z))_{|u|+1}$ and $(\theta^i(z))_{2|y|} = \bar{b} = \bar{c} = (u\theta^j(z))_{2|y|}$, which contradicts $b \neq \bar{b}$.

Case 3: $|y| + 1 \leq |u| \leq 2|y| - 1$. We obtain $(\theta^i(z))_{2|y|} = \bar{b} = \bar{c} = (u\theta^j(z))_{2|y|}$ and $(\theta^i(z))_{2|y|+1} = b = \bar{c} = (u\theta^j(z))_{2|y|+1}$ which contradicts $b \neq \bar{b}$.

Case 4: $2|y| \leq |u| \leq |z| - 1 = 3|y| - 1$. With this condition, necessarily we have $(\theta^i(z))_{|u|+1} = b = \bar{c} = (u\theta^j(z))_{|u|+1}$, therefore an integer $k \in [1, |y|]$ exists such that $\theta^i(z) = ub^k$ and $\theta^j(z) = b^kv$. $\square$

Lemma 5.2. *With the preceding notation, we have $A^+ZA^+ \cap ZX^*Z = \emptyset$.*

Proof. By contradiction, assume that $z_1, z_2, z_3 \in Z$, $x \in X^*$ and $u, v \in A^+$ exist such that $uz_1v = z_2xz_3$. By comparing the lengths of u, v with $|z|$, exactly one of the three following cases occurs:

Case 1: $|z| \leq |u|$ and $|z| \leq |v|$. With this condition, we have $z_2 \in P(u)$ and $z_3 \in S(v)$, therefore the word z_1 is a factor of x : this contradicts $Z \cap F(X^*) = \emptyset$.

Case 2: $|u| < |z| \leq |v|$. We have in fact $u \in P(z_2)$ and $z_3 \in S(v)$. We are in the condition of Lemma 5.1: the words z_2, z_1 overlap. Consequently, $u, z'_1 \in A^+$ and $b \in A$ exist such that $z_2 = ub^k$ and $z_1 = b^kz'_1$, with $1 \leq k \leq |y|$ and $|z'_1| = |u|$. But, by construction, we have $|uz_1| = |z_2xz_3| - |v|$. Since we assume $|v| \geq |z|$, this implies $|uz_1| \leq |z_2xz_3| - |z| = |z_2x|$, hence we obtain $uz_1 = ub^kz'_1 \in P(z_2x)$. It follows from $z_2 = ub^k$ that $z'_1 \in P(x)$. Since we have $z_1 \in Z$ and according to (3), $i \in \mathbb{Z}$ and $\bar{b} \in A$ exist such that we have $z_1 = b^kz'_1 = b^{|y|}\theta^i(y)\bar{b}^{|y|}$. Since by Lemma 5.1 we have $|z'_1| = |u| \geq 2|y|$, we obtain $\theta^i(y) \in F(z'_1)$, thus $\theta^i(y) \in F(x)$, which contradicts $y \notin F(X^*)$.

Case 3: $|v| < |z| \leq |u|$. Some similar arguments on the reversed words lead to a conclusion similar to that of Case 2.

Case 4: $|z| > |u|$ and $|z| > |v|$. With this condition, both the pairs of words z_2, z_1 and z_1, z_3

overlap. Once more we are in the condition of Lemma 5.1: letters c, d , words u, v, s, t , and integers h, k exist such that the two following properties hold:

$$z_2 = uc^h, \quad z_1 = c^h s, \quad |u| = |s| \geq 2|y|, \quad h \leq |y|, \quad (8)$$

$$z_1 = td^k, \quad z_3 = d^k v, \quad |v| = |t| \geq 2|y|, \quad k \leq |y|. \quad (9)$$

It follows from $uz_1v = z_2xz_3$ that $uz_1v = (uc^h)x(d^kv)$, thus $z_1 = c^hxd^k$. But according to (3), $i \in \mathbb{Z}$ and $\bar{c} \in A$ exist such that we have $z_1 = c^{|y|\theta^i(y)}\bar{c}^{|y|}$. Since we have $h, k \leq |y|$, this implies $d = \bar{c}$ moreover $\theta^i(y)$ is a factor of x . Once more, this contradicts $y \notin F(X^*)$. $\square$

As a direct consequence of Lemma 5.2, we obtain the following result:

Corollary 5.3. *With the preceding notation, X^*Z is a prefix code.*

Proof. Let $z_1, z_2 \in Z, x_1, x_2 \in X^*, u \in A^+$, such that $x_1z_1u = x_2z_2$. For any word $z_3 \in Z$, we have $(z_3x_1)z_1(u) = z_3x_2z_1$, a contradiction with Lemma 5.2. $\square$

5.3. The consequences for the set $X \cup T$

Lemma 5.4. *The set $X \cup T$ is a θ -invariant code.*

Proof. The fact that $X \cup T$ is θ -invariant comes from its construction. For proving that it is a code, we consider an arbitrary equation among the words in $X \cup T$. Since X is a code, and since $z \notin F(X^*)$, we may assume that at least one occurrence of a word in T appears in each side of the equation, therefore this equation takes the following form:

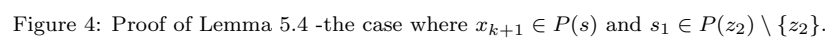
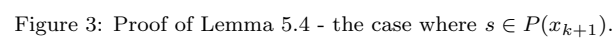
$$x_0t_0x_1t_1 \cdots t_{n-1}x_n = x'_0t'_0 \cdots t'_{p-1}x'_p, \quad (10)$$

with $x_i, x'_j \in X^*$ ($0 \leq i \leq n, 0 \leq j \leq p$) and $t_i, t'_j \in T$ ($0 \leq i \leq n-1, 0 \leq j \leq p-1$). Since by construction we have $T \subseteq W \subseteq ZA^*$, each side of the equation has a prefix in X^*Z . According to Corollary 5.3 and since all the words in Z have a common length, this implies $x_0 = x'_0$, therefore our equation is equivalent to:

$$t_0x_1t_1 \cdots t_{n-1}x_n = t'_0x'_1 \cdots t'_{p-1}x'_p. \quad (11)$$

Without loss of generality, we assume that $|t_0| \leq |t'_0|$; let k be the greatest non-negative integer such that a word s exists with $t_0x_1t_1 \cdots t_kx_{k+1}s = t'_0$, with $s \in A^*$. By contradiction, we assume $s \neq \varepsilon$. Let $z_0 \in Z$ ($z_1 \in Z$) be the unique word such that $t'_0 \in A^*z_0$ ($t_k \in A^*z_1$). According to the preceding property (3), an integer $i \in \mathbb{Z}$ and two letters $b, \bar{b}$ exist such that $z_0 = \bar{b}^{|y|\theta^i(y)}b^{|y|}$. Moreover, since we have $y \notin F(X^*)$, and since X is θ -invariant, we have $\theta^i(y) \notin F(X^*)$. By construction, the set with elements $t_0x_1 \cdots t_kx_{k+1}$ and t'_0 is not prefix; more precisely, exactly one of the two following main conditions holds:

1. At first, we assume that $t'_0 \in P(t_0x_1 \cdots t_kx_{k+1})$ that is, $s \in P(x_{k+1})$ (cf. Figure 3). By construction, at least one of the two words s, z_0 is a suffix of the other one. Actually, since we have $z_0 \notin F(X^*)$, necessarily s is a proper suffix of z_0 , therefore (z_1, z_0) is an overlapping pair of words. According to Lemma 5.1, necessarily we have $|s| \geq 2|y|$, which implies $\theta^i(y) \in F(s)$: a contradiction with $\theta^i(y) \notin F(X^*)$.
2. Now, we assume that $t_0x_1 \cdots t_kx_{k+1}$ is a proper prefix of t'_0 , thus $x_{k+1} \in P(s)$: we have $s = x_{k+1}s_1$, with $s_1 \neq \varepsilon$. Let $z_2 \in Z$ be the unique word such that $t_{k+1} \in z_2A^*$. By construction the set with elements $t_0x_1 \cdots t_kx_{k+1}z_2$ and t'_0 is not prefix. More precisely, exactly one of the two following cases occurs:



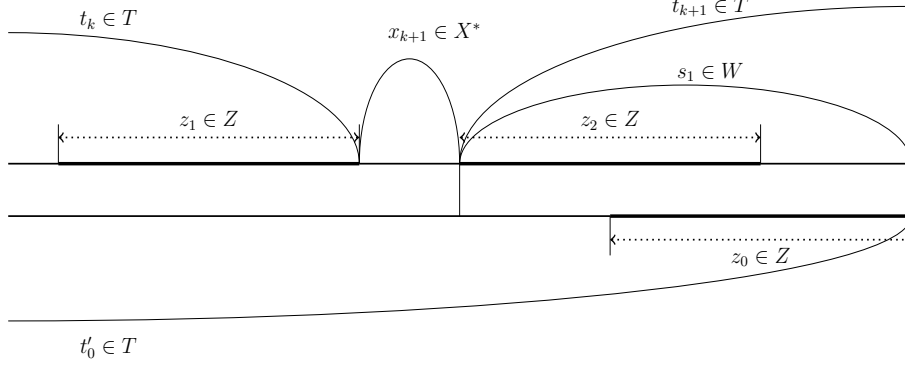


Figure 5: The case where $x_{k+1} \in P(S)$, with $z_2 \in P(s_1)$ in the proof of Lemma 5.4.

- 2.1 The first case corresponds to t'_0 being a proper prefix of $t_0x_1 \cdots t_kx_{k+1}z_2$, that is s_1 being a proper prefix of z_2 (cf. Figure 4). With this condition, the word z_0 is necessarily a factor of $z_1x_{k+1}z_2$. According to Lemma 5.2, since we have $s_1 \neq \varepsilon$, this implies $z_1 = z_0$, which contradicts $s \neq \varepsilon$.
- 2.2 It remains to consider the case where z_2 is a prefix of s_1 (cf. Figure 5). Actually, since $t_0x_1 \cdots t_kx_{k+1}z_2$ is a prefix of $t_0x_1 \cdots t_kx_{k+1}s_1 = t'_0$, with $|z_2| = |z_0|$, necessarily z_0 is a suffix of s_1 , hence we have $s_1 \in z_2A^* \cap A^*z_0$, thus $s_1 \in W$ according to (5). We obtain $t'_0 = t_0x_1 \cdots t_kx_{k+1}s_1 \in (TX^*)^+W$, thus $t'_0 \in (T \cup X)^+W$: this contradicts (6).

In each case we obtain a contradiction: as a consequence we have $s = \varepsilon$, thus $t'_0 = t_0x_1 \cdots x_k t_k$. Once more according to (6), it follows from $t'_0 \in T$ that we have $k = 0$, thus $t'_0 = t_0$. As a consequence, Equation (11) is equivalent to the following one:

$$x_1 t_1 \cdots t_{n-1} x_n = x'_1 \cdots t'_{p-1} x_p. \quad (12)$$

By iterating these arguments, we shall obtain: $n = p$ and $x_i = x'_i$, $t'_j = t_j$ ($0 \leq i \leq n$, $0 \leq j \leq n-1$), therefore $X \cup T$ is a code: this completes the proof of Lemma 5.4. $\square$

Lemma 5.5. *The code $X \cup T$ is complete.*

Proof. Let $w \in A^*$. According to the construction of W we have $ZwZ \subseteq W$. According to (7) this implies $ZwZ \subseteq (X \cup T)^*$, therefore we have $w \in F((X \cup T)^*)$. $\square$

In the case where A is a finite alphabet, the (anti-)automorphism θ is of finite order. If X is a regular code, in starting with $y \notin F(X^*)$, the construction in (4) leads to a finite set Z : this guarantees the regularity of the sets W and T . As a direct consequence, we obtain the following result:

Theorem 5.6. *Given a non-complete θ -invariant code $X \subseteq A^*$, the two following properties hold:*

- (i) *In any case, X can be embedded into a complete θ -invariant code in A^* .*
- (ii) *If A is finite and X regular, then X can be embedded into a regular complete θ -invariant code in A^* .*

Example 11. Let $A = \{a, b\}$, and θ be the anti-automorphism such that $\theta(a) = b$, $\theta(b) = a$, and let $X = \{a^4, a^2b^2, a^2b^4, a^4b^2, ba, ba^4, b^4a, b^4\}$

Trivially, X is θ -invariant. By applying Sardinas-Patterson algorithm [1, § 2.3], one can easily verify that X is a (non-prefix) code. It is non complete: by making use of the uniform Bernoulli distribution π , we obtain $\pi(X) < 1$.

Let $y = ba^3ba$. Firstly, we note that we have $y \notin F(X)$, hence $y \in F(X^*)$ implies $y = sp$, with $s \in S(X)$ and $p \in P(X^*)$. Secondly, we have $S(X) \cap P(y) = \{b, ba\}$, but since $\{a^3ba, a^2ba\} \cap P(X^*) = \emptyset$, necessarily we have $y \notin F(X^*)$. Thirdly, in view of obtaining an overlapping-free word, we substitute $by = b^2a^3ba$ to y (we have $by \notin F(X^*)$).

With the notation (2, 4), we have $z = a^7b^2a^3bab^7$, thus:

$Z = \bigcup_{i \in \mathbb{Z}} \{\theta^i(z)\} = \{a^7b^2a^3bab^7, a^7bab^3a^2b^7\}$. Moreover, the sets W and T shall be constructed according to (5,6).

Example 11 provides a (non-finite) regular complete θ -invariant code; in the sequel we give an example of a non-regular one:

Example 12. Let $A = \{a, b\}$, and θ be an arbitrary (anti-)automorphism of A^* . Consider the famous Dyck language $D_1^* = \{w \in A^* : |w|_a = |w|_b\}$. Each of its elements is classically represented by a so-called Dyck path in the grid $\mathbb{N} \times \mathbb{Z}$. To be more precise, with each word $w = w_1 \cdots w_n$ (with $w_i \in A$, for $1 \leq i \leq n$), a unique path is associated, namely $(i, y_i)_{0 \leq i \leq n}$, with $y_0 = y_n = 0$ and such that, for each $i \in [1, n]$:

$$w_i = a \implies y_i = y_{i-1} + 1 \quad \text{and} \quad w_i = b \implies y_i = y_{i-1} - 1.$$

By construction, D_1^* is a free submonoid of A^* . Its minimal generating set is the so-called Dyck code D_1 , whose elements are represented by those of the preceding non-empty paths which satisfy the following condition:

$$n \geq 2 \quad \text{and} \quad (\forall i \in [1, n-1]) \quad y_i \neq 0.$$

The code D_1 is well known for being a (non-thin) complete context-free language (e.g. [1, Example 2.5.3]). Moreover, according to Proposition 3.1, since D_1^* is θ -invariant, the same holds to the Dyck code.

As a consequence of Theorem 5.6, we obtain the following result, which states a property similar to [1, Theorem 2.5.16] in the framework of θ -invariant code:

Theorem 5.7. *Given a thin θ -invariant code $X \subseteq A^*$, the following conditions are equivalent:*

- (i) X is complete.
- (ii) X is a maximal code.
- (iii) X is maximal in the family θ -invariant codes.
- (iv) A positive Bernoulli distribution π exists such that $\pi(X) = 1$.
- (v) For any positive Bernoulli distribution π , we have $\pi(X) = 1$.

Proof. According to [1, Theorem 2.5.16], the conditions (i), (ii), (iv), (v) are equivalent. Trivially, Condition (ii) implies Condition (iii). By contradiction, we prove that Condition (iii) implies Condition (i). Starting with a non-complete θ -invariant code X , according to Theorem 5.6 the existence of a complete θ -invariant code that strictly contains X is guaranteed, thus X is not maximal in the family of θ -invariant codes: this completes the proof. $\square$

Aknowledgement

We would like to thank the anonymous reviewers for their fruitful suggestions and comments.

References

- [1] J. Berstel, D. Perrin, and C. Reutenauer. *Codes and Automata*. Cambridge University Press, 2010.
- [2] V. Bruyère. Automata and codes with a bounded deciphering delay. In I. Simon, editor, *LATIN'92*, volume 583, pages 99–107. Lect. Notes in Comp. Sci., 1992.
- [3] M. Bucci, A. de Luca, A. De Luca, and L. Q. Zamboni. On θ -episturmian words. *Eur. J. of Comb.*, 30:473–479, 2009.
- [4] V. Césari. Sur un algorithme donnant les codes bipréfixes finis. *Theory of Computing Systems*, 6:221–225, 1972.
- [5] E. Czeizler, E. Czeizler, L. Kari, and S. Seki. An extension of the Lyndon-Schützenberger result to pseudoperiodic words. *Inf. Comput.*, 209:717–730, 2011.
- [6] E. Czeizler, L. Kari, and S. Seki. On a special class of primitive words. *Theoret. Comp. Sci.*, 411:617–630, 2010.
- [7] C. Annal Deva Priya Darshini, V. Rajkumar Dare, I. Venkat., and K.G. Subramanian. Factors of words under an involution. *J. of Math. and Inf.*, 1:52–59, 2013–2014, ISSN: 2349-0632 (P), 2349-0640.
- [8] J.D. Day, P. Fleishmann, F. Manea F., and D. Nowotka. Equations enforcing repetitions under permutations. In S. Brlek, F. Dolce, C. Reutenauer, and E. Vandomme, editors, *Combinatorics on Words*, volume 10432, pages 72–84. Lect. Notes in Comp. Sci., 2017.
- [9] A. de Luca and A. De Luca. Pseudopalindrome closure operators in free monoids. *Theoret. Comp. Sci.*, 362:282–300, 2006.
- [10] A. Ehrenfeucht and S. Rozenberg. Each regular code is included in a regular maximal one. *RAIRO - Theor. Inform. Appl.*, 20:89–96, 1986.
- [11] C. De Felice. Construction of a family of finite maximal codes. *Theoret. Comput. Sci.*, 63:157–184, 1989.
- [12] C. De Felice and A. Restivo. Some results on finite maximal codes. *RAIRO - Theoret. Informatics and Appl.*, 19:383–403, 1985.
- [13] N.J. Fine and H. S. Wilf. Uniqueness theorem for periodic functions. *Proc. Am. Math. Soc.*, 16:109–114, 1965.
- [14] P. Gawrychowski, F. Manea, R. Mercas, D. Nowotka, and C. Tisceanu. Finding pseudo-repetitions. In N. Portier and T. Wilke, editors, *30th International Symposium on Theoretical Aspects of Computer Science (STACS 2013)*, volume 20 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 257–267, Dagstuhl, Germany, 2013. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [15] L. Kari and K. Mahalingam. Dna codes and their properties. In C. Mao and T. Yokomori, editors, *12th International Meeting on DNA Computing (DNA12)*, volume 4287, pages 127–142. Lect. Notes in Comp. Sci., June 2006.
- [16] L. Kari and K. Mahalingam. Watson-Crick conjugate and commutative words. In M.H. Garzon and H. Yan, editors, *DNA Computing. DNA 2007*, volume 4848, pages 273–283. Lect. Notes in Comp. Sci., 2008.
- [17] J.-L. Lassez. Circular codes and synchronization. *Internat. J. Computer Syst. Sci.*, 5:201–208, 1976.
- [18] M. Lothaire. *Combinatorics on Words*. Addison-Wesley Publishing Company (2nd edition Cambridge University Press 1997), 1983.
- [19] M. Lothaire. *Algebraic Combinatorics on Words*. Cambridge University Press, 2002.
- [20] F. Manea, R. Mercas, and D. Nowotka. Fine and Wilf’s theorem and pseudo-repetitions. In B. Rován, V. Sassone, and P. Widmayer, editors, *Acts of Mathematical Foundations of Computer Science 2012 (MFCS 2012)*, volume 7464, pages 668–680. Lect. Notes in Comp. Sci., 2012.
- [21] F. Manea, M. Müller, D. Nowotka, and S. Seki. Generalised Lyndon-Schützenberger equations. In E. Csuhaj-Varjú, M. Dietzfelbinger, and Z. Ésik, editors, *Mathematical Foundations of Computer Science 2014 (MFCS 2014)*, volume 8634, pages 402–413. Lect. Notes in Comp. Sci., 2014.
- [22] J. Néraud and C. Selmi. Invariance: a theoretical approach for coding sets of words modulo literal (anti)morphisms. In S. Brlek, F. Dolce, C. Reutenauer, and E. Vandomme, editors, *Combinatorics on Words*, volume 10432, pages 214–227. Lect. Notes in Comp. Sci., 2017.
- [23] A. Restivo. On codes having no finite completion. *Discr. Math.*, 17:309–316, 1977.
- [24] A. Restivo, S. Salemi, and T. Sportelli. Completing codes. *RAIRO - Theoret. Informatics and Appl.*, 23:135–147, 1989.
- [25] M.-P. Schützenberger. On a question concerning certain free submonoids. *J. Combin. Theory*, 1:437–442, 1966.