



# Heuristics and conjectures in direction of a p-adic Brauer–Siegel theorem

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# HEURISTICS AND CONJECTURES IN DIRECTION OF A $p$ -ADIC BRAUER–SIEGEL THEOREM

GEORGES GRAS

ABSTRACT. Let  $p$  be a fixed prime number. Let  $K$  be a totally real number field of discriminant  $D_K$  and let  $\mathcal{T}_K$  be the torsion group of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension of  $K$  (under Leopoldt’s conjecture). We conjecture the existence of a constant  $\mathcal{C}_p > 0$  such that  $\log(\#\mathcal{T}_K) \leq \mathcal{C}_p \cdot \log(\sqrt{D_K})$  when  $K$  varies in some specified families (e.g., fields of fixed degree). In some sense, we suggest the existence of a  $p$ -adic analogue, of the classical Brauer–Siegel Theorem, wearing here on the valuation of the residue at  $s = 1$  (essentially equal to  $\#\mathcal{T}_K$ ) of the  $p$ -adic  $\zeta$ -function  $\zeta_p(s)$  of  $K$ . We shall use a different definition than of Washington, given in the 1980’s, and approach this question via the arithmetical study of  $\mathcal{T}_K$  since  $p$ -adic analysis seems to fail because of possible abundant “Siegel zeros” of  $\zeta_p(s)$ , contrary to the classical framework. We give extensive numerical verifications for quadratic and cubic fields (cyclic or not) and publish the PARI/GP programs directly usable by the reader for numerical improvements. Such a conjecture (if exact) reinforces our conjecture that any fixed number field  $K$  is  $p$ -rational (i.e.,  $\mathcal{T}_K = 1$ ) for all  $p \gg 0$ .

## 1. ABELIAN $p$ -RAMIFICATION – MAIN DEFINITIONS AND NOTATIONS

Let  $K$  be a totally real number field of degree  $d$ , and let  $p \geq 2$  be a prime number fulfilling the Leopoldt conjecture in  $K$ . We denote by  $\mathcal{O}_K$  the  $p$ -class group of  $K$  (ordinary sense) and by  $E_K$  the group of  $p$ -principal global units  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$  of  $K$ .

Let’s recall from [9, 12] the diagram of the so called *abelian  $p$ -ramification theory*, in which  $K^c = K\mathbb{Q}^c$  is the cyclotomic  $\mathbb{Z}_p$ -extension of  $K$  (as compositum with that of  $\mathbb{Q}$ ),  $H_K$  the  $p$ -Hilbert class field and  $H_K^{\text{Pr}}$  the maximal abelian  $p$ -ramified (i.e., unramified outside  $p$ ) pro- $p$ -extension of  $K$ .

Let  $U_K := \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}^1$  be the  $\mathbb{Z}_p$ -module (of  $\mathbb{Z}_p$ -rank  $d$ ) of  $p$ -principal local units of  $K$ , where each  $U_{\mathfrak{p}}^1 := \{u \in K_{\mathfrak{p}}^{\times}, u \equiv 1 \pmod{\bar{\mathfrak{p}}}\}$  is the group of  $\bar{\mathfrak{p}}$ -principal units of the completion  $K_{\mathfrak{p}}$  of  $K$  at  $\mathfrak{p} \mid p$ , where  $\bar{\mathfrak{p}}$  is the maximal ideal of the ring of integers of  $K_{\mathfrak{p}}$ .

For any field  $k$ , let  $\mu_k$  be the group of roots of unity of  $k$  of  $p$ -power order. Then put  $W_K := \text{tor}_{\mathbb{Z}_p}(U_K) = \bigoplus_{\mathfrak{p}|p} \mu_{K_{\mathfrak{p}}}$  and  $\mathcal{W}_K := W_K / \mu_K$ , where  $\mu_K = \{1\}$  or  $\{\pm 1\}$ .

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Let  $\overline{E}_K$  be the closure in  $U_K$  of the diagonal image of  $E_K$ ; by class field theory this gives in the diagram  $\text{Gal}(H_K^{\text{pr}}/H_K) \simeq U_K/\overline{E}_K$ ; then let  $\mathcal{O}_K^c$  be the subgroup of  $\mathcal{O}_K$  corresponding to the subgroup  $\text{Gal}(H_K/K^c \cap H_K)$ .

Put (see [9, Chapter III, § 2 (a) & Theorem 2.5] with the set  $S$  of infinite places, to get the ordinary sense, and with the set  $T$  of  $p$ -places):

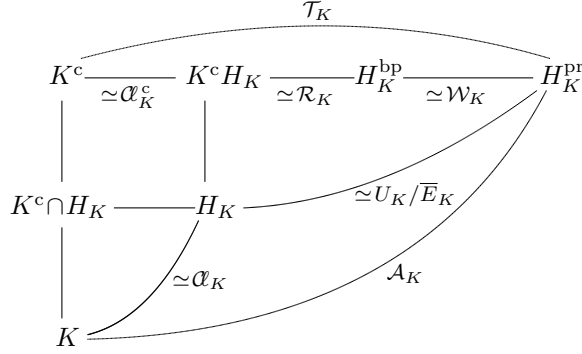
$$\mathcal{T}_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/K)) = \text{Gal}(H_K^{\text{pr}}/K^c).$$

As we know,  $\#\mathcal{T}_K$  is essentially the residue of the  $p$ -adic  $\zeta$ -function of  $K$  at  $s = 1$  [6, 34]; we will detail this in Subsection 2.2.

We have (because of Leopoldt's conjecture) the following exact sequence defining  $\mathcal{R}_K$ , where  $\log_p$  is the  $p$ -adic logarithm ([9, Lemma III.4.2.4 & Corollary III.3.6.3], [12, Lemma 3.1 & § 5]):

$$1 \rightarrow \mathcal{W}_K \rightarrow \text{tor}_{\mathbb{Z}_p}(U_K/\overline{E}_K) \xrightarrow{\log_p} \text{tor}_{\mathbb{Z}_p}(\log_p(U_K)/\log_p(\overline{E}_K)) =: \mathcal{R}_K \rightarrow 0.$$

The group  $\mathcal{R}_K$  (or its order) is called the *normalized  $p$ -adic regulator of  $K$*  and makes sense for any number field (provided one replaces  $K^c$  by the compositum  $\tilde{K}$  of the  $\mathbb{Z}_p$ -extensions):



The field  $H_K^{\text{bp}}$ , fixed by  $\mathcal{W}_K$ , is the Bertrandias–Payan field, i.e., the compositum of the  $p$ -cyclic extensions of  $K$  embeddable in  $p$ -cyclic extensions of arbitrary large degree.

## 2. $v$ -ADIC ANALYTIC PROSPECTS

Let  $\mathcal{K}_{\text{real}}$  (resp.  $\mathcal{K}_{\text{real}}^{(d)}$ ) be the set of totally real number fields  $K$  of any degree (resp. of fixed degree  $d$ ). For a fixed prime  $p$  and a random  $K \in \mathcal{K}_{\text{real}}$ , we have:

$$\#\mathcal{T}_K = \#\mathcal{O}_K^c \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K,$$

which may be equal to 1 (defining “ $p$ -rational fields”) or not, and it will be interesting to know if the  $p$ -adic valuation of  $\#\mathcal{T}_K$  can be bounded according, for instance, to the discriminant  $D_K$  of  $K$ . If so, this would be interpreted as a  $p$ -adic version of the archimedean Brauer–Siegel theorem, which is currently pure speculation, but we intend to experiment, algebraically, this context since  $p$ -adic analysis does not seem to succeed as explain by Washington in [40]:

*A Brauer–Siegel theorem using  $p$ -adic  $L$ -functions fails;*

in the same way, we have similar comments by Ivanov in [23, Section 1]:

*The  $p$ -adic analogue of Brauer–Siegel and hence also of Tsfasman–Vladuț fails.*

But this requires some explanation:

**2.1. The Siegel zeros.** In fact, there is a possible ambiguity about the definitions and the role of the discriminant in a  $p$ -adic Brauer–Siegel frame.

Let  $K \in \mathcal{K}_{\text{real}}$ , let  $h_K$  be its class number,  $R_{K,p}$  its classical  $p$ -adic regulator,  $D_K$  its discriminant; in [40, § 3], Washington considers a sequence of such number fields  $K$ , fulfilling the condition  $\frac{[K : \mathbb{Q}]}{v_p(\sqrt{D_K})} \rightarrow 0$ , and study the limit:

$$\lim_K \left( \frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})} \right),$$

where  $v_p$  denotes the  $p$ -adic valuation; thus the above condition implies that  $p$  must be “highly ramified” in the fields of the sequence, which eliminates for instance families of fields of constant degree  $d$ . So, with Washington’s definition,  $K$  belongs in general to some towers of number fields (e.g., the cyclotomic one).

Washington shows examples and counterexamples of the  $p$ -adic Brauer–Siegel property  $\frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})} \rightarrow 1$  ([40, Proposition 2 & Theorem 2]). In his Theorem 3,

he uses the formula of Coates [5, p. 364], which implies  $\liminf_K \left( \frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})} \right) \geq 1$  as  $\frac{[K : \mathbb{Q}]}{v_p(\sqrt{D_K})} \rightarrow 0$ . We shall consider instead  $\frac{v_p\left(h_K \cdot \frac{R_{K,p}}{\sqrt{D_K}}\right) \cdot \log_{\infty}(p)}{\log_{\infty}(\sqrt{D_K})}$ , where  $\log_{\infty}$  is the usual complex logarithm, or more precisely we shall study:

$$C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_{\infty}(p)}{\log_{\infty}(\sqrt{D_K})} = \frac{\log_{\infty}(\#\mathcal{T}_K)}{\log_{\infty}(\sqrt{D_K})}, \text{ for any } K \in \mathcal{K}_{\text{real}},$$

then the existence of  $\sup_{K \in \mathcal{K}} (C_p(K))$ , and of  $\limsup_{K \in \mathcal{K}} (C_p(K))$ , for any given infinite set  $\mathcal{K} \subseteq \mathcal{K}_{\text{real}}$ , and  $\sup_p (C_p(K))$ ,  $\limsup_p (C_p(K)) \in \{0, \infty\}$  for  $K$  fixed (see Conjectures 7.1, 7.2). However, there are some connections between the two definitions since the quantity  $v_p(h_K \cdot R_{K,p})$  appears in each of them; only the measure of the order of magnitude differs for the analysis of sequences of fields. It is therefore not surprising to find, for instance in [36, 40, 41], some allusions to the group  $\mathcal{T}_K$ .

Let’s finish these comments with a quote from Washington’s paper illustrating the crucial fact that a great  $v_p(\#\mathcal{T}_K)$  is related to the existence of zeros, of the  $p$ -adic  $\zeta$ -function, or of the  $L_p$ -functions (see [36, 41, 42, 43] for complements about these zeros and for some numerical data):

*In the proof of the classical Brauer–Siegel theorem, one needs the fact that there is at most one Siegel zero, that is, a zero close to 1. The fact that the Brauer–Siegel theorem fails  $p$ -adically could be taken as further evidence for the abundance of  $p$ -adic zeroes near 1.*

(...)

*Finally, we remark that the possible existence of  $p$ -adic Siegel zeroes and the failure of results such as the  $p$ -adic Brauer–Siegel theorem indicate that it could be difficult, if not impossible, to do analytic number theory with  $p$ -adic  $L$ -functions. For example, I do not know how to obtain estimates on  $\pi(x)$ , the number of primes less than or equal to  $x$ , using the fact that the  $p$ -adic zeta function has a pole at 1.*

**Remark 2.1.** One may explain what appens as follows, for simplicity in the case of a real quadratic field  $K$  of character  $\chi_K$ :

Roughly speaking,  $v_p(L_p(1, \chi_K))$  is closely related to  $v_p(\#\mathcal{T}_K)$  and  $v_p(L_p(0, \chi_K))$  is closely related to  $v_p(B_1(\omega^{-1} \chi_K))$  ( $\omega$  is the Teichmüller character and  $B_1(\omega^{-1} \chi_K)$  the generalized Bernoulli number of character  $\omega^{-1} \chi_K$ ), which is closely related to the order of a suitable component of the  $p$ -class group of the “mirror field  $K^*$ ” (e.g., for  $p = 3$  and  $K = \mathbb{Q}(\sqrt{m})$ ,  $K^* = \mathbb{Q}(\sqrt{-3m})$ ); but since  $\omega^{-1} \chi_K$  is odd, no unit intervenes and  $v_p(L_p(0, \chi_K))$  is usually “small” compared to  $v_p(\#\mathcal{T}_K)$  assumed to be “very large” (e.g.,  $m = 150094635296999122$  giving  $v_3(\#\mathcal{T}_K) = 19$  but  $v_3(\#\mathcal{C}_{K^*}) = 1$ ). Thus, there exist in general “Siegel zeros” of  $L_p(s, \chi_K)$ , i.e., very close to 1, which is an obstruction to a Brauer–Siegel strategy (see numerical illustrations for  $p = 2, 3$  in [41, 42, 43]).

Consequently we will adopt another point of view. Let  $K \in \mathcal{K}_{\text{real}}$  and let  $p \geq 2$  be any fixed prime number. As we have recalled it,  $\#\mathcal{T}_K$  is in close relationship with  $p$ -adic  $L$ -functions (at  $s = 1$ ) of even Dirichlet characters in the abelian case (Kubota–Leopoldt, Barsky, Amice–Fresnel,...), or more generally with the residue at  $s = 1$  of the  $p$ -adic  $\zeta$ -function of  $K$ , built or study by many authors (Coates, Shintani, Barsky, Serre, Cassou–Noguès, Deligne–Ribet, Katz, Colmez,...). Conversely, there is no *algebraic* invariant (like a Galois group) interpreting the residue of the complex  $\zeta$ -function, but we have in this (archimedean) case numerous inequalities. So, we shall compare the complex and  $p$ -adic cases to try to unify the set of all the points of view. For this, we define normalizations of the  $\zeta$ -functions of a totally real number field (from [5, 6], then [12] for the regulators).

**2.2. Definitions and normalizations.** Let  $K \in \mathcal{K}_{\text{real}}$  be of degree  $d$  and let:

$$\mathcal{P} := \{p_\infty, 2, 3, \dots, p, \dots\}$$

be the set of places of  $\mathbb{Q}$ , including the infinite place  $p_\infty$  (we also use the symbol  $\infty$  for real or complex functions, like log-function, in the same logic as for  $p$ -adic ones; for instance,  $R_{K,\infty}$  and  $R_{K,p}$  shall be the usual regulators built with  $\log_\infty$  and  $\log_p$ , respectively). We shall use, for any place  $v \in \mathcal{P}$ , subscripts  $(\bullet)_{K,v}$  for all invariants considered; when the context is clear, we omit  $v$  ( $p$ -adic in most cases).

**2.2.1.  $v$ -CYCLOTOMIC EXTENSIONS AND  $v$ -CONDUCTORS.** The  $p$ -cyclotomic  $\mathbb{Z}_p$ -extension is denoted  $\mathbb{Q}^{c,p}$  and we introduce  $\mathbb{Q}^{c,p_\infty} := \mathbb{Q}$  as the “ $p_\infty$ -cyclotomic extension”. We put  $\mathbb{Q}^{c,v} =: \mathbb{Q}^c$  for any  $v \in \mathcal{P}$  if there is no ambiguity. We attribute to the field  $\mathbb{Q}$  the “ $v$ -conductor”  $\mathfrak{f}_{\mathbb{Q},v} := p$  (resp. 4, 2) if  $v = p \neq 2$  (resp. 2,  $p_\infty$ ). We shall put  $\sim$  for equalities up to a  $p$ -adic unit.

**2.2.2. NORMALIZED  $\zeta$ -FUNCTIONS AT  $p_\infty$ .** We define at the infinite place  $p_\infty$ :

$$(2.1) \quad \tilde{\zeta}_{K,p_\infty}(s) := \frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \cdot \zeta_{K,p_\infty}(s) = \frac{1}{2^{d-1}} \cdot \zeta_{K,p_\infty}(s), \quad s \in \mathbb{C}$$

(see [9, Remark III.2.6.5 (ii)] for justifications about the factor  $\frac{1}{2^d}$ ); then, let  $h_K$  be the class number (ordinary sense),  $R_{K,\infty}$  the classical regulator,  $D_K$  the discriminant of  $K$ , and  $\mathcal{W}_{K,p_\infty} := \bigoplus_{w|p_\infty} \mu_{K_w} / \mu_K$ , of order  $2^{d-1}$  since  $K$  is totally real.

Then consider, with a perfect analogy with the  $p$ -adic case:

$$(2.2) \quad \#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{2^{d-1} \cdot \sqrt{D_K}} \cdot \#\mathcal{W}_{K,p_\infty} = h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} \cdot 1$$

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<sup>1</sup> The factor  $\frac{R_{K,\infty}}{2^{d-1} \cdot \sqrt{D_K}}$  is by definition the normalized regulator  $\mathcal{R}_{K,p_\infty}$  for  $v = p_\infty$ , using the normalized log-function  $\frac{1}{2} \cdot \log_\infty$  instead of  $\log_\infty$ ; from [1], it is defined without ambiguity.

Let  $\tilde{\kappa}_{K,p_\infty}$  be the residue at  $s = 1$  of  $\tilde{\zeta}_{K,p_\infty}(s)$ . From the so-called complex “analytic formula of the class number” of  $K$  (see, e.g., [39, Chap. 4]), we get:

$$(2.3) \quad \tilde{\kappa}_{K,p_\infty} = h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} = \#\mathcal{T}_{K,p_\infty}.$$

2.2.3. **NORMALIZED  $\zeta_p$ -FUNCTIONS AT  $v = p$ .** We define at a finite place  $p$ :

$$(2.4) \quad \tilde{\zeta}_{K,p}(s) := \frac{f_{K \cap \mathbb{Q}^c}}{2^d} \cdot \zeta_{K,p}(s), \quad s \in \mathbb{Z}_p,$$

where  $f_{K \cap \mathbb{Q}^c}$  is the conductor of  $K \cap \mathbb{Q}^c$  (if  $K \cap \mathbb{Q}^c$  is the  $n$ th stage in  $\mathbb{Q}^c$ , then  $f_{K \cap \mathbb{Q}^c} \sim 2p \cdot [K \cap \mathbb{Q}^c : \mathbb{Q}] \sim 2p^{n+1}$ ); since from [5, 6, 34], the residue of  $\zeta_{K,p}(s)$  at  $s = 1$  is  $\kappa_{K,p} \sim \frac{2^{d-1} \cdot h_K \cdot R_{K,p}}{\sqrt{D_K}}$ , we get the normalized  $p$ -adic residue:

$$(2.5) \quad \tilde{\kappa}_{K,p} = \frac{f_{K \cap \mathbb{Q}^c}}{2^d} \cdot \kappa_{K,p} \sim \#\mathcal{T}_{K,p} \text{ (see Subsection 2.4 for the abelian case).}$$

So, the residues of the normalized  $\zeta_v$ -functions of  $K$  are, for all  $v \in \mathcal{P}$ , such that:

$$\tilde{\kappa}_{K,v} := \lim_{s \rightarrow 1} (s-1) \cdot \frac{f_{K \cap \mathbb{Q}^c}}{2^d} \cdot \zeta_{K,v}(s) \sim \#\mathcal{T}_{K,v},$$

which is the order of an arithmetical invariant for finite places  $v = p$  and the measure of a real volume for  $v = p_\infty$  (see the last footnote).

2.3. **Abelian complex  $L$ -functions – Upper bounds.** In the abelian case:

$$(2.6) \quad \#\mathcal{T}_{K,p_\infty} = h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} = \prod_{\chi \neq 1} \frac{1}{2} L_{p_\infty}(1, \chi),$$

where  $\chi$  goes through all the corresponding Dirichlet characters of  $K$  with conductor  $f_\chi$ , and where  $L_{p_\infty}$  denotes the complex  $L$ -function. If  $K = \mathbb{Q}(\sqrt{m})$ , of fundamental unit  $\varepsilon_K$  and quadratic character  $\chi_K$ , one gets:

$$\#\mathcal{T}_{K,p_\infty} = h_K \cdot \frac{\log_\infty(\varepsilon_K)}{\sqrt{D_K}} = \frac{1}{2} \cdot L_{p_\infty}(1, \chi_K).$$

For each  $L_{p_\infty}(1, \chi)$  one has many upper bounds which are improvements of the classical inequality  $\frac{1}{2} \cdot L_{p_\infty}(1, \chi) \leq (1 + o(1)) \cdot \log_\infty(\sqrt{f_\chi})$ . In [32, Corollaire 1] one has, for even primitive characters:

$$\frac{1}{2} \cdot L_{p_\infty}(1, \chi) \leq \frac{1}{2} \cdot \log_\infty(\sqrt{f_\chi}),$$

giving from the previous definition (2.2) and formula (2.6):

$$(2.7) \quad \log_\infty(\#\mathcal{T}_{K,p_\infty}) \leq \mathcal{C}_{p_\infty} \cdot \log_\infty(\sqrt{D_K}),$$

with an explicit constant  $\mathcal{C}_{p_\infty}$  if  $K$  runs through the set of real abelian fields such that  $\frac{d}{\log_\infty(\sqrt{D_K})} \rightarrow 0$ , for instance in the simplest form of Brauer–Siegel theorem.

We shall give numerical complements in Subsection 7.2 by means of computations of lower and upper bounds of:

$$C_{p_\infty}(K) = \widetilde{BS}_K := \frac{\log_\infty(\#\mathcal{T}_{K,p_\infty})}{\log_\infty(\sqrt{D_K})}$$

(see Definition 7.1).

---

Thus, the factor  $\mathcal{W}_{K,p_\infty}$  does exist as in the  $p$ -adic case. The invariant  $\mathcal{T}_{K,p_\infty}$  is related to the Arakelov class group of  $K$  (see [33] and its bibliography), which gives the best interpretation.

*Remark 2.2.* For the sequel, we do not need any sophisticated upper bound (only the existence of  $\mathcal{C}_{p_\infty}$ ), but one may refer to [18, 26, 27, 30, 32] for other inequalities; for instance, one gets, for real abelian fields  $K$  of degree  $d$ , with our notations:

$$\#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} \leq \left( \frac{1}{2} \frac{\log_\infty(\sqrt{D_K})}{d-1} \right)^{d-1},$$

thus in the cases  $d = 2$  and  $d = 3$ :

$$\#\mathcal{T}_{K,p_\infty} = h_K \cdot \frac{\log_\infty(\varepsilon_K)}{\sqrt{D_K}} \leq \frac{1}{2} \log_\infty(\sqrt{D_K}), \quad \#\mathcal{T}_{K,p_\infty} = h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} \leq \frac{1}{16} \left( \log_\infty(\sqrt{D_K}) \right)^2,$$

respectively. In the quadratic and cubic cases one shows that:

$$(2.8) \quad h_K \leq \frac{1}{2} \cdot \sqrt{D_K}, \quad h_K \leq \frac{2}{3} \cdot \sqrt{D_K}, \quad \text{respectively.}$$

**2.4. Abelian  $L_p$ -functions.** The Kubota–Leopoldt  $p$ -adic  $L$ -functions give rise to the analytic formula [1, § 2.1 & Théorème 6, § 2.3]:

$$h_K \cdot \frac{R_{K,p}}{\sqrt{D_K}} \sim \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) \cdot \prod_{\chi \neq 1} \left( 1 - \frac{\chi(p)}{p} \right)^{-1}.$$

The “ $p$ -adic class number formula” for real abelian fields uses the formula of [5]:

$$\#\mathcal{T}_{K,p} \sim [K \cap \mathbb{Q}^c : \mathbb{Q}] \cdot \frac{p}{\prod_{\mathfrak{p}|p} \mathbf{N}\mathfrak{p}} \cdot h_K \cdot \frac{R_{K,p}}{\sqrt{D_K}}.$$

Thus, since  $\prod_{\chi} \left( 1 - \frac{\chi(p)}{p} \right)^{-1} = \prod_{\mathfrak{p}|p} (1 - \mathbf{N}\mathfrak{p}^{-1})^{-1} \sim \prod_{\mathfrak{p}|p} \mathbf{N}\mathfrak{p}$ , this yields:

$$(2.9) \quad \#\mathcal{T}_{K,p} \sim \frac{p}{\prod_{\mathfrak{p}|p} \mathbf{N}\mathfrak{p}} \cdot \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) \cdot \prod_{\chi \neq 1} \left( 1 - \frac{\chi(p)}{p} \right)^{-1} \sim \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) = \tilde{\kappa}_{K,p}.$$

But no upper bound of the  $p$ -adic valuation of this residue is known. So we must, on the contrary, try to study directly  $\#\mathcal{T}_{K,p}$  with arithmetic tools.

**2.5. Arithmetical study of  $\tilde{\kappa}_{K,p}$ .** To study this residue, consider (2.9) giving  $\tilde{\kappa}_{K,p} \sim \#\mathcal{T}_{K,p}$ . In  $\#\mathcal{T}_K = \#\mathcal{O}_K^c \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$ , the computation of  $\#\mathcal{W}_K$  is obvious. Then  $\#\mathcal{O}_K^c = \frac{\#\mathcal{O}_K}{[H_K \cap K^c : K]} = \#\mathcal{O}_K \cdot \frac{1}{e_p} \cdot (\langle -1 \rangle : \langle -1 \rangle \cap \mathbf{N}_{K/\mathbb{Q}}(U_K)) \cdot [K \cap \mathbb{Q}^c : \mathbb{Q}]$ , where  $e_p$  is the ramification index of  $p$  in  $K/\mathbb{Q}$  [9, Theorem III.2.6.4]. So, for  $p \gg 0$  we get  $\#\mathcal{O}_K^c \cdot \#\mathcal{W}_K = 1$ . Then the main factor is (whatever the field  $K$  and the prime  $p$  [12, Proposition 5.2]):

$$(2.10) \quad \#\mathcal{R}_K = \#\text{tor}_{\mathbb{Z}_p}(\log_p(U_K)/\log_p(\overline{E}_K)) \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_p : \log_p(\mathbf{N}_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}|p} \mathbf{N}\mathfrak{p}} \cdot \frac{R_{K,p}}{\sqrt{D_K}},$$

which is unpredictable and more complicate if  $p$  ramifies in  $K$  or if  $p = 2$ .

In the non-ramified case for  $p \neq 2$ , it is given by the classical determinant provided that one replaces  $\log_p$  by the “normalized logarithm”  $\frac{1}{p} \log_p$ .

*Remarks 2.3.* Let  $K = \mathbb{Q}(\sqrt{m})$  and let  $p \nmid D_K$  with residue degree  $f \in \{1, 2\}$ .

(i) For  $p \neq 2$ ,  $\#\mathcal{R}_K \sim \frac{1}{p} \log_p(\varepsilon_K) \sim p^{\delta_p(\varepsilon_K)}$ , where  $\delta_p(\varepsilon_K) = v_p\left(\frac{\varepsilon_K^{p^f-1} - 1}{p}\right)$ .

(ii) For  $p = 2$ , the good definition of the  $\delta_2$ -function is  $\delta_2(\varepsilon_K) := v_2\left(\frac{\varepsilon_K^2 - 1}{8}\right)$  if  $f = 1$  and  $v_2\left(\frac{\varepsilon_K^6 - 1}{4}\right)$  if  $f = 2$ , in which cases  $\#\mathcal{R}_K \sim 2^{\delta_2(\varepsilon_K)}$ .

(iii) The existence of an upper bound for  $v_p(\frac{1}{2} L_p(1, \chi_K))$  would be equivalent to an estimation of the order of magnitude of  $\delta_p(\eta_K)$  for the cyclotomic number

$\eta_K := \prod_{a, \chi(a)=1} (1 - \zeta_{D_K}^a)$ , where  $\zeta_{D_K}$  is a primitive  $D_K$ th root of unity (interpretation of the class number formula via cyclotomic units). The study given in [10, Théorème 1.1], and applied to the number  $\xi = 1 - \zeta_{D_K}$ , suggests that if  $p \rightarrow \infty$ , the probability of  $\delta_p(\eta_K) \geq 1$  for the  $\chi_K$ -component  $\langle \eta_K \rangle_{\mathbb{Z}} = \langle \xi \rangle^{\varepsilon_{\chi_K}}$ , of the Galois module generated by  $\xi$ , tends to 0 at least as  $O(1) \cdot p^{-1}$  and conjecturally as  $p^{-(\log(\log(p))/\log(c_0(\eta_K)) - O(1))}$ , where  $c_0(\eta_K) = |\eta_K| > 1$ ; this does not apply to small  $p$ . This explains the specific difficulties of the  $p$ -adic case, which is not surprising since the study of  $v_p(\#\mathcal{T}_K)$  represents a refinement of Leopoldt’s conjecture.

We intend to give estimations of  $v_p(\#\mathcal{T}_K)$  ( $p$  fixed) related to the discriminant  $D_K$  when  $K$  varies in a family  $\mathcal{K} \subseteq \mathcal{K}_{\text{real}}$  (as in [38], we call *family of number fields* any infinite set of non-isomorphic number fields  $K$ ; thus, the condition  $D_K \rightarrow \infty$  makes sense in  $\mathcal{K}$ ). In a numerical point of view, we shall analyse the set  $\mathcal{K}_{\text{real}}^{(2)}$  of real quadratic fields and the subset  $\mathcal{K}_{\text{ab}}^{(3)}$  of  $\mathcal{K}_{\text{real}}^{(3)}$  (totally real cubic fields), of cyclic cubic fields of conductor  $f$ , described by the polynomials (see, e.g., [7]):

$$(2.11) \quad \begin{aligned} P &= X^3 + X^2 - \frac{f-1}{3} \cdot X + \frac{1+f(a-3)}{27}, \text{ if } 3 \nmid f, \\ P &= X^3 - \frac{f}{3} \cdot X - \frac{fa}{27}, \text{ if } 3 \mid f, \end{aligned}$$

where  $f = \frac{a^2 + 27b^2}{4}$  with  $a \equiv 2 \pmod{3}$  (if  $3 \nmid f$ ),  $a \equiv 6 \pmod{9}$  &  $b \not\equiv 0 \pmod{3}$  (if  $3 \mid f$ ). Some non-cyclic cubic fields will also be considered.

In the forthcoming Sections, we deal only with finite places  $p$ ; so we simplify some notation in an obvious way.

### 3. DIRECT CALCULATION OF $v_p(\#\mathcal{T}_K)$ VIA PARI/GP

The programs shall try to verify a  $p$ -adic analogue of the relation (2.7), for quadratic and cubic fields; for each fixed  $p$ , they shall give the successive minima of the expression  $\Delta_p(K) := \frac{\log_{\infty}(\sqrt{D_K})}{\log_{\infty}(p)} - v_p(\#\mathcal{T}_K)$  and the successive maxima of:

$$(3.1) \quad C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_{\infty}(p)}{\log_{\infty}(\sqrt{D_K})},$$

when  $D_K$  increases in the selected family  $\mathcal{K}$ . It seems that a first minimum of  $\Delta_p(K)$  (on an interval  $I$  for  $D_K$ ) is rapidly obtained and is negative of small absolute value, giving  $C_p(K) > 1$ ; whence the interest of the computation of  $C_p(K)$  and the question of the existence of  $\mathcal{C}_p = \sup_{K \in \mathcal{K}} (C_p(K))$ . If  $\mathcal{C}_p = \infty$ , this means that (for example)  $v_p(\#\mathcal{T}_{K_i}) = \log_{\infty}(\sqrt{D_{K_i}}) \cdot O(\log_{\infty}(\log_{\infty}(\sqrt{D_{K_i}})))$  for infinitely many  $K_i \in \mathcal{K}$ , whence, in our opinion, the “excessive relations”  $\#\mathcal{T}_{K_i} \gg \sqrt{D_{K_i}}$ .

We shall observe that  $\sup_{D < x} (C_p(K))$  increases and stabilizes rapidly, for a rather small  $D_0$ ; this means that  $C_p(K)$  is locally decreasing for  $D_K \gg D_0$ , whence the interest of calculating  $C_p(K)$  for discriminants as large as possible to expect the existence of  $\limsup_{K \in \mathcal{K}} (C_p(K))$  of a different nature (see the very instructive example discussed in the § 4.2.3 (i)).

We shall adapt the following PARI program [13, § 3.2] (testing the  $p$ -rationality of *any number field*  $K$ ), that we recall for the convenience of the reader (for this, choose any monic irreducible polynomial  $P$  and any prime  $p$ ; the program gives in  $S$  the signature  $(r_1, r_2)$  of  $K$ , then  $r := r_2 + 1$ ; recall that from  $K = \text{bnfinit}(P, 1)$ , one gets  $D_K = \text{component}(\text{component}(K, 7), 3)$  and that from  $C8 = \text{component}(K, 8)$ ,

the structure of the class group, the regulator and a fundamental system of units are given by  $\text{component}(C8, 1)$ ,  $\text{component}(C8, 2)$ , and  $\text{component}(C8, 5)$ , respectively; whence the class number given by  $h_K = \text{component}(\text{component}(C8, 1), 1)$ :

```
{P=x^6-123*x^2+1;p=3;K=bnfinit(P,1);n=2;if(p==2,n=3);Kpn=bnrinit(K,p^n);
S=component(component(Kpn,1),7);r=component(component(S,2),2)+1;
print(p,"-rank of the compositum of the Z_",p,"-extensions: ",r);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);
R=0;for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));
print("Structure of the ",p,"-ray class group:",L);
if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational"));
if(R==r,print("rk(T)=",0," K is ",p,"-rational"))}
```

```
3-rank of the compositum of the Z_3-extensions: 2
Structure of the 3-ray class group: List([9, 9, 9])
rk(T)=1 K is not 3-rational
```

For any  $K \in \mathcal{K}_{\text{real}}$ , the  $p$ -invariants of  $\text{Gal}(K(p^n)/K)$ , where  $K(p^n)$  is the ray class field of modulus  $(p^n)$  for any  $n \geq 0$ , are given by the following simplest program (in which  $n = 0$  gives the structure of the  $p$ -class group):

```
{P=x^2-2*3*5*7*11*13*17;K=bnfinit(P,1);p=2;n=18;Kpn=bnrinit(K,p^n);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);
for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,
listinsert(L,p^valuation(c,p),1)));print(L)}
List([131072, 2, 2, 2, 2, 2])
```

For  $n = 0$  one gets  $\mathcal{C}_K \simeq [2, 2, 2, 2, 2]$ . Taking  $n$  large enough in the program allows us to compute directly the structure of  $\mathcal{T}_K$  as is done by a precise (but longer) program in [31]. This gives the  $p$ -valuation in  $\text{vptor}$  of  $\#\mathcal{T}_K$  as rapidly as possible; for this, explain some details about PARI (from [29]).

Let  $K \in \mathcal{K}_{\text{real}}$  be linearly disjoint from  $\mathbb{Q}^c$ ; let  $K(p^n)$  be the ray class field of modulus  $(p^n)$ ,  $n \geq 2$  (resp.  $n \geq 3$ ) if  $p \neq 2$  (resp.  $p = 2$ ); indeed, from [13, Theorem 2.1], these conditions on  $n$  are sufficient to give the  $p$ -rank  $t_K =: t$  of  $\mathcal{T}_K$ . Thus, for  $n$  large enough, the  $p$ -structure of  $\text{Gal}(K(p^n)/K)$  is of the form  $[p^a, p^{a_1}, \dots, p^{a_t}]$ , with  $a \geq a_1 \geq \dots \geq a_t$ , in  $Hpn := \text{component}(\text{component}(Kpn, 5), 2)$ , where  $Kpn = \text{bnrinit}(K, p^n)$  and  $p^a = [K(p^n) \cap K^c : K]$ .

Then  $\#\mathcal{T}_K = [K(p^n) : K] \times p^{-a}$  (up to a  $p$ -adic unit), where  $p^a$  is the largest component given in  $Hpn$  (whence the first one in the list, under the condition  $n \gg \max(a_1, \dots, a_t)$ ); so we have only to verify that  $p^n$  is much larger than the exponent  $\max(p^{a_1}, \dots, p^{a_t})$  of  $\mathcal{T}_K$ .

In practice, and to obtain fast programs, we must look at the order of magnitude of the results to increase  $n$  if necessary; in fact, once the part  $K = \text{bnfinit}(P, 1)$  of the program is completed, a large value of  $n$  does not significantly increase the execution time. For instance, with  $P = x^2 - 4194305$  and  $p = 2$ , one gets the successive structures for  $2 \leq n \leq 16$ :

|   |            |   |               |    |                 |    |                  |
|---|------------|---|---------------|----|-----------------|----|------------------|
| 2 | [2, 2]     | 6 | [32, 16, 2]   | 10 | [512, 256, 2]   | 14 | [8192, 2048, 2]  |
| 3 | [4, 2, 2]  | 7 | [64, 32, 2]   | 11 | [1024, 512, 2]  | 15 | [16384, 2048, 2] |
| 4 | [8, 4, 2]  | 8 | [128, 64, 2]  | 12 | [2048, 1024, 2] | 16 | [32768, 2048, 2] |
| 5 | [16, 8, 2] | 9 | [256, 128, 2] | 13 | [4096, 2048, 2] |    |                  |

showing that  $n$  must be at least 13 to give  $\mathcal{T}_K \simeq \mathbb{Z}/2^{11}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . In the forthcoming numerical results, if any doubt occurs for a specific field, it is sufficient to use the previous program with bigger  $n$ .

## 4. NUMERICAL INVESTIGATIONS FOR REAL QUADRATIC FIELDS

Let  $K = \mathbb{Q}(\sqrt{m})$ ,  $m > 0$  squarefree. We have  $\#\mathcal{W}_K = 2$  for  $p = 2$  &  $m \equiv \pm 1 \pmod{8}$ ,  $\#\mathcal{W}_K = 3$  for  $p = 3$  &  $m \equiv -3 \pmod{9}$ , and we are mainly concerned with the  $p$ -class group  $\mathcal{C}_K$  and the normalized regulator  $\mathcal{R}_K$ . When  $p > 2$  is unramified, we have  $v_p(\#\mathcal{R}_K) = \delta_p(\varepsilon)$  for the fundamental unit  $\varepsilon$  of  $K$  and if  $p = 2$  is unramified, we have  $\delta_2(\varepsilon) := v_2\left(\frac{\varepsilon^{2 \cdot (2^f - 1)} - 1}{2^{4-f}}\right)$  where  $f$  is the residue degree of 2 in  $K$  (see Remarks 2.3 (i), (ii)). So, we may compute  $v_p(\#\mathcal{T}_K)$  as  $v_p(\#\mathcal{C}_K^c) + \delta_p(\varepsilon) + v_p(\#\mathcal{W}_K)$  and we shall compare with the direct computation of the structure of  $\mathcal{T}_K$  as explain above. Remark that, for  $p = 2$ ,  $\#\mathcal{C}_K = 2 \cdot \#\mathcal{C}_K^c$  (instead of  $\#\mathcal{C}_K^c$ ) if and only if  $m \equiv 2 \pmod{8}$ , in which case  $H_K \cap K^c = K(\sqrt{2})$  is unramified over  $K$ .

We have the following result, about  $v_p(\#\mathcal{R}_K)$ , when  $p \geq 2$  ramifies:

**Proposition 4.1.** *For  $K = \mathbb{Q}(\sqrt{m})$  real and  $p \mid D_K$ ,  $v_p(\#\mathcal{R}_K)$  is given as follows:*

- (i) *For  $p \nmid 6$  ramified,  $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_p(\varepsilon)$  and  $v_p(\#\mathcal{R}_K) = \delta$  if  $v_p(\varepsilon^{p-1} - 1) = 1 + 2\delta$ , where  $\mathfrak{p} \mid p$ ,  $\delta \geq 0$ .*
- (ii) *For  $p = 3$  ramified,  $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_3(\varepsilon)$  (resp.  $\#\mathcal{R}_K \sim \frac{1}{3\sqrt{m}} \cdot \log_3(\varepsilon)$ ) if  $m \not\equiv -3 \pmod{9}$  (resp.  $m \equiv -3 \pmod{9}$ ). Then  $v_3(\#\mathcal{R}_K) = (v_{\mathfrak{p}}(\varepsilon^6 - 1) - 2 - \delta)/2$  where  $\mathfrak{p} \mid 3$  and  $\delta = 1$  (resp.  $\delta = 3$ ) if  $m \not\equiv -3 \pmod{9}$  (resp.  $m \equiv -3 \pmod{9}$ ).*
- (iii) *For  $p = 2$  ramified,  $\#\mathcal{R}_K \sim \frac{\log_2(\varepsilon)}{2\sqrt{m}}$  (resp.  $\frac{\log_2(\varepsilon)}{4}$ ) if  $m \not\equiv -1 \pmod{8}$  (resp.  $m \equiv -1 \pmod{8}$ ). Then,  $v_2(\#\mathcal{R}_K) = (v_{\mathfrak{p}}(\varepsilon^4 - 1) - 4 - \delta)/2$ , where  $\mathfrak{p} \mid 2$  and where  $\delta = 1, 2, 3, 4$  if  $m \equiv 2, 3, 6, 7 \pmod{8}$ , respectively.*

*Proof.* Exercise using the expression (2.10) of  $\#\mathcal{R}_K$  where  $N_{K/\mathbb{Q}}(U_K)$  is of index 2 in  $U_{\mathbb{Q}}$  (local class field theory), the fact that  $N_{K/\mathbb{Q}}(\varepsilon) = \pm 1$  (i.e.,  $\text{Tr}_{K/\mathbb{Q}}(\log_p(\varepsilon)) = 0$ ), and the classical computation of a  $p$ -adic logarithm.  $\square$

**Remark 4.2.** A first information is then the order of magnitude of  $\delta_p(\varepsilon)$  as  $D_K \rightarrow \infty$  ( $p$  fixed). Its non-nullity for  $p \gg 0$  ( $K$  fixed) is a deep problem for which we can only give some numerical experiments. For  $p \gg 0$  and any  $K \in \mathcal{K}_{\text{real}}$ , an extensive schedule is discussed in [10], for the study of  $p$ -adic regulators of an algebraic number  $\eta \in K^\times$  (giving “Frobenius determinants”), whose properties are characterized by the Galois  $\mathbb{Z}_p$ -module generated by its “Fermat quotient”  $\frac{1}{p}(\eta^{p^f-1} - 1)$ .

These questions, applied in our study to a “Minkowski unit”, are probably the explanation of the failure of the classical  $p$ -adic analysis of  $\zeta_p$ -functions (among many other subjects in number theory) since such Fermat quotients problems are neither easier nor more difficult than, for instance, the famous problem of Fermat quotients of the number 2, for which no one is able to say, so far, how much  $p$  are such that  $\frac{1}{p}(2^{p-1} - 1) \equiv 0 \pmod{p}$ .

**4.1. Maximal values of  $v_p(\#\mathcal{R}_K)$ .** Consider a prime  $p$  fixed and the family  $\mathcal{K}_{\text{real}}^{(2)}$ . The following programs find the successive maxima of  $\delta_p(\varepsilon)$  with the corresponding increasing  $D_K \in [bD, BD]$ ; the programs use the fact that for  $p$  unramified, in the inert case,  $\varepsilon^{p+1} \equiv N_{K/\mathbb{Q}}(\varepsilon) \pmod{p}$ , otherwise,  $\varepsilon^{p-1} \equiv 1 \pmod{p}$ .

We shall indicate if necessary the maximal value obtained for  $C_p(K)$  defined by the expression (3.1) by computing  $v_p(\#\mathcal{T}_K) = \delta_p(\varepsilon) + v_p(\#\mathcal{C}_K^c) + v_p(\#\mathcal{W}_K)$ .

4.1.1. PROGRAM FOR  $p = 2$  UNRAMIFIED. For  $p = 2$  unramified, we use the particular formula given in Remark 2.3 (ii).

```
{bD=5;BD=5*10^7;Max=0;for(D=bD,BD,if(core(D)!=D,next);ss=Mod(D,8);s=0;
if(ss==1,s=1);if(ss==5,s=-1);if(s==0,next);E=quadunit(D)^2;A=(E^(2-s)-1)/(2*s+6);
A=[component(A,2),component(A,3)];delta=valuation(A,2);
if(delta>Max,Max=delta;print("D=",D," delta=",delta))}}
D=21 delta=1 D=1185 delta=8 D=115005 delta=13 D=1051385 delta=19
D=41 delta=3 D=1201 delta=10 D=122321 delta=14 D=12256653 delta=21
D=469 delta=5 D=3881 delta=11 D=222181 delta=16 D=14098537 delta=22
D=645 delta=6 D=69973 delta=12 D=528077 delta=18 D=28527281 delta=25
```

The next discriminant in  $[5 \cdot 10^7, 5 \cdot 10^8]$  (two days of computer) is  $D_K = 214203013$ , where  $\delta_2(\varepsilon) = 26$ ,  $v_2(h_K) = 1$ ,  $v_2(\#\mathcal{W}_K) = 0$ ,  $v_2(\#\mathcal{T}_K) = 27$ ,  $C_2(K) = 1.951261$ .

4.1.2. PROGRAM FOR  $p = 2$  RAMIFIED. A similar program using Proposition 4.1(iii) gives analogous results for maximal values of  $\delta_2(\varepsilon)$ :

```
{bm=3;Bm=5*10^7;Max=0;for(m=bm,Bm,s=Mod(m,4);ss=Mod(m,8);
if(core(m)!=m || s==1,next);A=(quadunit(4*m)^4-1)/4;N=norm(A);v=valuation(N,2);
if(s==2,delta=v-3);if(ss==3,delta=v-2);if(ss==7,delta=v-4);delta=delta/2;
if(delta>Max,Max=delta;print("D=",4*m," delta=",delta))}}
D=28 delta=1 D=508 delta=6 D=28664 delta=13 D=15704072 delta=21
D=124 delta=2 D=1784 delta=7 D=81624 delta=17 D=29419592 delta=22
D=264 delta=3 D=10232 delta=8 D=1476668 delta=18 D=36650172 delta=23
D=456 delta=5 D=21980 delta=9 D=2692776 delta=19 D=80882380 delta=28
```

For  $D_K = 80882380 = 4 \cdot 5 \cdot 239 \cdot 16921$ ,  $\delta_2(\varepsilon) = 28$ ,  $v_2(h_K) = 2$ ,  $v_2(\#\mathcal{W}_K) = 0$ ,  $v_2(\#\mathcal{T}_K) = 30$ ,  $C_2(K) = 2.2840$ , whence the influence of genera theory on  $C_2(K)$ .

4.1.3. PROGRAM FOR ANY UNRAMIFIED  $p \geq 3$ . The program can be simplified:

```
{p=3;bD=5;BD=10^8;Max=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)|| (e==0 & Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);s=kronecker(D,p);
if(s==0,next);E=quadunit(D);nu=norm(E);u=(1+nu-nu*s)/2;
A=(E^(p-s)-u)/p;A=[component(A,2),component(A,3)];delta=valuation(A,p);
if(delta>Max,Max=delta;print("D=",D," delta=",delta))}}
D=29 delta=2 D=13861 delta=7 D=321253 delta=12 D=21242636 delta=16
D=488 delta=4 D=21713 delta=9 D=6917324 delta=13 D=71801701 delta=19
D=1213 delta=6 D=153685 delta=10 D=13495160 delta=14
```

which gives  $\delta_3(\varepsilon) \leq 19$  on the interval  $[2, 10^8]$ , obtained for  $D_K = 71801701$ , where  $v_3(h_K) = v_3(\#\mathcal{W}_K) = 0$ ,  $v_3(\#\mathcal{T}_K) = 19$ ,  $C_3(K) = 2.307828$ .

4.1.4. PROGRAMS FOR  $p = 3$  RAMIFIED. We obtain (cf. Proposition 4.1 (ii)):

```
{bD=5;BD=10^8;Max=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if(Mod(M,3)!=0|| (e==1||e>3)|| (e==0 & Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);
E=quadunit(D)^6;A=norm(E-1);v=valuation(A,3);if(Mod(D,9)!=-3,delta=(v-3)/2);
if(Mod(D,9)==-3,delta=(v-5)/2);if(delta>Max,Max=delta;
print("D=",D," delta=",delta))}}
D=93 delta=1 D=1896 delta=6 D=2354577 delta=11 D=104326449 delta=15
D=105 delta=2 D=102984 delta=8 D=6099477 delta=12 D=448287465 delta=18
D=492 delta=3 D=168009 delta=10 D=17157729 delta=13
```

4.1.5. PROGRAM FOR ANY RAMIFIED  $p > 3$ . Let's illustrate this case with a large  $p$ :

```
{p=1009;bD=5;BD=10^8;Max=0;for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if(Mod(M,p)!=0|| (e==1||e>3)|| (e==0 & Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);
E=quadunit(D)^(p-1);A=norm(E-1);delta=(valuation(A,p)-1)/2;
if(delta>Max,Max=delta;print("D=",D," delta=",delta))}}
D=1900956 delta=1
```

For large  $p$  (ramified or not) there are few solutions in a reasonable interval since we have, roughly speaking,  $\text{Prob}(\delta_p(\varepsilon) \geq \delta) \approx p^{-\delta}$ , otherwise, the solutions are often with  $\delta_p(\varepsilon) = 1$ , large  $D_K$ ,  $C_p(K)$  being rather small as we shall analyse now.

**4.2. Experiments for a conjectural upper bound - Quadratic fields.** We only assume  $K \neq \mathbb{Q}(\sqrt{2})$  when  $p = 2$  to always have  $K \cap \mathbb{Q}^c = \mathbb{Q}$ . We have given previously programs for the maximal values of  $v_p(\#\mathcal{R}_K)$ ; we now give the behaviour of the whole  $v_p(\#\mathcal{T}_K)$  for increasing discriminants; for this purpose, we compute:

$$\Delta_p(K) := \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} - v_p(\#\mathcal{T}_K) \quad \text{and} \quad C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}.$$

**4.2.1. PROGRAM FOR  $p = 2$ .** The numerical data are  $D_K$ ,  $v_p(\#\mathcal{T}_K)$  (in *vptor*; for this choose  $n$  large enough), the successive  $\Delta_p(K)$  (in *Ymin*) and the corresponding  $C_p(K)$  (in *Cp*); we omit the 2-rational fields (for them, *vptor* = 0):

```
{p=2;n=36;bd=5;BD=10^6;ymin=5;for(D=bd,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if((e==1||e>3)|| (e==0&&Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);
m=D;if(e!=0,m=D/4);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;
if(Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D)));
print("D=",D," m=",m," vptor=",vptor," Ymin=",Y," Cp=",Cp))}}
D=17      m=17      vptor=1      Ymin=1.04373142...    Cp=0.4893
D=28      m=7       vptor=2      Ymin=0.40367746...    Cp=0.8320
D=41      m=41      vptor=4      Ymin=-1.32122399...   Cp=1.4932
D=508     m=127     vptor=7      Ymin=-2.50565765...   Cp=1.5575
D=1185    m=1185    vptor=10     Ymin=-4.89466432...   Cp=1.9587
D=1201    m=1201    vptor=11     Ymin=-5.88498978...   Cp=2.1505
D=3881    m=3881    vptor=12     Ymin=-6.03889364...   Cp=2.0130
D=11985   m=11985    vptor=13     Ymin=-6.22552885...   Cp=1.9189
D=26377   m=26377    vptor=14     Ymin=-6.65650356...   Cp=1.9064
D=81624   m=20406    vptor=20     Ymin=-11.84164710...  Cp=2.4514
```

The larger computations in § 4.1.1 show the largest case  $D_K = 214203013$  with  $h_K = 2$  and  $\delta_2(\varepsilon) = 26$ , giving  $\Delta_2(K) \approx -13.1628$ , the best local minimum and gives  $C_2(K) = 1.951261$ . For the ramified case  $D_K = 4 \cdot 20220595$ , we obtained  $\delta_2(\varepsilon) = 28$ ,  $C_2(K) = 2.284033$ .

But the case  $D_K = 81624 = 8 \cdot 3 \cdot 19 \cdot 179$ , for which  $h_K = 8$ , with the valuation  $v_p(\#\mathcal{T}_K) = 20$ , gives  $C_2(K) = 2.4514$  and shows, once again, that genera theory may modify the results for  $p = 2$  and more generally for  $p \mid d$ . Note that in the above results, there is no solution  $D_K \in [20406, 10^6]$ . To illustrate this, we use the same program for  $D_K \in [81628, 5 \cdot 10^5]$ :

```
D=81628   m=20407    vptor=2      Ymin=6.15838824...    Cp=0.2451
D=81640   m=20410    vptor=4      Ymin=4.15849428...    Cp=0.4902
D=81713   m=81713     vptor=5      Ymin=3.15913899...    Cp=0.6128
D=81788   m=20447     vptor=7      Ymin=1.15980078...    Cp=0.8578
D=82684   m=20671     vptor=8      Ymin=0.16766028...    Cp=0.9794
D=83144   m=20786     vptor=9      Ymin=-0.82833773...   Cp=1.1013
D=84361   m=84361     vptor=10     Ymin=-1.81785571...   Cp=1.2221
D=86284   m=21571     vptor=11     Ymin=-2.80159728...   Cp=1.3417
D=100045  m=100045    vptor=14     Ymin=-5.69485522...   Cp=1.6857
D=115005  m=115005    vptor=16     Ymin=-7.59433146...   Cp=1.9034
D=376264  m=94066     vptor=17     Ymin=-7.73930713...   Cp=1.8357
D=495957  m=495957    vptor=19     Ymin=-9.54007224...   Cp=2.0084
D=1476668 m=369167    vptor=20     Ymin=-9.75304296...   Cp=1.9518
```

4.2.2. PROGRAM FOR  $p \in [3, 50]$ . In this case, genera theory does not intervene. We do not write the cases where  $v_p(\#T_K) = 0$  ( $p$ -rational fields). The constant  $C_p(K)$  has some variations for very small  $D_K$  but stabilizes and seems locally decreasing for larger  $D_K$ ; so we mention the maximal ones, but the last value is more significant to evaluate an upperbound:

```
{n=16;BD=5;BD=10^6;forprime(p=3,50,print(" ");print("p=",p);ymin=10;
for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3)|| (e==0 & Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);
m=D;if(e!=0,m=D/4);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(sqrt(D))/log(p)-vptor;if(Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D));
print("D=",D," m=",m," vptor=",vptor," Ymin=",Y," Cp=",Cp))})}
```

p=3

|          |          |          |                     |           |
|----------|----------|----------|---------------------|-----------|
| D=24     | m=6      | vptor=1  | Ymin=0.44639463...  | Cp=0.6913 |
| D=29     | m=29     | vptor=2  | Ymin=-0.46747762... | Cp=1.3050 |
| D=105    | m=105    | vptor=3  | Ymin=-0.88189136... | Cp=1.4163 |
| D=488    | m=122    | vptor=4  | Ymin=-1.18266604... | Cp=1.4197 |
| D=1213   | m=1213   | vptor=6  | Ymin=-2.76826302... | Cp=1.8565 |
| D=1896   | m=474    | vptor=7  | Ymin=-3.56498395... | Cp=2.0378 |
| D=13861  | m=13861  | vptor=8  | Ymin=-3.65959960... | Cp=1.8431 |
| D=21713  | m=21713  | vptor=10 | Ymin=-5.45532735... | Cp=2.2003 |
| D=168009 | m=168009 | vptor=11 | Ymin=-5.52410420... | Cp=2.0088 |
| D=321253 | m=321253 | vptor=12 | Ymin=-6.22909046... | Cp=2.0793 |

p=5

|           |           |          |                     |           |
|-----------|-----------|----------|---------------------|-----------|
| D=53      | m=53      | vptor=1  | Ymin=0.23344053...  | Cp=0.8107 |
| D=73      | m=73      | vptor=2  | Ymin=-0.66709383... | Cp=1.5005 |
| D=217     | m=217     | vptor=3  | Ymin=-1.32864091... | Cp=1.7949 |
| D=1641    | m=1641    | vptor=4  | Ymin=-1.70010976... | Cp=1.7392 |
| D=25037   | m=25037   | vptor=5  | Ymin=-1.85352571... | Cp=1.5890 |
| D=71308   | m=17827   | vptor=6  | Ymin=-2.52836443... | Cp=1.7283 |
| D=304069  | m=304069  | vptor=7  | Ymin=-3.07782014... | Cp=1.7847 |
| (...)     |           |          |                     |           |
| D=4788645 | m=4788645 | vptor=10 | Ymin=-5.22138818... | Cp=2.0926 |

p=7

|           |           |         |                     |           |
|-----------|-----------|---------|---------------------|-----------|
| D=24      | m=6       | vptor=1 | Ymin=-0.18340170... | Cp=1.2246 |
| D=145     | m=145     | vptor=2 | Ymin=-0.72123238... | Cp=1.5640 |
| D=797     | m=797     | vptor=3 | Ymin=-1.28335992... | Cp=1.7476 |
| D=30556   | m=7639    | vptor=4 | Ymin=-1.34640462... | Cp=1.5074 |
| D=92440   | m=23110   | vptor=5 | Ymin=-2.06196222... | Cp=1.7018 |
| D=287516  | m=71879   | vptor=6 | Ymin=-2.77039718... | Cp=1.8578 |
| (...)     |           |         |                     |           |
| D=4354697 | m=4354697 | vptor=7 | Ymin=-3.07207825... | Cp=1.7821 |

p=11

|          |          |         |                     |           |
|----------|----------|---------|---------------------|-----------|
| D=29     | m=29     | vptor=1 | Ymin=-0.29786428... | Cp=1.4242 |
| D=145    | m=145    | vptor=2 | Ymin=-0.96227041... | Cp=1.9272 |
| D=424    | m=106    | vptor=3 | Ymin=-1.73853259... | Cp=2.3781 |
| D=35068  | m=8767   | vptor=4 | Ymin=-1.81786877... | Cp=1.8330 |
| D=163873 | m=163873 | vptor=5 | Ymin=-2.49637793... | Cp=1.9971 |

p=13

|          |          |         |                     |           |
|----------|----------|---------|---------------------|-----------|
| D=8      | m=2      | vptor=1 | Ymin=-0.59464276... | Cp=2.4669 |
| D=2285   | m=2285   | vptor=3 | Ymin=-1.49234424... | Cp=1.9898 |
| D=98797  | m=98797  | vptor=4 | Ymin=-1.75808000... | Cp=1.7842 |
| D=382161 | m=382161 | vptor=5 | Ymin=-2.49437601... | Cp=1.9955 |

p=17

|        |        |         |                     |           |
|--------|--------|---------|---------------------|-----------|
| D=69   | m=69   | vptor=2 | Ymin=-1.25277309... | Cp=2.6765 |
| D=3209 | m=3209 | vptor=3 | Ymin=-1.57516648... | Cp=2.1055 |

|            |            |         |                     |           |
|------------|------------|---------|---------------------|-----------|
| D=8972     | m=2243     | vptor=4 | Ymin=-2.39372069... | Cp=2.4902 |
| D=1631753  | m=1631753  | vptor=5 | Ymin=-2.47545212... | Cp=1.9805 |
| p=19       |            |         |                     |           |
| D=109      | m=109      | vptor=1 | Ymin=-0.20335454... | Cp=1.2552 |
| D=193      | m=193      | vptor=2 | Ymin=-1.10633396... | Cp=2.2379 |
| D=2701     | m=2701     | vptor=3 | Ymin=-1.65825418... | Cp=2.2359 |
| (...)      |            |         |                     |           |
| D=1482837  | m=1482837  | vptor=4 | Ymin=-1.58706704... | Cp=1.6577 |
| D=6839105  | m=6839105  | vptor=5 | Ymin=-2.32747604... | Cp=1.8709 |
| D=8736541  | m=8736541  | vptor=5 | Ymin=-2.28589639... | Cp=1.8422 |
| p=23       |            |         |                     |           |
| D=140      | m=35       | vptor=1 | Ymin=-0.21198348... | Cp=1.2690 |
| D=493      | m=493      | vptor=2 | Ymin=-1.01123893... | Cp=2.0227 |
| D=10433    | m=10433    | vptor=3 | Ymin=-1.52451822... | Cp=2.0332 |
| D=740801   | m=740801   | vptor=4 | Ymin=-1.84475964... | Cp=1.8559 |
| p=29       |            |         |                     |           |
| D=33       | m=33       | vptor=1 | Ymin=-0.48081372... | Cp=1.9261 |
| D=41       | m=41       | vptor=2 | Ymin=-1.44858244... | Cp=3.6270 |
| D=53093    | m=53093    | vptor=4 | Ymin=-2.38448997... | Cp=2.4759 |
| D=30596053 | m=30596053 | vptor=5 | Ymin=-2.44061964... | Cp=1.9536 |
| p=31       |            |         |                     |           |
| D=8        | m=2        | vptor=1 | Ymin=-0.69722637... | Cp=3.3028 |
| D=6168     | m=1542     | vptor=2 | Ymin=-0.72930075... | Cp=1.5739 |
| D=90273    | m=90273    | vptor=3 | Ymin=-1.33857946... | Cp=1.8056 |
| D=1294072  | m=323518   | vptor=4 | Ymin=-1.95087990... | Cp=1.9520 |
| p=37       |            |         |                     |           |
| D=33       | m=33       | vptor=1 | Ymin=-0.51584228... | Cp=2.0654 |
| D=3340     | m=835      | vptor=2 | Ymin=-0.87650089... | Cp=1.7801 |
| D=124129   | m=124129   | vptor=3 | Ymin=-1.37588711... | Cp=1.8471 |
| p=41       |            |         |                     |           |
| D=73       | m=73       | vptor=1 | Ymin=-0.42232716... | Cp=1.7311 |
| D=2141     | m=2141     | vptor=2 | Ymin=-0.96743241... | Cp=1.9369 |
| D=187113   | m=187113   | vptor=3 | Ymin=-1.36552680... | Cp=1.8354 |
| p=43       |            |         |                     |           |
| D=88       | m=22       | vptor=1 | Ymin=-0.40479944... | Cp=1.6801 |
| D=6520     | m=1630     | vptor=2 | Ymin=-0.83246977... | Cp=1.7130 |
| D=283596   | m=70899    | vptor=3 | Ymin=-1.33094416... | Cp=1.7974 |
| p=47       |            |         |                     |           |
| D=301      | m=301      | vptor=1 | Ymin=-0.25884526... | Cp=1.3492 |
| D=26321    | m=26321    | vptor=2 | Ymin=-0.67821659... | Cp=1.5131 |
| D=368013   | m=368013   | vptor=3 | Ymin=-1.33566464... | Cp=1.8025 |

The interval  $[2, 10^6]$  was not always sufficient (see the cases  $p = 5, 7, 19, 29$ , above). For instance for  $p = 7$ , we ignore if the bound  $C_p(K) = 1.8578$  can be exceeded; we have computed up to  $D_K \leq 2 \cdot 10^7$ , where  $v_p(\#\mathcal{T}_K)$  takes at most the values 6 or 7 with  $C_p(K) < 1.7821$ . So  $v_p(\#\mathcal{T}_K) \geq 8$  does exist for greater discriminants, but  $\frac{8 \cdot \log_\infty(7)}{\log_\infty(\sqrt{2 \cdot 10^7})} \approx 1.8520$ , which is significant of the evolution of  $C_p(K)$  as  $D_K \rightarrow \infty$ .

The same program with  $p = 3$ ,  $n > 18$ , taking discriminants,  $D_K \in [10^6, 2.5 \cdot 10^7]$  then in  $[10^8, 5 \cdot 10^6]$  (two days of computer for each part), gives ( $p = 3$ ):

|           |           |         |                     |           |
|-----------|-----------|---------|---------------------|-----------|
| D=1000005 | m=1000005 | vptor=1 | Ymin=5.28771209...  | Cp=0.1590 |
| D=1000049 | m=1000049 | vptor=2 | Ymin=4.28773212...  | Cp=0.3180 |
| D=1000104 | m=250026  | vptor=3 | Ymin=3.28775715...  | Cp=0.4771 |
| D=1000133 | m=1000133 | vptor=4 | Ymin=2.28777034...  | Cp=0.6361 |
| D=1000169 | m=1000169 | vptor=5 | Ymin=1.28778673...  | Cp=0.7951 |
| D=1000380 | m=250095  | vptor=6 | Ymin=0.28788273...  | Cp=0.9542 |
| D=1001177 | m=1001177 | vptor=8 | Ymin=-1.71175481... | Cp=1.2722 |

|             |             |          |                     |           |
|-------------|-------------|----------|---------------------|-----------|
| D=1014693   | m=1014693   | vptor=9  | Ymin=-2.70565175... | Cp=1.4298 |
| D=1074760   | m=268690    | vptor=10 | Ymin=-3.67947724... | Cp=1.5821 |
| D=1185256   | m=296314    | vptor=11 | Ymin=-4.63493860... | Cp=1.7281 |
| D=2354577   | m=2354577   | vptor=12 | Ymin=-5.32254344... | Cp=1.7970 |
| D=6099477   | m=6099477   | vptor=13 | Ymin=-5.88934151... | Cp=1.8282 |
| D=13495160  | m=3373790   | vptor=14 | Ymin=-6.52791825... | Cp=1.8736 |
| D=21242636  | m=5310659   | vptor=16 | Ymin=-8.32143995... | Cp=2.0837 |
| (...)       |             |          |                     |           |
| D=100025621 | m=100025621 | vptor=13 | Ymin=-4.61627031... | Cp=1.5506 |
| D=104326449 | m=104326449 | vptor=16 | Ymin=-7.59711043... | Cp=1.9041 |

The case  $D_K = 21242636$  leads to  $C_3(K) = 2.0837$ ; but it is difficult to predict the behavior of  $C_3$  at infinity. In the second part, no data between the two discriminants, which suggests an irregular decreasing of  $C_3(K)$  as  $D_K \rightarrow \infty$ .

*Remark 4.3.* From these calculations in the quadratic case, one may consider, in an heuristic framework, that we have the good following lower bounds for  $C_p$ :

$$\begin{aligned} C'_3 &\approx 2.0837, C'_5 \approx 2.0926, C'_7 \approx 1.8578, C'_{11} \approx 1.9971, C'_{13} \approx 1.9955, C'_{17} \approx 1.9805, \\ C'_{19} &\approx 2.2379, C'_{23} \approx 1.8559, C'_{29} \approx 2.4759, C'_{31} \approx 1.9520, C'_{37} \approx 1.8471, C'_{41} \approx 1.8354, \\ C'_{43} &\approx 1.7974, C'_{47} \approx 1.8025. \end{aligned}$$

4.2.3. REMARKS AND HEURISTICS. Let  $\mathcal{K}_{\text{real}}^{(2)}$  be the family of real quadratic fields; we consider  $C_p(K)$  and try to understand its behavior regarding  $p$  and  $D_K$ :

(i) For  $p \gg 0$ , an estimation of  $\mathcal{C}_p^{(2)} := \sup_{K \in \mathcal{K}_{\text{real}}^{(2)}} (C_p(K))$  is more difficult and, a fortiori, for  $\limsup_{K \in \mathcal{K}_{\text{real}}^{(2)}} (C_p(K))$ ; for instance, we have found that for  $\mathbb{Q}(\sqrt{19})$  and  $p_0 = 13599893$ , one has  $v_{p_0}(\#\mathcal{T}_{\mathbb{Q}(\sqrt{19})}) = 1$ , whence  $\mathcal{C}_{p_0}^{(2)} \geq 7.5855$ . The following program can be used for *huge values* of  $p$  to find quadratic fields  $K$  such that  $v_p(\#\mathcal{R}_K) \geq 1$ ; in practice one never finds  $v_p(\#\mathcal{R}_K) \geq 2$  for “usual” discriminants. However, for these solutions, one must compute  $v_p(\#\mathcal{T}_K)$  with the classical program of Section 3 to be sure of the result (we treat separately the case  $p_0 \mid D_K$ ).

```
{p=13599893;pp=p^2;for(D=5,5*10^8,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)||e==0 & Mod(M,4)!=1)||e==2 & Mod(M,4)==1),next);s=kronoecker(D,p);
if(s==0,next);E=quadunit(D);nu=norm(E);u=(1+nu-nu*s)/2;P=component(E,1)+Mod(0,pp);
e1=component(E,2);e2=component(E,3);A=Mod(e1+e2*x,P)^(p-s)-u;if(A==0,print(D))}
```

The *next discriminants*  $D_K > 4 \cdot 19$ , up to  $5 \cdot 10^8$  (more that two days of computer), for which  $v_{p_0}(\#\mathcal{T}_K) \geq 1$  (in fact = 1), are:

37473505, 45304189, 104143053, 111800589, 112985161, 181148197, 239100989,  
288517452, 350532569, 387058008, 414929433, 477524401,  
giving  $C_{p_0}(K) = 1.8837, 1.8635, 1.7794, 1.7726, 1.7716, 1.7276, 1.7028, 1.6864,$   
 $1.6697, 1.6613, 1.6550, 1.6438$ , respectively.

Thus we notice, as expected, a significant decrease of the function  $C_{p_0}(K)$  since we did not find any  $v_{p_0}(\#\mathcal{T}_K) > 1$ , until  $D_K \leq 5 \cdot 10^8$ , knowing that other quadratic fields with *arbitrary*  $v_{p_0}(\#\mathcal{T}_K)$  exist with huge discriminants, as:

$$D_K = p_0^4 + 4 = 34209124997537575597791879605, \text{ for which } C_{p_0}(K) = 0.4999.$$

This field is the first element of families  $K = \mathbb{Q}(\sqrt{a^2 \cdot p_0^{2\rho} + b^2})$ ,  $a \geq 1$ ,  $b \in \{1, 2\}$ , described in Subsection 4.3, for which  $\delta_{p_0}(\varepsilon_K) = \rho - 1$ , whence  $v_{p_0}(\#\mathcal{T}_K) \geq \rho - 1$  and  $C_p(K) < 1 + o(1)$ . Note that for  $\rho - 1 = 10$  and  $p_0 = 13599893$ ,  $D_K \approx 10^{157}$ .

Unfortunately, we ignore what happens for  $5 \cdot 10^8 < D_K < p_0^4 + 4$  because of the order of magnitude; to get  $C_{p_0}(K) < 1.3$ , we must have for instance  $v_{p_0}(\#\mathcal{T}_K) = 1$  and  $D_K > 94334377272$ , then  $D_K > 9333929793774$  to get  $C_{p_0}(K) < 1.1$ .

We then have the following alternative: either  $C_{p_0}(K) < 7.5855$  for all  $D_K > 4 \cdot 19$ , whence  $\mathcal{C}_{p_0}^{(2)} = 7.5855$ , or  $\mathcal{C}_{p_0}^{(2)}$  is greater than 7.5855 or infinite.

The existence of infinitely many  $K \in \mathcal{K}_{\text{real}}^{(2)}$  such that  $C_{p_0}(K) > 7.5855$  remains possible but assumes the strong condition  $v_{p_0}(\#\mathcal{T}_K) > 0.4618 \cdot \log_{\infty}(\sqrt{D_K})$  for infinitely many  $K \in \mathcal{K}_{\text{real}}^{(2)}$ .

The most credible case should be that, for each  $p$ , *there exist finitely many*  $K \in \mathcal{K}_{\text{real}}^{(2)}$  for which  $v_p(\#\mathcal{T}_K) \gg \log_{\infty}(\sqrt{D_K})$ , whence  $C_p(K) \gg \log_{\infty}(p)$ ; so for “almost all”  $K \in \mathcal{K}_{\text{real}}^{(2)}$ , we would have  $C_p(K) \ll 1$  (and often 0 as explained in (iii)), except for some critical infinite families for which  $C_p(K) \leq 1 + o(1)$ ; if there is no other possibilities,  $\mathcal{C}_p^{(2)}$  does exist and is equal to  $\max_{D_K \leq D_0} (C_p(K))$  for a sufficiently large  $D_0$ .

(ii) The existence of  $\mathcal{C}_p$  (over  $\mathcal{K}_{\text{real}}$ ) essentially depends on  $v_p(\#\mathcal{R}_K)$  since the influence of  $v_p(\#\mathcal{C}_K^c)$  seems negligible, which is reinforced by classical heuristics on class groups [3, 4], or by specific results in suitable towers [38, Proposition 7.1], then, mainly, by strong conjectures (and partial proofs) in [8] as  $\#\mathcal{C}_K \ll_{\epsilon, p, d} (\sqrt{|D_K|})^{\epsilon}$  for any number field of degree  $d$ , i.e., for all  $\epsilon > 0$  the existence of  $C_{\epsilon, p, d}$  such that:

$$\log_{\infty}(\#\mathcal{C}_K) \leq \log_{\infty}(C_{\epsilon, p, d}) + \epsilon \cdot \log_{\infty}(\sqrt{|D_K|}),$$

strengthening the classical Brauer theorem (existence of an universal constant  $\mathcal{C}_0$  such that,  $\log_{\infty}(h_K) \leq \mathcal{C}_0 \cdot \log_{\infty}(\sqrt{|D_K|})$  for all number field  $K$ ); for quadratic and cyclic cubic fields,  $\mathcal{C}_0 = 1$  (Remark 2.2).

(iii) For any fixed  $p$ ,  $\liminf_{K \in \mathcal{K}_{\text{real}}^{(2)}} (C_p(K)) = 0$  (see Byeon [2, Theorem 1.1], after Ono,

where a lower bound of the density of  $p$ -rational fields is given for  $p > 3$ ). Indeed, as  $D_K \rightarrow \infty$ , statistically, “almost all” real quadratic fields  $K$  are such that  $\#\mathcal{T}_K = 1$ .

(iv) Now, if  $K$  is fixed and  $p \rightarrow \infty$ ,  $\liminf_p (C_p(K)) = 0$ . One may see this as an unproved generalization, for  $v_p(\#\mathcal{R}_K)$ , of theorems of Silverman [35], Graves–Murty [17] and others about Fermat quotients of rationals, showing the considerable difficulties of such subjects, despite the numerical obviousness since in practice, “for almost all  $p$ ”,  $v_p(\#\mathcal{T}_K) = 0$ . We have conjectured, after numerous calculations and heuristics, that, for  $K \in \mathcal{K}_{\text{real}}$  fixed, the set of primes  $p$ , such that  $\mathcal{T}_K \neq 1$ , is finite [10, Conjecture 8.11], i.e.,  $C_p(K) = 0$  for all  $p \gg 0$ ; otherwise  $\limsup_p (C_p(K)) = \infty$ .

If this conjecture is false for the field  $K$ , there exists an infinite set of prime numbers  $p_i$  such that  $v_{p_i}(\#\mathcal{T}_K) \geq 1$  giving  $C_{p_i}(K) \geq \frac{\log_{\infty}(p_i)}{\log_{\infty}(\sqrt{D_K})}$  arbitrary large as  $i \rightarrow \infty$ . But this is not incompatible with the existence, for each  $i$ , of  $\mathcal{C}_{p_i} < \infty$ ; indeed, in that case,  $C_{p_i}(K)$  may be very large with decreasing values of the  $C_{p_i}(K')$ , for  $D_{K'} \gg D_K$  as shown, for instance in  $\mathcal{K}_{\text{real}}^{(2)}$ , by the example given in (i).

If, on the contrary, the conjecture is true over  $\mathcal{K}_{\text{real}}^{(2)}$  (or more generally over  $\mathcal{K}_{\text{real}}$ ), for each fixed non- $p$ -rational field  $K$ , let  $p_K = \sup_{\mathcal{T}_K, p \neq 1} (p)$ ; then it will be interesting to have a great lot of  $C_{p_K}(K)$ , which is of course non-effective.

**4.3. A special family of quadratic fields.** Consider, for  $p$  fixed, the field:

$$K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1}), \text{ with } \rho \geq 2, a \geq 1, p \nmid a;$$

assuming that  $m := a^2 \cdot p^{2\rho} + 1$  is a squarefree integer, its fundamental unit is  $\varepsilon_K = a \cdot p^\rho + \sqrt{m}$  and  $D_K = m$  (for  $a \cdot p$  even) or  $4m$  (for  $a \cdot p$  odd); the case of  $m = a^2 \cdot p^{2\rho} + 4$  would be similar. From the formula (2.8), we have  $h_K < \frac{1}{2} \cdot \sqrt{D_K}$ , and an upper bound being  $a \cdot p^\rho$ , this allows to get  $v_p(\#\mathcal{C}_K) \leq \rho + \frac{\log_\infty(a)}{\log_\infty(p)}$  to take into account the possible (incredible) case where  $h_K$  is a maximal  $p$ th power. As  $\delta_p(\varepsilon_K) + v_2(\#\mathcal{W}_K) = \rho - 1$  for these fields, it follows:

$$\rho - 1 \leq v_p(\#\mathcal{T}_K) = v_p(\#\mathcal{C}_K) + \delta_p(\varepsilon_K) + v_p(\#\mathcal{W}_K) < 2\rho + \frac{\log_\infty(a)}{\log_\infty(p)}.$$

Thus, since  $\frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} \approx \rho + \frac{\log_\infty(a)}{\log_\infty(p)}$ , we have proved, in this particular case, that:

$$\frac{\rho - 1}{\rho + \frac{\log_\infty(2a)}{\log_\infty(p)}} \leq C_p(K) < \frac{2\rho + \frac{\log_\infty(a)}{\log_\infty(p)}}{\rho + \frac{\log_\infty(a)}{\log_\infty(p)}} \in [1, 2[.$$

We shall assume the conjecture that, for all  $p$ ,  $m := a^2 \cdot p^{2\rho} + 1$  is squarefree<sup>2</sup> for infinitely many integers  $\rho \geq 2$ . Whence the partial result:

**Theorem 4.4.** *Let  $\mathcal{K}_{\text{real}}^{(2)}$  be the family of real quadratic fields and let:*

$$C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}, \text{ for } K \in \mathcal{K}_{\text{real}}^{(2)} \text{ and } p \geq 2.$$

*Then, under the above conjecture on  $m := a^2 \cdot p^{2\rho} + 1$ ,  $\rho \geq 2$ , one has, for each fixed  $p$ ,  $C_p(K) \in [0, 2[$  for an infinite subset of  $\mathcal{K}_{\text{real}}^{(2)}$ .*

Moreover, if we consider the estimation of  $v_p(\#\mathcal{C}_K)$  largely excessive, as explained in the § 4.2.3 (ii), one may conjecture that, for the above family of fields  $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1})$ ,  $\rho \geq 2$ , one has:

$$\rho - 1 \leq v_p(\#\mathcal{T}_K) < \rho \cdot (1 + o(1)),$$

and the statement of the theorem becomes:

*For each  $p \geq 2$ ,  $C_p(K)$  is asymptotically equal to 1 for an infinite subset of  $\mathcal{K}_{\text{real}}^{(2)}$ .*

Indeed,  $v_p(\#\mathcal{T}_K)$  (in *vptor*) and  $v_p(\#\mathcal{C}_K)$  (in *vph*) are given by the following program, to illustrate the relation  $\rho - 1 \leq v_p(\#\mathcal{T}_K) < \rho \cdot (1 + o(1))$ .

We vary  $p$  and  $\rho$  in intervals such that, for instance,  $\log_\infty(m) < 40$  (just choose  $a$ ,  $n$  large enough, and copy and paste the program to get complete tables):

```
{a=1;B=40;n=26;forprime(p=2,20,for(rho=2,B/(2*log(p)),m=a^2*p^(2*rho)+1;
if(core(m)!=m,next);D=m;if(Mod(m,4)!=1,D=4*m);P=x^2-m;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Cp=vptor*log(p)/log(sqrt(D));h=component(component(component(K,8),1),1);
vph=valuation(h,p);
print("p=",p," m=",m," rho=",rho," vptor=",vptor," Cp=",Cp," vph=",vph)))}
```

```
a=1, p=2, D=m
m=17   rho=2   vptor=1 Cp=0.4893010842... vph=0
m=65   rho=3   vptor=3 Cp=0.9962858772... vph=1
```

<sup>2</sup> The conjecture is true for integers of the form  $n^2 + 1$  [20], but we ignore if this remains true for  $n = a \cdot p^\rho$ ,  $p$  prime,  $\rho \in \mathbb{N}$ ,  $a \geq 1$ ; but this is not so essential (see Remark 4.5).

```

(...)
m=4398046511105    rho=21    vptor=29    Cp=1.3809523809...    vph=10
m=17592186044417    rho=22    vptor=24    Cp=1.0909090909...    vph=3
(...)
m=18014398509481985    rho=27    vptor=29    Cp=1.074074074...    vph=6
m=72057594037927937    rho=28    vptor=26    Cp=0.9285714285...    vph=2

a=1, p=3, D=4*m
m=82    rho=2    vptor=1    Cp=0.3792886959...    vph=0
m=730    rho=3    vptor=3    Cp=0.8260927150...    vph=1
(...)
m=16677181699666570    rho=17    vptor=17    Cp=0.9642146068...    vph=1
m=150094635296999122    rho=18    vptor=19    Cp=1.0198095452...    vph=2

a=1, p=5, D=4*m
m=626    rho=2    vptor=1    Cp=0.4113240423...    vph=0
m=15626    rho=3    vptor=2    Cp=0.5829720101...    vph=0
(...)
m=2384185791015626    rho=11    vptor=11    Cp=0.9623227412...    vph=1
m=59604644775390626    rho=12    vptor=11    Cp=0.8849075871...    vph=0

a=2, p=3, D=m
m=2917    rho=3    vptor=3    Cp=0.8261991487...    vph=1
m=26245    rho=4    vptor=3    Cp=0.6478156494...    vph=0
(...)
m=66708726798666277    rho=17    vptor=16    Cp=0.9074961005...    vph=0
m=600378541187996485    rho=18    vptor=19    Cp=1.0198095452...    vph=2

a=2, p=5, D=m
m=2501    rho=2    vptor=1    Cp=0.4113870622...    vph=0
m=62501    rho=3    vptor=2    Cp=0.5829745440...    vph=0
(...)
m=9536743164062501    rho=11    vptor=10    Cp=0.8748388557...    vph=0
m=238418579101562501    rho=12    vptor=11    Cp=0.8849075871...    vph=0

For  $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 4})$ ,  $a$  odd,  $\varepsilon_K = \frac{a \cdot p^\rho + \sqrt{m}}{2}$ ,  $K$  is unramified at 2 giving
a maximal  $C_p(K) = 1.2222222215...$  (for  $a = 1$ ,  $p = 3$ ,  $\rho = 9$ ,  $vptor = 11$ ,  $vph = 3$ ):

a=1, p=3, D=m
m=85    rho=2    vptor=1    Cp=0.4945750747656077295917504    vph=0
m=733    rho=3    vptor=3    Cp=0.9991705549452351082457751    vph=1
(...)
m=109418989131512359213    rho=21    vptor=23    Cp=1.0952380952380952380943703    vph=3
m=984770902183611232885    rho=22    vptor=22    Cp=0.9999999999999999999999159    vph=1

a=1, p=5, D=m
m=629    rho=2    vptor=1    Cp=0.4995050064384236683280022    vph=0
m=15629    rho=3    vptor=2    Cp=0.6666489958698626477868625    vph=0
(...)
m=37252902984619140629    rho=14    vptor=13    Cp=0.9285714285714285714263589    vph=0
m=931322574615478515629    rho=15    vptor=16    Cp=1.06666666666666666666666666665717    vph=2

a=1, p=7, D=m
m=2405    rho=2    vptor=1    Cp=0.4998930943437939009946102    vph=0
m=117653    rho=3    vptor=2    Cp=0.6666647253436162691864834    vph=0
(...)
m=3909821048582988053    rho=11    vptor=10    Cp=0.9090909090909090908873656    vph=0
m=191581231380566414405    rho=12    vptor=12    Cp=0.9999999999999999999999529    vph=1

```

```

a=1, p=11, D=m
m=14645 rho=2 vptor=1 Cp=0.4999857604139424915125214 vph=0
m=1771565 rho=3 vptor=2 Cp=0.6666665620428398909421335 vph=0
(...)
m=5559917313492231485 rho=9 vptor=10 Cp=1.11111111111111110925908 vph=2
m=672749994932560009205 rho=10 vptor=9 Cp=0.89999999999999999999998884 vph=0

```

```

a=1, p=17, D=m
m=24137573 rho=3 vptor=2 Cp=0.6666666601676951315812133 vph=0
m=6975757445 rho=4 vptor=3 Cp=0.7499999999810259247791427 vph=0
(...)
m=168377826559400933 rho=7 vptor=6 Cp=0.8571428571428571423437840 vph=0
m=48661191875666868485 rho=8 vptor=8 Cp=0.999999999999999999999981866 vph=1

```

One sees, from these excerpts, the weak influence of  $vph = v_p(\mathcal{O}_K)$  giving very few  $C_p(K) = 1 + o(1)$ . Larger values of  $a, p$ , yields the same kind of results.

*Remark 4.5.* Without assuming that  $m = a^2 \cdot p^{2\rho} \pm 1$  (or  $m = a^2 \cdot p^{2\rho} \pm 4$ ) is squarefree (which is indeed impossible for minus signs), the same program gives always  $C_p(K)$  near 1 and in any case in  $[0, 2[$  as far as we have tested this property; of course, if  $m = b^2 m'$  with  $m'$  squarefree, the unit  $\varepsilon' = a \cdot p^\rho + b \cdot \sqrt{m'}$  is not necessarily fundamental so that  $\delta_p(\varepsilon_K) \leq \delta_p(\varepsilon')$  and  $D_K = m'$  or  $4m'$  may be very small (the program deals only with non-squarefree integers  $m$ ):

```

{B=60;for(a=1,18,forprime(p=2,19,for(rho=1,B/(2*log(p)),m=a^2*p^(2*rho)+1;
n=rho+6;if(core(m)!=m,P=x^2-m;K=bnfinit(P,1);D=component(component(K,7),3);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Cp=vptor*log(p)/log(sqrt(D));
print("a=",a," p=",p," m=",m," rho=",rho," vptor=",vptor," Cp=",Cp))}}

```

Then the biggest  $C_p(K)$  are for trivial cases ( $m = 5^2 \cdot 41$  and  $m = 250001 = 53^2 \cdot 89$ ):

|     |     |            |       |         |           |
|-----|-----|------------|-------|---------|-----------|
| a=1 | p=2 | D=m=1025   | rho=5 | vptor=4 | Cp=1.4932 |
| a=4 | p=5 | D=m=250001 | rho=3 | vptor=2 | Cp=1.4342 |

**4.4. Reciprocal study.** We fix  $p \geq 2, \rho \geq 2$ , and we try to build units of the form  $\eta = 1 + p^\rho \cdot (X + Y \cdot \sqrt{m})$ , where  $X, Y \in \mathbb{Z}$  and where  $m$  is a squarefree integer. It is not necessary to consider the case  $\frac{X+Y \cdot \sqrt{m}}{2}$ ,  $X$  and  $Y$  of same parity for  $m \equiv 1 \pmod{4}$ , since this only concerns the cases  $p = 2$  (in which case this can modify  $\rho$  into  $\rho - 1$ ) and  $p = 3$  (since any cube of unit is of the suitable form and this also modifies the choice of  $\rho$ ).

In  $K = \mathbb{Q}(\sqrt{m})$ ,  $\eta$  may be a  $p$ -power of the fundamental unit  $\varepsilon_K$ , but this goes in the good direction to get an upper bound of  $C_p(K)$ , if we use  $\delta_p(\eta)$  instead of  $\delta_p(\varepsilon_K)$  to compute  $v_p(\#T_K)$ , since  $\delta_p(\varepsilon_K) \leq \delta_p(\eta)$ .

**Lemma 4.6.** *The number  $\eta = 1 + p^\rho \cdot (X + Y \cdot \sqrt{m})$ ,  $X, Y \in \mathbb{Z}$ , is a unit of  $\mathbb{Q}(\sqrt{m})$  if and only if  $X = p^\rho \cdot a$  and  $a \cdot (2 + p^{2\rho} \cdot a) = m \cdot b^2$  (resp.  $a \cdot (1 + 2^{2\rho-2} \cdot a) = m \cdot b^2$ ) if  $p \neq 2$  (resp.  $p = 2$ ),  $a, b \in \mathbb{Z}$ .*

*Proof.* We have  $N_{K/\mathbb{Q}}(\eta) = \pm 1$  if and only if:

$$1 + p^\rho \cdot (X + Y \cdot \sqrt{m}) + p^\rho \cdot (X - Y \cdot \sqrt{m}) + p^{2\rho} \cdot (X^2 - m \cdot Y^2) = \pm 1$$

which is equivalent (since  $-1$  is absurd for  $\rho \geq 2$ ) to  $2 \cdot X + p^\rho \cdot X^2 = m \cdot p^\rho \cdot Y^2$ . For  $p \neq 2$ , this yields  $X = p^\rho \cdot a$ ,  $Y = b$ , such that  $a \cdot (2 + p^{2\rho} \cdot a) = m \cdot b^2$ . For  $p = 2$ , one must consider the relation  $a \cdot (1 + 2^{2\rho-2} \cdot a) = m \cdot b^2$ , whence in practice the relation  $a \cdot (1 + 2^{2\rho} \cdot a) = m \cdot b^2$  replacing  $\rho$  by  $\rho - 1$ .  $\square$

So, we shall fix  $\rho$  large enough, increase  $a$  in some interval and write  $a \cdot (2 + p^{2\rho} \cdot a)$  (resp.  $a \cdot (1 + 2^{2\rho} \cdot a)$ ) under the form  $m \cdot b^2$ ,  $m$  squarefree. We then compute the successive minima of  $D_K$  for  $K = \mathbb{Q}(\sqrt{m})$ , to try to get maximal values for  $C_p(K)$ :

```
{p=3;rho=21;n=rho+6;ba=10^8+1;Ba=2*10^8;pp=p^(2*rho);Dmin=10^100;d=2;
if(p==2,d=1);for(a=ba,Ba,B=a*(d+pp*a);m=core(B);D=m;if(Mod(m,4)!=1,D=4*m);
if(D<Dmin,Dmin=D;b=component(core(B,1),2);P=x^2-m;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(sqrt(D));
h=component(component(component(K,8),1),1);vph=valuation(h,p);
print("D=",D," a=",a," b=",b," vptor=",vptor," vph=",vph," Cp=",Cp))}
```

We have done a great lot of experimentations with very large discriminants without obtaining any  $C_p(K) > 2$ , except, for  $p = 2$  and the known case (see § 4.2.1):

D=81624    a=9728    b=557872    vptor=20    vph=3    Cp=2.45147522

which corresponds to a too small discriminant since the stabilisation of  $C_p(K)$  seems better and better as soon as  $D_K \gg 0$ . Moreover,  $v_2(\mathcal{O}_K) = 3$  in this example.

Let  $a \in [10^8 + 1, 2 \cdot 10^8]$  (an interval of negative values of  $a$  gives similar results):

| p=3, rho=21                           |           |         |       |     |        |  |
|---------------------------------------|-----------|---------|-------|-----|--------|--|
| D                                     | a         | b       | vptor | vph | Cp     |  |
| 4376759652795686111245843894049436844 | 100000001 | 1       | 22    | 2   | 0.5729 |  |
| 1094189935082719682370900209849436840 | 100000002 | 2       | 21    | 0   | 0.5560 |  |
| 6474496916274063005939132968034008    | 100000004 | 26      | 21    | 1   | 0.5926 |  |
| (...)                                 |           |         |       |     |        |  |
| 780348725011642441673212              | 100250343 | 2374203 | 21    | 0   | 0.8387 |  |
| 97192908950160977396761               | 100966886 | 3387724 | 21    | 1   | 0.8717 |  |

There is no solution  $a \in [10^8 + 966886, 2 \cdot 10^8]$  giving smaller discriminants.

| p=2, rho=30, n=2*rho                |           |         |       |     |        |  |
|-------------------------------------|-----------|---------|-------|-----|--------|--|
| D                                   | a         | b       | vptor | vph | Cp     |  |
| 11529215276652771834290899906846977 | 100000001 | 1       | 35    | 5   | 0.6186 |  |
| 17055053207700727651215465398745    | 100000004 | 26      | 42    | 11  | 0.8096 |  |
| (...)                               |           |         |       |     |        |  |
| 48025975228418415280613             | 100175668 | 490822  | 37    | 6   | 0.9821 |  |
| 28578131029527067857561             | 100311617 | 637139  | 34    | 4   | 0.9115 |  |
| 617974038061148975453               | 100469200 | 4339580 | 36    | 4   | 1.0424 |  |

Same remarks as for the case  $p = 3$ ; despite genera theory, it seems that  $C_p(K)$  remains close to 1 and is not increasing substantially in the process.

## 5. NUMERICAL INVESTIGATIONS FOR CYCLIC CUBIC FIELDS

For the computations in the set  $\mathcal{K}_{ab}^{(3)}$  of cyclic cubic fields, we shall use the direct calculation of  $\#\mathcal{T}_K$  from the program testing the  $p$ -rationality, taking  $n$  large enough.

See [22] for statistics on  $v_p(R_{K,p}) = v_p(\#\mathcal{R}_K) + 2$  (resp.  $v_p(\#\mathcal{R}_K) + 1$ ) in the non-ramified (resp. ramified) case for cyclic cubic fields of conductors up to  $10^8$ ; this gives, for cubic fields, the analogue of the computation of  $\delta_p(\varepsilon)$  for quadratic fields in Subsection 4.1.

Note that, due to Galois action, the integers  $v_p(\#\mathcal{T}_K)$  are even if  $p \equiv 2 \pmod{3}$  and arbitrary if not (same remark for  $v_p(\#\mathcal{O}_K)$  and  $v_p(\#\mathcal{R}_K)$ ); then  $v_2(\#\mathcal{W}_K) = 2$  if 2 splits in  $K$ , otherwise  $v_2(\#\mathcal{W}_K) = 0$  and  $v_p(\#\mathcal{W}_K) = 0$  for  $p > 2$ .

**5.1. Maximal values of  $v_p(\#\mathcal{T}_K)$ .** The program uses the well-known classification of cyclic cubic fields [7] with conductor  $f_K \leq Bf$  (see the formulas 2.11 giving the corresponding polynomials defining  $K$ ), and processes as for the quadratic case. We give first the case  $p = 3$  to see the influence of genera theory; we compute the successive maxima of  $v_p(\#\mathcal{T}_K)$  (in *vptor*) with the corresponding  $f_K$  and the polynomial defining the field of conductor  $f_K$ . We print in the first line the maximal value obtained for  $C_p(K)$  in the selected interval.

Recall that  $D_K = f_K^2$ , where  $f_K = f'_K$  or  $9 \cdot f'_K$  with  $f'_K = \ell_1 \cdots \ell_t$ , for distinct primes  $\ell_i \equiv 1 \pmod{3}$ :

```
{p=3;n=26;bf=7;Bf=10^7;Max=0;for(f=bf,Bf,e=valuation(f,3);if(e!=0 & e!=2,next);
F=f/3^e;if(Mod(F,3)!=1 || core(F)!=F,next);F=factor(F);Div=component(F,1);
d=component(matsize(F),1);for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;
if(issquare(A,&a)==1,if(e==0,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(f);
if(vptor>Max,Max=vptor;print("f=",f," vptor=",vptor," P=",P," Cp=",Cp))))}
p=3 Cp=1.1492
f=19 vptor=1 P=x^3 + x^2 - 6*x - 7
f=199 vptor=2 P=x^3 + x^2 - 66*x + 59
f=427 vptor=4 P=x^3 + x^2 - 142*x - 680
f=1843 vptor=5 P=x^3 + x^2 - 614*x + 3413
f=2653 vptor=6 P=x^3 + x^2 - 884*x - 8352
f=17353 vptor=7 P=x^3 + x^2 - 5784*x - 145251
f=30121 vptor=8 P=x^3 + x^2 - 10040*x + 306788
f=114079 vptor=9 P=x^3 + x^2 - 38026*x + 2822399
f=126369 vptor=10 P=x^3 - 42123*x + 3046897
f=355849 vptor=11 P=x^3 + x^2 - 118616*x - 15235609
f=371917 vptor=12 P=x^3 + x^2 - 123972*x + 15854684
f=1687987 vptor=15 P=x^3 + x^2 - 562662*x - 116533621
p=2, n=36 Cp=1.2475
f=31 vptor=2 P=x^3 + x^2 - 10*x - 8
f=171 vptor=6 P=x^3 - 57*x - 152
f=2689 vptor=8 P=x^3 + x^2 - 896*x + 5876
f=6013 vptor=12 P=x^3 + x^2 - 2004*x - 32292
f=6913 vptor=13 P=x^3 + x^2 - 2304*x - 256
f=311023 vptor=16 P=x^3 + x^2 - 103674*x + 5068523
f=544453 vptor=18 P=x^3 + x^2 - 181484*x - 19862452
f=618093 vptor=24 P=x^3 - 206031*x + 21289870
p=7 Cp=1.3955
f=9 vptor=1 P=x^3 - 3*x + 1
f=313 vptor=2 P=x^3 + x^2 - 104*x + 371
f=721 vptor=3 P=x^3 + x^2 - 240*x - 988
f=1381 vptor=4 P=x^3 + x^2 - 460*x - 1739
f=29467 vptor=6 P=x^3 + x^2 - 9822*x - 20736
f=177541 vptor=7 P=x^3 + x^2 - 59180*x + 3051075
f=1136587 vptor=10 P=x^3 + x^2 - 378862*x + 58428991
```

**5.2. Experiments for a conjectural upper bound – Cubic fields.** In the same way as for quadratic fields, we give, for each prime  $p$ , the successive minima of  $\Delta_p(K) = \frac{\log_\infty(f_K)}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$  (in *Ymin*) with the value of  $C_p(K) = \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(f_K)}$  (in *Cp*), obtained for some polynomial  $P$  and the corresponding conductor  $f_K$ :

```

{n=36;bf=7;Bf=5*10^6;forprime(p=2,50,ymin=10;print("p="p);for(f=bf,Bf,
e=valuation(f,3);if(e!=0 & e!=2,next);F=f/3^e;if(Mod(F,3)!=1||core(F)!=F,next);
F=factor(F);Div=component(F,1);d=component(matsize(F),1);
for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;
if(issquare(A,&a)==1,if(e==0,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(f)/log(p)-vptor;if(Y<ymin,ymin=Y;print(P);Cp=vptor*log(p)/log(f);
print("f=",f," vptor=",vptor," Ymin=",Y," Cp=",Cp))))))}

```

The first minimum occurs for  $f := f_K = 7$  and  $vptor := v_p(\#T_K) = 0$ ; we omit these cases of  $p$ -rationality. For some  $p$ , we have been obliged to consider larger conductors  $f$  to get significant solutions, especially for  $p = 11$  for which the first non-trivial example is for  $f = 5000059$  and  $P = x^3 + x^2 - 1666686x - 40852339$ .

```

p=2, Cp=1.247565
P=x^3 - 57*x - 152
f=171 vptor=6 Ymin=1.41785251... Cp=0.8088
P=x^3 + x^2 - 2004*x - 32292
f=6013 vptor=12 Ymin=0.55386924... Cp=0.9559
P=x^3 + x^2 - 2304*x - 256
f=6913 vptor=14 Ymin=-1.24490378... Cp=1.0976
P=x^3 - 206031*x + 21289870
f=618093 vptor=24 Ymin=-4.76253559... Cp=1.2475
p=3, Cp=1.149252
P=x^3 + x^2 - 6*x - 7
f=19 vptor=1 Ymin=1.68014385... Cp=0.3731
P=x^3 + x^2 - 142*x - 680
f=427 vptor=4 Ymin=1.51312239... Cp=0.7255
P=x^3 + x^2 - 884*x - 8352
f=2653 vptor=6 Ymin=1.17582211... Cp=0.8361
P=x^3 - 42123*x + 3046897
f=126369 vptor=10 Ymin=0.69254513... Cp=0.9352
P=x^3 + x^2 - 118616*x - 15235609
f=355849 vptor=11 Ymin=0.63491606... Cp=0.9454
P=x^3 + x^2 - 123972*x + 15854684
f=371917 vptor=12 Ymin=-0.32488392... Cp=1.0278
P=x^3 + x^2 - 562662*x - 116533621
f=1687987 vptor=15 Ymin=-1.94803671... Cp=1.1492
p=5, Cp=1.462906
P=x^3 + x^2 - 50*x - 123
f=151 vptor=2 Ymin=1.11741123... Cp=0.6415
P=x^3 + x^2 - 1002*x + 6905
f=3007 vptor=4 Ymin=0.97608396... Cp=0.8038
P=x^3 + x^2 - 2214*x + 19683
f=6643 vptor=8 Ymin=-2.53143306... Cp=1.4629
p=7, Cp=1.395563
P=x^3 - 3*x + 1
f=9 vptor=1 Ymin=0.12915006... Cp=0.8856
P=x^3 + x^2 - 460*x - 1739
f=1381 vptor=4 Ymin=-0.28422558... Cp=1.0765
P=x^3 + x^2 - 9822*x - 20736
f=29467 vptor=6 Ymin=-0.71145865... Cp=1.1345
P=x^3 + x^2 - 59180*x + 3051075
f=177541 vptor=7 Ymin=-0.78853291... Cp=1.1269
P=x^3 + x^2 - 378862*x + 58428991

```

$f=1136587$      $vptor=10$      $Ymin=-2.83443766\dots$      $Cp=1.3955$   
 $p=11$ ,  $Cp=0.621490$   
 $P=x^3 + x^2 - 1666686x - 408523339$   
 $f=5000059$      $vptor=2$      $Ymin=4.43270806\dots$      $Cp=0.3109$   
 $P=x^3 - 1680483x - 503584739$   
 $f=5041449$      $vptor=4$      $Ymin=2.43614601\dots$      $Cp=0.6215$   
 $p=13$ ,  $Cp=1.632521$   
 $P=x^3 + x^2 - 20x - 9$   
 $f=61$      $vptor=1$      $Ymin=0.60271151\dots$      $Cp=0.6239$   
 $P=x^3 + x^2 - 196x - 349$   
 $f=589$      $vptor=2$      $Ymin=0.48676495\dots$      $Cp=0.8042$   
 $P=x^3 + x^2 - 1064x + 12299$   
 $f=3193$      $vptor=3$      $Ymin=0.14576042\dots$      $Cp=0.9536$   
 $P=x^3 + x^2 - 1824x + 8919$   
 $f=5473$      $vptor=4$      $Ymin=-0.64415121\dots$      $Cp=1.1919$   
 $P=x^3 + x^2 - 19920x + 615317$   
 $f=59761$      $vptor=7$      $Ymin=-2.71215372\dots$      $Cp=1.6325$   
 $p=17$ ,  $Cp=0.910481$   
 $P=x^3 - 399x - 3059$   
 $f=1197$      $vptor=2$      $Ymin=0.50160254\dots$      $Cp=0.7994$   
 $P=x^3 - 84837x + 1046323$   
 $f=254511$      $vptor=4$      $Ymin=0.39327993\dots$      $Cp=0.9105$   
 $p=19$ ,  $Cp=0.974463$   
 $P=x^3 + x^2 - 30x + 27$   
 $f=91$      $vptor=1$      $Ymin=0.53199286\dots$      $Cp=0.6527$   
 $P=x^3 + x^2 - 404x + 629$   
 $f=1213$      $vptor=2$      $Ymin=0.41161455\dots$      $Cp=0.8293$   
 $P=x^3 - 3477x - 26657$   
 $f=10431$      $vptor=3$      $Ymin=0.14237703\dots$      $Cp=0.9547$   
 $P=x^3 + x^2 - 1213944x - 503921781$   
 $f=3641833$      $vptor=5$      $Ymin=0.13102760\dots$      $Cp=0.9744$   
 $p=23$ ,  $Cp=0.880087$   
 $P=x^3 + x^2 - 1060x - 11428$   
 $f=3181$      $vptor=2$      $Ymin=0.57214663\dots$      $Cp=0.7775$   
 $P=x^3 + x^2 - 515154x - 19633104$   
 $f=1545463$      $vptor=4$      $Ymin=0.54500411\dots$      $Cp=0.8801$   
 $p=29$ ,  $Cp=1.569666$   
 $P=x^3 + x^2 - 24x - 27$   
 $f=73$      $vptor=2$      $Ymin=-0.72584422\dots$      $Cp=1.5696$   
 $p=31$ ,  $Cp=0.981745$   
 $P=x^3 + x^2 - 30x + 27$   
 $f=91$      $vptor=1$      $Ymin=0.31359240\dots$      $Cp=0.7613$   
 $P=x^3 - 12027x + 388873$   
 $f=36081$      $vptor=3$      $Ymin=0.05578357\dots$      $Cp=0.9817$   
 $p=37$ ,  $Cp=1.119764$   
 $P=x^3 - 39x - 26$   
 $f=117$      $vptor=1$      $Ymin=0.31882641\dots$      $Cp=0.7582$   
 $P=x^3 + x^2 - 5300x + 119552$   
 $f=15901$      $vptor=3$      $Ymin=-0.32086480\dots$      $Cp=1.1197$   
 $p=41$ ,  $Cp=0.976052$   
 $P=x^3 + x^2 - 672x - 2764$   
 $f=2017$      $vptor=2$      $Ymin=0.04906930\dots$      $Cp=0.9760$   
 $p=43$ ,  $Cp=0.914939$   
 $P=x^3 + x^2 - 20x - 9$   
 $f=61$      $vptor=1$      $Ymin=0.09296866\dots$      $Cp=0.9149$   
 $p=47$ ,  $Cp=0.878952$   
 $P=x^3 + x^2 - 2126x + 11813$   
 $f=6379$      $vptor=2$      $Ymin=0.27543656\dots$      $Cp=0.8789$

## 6. EXAMPLES OF NON-GALOIS TOTALLY REAL NUMBER FIELDS

We shall consider (non necessarily Galois) cubic fields, with an approach using randomness. The tested polynomials of degree 3 define almost always Galois groups isomorphic to  $S_3$ . It is more difficult to find non- $p$ -rational fields for large  $p$  and to obtain a lower bound of  $\mathcal{C}_p^{(3)}$  for the family  $\mathcal{K}_{\text{real}}^{(3)}$  of totally real cubic fields.

**6.1. Program for a given cubic polynomial and increasing  $p$ .** The program concerns fields  $K$  defined by  $P = x^3 + ax^2 + bx + 1$ , for random  $a, b$  and increasing  $p$  in  $[2, 10^5]$ . It tests the irreducibility of  $P$  and that  $D_K > 0$  (real roots). We give only the non- $p$ -rational cases for which one prints the corresponding  $C_p(K)$ .

```
{n=4;N=100;bp=2;Bp=10^5;Ymin=10;a=random(N);b=random(N);P=x^3+a*x^2+b*x+1;
if(polisirreducible(P)==1 & poldisc(P)>0,print(P);K=bnfinit(P,1);
D=component(component(K,7),3);forprime(p=bp,Bp,Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;
if(vptor > 0 & Y<Ymin,Ymin=Y;Cp=vptor*log(p)/log(sqrt(D));
print("p=",p," vptor=",vptor," Ymin=",Y," Cp=",Cp))})
```

We obtain, after several tries and  $p$  up to  $10^5$ , omitting the small values of  $C_p(K)$ :

|                           |         |                     |            |
|---------------------------|---------|---------------------|------------|
| $P=x^3 + 21x^2 + 47x + 1$ |         |                     |            |
| p=11                      | vptor=1 | Ymin=1.75210757...  |            |
| p=523                     | vptor=1 | Ymin=0.05426629...  |            |
| p=3517                    | vptor=1 | Ymin=-0.19179768... | Cp=1.2373  |
| p=173483                  | vptor=1 | Ymin=-0.45297114... | Cp=1.8280  |
| $P=x^3 + 19x^2 + 51x + 1$ |         |                     |            |
| p=487                     | vptor=1 | Ymin=-0.40614414... | Cp=1.6839  |
| $P=x^3 + 92x^2 + 52x + 1$ |         |                     |            |
| p=18637                   | vptor=1 | Ymin=-0.14697706... | Cp=1.1723  |
| $P=x^3 + 99x^2 + 23x + 1$ |         |                     |            |
| p=73                      | vptor=1 | Ymin=0.47867182...  |            |
| p=15803                   | vptor=1 | Ymin=-0.34379282... | Cp=1.5239  |
| p=145259                  | vptor=1 | Ymin=-0.46625984... | Cp=1.8735  |
| p=622519                  | vptor=1 | Ymin=-0.52447869... | Cp=2.1029  |
| $P=x^3 + 98x^2 + 62x + 1$ |         |                     |            |
| p=3                       | vptor=2 | Ymin=3.86940839...  |            |
| p=61                      | vptor=1 | Ymin=0.56857262...  |            |
| p=37549                   | vptor=1 | Ymin=-0.38783270... | Cp=1.63354 |
| $P=x^3 + 87x^2 + 74x + 1$ |         |                     |            |
| p=5441                    | vptor=1 | Ymin=0.01344518...  | Cp=0.9867  |
| $P=x^3 + 73x^2 + 67x + 1$ |         |                     |            |
| p=6133                    | vptor=1 | Ymin=-0.03273397... | Cp=1.0338  |
| $P=x^3 + 19x^2 + 83x + 1$ |         |                     |            |
| p=61                      | vptor=1 | Ymin=-0.52664318... | Cp=2.1126  |
| p=5419                    | vptor=1 | Ymin=-0.77366996... | Cp=4.4183  |
| p=12703                   | vptor=1 | Ymin=-0.79407472... | Cp=4.8561  |

Note that by accident,  $P = x^3 + 19x^2 + 83x + 1$ , with a large  $C_{12703}(K) \approx 4.8561$ , defines the cyclic cubic field  $K$  of conductor 7 (in some sense, an analogue of  $K = \mathbb{Q}(\sqrt{19})$  with  $p_0 = 13599893$  for which  $C_{p_0}(K) \approx 7.5856$ , see § 4.2.3 (i)).

But the forthcoming conductors  $f > 7$ , up to  $4 \cdot 10^6$ , give decreasing  $C_{12703}(K)$ , as shown by the following excerpts, where no  $v_p(\# \mathcal{T}_K) \geq 2$  were found with  $p = 12703$ :

|          |         |                                |             |
|----------|---------|--------------------------------|-------------|
| f=7      | vptor=1 | $x^3 + x^2 - 2x - 1$           | Cp=4.856130 |
| f=17767  | vptor=1 | $x^3 + x^2 - 5922x + 17109$    | Cp=0.965712 |
| f=54649  | vptor=1 | $x^3 + x^2 - 18216x - 931057$  | Cp=0.866244 |
| f=101839 | vptor=1 | $x^3 + x^2 - 33946x + 1059880$ | Cp=0.819484 |

```
(...)
f=497647    vptor=1    x^3 + x^2 - 165882*x + 7114509    Cp=0.720372
f=547903    vptor=1    x^3 + x^2 - 182634*x - 12804696    Cp=0.715127
(...)
f=859621    vptor=1    x^3 + x^2 - 286540*x + 49348613    Cp=0.691556
f=865189    vptor=1    x^3 + x^2 - 288396*x - 7818745    Cp=0.691229
(...)
f=1680543    vptor=1    x^3 - 560181*x + 55084465    Cp=0.659214
f=1744477    vptor=1    x^3 + x^2 - 581492*x - 143305555    Cp=0.657501
(...)
f=2477313    vptor=1    x^3 - 825771*x + 262870435    Cp=0.641839
f=2486871    vptor=1    x^3 - 828957*x - 138988457    Cp=0.641671
(...)
f=3616141    vptor=1    x^3 + x^2 - 1205380*x + 483625376    Cp=0.625762
f=3628081    vptor=1    x^3 + x^2 - 1209360*x - 96883200    Cp=0.625626
(...)
f=4036591    vptor=1    x^3 + x^2 - 1345530*x + 122293757    Cp=0.621237
f=4037779    vptor=1    x^3 + x^2 - 1345926*x - 499488217    Cp=0.621225
```

**6.2. Program for a given  $p$  and random cubic polynomials.** The program tries polynomials in a random way, so that the discriminants are not obtained in the natural order; we then write, in the first line, the largest  $C_p(K)$  obtained:

```
{p=3;N=1000;n=18;ymax=10;for(k=1,10^6,a=random(N);b=random(N);c=random(N);
P=x^3+a*x^2+b*x+c;if(polisirreducible(P)==1 & poldisc(P)>0,K=bnfinit(P,1);
D=component(component(K,7),3);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;
if(vptor>0 & Y<ymax,ymax=Y;Cp=vptor*log(p)/log(sqrt(D)));
print("P=",P," vptor=",vptor," Ymin=",Y," Cp=",Cp))}}
p=2      Cp=1.497370
P=x^3 + 315*x^2 + 151*x + 13      vptor=6      Ymin=4.62049695...
P=x^3 + 44*x^2 + 388*x + 962      vptor=7      Ymin=2.65795067...
P=x^3 + 78*x^2 + 498*x + 584      vptor=6      Ymin=2.33817139...
P=x^3 + 473*x^2 + 759*x + 90      vptor=12     Ymin=1.79924824...
P=x^3 + 176*x^2 + 760*x + 472     vptor=14     Ymin=-0.65040380...
P=x^3 + 30*x^2 + 165*x + 220      vptor=12     Ymin=-3.98594984...
p=3      Cp=1.042763
P=x^3 + 57*x^2 + 251*x + 70      vptor=4      Ymin=2.95145981...
P=x^3 + 93*x^2 + 396*x + 396      vptor=4      Ymin=2.08419811...
P=x^3 + 53*x^2 + 602*x + 140      vptor=6      Ymin=1.91171871...
P=x^3 + 143*x^2 + 672*x + 617     vptor=8      Ymin=1.71414906...
P=x^3 + 360*x^2 + 698*x + 132     vptor=4      Ymin=1.11320078...
P=x^3 + 194*x^2 + 649*x + 440     vptor=7      Ymin=1.02340828...
P=x^3 + 38*x^2 + 343*x + 722      vptor=6      Ymin=0.41712275...
P=x^3 + 77*x^2 + 512*x + 874      vptor=8      Ymin=-0.32807458...
p=5      Cp=1.238605
P=x^3 + 177*x^2 + 590*x + 456     vptor=1      Ymin=4.94615149...
P=x^3 + 222*x^2 + 789*x + 180     vptor=2      Ymin=1.62797441...
P=x^3 + 45*x^2 + 362*x + 772     vptor=3      Ymin=1.32811388...
P=x^3 + 83*x^2 + 400*x + 251     vptor=2      Ymin=1.22069007...
P=x^3 + 197*x^2 + 718*x + 508     vptor=8      Ymin=-1.54112474...
p=7      Cp=1.201178
P=x^3 + 784*x^2 + 964*x + 288     vptor=1      Ymin=3.97483926...
P=x^3 + 505*x^2 + 710*x + 134     vptor=2      Ymin=2.57552488...
P=x^3 + 73*x^2 + 492*x + 196     vptor=3      Ymin=1.85163167...
P=x^3 + 57*x^2 + 695*x + 263     vptor=1      Ymin=1.35093638...
P=x^3 + 95*x^2 + 839*x + 252     vptor=5      Ymin=0.64570147...
```

|                               |           |                         |
|-------------------------------|-----------|-------------------------|
| $P=x^3 + 114x^2 + 804x + 142$ | $vptor=2$ | $Ymin=-0.37221306\dots$ |
| $P=x^3 + 97x^2 + 829x + 122$  | $vptor=5$ | $Ymin=-0.83742084\dots$ |
| $p=19 \quad C_p=1.139412$     |           |                         |
| $P=x^3 + 50x^2 + 631x + 470$  | $vptor=1$ | $Ymin=1.58556226\dots$  |
| $P=x^3 + 57x^2 + 777x + 801$  | $vptor=1$ | $Ymin=1.54028119\dots$  |
| $P=x^3 + 549x^2 + 732x + 39$  | $vptor=3$ | $Ymin=0.69038895\dots$  |
| $P=x^3 + 93x^2 + 891x + 383$  | $vptor=2$ | $Ymin=0.64611301\dots$  |
| $P=x^3 + 123x^2 + 375x + 217$ | $vptor=1$ | $Ymin=0.46422353\dots$  |
| $P=x^3 + 226x^2 + 777x + 408$ | $vptor=2$ | $Ymin=0.20875475\dots$  |
| $P=x^3 + 196x^2 + 849x + 918$ | $vptor=2$ | $Ymin=-0.24470848\dots$ |
| $p=1009 \quad C_p=1.227512$   |           |                         |
| $P=x^3 + 171x^2 + 667x + 604$ | $vptor=1$ | $Ymin=0.49598190\dots$  |
| $P=x^3 + 89x^2 + 567x + 36$   | $vptor=1$ | $Ymin=0.37961552\dots$  |
| $P=x^3 + 54x^2 + 435x + 719$  | $vptor=1$ | $Ymin=0.29433117\dots$  |
| $P=x^3 + 93x^2 + 636x + 944$  | $vptor=1$ | $Ymin=0.07490160\dots$  |
| $P=x^3 + 432x^2 + 347x + 19$  | $vptor=1$ | $Ymin=-0.06432442\dots$ |
| $P=x^3 + 130x^2 + 942x + 899$ | $vptor=1$ | $Ymin=-0.06692434\dots$ |
| $P=x^3 + 70x^2 + 553x + 735$  | $vptor=1$ | $Ymin=-0.18534377\dots$ |

*Remarks 6.1.* (i) The case  $p = 2$  with  $P = x^3 + 30x^2 + 165x + 220$ , where:

$$v_2(\#\mathcal{T}_K) = 12 \quad \& \quad \Delta_2(K) \approx -3.98595,$$

seems exceptional, but the discriminant  $D_K = 66825$  is rather small. The Galois closure  $L$  of  $K$  contains  $\mathbb{Q}(\sqrt{33})$  and is defined by the polynomial:

$$Q = x^6 - 60x^5 + 1131x^4 - 6380x^3 - 15708x^2 + 145200x + 170368;$$

then  $v_2(\#\mathcal{T}_L) = 25$ , giving  $C_2(L) \approx 1.3476$  instead of  $C_2(K) \approx 1.4973$ .

(ii) For  $p = 5$  and  $P = x^3 + 197x^2 + 718x + 508$ ,  $v_5(\#\mathcal{T}_K) = 8$  is large, but  $D_K = 1069350637 = 769 \cdot 1390573$  is rather large, giving  $C_5(K) \approx 1.2386$ .

(iii) For  $p = 7$ ,  $P = x^3 + 95x^2 + 839x + 252$ ,  $v_7(\#\mathcal{T}_K) = 5$ , with  $C_7(K) \approx 0.8856$ , but  $D_K = 3486121421$ , while for  $P = x^3 + 114x^2 + 804x + 142$ ,  $v_7(\#\mathcal{T}_K) = 2$  with  $C_7(K) \approx 1.2286$ , but  $D_K = 564$ .

(iv) We have computed  $C_p(L)$  for the Galois closure  $L$  of the above fields  $K$  (Galois group  $S_3$ ). The values  $C_p(L)$  are smaller, although the  $v_p(\#\mathcal{T}_L)$  are roughly speaking twice of  $v_p(\#\mathcal{T}_K)$  (cf. Example (i)). This reinforces the idea that extensions  $L/K$  may give in general values of  $C_p(L)$  smaller than those of  $C_p(K)$ .

## 7. CONJECTURES ON $v_p(\#\mathcal{T}_K)$

**7.1.  $p$ -adic statements.** The numerical results (quadratic and cubic cases, with the particular family of quadratic fields studied in Subsections 4.2, 4.3, 4.4) suggest the following conjecture that we state in its strongest form; we shall discuss about some conditions of application of such a conjecture, for instance assuming that the fields  $K$  are of given degree or are elements of specified families.

The points (i) and (ii) are equivalent statements:

**Conjecture 7.1.** *Let  $K \in \mathcal{K}_{\text{real}}$  (or any element of a specified family  $\mathcal{K} \subseteq \mathcal{K}_{\text{real}}$ ), and let  $p \geq 2$  be a prime number. Let  $\mathcal{T}_K$  be the torsion group of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension of  $K$  (under Leopoldt's conjecture).*

(i) *There exists a constant  $\mathcal{C}_p(\mathcal{K}) =: \mathcal{C}_p$ , independent of  $K \in \mathcal{K}$ , such that:*

$$v_p(\#\mathcal{T}_K) \leq \mathcal{C}_p \cdot \frac{\log_{\infty}(\sqrt{D_K})}{\log_{\infty}(p)}, \text{ for all } K \in \mathcal{K}.$$

(ii) The residue  $\tilde{\kappa}_{K,p}$  of the normalized  $\zeta$ -function  $\tilde{\zeta}_{K,p}(s) = \frac{p \cdot [K \cap \mathbb{Q}^c : \mathbb{Q}]}{2^{d-1}} \zeta_{K,p}(s)$  at  $s = 1$  (see Subsection 2.2), is conjecturally such that:

$$v_p(\tilde{\kappa}_{K,p}) \leq \mathcal{C}_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}, \text{ for all } K \in \mathcal{K}.$$

We may propose the following conjecture which takes into account the numerical behaviour of the  $\mathcal{C}_p(K)$  that we have observed; but unfortunately, this would need inaccessible computations to be more convincing:

**Conjecture 7.2.** Let  $\mathcal{K}_{\text{real}}$  be the set of all totally real number fields and let  $p \geq 2$  be any fixed prime number. Then  $\limsup_{K \in \mathcal{K}_{\text{real}}, D_K \rightarrow \infty} \left( \frac{v_p(\# \mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} \right) = 1$ .

**Theorem 7.3.** Let  $d$  be a fixed positive integer and let  $p \nmid d$ . Let  $\mathcal{K}_{\text{ab}}^{(d)}$  be the set of real abelian extension of  $\mathbb{Q}$  whose degree divides  $d$ . Then the conjecture 7.1 is true for  $\mathcal{K}_{\text{ab}}^{(d)}$  if and only if it is true for the subset of cyclic extensions of  $\mathcal{K}_{\text{ab}}^{(d)}$ .

*Proof.* Let  $K \in \mathcal{K}_{\text{ab}}^{(d)}$ . As  $p \nmid [K : \mathbb{Q}]$ ,  $\mathcal{T}_K \simeq \bigoplus_\chi \mathcal{T}_K^{e_\chi}$ , where  $\chi$  runs through the set of irreducible rational characters of  $\text{Gal}(K/\mathbb{Q})$  (a set which is in bijection with that of cyclic subfields of  $K$ ),  $e_\chi$  being the corresponding idempotent; then  $\mathcal{T}_K^{e_\chi}$  is isomorphic to a submodule of  $\mathcal{T}_{k_\chi}$ , where  $k_\chi$  (cyclic) is the subfield of  $K$  fixed by the kernel of  $\chi$ , and  $v_p(\# \mathcal{T}_K) = \sum_\chi v_p(\# \mathcal{T}_K^{e_\chi})$ . We have:

$$\mathcal{C}_p(K) = \frac{v_p(\# \mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} = \sum_\chi \frac{v_p(\# \mathcal{T}_K^{e_\chi}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} \leq \sum_\chi \frac{v_p(\# \mathcal{T}_{k_\chi}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})};$$

but  $D_K = D_{k_\chi}^{[K:k_\chi]} \cdot N_{k_\chi/\mathbb{Q}}(D_{K/k_\chi})$  yields  $\log_\infty(\sqrt{D_K}) \geq [K : k_\chi] \cdot \log_\infty(\sqrt{D_{k_\chi}})$  for all  $\chi$ . Thus, if we have the inequalities  $\mathcal{C}_p(k_\chi) = \frac{v_p(\# \mathcal{T}_{k_\chi}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_{k_\chi}})} \leq \mathcal{C}_p$  for all  $\chi$ ,

the theorem follows with a constant  $\mathcal{C}'_p$ , depending on the maximal number of cyclic subfields for elements of the set  $\mathcal{K}_{\text{ab}}^{(d)}$ , which may be explicated.  $\square$

Let's illustrate this by means of random real biquadratic fields  $K$  for which we compute the invariants of  $K$  and its subfields (then  $v_{\text{ptor}} = v_1 + v_2 + v_3$  for  $p \neq 2$ ):

```
{p=3;n=18;N=2*10^2;B=10^6;vmax=0;for(j=1,B,m1=random(N)+1;m2=random(N)+1;
P1=x^2-m1;P2=x^2-m2;P3=x^2-m1*m2;P=component(polcompositum(P1,P2),1);
if(poldegree(P)!=4,next);D1=nfdisc(P1);D2=nfdisc(P2);D3=nfdisc(P3);D=nfdisc(P);
K1=bnfinit(P1,1);Kpn=bnrinit(K1,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);v1=valuation(Hpn0/Hpn1,p);
K2=bnfinit(P2,1);Kpn=bnrinit(K2,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);v2=valuation(Hpn0/Hpn1,p);
K3=bnfinit(P3,1);Kpn=bnrinit(K3,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);v3=valuation(Hpn0/Hpn1,p);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Cp1=v1*log(p)/log(sqrt(D1));Cp2=v2*log(p)/log(sqrt(D2));
Cp3=v3*log(p)/log(sqrt(D3));Cp=vptor*log(p)/log(sqrt(D));
if(vptor>vmax,vmax=vptor;print(D1," ",D2," ",D3," ",D," ",
v1," ",v2," ",v3," ",vptor," ",Cp1," ",Cp2," ",Cp3," ",Cp))}
```

| D1 | D2   | D3    | D          | v1 | v2 | v3 | vptor | Cp1 | Cp2    | Cp3    | Cp     |
|----|------|-------|------------|----|----|----|-------|-----|--------|--------|--------|
| 41 | 840  | 34440 | 1186113600 | 0  | 0  | 2  | 2     | 0   | 0      | 0.4206 | 0.2103 |
| 12 | 1896 | 632   | 14379264   | 0  | 7  | 0  | 7     | 0   | 2.0378 | 0      | 0.9332 |

|      |      |        |              |   |   |   |    |        |        |        |        |
|------|------|--------|--------------|---|---|---|----|--------|--------|--------|--------|
| 1896 | 1096 | 32469  | 67471101504  | 7 | 0 | 1 | 8  | 2.0378 | 0      | 0.2115 | 0.7049 |
| 1896 | 13   | 24648  | 607523904    | 7 | 0 | 2 | 9  | 2.0378 | 0      | 0.4345 | 0.9777 |
| 1976 | 1896 | 234156 | 877264517376 | 2 | 7 | 1 | 10 | 0.5790 | 2.0378 | 0.1777 | 0.7989 |
| 1896 | 824  | 97644  | 152549611776 | 7 | 4 | 0 | 11 | 2.0378 | 1.3090 | 0      | 0.9385 |
| 1896 | 488  | 14457  | 13376310336  | 7 | 4 | 1 | 12 | 2.0378 | 1.4197 | 0.2293 | 1.1308 |
| 449  | 1896 | 851304 | 724718500416 | 1 | 7 | 5 | 13 | 0.3597 | 2.0378 | 0.8045 | 1.0459 |

For two random discriminants of quadratic fields, taken up to  $2 \cdot 10^2$ , the program did not find any  $v_3(\#\mathcal{T}_K) > 13$ . We have  $C_p(K) < \max(C_p(K_1), C_p(K_2), C_p(K_3))$  (obvious for the biquadratic case). It is likely that the compositum  $K$  of two fields  $K_1, K_2$ , gives in general *smaller*  $C_p(K)$ , except if  $v_p(\#\mathcal{T}_{K_1})$  and  $v_p(\#\mathcal{T}_{K_2})$  are small regarding  $v_p(\#\mathcal{T}_K)$  and if the number of subfields of  $K$  is important, but in that case  $C_p(K)$  remains very small, as is shown by the following rare examples obtained as compositum of two random non-Galois cubic fields giving large  $v_p(\#\mathcal{T}_K)$  (the last line gives  $v1, v2, vptor, Cp1, Cp2, Cp$ ):

```

p=2
P1=x^3-45*x^2+24*x-1,P2=x^3-36*x^2+27*x-1,P=x^9+27*x^8-2844*x^7-54486*x^6
+2141829*x^5+20969253*x^4-10466577*x^3-5546475*x^2+1542807*x+10233
766017 77433 23187342173591131003005670474209
1 1 9 0.102317 0.123147 0.172756
P1=x^3-12*x^2+9*x-1,P2=x^3-20*x^2+23*x-1,P=x^9-24*x^8-192*x^7+5728*x^6
+10131*x^5-301710*x^4+238483*x^3+148968*x^2-83460*x- 8520
3753 15465 21724158202972986227625
1 1 10 0.168437 0.143712 0.269535
P1=x^3-23*x^2+22*x-1,P2=x^3-19*x^2+42*x-1,P=x^9+12*x^8-634*x^7-4844*x^6
+112245*x^5+317540*x^4-1892181*x^3+376428*x^2+2193504*x+51904
173857 1937 38191384824694383099923729
1 1 8 0.114892 0.183156 0.188276
P1=x^3-27*x^2+35*x-1,P2=x^3-11*x^2+8*x-1,P=x^9+48*x^8+303*x^7-10953*x^6
-72549*x^5+825678*x^4+1083824*x^3-357201*x^2-414609*x+57421
10309 1929 7864050646576255644981
2 1 11 0.300038 0.183256 0.302464
P1=x^3-18*x^2+31*x-1,P2=x^3-30*x^2+43*x-1,P=x^9-36*x^8-426*x^7+18708*x^6
+66213*x^5-2207940*x^4-1980725*x^3+5522748*x^2+2482560*x+22464
178889 1261265 11486029882117782845780928107151625
2 3 17 0.229243 0.296055 0.300498
p=3
P1=x^3-47*x^2+27*x-1,P2=x^3-14*x^2+26*x-1,P=x^9+99*x^8+2110*x^7-39581*x^6
-841754*x^5+12433359*x^4-31915251*x^3+12891832*x^2+16161948*x+8084
284788 57741 4446496553844548173991089269312
1 1 7 0.174945 0.200408 0.217948
P1=x^3-31*x^2+25*x-1,P2=x^3-24*x^2+38*x-1,P=x^9+21*x^8-1152*x^7-17265*x^6
+370464*x^5+2658657*x^4-5851191*x^3-1210464*x^2+3554288*x+55138
432884 573349 15288742990049019447046087884332096
1 1 10 0.169300 0.165712 0.279145
P1=x^3-22*x^2+41*x-1,P2=x^3-9*x^2+18*x-1,P=x^9+39*x^8+288*x^7-3470*x^6
-23571*x^5+176589*x^4-88881*x^3-684987*x^2+578139*x-18043
511537 321 4427374441992552457143633
2 1 14 0.334301 0.380706 0.542048
P1=x^3-23*x^2+35*x-1,P2=x^3-24*x^2+30*x-1,P=x^9-3*x^8-906*x^7+1667*x^6
+206130*x^5-144453*x^4-552539*x^3+378690*x^2+168384*x-876
110580 368037 748965293428340819016772904000
1 1 7 0.189195 0.171444 0.216350
P1=x^3-23*x^2+17*x-1,P2=x^3-36*x^2+27*x-1,P=x^9-39*x^8-1017*x^7+37436*x^6
+322812*x^5-7556721*x^4-95099*x^3+3294255*x^2-9906*x-2367
91572 77433 39611733265845206525895660864
1 1 8 0.192319 0.195184 0.266941

```

**Theorem 7.4.** *Let  $K$  be a totally real number field and let  $\mathcal{K}^c$  be the set of subfields  $K_n$  of the  $p$ -cyclotomic tower  $K^c$  of  $K$  (with  $[K_n : K] = p^n$ , for all  $n \geq 0$ ). Then, under the Leopoldt conjecture in  $K^c$ ,  $C_p(K_n) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* From [40, § 3, Proposition 2], we get  $\sqrt{D_{K_n}} \geq p^{\alpha \cdot n \cdot p^n + O(p^n)}$  with  $\alpha > 0$ ; then from Iwasawa's theory, there exist  $\lambda, \mu \in \mathbb{N}$  and  $\nu \in \mathbb{Z}$  such that  $\#\mathcal{T}_{K_n} = p^{\lambda n + \mu p^n + \nu}$  for  $n \gg 0$ . So we obtain  $C_p(K_n) \leq \frac{\lambda n + \mu p^n + \nu}{\alpha \cdot n \cdot p^n + O(p^n)}$  for  $n \gg 0$ , where the limit of the upper bound is 0; whence the result giving an example of family  $(\mathcal{K}^c)$  for which the Conjecture 7.1 is verified.  $\square$

Note that if  $K \in \mathcal{K}_{\text{real}}$  is  $p$ -rational (i.e.,  $C_p(K) = 0$ ), then  $C_p(K_n) = 0$  for all  $n \geq 0$ : see [9], Proposition IV.3.4.6 from the formula of invariants (Theorem 3.3) giving  $C_p(L) = 0$  for any  $p$ -primitively ramified  $p$ -extension  $L$  of  $K$  (Definition 3.4).

*Remark 7.5.* In [21], Hajir and Maire define, in the spirit of an algebraic  $p$ -adic Brauer–Siegel theorem, the *logarithmic mean exponent* of a finite  $p$ -group  $A \simeq \prod_{i=1}^r \mathbb{Z}/p^{a_i}\mathbb{Z}$ , by the formula  $\mathbb{M}_p(A) := \frac{1}{r} \cdot \frac{\log_{\infty}(\#A)}{\log_{\infty}(p)} = \frac{1}{r} \sum_{i=1}^r a_i = \frac{1}{r} \cdot v_p(\#A)$ , and applied to tame generalized class groups. In the case of  $\mathcal{T}_K$ , we get  $v_p(\#\mathcal{T}_K) = \text{rk}_p(\mathcal{T}_K) \cdot \mathbb{M}_p(\mathcal{T}_K)$ , and we would have conjecturally, for any  $K \in \mathcal{K}_{\text{real}}$ :

$$\mathbb{M}_p(\mathcal{T}_K) \leq \mathcal{C}_p \cdot \frac{1}{\text{rk}_p(\mathcal{T}_K)} \cdot \frac{\log_{\infty}(\sqrt{D_K})}{\log_{\infty}(p)} \leq \mathcal{C}_p \cdot \frac{\log_{\infty}(\sqrt{D_K})}{\log_{\infty}(p)}.$$

But in [21, Theorems 0.1, 1.1, Proposition 2.2], this function  $\mathbb{M}_p$  is essentially used for class groups in particular infinite towers with tame restricted ramification for which some explicit upper bounds are obtained.

In this context, we can suggest the following direction of search:

**Proposition 7.6.** *Let  $K$  be a totally real number field and let  $L$  be the (totally real)  $p$ -Hilbert tower of  $K$ ; we assume that  $L/K$  is infinite. Let  $\mathcal{K}$  be a set of subfields  $K_n$  of  $L$ , with  $K_n \subset K_{n+1}$  and  $[K_n : K] = p^n$  for all  $n \geq 0$ .*

*Then  $C_p(K_n) = \frac{v_p(\#\mathcal{T}_{K_n}) \cdot \log_{\infty}(p)}{p^n \cdot \log_{\infty}(\sqrt{D_K})}$ , and Conjecture 7.1 is true for  $\mathcal{K}$  as soon as  $v_p(\#\mathcal{T}_{K_n})$  is “essentially” a linear function of the degree  $[K_n : K] = p^n$  as  $n \rightarrow \infty$  (i.e.,  $v_p(\#\mathcal{T}_{K_n}) = \alpha n + \beta p^n + \gamma$  for all  $n \gg 0$ ,  $\alpha, \beta \in \mathbb{N}$ ,  $\gamma \in \mathbb{Z}$ ).*

*Proof.* Since  $K_n/K$  is unramified,  $D_{K_n} = D_K^{[K_n:K]} \cdot N_{K/\mathbb{Q}}(D_{K_n/K}) = D_K^{p^n}$ . So, for all  $n \gg 0$ ,  $C_p(K_n) = \frac{(\alpha n + \beta p^n + \gamma) \cdot \log_{\infty}(p)}{p^n \cdot \log_{\infty}(\sqrt{D_K})}$ , equivalent to the constant  $\frac{\beta \cdot \log_{\infty}(p)}{\log_{\infty}(\sqrt{D_K})}$  at infinity. Whence the existence of  $\mathcal{C}_p$  over  $\mathcal{K}$ . If  $\beta = 0$ , then  $C_p(K_n) \rightarrow 0$ .  $\square$

The orders  $\#\mathcal{A}_{K_n}$  have this property of “linearity” and  $\text{rk}_p(\mathcal{A}_{K_n}) \rightarrow \infty$  under some conditions [19, Theorem A]; thus, it would remain the question of a similar linearity for the valuations, according to  $[K_n : K]$ , of the normalized regulators  $\mathcal{R}_{K_n}$ .

**7.2. Comparison “archimedean” versus “ $p$ -adic”.** The above considerations are, in some sense, a  $p$ -adic approach of some deep results (Brauer–Siegel–Tsfasman–Vladuț theorems [38, 44] and broad generalizations in [37], then [25] for quantitative bounds from the Brauer–Siegel theorem) on the behavior, in a tower  $L := \bigcup_{n \geq 0} K_n$  of finite extensions  $K_n/K$ , of the quotient  $BS_{K_n} := \frac{\log_{\infty}(h_{K_n} \cdot R_{K_n, \infty})}{\log_{\infty}(\sqrt{D_{K_n}})}$ .

Of course, in order to infer the  $p$ -adic case, our purpose is to deal, in the archimedean one, with any  $K \in \mathcal{K}_{\text{real}}$  or with families  $\mathcal{K}$  fulfilling some specific conditions (e.g.,

$[K : \mathbb{Q}] = d$ ,  $\frac{[K : \mathbb{Q}]}{\log_\infty(\sqrt{D_K})} \rightarrow 0$ ), which is possible thanks to [44, Theorem 1], at least for Galois fields. For any  $K \in \mathcal{K}_{\text{real}}$ , let  $BS_K := \frac{\log_\infty(h_K \cdot R_{K,\infty})}{\log_\infty(\sqrt{D_K})}$ .

We shall consider the following *normalized quotient*  $\widetilde{BS}_K = BS_K - 1$  using  $\#\mathcal{T}_{K,p_\infty}$  instead of  $h_K \cdot R_{K,\infty}$ :

$$(7.1) \quad \widetilde{BS}_K := \frac{\log_\infty\left(h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}}\right)}{\log_\infty(\sqrt{D_K})} = \frac{\log_\infty(\#\mathcal{T}_{K,p_\infty})}{\log_\infty(\sqrt{D_K})}, \quad K \in \mathcal{K} \text{ (from formula (2.2))},$$

and presume that this function is bounded over  $\mathcal{K}$ . When the degree is constant in the family, the classical Brauer–Siegel theorem applies since  $\frac{[K : \mathbb{Q}]}{\log_\infty(\sqrt{D_K})} \rightarrow 0$ .

The following program gives, for the family  $\mathcal{K}_{\text{real}}^{(2)}$  of real quadratic fields of discriminants  $D$ , consistent verifications for the original function  $BS$ :

```
{Max=0;Min=1;for(D=10^8,10^8+10^6,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)|| (e==0 & Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);P=x^2-D;
K=bnfinit(P,1);C8=component(K,8);h=component(component(C8,1),1);
reg=component(C8,2);BS=log(h*reg)/log(sqrt(D));if(BS<Min,Min=BS;
print(D," ",Min," ",Max));if(BS>Max,Max=BS;print(D," ",Min," ",Max)))}
```

$0.647 < BS < 1.155$  for  $D \in [10^5, 2 \cdot 10^5]$ ,  $0.734 < BS < 1.136$  for  $D \in [10^7, 10^7 + 10^5]$ ,  $0.7657 < BS < 1.1239$  for  $D \in [10^8, 10^8 + 10^5]$ , and  $0.75738 < BS < 1.12713$  for  $D \in [10^8, 10^8 + 10^6]$  (more than two days of computer), showing:  $\widetilde{BS}_K = O(1) < 1$ . Then  $0.773 < BS < 1.113$  for the family  $K = \mathbb{Q}(\sqrt{a^2 + 1})$ ,  $a \in [10^4, 2 \cdot 10^4]$ .

In the same way, the family  $\mathcal{K}_{\text{ab}}^{(3)}$  of cyclic cubic fields of conductors  $f$ , gives:  $0.6653 \leq BS \leq 1.1478$  for  $f \in [10^4, 10^6]$ ,  $0.7547 \leq BS \leq 1.1385$  for  $f \in [10^6, 2 \cdot 10^6]$ .

*Remarks 7.7.* (i) In the archimedean viewpoint, we have  $C_{p_\infty}(K) = \frac{\log_\infty(\#\mathcal{T}_{K,p_\infty})}{\log_\infty(\sqrt{D_K})}$ , giving, from the expression (7.1) of  $\widetilde{BS}_K$ ,  $\log_\infty(\#\mathcal{T}_{K,p_\infty}) = \widetilde{BS}_K \cdot \log_\infty(\sqrt{D_K})$ ; thus we obtain about the above calculations for the examples of fixed families  $\mathcal{K}$ :

$\log_\infty(\#\mathcal{T}_{K,p_\infty}) \leq O(1) \cdot \log_\infty(\sqrt{D_K})$  written  $\log_\infty(\#\mathcal{T}_{K,p_\infty}) \leq \mathcal{C}_{p_\infty} \cdot \log_\infty(\sqrt{D_K})$ , giving, in some sense, the inequality of the  $p$ -adic Conjecture 7.1 with the audacious convention for the *infinite place*  $p_\infty$  and  $\mathcal{T}_{K,p_\infty} = h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}}$ :

$$\log_\infty(p_\infty) = 1 \quad \& \quad v_{p_\infty}(\#\mathcal{T}_{K,p_\infty}) = \log_\infty(\#\mathcal{T}_{K,p_\infty}).$$

in which case, the constant  $\mathcal{C}_{p_\infty}$  is the maximal value reached by  $\widetilde{BS}_K = BS_K - 1$  over the given family  $\mathcal{K}$ .

(ii) One may wonder about the differences of behaviour and properties between  $C_{p_\infty}(K)$  and  $C_p(K)$ , as  $D_K \rightarrow \infty$ , because of the choosen normalizations and the role of the discriminant in the definitions. The only change could be to define:

$$\mathcal{T}'_{K,p_\infty} = h_K \cdot R_{K,\infty} \quad \text{and} \quad \mathcal{C}'_{p_\infty}(K) = \frac{\log_\infty(\mathcal{T}'_{K,p_\infty})}{\log_\infty(\sqrt{D_K})} = C_{p_\infty}(K) + 1 = BS_K,$$

by reference to Brauer–Siegel context, but in that case, we should have (from (2.3))  $\mathcal{T}'_{K,p_\infty} = \tilde{\kappa}_{K,p_\infty} \cdot \sqrt{D_K}$ , with  $\tilde{\kappa}_{K,p_\infty} = \frac{1}{2^{d-1}} \cdot \kappa_{K,p_\infty}$ , which cannot be a suitable normalization of the  $\zeta$ -function and its residue; indeed, on the interval  $[2, 10^6]$  of discriminants of real quadratic fields, the local maxima of  $(\kappa_{K,p_\infty}, \kappa_{K,p_\infty} \cdot \sqrt{D_K})$  increase excessively from (0.215204, 0.481211) to (2.732814, 2705.305810).

But the comparison must take into account the difference of nature of the sets of values of the functions  $C_{p_\infty}$  and  $C_p$ :

The first one takes its values in an explicitly bounded interval of  $\mathbb{R}$ , containing 0, given by the Brauer–Siegel–Tsfasman–Vladuț–Zykin results:

$$S_{p_\infty} = \left\{ v_{p_\infty}(\#\mathcal{T}_{K,\infty}) \cdot \frac{\log_\infty(p_\infty)}{\log_\infty(\sqrt{D_K})}, K \in \mathcal{K} \right\} \subseteq \mathbb{R} \cdot \frac{\log_\infty(p_\infty)}{\log_\infty(\sqrt{D_K})},$$

while the second one takes its values in a discrete set of the form:

$$S_p = \left\{ v_p(\#\mathcal{T}_{K,p}) \cdot \frac{\log_\infty(p)}{\log_\infty(\sqrt{D_K})}, K \in \mathcal{K} \right\} \subseteq \mathbb{N} \cdot \frac{\log_\infty(p)}{\log_\infty(\sqrt{D_K})},$$

so that  $v_{p_\infty}(\#\mathcal{T}_{K,\infty}) = \log_\infty(\#\mathcal{T}_{K,\infty})$  is never 0 (except if  $K = \mathbb{Q}$ ) while  $v_p(\#\mathcal{T}_{K,p})$  is equal to 0 for infinitely many fields  $K$ , probably with a positive density which increases significantly as  $p \rightarrow \infty$ ; but, symmetrically, we have seen that the integers  $v_p(\#\mathcal{T}_{K,p})$  take infinitely many strictly positive values for huge discriminants.

To compare the two situations one must probably compute some “integrals” when  $D_K$  varies in some intervals. Whatever the choice of the family  $\mathcal{K}$ , the sets of real coefficients  $\frac{\log_\infty(p_v)}{\log_\infty(\sqrt{D_K})}$  are homothetic discrete subsets of  $\mathbb{R}_+$  as  $v$  varies, so that the comparison is based on the coefficients  $v_{p_\infty}(\#\mathcal{T}_{K,\infty})$  &  $v_p(\#\mathcal{T}_{K,p})$ , respectively.

The following programs compute the means of  $C_v(K)$  on intervals of discriminants  $D_K$ ,  $K \in \mathcal{K}_{\text{real}}^{(2)}$ , for  $p_\infty$  and  $p \geq 2$ , but many other means may be interesting:

```
{Sinfy=0.0;N=0;for(D=10^5,2*10^5,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)|| (e==0 & Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);P=x^2-D;
N=N+1;K=bnfinit(P,1);C8=component(K,8);h=component(component(C8,1),1);
reg=component(C8,2);Cp=log(h*reg)/log(sqrt(D))-1;Sinfy=Sinfy+Cp);print(Sinfy/N)}

{p=3;n=18;Sp=0.0;N=0;for(D=10^5,2*10^5,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3)|| (e==0 & Mod(M,4)!=1)|| (e==2 & Mod(M,4)==1),next);P=x^2-D;
N=N+1;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(sqrt(D));Sp=Sp+Cp);print(Sp/N)}
```

$v = p_\infty$  gives  $M_\infty = -0.08072025$  for  $D \in [5, 10^6]$ ,  $M_\infty = -0.05566364$  for  $D \in [10^8, 10^8 + 10^5]$   
 $M_\infty = -0.06817971$  for  $D \in [5, 10^7]$ ,  $M_\infty = -0.05562784$  for  $D \in [10^8, 10^8 + 10^6]$   
 $M_\infty = -0.04947600$  for  $D \in [10^9, 10^9 + 10^4]$   
 $p = 3$  ( $n = 18$ ) gives  $M_3 = 0.12656432$  for  $D \in [5, 10^6]$ ,  $M_3 = 0.10463765$  for  $D \in [10^7, 10^7 + 10^5]$   
 $p = 5$  ( $n = 12$ ) gives  $M_5 = 0.07257764$  for  $D \in [5, 10^6]$ ,  $M_5 = 0.05897703$  for  $D \in [10^7, 10^7 + 10^5]$   
 $p = 7$  ( $n = 10$ ) gives  $M_7 = 0.05647554$  for  $D \in [5, 10^6]$ ,  $M_7 = 0.04649732$  for  $D \in [10^7, 10^7 + 10^5]$   
 $p = 29$  ( $n = 6$ ) gives  $M_{29} = 0.01901355$  for  $D \in [5, 10^6]$ ,  $M_{29} = 0.01572121$  for  $D \in [10^7, 10^7 + 10^5]$

giving obvious heuristics about the behaviour of each mean.

## 8. CONCLUSIONS

The analysis of the archimedean case, depending on the properties of the complex  $\zeta$ -function of  $K$ , is sufficiently significant to hope the relevance of the  $p$ -adic one for which we give some observations, despite the lack of proofs:

(a) In the  $p$ -adic Conjecture 7.1, the most important term is  $v_p(\#\mathcal{R}_{K,p})$ , the valuation of the normalized  $p$ -adic regulator, the contribution of  $v_p(\#\mathcal{A}_{K,p})$  being probably negligible compared to  $v_p(\#\mathcal{R}_{K,p})$  as shown, among other, by classical heuristics [3, 4], and reinforced by the recent conjectures cited in the § 4.2.3 (ii).

Furthermore, for  $K$  fixed,  $v_p(\#\mathcal{A}_{K,p}) \geq 1$  for finitely many primes  $p$ , but the case of  $v_p(\#\mathcal{R}_{K,p})$  is an out of reach conjecture [10, Conjecture 8.11].

(b) The family of Subsection 4.3 shows that  $p$ -adic regulators may tend  $p$ -adically to 0, even in simplest cases, and it should be of great interest to find other such critical sub-families of units, depending on arbitrary large  $p$ -powers, to precise the relation between  $v_p(\#\mathcal{R}_{K,p})$  and  $\log_\infty(\sqrt{D_K})$ ,  $K \in \mathcal{K}_{\text{real}}^{(d)}$ , for degrees  $d > 2$ .

After the writing of this paper we have found the reference [43] about the family of cyclic cubic fields  $K$  defined by  $P = x^3 - (N^3 - 2N^2 + 3N - 3)x^2 - N^2x - 1$  for any  $N \in \mathbb{Z}$ ,  $N \neq 1$ , near 1 in  $\mathbb{Z}_3$ ; this paper of Washington deals with  $p = 3$ , to obtain 3-adic  $L$ -functions with zeros arbitrarily close to 1, but we observed that any  $p \geq 2$  gives interesting non- $p$ -rational fields with large  $v_p(\#\mathcal{T}_{K,p})$  and  $C_p(K) < 1$  for all. The reader may play with the following program (choose  $p \geq 2$ , the intervals defining  $N = 1 + ap^k$ , a lower bound  $vp$  for  $vptor$  and  $n$  large enough):

```
{p=2;bk=2;Bk=10;ba=1;Ba=12;vp=10;n=36;print("p=",p);for(k=bk,Bk,for(a=ba,Ba,
if(Mod(a,p)==0,next);N=1+a*p^k;P=x^3-(N^3-2*N^2+3*N-3)*x^2-N^2*x-1;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
if(vptor>vp,D=component(component(K,7),3);Cp=vptor*log(p)/log(sqrt(D));
print("a=",a," k=",k," D=",D," vptor=",vptor," Cp=",Cp);print("P=",P))}}
giving for instance the interesting cases with  $a = 1$  ( $p = 2, 3, 5$ ):
p=2 k=9 D=17213619969^2 vptor=28 Cp=0.8234 P=x^3-134480895*x^2-263169*x-1
p=3 k=9 D=150102262056706213^2 vptor=23 Cp=0.6388
P=x^3-7625984944841*x^2-387459856*x-1
p=5 k=5 D=95397978509379^2 vptor=10 Cp=0.4999 P=x^3-30527349999*x^2-9771876*x-1
```

(c) Consider, for any  $p \geq 2$  and any  $K \in \mathcal{K}_{\text{real}}$ :

$$C_p(K) := \frac{v_p(\#\mathcal{T}_{K,p}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}, \quad C_p := \sup_K (C_p(K)), \quad C_K := \sup_p (C_p(K)).$$

(i) The existence of  $C_K < \infty$ , for a given  $K$ , only says that the conjecture proposed in [10, Conjecture 8.11], claiming that any number field is  $p$ -rational for all  $p \gg 0$ , is true for the field  $K$ ; for this field,  $\limsup_p (C_p(K)) = 0$ .

(ii) If  $C_p$  does exist for a given  $p$ , we have an universal  $p$ -adic analog of Brauer–Siegel theorem (Conjecture 7.1). The existence of  $C_p < \infty$  may be true taking instead  $\sup_{K \in \mathcal{K}} (C_p(K))$ , for particular families  $\mathcal{K}$  (e.g., extensions of fixed degree or subfields of some infinite towers as in [21, 23, 38, 44]); but we must mention that for the invariants  $\mathcal{T}_{K,p}$ , the transfer map  $\mathcal{T}_{K,p} \rightarrow \mathcal{T}_{L,p}$  is injective in any extension  $L/K$  in which Leopoldt’s conjecture is assumed [9, Theorem IV.2.1], which leads to a major difference from the case of  $p$ -class groups.

(iii) Furthermore, it seems that  $\limsup_{K \in \mathcal{K}} (C_p(K))$  may be  $\leq 1$  for any  $p$ ; then  $\limsup_p (C_p(K)) = \infty$  or 0, for any  $K$ , depends on [10, Conjecture 8.11]. But computations for very large discriminants (of a great lot of quadratic fields for instance) is out of reach (see the Remarks of the § 4.2.3).

(d) When  $p$  and  $D_K$  are not independent, this yields some interesting potential results as the following one: let  $\mathcal{K}_{\text{real}}(p^e)$  be the set of fields  $K \in \mathcal{K}_{\text{real}}$  of discriminant  $D_K = p^e$ , for any fixed  $p$ -power  $p^e$ ,  $e \geq 1$ ; then, as soon as  $C_p(K) < \frac{2}{e}$  for all  $K$  in some subfamily  $\mathcal{K}(p^e)$  of  $\mathcal{K}_{\text{real}}(p^e)$ ,  $K$  is  $p$ -rational since then  $C_p(K) = \frac{2}{e} \cdot v_p(\#\mathcal{T}_K)$ . For instance, if we were able to prove that  $C_p(K) < 2$  for all  $K \in \mathcal{K}_{\text{real}}^{(2)}(p)$  (quadratic fields  $K = \mathbb{Q}(\sqrt{p})$ ,  $p \equiv 1 \pmod{4}$ ), this would imply the conjecture of Ankeny–Artin–Chowla (see [39, § 5.6]), affirming that  $\varepsilon_K =: u + v\sqrt{p}$  is such that  $v \neq 0$

(mod  $p$ ), which is equivalent, since  $\mathcal{C}_K = 1$ , to  $\mathcal{R}_K \sim 1$  (indeed,  $\varepsilon_K^p \equiv u \equiv \varepsilon_K - v\sqrt{p}$  (mod  $p$ ), whence  $\varepsilon_K^{p-1} \equiv 1 + \varepsilon_K^{-1}v\sqrt{p}$  (mod  $p$ )).

The cyclic quartic fields of conductor  $p$  (i.e.,  $K \in \mathcal{K}_{\text{real}}^{(4)}(p^3)$ ) give no solution in the selected interval, although  $C_p(K) = \frac{2}{3}v_p(\#\mathcal{T}_K)$ . The case of  $K \in \mathcal{K}_{\text{real}}^{(3)}(p^2)$  (cyclic cubic fields of conductor  $p$ ) is interesting since, in this case,  $C_p(K) = v_p(\#\mathcal{T}_K)$ , for which  $v_p(\#\mathcal{T}_K) = 1$  is more credible if we consider that for instance  $C_p(K) < 2$  over  $\mathcal{K}_{\text{real}}^{(3)}(p^2)$ ; indeed we have found only two examples up to  $p \leq 10^8$ :

```
p=5479      vptor=Cp=1      P=x^3 + x^2 - 1826x + 13799
p=15646243  vptor=Cp=1      P=x^3 + x^2 - 5215414x - 311765879
```

Let's give few examples in degrees  $d = 5, 7, 9$  using *polsubcyclo*( $p, d$ ) (cyclic fields of conductor  $p$ ) since  $C_p(K) = \frac{2}{d-1}v_p(\#\mathcal{T}_K)$  (for all,  $v_p(\#\mathcal{T}_K) = 1$ ,  $v_p(\#\mathcal{C}_K) = 0$ ):

```
p=130811    Cp=0.5000    P=x^5+x^4-52324*x^3-429060*x^2+575263872*x+3600157696
p=421       Cp=0.3333    P=x^7+x^6-180*x^5-103*x^4+6180*x^3+11596*x^2-25209*x-49213
p=44563     Cp=0.3333    P=x^7+x^6-19098*x^5-87307*x^4+73981206*x^3-1061790574*x^2
                                -13438850605*x-28465212577
p=37        Cp=0.2500    P=x^9+x^8-16*x^7-11*x^6+66*x^5+32*x^4-73*x^3-7*x^2+7*x+1
p=13411     Cp=0.2500    P=x^9+x^8-5960*x^7+117167*x^6+5761671*x^5-114461957*x^4
                                -2103829198*x^3+33776243778*x^2+244391306047*x-3339737282887
```

In other words, a more general “Ankeny–Artin–Chowla Conjecture” should be that the set of non- $p$ -rational  $K \in \mathcal{K}_{\text{real}}^{(d)}(p^e)$  (or any suitable subfamily) is finite. Thus the existence (if so), and then the order of magnitude of  $\mathcal{C}_p$ , would govern many obstructions and/or finiteness theorems in number theory.

(e) On another hand, the difficult Greenberg’s conjecture [15], on the triviality of the Iwasawa invariants  $\lambda$ ,  $\mu$  for the  $p$ -class groups in  $K^c$ , in the totally real case, goes in the sense of rarity of large  $p$ -class groups as we have mentioned at the § 4.2.3 (ii), and this conjecture also depends on Fermat’s quotients of algebraic numbers ([11, § 7.7], [14, § 4.2]) or of a similar logarithmic framework as in [24]. In the same way, some other conjectures of Greenberg [16] depend, in a crucial manner, of the existence of  $p$ -rational fields with given Galois groups.

(f) But all this is far to be proved because of a terrible lack of knowledge of  $p$ -Fermat quotients of algebraic numbers, a notion which gives a *weaker information* than the  $p$ -adic logarithms or regulators, but which governs many deep arithmetical problems, even assuming the Leopoldt conjecture which appears as a rough step in the study of  $\text{Gal}(H_K^{\text{pf}}/K)$ ; indeed, if Leopoldt’s conjecture is not fulfilled in a given field  $K$ , there exists a sequence  $\varepsilon_i \in E_K$ ,  $\varepsilon_i \notin E_K^p$ , such that  $\delta_p(\varepsilon_i) \rightarrow \infty$  with  $i$ , which shows the extreme uncertainty about the  $\mathcal{T}_{K,p}$  groups.

(g) Recall to finish that  $\mathcal{T}_{K,p}$  is the dual of  $H^2(G_p(K), \mathbb{Z}_p)$  ([28, Chapitre 1], then [9, Appendix, Theorem 2.2]), where  $G_p(K)$  is the Galois group of the maximal  $p$ -ramified pro- $p$ -extension of  $K$  (for which  $G_p(K)^{\text{ab}} \simeq \mathbb{Z}_p \times \mathcal{T}_{K,p}$  in the totally real case, under Leopoldt’s conjecture), and can be considered as the first of the still mysterious non positive twists  $H^2(G_p(K), \mathbb{Z}_p(i))$  of the motivic cohomology (whereas the positive twists can be dealt with using K-theory thanks to the Quillen–Lichtenbaum conjecture, now a theorem of Voevodsky–Rost and al.).

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