



HAL
open science

Heuristics in direction of a p-adic Brauer–Siegel theorem

Georges Gras

► **To cite this version:**

| Georges Gras. Heuristics in direction of a p-adic Brauer–Siegel theorem. 2018. hal-01682891v2

HAL Id: hal-01682891

<https://hal.science/hal-01682891v2>

Preprint submitted on 23 Jan 2018 (v2), last revised 13 Mar 2018 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HEURISTICS IN DIRECTION OF A p -ADIC BRAUER–SIEGEL THEOREM

GEORGES GRAS

ABSTRACT. Let p be a fixed prime number. Let K be a totally real number field of discriminant D_K and let \mathcal{T}_K be the torsion group of the Galois group of the maximal abelian p -ramified pro- p -extension of K (under Leopoldt’s conjecture). We conjecture the existence of a constant $C_p > 0$ such that $\log(\#\mathcal{T}_K) \leq C_p \cdot \log(\sqrt{D_K})$ when K varies in some specified families (e.g., fields of fixed degree). In some sense, we try to suggest a p -adic analogue, of the classical Brauer–Siegel Theorem, wearing on the valuation of the residue, at $s = 1$, of the p -adic ζ -function of K . We shall use a different definition that of Washington, given in the 1980’s, to approach this question via the arithmetical study of \mathcal{T}_K since p -adic analysis seems to fail contrary to the classical framework. We give extensive numerical verifications for quadratic and cubic fields (cyclic or not) and publish the PARI/GP programs. Such a conjecture (if exact) reinforces our conjecture that any number field is p -rational (i.e., $\mathcal{T}_K = 1$) for all $p \gg 0$.

1. ABELIAN p -RAMIFICATION – MAIN DEFINITIONS AND NOTATIONS

Let K be a totally real number field of degree d , and let $p \geq 2$ be a prime number fulfilling the Leopoldt conjecture in K . We denote by \mathcal{C}_K the p -class group of K in the ordinary sense and by E_K its group of p -principal global units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$.

Let us recall from [9, 11, 12, 14] the diagram of the so called *abelian p -ramification theory*, in which $K^c = K\mathbb{Q}^c$ is the cyclotomic \mathbb{Z}_p -extension of K (as compositum with that of \mathbb{Q}), H_K the p -Hilbert class field and H_K^{pr} the maximal abelian p -ramified (i.e., unramified outside p) pro- p -extension of K .

Let $U_K := \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}^1$ be the \mathbb{Z}_p -module (of \mathbb{Z}_p -rank d) of p -principal local units of K , where each $U_{\mathfrak{p}}^1 := \{u \in K_{\mathfrak{p}}^{\times}, u \equiv 1 \pmod{\bar{\mathfrak{p}}}\}$ is the group of $\bar{\mathfrak{p}}$ -principal units of the completion $K_{\mathfrak{p}}$ of K at $\mathfrak{p} | p$, where $\bar{\mathfrak{p}}$ is the maximal ideal of the ring of integers of $K_{\mathfrak{p}}$.

For any field k , let μ_k be the group of roots of unity of k of p -power order. Then put $W_K := \text{tor}_{\mathbb{Z}_p}(U_K) = \bigoplus_{\mathfrak{p}|p} \mu_{K_{\mathfrak{p}}}$ and $\mathcal{W}_K := W_K / \mu_K$.

Let \bar{E}_K be the closure in U_K of the diagonal image of E_K ; by class field theory this gives in the diagram $\text{Gal}(H_K^{\text{pr}}/H_K) \simeq U_K/\bar{E}_K$; then let \mathcal{C}_K^c be the subgroup of \mathcal{C}_K corresponding to the subgroup $\text{Gal}(H_K/K^c \cap H_K)$.

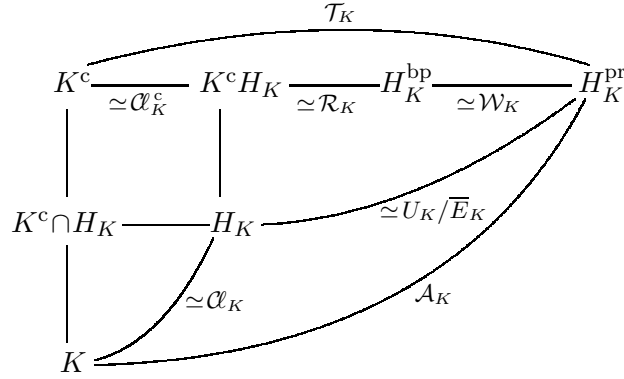
Put ([9, Chapter III, § 2 (a) & Theorem 2.5] with the set S of infinite places (ordinary sense) and the set T of p -places):

$$\mathcal{T}_{K,p} = \mathcal{T}_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{pr}}/K)) = \text{Gal}(H_K^{\text{pr}}/K^c).$$

We have (because of Leopoldt’s conjecture) the following exact sequence defining $\mathcal{R}_{K,p} = \mathcal{R}_K$ (see [9, Lemma III.4.2.4 with Corollary III.3.6.3] or [12, Lemma 3.1 & § 5]):

$$1 \rightarrow \mathcal{W}_K \rightarrow \text{tor}_{\mathbb{Z}_p}(U_K/\bar{E}_K) \xrightarrow{\log_p} \text{tor}_{\mathbb{Z}_p}(\log_p(U_K)/\log_p(\bar{E}_K)) =: \mathcal{R}_K \rightarrow 0.$$

The group \mathcal{R}_K (or its order) is called the *normalized p -adic regulator of K* and makes sense for any number field:



The field H_K^{bp} , fixed by \mathcal{W}_K , is the Bertrandias–Payan field, i.e., the compositum of the p -cyclic extensions of K embeddable in p -cyclic extensions of arbitrary large degree.

2. v -ADIC ANALYTIC PROSPECTS

For a random totally real number field K and a fixed prime p , $\#\mathcal{T}_K = \#\mathcal{O}_K^c \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$, may be equal to 1 (defining « p -rational fields») or not, and it will be interesting to know if the p -adic valuation of $\#\mathcal{T}_K$ can be bounded according, for instance, to the discriminant D_K of K . If so, this would be interpreted as a p -adic version of the archimedean Brauer–Siegel theorem, which is currently pure speculation, but we intend to experiment, algebraically, this context since p -adic analysis does not seem to succeed as explain by Washington in [37]:

A Brauer–Siegel theorem using p -adic L -functions fails;

in the same way, we have similar comments by Ivanov in [22, Section 1]:

The p -adic analogue of Brauer–Siegel and hence also of Tsfasman–Vladuț fails.

But this requires some explanation:

2.1. The Siegel zeroes. In fact, there is a confusion about the definitions and the role of the discriminant in a Brauer–Siegel frame. Let K be a totally real number field, let h_K be its class number, $R_{K,p}$ its classical p -adic regulator, D_K its discriminant; in [37, § 3], Washington considers a sequence of number fields K and the limit of:

$$\frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})}, \text{ under the condition } \frac{[K : \mathbb{Q}]}{v_p(D_K)} \rightarrow 0,$$

where v_p denotes the p -adic valuation; thus this condition implies that p must be «highly ramified» in the fields K if $[K : \mathbb{Q}] \rightarrow \infty$, which eliminates for instance families of fields of constant degree. So, with Washington’s definition, K belong in general to some towers of number fields (e.g., the cyclotomic one). Washington shows examples and counterexamples of the p -adic Brauer–Siegel property $\frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})} \rightarrow 1$ ([37, Proposition 2 & Theorem 2]). In its Theorem 3, he uses the formula of Coates [5, p. 364], which implies $\liminf_K \left(\frac{v_p(h_K \cdot R_{K,p})}{v_p(\sqrt{D_K})} \right) \geq 1$ as $\frac{[K : \mathbb{Q}]}{v_p(D_K)} \rightarrow 0$.

We shall consider instead $\frac{v_p\left(h_K \cdot \frac{R_{K,p}}{\sqrt{D_K}}\right) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$ or more precisely:

$$C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})} = \frac{\log_\infty(\#\mathcal{T}_K)}{\log_\infty(\sqrt{D_K})}, \text{ for any } K,$$

where \log_∞ is the usual complex logarithm.

However, there are some connections between the two definitions since the quantity $v_p(h_K \cdot R_{K,p})$ appears in each of them; only the order of magnitude differs for the analyse of sequences of fields. It is therefore not surprising to find, for instance in [37, 38, 33], some allusions to the group \mathcal{T}_K .

Let's finish these comments with a quote in Washington's paper illustrating the interesting fact that a great $v_p(\#\mathcal{T}_K)$ is related to the existence of zeroes, of the p -adic L -functions, close to 1 (see [38, 33] for complements about these zeroes and for some numerical data):

In the proof of the classical Brauer–Siegel theorem, one needs the fact that there is at most one Siegel zero, that is, a zero close to 1. The fact that the Brauer–Siegel theorem fails p -adically could be taken as further evidence for the abundance of p -adic zeroes near 1.

(...)

Finally, we remark that the possible existence of p -adic Siegel zeroes and the failure of results such as the p -adic Brauer–Siegel theorem indicate that it could be difficult, if not impossible, to do analytic number theory with p -adic L -functions. For example, I do not know how to obtain estimates on $\pi(x)$, the number of primes less than or equal to x , using the fact that the p -adic zeta function has a pole at 1.

Remark 2.1. One may explain what appens as follows, for simplicity in the case of a real quadratic field K of character χ_K .

Roughly speaking, $v_p(L_p(1, \chi_K))$ is closely related to $v_p(\#\mathcal{T}_K)$ and $v_p(L_p(0, \chi_K))$ is closely related to $v_p(B_1(\omega^{-1} \chi_K))$ (where ω is the Teichmüller character and $B_1(\omega^{-1} \chi_K)$ the generalized Bernoulli number of character $\omega^{-1} \chi_K$) which is closely related to the order a suitable component of the p -class group of the “mirror field K^* ” (e.g., for $p = 3$ and $K = \mathbb{Q}(\sqrt{m})$, $K^* = \mathbb{Q}(\sqrt{-3m})$); but since $\omega^{-1} \chi_K$ is odd, no unit intervenes and $v_p(L_p(0, \chi_K))$ is usually “small” compared to $v_p(\#\mathcal{T}_K)$ assumed to be “very large”.

So there exist in general “Siegel zeroes” of $L_p(s, \chi_K)$ very close to 1, which is an obstruction to a Brauer–Siegel strategy (see numerical illustrations for $p = 2, 3$ in [38, 39]).

So we will adopt another point of view.

We still suppose that K is a totally real number field and $p \geq 2$ any fixed prime number. As we have recalled it, for (abelian) fields K , $\#\mathcal{T}_K$ is in close relationship with the Kubota–Leopoldt p -adic L -functions (at $s = 1$) of even Dirichlet characters (Kubota–Leopoldt, Barsky, Amice–Fresnel,...), or more generally with the residue at $s = 1$ of the p -adic ζ -function of K built or study by many authors (Amice–Fresnel, Coates, Shintani, Barsky, Serre, Cassou–Noguès, Deligne–Ribet, Katz, Colmez,...).

Conversely, there is no *algebraic* invariant (like a Galois group) interpreting the residue of the complex ζ -function, but we have in this (archimedean) case numerous inequalities.

So, we shall compare the complex and p -adic cases to try to unify the set of all the points of view. For this, we define normalizations of the ζ -functions of a totally real number field (from [5, 6], and [12] for the regulators).

2.2. Definitions and normalizations. Let K be a totally real number field of degree d and let:

$$\mathcal{P} := \{p_\infty, 2, 3, \dots, p, \dots\}$$

be the set of all places of \mathbb{Q} (we also use the symbol ∞ for complex functions, like L -functions, ζ -functions, log-functions, in the same logic as for p -adic functions; for instance, $R_{K,\infty}$ and $R_{K,p}$ shall be the usual regulators built with \log_∞ and \log_p , respectively). We shall use indices $(\)_{K,v}$ for any place $v \in \mathcal{P}$; when the context is clear, we omit the place v (p -adic in most cases).

2.2.1. *v*-CYCLOTOMIC EXTENSIONS AND *v*-CONDUCTORS. The *p*-cyclotomic \mathbb{Z}_p -extension is denoted $\mathbb{Q}^{c,p}$ and we introduce the p_∞ -cyclotomic extension $\mathbb{Q}^{c,p_\infty} =: \mathbb{Q}$. We put $\mathbb{Q}^{c,v} =: \mathbb{Q}^c$ for all $v \in \mathcal{P}$ if there is no ambiguity.

We attribute to the field \mathbb{Q} the « conductor » $\mathfrak{f}_{\mathbb{Q}} := p$ (resp. 4, 2) if $v = p$ (resp. 2, p_∞). We shall put \sim for equalities up to a *p*-adic unit.

2.2.2. NORMALIZED ζ -FUNCTIONS AT p_∞ . We define at the infinite place p_∞ :

$$(2.1) \quad \tilde{\zeta}_{K,p_\infty} := \frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \cdot \zeta_{K,p_\infty} = \frac{1}{2^{d-1}} \cdot \zeta_{K,p_\infty}.$$

$$(2.2) \quad \#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}},^1$$

where h_K is the class number (ordinary sense), $R_{K,\infty}$ the classical regulator and D_K the discriminant of K (see [9, Remark III.2.6.5 (ii)] for justifications about the factor $\frac{1}{2^d}$).

Denote by $\tilde{\kappa}_{K,p_\infty}$ the residue at $s = 1$ of $\tilde{\zeta}_{K,p_\infty}$. From classical formulas of the so-called complex « analytic formula of the class number » of K (see, e.g., [36, Chap. 4]), we get:

$$(2.3) \quad \tilde{\kappa}_{K,p_\infty} = h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} = \#\mathcal{T}_{K,p_\infty}.$$

2.2.3. NORMALIZED ζ -FUNCTIONS AT $v = p$. We define at a finite place p :

$$(2.4) \quad \tilde{\zeta}_{K,p} := \frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \cdot \zeta_{K,p},$$

where $\mathfrak{f}_{K \cap \mathbb{Q}^c}$ is the conductor of $K \cap \mathbb{Q}^c$ ($\mathfrak{f}_{K \cap \mathbb{Q}^c} \sim 2p \cdot [K \cap \mathbb{Q}^c : \mathbb{Q}]$ if $K \cap \mathbb{Q}^c \neq \mathbb{Q}$ and $\mathfrak{f}_{K \cap \mathbb{Q}^c} \sim 2p$ if $K \cap \mathbb{Q}^c = \mathbb{Q}$), and for which we have the *p*-adic residue [5, 6, 31]:

$$(2.5) \quad \tilde{\kappa}_{K,p} = \frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \cdot \kappa_{K,p} \sim \#\mathcal{T}_{K,p}.$$

So, the residues of the normalized ζ -functions of K are such that:

$$\tilde{\kappa}_{K,v} \sim \#\mathcal{T}_{K,v}, \text{ for all } v \in \mathcal{P},$$

which is the order of an arithmetical invariant for finite places $v = p$ and the measure of a real volume for p_∞ .

2.3. **Abelian complex *L*-functions – Upper bounds.** In the abelian case, we have:

$$(2.6) \quad \#\mathcal{T}_{K,p_\infty} = h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} = \prod_{\chi \neq 1} \frac{1}{2} L_{p_\infty}(1, \chi),$$

where χ goes through all of the corresponding Dirichlet characters of K with conductor f_χ , and where L_{p_∞} denotes the complex « *L*-function ». If $K = \mathbb{Q}(\sqrt{m})$, of fundamental unit ε_K and quadratic character χ_K , one gets:

$$\#\mathcal{T}_{K,p_\infty} = h_K \cdot \frac{\log_\infty(\varepsilon_K)}{\sqrt{D_K}} = \frac{1}{2} \cdot L_{p_\infty}(1, \chi_K).$$

For each $L_{p_\infty}(1, \chi)$ one has many upper bounds which are improvements of the inequality:

$$\frac{1}{2} \cdot L_{p_\infty}(1, \chi) \leq (1 + o(1)) \cdot \log_\infty(\sqrt{f_\chi}).$$

In [29, Corollaire 1] one has, for even primitive characters:

$$\frac{1}{2} \cdot L_{p_\infty}(1, \chi) \leq \frac{1}{2} \cdot \log_\infty(\sqrt{f_\chi}),$$

¹ The factor $\frac{R_{K,\infty}}{\sqrt{D_K}}$ may be interpreted as the normalized regulator $\mathcal{R}_{K,p_\infty} = \mathcal{R}_K$ attached to $v = p_\infty$; from [1], it is defined without ambiguity. The invariant \mathcal{T}_{K,p_∞} is related to the Arakelov class group of K (see [30] and its bibliography), which gives a best interpretation.

giving from the previous definition (2.2):

$$(2.7) \quad \#\mathcal{T}_{K,p_\infty} \leq C_{p_\infty} \cdot \sqrt{D_K},$$

with an explicit constant C_{p_∞} if K runs through the set of real abelian fields of given degree d .

Remark 2.2. We do not need any sophisticated upper bound (only the existence of C_{p_∞}), but one may refer to [18, 24, 25, 27, 29] for other inequalities; for instance, one gets, in the real abelian case, with our notations:

$$h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} \leq \left(\frac{1}{2} \frac{\log_\infty(\sqrt{D_K})}{d-1} \right)^{d-1} \quad \& \quad R_{K,\infty} \leq \sqrt{D_K} \cdot \left(\frac{1}{2} \frac{\log_\infty(\sqrt{D_K})}{d-1} \right)^{d-1},$$

thus in the cases $d = 2$ and $d = 3$ for K :

$$\#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{\log_\infty(\varepsilon_K)}{\sqrt{D_K}} \leq \frac{1}{2} \log_\infty(\sqrt{D_K}), \quad \#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} \leq \left(\frac{1}{4} \log_\infty(\sqrt{D_K}) \right)^2,$$

respectively, and, for real quadratic and cyclic cubic fields:

$$(2.8) \quad h_K \leq \frac{1}{2} \cdot \sqrt{D_K}, \quad h_K \leq \frac{2}{3} \cdot \sqrt{D_K}, \quad \text{respectively.}$$

2.4. Abelian L_p -functions. The Kubota–Leopoldt p -adic L -functions gives rise to the analytic formula [1, § 2.1 & Théorème 6, § 2.3]:

$$\#\mathcal{C}_K \cdot \frac{R_{K,p}}{\sqrt{D_K}} \sim \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) \cdot \prod_{\chi \neq 1} \left(1 - \frac{\chi(p)}{p} \right)^{-1}.$$

The « p -adic class number formula» for real abelian fields uses the first known formula:

$$\#\mathcal{T}_{K,p} \sim [K \cap \mathbb{Q}^c : \mathbb{Q}] \cdot \frac{p}{\prod_{\mathfrak{p}|p} \mathfrak{N}\mathfrak{p}} \cdot \#\mathcal{C}_K \cdot \frac{R_{K,p}}{\sqrt{D_K}} \quad (\text{see [5]}).$$

Thus in the abelian case, since $\prod_{\chi} \left(1 - \frac{\chi(p)}{p} \right)^{-1} = \prod_{\mathfrak{p}|p} (1 - \mathfrak{N}\mathfrak{p}^{-1})^{-1} \sim \prod_{\mathfrak{p}|p} \mathfrak{N}\mathfrak{p}$:

$$(2.9) \quad \#\mathcal{T}_{K,p} \sim \frac{p}{\prod_{\mathfrak{p}|p} \mathfrak{N}\mathfrak{p}} \cdot \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) \cdot \prod_{\chi \neq 1} \left(1 - \frac{\chi(p)}{p} \right)^{-1} \sim \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) = \tilde{\kappa}_{K,p}.$$

But no upper bound of the p -adic valuation of this residue is known. So we must, on the contrary, try to study directly $\#\mathcal{T}_{K,p}$ with arithmetic tools.

2.5. Arithmetical study of $\tilde{\kappa}_{K,p}$. To study this residue, we start from the general relation (2.5) for totally real number fields. In $\#\mathcal{T}_K = \#\mathcal{C}_K^c \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$, the computation of $\#\mathcal{W}_K$ is obvious, that of:

$$(2.10) \quad \#\mathcal{R}_K = \#\text{tor}_{\mathbb{Z}_p}(\log_p(U_K)/\log_p(\overline{E}_K)) \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_p : \log(\mathfrak{N}_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}|p} \mathfrak{N}\mathfrak{p}} \cdot \frac{R_{K,p}}{\sqrt{D_K}}$$

(see [12, Proposition 5.2]), is unpredictable, and more complicate if p ramifies in K . In the non-ramified case, it is given by the classical determinant provided that one replaces \log_p by the «normalized logarithm» $\frac{1}{p} \log_p$; then $\#\mathcal{C}_K^c = \frac{\#\mathcal{C}_K}{[H_K \cap K^c : K]}$, where the degree $[H_K \cap K^c : K]$ is equal, up to a p -adic unit, to:

$$\frac{e_p}{\langle \langle -1 \rangle : \langle -1 \rangle \cap \mathfrak{N}_{K/\mathbb{Q}}(U_K) \rangle \cdot [K \cap \mathbb{Q}^c : \mathbb{Q}]},$$

where e_p is the ramification index of p in K/\mathbb{Q} [9, Theorem III.2.6.4].

Remarks 2.3. Let $K = \mathbb{Q}(\sqrt{m})$ and let $p \nmid D_K$ with residue degree $f \in \{1, 2\}$.

(i) For $p \neq 2$, we have $\#\mathcal{R}_K \sim \frac{1}{p} \log_p(\varepsilon_K) \sim p^{\delta_p(\varepsilon_K)}$, where $\delta_p(\varepsilon_K) = v_p\left(\frac{\varepsilon_K^{p^f-1} - 1}{p}\right)$.

(ii) For $p = 2$, the good definition of the exponent δ_2 is $\delta_2(\varepsilon_K) := v_2\left(\frac{\varepsilon_K^{2 \cdot (2^f-1)} - 1}{8}\right)$ if $f = 1$ and $v_2\left(\frac{\varepsilon_K^{2 \cdot (2^f-1)} - 1}{4}\right)$ if $f = 2$, in which case $\#\mathcal{R}_K \sim 2^{\delta_2(\varepsilon_K)}$. This gives the non-trivial $v_2(\#\mathcal{R}_K) = \delta_2(\varepsilon_K) = 1, 3, 1, 2, 2, 1$ for $D_K = m = 21, 41, 65, 69, 77, 93$, respectively, up to $D_K \leq 100$.

(iii) The existence of an upper bound for $v_p(\frac{1}{2}L_p(1, \chi_K))$ would be equivalent to the estimation of the order of magnitude of $\delta_p(\eta_K)$ for the cyclotomic number $\eta_K := \prod_{a, \chi(a)=1} (1 - \zeta_{D_K}^a)$, where ζ_{D_K} is a primitive D_K th root of unity (interpretation via cyclotomic units of the class number formula). The study given in [10, Théorème 1.1], and applied to the element $\xi = 1 - \zeta_{D_K}$, suggests that if $p \rightarrow \infty$, the probability of $\delta_p(\eta_K) \geq 1$ for the χ_K -component $\langle \eta_K \rangle_{\mathbb{Z}} = \langle \xi \rangle^{e_{\chi_K}}$, of the Galois module generated by ξ , tends to 0 at least as $\frac{O(1)}{p}$ and conjecturally as $\frac{1}{p^{\log_2(p)/\log(c_0(\eta_K))}}$, where $c_0(\eta_K) = |\eta_K| > 1$ and $\log_2 = \log \circ \log$. This does not apply to small p . Which explain the specific difficulties of the p -adic case, which is natural since the study of $v_p(\#\mathcal{T}_K)$ represents a refinement of the Leopoldt conjecture.

We intend to give an estimation of $v_p(\#\mathcal{T}_K)$ related to the discriminant D_K (p fixed). In a numerical point of view, we shall analyse the case of real quadratic fields $K = \mathbb{Q}(\sqrt{m})$ and that of cyclic cubic fields which are described by the polynomials (see, e.g., [7]):

$$P = X^3 + X^2 + \frac{1-f}{3} \cdot X + \frac{1+f(a-3)}{27}, \text{ if } 3 \nmid f,$$

$$P = X^3 - \frac{f}{3} \cdot X - \frac{fa}{27}, \text{ if } 3 \mid f,$$

where the conductor f is $\frac{a^2 + 27b^2}{4}$, with $b > 0$, $a \equiv 2 \pmod{3}$ (for $3 \nmid f$), $a \equiv 6 \pmod{9}$ & $b \not\equiv 0 \pmod{3}$ (for $3 \mid f$). Some non-cyclic cubic fields will be also considered.

In the forthcoming Sections, we deal only with finite places p ; so we simplify some notation in an obvious way.

3. DIRECT CALCULATION OF $v_p(\#\mathcal{T}_K)$

The programs shall try to verify a p -adic analogue of the relation (2.7), for quadratic and cubic fields K ; for each fixed p , they shall give the successive minima of the expression $\Delta_p(K) := \frac{\log_{\infty}(\sqrt{D_K})}{\log_{\infty}(p)} - v_p(\#\mathcal{T}_K)$ and:

$$(3.1) \quad C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_{\infty}(p)}{\log_{\infty}(\sqrt{D_K})},$$

when D_K increases,

It seems that, for a fixed degree d , the absolute minimum of $\Delta_p(K)$ (on the selected interval), is rapidly obtained and is in general negative of small absolute value; whence the interest of the computation of $C_p(K)$ and the question of the existence of a constant $C_p > 0$ as upper bound of the $C_p(K)$ when K runs through such a family of number fields. So in this case $C_p(K)$ has some variations for small discriminants and stabilizes at a maximal value (on the selected interval).

We shall adapt the following program [13, § 3.2] (testing the p -rationality of any number field K), that we recall for the convenience of the reader (choose any monic irreducible P and any prime p ; the program gives the signature (r_1, r_2) of K in S and $r := r_2 + 1$; recall

that from $K = \text{bnfinit}(P, 1)$ and $C8 = \text{component}(K, 8)$, the structure of the class group, the regulator and a fundamental system of units of K are given by $\text{component}(C8, 1)$, $\text{component}(C8, 2)$, and $\text{component}(C8, 5)$, respectively; whence the class number given by $h_K = \text{component}(\text{component}(C8, 1), 1)$:

```
{P=x^6-123*x+1;p=3;K=bnfinit(P,1);n=2;if(p==2,n=3);Kpn=bnrinit(K,p^n);
S=component(component(Kpn,1),7);r=component(component(S,2),2)+1;
print(p,"-rank of the compositum of the Z_",p,"-extensions: ",r);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);R=0;
for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1));
print("Structure of the ",p,"-ray class group: ",L);
if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational"));
if(R==r,print("rk(T)=",0," K is ",p,"-rational"))}
```

For totally real fields the p -invariants of $\text{Gal}(K(p^n)/K)$, where $K(p^n)$ is the ray class field of modulus (p^n) , for any $n \geq 0$ ($n = 0$ corresponding to the Hilbert class field), are given by the simplest program:

```
{P=x^2-2*3*5*7*11*13*17;K=bnfinit(P,1);p=2;n=18;Kpn=bnrinit(K,p^n);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);R=0;
for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1));
print(L)}
```

Then, taking n large enough in the program allows us to compute directly the structure of \mathcal{T}_K as is done by a precise (but longer) program in [28]. This gives the p -valuation v_{ptor} of $\#\mathcal{T}_K$ as rapidly as possible; for this, explain some details about PARI [26]:

Let K be a totally real number field, linearly disjoint from \mathbb{Q}^c ; let $K(p^n)$ be the ray class field of modulus (p^n) , $n \geq 2$ (resp. $n \geq 3$) if $p \neq 2$ (resp. $p = 2$); indeed, from [13, Theorem 2.1], these conditions on n are sufficient to give the p -rank $t_K =: t$ of \mathcal{T}_K .

Thus, for n large enough, the p -structure of $\text{Gal}(K(p^n)/K)$, given by PARI, is of the form $[p^a, p^{a_1}, \dots, p^{a_t}]$, with $a \geq a_1 \geq \dots \geq a_t$, in $Hpn := \text{component}(\text{component}(Kpn, 5), 2)$, where $Kpn = \text{bnrinit}(K, p^n)$ and $p^a = [K(p^n) \cap \mathbb{Q}^c : K]$.

Then the order of \mathcal{T}_K is $[K(p^n) : K] \times p^{-a}$ (up to a p -adic unit), where p^a is the largest component given in Hpn (whence the first one in the list, under the condition $n \gg \max(a_1, \dots, a_t)$); so we have only to verify that p^n is much larger than the exponent $\max(p^{a_1}, \dots, p^{a_t})$ of \mathcal{T}_K .

In practice, and to obtain fast programs, we must look at the order of magnitude of the results to increase n if necessary; in fact, once the part $K = \text{bnfinit}(P, 1)$ of the program is completed, a large value of n does not significantly increase the execution time.

For instance, with $P = x^2 - 4194305$, one gets the successive structures for $2 \leq n \leq 16$:

2	[2, 2]	7	[64, 32, 2]	12	[2048, 1024, 2]
3	[4, 2, 2]	8	[128, 64, 2]	13	[4096, 2048, 2]
4	[8, 4, 2]	9	[256, 128, 2]	14	[8192, 2048, 2]
5	[16, 8, 2]	10	[512, 256, 2]	15	[16384, 2048, 2]
6	[32, 16, 2]	11	[1024, 512, 2]	16	[32768, 2048, 2]

showing that n must be at least 13 to give $\mathcal{T}_K \simeq \mathbb{Z}/2^{11}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In the forthcoming numerical results, if any doubt occurs for a specific field, it is sufficient to use the previous program with bigger n ; we have done these verifications for some large $v_p(\#\mathcal{T}_K)$.

4. NUMERICAL INVESTIGATIONS FOR REAL QUADRATIC FIELDS

We have $\#\mathcal{W}_K = 2$ for $p = 2$ & $m \equiv \pm 1 \pmod{8}$, $\#\mathcal{W}_K = 3$ for $p = 3$ & $m \equiv -3 \pmod{9}$, and we are concerned with the p -class group \mathcal{C}_K and the normalized regulator \mathcal{R}_K .

When $p \nmid D_K$ and $p > 2$, we have $v_p(\#\mathcal{R}_K) = \delta_p(\varepsilon)$ for the fundamental unit ε of K ; if $p = 2$ is unramified, we have $\delta_2(\varepsilon) := v_2\left(\frac{\varepsilon^{2 \cdot (2^f - 1)} - 1}{2^{4-f}}\right)$ where f is the residue degree of 2 in K (see Remark 2.3).

So, we may compute $v_p(\#\mathcal{T}_K)$ as $v_p(\#\mathcal{C}_K) + \delta_p(\varepsilon) + v_p(\#\mathcal{W}_K)$ and we shall compare with the direct computation of the structure of \mathcal{T}_K as explain above.

We have the following result, especially for the computation of $v_p(\#\mathcal{R}_K)$ when $p \geq 2$ ramifies:

Proposition 4.1. *For real quadratic fields K and $p \mid D_K$, $v_p(\#\mathcal{R}_K)$ is given as follows:*

(i) *For $p \nmid 6$ ramified, $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_p(\varepsilon)$ and $v_p(\#\mathcal{R}_K) = \delta$ if $v_p(\varepsilon^{p-1} - 1) = 1 + 2\delta$, $\mathfrak{p} \mid p$, $\delta \geq 0$.*

(ii) *For $p = 3$ ramified, $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_3(\varepsilon)$ (resp. $\#\mathcal{R}_K \sim \frac{1}{3\sqrt{m}} \cdot \log_3(\varepsilon)$) if $m \not\equiv -3 \pmod{9}$ (resp. if $m \equiv -3 \pmod{9}$). Then $v_3(\mathcal{R}_K) = (v_p(\varepsilon^6 - 1) - 2 - \delta)/2$ where $\mathfrak{p} \mid 3$ and $\delta = 1$ (resp. $\delta = 3$) if $m \not\equiv -3 \pmod{9}$ (resp. if $m \equiv -3 \pmod{9}$).*

(iii) *For $p = 2$ ramified, $\mathcal{R}_K \sim \frac{\log_2(\varepsilon)}{2\sqrt{m}}$ (resp. $\frac{\log_2(\varepsilon)}{4}$) if $m \not\equiv -1 \pmod{8}$ (resp. $m \equiv -1 \pmod{8}$). Then, $v_2(\mathcal{R}_K) = (v_p(\varepsilon^4 - 1) - \delta)/2$, where $\mathfrak{p} \mid 2$ and $\delta = 2$ (resp. $\delta = 3$, $\delta = 4$) if $m \equiv 3 \pmod{8}$ (resp. $m \equiv 2, 6 \pmod{8}$, $m \equiv 7 \pmod{8}$).*

Proof. Exercise using the expression (2.10) of $\#\mathcal{R}_K$ where $N_{K/\mathbb{Q}}(U_K)$ is of index 2 in $U_{\mathbb{Q}}$, the fact that $N_{K/\mathbb{Q}}(\varepsilon) = \pm 1$ (i.e., $\text{Tr}_{K/\mathbb{Q}}(\log_p(\varepsilon)) = 0$), and the classical computation of a p -adic logarithm. \square

Remark 4.2. A first information is then the order of magnitude of $\delta_p(\varepsilon)$ for fixed p as $D_K \rightarrow \infty$. Its non-nullity for $p \gg 0$ (K fixed) is a deep problem for which we can only give some experiments with PARI programs (an extensive schedule is discussed in [10], for a general study of p -adic regulators as Frobenius determinants when $p \gg 0$).

Note that these questions of « Fermat quotients », applied to algebraic numbers, are probably the explanation of the failure of the classical p -adic analysis of ζ -functions (among many other subjects) since such Fermat quotients problems are neither easier nor more difficult than the famous problem of Fermat quotients of 2, for instance, for which no one is able to say, so far, how much p are such that $\frac{2^{p-1}-1}{p} \equiv 0 \pmod{p}$.

4.1. Maximal values of $v_p(\mathcal{R}_K)$. Consider a prime p fixed and the family of real quadratic fields $K = \mathbb{Q}(\sqrt{m})$. The following programs find the successive maxima of $\delta_p(\varepsilon)$ with the corresponding increasing $D_K \in [bD, BD]$; the program uses the fact that for p unramified, in the inert case $\varepsilon^{p+1} \equiv N_{K/\mathbb{Q}}(\varepsilon) \pmod{p}$, otherwise, $\varepsilon^{p-1} \equiv 1 \pmod{p}$. We shall indicate the final value of $C_p(K)$ defined by expression (3.1) by computing for instance $v_p(h_K)$, then $v_p(\mathcal{T}_K) = \delta_p(\varepsilon) + v_p(h_K) + v_p(\mathcal{W}_K)$:

4.1.1. PROGRAM FOR $p = 2$ UNRAMIFIED. For $p = 2$ unramified, we use the particular formula given in Remark 2.3 (ii).

```
{p=2;bm=3;Bm=5*10^8;Max=0;for(m=bm,Bm,ss=Mod(m,8);s=0;if(ss==1,s=1);
if(ss==5,s=-1);if(core(m)==m & s!=0,E=quadunit(m);A=(E^(p-s)-1)/p;
A1=[component(A,2),component(A,3)];A2=[component(A,2)+1,component(A,3)];
delta1=valuation(A1,p);delta2=valuation(A2,p);delta=max(delta1,delta2);
if(delta>Max,Max=delta;print("D=",m," delta=",delta))}}
```

D=21	delta=1	D=3881	delta=11	D=1051385	delta=19
D=41	delta=3	D=69973	delta=12	D=12256653	delta=21
D=469	delta=5	D=115005	delta=13	D=14098537	delta=22
D=645	delta=6	D=122321	delta=14	D=28527281	delta=25
D=1185	delta=8	D=222181	delta=16		
D=1201	delta=10	D=528077	delta=18		

The next discriminant is $D_K = 214203013$, where $\delta_2(\varepsilon) = 26$, $v_2(h_K) = 1$, $v_2(\mathcal{T}_K) = 27$, $C_2(K) = 1.951261$; the interval was $[2, 5 \cdot 10^8]$ (two days of computer).

4.1.2. PROGRAM FOR $p = 2$ RAMIFIED. A similar program using Proposition 4.1 (ii) gives analogous results for maximal values of $\delta_2(\varepsilon)$ (in *delta*):

```
{p=2;bm=3;Bm=5*10^8;Max=0;for(m=bm,Bm,s=Mod(m,4);ss=Mod(m,8);
if(core(m)==m & s!=1,A=(quadunit(4*m)^4-1)/4;N=norm(A);v=valuation(N,2);
if(s==2,delta=v-3);if(s==3,delta=v-2);if(ss==7,delta=v-4);delta=delta/2;
if(delta>Max,Max=delta;print("D=",4*m," delta=",delta))}}
D=28    delta=1      D=10232   delta=8      D=15704072  delta=20
D=124   delta=2      D=21980   delta=9      D=29419592  delta=21
D=456   delta=4      D=28664   delta=13     D=36650172  delta=23
D=508   delta=6      D=81624   delta=17     D=80882380  delta=28
D=1784  delta=7      D=1476668 delta=18
```

For $D = 80882380 = 4 \cdot 5 \cdot 239 \cdot 16921$, $\delta_2(\varepsilon) = 27$, $v_2(h_K) = 2$, $v_2(\mathcal{W}_K) = 1$, and $C_2(K) = 2.2840$, whence the influence of genera theory on $C_2(K)$.

4.1.3. PROGRAM FOR $p \geq 3$ UNRAMIFIED. In this case, the program can be simplified:

```
{p=3;bd=5;BD=10^8;Max=0;for(D=bd,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3) || (e==0 & Mod(M,4)!=1) || (e==2 & Mod(M,4)==1),next);
s=kroncker(D,p);if(s!=0,E=quadunit(D);nu=norm(E);u=(1+nu-nu*s+s)/2;A=(E^(p-s)-u)/p;
A=[component(A,2),component(A,3)];delta=valuation(A,p);
if(delta>Max,Max=delta;print("D=",D," delta=",delta))}}
D=29    delta=2      D=21713   delta=9      D=13495160  delta=14
D=488   delta=4      D=153685  delta=10     D=21242636  delta=16
D=1213  delta=6      D=321253  delta=12     D=71801701  delta=19
D=13861 delta=7      D=6917324 delta=13
```

Which gives $\delta_3(\varepsilon) \leq 19$ on the interval $[2, 10^8]$, obtained for $D_K = 71801701$ ($h = 4$, $C_p(K) = 2.307828$).

4.1.4. PROGRAM FOR $p \geq 3$ RAMIFIED. We obtain (cf. formula of Proposition 4.1 (ii)):

```
{p=3;bd=5;BD=10^8;Max=0;for(D=bd,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if(Mod(M,3)!=0 || (e==1 || e>3) || (e==0 & Mod(M,4)!=1) || (e==2 & Mod(M,4)==1),next);
E=quadunit(D)^6;A=norm(E-1);v=valuation(A,3);if(Mod(D,9)!=-3,delta=(v-3)/2);
if(Mod(D,9)==-3,delta=(v-5)/2);if(delta>Max,Max=delta;print("D=",D," delta=",delta))}}
D=93    delta=1      D=102984  delta=8      D=17157729  delta=13
D=105   delta=2      D=168009  delta=10     D=104326449 delta=15
D=492   delta=3      D=2354577 delta=11     D=448287465 delta=18
D=1896  delta=6      D=6099477 delta=12
```

For very large p there is in general no solution in a reasonable interval since we have at least, roughly speaking, $\text{Prob}(\delta_p(\varepsilon) \geq \delta) \approx p^{-\delta}$, otherwise, the solutions are often with $\delta_p(\varepsilon) = 1$, large discriminants, $C_p(K)$ being rather small as we shall analyse now.

4.2. **Experiments for a conjectural upper bound - Quadratic fields.** We only assume $K \neq \mathbb{Q}(\sqrt{2})$ when $p = 2$ to always have $K \cap \mathbb{Q}^c = \mathbb{Q}$.

We have given previously programs for the maximal values of $v_p(\mathcal{R}_K)$; we now give the behaviour of the whole $v_p(\#\mathcal{T}_K)$ for increasing discriminants; for this purpose, we compute $\Delta_p(K) := \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$ and $C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$.

4.2.1. PROGRAM FOR $p = 2$. The numerical data are D_K , $v_p(\#\mathcal{T}_K)$ (in *vptor*) and the successive $\Delta_p(K)$ (in *Ymin*) and $C_p(K)$ (in *Cp*); we omit the 2-rational fields (for them, *vptor* = 0):

```
{p=2;n=36;bd=5;BD=10^6;ymin=5;for(D=bd,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)||e==0&Mod(M,4)!=1)||e==2&Mod(M,4)==1),next);m=D;if(e!=0,m=D/4);
P=x^2-D;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(sqrt(D))/log(p)-vptor;if(Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D)));
print("D=",D," m=",m," vptor=",vptor," Ymin=",Y," Cp=",Cp)}}
p=2
```

D=17	m=17	vptor=1	Ymin=1.04373142...	Cp=0.4893
D=28	m=7	vptor=2	Ymin=0.40367746...	Cp=0.8320
D=41	m=41	vptor=4	Ymin=-1.32122399...	Cp=1.4932
D=508	m=127	vptor=7	Ymin=-2.50565765...	Cp=1.5575
D=1185	m=1185	vptor=10	Ymin=-4.89466432...	Cp=1.9587
D=1201	m=1201	vptor=11	Ymin=-5.88498978...	Cp=2.1505
D=3881	m=3881	vptor=12	Ymin=-6.03889364...	Cp=2.0130
D=11985	m=11985	vptor=13	Ymin=-6.22552885...	Cp=1.9189
D=26377	m=26377	vptor=14	Ymin=-6.65650356...	Cp=1.9064
D=81624	m=20406	vptor=20	Ymin=-11.84164710...	Cp=2.4514

The larger computations in §4.1.1 show the largest case $D_K = 214203013$ with $h = 2$ and $\delta_2(\varepsilon_K) = 26$, giving $\Delta_2(K) = \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(2)} - v_p(\#\mathcal{T}_K) \approx -13.1628$, the best local minimum and gives $C_2 \geq 1.951261$. For the ramified case $D_K = 4 \cdot 20220595$, we obtained $\delta_2(\varepsilon) = 28$, $C_2(K) = 2.284033$.

But the case $D_K = 81624 = 8 \cdot 3 \cdot 19 \cdot 179$, for which $h_K = 8$, with the valuation $vptor = 20$, gives $C_2 \geq 2.4514$ and shows, once again, that genera theory may modify the results for $p = 2$ and more generally for $p \mid d$. So the main conjecture that we shall consider would perhaps be modified for $p \mid d$. Note that in the above results, there is no solution $D \in [20406, 10^6]$. To illustrate this, we use the same program for $D \in [81628, 0.5 \cdot 10^6]$:

```
p=2
D=81628 m=20407 vptor=2 Ymin=6.15838824... Cp=0.2451
D=81640 m=20410 vptor=4 Ymin=4.15849428... Cp=0.4902
D=81713 m=81713 vptor=5 Ymin=3.15913899... Cp=0.6128
D=81788 m=20447 vptor=7 Ymin=1.15980078... Cp=0.8578
D=82684 m=20671 vptor=8 Ymin=0.16766028... Cp=0.9794
D=83144 m=20786 vptor=9 Ymin=-0.82833773... Cp=1.1013
D=84361 m=84361 vptor=10 Ymin=-1.81785571... Cp=1.2221
D=86284 m=21571 vptor=11 Ymin=-2.80159728... Cp=1.3417
D=100045 m=100045 vptor=14 Ymin=-5.69485522... Cp=1.6857
D=115005 m=115005 vptor=16 Ymin=-7.59433146... Cp=1.9034
D=376264 m=94066 vptor=17 Ymin=-7.73930713... Cp=1.8357
D=495957 m=495957 vptor=19 Ymin=-9.54007224... Cp=2.0084
D=1476668 m=369167 vptor=20 Ymin=-9.75304296... Cp=1.9518
```

4.2.2. PROGRAM FOR $p \in [3, 50]$. In this case, genera theory does not intervene. We do not write the cases where $vptor = 0$ (p -rational fields). The constant $C_p(K)$ has some variations for very small D_K but stabilizes and seems to decrease for larger D_K ; so we mention the maximal ones, but the last value is more significant to evaluate C_p :

```
{n=16;bd=5;BD=10^6;forprime(p=3,50,print(" ");print("p=",p);ymin=10;
for(D=bd,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)||e==0&Mod(M,4)!=1)||e==2&Mod(M,4)==1),next);m=D;if(e!=0,m=D/4);
P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(sqrt(D))/log(p)-vptor;if(Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D)));
print("D=",D," m=",m," vptor=",vptor," Ymin=",Y," Cp=",Cp))}}
p=3
```

D=24	m=6	vptor=1	Ymin=0.44639463...	Cp=0.6913
D=29	m=29	vptor=2	Ymin=-0.46747762...	Cp=1.3050
D=105	m=105	vptor=3	Ymin=-0.88189136...	Cp=1.4163

D=488	m=122	vptor=4	Ymin=-1.18266604...	Cp=1.4197
D=1213	m=1213	vptor=6	Ymin=-2.76826302...	Cp=1.8565
D=1896	m=474	vptor=7	Ymin=-3.56498395...	Cp=2.0378
D=13861	m=13861	vptor=8	Ymin=-3.65959960...	Cp=1.8431
D=21713	m=21713	vptor=10	Ymin=-5.45532735...	Cp=2.2003
D=168009	m=168009	vptor=11	Ymin=-5.52410420...	Cp=2.0088
D=321253	m=321253	vptor=12	Ymin=-6.22909046...	Cp=2.0793
p=5				
D=53	m=53	vptor=1	Ymin=0.23344053...	Cp=0.8107
D=73	m=73	vptor=2	Ymin=-0.66709383...	Cp=1.5005
D=217	m=217	vptor=3	Ymin=-1.32864091...	Cp=1.7949
D=1641	m=1641	vptor=4	Ymin=-1.70010976...	Cp=1.7392
D=25037	m=25037	vptor=5	Ymin=-1.85352571...	Cp=1.5890
D=71308	m=17827	vptor=6	Ymin=-2.52836443...	Cp=1.7283
D=304069	m=304069	vptor=7	Ymin=-3.07782014...	Cp=1.7847
p=7				
D=24	m=6	vptor=1	Ymin=-0.18340170...	Cp=1.2246
D=145	m=145	vptor=2	Ymin=-0.72123238...	Cp=1.5640
D=797	m=797	vptor=3	Ymin=-1.28335992...	Cp=1.7476
D=30556	m=7639	vptor=4	Ymin=-1.34640462...	Cp=1.5074
D=92440	m=23110	vptor=5	Ymin=-2.06196222...	Cp=1.7018
D=287516	m=71879	vptor=6	Ymin=-2.77039718...	Cp=1.8578
p=11				
D=29	m=29	vptor=1	Ymin=-0.29786428...	Cp=1.4242
D=145	m=145	vptor=2	Ymin=-0.96227041...	Cp=1.9272
D=424	m=106	vptor=3	Ymin=-1.73853259...	Cp=2.3781
D=35068	m=8767	vptor=4	Ymin=-1.81786877...	Cp=1.8330
D=163873	m=163873	vptor=5	Ymin=-2.49637793...	Cp=1.9971
p=13				
D=8	m=2	vptor=1	Ymin=-0.59464276...	Cp=2.4669
D=2285	m=2285	vptor=3	Ymin=-1.49234424...	Cp=1.9898
D=98797	m=98797	vptor=4	Ymin=-1.75808000...	Cp=1.7842
D=382161	m=382161	vptor=5	Ymin=-2.49437601...	Cp=1.9955
p=17				
D=69	m=69	vptor=2	Ymin=-1.25277309...	Cp=2.6765
D=3209	m=3209	vptor=3	Ymin=-1.57516648...	Cp=2.1055
D=8972	m=2243	vptor=4	Ymin=-2.39372069...	Cp=2.4902
D=1631753	m=1631753	vptor=5	Ymin=-2.47545212...	Cp=1.9805
p=19				
D=109	m=109	vptor=1	Ymin=-0.20335454...	Cp=1.2552
D=193	m=193	vptor=2	Ymin=-1.10633396...	Cp=2.2379
D=2701	m=2701	vptor=3	Ymin=-1.65825418...	Cp=2.2359
p=23				
D=140	m=35	vptor=1	Ymin=-0.21198348...	Cp=1.2690
D=493	m=493	vptor=2	Ymin=-1.01123893...	Cp=2.0227
D=10433	m=10433	vptor=3	Ymin=-1.52451822...	Cp=2.0332
D=740801	m=740801	vptor=4	Ymin=-1.84475964...	Cp=1.8559
p=29				
D=33	m=33	vptor=1	Ymin=-0.48081372...	Cp=1.9261
D=41	m=41	vptor=2	Ymin=-1.44858244...	Cp=3.6270
D= 53093	m=53093	vptor=4	Ymin=-2.38448997...	Cp=2.4759
p=31				
D=8	m=2	vptor=1	Ymin=-0.69722637...	Cp=3.3028
D=6168	m=1542	vptor=2	Ymin=-0.72930075...	Cp=1.5739
D=90273	m=90273	vptor=3	Ymin=-1.33857946...	Cp=1.8056
D=1294072	m=323518	vptor=4	Ymin=-1.95087990...	Cp=1.9520
p=37				
D=33	m=33	vptor=1	Ymin=-0.51584228...	Cp=2.0654
D=3340	m=835	vptor=2	Ymin=-0.87650089...	Cp=1.7801
D= 124129	m=124129	vptor=3	Ymin=-1.37588711...	Cp=1.8471

p=41				
D=73	m=73	vptor=1	Ymin=-0.42232716...	Cp=1.7311
D=2141	m=2141	vptor=2	Ymin=-0.96743241...	Cp=1.9369
D=187113	m=187113	vptor=3	Ymin=-1.36552680...	Cp=1.8354
p=43				
D=88	m=22	vptor=1	Ymin=-0.40479944...	Cp=1.6801
D=6520	m=1630	vptor=2	Ymin=-0.83246977...	Cp=1.7130
D=283596	m=70899	vptor=3	Ymin=-1.33094416...	Cp=1.7974
p=47				
D=301	m=301	vptor=1	Ymin=-0.25884526...	Cp=1.3492
D=26321	m=26321	vptor=2	Ymin=-0.67821659...	Cp=1.5131
D=368013	m=368013	vptor=3	Ymin=-1.33566464...	Cp=1.8025

The same program with $p = 3$ gives, taking larger discriminants, $D \in [10^6, 2.5 \cdot 10^7]$ (two days of computer):

p=3				
D=1000005	m=1000005	vptor=1	Ymin=5.28771209...	Cp=0.1590
D=1000049	m=1000049	vptor=2	Ymin=4.28773212...	Cp=0.3180
D=1000104	m=250026	vptor=3	Ymin=3.28775715...	Cp=0.4771
D=1000133	m=1000133	vptor=4	Ymin=2.28777034...	Cp=0.6361
D=1000169	m=1000169	vptor=5	Ymin=1.28778673...	Cp=0.7951
D=1000380	m=250095	vptor=6	Ymin=0.28788273...	Cp=0.9542
D=1001177	m=1001177	vptor=8	Ymin=-1.71175481...	Cp=1.2722
D=1014693	m=1014693	vptor=9	Ymin=-2.70565175...	Cp=1.4298
D=1074760	m=268690	vptor=10	Ymin=-3.67947724...	Cp=1.5821
D=1185256	m=296314	vptor=11	Ymin=-4.63493860...	Cp=1.7281
D=2354577	m=2354577	vptor=12	Ymin=-5.32254344...	Cp=1.7970
D=6099477	m=6099477	vptor=13	Ymin=-5.88934151...	Cp=1.8282
D=13495160	m=3373790	vptor=14	Ymin=-6.52791825...	Cp=1.8736
D=21242636	m=5310659	vptor=16	Ymin=-8.32143995...	Cp=2.0837

The last case leads to $C_3(K) = 2.0837$; but it is difficult to predict the behavior of C_3 at infinity even if we notice a very low growth which remains in the same order of magnitude as for smaller discriminants.

The interval $[2, 10^6]$ is not always sufficient; for instance, we have obtained the very similar data:

p=5	D=m=4788645	vptor=10	Ymin=-5.22138818...	Cp=2.0926
p=7	D=m=4354697	vptor=7	Ymin=-3.07207825...	Cp=1.7821

But for $p = 7$, we ignore if the bound $C_p(K) = 1.8578$ can be exceeded.

From these calculations in the quadratic case, one may consider, in an heuristic framework, that we have the good following lower bounds for C_p (if it exists):

$$C'_3 \approx 2.0837, C'_5 \approx 2.0926, C'_7 \approx 1.8578, C'_{11} \approx 1.9971, C'_{13} \approx 1.9955, C'_{17} \approx 1.9805, C'_{19} \approx 2.2379, C'_{23} \approx 1.8559, C'_{29} \approx 2.4759, C'_{31} \approx 1.9520, C'_{37} \approx 1.8471, C'_{41} \approx 1.8354, C'_{43} \approx 1.7974, C'_{47} \approx 1.8025.$$

Remarks 4.3. (i) For $p \gg 0$, an estimation of C_p is more difficult; for instance, we have found that for $K = \mathbb{Q}(\sqrt{19})$ and $p_0 = 13599893$, $v_{p_0}(\#\mathcal{T}_K) = 1$, whence $C_{p_0} \geq 7.585585$. Unfortunately, for $D_K > 4 \cdot 19$, we did not find any example of non- p_0 -rational field $\mathbb{Q}(\sqrt{m})$ up to 10^8 , knowing that such fields exist with huge discriminants, as:

$$D = p_0^2 + 4 = 34209124997537575597791879605$$

for which $v_{p_0}(\#\mathcal{T}_K) = 1$, $C_{p_0}(K) = 0.4999$ and $v_{p_0}(\#\mathcal{C}_K) = 0$, since this field is an element of families $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + b^2})$, $b \in \{1, 2\}$, that we will describe in the next subsection 4.3, and which is by definition such that $\delta_{p_0}(\varepsilon_K) = \rho - 1$ and $C_p(K) < 1 + o(1)$.

We then have the following alternative: either $C_{p_0}(K) < 7.585585$ for all $D > 4 \cdot 19$, whence $C_{p_0} = 7.585585$, or the maximum C_{p_0} of the $C_{p_0}(K)$ is greater than 7.585585 or

infinite. Note that to get $C_{p_0}(K) < 1.1$, we must have for instance $v_{p_0}(\#\mathcal{T}_K) = 1$ and $D_K > 9333929793774$, $D_K > 94334377272$ for $C_{p_0}(K) < 1.3$, etc.

The existence of infinitely many quadratic fields K such that $C_{p_0}(K) > 7.585585$ remains possible but assumes the strong condition $v_{p_0}(\#\mathcal{T}_K) > 0.4618 \cdot \log_\infty(\sqrt{D_K})$ for infinitely many K . This is incredible since the probabilities $\text{Prob}(v_p(\#\mathcal{T}_K) \geq 1)$ are at most $\frac{1}{p}$ for any p and K ; the most credible « unfavorable » case should be that, for a given p , there exist infinitely many quadratic fields K for which $v_p(\#\mathcal{T}_K) = 1$ for $D_K \gg 0$, whence $C_p(K) \rightarrow 0$ (instead of $C_p(K) = 0$ for $D_K \gg 0$); under this assumption, the constant C_p does exist for real quadratic fields and is equal to $\max_{5 \leq D_K \leq D_0} (C_p(K))$ for a sufficiently large D_0 .

(ii) The existence of C_p essentially depends on $v_p(\mathcal{R}_K)$ since the influence of $v_p(\#\mathcal{C}_K)$ seems negligible, which is reinforced by classical heuristics on class groups [3, 4], or by specific results in suitable towers [35, Proposition 7.1], and mainly by strong conjectures in [8] as $\#\mathcal{C}_K \ll_{\epsilon, p, d} D_K^\epsilon$ for any number field of degree d , strengthening the Brauer theorem (existence of a constant $C_0 > 0$ such that, $\log_\infty(\#\mathcal{C}_K) \leq C_0 \cdot \log_\infty(\sqrt{D_K})$ for all number field K). For quadratic and cyclic cubic fields, $C_0 = 1$ (Remark 2.2).

(iii) For any fixed p , $\inf_K (C_p(K)) = 0$ (see Byeon [2, Theorem 1.1], after Ono, where a lower bound of the density of p -rational fields is given for $p > 3$). Indeed, as $D_K \rightarrow \infty$, there are, statistically, most of the real quadratic fields K such that $\#\mathcal{T}_K = 1$.

(iv) Now, if K is fixed and $p \rightarrow \infty$, $\inf_p (C_p(K)) = 0$ (one may see this as an unproved generalization, for $v_p(\mathcal{R}_K)$, of theorems of Silverman [32], Graves–Murty [17] and others, showing the considerable difficulties of such subjects). We have conjectured, after numerous calculations and heuristics, that the set of primes p such that K is non- p -rational (i.e., $\mathcal{T}_K \neq 1$) is finite [10, Conjecture 8.11], i.e., $C_p(K) = 0$ for all $p \gg 0$.

4.3. A special family of quadratic fields. Consider, for p fixed, the quadratic field:

$$K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1}), \text{ with } \rho \rightarrow \infty, a \geq 1, p \nmid a;$$

assuming that $m := a^2 \cdot p^{2\rho} + 1$ is a squarefree integer, its fundamental unit is $\varepsilon_K = a \cdot p^\rho + \sqrt{m}$ and $D_K = m$ (for $a \cdot p$ even) or $4m$ (for $a \cdot p$ odd). Its p -class number is unknown, but the formula (2.8) implies $h \leq \frac{1}{2} \cdot \sqrt{D_K}$, and an upper bound being $a \cdot p^\rho$, this allows to get $v_p(\#\mathcal{C}_K) \leq \rho + \frac{\log(a)}{\log(p)}$ to take into account the possible case where h is a maximal p th power. As $\delta_p(\varepsilon_K) = \rho - 1$ and $\mathcal{W}_K = 1$ for these fields, it follows:

$$\rho - 1 \leq v_p(\#\mathcal{T}_K) = v_p(\#\mathcal{C}_K) + \delta_p(\varepsilon_K) \leq 2\rho - 1 + \frac{\log(a)}{\log(p)}.$$

Thus, since $\frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} \approx \rho + \frac{\log(a)}{\log(p)}$, we have proved, in this particular case, that:

$$\frac{\rho - 1}{\rho + \frac{\log(2a)}{\log(p)}} \leq C_p(K) \leq \frac{2\rho - 1 + \frac{\log(a)}{\log(p)}}{\rho + \frac{\log(a)}{\log(p)}} \in [1, 2[.$$

These inequalities strengthen the fact that the upper bound cannot be decreased substantially.

We shall assume the conjecture that, for all p , $m := a^2 \cdot p^{2\rho} + 1$ is squarefree for infinitely many integers $\rho \geq 1$.² Whence the partial result:

Theorem 4.4. *Let \mathcal{K}_0 be the set of real quadratic fields and $C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$, for $K \in \mathcal{K}_0$ and $p \geq 2$. Then, under the above conjecture on $m := a^2 \cdot p^{2\rho} + 1$, for each fixed p one has $C_p(K) \in [0, 2[$ for an infinite subset of \mathcal{K}_0 .*

²The conjecture is true for integers of the form $n^2 + 1$ [19], but we ignore if this remains true for $n = a \cdot p^\rho$, p prime, $\rho \in \mathbb{N}$, $a \geq 1$; but this is not so essential (see Remark 4.5).

Moreover, if we consider the estimation of $v_p(\#\mathcal{C}_K)$ largely excessive, as explained in Remark 4.3 (i), one may conjecture that, for the above family $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1})$, and $p > 2$, one has:

$$\rho - 1 \leq v_p(\#\mathcal{T}_K) \leq \rho \cdot (1 + o(1)),$$

and the statement of the theorem becomes:

For each fixed $p > 2$, $C_p(K)$ is asymptotically equal to 1 for an infinite subset of \mathcal{K}_0 .

Indeed, the small p -valuations of the class number are the following, to illustrate the relation $\rho - 1 \leq v_p(\#\mathcal{T}_K) \leq \rho \cdot (1 + o(1))$. We suppose $p > 2$ because of genera theory; we vary p and ρ in intervals such that $\log(m) < 40$:

```
{a=1;B=40;n=26;forprime(p=3,20,for(rho=1,B/(2*log(p)),m=a^2*p^(2*rho)+1;
if(core(m)==m,D=m;if(Mod(m,4)!=1,D=4*m);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;Cp=vptor*log(p)/log(sqrt(D));
h=component(component(component(K,8),1),1);vph=valuation(h,p);
print("p=",p," m=",m," rho=",rho," vptor=",vptor," Cp=",Cp," vph=",vph))}}
a=1, p=3, D=4*m
m=82 rho=2 vptor=1 Cp=0.3792886959... vph=0
m=730 rho=3 vptor=3 Cp=0.8260927150... vph=1
m=6562 rho=4 vptor=3 Cp=0.6478083724... vph=0
m=531442 rho=6 vptor=6 Cp=0.9048503676... vph=1
m=4782970 rho=7 vptor=7 Cp=0.9173194012... vph=1
m=43046722 rho=8 vptor=7 Cp=0.8110366080... vph=0
m=387420490 rho=9 vptor=10 Cp=1.0383213515... vph=2
m=3486784402 rho=10 vptor=10 Cp=0.9406514981... vph=1
m=31381059610 rho=11 vptor=10 Cp=0.8597764935... vph=0
m=282429536482 rho=12 vptor=13 Cp=1.0292195629... vph=2
m=2541865828330 rho=13 vptor=12 Cp=0.8803508063... vph=0
m=22876792454962 rho=14 vptor=14 Cp=0.9568769883... vph=1
m=1853020188851842 rho=16 vptor=15 Cp=0.9019339400... vph=0
m=16677181699666570 rho=17 vptor=17 Cp=0.9642146068... vph=1
m=150094635296999122 rho=18 vptor=19 Cp=1.0198095452... vph=2
a=1, p=5, D=4*m
m=626 rho=2 vptor=1 Cp=0.4113240423... vph=0
m=15626 rho=3 vptor=2 Cp=0.5829720101... vph=0
m=390626 rho=4 vptor=4 Cp=0.9027964983... vph=1
m=9765626 rho=5 vptor=5 Cp=0.9206955923... vph=1
m=244140626 rho=6 vptor=5 Cp=0.7775231663... vph=0
m=6103515626 rho=7 vptor=6 Cp=0.8074634864... vph=0
m=152587890626 rho=8 vptor=7 Cp=0.8303010976... vph=0
m=3814697265626 rho=9 vptor=9 Cp=0.9543323795... vph=1
m=95367431640626 rho=10 vptor=10 Cp=0.9587105826... vph=1
m=2384185791015626 rho=11 vptor=11 Cp=0.9623227412... vph=1
m=59604644775390626 rho=12 vptor=11 Cp=0.8849075871... vph=0
a=2, p=3, D=m
m=2917 rho=3 vptor=3 Cp=0.8261991487... vph=1
m=26245 rho=4 vptor=3 Cp=0.6478156494... vph=0
m=236197 rho=5 vptor=5 Cp=0.8879525243... vph=1
m=2125765 rho=6 vptor=6 Cp=0.9048504552... vph=1
m=19131877 rho=7 vptor=6 Cp=0.7862737798... vph=0
m=172186885 rho=8 vptor=7 Cp=0.8110366087... vph=0
m=1549681957 rho=9 vptor=10 Cp=1.0383213516... vph=2
m=13947137605 rho=10 vptor=10 Cp=0.9406514982... vph=1
m=10167463313317 rho=13 vptor=15 Cp=1.1004385079... vph=3
m=823564528378597 rho=15 vptor=15 Cp=0.9596358141... vph=1
m=7412080755407365 rho=16 vptor=15 Cp=0.9019339400... vph=0
m=66708726798666277 rho=17 vptor=16 Cp=0.9074961005... vph=0
m=600378541187996485 rho=18 vptor=19 Cp=1.0198095452... vph=2
```

$a=2, p=5, D=m$
 $m=2501 \quad \rho=2 \quad \text{vptor}=1 \quad C_p=0.4113870622\dots \quad \text{vph}=0$
 $m=62501 \quad \rho=3 \quad \text{vptor}=2 \quad C_p=0.5829745440\dots \quad \text{vph}=0$
 $m=1562501 \quad \rho=4 \quad \text{vptor}=4 \quad C_p=0.9027966199\dots \quad \text{vph}=1$
 $m=39062501 \quad \rho=5 \quad \text{vptor}=5 \quad C_p=0.9206955964\dots \quad \text{vph}=1$
 $m=24414062501 \quad \rho=7 \quad \text{vptor}=6 \quad C_p=0.8074634864\dots \quad \text{vph}=0$
 $m=610351562501 \quad \rho=8 \quad \text{vptor}=7 \quad C_p=0.8303010976\dots \quad \text{vph}=0$
 $m=15258789062501 \quad \rho=9 \quad \text{vptor}=8 \quad C_p=0.8482954484\dots \quad \text{vph}=0$
 $m=381469726562501 \quad \rho=10 \quad \text{vptor}=11 \quad C_p=1.0545816408\dots \quad \text{vph}=2$
 $m=9536743164062501 \quad \rho=11 \quad \text{vptor}=10 \quad C_p=0.8748388557\dots \quad \text{vph}=0$
 $m=238418579101562501 \quad \rho=12 \quad \text{vptor}=11 \quad C_p=0.8849075871\dots \quad \text{vph}=0$

For the similar family $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 4})$, a odd and $p > 2$, where $\varepsilon = \frac{a \cdot p^\rho + \sqrt{m}}{2}$, one gets quadratic fields unramified at 2 giving a maximal $C_p = 1.2222222215\dots$

$a=1, p=3, D=m$
 $m=85 \quad \rho=2 \quad \text{vptor}=1 \quad C_p=0.4945750747656077295917504 \quad \text{vph}=0$
 $m=733 \quad \rho=3 \quad \text{vptor}=3 \quad C_p=0.9991705549452351082457751 \quad \text{vph}=1$
 $m=6565 \quad \rho=4 \quad \text{vptor}=3 \quad C_p=0.7499479938973711320785265 \quad \text{vph}=0$
 $m=59053 \quad \rho=5 \quad \text{vptor}=4 \quad C_p=0.7999950674034869040547426 \quad \text{vph}=0$
 $m=531445 \quad \rho=6 \quad \text{vptor}=7 \quad C_p=1.166666005901177774552383 \quad \text{vph}=2$
 $m=4782973 \quad \rho=7 \quad \text{vptor}=7 \quad C_p=0.9999999456261941952146777 \quad \text{vph}=1$
 $m=387420493 \quad \rho=9 \quad \text{vptor}=11 \quad C_p=1.222222215840900774996624 \quad \text{vph}=3$
 $m=3486784405 \quad \rho=10 \quad \text{vptor}=10 \quad C_p=0.999999999477891878949469 \quad \text{vph}=1$
 $m=31381059613 \quad \rho=11 \quad \text{vptor}=10 \quad C_p=0.9090909090861147096295570 \quad \text{vph}=0$
 $m=282429536485 \quad \rho=12 \quad \text{vptor}=13 \quad C_p=1.083333333327514227042307 \quad \text{vph}=2$
 $m=2541865828333 \quad \rho=13 \quad \text{vptor}=12 \quad C_p=0.9230769230768722226487080 \quad \text{vph}=0$
 $m=22876792454965 \quad \rho=14 \quad \text{vptor}=14 \quad C_p=0.999999999999943158912907 \quad \text{vph}=1$
 $m=205891132094653 \quad \rho=15 \quad \text{vptor}=17 \quad C_p=1.13333333333326652751245 \quad \text{vph}=3$
 $m=1853020188851845 \quad \rho=16 \quad \text{vptor}=15 \quad C_p=0.93749999999999424352416 \quad \text{vph}=0$
 $m=16677181699666573 \quad \rho=17 \quad \text{vptor}=16 \quad C_p=0.9411764705882352880741958 \quad \text{vph}=0$
 $m=1350851717672992093 \quad \rho=19 \quad \text{vptor}=19 \quad C_p=0.999999999999999290709 \quad \text{vph}=1$
 $m=12157665459056928805 \quad \rho=20 \quad \text{vptor}=19 \quad C_p=0.949999999999999928873 \quad \text{vph}=0$
 $m=109418989131512359213 \quad \rho=21 \quad \text{vptor}=23 \quad C_p=1.0952380952380952380943703 \quad \text{vph}=3$
 $m=984770902183611232885 \quad \rho=22 \quad \text{vptor}=22 \quad C_p=0.9999999999999999159 \quad \text{vph}=1$
 $a=1, p=5, D=m$
 $m=629 \quad \rho=2 \quad \text{vptor}=1 \quad C_p=0.4995050064384236683280022 \quad \text{vph}=0$
 $m=15629 \quad \rho=3 \quad \text{vptor}=2 \quad C_p=0.6666489958698626477868625 \quad \text{vph}=0$
 $m=390629 \quad \rho=4 \quad \text{vptor}=3 \quad C_p=0.7499994035219911691252645 \quad \text{vph}=0$
 $m=9765629 \quad \rho=5 \quad \text{vptor}=5 \quad C_p=0.999999745501269402681349 \quad \text{vph}=1$
 $m=244140629 \quad \rho=6 \quad \text{vptor}=5 \quad C_p=0.833333326263922586253773 \quad \text{vph}=0$
 $m=6103515629 \quad \rho=7 \quad \text{vptor}=6 \quad C_p=0.8571428571179266492305481 \quad \text{vph}=0$
 $m=152587890629 \quad \rho=8 \quad \text{vptor}=7 \quad C_p=0.874999999991092542378279 \quad \text{vph}=0$
 $m=3814697265629 \quad \rho=9 \quad \text{vptor}=8 \quad C_p=0.88888888888567152147980 \quad \text{vph}=0$
 $m=95367431640629 \quad \rho=10 \quad \text{vptor}=10 \quad C_p=0.99999999999986969661993 \quad \text{vph}=1$
 $m=2384185791015629 \quad \rho=11 \quad \text{vptor}=10 \quad C_p=0.90909090909090478335933 \quad \text{vph}=0$
 $m=59604644775390629 \quad \rho=12 \quad \text{vptor}=11 \quad C_p=0.9166666666666665074069 \quad \text{vph}=0$
 $m=1490116119384765629 \quad \rho=13 \quad \text{vptor}=12 \quad C_p=0.9230769230769230768638621 \quad \text{vph}=0$
 $m=37252902984619140629 \quad \rho=14 \quad \text{vptor}=13 \quad C_p=0.9285714285714285714263589 \quad \text{vph}=0$
 $m=931322574615478515629 \quad \rho=15 \quad \text{vptor}=16 \quad C_p=1.0666666666666666665717 \quad \text{vph}=2$
 $a=1, p=7, D=m$
 $m=2405 \quad \rho=2 \quad \text{vptor}=1 \quad C_p=0.4998930943437939009946102 \quad \text{vph}=0$
 $m=117653 \quad \rho=3 \quad \text{vptor}=2 \quad C_p=0.666647253436162691864834 \quad \text{vph}=0$
 $m=5764805 \quad \rho=4 \quad \text{vptor}=3 \quad C_p=0.749999665709531099157508 \quad \text{vph}=0$
 $m=282475253 \quad \rho=5 \quad \text{vptor}=4 \quad C_p=0.799999994178340620762893 \quad \text{vph}=0$
 $m=13841287205 \quad \rho=6 \quad \text{vptor}=6 \quad C_p=0.999999999876240233338026 \quad \text{vph}=1$
 $m=678223072853 \quad \rho=7 \quad \text{vptor}=7 \quad C_p=0.99999999997835106122198 \quad \text{vph}=1$
 $m=33232930569605 \quad \rho=8 \quad \text{vptor}=7 \quad C_p=0.87499999999966173533159 \quad \text{vph}=0$
 $m=1628413597910453 \quad \rho=9 \quad \text{vptor}=8 \quad C_p=0.88888888888888265516646 \quad \text{vph}=0$
 $m=79792266297612005 \quad \rho=10 \quad \text{vptor}=9 \quad C_p=0.89999999999999988407184 \quad \text{vph}=0$
 $m=3909821048582988053 \quad \rho=11 \quad \text{vptor}=10 \quad C_p=0.90909090909090908873656 \quad \text{vph}=0$

So we shall fix ρ large enough, and increase a in some interval and write $a \cdot (2 + p^{2\rho} \cdot a)$ (resp. $a \cdot (1 + p^{2\rho} \cdot a)$) under the form $m \cdot b^2$, m squarefree. We then compute the successive minima of D_K for $K = \mathbb{Q}(\sqrt{m})$ to try to get maximal values of $C_p(K)$:

```
{p=3;rho=21;n=rho+6;ba=10^8+1;Ba=2*10^8;pp=p^(2*rho);Dmin=10^100;d=2;
if (p==2,d=1);for (a=ba,Ba,B=a*(d+pp*a);m=core(B);D=m;if (Mod(m,4)!=1,D=4*m);
if (D<Dmin,Dmin=D;b=component(core(B,1),2);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(sqrt(D));
h=component(component(component(K,8),1),1);vph=valuation(h,p);
print("D=",D," a=",a," b=",b," vptor=",vptor," vph=",vph," Cp=",Cp))}
```

Since the initial value of a has some importance, one may replace the lower bound of the interval by $random(10^8)$ for instance. We have done a great lot of experimentations with very large discriminants without obtaining any $C_p(K) > 2$, except, for $p = 2$, the non-characteristic:

```
D=81624 a=9728 b=557872 vptor=20 vph=3 Cp=2.45147522
```

which is known (see § 4.2.1) and corresponds to a too small discriminant since the stabilisation of $C_p(K)$ seems better and better as soon as $D_K \gg 0$. Let $a \in [10^8 + 1, 2 \cdot 10^8]$:

```
p=3, rho=21
```

D	a	b	vptor	vph	Cp
4376759652795686111245843894049436844	100000001	1	22	2	0.5729
1094189935082719682370900209849436840	100000002	2	21	0	0.5560
6474496916274063005939132968034008	100000004	26	21	1	0.5926
844282458630797226331079592122677	100000008	36	22	0	0.6375
243749276912797619302002823856008	100000028	134	20	0	0.5892
229823745894232258887149684378792	100000044	138	21	0	0.6192
118727690905177855319225723684396	100000206	192	21	0	0.6247
31970646085359206852554564577080	100000250	370	21	1	0.6360
4248380707363826430616087080556	100000325	1015	20	0	0.6231
1265114522670559167676904722168	100000350	1860	21	0	0.6657
656519580808913341291735184520	100000860	2582	21	0	0.6720
418229777412778226609667972540	100001247	3235	21	0	0.6765
372030701452432299153951624968	100001650	3430	20	0	0.6454
13706281694543771696481156792	100001850	17870	21	0	0.7122
8423316693760059792026840684	100005496	22796	20	0	0.6834
95797673387147609199079608	100006350	213760	21	0	0.7712
38174378342984600017332360	100020480	338672	21	0	0.7833
780348725011642441673212	100250343	2374203	21	0	0.8387

Remarkably, there is no solution $a \in [10^8 + 250343, 2 * 10^8]$ giving smaller discriminants.

```
p=2, rho=30, n=2*rho
```

D	a	b	vptor	vph	Cp
11529215276652771834290899906846977	100000001	1	35	5	0.6186
17055053207700727651215465398745	100000004	26	42	11	0.8096
1045734313893579444006826247561	100000025	105	35	5	0.7019
219851764946410623412729649352	100000136	458	41	10	0.8412
13770844779037795467289609685	100000349	915	33	3	0.7060
10261042435006502090152345689	100000400	1060	36	4	0.7737
669437189125078856962747773	100000724	4150	36	5	0.8079
314916414589340809658878597	100005741	6051	35	5	0.7952
117056047025426780994615381	100006349	9925	35	5	0.8083
115315210043698339146366977	100020001	10001	31	2	0.7161
10110802585632544739300412	100023296	67552	43	8	1.0353
2465588264382971658630552	100054616	136838	35	4	0.8638
1175830296704323573471548	100097399	198235	36	6	0.9004
48025975228418415280613	100175668	490822	37	6	0.9821
28578131029527067857561	100311617	637139	34	4	0.9115
617974038061148975453	100469200	4339580	36	4	1.0424

Same remarks as for the case $p = 3$; despite genera theory, it seems that the $C_p(K)$ remains close to 1 and is not increasing substantially as $D_K \rightarrow \infty$.

5. NUMERICAL INVESTIGATIONS FOR CYCLIC CUBIC FIELDS

For the computations with cubic fields, we shall use the direct calculation of $\#\mathcal{T}_K$ from the program testing the p -rationality, taking n large enough.

See [21] for statistics on $v_p(R_{K,p}) = v_p(\mathcal{R}_K) + 2$ (resp. $v_p(\mathcal{R}_K) + 1$) in the non-ramified (resp. ramified) case for cyclic cubic fields of conductors up to 10^8 ; this gives, for cubic fields, the analogue of the computation of $\delta_p(\varepsilon)$ for quadratic fields in §4.1. Note that, due to Galois action, the integers $v_p(\#\mathcal{T}_K)$ are even if $p \equiv 1 \pmod{3}$ and arbitrary if not.

5.1. Maximal values of $v_p(\#\mathcal{T}_K)$. The program uses the well-known classification of cyclic cubic fields (from [7]) with conductor $f_K \leq Bf$, and processes as for the quadratic case. We give first the case $p = 3$ to see the influence of genera theory; we compute the successive maxima of $v_p(\#\mathcal{T}_K)$ (in *vptor*) with the corresponding f_K and the polynomial defining the field of conductor f_K . We print the maximal value obtained for $C_p(K)$.

Recall that $D_K = f_K^2$, where $f_k = f'_K$ or $f_K = 9 \cdot f'_K$ with $f'_K = \ell_1 \cdots \ell_t$, for primes $\ell_i \equiv 1 \pmod{3}$:

```
{p=3;n=20;bf=7;Bf=10^7;Max=0;for(f=bf,Bf,e=valuation(f,3);if(e!=0 & e!=2,next);
F=f/3^e;if(Mod(F,3)!=1 || core(F)!=F,next);F=factor(F);Div=component(F,1);
d=component(matsize(F),1);for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(e==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(f);
if(vptor>Max,Max=vptor;print("f=",f," vptor=", vptor," ",P," Cp=",Cp))}}}
```

```
p=3
f=19      vptor=1      P=x^3 + x^2 - 6*x - 7
f=199     vptor=2      P=x^3 + x^2 - 66*x + 59
f=427     vptor=4      P=x^3 + x^2 - 142*x - 680
f=1843    vptor=5      P=x^3 + x^2 - 614*x + 3413
f=2653    vptor=6      P=x^3 + x^2 - 884*x - 8352
f=17353   vptor=7      P=x^3 + x^2 - 5784*x - 145251
f=30121   vptor=8      P=x^3 + x^2 - 10040*x + 306788
f=114079  vptor=9      P=x^3 + x^2 - 38026*x + 2822399
f=126369  vptor=10     P=x^3 - 42123*x + 3046897
f=355849  vptor=11     P=x^3 + x^2 - 118616*x - 15235609
f=371917  vptor=12     P=x^3 + x^2 - 123972*x + 15854684
f=1687987 vptor=15     P=x^3 + x^2 - 562662*x - 116533621   Cp=1.1492

p=2, n=36
f=31      vptor=2      P=x^3 + x^2 - 10*x - 8
f=171     vptor=6      P=x^3 - 57*x - 152
f=2689    vptor=8      P=x^3 + x^2 - 896*x + 5876
f=6013    vptor=12     P=x^3 + x^2 - 2004*x - 32292
f=6913    vptor=13     P=x^3 + x^2 - 2304*x - 256
f=311023  vptor=16     P=x^3 + x^2 - 103674*x + 5068523
f=544453  vptor=18     P=x^3 + x^2 - 181484*x - 19862452
f=618093  vptor=24     P=x^3 - 206031*x + 21289870          Cp=1.2475

p=7
f=9       vptor=1      P=x^3 - 3*x + 1
f=313    vptor=2      P=x^3 + x^2 - 104*x + 371
f=721    vptor=3      P=x^3 + x^2 - 240*x - 988
f=1381   vptor=4      P=x^3 + x^2 - 460*x - 1739
f=29467  vptor=6      P=x^3 + x^2 - 9822*x - 20736
f=177541 vptor=7      P=x^3 + x^2 - 59180*x + 3051075
```

f=1136587 vptor=10 P=x^3 + x^2 - 378862*x + 58428991 Cp=1.3955

5.2. Experiments for a conjectural upper bound – Cubic fields. In the same way as for quadratic fields, we give, for each prime p , the successive minima of the difference $\Delta_p(K) = \frac{\log_\infty(f_K)}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$ and the value of $C_p(K) = \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(f_K)}$, obtained for some polynomial P of conductor f :

```
{n=12;bf=7;Bf=5*10^6;forprime(p=2,50,ymin=10;print("p="p);for(f=bf,Bf,
e=valuation(f,3);if(e!=0 & e!=2,next);F=f/3^e;if(Mod(F,3)!=1 || core(F)!=F,next);
F=factor(F);Div=component(F,1);d=component(matsize(F),1);
for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(e==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(f)/log(p)-vptor;if(Y<ymin,ymin=Y;print(P);Cp=vptor*log(p)/log(f);
print("f=",f," ",vptor=" ",vptor," Ymin=",Y," Cp=",Cp))))}
```

The first minimum occurs for $f = 7$ and $vptor = 0$; we omit the cases of p -rationality. For $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ we have been obliged to consider larger conductors f to get significant solutions, especially for $p = 11$ for which the first non-trivial example is for $f = 5000059$ and $P = x^3 + x^2 - 1666686x - 408523339$.

For convenience, we print, with each p , the final value of $C_p(K)$ obtained for the tested fields K , then a polynomial P , its conductor f , $vptor = v_p(\#\mathcal{T}_K)$, $Ymin$, Cp .

```
p=2, Cp=1.247565
P=x^3 - 57*x - 152
f=171 vptor=6 Ymin=1.41785251... Cp=0.8088
P=x^3 + x^2 - 2004*x - 32292
f=6013 vptor=12 Ymin=0.55386924... Cp=0.9559
P=x^3 + x^2 - 2304*x - 256
f=6913 vptor=14 Ymin=-1.24490378... Cp=1.0976
P=x^3 - 206031*x + 21289870
f=618093 vptor=24 Ymin=-4.76253559... Cp=1.2475
p=3, Cp=1.149252
P=x^3 + x^2 - 6*x - 7
f=19 vptor=1 Ymin=1.68014385... Cp=0.3731
P=x^3 + x^2 - 142*x - 680
f=427 vptor=4 Ymin=1.51312239... Cp=0.7255
P=x^3 + x^2 - 884*x - 8352
f=2653 vptor=6 Ymin=1.17582211... Cp=0.8361
P=x^3 - 42123*x + 3046897
f=126369 vptor=10 Ymin=0.69254513... Cp=0.9352
P=x^3 + x^2 - 118616*x - 15235609
f=355849 vptor=11 Ymin=0.63491606... Cp=0.9454
P=x^3 + x^2 - 123972*x + 15854684
f=371917 vptor=12 Ymin=-0.32488392... Cp=1.0278
P=x^3 + x^2 - 562662*x - 116533621
f=1687987 vptor=15 Ymin=-1.94803671... Cp=1.1492
p=5, Cp=1.462906
P=x^3 + x^2 - 50*x - 123
f=151 vptor=2 Ymin=1.11741123... Cp=0.6415
P=x^3 + x^2 - 1002*x + 6905
f=3007 vptor=4 Ymin=0.97608396... Cp=0.8038
P=x^3 + x^2 - 2214*x + 19683
f=6643 vptor=8 Ymin=-2.53143306... Cp=1.4629
p=7, Cp=1.395563
P=x^3 - 3*x + 1
```

f=9	vptor=1	Ymin=0.12915006...	Cp=0.8856
P=x ³ + x ² - 460*x - 1739			
f=1381	vptor=4	Ymin=-0.28422558...	Cp=1.0765
P=x ³ + x ² - 9822*x - 20736			
f=29467	vptor=6	Ymin=-0.71145865...	Cp=1.1345
P=x ³ + x ² - 59180*x + 3051075			
f=177541	vptor=7	Ymin=-0.78853291...	Cp=1.1269
P=x ³ + x ² - 378862*x + 58428991			
f=1136587	vptor=10	Ymin=-2.83443766...	Cp=1.3955
p=11, Cp=0.621490			
P=x ³ + x ² - 1666686*x - 408523339			
f=5000059	vptor=2	Ymin=4.43270806...	Cp=0.3109
P=x ³ - 1680483*x - 503584739			
f=5041449	vptor=4	Ymin=2.43614601...	Cp=0.6215
p=13, Cp=1.632521			
P=x ³ + x ² - 20*x - 9			
f=61	vptor=1	Ymin=0.60271151...	Cp=0.6239
P=x ³ + x ² - 196*x - 349			
f=589	vptor=2	Ymin=0.48676495...	Cp=0.8042
P=x ³ + x ² - 1064*x + 12299			
f=3193	vptor=3	Ymin=0.14576042...	Cp=0.9536
P=x ³ + x ² - 1824*x + 8919			
f=5473	vptor=4	Ymin=-0.64415121...	Cp=1.1919
P=x ³ + x ² - 19920*x + 615317			
f=59761	vptor=7	Ymin=-2.71215372...	Cp=1.6325
p=17, Cp=0.91048147			
P=x ³ - 399*x - 3059			
f=1197	vptor=2	Ymin=0.50160254...	Cp=0.7994
P=x ³ - 84837*x + 1046323			
f=254511	vptor=4	Ymin=0.39327993...	Cp=0.9105
p=19, Cp=0.97446367			
P=x ³ + x ² - 30*x + 27			
f=91	vptor=1	Ymin=0.53199286...	Cp=0.6527
P=x ³ + x ² - 404*x + 629			
f=1213	vptor=2	Ymin=0.41161455...	Cp=0.8293
P=x ³ - 3477*x - 26657			
f=10431	vptor=3	Ymin=0.14237703...	Cp=0.9547
P=x ³ + x ² - 1213944*x - 503921781			
f=3641833	vptor=5	Ymin=0.13102760...	Cp=0.9744
p=23, Cp=0.88008721			
P=x ³ + x ² - 1060*x - 11428			
f=3181	vptor=2	Ymin=0.57214663...	Cp=0.7775
P=x ³ + x ² - 515154*x - 19633104			
f=1545463	vptor=4	Ymin=0.54500411...	Cp=0.8801
p=29, Cp=1.569666			
P=x ³ + x ² - 24*x - 27			
f=73	vptor=2	Ymin=-0.72584422...	Cp=1.5696
p=31, Cp=0.981745			
P=x ³ + x ² - 30*x + 27			
f=91	vptor=1	Ymin=0.31359240...	Cp=0.7613
P=x ³ - 12027*x + 388873			
f=36081	vptor=3	Ymin=0.05578357...	Cp=0.9817
p=37, Cp=1.119764			
P=x ³ - 39*x - 26			
f=117	vptor=1	Ymin=0.31882641...	Cp=0.7582
P=x ³ + x ² - 5300*x + 119552			
f=15901	vptor=3	Ymin=-0.32086480...	Cp=1.1197
p=41, Cp=0.976052			
P=x ³ + x ² - 672*x - 2764			
f=2017	vptor=2	Ymin=0.04906930...	Cp=0.9760

```

p=43, Cp=0.914939
P=x^3 + x^2 - 20*x - 9
f=61      vptor=1      Ymin=0.09296866...   Cp=0.9149
p=47, Cp=0.878952
P=x^3 + x^2 - 2126*x + 11813
f=6379    vptor=2      Ymin=0.27543656...   Cp=0.8789

```

6. EXAMPLES OF NON-GALOIS TOTALLY REAL NUMBER FIELDS

We shall consider (non necessarily Galois) cubic fields, with an approach using randomness. The tested polynomials of degree 3 define almost always Galois groups S_3 . It is more difficult to find non- p -rational fields for large p and to obtain an estimation of C_p for the families of cubic fields.

6.1. Program for a given cubic polynomial and increasing p . The program concerns fields K defined by $P = x^3 + ax^2 + bx + 1$ for random a, b , and increasing p in $[2, 10^5]$. It tests the irreducibility of P and that $D_K > 0$ (real roots). We give only the non- p -rational cases for which one prints only the corresponding $C_p(K)$.

```

{n=8;bm=2;Bm=10^5;ymin=10;a=random(100);b=random(100);P=x^3+a*x^2+b*x+1;
if(polisirreducible(P)==1 & poldisc(P)>0,print(P);K=bnfinit(P,1);
D=component(component(K,7),3);forprime(p=bm,Bm,Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;
if(vptor > 0 & Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D));
print("p=",p," ", "vptor=",vptor," Ymin=",Y," Cp=",Cp))}}

```

We obtain, after several tries and p up to 10^5 , omitting the small values of p and $C_p(K)$:

```

P=x^3 + 21*x^2 + 47*x + 1
p=11      vptor=1      Ymin=1.75210757...
p=523     vptor=1      Ymin=0.05426629...
p=3517    vptor=1      Ymin=-0.19179768... Cp=1.2373
p=173483  vptor=1      Ymin =-0.45297114... Cp=1.8280
P=x^3 + 19*x^2 + 51*x + 1
p=487     vptor=1      Ymin=-0.40614414... Cp=1.6839
P=x^3 + 92*x^2 + 52*x + 1
p=18637   vptor=1      Ymin=-0.14697706... Cp=1.1723
P=x^3 + 99*x^2 + 23*x + 1
p=73      vptor=1      Ymin=0.47867182...
p=15803   vptor=1      Ymin=-0.34379282... Cp=1.5239
p=145259  vptor=1      Ymin =-0.46625984... Cp=1.8735
p=622519  vptor=1      Ymin =-0.52447869... Cp=2.1029
P=x^3 + 98*x^2 + 62*x + 1
p=3       vptor=2      Ymin=3.86940839...
p=61     vptor=1      Ymin=0.56857262...
p=37549  vptor=1      Ymin=-0.38783270... Cp=1.63354
P=x^3 + 87*x^2 + 74*x + 1
p=5441   vptor=1      Ymin=0.01344518... Cp=0.9867
P=x^3 + 73*x^2 + 67*x + 1
p=6133   vptor=1      Ymin=-0.03273397... Cp=1.0338
P=x^3 + 19*x^2 + 83*x + 1
p=61     vptor=1      Ymin=-0.52664318...
p=5419   vptor=1      Ymin=-0.77366996...
p=12703  vptor=1      Ymin=-0.79407472... Cp=4.8561

```

Note that by accident, $P = x^3 + 19x^2 + 83x + 1$, with a large $C_{12703}(K) \approx 4.8561$, defines the cyclic cubic field K of conductor 7 (in some sense, an analogue of $K = \mathbb{Q}(\sqrt{19})$ with $p = 13599893$ for which $C_p(K) \approx 7.5856$).

But the forthcoming conductors $f > 7$, up to $4 \cdot 10^6$, give decreasing $C_p(K)$, as shown by the following excerpts, where no $vptor \geq 2$ were found:

$P=x^3 + 784x^2 + 964x + 288$	$vptor=1$	$Ymin=3.97483926\dots$
$P=x^3 + 505x^2 + 710x + 134$	$vptor=2$	$Ymin=2.57552488\dots$
$P=x^3 + 73x^2 + 492x + 196$	$vptor=3$	$Ymin=1.85163167\dots$
$P=x^3 + 57x^2 + 695x + 263$	$vptor=1$	$Ymin=1.35093638\dots$
$P=x^3 + 95x^2 + 839x + 252$	$vptor=5$	$Ymin=0.64570147\dots$
$P=x^3 + 114x^2 + 804x + 142$	$vptor=2$	$Ymin=-0.37221306\dots$
$P=x^3 + 97x^2 + 829x + 122$	$vptor=5$	$Ymin=-0.83742084\dots$
$Cp=1.201178$		
$p=19$		
$P=x^3 + 50x^2 + 631x + 470$	$vptor=1$	$Ymin=1.58556226\dots$
$P=x^3 + 57x^2 + 777x + 801$	$vptor=1$	$Ymin=1.54028119\dots$
$P=x^3 + 549x^2 + 732x + 39$	$vptor=3$	$Ymin=0.69038895\dots$
$P=x^3 + 93x^2 + 891x + 383$	$vptor=2$	$Ymin=0.64611301\dots$
$P=x^3 + 123x^2 + 375x + 217$	$vptor=1$	$Ymin=0.46422353\dots$
$P=x^3 + 226x^2 + 777x + 408$	$vptor=2$	$Ymin=0.20875475\dots$
$P=x^3 + 196x^2 + 849x + 918$	$vptor=2$	$Ymin=-0.24470848\dots$
$Cp=1.139412$		
$p=1009$		
$P=x^3 + 171x^2 + 667x + 604$	$vptor=1$	$Ymin=0.49598190\dots$
$P=x^3 + 89x^2 + 567x + 36$	$vptor=1$	$Ymin=0.37961552\dots$
$P=x^3 + 54x^2 + 435x + 719$	$vptor=1$	$Ymin=0.29433117\dots$
$P=x^3 + 93x^2 + 636x + 944$	$vptor=1$	$Ymin=0.07490160\dots$
$P=x^3 + 432x^2 + 347x + 19$	$vptor=1$	$Ymin=-0.06432442\dots$
$P=x^3 + 130x^2 + 942x + 899$	$vptor=1$	$Ymin=-0.06692434\dots$
$P=x^3 + 70x^2 + 553x + 735$	$vptor=1$	$Ymin=-0.18534377\dots$
$Cp=1.227512$		

Remarks 6.1. (i) The case $p = 2$ with $P = x^3 + 30x^2 + 165x + 220$, where:

$$v_2(\#\mathcal{T}_K) = 12 \quad \& \quad \Delta_2(K) = \log(\sqrt{D_K})/\log(2) - 12 \approx -3.98595,$$

seems pathological, but the discriminant is rather small ($D_K = 66825$ and the algebraic closure of K contains $\mathbb{Q}(\sqrt{33})$). The Galois closure L of K is defined by the polynomial $Q = x^6 - 60x^5 + 1131x^4 - 6380x^3 - 15708x^2 + 145200x + 170368$; then $v_2(\#\mathcal{T}_L) = 25$, giving $C_2(L) \approx 1.347687$ instead of $C_2(K) \approx 1.497370$.

(ii) For $p = 5$ and $P = x^3 + 197x^2 + 718x + 508$, $v_5(\#\mathcal{T}_K) = 8$ is exceptionally large, but the discriminant is also large: $D_K = 1069350637 = 769 \cdot 1390573$ and $C_5(K) \approx 1.238605$.

(iii) For $p = 7$, $P = x^3 + 95x^2 + 839x + 252$, $v_7(\#\mathcal{T}_K) = 5$, with $C_7(K) \approx 0.885629$, but $D_K = 3486121421$, while for $P = x^3 + 114x^2 + 804x + 142$, $v_7(\#\mathcal{T}_K) = 2$ with $C_7(K) \approx 1.228662$, but $D_K = 564$.

(iv) We have computed $C_p(L)$ for the Galois closure L of the above fields K (Galois group D_6). The values $C_p(L)$ are smaller, although the $v_p(\#\mathcal{T}_L)$ are roughly speaking twice of $v_p(\#\mathcal{T}_K)$. For instance, for $p = 2$, $P = x^3 + 30x^2 + 165x + 220$, $v_2(\#\mathcal{T}_K) = 12$, $C_2(K) = 1.497370$ but $v_2(\#\mathcal{T}_L) = 25$ and $C_2(L) = 1.347687$.

7. CONJECTURE ON $v_p(\#\mathcal{T}_K)$

7.1. p -adic statement. The numerical results (quadratic and cubic cases, with the particular family of quadratic fields studied in §§ 4.2, 4.3, 4.4) suggest the following conjecture that we state in its strongest form; we shall discuss about some conditions of application of such a conjecture, for instance assuming that the fields K are of given degree or are elements of specified families³:

Conjecture 7.1. *Let K be any totally real number field (or any element of a specified family \mathcal{K}), and let $p \geq 2$ be a prime number.*

³As in [35], we call *family of number fields*, \mathcal{K} , any infinite set of non-isomorphic number fields K . Thus, the condition $D_K \rightarrow \infty$ makes sense in \mathcal{K} . We restrict ourselves to totally real fields.

(i) There exists a constant C_p (or $C_p(\mathcal{K})$), independent of K (or of $K \in \mathcal{K}$), such that:

$$v_p(\#\mathcal{T}_K) \leq C_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}, \text{ for all } K \text{ (or } K \in \mathcal{K}\text{)}.$$

(ii) The residue $\tilde{\kappa}_{K,p}$, at $s = 1$, of the normalized p -adic ζ -function $\frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \zeta_{K,p}(s)$ of K (see § 2.2), is conjecturally such that:

$$v_p(\tilde{\kappa}_{K,p}) \leq C_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}, \text{ for all } K \text{ (or } K \in \mathcal{K}\text{)}.$$

Theorem 7.2. Let K be a totally real number field and let \mathcal{K} be the set of subfields K_n of the p -cyclotomic tower K^c of K (with $[K_n : K] = p^n$, for all $n \geq 0$). Then, under the Leopoldt conjecture in K^c , $C_p(K_n) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. From [37, § 3, Proposition 2], we get $\sqrt{D_{K_n}} \geq p^{\alpha \cdot n \cdot p^n + O(p^n)}$ with $\alpha > 0$; then from Iwasawa's theory, there exist $\lambda, \mu \in \mathbb{N}$ and $\nu \in \mathbb{Z}$ such that $\#\mathcal{T}_{K_n} = p^{\lambda n + \mu p^n + \nu}$. So we obtain, for $n \gg 0$, $C_p(K_n) \leq \frac{\lambda n + \mu p^n + \nu}{\alpha \cdot n \cdot p^n + O(p^n)}$, where the limit of the upper bound is 0; whence the result giving an example for which the conjecture is fulfilled. \square

Remark 7.3. In [20], Hajir and Maire define, in the spirit of an algebraic p -adic Brauer–Siegel theorem, the *logarithmic mean exponent* of a finite p -group $A \simeq \prod_{i=1}^r \mathbb{Z}/p^{a_i}\mathbb{Z}$, by

means of the formula $\mathbb{M}_p(A) := \frac{1}{r} \cdot \frac{\log_\infty(\#A)}{\log_\infty(p)} = \frac{1}{r} \sum_{i=1}^r a_i = \frac{1}{r} \cdot v_p(\#A)$, and applied to tame generalized class groups. In the case of \mathcal{T}_K , we get $v_p(\#\mathcal{T}_K) = \text{rk}_p(\mathcal{T}_K) \cdot \mathbb{M}_p(\mathcal{T}_K)$, and we would have conjecturally, for totally real fields:

$$\mathbb{M}_p(\mathcal{T}_K) \leq C_p \cdot \frac{1}{\text{rk}_p(\mathcal{T}_K)} \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} \leq C_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}.$$

But in [20, Theorems 0.1, 1.1, Proposition 2.2], this function \mathbb{M}_p is essentially used for class groups in particular infinite towers with tame restricted ramification for which some explicit upper bounds are obtained.

7.2. Comparison « archimedean » versus « p -adic ». The above considerations are, in some sense, a p -adic approach of some deep and very general results (Brauer–Siegel–Tsfasman–Vladuț theorems [35, 40] and broad generalizations in [34]; see [23] for quantitative bounds from the Brauer–Siegel theorem) on the behavior, in a tower $L := \bigcup_{n \geq 0} K_n$ of finite extensions K_n/K , of the quotient $BS_{K_n} := \frac{\log_\infty(h_{K_n} \cdot R_{K_n})}{\log_\infty(\sqrt{D_{K_n}})}$.

Of course, in order to infer the p -adic case, our purpose is to deal, in the archimedean one, with any totally real number field K or with families \mathcal{K} fulfilling some conditions (e.g., $[K : \mathbb{Q}] = d$, $\frac{[K : \mathbb{Q}]}{\log_\infty(\sqrt{D_n})} \rightarrow 0$), which is possible thanks to [40, Theorem 1].

For any totally real field K , let:

$$BS_K := \frac{\log_\infty(h_K \cdot R_{K,\infty})}{\log_\infty(\sqrt{D_K})}.$$

We shall also consider the normalized quotient (which does not matter for the existence of lower and upper bounds since $\widetilde{BS}_K = BS_K - 1$):

$$(7.1) \quad \widetilde{BS}_K := \frac{\log_\infty\left(h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}}\right)}{\log_\infty(\sqrt{D_K})} = \frac{\log_\infty(\mathcal{T}_{K,p,\infty})}{\log_\infty(\sqrt{D_K})}, \quad K \in \mathcal{K} \text{ (from formula (2.2))},$$

and presume that this function is bounded over \mathcal{K} . When the degree is constant in the family, the classical Brauer–Siegel theorem applies since $\frac{[K:\mathbb{Q}]}{\log_\infty(\sqrt{D_n})} \rightarrow 0$.

The following program gives, for the family \mathcal{K}_0 of real quadratic fields, consistent verifications for the original function BS :

```
{Max=0;Min=1;for(D=10^8,10^8+10^6,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1||e>3)|| (e==0&Mod(M,4)!=1)|| (e==2&Mod(M,4)==1),next);P=x^2-D;K=bnfinit(P,1);
C8=component(K,8);h=component(component(C8,1),1);reg=component(C8,2);
BS=log(h*reg)/log(sqrt(D));if(BS<Min,Min=BS;print(D," ",Min," ",Max));
if(BS>Max,Max=BS;print(D," ",Min," ",Max))}
```

$0.647 < BS < 1.155$ for $m \in [10^5, 2 \cdot 10^5]$, $0.734 < BS < 1.136$ for $m \in [10^7, 10^7 + 10^5]$, $0.7657 < BS < 1.1239$ for $m \in [10^8, 10^8 + 10^5]$, and $0.75738 < BS < 1.12713$ for $m \in [10^8, 10^8 + 10^6]$ (two days of computer), showing: $BS_K = 1 + o(1)$.

In the same way for cyclic cubic fields, one obtains:

$0.6653 \leq BS \leq 1.1478$ for $f \in [10^4, 10^6]$, $0.7547 \leq BS \leq 1.1385$ for $f \in [10^6, 2 \cdot 10^6]$.

Remark 7.4. In the archimedean point of view, we have put $\#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}}$, giving, from the expression (7.1) of \widetilde{BS}_K :

$$\log_\infty(\#\mathcal{T}_{K,p_\infty}) = \widetilde{BS}_K \cdot \log_\infty(\sqrt{D_K});$$

thus we obtain about the above calculations for examples of fixed family \mathcal{K} :

$$\log_\infty(\#\mathcal{T}_{K,p_\infty}) \leq O(1) \cdot \log_\infty(\sqrt{D_K}) \quad \text{written} \quad \log_\infty(\#\mathcal{T}_{K,p_\infty}) \leq C_{p_\infty} \cdot \log_\infty(\sqrt{D_K}),$$

giving, in some sense, the inequality of the p -adic Conjecture 7.1 with the audacious convention for the *infinite place* p_∞ :

$$\log_\infty(p_\infty) = 1 \quad \& \quad v_{p_\infty}(\#\mathcal{T}_{K,p_\infty}) = \log_\infty(\#\mathcal{T}_{K,p_\infty}).$$

in which case, the constant C_{p_∞} exists and is the maximal value reached by $\widetilde{BS}_K = BS_K - 1$ over the given family \mathcal{K} .

8. CONCLUSIONS

The analysis of the archimedean case is sufficiently significant to hope the relevance of the p -adic one for which we give some observations despite the lack of proofs:

(a) We note that in the p -adic Conjecture 7.1, the most important term is the valuation $v_p(\mathcal{R}_{K,p})$ of the normalized p -adic regulator, the contribution of $v_p(\#\mathcal{C}_{K,p})$ being probably negligible compared to $v_p(\#\mathcal{R}_{K,p})$ as shown, among other, by classical heuristics [3, 4], giving for quadratic fields, $\text{Prob}(p^v \parallel h_K) = \prod_{\alpha \leq v} \left(1 - \frac{(p)_\infty}{(p)_1}\right) \sum_{0 \leq \beta \leq \alpha} \frac{1}{p^{2\beta} \cdot (p)_\beta}$, where $(p)_\gamma = \prod_{1 \leq k \leq \gamma} \left(1 - \frac{1}{p^k}\right)$, $\gamma \in \mathbb{N} \cup \{\infty\}$; this gives for instance 0.019779 for the divisibility by $p^v = 9$; which is clear from our numerical calculations. This is reinforced by the more recent conjectures given in the Remark 4.3 (ii). Furthermore, for K fixed, $v_p(\#\mathcal{C}_{K,p}) \geq 1$ for finitely many primes p , but the case of $v_p(\#\mathcal{R}_{K,p})$ is only an out of reach conjecture.

(b) The family of § 4.3 shows that p -adic regulators may tend to 0, even in simplest cases, and it should be necessary to find other special families of units with parameters depending of arbitrary large p -powers p^ρ , $\rho \in \mathbb{N}$, to precise the relation between $v_p(\#\mathcal{R}_{K,p})$ and $\log_\infty(\sqrt{D_K})$, $K \in \mathcal{K}$. It would be of great interest to know if, for each fixed p , there exist critical sub-families of fields $(K_\rho)_{\rho \in \mathbb{N}} \subset \mathcal{K}$ such that $v_p(\#\mathcal{R}_{K_\rho,p}) \rightarrow \infty$ as $\rho \rightarrow \infty$. This will of course depend on \mathcal{K} , but one may choose the families of real extensions with given Galois group G .

(c) Consider (for any $p \geq 2$ and any totally real number field K):

$$C_p(K) := \frac{v_p(\#\mathcal{T}_{K,p}) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}, \quad \mathcal{C}(p) := \sup_K(C_p(K)), \quad \mathcal{C}(K) := \sup_p(C_p(K)).$$

The existence of $\mathcal{C}(K) < \infty$, for a given K , only says that the conjecture [10, Conjecture 8.11], claiming that any number field is p -rational for $p \gg 0$, is true for the field K .

If $\mathcal{C}(p)$ does exist for a given p , we have a p -adic analog of Brauer–Siegel theorem (Conjecture 7.1).

The existence of $\mathcal{C}(K)$, is more credible by reference to our conjecture, but that of $\mathcal{C}(p)$ may be exact taking instead $\sup_{K \in \mathcal{K}}$, for particular families \mathcal{K} (e.g., extensions of fixed degree or subfields of some infinite towers as in [20, 22, 35, 40]); but we must mention that for the invariants $\mathcal{T}_{K,p}$, the transfer map $\mathcal{T}_{K,p} \rightarrow \mathcal{T}_{L,p}$ is injective in any extension L/K (in which Leopoldt’s conjecture is assumed, [9, Theorem IV.2.1]), which leads to a major difference from the case of p -class groups.

Furthermore, it seems that $\limsup_{K \in \mathcal{K}}(C_p(K))$ may be ≤ 1 for any fixed p and that we may have $\limsup_p(C_p(K)) < \infty$ and possibly equal to 0 for any fixed K (see Remarks 4.3); but computations for very large discriminants (of a great lot of quadratic fields for instance) is out of reach.

(d) In another direction, the difficult Greenberg’s conjecture [15], on the triviality of the Iwasawa invariants λ, μ for K^c , in the totally real case, goes in the sense of rarity for p -class groups as we have mention in Remark 4.3 (i), and this conjecture also depends on Fermat’s quotients of algebraic numbers ([11, § 7.7], [14, § 4.2]). Similarly, some conjectures of Greenberg in [16] depend on the existence of some p -rational fields with given Galois groups,

But all this is far to be proved because of a terrible lack of knowledge of p -Fermat quotients of algebraic numbers, a notion which gives a *weaker information* than the p -adic logarithm, but which governs many deep arithmetical problems, even assuming the Leopoldt conjecture which appears as a rough step in the study of $\text{Gal}(H_K^{\text{pr}}/K)$; indeed, if Leopoldt’s conjecture is not fulfilled, there exists a sequence $\varepsilon_i \in E_K$, $\varepsilon_i \notin E_K^p$, such that $\delta_p(\varepsilon_i) \rightarrow \infty$ with i , which shows the extreme uncertainty about the \mathcal{T}_p groups.

Acknowledgments. I thank Christian Maire for many discussions about some aspects of Brauer–Siegel–Tsfasman–Vladuț theorems, and Stéphane Louboutin for information and references on complex ζ -functions.

REFERENCES

- [1] Y. Amice et J. Fresnel, *Fonctions zêta p -adiques des corps de nombres abéliens réels*, Acta Arithmetica **20** (1972), no. 4, 353–384. <http://matwbn.icm.edu.pl/ksiazki/aa/aa20/aa2043.pdf>
- [2] D. Byeon, *Indivisibility of class numbers and Iwasawa λ -invariants of real quadratic fields*, Compositio Mathematica **126** (2001), no. 3, 249–256. http://www.math.snu.ac.kr/~dhbyeon/11_byeon-compositio.pdf
- [3] H. Cohen and H.W. Lenstra, *Heuristics on class groups of number fields*, In: H. Jager (eds), Number Theory Noordwijkerhout 1983, Lecture Notes in Mathematics, vol. 1068, Springer, Berlin, Heidelberg (1984), 33–62. <https://link.springer.com/chapter/10.1007/BFb0099440>
- [4] H. Cohen and J. Martinet, *Class groups of number fields: Numerical heuristics*, Math. Comp. **48**(177) (1987), 123–137. <http://www.ams.org/journals/mcom/1987-48-177/S0025-5718-1987-0866103-4/>
- [5] J. Coates, *p -adic L -functions and Iwasawa’s theory*, Algebraic number fields: L -functions and Galois properties (Proc. Sympos., Univ. Durham (1975)), Academic Press, London (1977), 269–353.
- [6] P. Colmez, *Résidu en $s = 1$ des fonctions zêta p -adiques*, Invent. Math. **91** (1988), 371–389. <http://gdz.sub.uni-goettingen.de/dms/load/img/?PID=GDZPPN002104911>
- [7] V. Ennola and R. Turunen – *On Cyclic Cubic Fields*, Math. of Computation **45** (1985), no. 172, 585–589. <http://www.ams.org/journals/mcom/1985-45-172/S0025-5718-1985-0804947-3/>
- [8] J. S. Ellenberg and A. Venkatesh, *Reflection principles and bounds for class group torsion*, Int. Math. Res. Not. IMRN (1) (2007). <http://math.stanford.edu/~akshay/research/sch.pdf>

- [9] G. Gras, *Class Field Theory: from theory to practice*, SMM, Springer-Verlag 2003; second corrected printing 2005. <https://www.researchgate.net/publication/268005797>
- [10] G. Gras, *Les θ -régulateurs locaux d'un nombre algébrique : Conjectures p -adiques*, Canadian Journal of Mathematics **68**(3) (2016), 571–624. <http://dx.doi.org/10.4153/CJM-2015-026-3> <https://arxiv.org/pdf/1701.02618.pdf>
- [11] G. Gras, *Approche p -adique de la conjecture de Greenberg pour les corps totalement réels*, Annales Mathématiques Blaise Pascal **24** (2017), no. 2, p. 235–291. http://ambp.cedram.org/cedram-bin/article/AMBP_2017_24_2_235_0.pdf
- [12] G. Gras, *The p -adic Kummer-Leopoldt Constant – Normalized p -adic Regulator*, Int. Journal of Number Theory, 2018 (to appear). <https://arxiv.org/abs/1701.06857>
- [13] G. Gras, *On p -rationality of number fields. Applications – PARI/GP programs* (preprint). <https://arxiv.org/pdf/1709.06388.pdf>
- [14] G. Gras, *Normes d'idéaux dans la tour cyclotomique et conjecture de Greenberg* (preprint 2017). <https://arxiv.org/pdf/1706.08784.pdf>
- [15] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math. **98** (1976), no. 1, p. 263–284. http://www.jstor.org/stable/2373625?seq=1#page_scan_tab_contents
- [16] R. Greenberg, *Galois representations with open image*, Annales de Mathématiques du Québec, special volume in honor of Glenn Stevens, **40**(1) (2016), 83–119. <https://link.springer.com/article/10.1007/s40316-015-0050-6>
- [17] H. Graves and M.R. Murty, *The abc conjecture and non-Wieferich primes in arithmetic progressions*, Journal of Number Theory **133** (2013), 1809–1813.
- [18] A. Granville and K. Soundararajan, *Upper bounds for $|L(1, \chi)|$* , Quarterly Journal of Mathematics **53** (2002), 265–284. <http://www.dms.umontreal.ca/~andrew/PDF/sizeL1chi.pdf>
- [19] D.R. Heath-Brown, *Square-Free Values of $n^2 + 1$* , Acta Arithmetica **155** (2012), no. 1, 1–13. <https://arxiv.org/pdf/1010.6217.pdf>
- [20] F. Hajir and C. Maire, *On the invariant factors of class groups in towers of number fields*, Canadian Journal of Mathematics (Published: 2017-11-02), 31 pages. <http://dx.doi.org/10.4153/CJM-2017-032-9>
- [21] T. Hofmann and Y. Zhang – *Valuations of p -adic regulators of cyclic cubic fields*, Journal of Number Theory **169** (2016), 86–102. <https://doi.org/10.1016/j.jnt.2016.05.016>
- [22] A. Ivanov, *Reconstructing decomposition subgroups in arithmetic fundamental groups using regulators*. <https://arxiv.org/pdf/1409.4909.pdf>
- [23] S. Louboutin – *The Brauer–Siegel Theorem*, J. London Math. Soc. **2** (2005), no. 72, 40–52. <https://www.researchgate.net/publication/266514633>
- [24] S. Louboutin – *Upper bounds for residues of Dedekind zeta functions and class numbers of cubic and quartic number fields*, Math. of Computation **80** (2011), no. 275, 1813–1822. <http://www.jstor.org/stable/23075379>
- [25] S. Louboutin – *Explicit upper bounds for residues of Dedekind zeta functions*, Moscow Mathematical Journal **15** (2015), no. 4, 727–740. <http://www.mathjournals.org/mmj/2015-015-004/2015-015-004-009.html>
- [26] THE PARI GROUP – PARI/GP, version 2.9.0, Université de Bordeaux (2016). <http://pari.math.u-bordeaux.fr/http://www.sagemath.org/fr/telecharger.html>
- [27] J. Pintz, *Elementary methods in the theory of L -functions VII, Upper bounds for $L(1, \chi)$* , Acta Arithmetica **32** (1977), 397–406. <http://matwbn.icm.edu.pl/ksiazki/aa/aa32/aa3246.pdf>
- [28] F. Pitoun and F. Varescon, *Computing the torsion of the p -ramified module of a number field*, Math. Comp. **84**(291) (2015), 371–383. <http://www.ams.org/journals/mcom/2015-84-291/S0025-5718-2014-02838-X/S0025-5718-2014-02838-X.pdf>
- [29] O. Ramaré, *Approximate formulae for $L(1, \chi)$* , Acta Arith. **100** (2001), no. 3, 245–266. <http://iml.univ-mrs.fr/~ramare/Maths/LOneChi2.pdf>
- [30] R. Schoof, *Computing Arakelov class groups*, Algorithmic Number Theory, MSRI Publications **44** (2008), 447–495. <https://www.mat.uniroma2.it/~schoof/14schoof.pdf>
- [31] J-P. Serre – *Sur le résidu de la fonction zêta p -adique d'un corps de nombres*, C.R. Acad. Sci. Paris **287** (1978), Série I, p. 183–188.
- [32] J.H. Silverman, *Wieferich's criterion and the abc-conjecture*, Journal of Number Theory **30** (1988), 226–237.
- [33] D.C. Shanks, P.J. Sime and L.C. Washington, *Zeros of 2-adic L -functions and congruences for class numbers and fundamental units*, Math. Comp. **68**(227) (1987), 1243–1255. <http://www.ams.org/journals/mcom/1999-68-227/S0025-5718-99-01046-7/S0025-5718-99-01046-7.pdf>
- [34] J. Tsimerman, *Brauer–Siegel for arithmetic tori and lower bounds for Galois orbits of special points*, J. Amer. Math. Soc. **25** (2012), no. 4, 1091–1117. <https://arxiv.org/abs/1103.5619v3>

- [35] M. Tsfasman and S. Vladuț, *Infinite global fields and the generalized Brauer–Siegel theorem*, Dedicated to Yuri I. Manin on the occasion of his 65th birthday, *Moscow Math. J.* **2** (2002), no. 2, 329–402. <http://www.ams.org/distribution/mmj/vol2-2-2002/tsfasman-vladuts.pdf>
- [36] L.C. Washington – *Introduction to Cyclotomic Fields*, Graduate Texts in Math. 83, Springer enlarged second edition 1997.
- [37] L.C. Washington – *Zeroes of p -adic L -functions*, Séminaire de théorie des nombres, Paris, 1980–81 (Sém. Delange–Pisot–Poitou), Birkhäuser, Boston, 1982, pp. 337–357. <http://plouffe.fr/simon/math/Seminaire%20de%20Theorie%20des%20Nombres,%20Paris,%201980-1981.pdf>
- [38] L.C. Washington – *Siegel zeroes for 2-adic L -functions*, Number Theory, Halifax, NS 1994), 393–396. CMS Conf. Proc., American Mathematical Society, Providence, RI, 1995. https://books.google.fr/books?id=gzRti1o1bXkC&printsec=frontcover&hl=fr&source=gbs_ge_summary_r&cad=0#v=onepage&q&
- [39] L.C. Washington – *Zeroes of p -adic L -functions*, Séminaire de Théorie des Nombres de Bordeaux (1980–1981), pp. 1–4. <http://www.jstor.org/stable/44166382>
- [40] A.I. Zykin, *Brauer–Siegel and Tsfasman–Vladuț theorems for almost normal extensions of global fields*, *Moscow Mathematical Journal* **5** (2005), no. 4, 961–968. <http://www.ams.org/distribution/mmj/vol5-4-2005/zykin.pdf>

VILLA LA GARDETTE, CHEMIN CHÂTEAU GAGNIÈRE F–38520 LE BOURG D’OISANS, FRANCE https://www.researchgate.net/profile/Georges_Gras
E-mail address: g.mn.gras@wanadoo.fr