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HEURISTICS IN DIRECTION OF A p -ADIC BRAUER–SIEGEL THEOREM

GEORGES GRAS

ABSTRACT. Let p be a fixed prime number. Let K be a totally real number field of discriminant D_K and let \mathcal{T}_K be the torsion group of the Galois group of the maximal abelian p -ramified pro- p -extension of K (under Leopoldt’s conjecture). We conjecture the existence of a constant $C_p > 0$ such that $\log(\#\mathcal{T}_K) \leq C_p \cdot \log(\sqrt{D_K})$ when K varies in some specified families (e.g., fields of fixed degree). In some sense, we try to suggest a p -adic analogue, of the classical Brauer–Siegel Theorem, wearing on the valuation of the residue, at $s = 1$, of the p -adic ζ -function of K . We approach this question via the arithmetical study of \mathcal{T}_K since p -adic analysis fails contrary to the classical framework. We give extensive numerical verifications for quadratic and cubic fields (cyclic or not) and publish the PARI programs. Such a conjecture (if exact) reinforces our conjecture that any number field is p -rational (i.e., $\mathcal{T}_K = 1$) for all $p \gg 0$.

1. ABELIAN p -RAMIFICATION – MAIN DEFINITIONS AND NOTATIONS

Let K be a totally real number field of degree d , and let $p \geq 2$ be a prime number fulfilling the Leopoldt conjecture in K . We denote by \mathcal{C}_K the p -class group of K in the ordinary sense and by E_K its group of p -principal global units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$.

Let us recall from [9, 11, 12, 14] the diagram of the so called *abelian p -ramification theory*, in which $K^c = K\mathbb{Q}^c$ is the cyclotomic \mathbb{Z}_p -extension of K (as compositum with that of \mathbb{Q}), H_K the p -Hilbert class field and H_K^{PF} the maximal abelian p -ramified (i.e., unramified outside p) pro- p -extension of K .

Let $U_K := \bigoplus_{\mathfrak{p}|p} U_{\mathfrak{p}}^1$ be the \mathbb{Z}_p -module (of \mathbb{Z}_p -rank d) of p -principal local units of K , where each $U_{\mathfrak{p}}^1 := \{u \in K_{\mathfrak{p}}^{\times}, u \equiv 1 \pmod{\bar{\mathfrak{p}}}\}$ is the group of $\bar{\mathfrak{p}}$ -principal units of the completion $K_{\mathfrak{p}}$ of K at $\mathfrak{p} | p$, where $\bar{\mathfrak{p}}$ is the maximal ideal of the ring of integers of $K_{\mathfrak{p}}$.

For any field k , let μ_k be the group of roots of unity of k of p -power order. Then put $W_K := \text{tor}_{\mathbb{Z}_p}(U_K) = \bigoplus_{\mathfrak{p}|p} \mu_{K_{\mathfrak{p}}}$ and $\mathcal{W}_K := W_K/\mu_K$.

Let \bar{E}_K be the closure in U_K of the diagonal image of E_K ; by class field theory this gives in the diagram $\text{Gal}(H_K^{\text{PF}}/H_K) \simeq U_K/\bar{E}_K$; then let \mathcal{C}_K^c be the subgroup of \mathcal{C}_K corresponding to the subgroup $\text{Gal}(H_K/K^c \cap H_K)$.

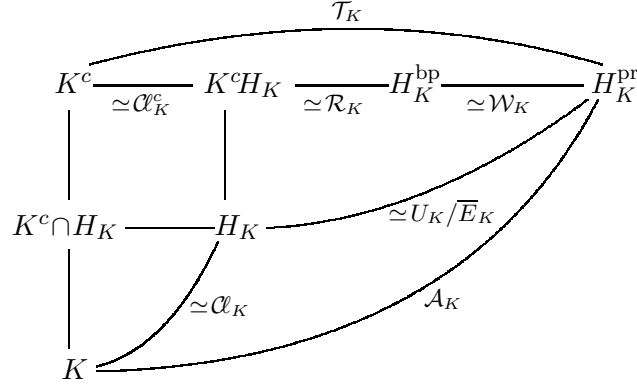
Put ([9, Chapter III, § 2 (a) & Theorem 2.5] with the set S of infinite places (ordinary sense) and the set T of p -places):

$$\mathcal{T}_{K,p} = \mathcal{T}_K := \text{tor}_{\mathbb{Z}_p}(\text{Gal}(H_K^{\text{PF}}/K)) = \text{Gal}(H_K^{\text{PF}}/K^c).$$

We have (because of Leopoldt’s conjecture) the following exact sequence defining $\mathcal{R}_{K,p} = \mathcal{R}_K$ (see [9, Lemma III.4.2.4 with Corollary III.3.6.3] or [12, Lemma 3.1 & § 5]):

$$(1.1) \quad 1 \rightarrow \mathcal{W}_K \rightarrow \text{tor}_{\mathbb{Z}_p}(U_K/\bar{E}_K) \xrightarrow{\log_p} \text{tor}_{\mathbb{Z}_p}(\log_p(U_K)/\log_p(\bar{E}_K)) =: \mathcal{R}_K \rightarrow 0.$$

The group \mathcal{R}_K (or its order) is called the *normalized p -adic regulator of K* and makes sense for any number field:



The field H_K^{bp} , fixed by \mathcal{W}_K , is the Bertrandias–Payan field, i.e., the compositum of the p -cyclic extensions of K embeddable in p -cyclic extensions of arbitrary large degree.

2. v -ADIC ANALYTIC PROSPECTS

For a random totally real number field K and a fixed prime p , $\#\mathcal{T}_K = \#\mathcal{C}_K^c \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$, may be equal to 1 (defining « p -rational fields») or not, and it will be interesting to know if the p -adic valuation of $\#\mathcal{T}_K$ can be bounded according, for instance, to the discriminant of K . If so, this would be interpreted as a p -adic version of the archimedean Brauer–Siegel theorem, which is currently pure speculation, but we intend to experiment, algebraically, this context since p -adic analysis does not seem to succeed as explain by Washington in [34]:

A Brauer–Siegel theorem using p -adic L -functions fails;

in the same way, we have similar comments by Ivanov in [20, Section 1]:

The p -adic analogue of Brauer–Siegel and hence also of Tsfasman–Vladuț fails.

We still suppose that K is a totally real number field and $p \geq 2$ any fixed prime number. As we shall recall it, for abelian fields K , $\#\mathcal{T}_K$ is in close relationship with the Kubota–Leopoldt p -adic L -functions (at $s = 1$) of Dirichlet characters (Kubota–Leopoldt, Barsky, Amice–Fresnel,...), or more generally with the residue at $s = 1$ of the p -adic ζ -function of K built or study by many authors (Amice–Fresnel, Coates, Shintani, Barsky, Serre, Cassou–Noguès, Deligne–Ribet, Katz, Colmez,...).

Conversely, there is no *algebraic* invariant (like a Galois group) interpreting the residue of the complex ζ -function, but we have in this (archimedean) case numerous inequalities. So, we shall compare the complex and p -adic cases to try to find analogies. For this, we define normalizations of the ζ -functions of a totally real number field (from [5, 6], and [12] for the regulators).

2.1. Definitions and normalizations. Let K be a totally real number field of degree d and let $\mathcal{P} := \{p_\infty, 2, 3, \dots, p, \dots\}$ be the set of all places of \mathbb{Q} (we also use the symbol ∞ for complex functions, like L -functions, ζ -functions, log-functions, in the same logic as for p -adic functions; for instance, $R_{K,\infty}$ and $R_{K,p}$ shall be the usual regulators built with \log_∞ and \log_p , respectively). We shall use indices $(\)_{K,v}$ for any place $v \in \mathcal{P}$; when the context is clear, we omit the place v (p -adic in most cases).

2.1.1. v -CYCLOTOMIC EXTENSIONS AND v -CONDUCTORS. The p -cyclotomic \mathbb{Z}_p -extension is denoted $\mathbb{Q}^{c,p}$ and we introduce the p_∞ -cyclotomic extension $\mathbb{Q}^{c,p_\infty} =: \mathbb{Q}$. We put $\mathbb{Q}^{c,v} =: \mathbb{Q}^c$ for all $v \in \mathcal{P}$ if there is no ambiguity.

We attribute to the field \mathbb{Q} the « conductor » $\mathfrak{f}_{\mathbb{Q}} := p$ (resp. $4, 2$) if $v = p$ (resp. $2, p_{\infty}$). We shall put \sim for equalities up to a p -adic unit.

2.1.2. NORMALIZED ζ -FUNCTIONS AT p_{∞} . We define at the infinite place p_{∞} :

$$(2.1) \quad \tilde{\zeta}_{K, p_{\infty}} := \frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \cdot \zeta_{K, p_{\infty}} = \frac{1}{2^{d-1}} \cdot \zeta_{K, p_{\infty}}.$$

$$(2.2) \quad \#\mathcal{T}_{K, p_{\infty}} := h_K \cdot \frac{R_{K, \infty}}{\sqrt{D_K}},$$

where h_K is the class number (ordinary sense), $R_{K, \infty}$ the classical regulator and D_K the discriminant of K (see [9, Remark III.2.6.5 (ii)] for justifications about the factor $\frac{1}{2^d}$).

Denote by $\tilde{\kappa}_{K, p_{\infty}}$ the residue at $s = 1$ of $\tilde{\zeta}_{K, p_{\infty}}$. From classical formulas of the so-called complex « analytic formula of the class number » of K (see, e.g., [33, Chap. 4]), we get:

$$(2.3) \quad \tilde{\kappa}_{K, p_{\infty}} = h_K \cdot \frac{R_{K, \infty}}{\sqrt{D_K}} = \#\mathcal{T}_{K, p_{\infty}}.$$

2.1.3. NORMALIZED ζ -FUNCTIONS AT $v = p$. We define at a finite place p :

$$(2.4) \quad \tilde{\zeta}_{K, p} := \frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \cdot \zeta_{K, p},$$

where $\mathfrak{f}_{K \cap \mathbb{Q}^c}$ is the conductor of $K \cap \mathbb{Q}^c$ ($\mathfrak{f}_{K \cap \mathbb{Q}^c} \sim 2p \cdot [K \cap \mathbb{Q}^c : \mathbb{Q}]$ if $K \cap \mathbb{Q}^c \neq \mathbb{Q}$ and $\mathfrak{f}_{K \cap \mathbb{Q}^c} \sim 2p$ if $K \cap \mathbb{Q}^c = \mathbb{Q}$), and for which we have the p -adic residue [5, 6, 29]:

$$(2.5) \quad \tilde{\kappa}_{K, p} = \frac{\mathfrak{f}_{K \cap \mathbb{Q}^c}}{2^d} \cdot \kappa_{K, p} \sim \#\mathcal{T}_K = \#\mathcal{C}_K^c \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K.$$

So, the residues of the normalized ζ -functions of K are such that:

$$\tilde{\kappa}_{K, v} \sim \#\mathcal{T}_{K, v}, \text{ for all } v \in \mathcal{P},$$

which is the order of an arithmetical invariant for finite places $v = p$ and the measure of a real volume for p_{∞} .

2.2. **Abelian complex L -functions – Upper bounds.** In the abelian case, we have:

$$(2.6) \quad \#\mathcal{T}_{K, p_{\infty}} = h_K \cdot \frac{R_{K, \infty}}{\sqrt{D_K}} = \prod_{\chi \neq 1} \frac{1}{2} L_{p_{\infty}}(1, \chi),$$

where χ goes through all of the corresponding Dirichlet characters of K with conductor f_{χ} , and where $L_{p_{\infty}}$ denotes the complex « L -function ». If $K = \mathbb{Q}(\sqrt{m})$, of fundamental unit ε_K and quadratic character χ_K , one gets:

$$\#\mathcal{T}_{K, p_{\infty}} = h_K \cdot \frac{\log_{\infty}(\varepsilon_K)}{\sqrt{D_K}} = \frac{1}{2} \cdot L_{p_{\infty}}(1, \chi_K).$$

For each $L_{p_{\infty}}(1, \chi)$ one has many upper bounds which are improvements of the inequality:

$$(2.7) \quad \frac{1}{2} \cdot L_{p_{\infty}}(1, \chi) \leq (1 + o(1)) \cdot \log_{\infty}(\sqrt{f_{\chi}}).$$

In [27, Corollaire 1] one has, for even primitive characters:

$$\frac{1}{2} \cdot L_{p_{\infty}}(1, \chi) \leq \frac{1}{2} \cdot \log_{\infty}(\sqrt{f_{\chi}}),$$

giving from the previous definition (2.2):

$$(2.8) \quad \#\mathcal{T}_{K, p_{\infty}} \leq C_{p_{\infty}} \cdot \sqrt{D_K},$$

with an explicit constant $C_{p_{\infty}}$ if K runs through the set of real abelian fields of given degree d .

¹ The factor $\frac{R_{K, \infty}}{\sqrt{D_K}}$ may be interpreted as the normalized regulator $\mathcal{R}_{K, p_{\infty}} = \mathcal{R}_K$ attached to $v = p_{\infty}$; from [1], it is defined without ambiguity. The invariant $\mathcal{T}_{K, p_{\infty}}$ is related to the Arakelov class group of K (see [28] and its bibliography).

Remark 2.1. We do not need any sophisticated upper bound (only the existence of C_{p_∞}), but one may refer to [16, 22, 23, 25, 27] for other inequalities; for instance, one gets, in the real abelian case, with our notations:

$$h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} \leq \left(\frac{1}{2} \frac{\log_\infty(\sqrt{D_K})}{d-1} \right)^{d-1} \quad \& \quad R_{K,\infty} \leq \sqrt{D_K} \cdot \left(\frac{1}{2} \frac{\log_\infty(\sqrt{D_K})}{d-1} \right)^{d-1},$$

thus in the cases $d = 2$ and $d = 3$ for K :

$$\#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{\log_\infty(\varepsilon_K)}{\sqrt{D_K}} \leq \frac{1}{2} \log_\infty(\sqrt{D_K}), \quad \#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}} \leq \left(\frac{1}{4} \log_\infty(\sqrt{D_K}) \right)^2,$$

respectively, and, for real quadratic and cyclic cubic fields:

$$(2.9) \quad h_K \leq \frac{1}{2} \cdot \sqrt{D_K}, \quad h_K \leq \frac{2}{3} \cdot \sqrt{D_K}, \quad \text{respectively.}$$

2.3. Abelian L_p -functions – Algebraic interpretation. The Kubota–Leopoldt p -adic L -functions gives rise to the analytic formula [1, §2.1 & Théorème 6, §2.3]:

$$(2.10) \quad \#\mathcal{C}_K \cdot \frac{R_{K,p}}{\sqrt{D_K}} \sim \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) \cdot \prod_{\chi \neq 1} \left(1 - \frac{\chi(p)}{p} \right)^{-1}.$$

But no upper bound of the p -adic valuation of this residue is known. So we must, on the contrary, try to study directly $\#\mathcal{T}_K$ with arithmetic tools. For this recall the analytical expression of $\#\mathcal{T}_K$; the « p -adic class number formula » for real abelian fields uses the formula [5]:

$$(2.11) \quad \#\mathcal{T}_K \sim [K \cap \mathbb{Q}^c : \mathbb{Q}] \cdot \prod_{\mathfrak{p}|p} \frac{p}{N\mathfrak{p}} \cdot \#\mathcal{C}_K \cdot \frac{R_{K,p}}{\sqrt{D_K}}.$$

Thus in the abelian case, since $\prod_{\chi} \left(1 - \frac{\chi(p)}{p} \right)^{-1} = \prod_{\mathfrak{p}|p} (1 - N\mathfrak{p}^{-1})^{-1} \sim \prod_{\mathfrak{p}|p} N\mathfrak{p}$:

$$(2.12) \quad \#\mathcal{T}_K \sim \prod_{\mathfrak{p}|p} \frac{p}{N\mathfrak{p}} \cdot \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi) \cdot \prod_{\chi \neq 1} \left(1 - \frac{\chi(p)}{p} \right)^{-1} \sim \prod_{\chi \neq 1} \frac{1}{2} L_p(1, \chi).$$

In $\#\mathcal{T}_K = \#\mathcal{C}_K^c \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K$, the computation of $\#\mathcal{W}_K$ is obvious, that of:

$$\#\mathcal{R}_K = \#\text{tor}_{\mathbb{Z}_p}(\log_p(U_K)/\log_p(\overline{E}_K)) \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_p : \log(N_{K/\mathbb{Q}}(U_K)))}{\#\mathcal{W}_K \cdot \prod_{\mathfrak{p}|p} N\mathfrak{p}} \cdot \frac{R_{K,p}}{\sqrt{D_K}}$$

(see [12, Proposition 5.2]), is unpredictable, and more complicate if p ramifies in K . In the non-ramified case, it is given by the classical determinant provided that one replaces \log_p by the « normalized logarithm » $\frac{1}{p} \log_p$ (see [10] for a general study of such regulators as Frobenius determinants when $p \gg 0$); then $\#\mathcal{C}_K^c = \frac{\#\mathcal{C}_K}{[H_K \cap K^c : K]}$, where the degree $[H_K \cap K^c : K]$ is equal, up to a p -adic unit, to $\frac{e_p}{\langle (-1) : (-1) \cap N_{K/\mathbb{Q}}(U_K) \rangle \cdot [K \cap \mathbb{Q}^c : \mathbb{Q}]}$, where e_p is the ramification index of p in K/\mathbb{Q} [9, Theorem III.2.6.4].

Remark 2.2. Recal that in the case $p > 2$ unramified in a quadratic field K , we have $\#\mathcal{R}_K \sim \frac{1}{p} \log_p(\varepsilon_K) \sim p^{\delta_p(\varepsilon_K)}$, where $\delta_p(\varepsilon_K) = v_p\left(\frac{\varepsilon_K^{p^f-1} - 1}{p}\right)$ ($f \in \{1, 2\}$ is the residue degree of p).

The existence of an upper bound for $v_p(\frac{1}{2}L_p(1, \chi_K))$ would be equivalent to estimate the order of magnitude of $\delta_p(\eta_K)$ for the cyclotomic number $\eta_K := \prod_{a, \chi(a)=1} (1 - \zeta_{D_K}^a)$, where ζ_{D_K} is a primitive D_K th root of unity (interpretation via cyclotomic units of the class number formula).

The study given in [10, Théorème 1.1], and applied to the element $\xi = 1 - \zeta_{D_K}$, suggests that if $p \rightarrow \infty$, the probability of $\delta_p(\eta_K) \geq 1$ for the χ_K -component $\langle \eta_K \rangle_{\mathbb{Z}} = \langle \xi \rangle^{\varepsilon_{\chi_K}}$, of the Galois module generated by ξ , tends to 0 at least as $\frac{O(1)}{p}$ and conjecturally as

$\frac{1}{p^{\log_2(p)/\log(c_0(\eta_K))}}$, where $c_0(\eta_K) = |\eta_K| > 1$ and $\log_2 = \log \circ \log$. This does not apply to small p . Which explain the specific difficulties of the p -adic case, which is natural since the general study of $v_p(\#\mathcal{T}_K)$ represents a refinement of the Leopoldt conjecture.

We intend to give an estimation of $v_p(\#\mathcal{T}_K)$ related to the discriminant D_K (p fixed). In a numerical point of view, we shall analyse the case of real quadratic fields $K = \mathbb{Q}(\sqrt{m})$ and that of cyclic cubic fields which are described by the polynomials (see, e.g., [7]):

$$(2.13) \quad \begin{aligned} P &= X^3 + X^2 + \frac{1-f}{3} \cdot X + \frac{1+f(a-3)}{27}, \text{ if } 3 \nmid f, \\ P &= X^3 - \frac{f}{3} \cdot X - \frac{fa}{27}, \text{ if } 3 \mid f, \end{aligned}$$

where the conductor f is $\frac{a^2+27b^2}{4}$, with $b > 0$, $a \equiv 2 \pmod{3}$ (for $3 \nmid f$), $a \equiv 6 \pmod{9}$ & $b \not\equiv 0 \pmod{3}$ (for $3 \mid f$). Some non-cyclic cubic fields will be also considered.

In the forthcoming Sections, we deal only with finite places p ; so we simplify some notation in an obvious way.

3. DIRECT CALCULATION OF $v_p(\#\mathcal{T}_K)$

The programs shall try to verify a p -adic analogue of the relation (2.8), for quadratic and cubic fields; they shall give, for each fixed p , the successive minima of the expression $\Delta_p(K) := \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$, when D_K increases, and compute for each K :

$$(3.1) \quad C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}.$$

It seems that the absolute minimum of $\Delta_p(K)$ (on the selected interval), is rapidly obtained for a fixed degree d and is in general negative of small absolute value since one must probably introduce a coefficient in the expression of $\Delta_p(K)$, so that the best heuristic should be the study of the positivity of:

$$\tilde{\Delta}_p(K) := C_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} - v_p(\#\mathcal{T}_K),$$

for a constant $C_p \gg 0$ being possibly an upper bound of the $C_p(K)$. We shall adapt the following program [13, § 3.2] (testing the p -rationality of any number field K), that we recall for the convenience of the reader (choose any monic irreducible P and any prime p ; the program gives the signature (r_1, r_2) of K in S and $r := r_2 + 1$; recall that from $K = \text{bnfinit}(P, 1)$ and $C8 = \text{component}(K, 8)$, the class number and the regulator of K are given by: $h = \text{component}(\text{component}(C8, 1), 1)$ and $\text{reg} = \text{component}(C8, 2)$, respectively).

```
{P=x^6-123*x+1;p=3;K=bnfinit(P,1);n=2;if(p==2,n=3);Kpn=bnrinit(K,p^n);
S=component(component(Kpn,1),7);r=component(component(S,2),2)+1;
print(p,"-rank of the compositum of the Z_",p,"-extensions: ",r);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);R=0;
for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1));
print("Structure of the ",p,"-ray class group: ",L);
if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational"));
if(R==r,print("rk(T)=",0," K is ",p,"-rational"))}
```

For totally real fields the p -invariants of $\text{Gal}(K(p^n)/K)$, where $K(p^n)$ is the ray class field of modulus (p^n) , for any $n \geq 0$ ($n = 0$ corresponding to the Hilbert class field), are given by the simplest program:

```
{P=x^2-2*3*5*7*11*13*17;K=bnfinit(P,1);p=2;n=18;Kpn=bnrinit(K,p^n);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);R=0;
for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1));
print(L)}
```

Then, taking n large enough in the program allows us to compute directly the structure of \mathcal{T}_K as is done by a precise (but longer) program in [26]. This gives the p -valuation v_{ptor} of $\#\mathcal{T}_K$ as rapidly as possible; for this explain some details about PARI [24]:

Let K be a totally real number field, linearly disjoint from \mathbb{Q}^c ; let $K(p^n)$ be the ray class field of modulus (p^n) , $n \geq 2$ (resp. $n \geq 3$) if $p \neq 2$ (resp. $p = 2$); indeed, from [13, Theorem 2.1], these conditions on n are sufficient to give the p -rank $t_K =: t$ of \mathcal{T}_K .

Thus, for n large enough, the p -structure of $\text{Gal}(K(p^n)/K)$, given by PARI, is of the form $[p^a, p^{a_1}, \dots, p^{a_t}]$, with $a \geq a_1 \geq \dots \geq a_t$, in $Hpn := \text{component}(\text{component}(Kpn, 5), 2)$, where $p^a = [K(p^n) \cap \mathbb{Q}^c : K]$.

Then the order of \mathcal{T}_K is $[K(p^n) : K] \times p^{-a}$ (up to a p -adic unit), where p^a is the largest component given in Hpn (whence the first one in the list, under the condition $n \gg \max(a_1, \dots, a_t)$); so we have only to verify that p^n is much larger than the exponent $\max(p^{a_1}, \dots, p^{a_t})$ of \mathcal{T}_K .

In practice, and to obtain fast programs, we must look at the order of magnitude of the results to increase n if necessary; in fact, once the part $K = \text{bnfinit}(P, 1)$ of the program is completed, a large value of n does not significantly increase the execution time.

For instance, with $P = x^2 - 4194305$, one gets the successive structures for $2 \leq n \leq 16$:

2	[2, 2]	7	[64, 32, 2]	12	[2048, 1024, 2]
3	[4, 2, 2]	8	[128, 64, 2]	13	[4096, 2048, 2]
4	[8, 4, 2]	9	[256, 128, 2]	14	[8192, 2048, 2]
5	[16, 8, 2]	10	[512, 256, 2]	15	[16384, 2048, 2]
6	[32, 16, 2]	11	[1024, 512, 2]	16	[32768, 2048, 2]

showing that n must be at least 13 to give $\mathcal{T}_K \simeq \mathbb{Z}/2^{11}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In the forthcoming numerical results, if any doubt occurs for a specific field, it is sufficient to use the previous program with bigger n ; we have done these verifications for some large $v_p(\#\mathcal{T}_K)$.

4. NUMERICAL INVESTIGATIONS FOR REAL QUADRATIC FIELDS

We have $\#\mathcal{W}_K = 2$ for $p = 2$ & $m \equiv \pm 1 \pmod{8}$, $\#\mathcal{W}_K = 3$ for $p = 3$ & $m \equiv -3 \pmod{9}$, and we are concerned with the p -class group \mathcal{C}_K and the normalized regulator \mathcal{R}_K . When $p \nmid D_K$, $p > 2$, $v_p(\#\mathcal{R}_K) = \delta_p(\varepsilon)$ for the fundamental unit $\varepsilon_K =: \varepsilon$ of K (Remark 2.2); so, if $p \nmid D_K$, we may compute $v_p(\#\mathcal{T}_K)$ as $v_p(\#\mathcal{C}_K) + \delta_p(\varepsilon) + v_p(\#\mathcal{W}_K)$ and we shall compare with the direct computation of the structure of \mathcal{T}_K as explain above. We have the following result, especially for the computation of $v_p(\#\mathcal{R}_K)$ when $p \geq 2$ ramifies:

Proposition 4.1. *For real quadratic fields K and $p \mid D_K$, $v_p(\#\mathcal{R}_K)$ is given as follows:*

(i) *For $p \nmid 6$ ramified, $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_p(\varepsilon)$ and $v_p(\#\mathcal{R}_K) = \delta$ if $v_{\mathfrak{p}}(\varepsilon^{p-1} - 1) = 1 + 2\delta$, $\mathfrak{p} \mid p$, $\delta \geq 0$.*

(ii) *For $p = 3$ ramified, $\#\mathcal{R}_K \sim \frac{1}{\sqrt{m}} \cdot \log_3(\varepsilon)$ (resp. $\#\mathcal{R}_K \sim \frac{1}{3\sqrt{m}} \cdot \log_3(\varepsilon)$) if $m \not\equiv -3 \pmod{9}$ (resp. if $m \equiv -3 \pmod{9}$). Then $v_3(\mathcal{R}_K) = (v_{\mathfrak{p}}(\varepsilon^6 - 1) - 2 - \delta)/2$ where $\mathfrak{p} \mid 3$ and $\delta = 1$ (resp. $\delta = 3$) if $m \not\equiv -3 \pmod{9}$ (resp. if $m \equiv -3 \pmod{9}$).*

(iii) *For $p = 2$ ramified, $\mathcal{R}_K \sim \frac{\log_2(\varepsilon)}{2\sqrt{m}}$ (resp. $\frac{\log_2(\varepsilon)}{4}$) if $m \not\equiv -1 \pmod{8}$ (resp. $m \equiv -1 \pmod{8}$). Then, $v_2(\mathcal{R}_K) = (v_{\mathfrak{p}}(\varepsilon^4 - 1) - \delta)/2$, where $\mathfrak{p} \mid 2$ and $\delta = 2$ (resp. $\delta = 3$, $\delta = 4$) if $m \equiv 3 \pmod{8}$ (resp. $m \equiv 2, 6 \pmod{8}$, $m \equiv 7 \pmod{8}$).*

Proof. Exercise using the expression of $\#\mathcal{R}_K$ in [12, Proposition 5.2], the fact that $N_{K/\mathbb{Q}}(\varepsilon) = \pm 1$ (i.e., $\text{Tr}_{K/\mathbb{Q}}(\log_p(\varepsilon)) = 0$), and the classical computation of a p -adic logarithm. \square

A first information is then the order of magnitude of $\delta_p(\varepsilon)$ for fixed p as $D_K \rightarrow \infty$. Its non-nullity for $p \gg 0$ (K fixed) is a deep problem for which we can only give some

experiments with PARI programs. Note that these questions of « Fermat quotients », applied to algebraic numbers, is probably the explanation of the failure of the classical p -adic analysis on ζ -functions since such Fermat quotients problems are neither easier nor more difficult than the famous problem of Fermat quotients of 2 for instance.

4.1. Maximal values of $\delta_p(\varepsilon)$. Consider a prime p fixed and the family of real quadratic fields $K = \mathbb{Q}(\sqrt{m})$. The following programs find the successive maxima of $\delta_p(\varepsilon)$ with the corresponding $m \in [bm, Bm]$; the program uses the fact that for p unramified, in the inert case $\varepsilon^{p+1} \equiv N_{K/\mathbb{Q}}(\varepsilon) \pmod{p}$, otherwise, $\varepsilon^{p-1} \equiv 1 \pmod{p}$. We shall indicate the final value of $C_p(K)$ defined by expression (3.1):

4.1.1. PROGRAM FOR $p = 2$ UNRAMIFIED. For $p = 2$ we must take into account the fact that ε and $-\varepsilon$ may have distinct δ_2 ; so if $\varepsilon^{2\pm 1} = 1 + 2 \cdot 2^\delta \alpha$, $(-\varepsilon)^{2\pm 1} = -1 - 2 \cdot 2^\delta \alpha = 1 + (-2 - 2 \cdot 2^\delta \alpha) = 1 + 2 \cdot (-1 - 2^\delta \alpha)$ which may give another value when $\delta = 0$.

```
{p=2;bm=3;Bm=5*10^8;Max=0;for(m=bm,Bm,ss=Mod(m,8);s=0;if(ss==1,s=1);
if(ss==5,s=-1);if(core(m)==m & s!=0,E=quadunit(m);A=(E^(p-s)-1)/p;
A1=[component(A,2),component(A,3)];A2=[component(A,2)+1,component(A,3)];
delta1=valuation(A1,p);delta2=valuation(A2,p);delta=max(delta1,delta2);
if(delta>Max,Max=delta;print("m=",m," delta=",delta))}}
```

m=21	delta=1	m=1185	delta=9	m=528077	delta=18
m=69	delta=2	m=11985	delta=11	m=7916761	delta=19
m=161	delta=4	m=27313	delta=12	m=12256653	delta=21
m=469	delta=5	m=115005	delta=13	m=14098537	delta=23
m=645	delta=6	m=222181	delta=16	m=214203013	delta=26

Which gives $\delta_2(\varepsilon) = 26$, $C_2(K) = 1.878992$, obtained for $D_K = m = 214203013$ and a too large unit; the interval was $[2, 5 \cdot 10^8]$ (two days of computer).

4.1.2. PROGRAM FOR $p = 2$ RAMIFIED. A similar program gives analogous results for maximal values of $v_2(\mathcal{R}_K)$ (in *vptor*) distinct from $\delta_2(\varepsilon)$ because of formulas of Proposition 4.1 (iii):

```
{p=2;bm=3;Bm=5*10^8;Max=0;for(m=bm,Bm,s=Mod(m,4);ss=Mod(m,8);
if(core(m)==m & s!=1,A=(quadunit(4*m)^4-1)/4;N=norm(A);v=valuation(N,2);
if(s==2,vptor=v-3);if(s==3,vptor=v-2);if(ss==7,vptor=v-4);vptor=vptor/2;
if(vptor>Max,Max=vptor;M=m;print("m=",m," delta=", vptor))}}
```

m=7	delta=1	m=2558	delta=8	m=3926018	delta=20
m=31	delta=2	m=5495	delta=9	m=7354898	delta=21
m=114	delta=4	m=7166	delta=13	m=9162543	delta=23
m=127	delta=6	m=20406	delta=17	m=20220595	delta=27
m=446	delta=7	m=369167	delta=18		

where $\delta_2(\varepsilon) = 27$, $C_2(K) = 2.055629$, for $m = 20220595 = 5 \cdot 239 \cdot 16921$, whence influence of genera theory on $C_2(K)$ ($h = 4$).

4.1.3. PROGRAM FOR $p \geq 3$ UNRAMIFIED. In this case, the program can be simplified:

```
{p=3;bd=5;BD=10^6;Max=0;for(D=bd,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3) || (e==0 & Mod(M,4)!=1) || (e==2 & Mod(M,4)==1),next);
s=kronecker(D,p);if(s!=0,E=quadunit(D);nu=norm(E);u=(1+nu-nu*s+s)/2;A=(E^(p-s)-u)/p;
A=[component(A,2),component(A,3)];delta=valuation(A,p);
if(delta>Max,Max=delta;print("D=",D," delta=",delta))}}
```

D=29	delta=2	D=21713	delta=9	D=13495160	delta=14
D=488	delta=4	D=153685	delta=10	D=21242636	delta=16
D=1213	delta=6	D=321253	delta=12	D=71801701	delta=19
D=13861	delta=7	D=6917324	delta=13		

Which gives $\delta_3(\varepsilon) \leq 19$ on the interval $[2, 10^8]$, obtained for $m = 71801701$.

We observe in this numerical data a regularity of the increasing of the pairs (D, δ_p) and the fact that large $\delta_p(\varepsilon)$ correspond to large discriminants. For very large p there is in general no solution in a reasonable interval since we have at least, roughly speaking, $\text{Prob}(\delta_p(\varepsilon) \geq \delta) \approx p^{-\delta}$, otherwise, the solutions are often with $\delta_p(\varepsilon) = 1$, large discriminants, $C_p(K)$ being rather small as we shall analyse now.

4.2. Experiments for a conjectural upper bound - Quadratic fields. We only assume $K \neq \mathbb{Q}(\sqrt{2})$ when $p = 2$ to always have $K \cap \mathbb{Q}^c = \mathbb{Q}$.

We have given previously programs for the maximal values of $v_p(\mathcal{R}_K)$; we now give the behaviour of the whole $v_p(\#\mathcal{T}_K)$ for increasing discriminants; for this purpose, we compute $\Delta_p(K) := \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$ and $C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$.

4.2.1. PROGRAM FOR $p = 2$. The numerical data are D_K , $v_p(\#\mathcal{T}_K)$ (in *vptor*) and the successive $\Delta_p(K)$ (in *Ymin*) and $C_p(K)$ (in *Cp*); we omit the 2-rational fields (for them, *vptor* = 0):

```
{p=2;n=36;bd=5;BD=10^6;ymin=10;for(D=bd,BD,e=valuation(D,2);M=D/2^e;
if(core(M)!=M,next);if((e==1||e>3)|| (e==0&Mod(M,4)!=1)|| (e==2&Mod(M,4)==1),next);
P=x^2-D;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(sqrt(D))/log(p)-vptor;if(Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D)));
print("D=",D," vptor=",vptor," Ymin=",Y," Cp=",Cp))}
p=2
```

D=17	m=17	vptor=1	Ymin=1.04373142...	Cp=0.4893
D=28	m=7	vptor=2	Ymin=0.40367746...	Cp=0.8320
D=41	m=41	vptor=4	Ymin=-1.32122399...	Cp=1.4932
D=508	m=127	vptor=7	Ymin=-2.50565765...	Cp=1.5575
D=1185	m=1185	vptor=10	Ymin=-4.89466432...	Cp=1.9587
D=1201	m=1201	vptor=11	Ymin=-5.88498978...	Cp=2.1505
D=3881	m=3881	vptor=12	Ymin=-6.03889364...	Cp=2.0130
D=11985	m=11985	vptor=13	Ymin=-6.22552885...	Cp=1.9189
D=26377	m=26377	vptor=14	Ymin=-6.65650356...	Cp=1.9064
D=81624	m=20406	vptor=20	Ymin=-11.84164710...	Cp=2.4514

The larger computations in §4.1.1 show the largest case $m = 214203013$ with $h = 2$ and $\delta_2(\varepsilon_K) = 26$, giving $\Delta_2(K) = \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(2)} - (v_p(\#\mathcal{C}_K) + \delta_2(\varepsilon_K)) \approx -13.1628$, the best local minimum and gives $C_2 \geq 1.878992$. For the ramified case $m = 20220595$, we obtained $\delta_2(\varepsilon) = 27$, $C_2(K) = 2.055629$.

But the case of $m = 20406 = 2 \cdot 3 \cdot 19 \cdot 179$, for which $h_K = 8$, with the valuation 20, gives $C_2 \geq 2.4514$ and shows, once again, that genera theory for class groups may modify the results for $p = 2$ and more generally for $p \mid d$. So the main conjecture that we shall consider would be modified for $p \mid d$. Note that in the above results, there is no solution $m \in [20406, 10^6]$. To illustrate this, we use the same program for $m \in [20407, 0.5 \cdot 10^6]$:

```
p=2
D=81628 m=20407 vptor=2 Ymin=6.15838824... Cp=0.2451
D=81640 m=20410 vptor=4 Ymin=4.15849428... Cp=0.4902
D=81713 m=81713 vptor=5 Ymin=3.15913899... Cp=0.6128
D=81788 m=20447 vptor=7 Ymin=1.15980078... Cp=0.8578
D=82684 m=20671 vptor=8 Ymin=0.16766028... Cp=0.9794
D=83144 m=20786 vptor=9 Ymin=-0.82833773... Cp=1.1013
D=84361 m=84361 vptor=10 Ymin=-1.81785571... Cp=1.2221
D=86284 m=21571 vptor=11 Ymin=-2.80159728... Cp=1.3417
D=100045 m=100045 vptor=14 Ymin=-5.69485522... Cp=1.6857
D=115005 m=115005 vptor=16 Ymin=-7.59433146... Cp=1.9034
D=376264 m=94066 vptor=17 Ymin=-7.73930713... Cp=1.8357
```

D=495957 m=495957 vptor=19 Ymin=-9.54007224... Cp=2.0084
D=1476668 m=369167 vptor=20 Ymin=-9.75304296... Cp=1.9518

4.2.2. PROGRAM FOR $p \in [3, 50]$. In this case, genera theory does not intervene. We do not write the cases where $vptor = 0$ (p -rational fields). The constant $C_p(K)$ has some variations for very small D_K but stabilizes and seems to decrease for larger D_K ; so we mention the maximal ones, but the last value is more significant to evaluate C_p :

```
{n=16;BD=5;BD=10^6;forprime(p=3,50,print(" ");print("p=",p);ymin=10;
for(D=bD,BD,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3) || (e==0 & Mod(M,4)!=1) || (e==2 & Mod(M,4)==1),next);
P=x^2-D;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(sqrt(D))/log(p)-vptor;if(Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D));
print("D=",D," vptor=",vptor," Ymin=",Y," Cp=",Cp))}}}
```

p=3

D=24	m=6	vptor=1	Ymin=0.44639463...	Cp=0.6913
D=29	m=29	vptor=2	Ymin=-0.46747762...	Cp=1.3050
D=105	m=105	vptor=3	Ymin=-0.88189136...	Cp=1.4163
D=488	m=122	vptor=4	Ymin=-1.18266604...	Cp=1.4197
D=1213	m=1213	vptor=6	Ymin=-2.76826302...	Cp=1.8565
D=1896	m=474	vptor=7	Ymin=-3.56498395...	Cp=2.0378
D=13861	m=13861	vptor=8	Ymin=-3.65959960...	Cp=1.8431
D=21713	m=21713	vptor=10	Ymin=-5.45532735...	Cp=2.2003
D=168009	m=168009	vptor=11	Ymin=-5.52410420...	Cp=2.0088
D=321253	m=321253	vptor=12	Ymin=-6.22909046...	Cp=2.0793

p=5

D=53	m=53	vptor=1	Ymin=0.23344053...	Cp=0.8107
D=73	m=73	vptor=2	Ymin=-0.66709383...	Cp=1.5005
D=217	m=217	vptor=3	Ymin=-1.32864091...	Cp=1.7949
D=1641	m=1641	vptor=4	Ymin=-1.70010976...	Cp=1.7392
D=25037	m=25037	vptor=5	Ymin=-1.85352571...	Cp=1.5890
D=71308	m=17827	vptor=6	Ymin=-2.52836443...	Cp=1.7283
D=304069	m=304069	vptor=7	Ymin=-3.07782014...	Cp=1.7847

p=7

D=24	m=6	vptor=1	Ymin=-0.18340170...	Cp=1.2246
D=145	m=145	vptor=2	Ymin=-0.72123238...	Cp=1.5640
D=797	m=797	vptor=3	Ymin=-1.28335992...	Cp=1.7476
D=30556	m=7639	vptor=4	Ymin=-1.34640462...	Cp=1.5074
D=92440	m=23110	vptor=5	Ymin=-2.06196222...	Cp=1.7018
D=287516	m=71879	vptor=6	Ymin=-2.77039718...	Cp=1.8578

p=11

D=29	m=29	vptor=1	Ymin=-0.29786428...	Cp=1.4242
D=145	m=145	vptor=2	Ymin=-0.96227041...	Cp=1.9272
D=424	m=106	vptor=3	Ymin=-1.73853259...	Cp=2.3781
D=35068	m=8767	vptor=4	Ymin=-1.81786877...	Cp=1.8330
D=163873	m=163873	vptor=5	Ymin=-2.49637793...	Cp=1.9971

p=13

D=8	m=2	vptor=1	Ymin=-0.59464276...	Cp=2.4669
D=2285	m=2285	vptor=3	Ymin=-1.49234424...	Cp=1.9898
D=98797	m=98797	vptor=4	Ymin=-1.75808000...	Cp=1.7842
D=382161	m=382161	vptor=5	Ymin=-2.49437601...	Cp=1.9955

p=17

D=69	m=69	vptor=2	Ymin=-1.25277309...	Cp=2.6765
D=3209	m=3209	vptor=3	Ymin=-1.57516648...	Cp=2.1055
D=8972	m=2243	vptor=4	Ymin=-2.39372069...	Cp=2.4902
D=1631753	m=1631753	vptor=5	Ymin=-2.47545212...	Cp=1.9805

p=19*

D=109	m=109	vptor=1	Ymin=-0.20335454...	Cp=1.2552
D=193	m=193	vptor=2	Ymin=-1.10633396...	Cp=2.2379

D=2701	m=2701	vptor=3	Ymin=-1.65825418...	Cp=2.2359
p=23				
D=140	m=35	vptor=1	Ymin=-0.21198348...	Cp=1.2690
D=493	m=493	vptor=2	Ymin=-1.01123893...	Cp=2.0227
D=10433	m=10433	vptor=3	Ymin=-1.52451822...	Cp=2.0332
D=740801	m=740801	vptor=4	Ymin=-1.84475964...	Cp=1.8559
p=29				
D=33	m=33	vptor=1	Ymin=-0.48081372...	Cp=1.9261
D=41	m=41	vptor=2	Ymin=-1.44858244...	Cp=3.6270
D= 53093	m=53093	vptor=4	Ymin=-2.38448997...	Cp=2.4759
p=31				
D=8	m=2	vptor=1	Ymin=-0.69722637...	Cp=3.3028
D=6168	m=1542	vptor=2	Ymin=-0.72930075...	Cp=1.5739
D=90273	m=90273	vptor=3	Ymin=-1.33857946...	Cp=1.8056
D=1294072	m=323518	vptor=4	Ymin=-1.95087990...	Cp=1.9520
p=37				
D=33	m=33	vptor=1	Ymin=-0.51584228...	Cp=2.0654
D=3340	m=835	vptor=2	Ymin=-0.87650089...	Cp=1.7801
D= 124129	m=124129	vptor=3	Ymin=-1.37588711...	Cp=1.8471
p=41				
D=73	m=73	vptor=1	Ymin=-0.42232716...	Cp=1.7311
D=2141	m=2141	vptor=2	Ymin=-0.96743241...	Cp=1.9369
D=187113	m=187113	vptor=3	Ymin=-1.36552680...	Cp=1.8354
p=43				
D=88	m=22	vptor=1	Ymin=-0.40479944...	Cp=1.6801
D=6520	m=1630	vptor=2	Ymin=-0.83246977...	Cp=1.7130
D=283596	m=70899	vptor=3	Ymin=-1.33094416...	Cp=1.7974
p=47				
D=301	m=301	vptor=1	Ymin=-0.25884526...	Cp=1.3492
D=26321	m=26321	vptor=2	Ymin=-0.67821659...	Cp=1.5131
D=368013	m=368013	vptor=3	Ymin=-1.33566464...	Cp=1.8025

The same program with $p = 3$ gives, taking larger discriminants, $D \in [10^6, 2.5 \cdot 10^7]$ (two days of computer):

p=3				
D=1000005	m=1000005	vptor=1	Ymin=5.28771209...	Cp=0.1590
D=1000049	m=1000049	vptor=2	Ymin=4.28773212...	Cp=0.3180
D=1000104	m=250026	vptor=3	Ymin=3.28775715...	Cp=0.4771
D=1000133	m=1000133	vptor=4	Ymin=2.28777034...	Cp=0.6361
D=1000169	m=1000169	vptor=5	Ymin=1.28778673...	Cp=0.7951
D=1000380	m=250095	vptor=6	Ymin=0.28788273...	Cp=0.9542
D=1001177	m=1001177	vptor=8	Ymin=-1.71175481...	Cp=1.2722
D=1014693	m=1014693	vptor=9	Ymin=-2.70565175...	Cp=1.4298
D=1074760	m=268690	vptor=10	Ymin=-3.67947724...	Cp=1.5821
D=1185256	m=296314	vptor=11	Ymin=-4.63493860...	Cp=1.7281
D=2354577	m=2354577	vptor=12	Ymin=-5.32254344...	Cp=1.7970
D=6099477	m=6099477	vptor=13	Ymin=-5.88934151...	Cp=1.8282
D=13495160	m=3373790	vptor=14	Ymin=-6.52791825...	Cp=1.8736
D=21242636	m=5310659	vptor=16	Ymin=-8.32143995...	Cp=2.0837

The last case leads to $C_3(K) = 2.0837$; but it is difficult to predict the behavior of C_3 at infinity even if we notice a very low growth which remains in the same order of magnitude as for smaller discriminants.

The interval $[2, 10^6]$ is not always sufficient; for instance, we have obtained the very similar data:

p=5				
D=m=4788645		vptor=10	Ymin=-5.22138818...	Cp=2.0926
p=7				
D=m=4354697		vptor=7	Ymin=-3.07207825	Cp=1.7821

From these calculations in the quadratic case, one may consider, in an heuristic framework, that we have the good following lower bounds for C_p (if it exists):

$$C'_3 \approx 2.0837, C'_5 \approx 2.0926, C'_7 \approx 1.8578, C'_{11} \approx 1.9971, C'_{13} \approx 1.9955, C'_{17} \approx 1.9805, \\ C'_{19} \approx 2.2379, C'_{23} \approx 1.8559, C'_{29} \approx 2.4759, C'_{31} \approx 1.9520, C'_{37} \approx 1.8471, C'_{41} \approx 1.8354, \\ C'_{43} \approx 1.7974, C'_{47} \approx 1.8025.$$

Remarks 4.2. (i) For $p \gg 0$, an estimation of C_p is more difficult; for instance, we have found that for $K = \mathbb{Q}(\sqrt{19})$ and $p_0 = 13599893$, $v_{p_0}(\#\mathcal{T}_K) = 1$, whence $C_{p_0} \geq 7.585585$. Unfortunately, for $m > 19$, we did not find any example of non- p_0 -rational field $\mathbb{Q}(\sqrt{m})$.

We then have the following alternative: either $C_{p_0}(K) < 7.585585$ for all $m > 19$, whence $C_{p_0} = 7.585585$, or the maximum C_{p_0} of the $C_{p_0}(K)$ is greater than 7.585585 or infinite.

The existence of infinitely many K such that $v_{p_0}(\#\mathcal{T}_K) \gg 1$ with $C_{p_0}(K) > 7.585585$ remains possible but assume $v_{p_0}(\#\mathcal{T}_K) = O(\log_\infty(\sqrt{D_K}))$ for infinitely many K .

(ii) The existence of C_p essentially depends on $v_p(\mathcal{R}_K)$ since the influence of $v_p(\#\mathcal{C}_K)$ seems negligible, which is reinforced by the classical heuristics on class groups [3, 4], or by specific results in suitable towers [32, Proposition 7.1], and mainly by strong conjectures in [8] as $\#\mathcal{C}_K \ll_{\epsilon,p,d} D_K^\epsilon$ for any number field of degree d , strengthening the Brauer theorem (existence of a constant $C_0 > 0$ such that, $\log_\infty(\#\mathcal{C}_K) \leq C_0 \cdot \log_\infty(\sqrt{D_K})$ for all number field K). For quadratic and cyclic cubic fields, $C_0 = 1$ (Remark 2.1).

(iii) For any fixed p , $\text{Inf}_K(C_p(K)) = 0$ (see Byeon [2, Theorem 1.1], after Ono, where a lower bound of the density of p -rational fields is given for $p > 3$). Indeed, as $D_K \rightarrow \infty$, there are, statistically, most of the real quadratic fields K such that $\#\mathcal{T}_K = 1$.

(iv) Now, if K is fixed and $p \rightarrow \infty$, $\text{Inf}_p(C_p(K)) = 0$ (one may see this as an unproved generalization, for $v_p(\mathcal{R}_K)$, of theorems of Silverman [30], Graves–Murty [15] and others, showing the considerable difficulties of such subjects). We have conjectured, after numerous calculations and heuristics, that the set of primes p such that K is non- p -rational (i.e., $\mathcal{T}_K \neq 1$) is finite [10, Conjecture 8.11], i.e., $C_p(K) = 0$ for all $p \gg 0$.

4.3. A special family of quadratic fields. Consider, for p fixed, the quadratic field:

$$K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1}), \text{ with } \rho \rightarrow \infty, a \geq 1, p \nmid a;$$

assuming that $m := a^2 \cdot p^{2\rho} + 1$ is a squarefree integer, its fundamental unit is $\varepsilon_K = a \cdot p^\rho + \sqrt{m}$ and $D_K = m$ (for $a \cdot p$ even) or $4m$ (for $a \cdot p$ odd). Its p -class number is unknown, but the formula (2.9) implies $h \leq \frac{1}{2} \cdot \sqrt{D_K}$, and an upper bound being $a \cdot p^\rho$, this allows to get $v_p(\#\mathcal{C}_K) \leq \rho + \frac{\log(a)}{\log(p)}$ to take into account the possible case where h is a maximal p th power. As $\delta_p(\varepsilon_K) = \rho - 1$ and $\mathcal{W}_K = 1$ for these fields, it follows:

$$\rho - 1 \leq v_p(\#\mathcal{T}_K) = v_p(\#\mathcal{C}_K) + \delta_p(\varepsilon_K) \leq 2\rho - 1 + \frac{\log(a)}{\log(p)}.$$

Thus, since $\frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} \approx \rho + \frac{\log(a)}{\log(p)}$, we have proved, in this particular case, that:

$$\frac{\rho - 1}{\rho + \frac{\log(2a)}{\log(p)}} \leq C_p(K) \leq \frac{2\rho - 1 + \frac{\log(a)}{\log(p)}}{\rho + \frac{\log(a)}{\log(p)}} \in [1, 2].$$

These inequalities strengthen the fact that the upper bound cannot be decreased substantially.

We shall assume the conjecture that, for all p , $m := a^2 \cdot p^{2\rho} + 1$ is squarefree for infinitely many integers $\rho \geq 1$.² Whence the partial result:

²The conjecture is true for integers of the form $n^2 + 1$ [17], but we ignore if this remains true for $n = a \cdot p^\rho$, p prime, $\rho \in \mathbb{N}$, $a \geq 1$; but this is not so essential (see Remark 4.4).

Theorem 4.3. *Let \mathcal{K}_0 be the set of real quadratic fields and $C_p(K) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}$, for $K \in \mathcal{K}_0$ and $p \geq 2$. Then, under the above conjecture on $m := a^2 \cdot p^{2\rho} + 1$, for each fixed p one has $C_p(K) \in [0, 2[$ for an infinite subset of \mathcal{K}_0 .*

Moreover, if we consider the estimation of $v_p(\#\mathcal{C}_K)$ largely excessive, as explained in Remark 4.2 (i), one may conjecture that, for the above family $K = \mathbb{Q}(\sqrt{a^2 \cdot p^{2\rho} + 1})$, and $p > 2$, one has:

$$\rho - 1 \leq v_p(\#\mathcal{T}_K) \leq \rho \cdot (1 + o(1)),$$

and the statement of the theorem becomes:

For each fixed $p > 2$, $C_p(K)$ is asymptotically equal to 1 for an infinite subset of \mathcal{K}_0 .

Indeed, the small p -valuations of the class number are the following, to illustrate the relation $\rho - 1 \leq v_p(\#\mathcal{T}_K) \leq \rho \cdot (1 + o(1))$. We suppose $p > 2$ because of genera theory; we vary p and ρ in intervals such that $\log(m) < 40$:

```
{a=1;B=40;n=26;forprime(p=3,20,for(rho=1,B/(2*log(p)),m=a^2*p^(2*ro)+1;
if(core(m)==m,D=m;if(Mod(m,4)!=1,D=4*m);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;Cp=vptor*log(p)/log(sqrt(D));
h=component(component(component(K,8),1),1);vph=valuation(h,p);
print("p=",p," m=",m," rho=",rho," vptor=",vptor," Cp=",Cp," vph=",vph))}}
a=1, p=3, D=4*m
m=82 rho=2 vptor=1 Cp=0.3792886959... vph=0
m=730 rho=3 vptor=3 Cp=0.8260927150... vph=1
m=6562 rho=4 vptor=3 Cp=0.6478083724... vph=0
m=531442 rho=6 vptor=6 Cp=0.9048503676... vph=1
m=4782970 rho=7 vptor=7 Cp=0.9173194012... vph=1
m=43046722 rho=8 vptor=7 Cp=0.8110366080... vph=0
m=387420490 rho=9 vptor=10 Cp=1.0383213515... vph=2
m=3486784402 rho=10 vptor=10 Cp=0.9406514981... vph=1
m=31381059610 rho=11 vptor=10 Cp=0.8597764935... vph=0
m=282429536482 rho=12 vptor=13 Cp=1.0292195629... vph=2
m=2541865828330 rho=13 vptor=12 Cp=0.8803508063... vph=0
m=22876792454962 rho=14 vptor=14 Cp=0.9568769883... vph=1
m=1853020188851842 rho=16 vptor=15 Cp=0.9019339400... vph=0
m=16677181699666570 rho=17 vptor=17 Cp=0.9642146068... vph=1
m=150094635296999122 rho=18 vptor=19 Cp=1.0198095452... vph=2
a=1, p=5, D=4*m
m=626 rho=2 vptor=1 Cp=0.4113240423... vph=0
m=15626 rho=3 vptor=2 Cp=0.5829720101... vph=0
m=390626 rho=4 vptor=4 Cp=0.9027964983... vph=1
m=9765626 rho=5 vptor=5 Cp=0.9206955923... vph=1
m=244140626 rho=6 vptor=5 Cp=0.7775231663... vph=0
m=6103515626 rho=7 vptor=6 Cp=0.8074634864... vph=0
m=152587890626 rho=8 vptor=7 Cp=0.8303010976... vph=0
m=3814697265626 rho=9 vptor=9 Cp=0.9543323795... vph=1
m=95367431640626 rho=10 vptor=10 Cp=0.9587105826... vph=1
m=2384185791015626 rho=11 vptor=11 Cp=0.9623227412... vph=1
m=59604644775390626 rho=12 vptor=11 Cp=0.8849075871... vph=0
a=2, p=3, D=m
m=2917 rho=3 vptor=3 Cp=0.8261991487... vph=1
m=26245 rho=4 vptor=3 Cp=0.6478156494... vph=0
m=236197 rho=5 vptor=5 Cp=0.8879525243... vph=1
m=2125765 rho=6 vptor=6 Cp=0.9048504552... vph=1
m=19131877 rho=7 vptor=6 Cp=0.7862737798... vph=0
m=172186885 rho=8 vptor=7 Cp=0.8110366087... vph=0
m=1549681957 rho=9 vptor=10 Cp=1.0383213516... vph=2
m=13947137605 rho=10 vptor=10 Cp=0.9406514982... vph=1
```

```

m=10167463313317   rho=13   vptor=15   Cp=1.1004385079...   vph=3
m=823564528378597   rho=15   vptor=15   Cp=0.9596358141...   vph=1
m=7412080755407365   rho=16   vptor=15   Cp=0.9019339400...   vph=0
m=66708726798666277   rho=17   vptor=16   Cp=0.9074961005...   vph=0
m=600378541187996485   rho=18   vptor=19   Cp=1.0198095452...   vph=2
a=2, p=5, D=m
m=2501   rho=2   vptor=1   Cp=0.4113870622...   vph=0
m=62501   rho=3   vptor=2   Cp=0.5829745440...   vph=0
m=1562501   rho=4   vptor=4   Cp=0.9027966199...   vph=1
m=39062501   rho=5   vptor=5   Cp=0.9206955964...   vph=1
m=24414062501   rho=7   vptor=6   Cp=0.8074634864...   vph=0
m=610351562501   rho=8   vptor=7   Cp=0.8303010976...   vph=0
m=15258789062501   rho=9   vptor=8   Cp=0.8482954484...   vph=0
m=381469726562501   rho=10   vptor=11   Cp=1.0545816408...   vph=2
m=9536743164062501   rho=11   vptor=10   Cp=0.8748388557...   vph=0
m=238418579101562501   rho=12   vptor=11   Cp=0.8849075871...   vph=0

```

One sees from these excerpts the weak influence of the p -class group (in $vph = v_p(\mathcal{C}_K)$) giving very few $C_p(K) = 1 + o(1)$. Larger values of a, p , give the same kind of results.

Remark 4.4. Without assuming that $m := a^2 \cdot p^{2\rho} + 1$ is squarefree, the same program gives always $C_p(K)$ near 1 and in any case in $[0, 2[$ as far as we have tested this property; of course, if $m = b^2 m'$, the unit $\varepsilon = a \cdot p^\rho + b \cdot \sqrt{m'}$ is not necessarily fundamental and D_K may be small:

```

{B=60;for(a=1,18,forprime(p=2,19,for(rho=1,B/(2*log(p)),m=a^2*p^(2*rho)+1;n=rho+6;
if(core(m)!=m,P=x^2-m;K=bnfinit(P,1);D=component(component(K,7),3);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn1=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;Cp=2*vptor*log(p)/log(D);
print("a=",a," p=",p," m=",m," rho=",rho," vptor=",vptor," Cp=",Cp))}}

```

The maximal values of C_p are for trivial cases ($m = 5^2 \cdot 41$ and $250001 = 53^2 \cdot 89$):

```

a=1   p=2   D=m=1025   rho=5   vptor=4   Cp=1.4932
a=4   p=5   D=m=250001   rho=3   vptor=2   Cp=1.4342

```

4.4. Reciprocal study. We fix $p \geq 2$ and ρ large enough and we try to build units of the form $\eta = 1 + p^\rho \cdot (X + Y \cdot \sqrt{m})$, where $X, Y \in \mathbb{Z}$ and where m is a squarefree integer. It is not necessary to consider the case $\frac{X+Y \cdot \sqrt{m}}{2}$, X and Y of same parity, since this only concerns the cases $p = 2$ (in which case this modifies ρ into $\rho - 1$) and $p = 3$ (since any cube of unit is of the suitable form and this also modifies the choice of ρ).

The quadratic field $K = \mathbb{Q}(\sqrt{m})$ is such that η may be a p th power of the fundamental unit ε_K of K , but this goes in the good direction to get an upper bound of $C_p(K)$.

Lemma 4.5. *The number $\eta = 1 + p^\rho \cdot (X + Y \cdot \sqrt{m})$, $X, Y \in \mathbb{Z}$, is a unit of $\mathbb{Q}(\sqrt{m})$ if and only if $X = p^\rho \cdot a$ and $a \cdot (2 + p^{2\rho} \cdot a) = m \cdot b^2$ (resp. $a \cdot (1 + 2^{2\rho-2} \cdot a) = m \cdot b^2$) if $p \neq 2$ (resp. $p = 2$), $a, b \in \mathbb{Z}$.*

Proof. We have $N_{K/\mathbb{Q}}(\eta) = \pm 1$ if and only if:

$$1 + p^\rho \cdot (X + Y \cdot \sqrt{m}) + p^\rho \cdot (X - Y \cdot \sqrt{m}) + p^{2\rho} \cdot (X^2 - m \cdot Y^2) = \pm 1$$

which is equivalent (since -1 is absurd for $\rho > 0$) to $2 \cdot X + p^\rho \cdot X^2 = m \cdot p^\rho \cdot Y^2$. For $p \neq 2$, this yields $X = p^\rho \cdot a$, $Y = b$, such that $a \cdot (2 + p^{2\rho} \cdot a) = m \cdot b^2$. For $p = 2$, one must consider the relation $a \cdot (1 + 2^{2\rho-2} \cdot a) = m \cdot b^2$ or in practice the relation $a \cdot (1 + 2^{2\rho} \cdot a) = m \cdot b^2$ replacing ρ by $\rho - 1$. \square

So we shall fix ρ large enough, and increase a in some interval and write $a \cdot (2 + p^{2\rho} \cdot a)$ (resp. $a \cdot (1 + p^{2\rho} \cdot a)$) under the form $m \cdot b^2$, m squarefree. We then compute the successive minima of D_K for $K = \mathbb{Q}(\sqrt{m})$ to try to get maximal values of $C_p(K)$:

```
{p=3;rho=21;n=rho+6;ba=10^8+1;Ba=2*10^8;pp=p^(2*rho);Dmin=10^100;
for(a=ba,Ba,B=a*(2+pp*a);if(p==2,B=a*(1+pp*a));m=core(B);D=m;if(Mod(m,4)!=1,D=4*m);
if(D<Dmin,Dmin=D;b=component(core(B,1),2);P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(sqrt(D));
h=component(component(component(K,8),1),1);vph=valuation(h,p);
print("D=",D," a=",a," b=",b," vptor=",vptor," vph=",vph," Cp=",Cp))}
```

Since the initial value of a has some importance, one may replace the lower bound of the interval by $random(10^8)$ for instance. We have done a great lot of experimentations with very large discriminants without obtaining any $C_p(K) > 2$, except, for $p = 2$, the non-characteristic:

D=81624 a=9728 b=557872 vptor=20 vph=3 Cp=2.45147522

which is known (see §4.2.1) and corresponds to a too small discriminant since the stabilisation of $C_p(K)$ seems better and better as soon as $D_K \gg 0$.

Let $a \in [10^8 + 1, 2 \cdot 10^8]$:

p=3, rho=21

D	a	b	vptor	vph	Cp
4376759652795686111245843894049436844	100000001	1	22	2	0.5729
1094189935082719682370900209849436840	100000002	2	21	0	0.5560
6474496916274063005939132968034008	100000004	26	21	1	0.5926
844282458630797226331079592122677	100000008	36	22	0	0.6375
243749276912797619302002823856008	100000028	134	20	0	0.5892
229823745894232258887149684378792	100000044	138	21	0	0.6192
118727690905177855319225723684396	100000206	192	21	0	0.6247
31970646085359206852554564577080	100000250	370	21	1	0.6360
4248380707363826430616087080556	100000325	1015	20	0	0.6231
1265114522670559167676904722168	100000350	1860	21	0	0.6657
656519580808913341291735184520	100000860	2582	21	0	0.6720
418229777412778226609667972540	100001247	3235	21	0	0.6765
372030701452432299153951624968	100001650	3430	20	0	0.6454
13706281694543771696481156792	100001850	17870	21	0	0.7122
8423316693760059792026840684	100005496	22796	20	0	0.6834
95797673387147609199079608	100006350	213760	21	0	0.7712
38174378342984600017332360	100020480	338672	21	0	0.7833
780348725011642441673212	100250343	2374203	21	0	0.8387

Remarkably, there is no solution $a \in [10^8 + 250343, 2 * 10^8]$ giving smaller discriminants.

p=2, rho=30, n=2*rho

D	a	b	vptor	vph	Cp
11529215276652771834290899906846977	100000001	1	35	5	0.6186
17055053207700727651215465398745	100000004	26	42	11	0.8096
1045734313893579444006826247561	100000025	105	35	5	0.7019
219851764946410623412729649352	100000136	458	41	10	0.8412
13770844779037795467289609685	100000349	915	33	3	0.7060
10261042435006502090152345689	100000400	1060	36	4	0.7737
669437189125078856962747773	100000724	4150	36	5	0.8079
314916414589340809658878597	100005741	6051	35	5	0.7952
117056047025426780994615381	100006349	9925	35	5	0.8083
115315210043698339146366977	100020001	10001	31	2	0.7161
10110802585632544739300412	100023296	67552	43	8	1.0353
2465588264382971658630552	100054616	136838	35	4	0.8638
1175830296704323573471548	100097399	198235	36	6	0.9004
48025975228418415280613	100175668	490822	37	6	0.9821
28578131029527067857561	100311617	637139	34	4	0.9115
617974038061148975453	100469200	4339580	36	4	1.0424

Same remarks as for the case $p = 3$; despite genera theory, it seems that the $C_p(K)$ remains close to 1 and is not increasing substantially as $D_K \rightarrow \infty$.

5. NUMERICAL INVESTIGATIONS FOR CYCLIC CUBIC FIELDS

For the computations with cubic fields, we shall use the direct calculation of $\#\mathcal{T}_K$ from the program testing the p -rationality, taking n large enough.

See [19] for statistics on $v_p(R_{K,p}) = v_p(\mathcal{R}_K) + 2$ (resp. $v_p(\mathcal{R}_K) + 1$) in the non-ramified (resp. ramified) case, for cyclic cubic fields of conductors up to 10^8 ; this gives, for the cubic fields, the analogue of the computation of $\delta_p(\varepsilon)$ for quadratic fields in § 4.1. Note that, due to Galois action, the integers $v_p(\#\mathcal{T}_K)$ are even if $p \equiv 1 \pmod{3}$ and arbitrary if not.

5.1. Maximal values of $v_p(\#\mathcal{T}_K)$. The program uses the well-known classification of cyclic cubic fields (from [7]) with conductor $f_K \leq Bf$, and processes as for the quadratic case. We give first the case $p = 3$ to see the influence of genera theory; we compute the successive maxima of $v_p(\#\mathcal{T}_K)$ (in *vptor*) with the corresponding f_K and the polynomial defining the field of conductor f_K .

Recall that $D_K = f_K^2$, where $f_k = f'_K$ or $f_K = 9 \cdot f'_K$ with $f'_K = \ell_1 \cdots \ell_t$, for primes $\ell_i \equiv 1 \pmod{3}$:

```
{p=3;n=20;bf=7;Bf=10^7;Max=0;for(f=bf,Bf,e=valuation(f,3);if(e!=0 & e!=2,next);
F=f/3^e;if(Mod(F,3)!=1 || core(F)!=F,next);F=factor(F);Div=component(F,1);
d=component(matsize(F),1);for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(e==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Cp=vptor*log(p)/log(f);
if(vptor>Max,Max=vptor;M=f;print("f=",M," vptor=",Max," ",P," Cp=",Cp))}}}
```

p=3			
f=19	vptor=1	$P=x^3 + x^2 - 6x - 7$	
f=199	vptor=2	$P=x^3 + x^2 - 66x + 59$	
f=427	vptor=4	$P=x^3 + x^2 - 142x - 680$	
f=1843	vptor=5	$P=x^3 + x^2 - 614x + 3413$	
f=2653	vptor=6	$P=x^3 + x^2 - 884x - 8352$	
f=17353	vptor=7	$P=x^3 + x^2 - 5784x - 145251$	
f=30121	vptor=8	$P=x^3 + x^2 - 10040x + 306788$	
f=114079	vptor=9	$P=x^3 + x^2 - 38026x + 2822399$	
f=126369	vptor=10	$P=x^3 - 42123x + 3046897$	
f=355849	vptor=11	$P=x^3 + x^2 - 118616x - 15235609$	
f=371917	vptor=12	$P=x^3 + x^2 - 123972x + 15854684$	
f=1687987	vptor=15	$P=x^3 + x^2 - 562662x - 116533621$	Cp=1.1492
p=2, n=36			
f=31	vptor=2	$P=x^3 + x^2 - 10x - 8$	
f=171	vptor=6	$P=x^3 - 57x - 152$	
f=2689	vptor=8	$P=x^3 + x^2 - 896x + 5876$	
f=6013	vptor=12	$P=x^3 + x^2 - 2004x - 32292$	
f=6913	vptor=13	$P=x^3 + x^2 - 2304x - 256$	
f=311023	vptor=16	$P=x^3 + x^2 - 103674x + 5068523$	
f=544453	vptor=18	$P=x^3 + x^2 - 181484x - 19862452$	
f=618093	vptor=24	$P=x^3 - 206031x + 21289870$	Cp=1.2475
p=7			
f=9	vptor=1	$P=x^3 - 3x + 1$	
f=313	vptor=2	$P=x^3 + x^2 - 104x + 371$	
f=721	vptor=3	$P=x^3 + x^2 - 240x - 988$	
f=1381	vptor=4	$P=x^3 + x^2 - 460x - 1739$	
f=29467	vptor=6	$P=x^3 + x^2 - 9822x - 20736$	
f=177541	vptor=7	$P=x^3 + x^2 - 59180x + 3051075$	
f=1136587	vptor=10	$P=x^3 + x^2 - 378862x + 58428991$	Cp=1.3955

5.2. Experiments for a conjectural upper bound – Cubic fields. In the same way as for quadratic fields, we give, for each prime p , the successive minima of the difference $\Delta_p(K) = \frac{\log_\infty(f_K)}{\log_\infty(p)} - v_p(\#\mathcal{T}_K)$ and $C_p(K) = \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(f_K)}$, obtained for some f_K and the corresponding polynomial:

```
{n=12;bf=7;Bf=5*10^6;forprime(p=2,50,ymin=10;print("p="p);for(f=bf,Bf,
e=valuation(f,3);if(e!=0 & e!=2,next);F=f/3^e;if(Mod(F,3)!=1 || core(F)!=F,next);
F=factor(F);Div=component(F,1);d=component(matsize(F),1);
for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(e==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);Hpn0=component(C5,1);
Hpn=component(C5,2);Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);
Y=log(f)/log(p)-vptor;if(Y<ymin,ymin=Y;print(P);Cp=vptor*log(p)/log(f);
print("f=",f," ", "vptor=", vptor," Ymin=",Y," Cp=",Cp))))))}
```

The first minimum occurs for $f = 7$ and $vptor = 0$; we omit the cases of p -rationality. For $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ we have been obliged to consider larger conductors f to get significant solutions, especially for $p = 11$ for which the first non-trivial example is for $f = 5000059$ and $P = x^3 + x^2 - 1666686x - 408523339$.

For convenience, we print the final value of $C_p(K)$ at the beginning of the sequence of results for each p , then a polynomial P , its conductor f , $vptor = v_p(\#\mathcal{T}_K)$, $Ymin$. The final value obtained for $C_p(K)$ is the best lower bound for C_p on the selected interval, but in general, $C_p(K)$ stabilizes between 1 and 2.

```
p=2, Cp=1.247565
P=x^3 - 57*x - 152
f=171 vptor=6 Ymin=1.41785251... Cp=0.8088
P=x^3 + x^2 - 2004*x - 32292
f=6013 vptor=12 Ymin=0.55386924... Cp=0.9559
P=x^3 + x^2 - 2304*x - 256
f=6913 vptor=14 Ymin=-1.24490378... Cp=1.0976
P=x^3 - 206031*x + 21289870
f=618093 vptor=24 Ymin=-4.76253559... Cp=1.2475
p=3, Cp=1.14925239
P=x^3 + x^2 - 6*x - 7
f=19 vptor=1 Ymin=1.68014385... Cp=0.3731
P=x^3 + x^2 - 142*x - 680
f=427 vptor=4 Ymin=1.51312239... Cp=0.7255
P=x^3 + x^2 - 884*x - 8352
f=2653 vptor=6 Ymin=1.17582211... Cp=0.8361
P=x^3 - 42123*x + 3046897
f=126369 vptor=10 Ymin=0.69254513... Cp=0.9352
P=x^3 + x^2 - 118616*x - 15235609
f=355849 vptor=11 Ymin=0.63491606... Cp=0.9454
P=x^3 + x^2 - 123972*x + 15854684
f=371917 vptor=12 Ymin=-0.32488392... Cp=1.0278
P=x^3 + x^2 - 562662*x - 116533621
f=1687987 vptor=15 Ymin=-1.94803671... Cp=1.1492
p=5, Cp=1.462906
P=x^3 + x^2 - 50*x - 123
f=151 vptor=2 Ymin=1.11741123... Cp=0.6415
P=x^3 + x^2 - 1002*x + 6905
f=3007 vptor=4 Ymin=0.97608396... Cp=0.8038
P=x^3 + x^2 - 2214*x + 19683
f=6643 vptor=8 Ymin=-2.53143306... Cp=1.4629
p=7, Cp=1.395563
P=x^3 - 3*x + 1
```

f=9	vptor=1	Ymin=0.12915006...	Cp=0.8856
P=x ³ + x ² - 460*x - 1739			
f=1381	vptor=4	Ymin=-0.28422558...	Cp=1.0765
P=x ³ + x ² - 9822*x - 20736			
f=29467	vptor=6	Ymin=-0.71145865...	Cp=1.1345
P=x ³ + x ² - 59180*x + 3051075			
f=177541	vptor=7	Ymin=-0.78853291...	Cp=1.1269
P=x ³ + x ² - 378862*x + 58428991			
f=1136587	vptor=10	Ymin=-2.83443766...	Cp=1.3955
p=11, Cp=0.621490			
P=x ³ + x ² - 1666686*x - 408523339			
f=5000059	vptor=2	Ymin=4.43270806...	Cp=0.3109
P=x ³ - 1680483*x - 503584739			
f=5041449	vptor=4	Ymin=2.43614601...	Cp=0.6215
p=13, Cp=1.632521			
P=x ³ + x ² - 20*x - 9			
f=61	vptor=1	Ymin=0.60271151...	Cp=0.6239
P=x ³ + x ² - 196*x - 349			
f=589	vptor=2	Ymin=0.48676495...	Cp=0.8042
P=x ³ + x ² - 1064*x + 12299			
f=3193	vptor=3	Ymin=0.14576042...	Cp=0.9536
P=x ³ + x ² - 1824*x + 8919			
f=5473	vptor=4	Ymin=-0.64415121...	Cp=1.1919
P=x ³ + x ² - 19920*x + 615317			
f=59761	vptor=7	Ymin=-2.71215372...	Cp=1.6325
p=17, Cp=0.91048147			
P=x ³ - 399*x - 3059			
f=1197	vptor=2	Ymin=0.50160254...	Cp=0.7994
P=x ³ - 84837*x + 1046323			
f=254511	vptor=4	Ymin=0.39327993...	Cp=0.9105
p=19, Cp=0.97446367			
P=x ³ + x ² - 30*x + 27			
f=91	vptor=1	Ymin=0.53199286...	Cp=0.6527
P=x ³ + x ² - 404*x + 629			
f=1213	vptor=2	Ymin=0.41161455...	Cp=0.8293
P=x ³ - 3477*x - 26657			
f=10431	vptor=3	Ymin=0.14237703...	Cp=0.9547
P=x ³ + x ² - 1213944*x - 503921781			
f=3641833	vptor=5	Ymin=0.13102760...	Cp=0.9744
p=23, Cp=0.88008721			
P=x ³ + x ² - 1060*x - 11428			
f=3181	vptor=2	Ymin=0.57214663...	Cp=0.7775
P=x ³ + x ² - 515154*x - 19633104			
f=1545463	vptor=4	Ymin=0.54500411...	Cp=0.8801
p=29, Cp=1.569666			
P=x ³ + x ² - 24*x - 27			
f=73	vptor=2	Ymin=-0.72584422...	Cp=1.5696
p=31, Cp=0.981745			
P=x ³ + x ² - 30*x + 27			
f=91	vptor=1	Ymin=0.31359240...	Cp=0.7613
P=x ³ - 12027*x + 388873			
f=36081	vptor=3	Ymin=0.05578357...	Cp=0.9817
p=37, Cp=1.119764			
P=x ³ - 39*x - 26			
f=117	vptor=1	Ymin=0.31882641...	Cp=0.7582
P=x ³ + x ² - 5300*x + 119552			
f=15901	vptor=3	Ymin=-0.32086480...	Cp=1.1197
p=41, Cp=0.976052			
P=x ³ + x ² - 672*x - 2764			
f=2017	vptor=2	Ymin=0.04906930...	Cp=0.9760

```

p=43, Cp=0.914939
P=x^3 + x^2 - 20*x - 9
f=61      vptor=1      Ymin=0.09296866...   Cp=0.9149
p=47, Cp=0.878952
P=x^3 + x^2 - 2126*x + 11813
f=6379   vptor=2      Ymin=0.27543656...   Cp=0.8789

```

6. EXAMPLES OF NON-GALOIS TOTALLY REAL NUMBER FIELDS

We shall consider (non necessarily Galois) cubic and quartic fields, with an approach using randomness. The tested polynomials of degree 3 (resp. 4) define almost always Galois groups S_3 (resp. S_4). It is difficult in these cases to find non- p -rational fields for large p and to obtain an estimation of C_p for the families of cubic and quartic fields.

6.1. Program for a given cubic polynomial and increasing p . The program concerns fields K defined by $P = x^3 + ax^2 + bx + 1$ for random a, b , and increasing p in $[2, 10^5]$. It tests the irreducibility of P and that $D_K > 0$. We give only the non- p -rational cases for which one prints only the lowest difference $Ymin$ and the corresponding $C_p(K)$.

```

{m=8;bm=2;Bm=10^5;ymin=10;a=random(100);b=random(100);P=x^3+a*x^2+b*x+1;
if(polisirreducible(P)==1 & poldisc(P)>0,print(P);K=bnfinit(P,1);
D=component(component(K,7),3);forprime(p=bm,Bm,Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;
if(vptor > 0 & Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D));
print("p=",p," ", "vptor=",vptor," Ymin=",Y," Cp=",Cp))}}

```

We obtain, after several tries and p up to 10^5 :

```

P=x^3 + 21*x^2 + 47*x + 1
p=11      vptor=1      Ymin=1.75210757...
p=523    vptor=1      Ymin=0.05426629...
p=3517   vptor=1      Ymin=-0.19179768...   Cp=1.2373
P=x^3 + 19*x^2 + 51*x + 1
p=487    vptor=1      Ymin=-0.40614414...   Cp=1.6839
P=x^3 + 92*x^2 + 52*x + 1
p=18637  vptor=1      Ymin=-0.14697706...   Cp=1.1723
P=x^3 + 99*x^2 + 23*x + 1
p=73     vptor=1      Ymin=0.47867182...
p=15803  vptor=1      Ymin=-0.34379282...   Cp=1.5239
P=x^3 + 98*x^2 + 62*x + 1
p=3      vptor=2      Ymin=3.86940839...
p=61     vptor=1      Ymin=0.56857262...
p=37549  vptor=1      Ymin=-0.38783270...   Cp=1.63354
P=x^3 + 87*x^2 + 74*x + 1
p=5441   vptor=1      Ymin=0.01344518...   Cp=0.9867
P=x^3 + 73*x^2 + 67*x + 1
p=6133   vptor=1      Ymin=-0.03273397...   Cp=1.0338
P=x^3 + 19*x^2 + 83*x + 1
p=61     vptor=1      Ymin=-0.52664318...
p=5419   vptor=1      Ymin=-0.77366996...
p=12703  vptor=1      Ymin=-0.79407472...   Cp=4.8561

```

Note that by accident, $P = x^3 + 19x^2 + 83x + 1$, with a large $C_{12703}(K) \approx 4.8561$, defines the cyclic cubic field K of conductor 7 (in some sense, an analogue of $K = \mathbb{Q}(\sqrt{19})$ with $p = 13599893$ for which $C_p(K) \approx 7.5856$).

6.2. Program for a given p and random cubic polynomials. The program tries a great lot of polynomials (but only in a random way), so that the discriminants are not classified in the natural order; we then write in C_p the largest $C_p(K)$ obtained:

```
{p=3;n=18;ymin=10;for(k=1,10^6,a=random(1000);b=random(1000);c=random(1000);
P=x^3+a*x^2+b*x+c;if(polisirreducible(P)==1 & poldisc(P)>0,K=bnfinit(P,1);
D=component(component(K,7),3);Kpn=bnrinit(K,p^n);C5=component(Kpn,5);
Hpn0=component(C5,1);Hpn=component(C5,2);Hpn1=component(Hpn,1);
vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;
if(vptor > 0 & Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D));
print("P=",P," ", "vptor=",vptor," Ymin=" ,Y," Cp=",Cp))}}}
```

```
p=2
P=x^3 + 315*x^2 + 151*x + 13      vptor=6      Ymin=4.62049695...
P=x^3 + 44*x^2 + 388*x + 96      vptor=7      Ymin=2.65795067...
P=x^3 + 78*x^2 + 498*x + 584     vptor=6      Ymin=2.33817139...
P=x^3 + 473*x^2 + 759*x + 90     vptor=12     Ymin=1.79924824...
P=x^3 + 176*x^2 + 760*x + 472    vptor=14     Ymin=-0.65040380...
P=x^3 + 30*x^2 + 165*x + 220     vptor=12     Ymin=-3.98594984...
Cp=1.497370
p=3
P=x^3 + 57*x^2 + 251*x + 70      vptor=4      Ymin=2.95145981...
P=x^3 + 93*x^2 + 396*x + 396     vptor=4      Ymin=2.08419811...
P=x^3 + 53*x^2 + 602*x + 140     vptor=6      Ymin=1.91171871...
P=x^3 + 143*x^2 + 672*x + 617    vptor=8      Ymin=1.71414906...
P=x^3 + 360*x^2 + 698*x + 132    vptor=4      Ymin=1.11320078...
P=x^3 + 194*x^2 + 649*x + 440    vptor=7      Ymin=1.02340828...
P=x^3 + 38*x^2 + 343*x + 722     vptor=6      Ymin=0.41712275...
P=x^3 + 77*x^2 + 512*x + 874     vptor=8      Ymin=-0.32807458...
Cp=1.042763
p=5
P=x^3 + 177*x^2 + 590*x + 456    vptor=1      Ymin=4.94615149...
P=x^3 + 222*x^2 + 789*x + 180    vptor=2      Ymin=1.62797441...
P=x^3 + 45*x^2 + 362*x + 772     vptor=3      Ymin=1.32811388...
P=x^3 + 83*x^2 + 400*x + 251     vptor=2      Ymin=1.22069007...
P=x^3 + 197*x^2 + 718*x + 508    vptor=8      Ymin=-1.54112474...
Cp=1.238605
p=7
P=x^3 + 784*x^2 + 964*x + 288    vptor=1      Ymin=3.97483926...
P=x^3 + 505*x^2 + 710*x + 134    vptor=2      Ymin=2.57552488...
P=x^3 + 73*x^2 + 492*x + 196     vptor=3      Ymin=1.85163167...
P=x^3 + 57*x^2 + 695*x + 263     vptor=1      Ymin=1.35093638...
P=x^3 + 95*x^2 + 839*x + 252     vptor=5      Ymin=0.64570147...
P=x^3 + 114*x^2 + 804*x + 142    vptor=2      Ymin=-0.37221306...
P=x^3 + 97*x^2 + 829*x + 122     vptor=5      Ymin=-0.83742084...
Cp=1.201178
p=19
P=x^3 + 50*x^2 + 631*x + 470     vptor=1      Ymin=1.58556226...
P=x^3 + 57*x^2 + 777*x + 801     vptor=1      Ymin=1.54028119...
P=x^3 + 549*x^2 + 732*x + 39     vptor=3      Ymin=0.69038895...
P=x^3 + 93*x^2 + 891*x + 383     vptor=2      Ymin=0.64611301...
P=x^3 + 123*x^2 + 375*x + 217    vptor=1      Ymin=0.46422353...
P=x^3 + 226*x^2 + 777*x + 408    vptor=2      Ymin=0.20875475...
P=x^3 + 196*x^2 + 849*x + 918    vptor=2      Ymin=-0.24470848...
Cp=1.139412
p=1009
P=x^3 + 171*x^2 + 667*x + 604    vptor=1      Ymin=0.49598190...
P=x^3 + 89*x^2 + 567*x + 36      vptor=1      Ymin=0.37961552...
P=x^3 + 54*x^2 + 435*x + 719     vptor=1      Ymin=0.29433117...
P=x^3 + 93*x^2 + 636*x + 944     vptor=1      Ymin=0.07490160...
P=x^3 + 432*x^2 + 347*x + 19     vptor=1      Ymin=-0.06432442...
P=x^3 + 130*x^2 + 942*x + 899    vptor=1      Ymin=-0.06692434...
P=x^3 + 70*x^2 + 553*x + 735     vptor=1      Ymin=-0.18534377...
Cp=1.227512
```

Remarks 6.1. (i) The case $p = 2$ with $P = x^3 + 30x^2 + 165x + 220$, where:

$$\Delta_2(K) = \log(\sqrt{D_K})/\log(2) - 12 \approx -3.98595,$$

seems pathological, but the discriminant is rather small ($D_K = 66825$ and the algebraic closure of K contains $\mathbb{Q}(\sqrt{33})$). The Galois closure L of K is defined by the polynomial $Q = x^6 - 60x^5 + 1131x^4 - 6380x^3 - 15708x^2 + 145200x + 170368$; then $v_2(\#\mathcal{T}_L) = 25$, giving $C_2(L) \approx 1.347687$ instead of $C_2(K) \approx 1.497370$.

(ii) For $p = 5$ and $P = x^3 + 197x^2 + 718x + 508$, $v_5(\#\mathcal{T}_K) = 8$ is exceptionally large, but the discriminant is also large: $D_K = 1069350637 = 769 \cdot 1390573$ and $C_5(K) \approx 1.238605$.

(iii) For $p = 7$, $P = x^3 + 95x^2 + 839x + 252$, $v_7(\#\mathcal{T}_K) = 5$, with $C_7(K) \approx 0.885629$, but $D_K = 3486121421$, while for $P = x^3 + 114x^2 + 804x + 142$, $v_7(\#\mathcal{T}_K) = 2$ with $C_7(K) \approx 1.228662$, but $D_K = 564$.

(iv) For $p = 19$ we have performed three executions of the program to obtain one negative $Ymin$ (but with a small $D_K = 30844$).

6.3. Program for a given quartic polynomial and increasing p . We have obtained few examples with a program computing quartic polynomials with random real roots:

```
{n=4;bm=2;Bm=10^5;ymin=10;Q=1;for(k=1,4,Q=Q*(x-(random(25)-10)*Pi));P=0;
for(j=0,4,P=P+x^j*floor(component(Q,j+1)));if(polisirreducible(P)==1,K=bnfinit(P,1);
S=component(component(K,7),2);r=component(S,2);if(r==0,print(P);
D=component(component(K,7),3);forprime(p=bm,Bm,Kpn=bnrinit(K,p^n);
C5=component(Kpn,5);Hpn0=component(C5,1);Hpn=component(C5,2);
Hpn1=component(Hpn,1);vptor=valuation(Hpn0/Hpn1,p);Y=log(sqrt(D))/log(p)-vptor;
if(vptor>0 & Y<ymin,ymin=Y;Cp=vptor*log(p)/log(sqrt(D));
print("p=",p," ", "vptor=",vptor," Ymin=",Y," Cp=",Cp))}}}
```

```
P=x^4 - 7*x^3 - 326*x^2 + 558*x + 21040
p=98993 vptor=1 Ymin=0.31495485...
p=315907 vptor=1 Ymin=0.19445832... Cp=0.8372
P=x^4 - 19*x^3 - 198*x^2 + 4650*x - 12177
p=32251 vptor=1 Ymin=0.42617392... Cp=0.7012
P=x^4 + 18*x^3 - 1372*x^2 - 37952*x - 227938
p=379 vptor=1 Ymin=2.09588406...
p=15121 vptor=1 Ymin=0.91004049...
p=22907 vptor=1 Ymin=0.83101519...
p=38501 vptor=1 Ymin=0.74096974... Cp=0.5744
P=x^4 + 12*x^3 - 701*x^2 - 11349*x - 42081
p=63197 vptor=1 Ymin=0.39705480... Cp=0.7158
P=x^4 - 44*x^3 + 365*x^2 + 2604*x - 10521
p=26633 vptor=1 Ymin=0.39345331...
p=65053 vptor=1 Ymin=0.28117054... Cp=0.78053
```

7. CONJECTURE ON $v_p(\#\mathcal{T}_K)$

7.1. p -adic statement. The numerical results (quadratic and cubic cases, with the particular family of quadratic fields that we have studied in §§ 4.2, 4.3, 4.4) suggest the following conjecture that we state in its strongest form; we shall discuss about some conditions of application of such a conjecture, for instance assuming that the fields K are of given degree:

Conjecture 7.1. *Let K be any totally real number field (or any element of a specified family \mathcal{K}), and let $p \geq 2$ be a prime number.*

(i) *Then there exists a constant C_p (or $C_p(\mathcal{K})$), independent of K (or of $K \in \mathcal{K}$), such that:*

$$v_p(\#\mathcal{T}_K) \leq C_p \cdot \frac{\log_{\infty}(\sqrt{D_K})}{\log_{\infty}(p)}, \text{ for all } K (K \in \mathcal{K}).$$

(ii) The residue $\tilde{\kappa}_{K,p}$, at $s = 1$, of the normalized p -adic ζ -function $\frac{f_{K \cap \mathbb{Q}^c}}{2^d} \zeta_{K,p}(s)$ of K (see § 2.1), is conjecturally such that $v_p(\tilde{\kappa}_{K,p}) \leq C_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}$, for all K ($K \in \mathcal{K}$).

Remark 7.2. In [18], Hajir and Maire define, in the spirit of an algebraic p -adic Brauer–Siegel theorem, the *logarithmic mean exponent* of a finite p -group $A \simeq \prod_{i=1}^r \mathbb{Z}/p^{a_i}\mathbb{Z}$, by means of the formula $\mathbb{M}_p(A) := \frac{1}{r} \cdot \frac{\log_\infty(\#A)}{\log_\infty(p)} = \frac{1}{r} \sum_{i=1}^r a_i = \frac{1}{r} \cdot v_p(\#A)$, and applied to tame generalized class groups. In the case of \mathcal{T}_K , we get $v_p(\#\mathcal{T}_K) = \text{rk}_p(\mathcal{T}_K) \cdot \mathbb{M}_p(\mathcal{T}_K)$, and we would have conjecturally, for totally real fields:

$$\mathbb{M}_p(\mathcal{T}_K) \leq C_p \cdot \frac{1}{\text{rk}_p(\mathcal{T}_K)} \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)} \leq C_p \cdot \frac{\log_\infty(\sqrt{D_K})}{\log_\infty(p)}.$$

But in [18, Theorems 0.1, 1.1, Proposition 2.2], this function \mathbb{M}_p is essentially used in particular infinite towers with tame restricted ramification.

7.2. Comparison « archimedean » versus « p -adic ». The above considerations are, in some sense, a p -adic approach of some deep and very general results (Brauer–Siegel–Tsfasman–Vladuț theorems [32, 35] and broad generalizations in [31]; see also [21] for quantitative bounds about the Brauer–Siegel theorem) on the behavior, in a tower $L := \bigcup_{n \geq 0} K_n$ of finite extensions K_n/K , of the quotient:

$$BS_n := \frac{\log_\infty(h_n \cdot R_n)}{\log_\infty(\sqrt{D_n})},$$

with obvious notations for each K_n . Of course, in order to infer the p -adic case, our purpose is to deal, in the archimedean one, with any totally real number field K or with sub-families \mathcal{K} fulfilling some conditions (e.g., $[K : \mathbb{Q}] = d$, $D_K \rightarrow \infty$, $\frac{[K : \mathbb{Q}]}{\log_\infty(\sqrt{D_n})} \rightarrow 0$).

For any totally real field K , let:

$$BS_K := \frac{\log_\infty(h_K \cdot R_{K,\infty})}{\log_\infty(\sqrt{D_K})}.$$

We shall also consider the normalized quotient (which does not matter for the existence of lower and upper bounds since $\widetilde{BS}_K = BS_K - 1$):

$$(7.1) \quad \widetilde{BS}_K := \frac{\log_\infty\left(h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}}\right)}{\log_\infty(\sqrt{D_K})} = \frac{\log_\infty(\mathcal{T}_{K,p,\infty})}{\log_\infty(\sqrt{D_K})}, \quad K \in \mathcal{K} \text{ (from formula (2.2))},$$

and presume that this function is bounded over \mathcal{K} . This is a broad purpose, but the literature is rich on the subject and we shall give few numerical heuristics.

The following program gives, for the set \mathcal{K}_0 of quadratic fields, consistent results for the original function BS :

```
{Max=0;Min=1;for(D=10^6,10^6+10^4,e=valuation(D,2);M=D/2^e;if(core(M)!=M,next);
if((e==1 || e>3) || (e==0 & Mod(M,4)!=1) || (e==2 & Mod(M,4)==1),next);P=x^2-D;
K=bnfinit(P,1);C8=component(K,8);h=component(component(C8,1),1);reg=component(C8,2);
BS=2*log(h*reg)/log(D);if(BS<Min,Min=BS);if(BS>Max,Max=BS));print(Min," ",Max)}
```

with $0.647 < BS < 1.155$ for $m \in [10^5, 2 \cdot 10^5]$, $0.734 < BS < 1.136$ for $m \in [10^7, 10^7 + 10^5]$, and $0.765 < BS < 1.124$ for $m \in [10^8, 10^8 + 10^5]$, showing, for real quadratic fields K , the probable heuristic $BS_K \leq 1 + O(1)$.

In the same way for cyclic cubic fields (in the program, h denotes the class number h_K and reg the classical regulator $R_{K,\infty}$):

```
{bf=10^6;Bf=2*10^6;Max=0;Min=1;for(f=bf,Bf,e=valuation(f,3);if(e!=0&&e!=2,next);
F=f/3^e;if(Mod(F,3)!=1 || core(F)!=F,next);F=factor(F);Div=component(F,1);
d=component(matsize(F),1);for(j=1,d-1,D=component(Div,j);if(Mod(D,3)!=1,break));
for(b=1,sqrt(4*f/27),if(e==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(e==0,if(Mod(a,3)==1,a=-a);P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(e==2,if(Mod(a,9)==3,a=-a);P=x^3-f/3*x-f*a/27);K=bnfinit(P,1);
C8=component(K,8);h=component(component(C8,1),1);reg=component(C8,2);
BS=log(h*reg)/log(f);if(BS<Min,Min=BS);if(BS>Max,Max=BS));print(Min," ",Max)}
```

with $0.6653 \leq BS \leq 1.1478$ for $f \in [10^4, 10^6]$, $0.7547 \leq BS \leq 1.1385$ for $f \in [10^6, 2 \cdot 10^6]$, showing, for cubic fields, the same heuristic $BS_K \leq 1 + O(1)$.

This is in excellent agreement with the results of [32, Theorem H] giving, in a well-defined context of infinite towers, the universal upper bound 1.1588... and the universal lower bound 0.4087....

Remark 7.3. In the archimedean point of view (using the notation p_∞ for the infinite place instead of ∞ for complex functions as $\log_\infty =: \ln$ or $R_{K,\infty}$ built with \log_∞), we have put $\#\mathcal{T}_{K,p_\infty} := h_K \cdot \frac{R_{K,\infty}}{\sqrt{D_K}}$, giving from the expression (7.1) of \widetilde{BS}_K :

$$\log_\infty(\#\mathcal{T}_{K,p_\infty}) = \log_\infty(h_K) + \log_\infty(R_{K,\infty}) - \log_\infty(\sqrt{D_K}) = \widetilde{BS}_K \cdot \log_\infty(\sqrt{D_K});$$

thus we obtain about the above calculations for examples of fixed family \mathcal{K} :

$$\log_\infty(\#\mathcal{T}_{K,p_\infty}) \leq O(1) \cdot \log_\infty(\sqrt{D_K}) \quad \text{written} \quad \log_\infty(\#\mathcal{T}_{K,p_\infty}) \leq C_{p_\infty} \cdot \log_\infty(\sqrt{D_K}),$$

giving, in some sense, the inequality of the p -adic Conjecture 7.1 with the audacious convention for the *infinite place* p_∞ :

$$\log_\infty(p_\infty) = 1 \quad \& \quad v_{p_\infty}(\#\mathcal{T}_{K,p_\infty}) = \log_\infty(\#\mathcal{T}_{K,p_\infty}).$$

in which case, the constant C_{p_∞} should be the maximal value reached by $\widetilde{BS}_K = BS_K - 1$ over the given family \mathcal{K} .

8. CONCLUSIONS

The analysis of the archimedean case is sufficiently significant to hope the relevance of the p -adic one for which we give some observations:

(a) We note that in the p -adic Conjecture 7.1, the most important term is the valuation $v_p(\mathcal{R}_K)$ of the normalized p -adic regulator, the contribution of $v_p(\#\mathcal{C}_K)$ being probably negligible compared to $v_p(\#\mathcal{R}_K)$ as shown, among other, by classical heuristics [3, 4], giving for quadratic fields, $\text{Prob}(p^v \parallel h_K) = \prod_{\alpha \leq v} \left(1 - \frac{(p)_\infty}{(p)_1}\right) \sum_{0 \leq \beta \leq \alpha} \frac{1}{p^{2\beta} \cdot (p)_\beta}$, where $(p)_\gamma = \prod_{1 \leq k \leq \gamma} \left(1 - \frac{1}{p^k}\right)$, $\gamma \in \mathbb{N} \cup \{\infty\}$; this gives for instance 0.019779 for the divisibility by $p^v = 9$; which is clear from our numerical calculations. This is reinforced by more recent conjectures given in the Remark 4.2 (i). Furthermore, for K fixed, $v_p(\#\mathcal{C}_K) \geq 1$ for finitely many primes p , but the case of $v_p(\mathcal{R}_K)$ is only an out of reach conjecture, all this assuming the still unproved Leopoldt conjecture.

(b) The family of §4.3 shows that p -adic regulators may tend to 0 even in simplest cases; it should be interesting to find other special families of units with parameters depending of a p -power p^n , $n \rightarrow \infty$, to confirm the relation between $v_p(\#\mathcal{R}_K)$ and $\log_\infty(\sqrt{D_K})$ (i.e., the existence of the constant C_p). This would be usefull to see if $\sqrt{D_K}$ is the best normalization for the definition of $C_p(K)$; one can think for instance about the «root discriminant» $D_K^{1/[K:\mathbb{Q}]}$.

(c) Consider (for any $p \geq 2$ and any totally real K):

$$\mathcal{C}(K, p) := \frac{v_p(\#\mathcal{T}_K) \cdot \log_\infty(p)}{\log_\infty(\sqrt{D_K})}, \quad \mathcal{C}(p) := \text{Sup}_K(\mathcal{C}(K, p)), \quad \mathcal{C}(K) := \text{Sup}_p(\mathcal{C}(K, p)).$$

The existence of $\mathcal{C}(K)$, for a given K , only says that the conjecture [10, Conjecture 8.11], claiming that any number field is p -rational for $p \gg 0$, is true for the fixed field K .

We have a p -adic analog of Brauer–Siegel theorem (i.e., Conjecture 7.1) if $\mathcal{C}(p)$ does exist, for fixed p ; this does not imply the existence of an universal bound C for all K & p .

The existence of $\mathcal{C}(K)$, is more credible by reference to our conjecture, but that of $\mathcal{C}(p)$ may be exact taking instead $\text{Sup}_{K \in \mathcal{K}}$, for particular families \mathcal{K} (e.g., extensions of fixed degree or some infinite towers as in [18, 20, 32, 35]); but we must mention that for the invariants \mathcal{T}_K , the transfer map $\mathcal{T}_K \rightarrow \mathcal{T}_{L,p}$ is injective in any extension L/K (in which Leopoldt’s conjecture is assumed, [9, TheoremIV.2.1]), which leads to a major difference from the case of p -class groups.

Furthermore, it seems that $\text{Lim Sup}_{K \in \mathcal{K}}(C_p(K))$ may be 1 or < 1 ; but computations for very large discriminants (of a great lot of quadratic fields for instance) is out of reach.

Moreover, the tricky Greenberg’s conjecture, on the triviality of the Iwasawa p -invariants λ , μ for K^c , in the totally real case, depends also on Fermat’s quotients of algebraic numbers [11, § 7.7], [14, § 4.2] and goes in the sense of rarity for p -class groups as we have mention in Remark 4.2 (i), but the discussion about \mathcal{T}_p groups in K^c remains open.

But all this is far to be proved because of a terrible lack of knowledge of p -Fermat quotients of algebraic numbers, a notion which gives a *weaker information* than the p -adic logarithm, but which governs many deep arithmetical problems, even assuming Leopoldt’s conjecture.

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