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Perceiving the topological structure of space through sensorimotor interaction

Valentin Marcel, Sylvain Argentiari and Bruno Gas

Abstract—This paper deals with the perception of mobile robotic systems within the framework of interactive perception, and inspired by the sensorimotor contingencies theory. These approaches state that perception is an ability built from the motor and sensory data that an agent/robot experiences through its environment exploration. In a recent article, the authors have shown that the perception of space, provided it is limited to the agent’s own body, finds a solid mathematical basis in the SMC theory. An extension of these results to the agent working space is proposed in this paper. More precisely, it is demonstrated that a specific motor quotient space constitutes one possible support for a good representation of space. By defining a partial order relation over all the set of partitions of the motor space, a refinement process is defined on the basis on the sensorimotor invariants. The finest motor partition, obtained after having experienced all environment states, is then shown to be homeomorphic to the robot working space. A very simple algorithm implementing these ideas is proposed together with mathematical proofs establishing its convergence. Simulations show the properties of the obtained spatial representation for different scenarios.

I. INTRODUCTION

Space perception is a central issue in mobile robotics. Indeed, many abilities depend on it, as moving, trajectory planning or obstacle avoidance. Traditional approaches consider that space is something that exists objectively. But the sensorimotor contingencies theory (SMC) [1], [2] claims that it has not to be the case. Space has not to be an established substrate *per se*, but something that an agent may experiences via the determination of sensorimotor invariants called *contingencies*. In other words the discovery, at first, and then the use of such contingencies is enough to make an agent realize actions without the need of having an internal, local or global representation, analytic or not, of space. Terekhov et al. [3] have shown it unambiguously by putting in obviousness a φ function that can be learned only from the sensorimotor flow, and able to represent any translation, whatever its origin and whatever the state of the environment. Le Clech et al. [4] have extended this idea by proposing a representation of the group of the two-dimensional space transformations. The underlying idea is based on the notion of *compensable sensory changes* proposed by Poincaré [5], [6]. The mathematician was the first to formalize the idea that one can obtain informations about the geometric space in which we are immersed, by only comparing signals sent to our muscles with signals coming from our sensory organs. Our performed actions induce sensory variations through the environment, which can then be perceived,

provided the so called contingencies are detected. Poincaré has defined a compensable transformation as an action that a mobile agent can do in order to retrieve an initial sensation previously modified by a transformation of the environment. The set of these transformations forms a group which is at all points identical to the group of geometric transformations of the space. Since then, substantial works have been published about considering action in the structuring of perception [7]–[11], some of them aiming to verify the Poincaré idea, but more recently with a growing interest for robotics applications. For instance, [12] introduces very recently “interactive perception” as a set of approaches in robotics concerned with the implication of action in perception.

In the early 2000s, Philipona [13] proposed a first mathematical formalism by defining the sensorimotor law as $\mathbf{s} = \Psi(\mathbf{e}, \mathbf{m})$ where \mathbf{s} denotes the sensation vector, \mathbf{e} the vector representing the state of the environment and \mathbf{m} the motor state of the agent. By analyzing the sensory data received by a simulated agent in a three-dimensional space (a rat equipped with extremely simple visual, auditory and proprioceptive sensors), he succeeded in extracting the order of the group of compensable transformations, and thus the dimension of the geometric space in which the simulated rat navigated. These results corroborate Poincaré’s intuition, but only in a limited way in that they are limited to infinitesimal movements of the agent. A constraint that can be explained by the linear nature of the analysis tools used. In 2012, Laflaquière [14] has taken up the idea using a bootstrap technique and a curvilinear component analysis of data (CCA) [15]. By this way, he confirmed the ideas of Poincaré for more realistic movements amplitude (up to ten degrees for rotations).

The approach initiated by Philipona, however, requires a strong assumption that contradicts the initial postulate of the SMC theory. The agent must be able to determine by its own whether the environment is stationary or not. The authors have not proposed a clear solution to this problem, but Roschin and Frolov [16] have raised this objection and have suggested another formulation of the problem. They consider an agent endowed with a tactile skin completely covering it, and a redundant robotic arm equipped at its end with an optical sensor. The arm end-effector has to touch the body itself so as to ensure a stable perception, thus obtained without any prior hypothesis on the environmental state. Such an agent may be of any non-deformable shape, spherical for example. They thus succeeded in obtaining an internal representation allowing them to determine the dimension of the space delimited by the body of the agent, in a multimodal context (the tactile modality in addition to the arm end-effector sensor). Of course

this approach temporarily avoids the problem of unstable environment. But it remains crucial since the question of the agent perception by itself constitutes a central point of perception in general [17]. In the same vein, Laflaquière [18] more recently proposed to analyze the perception of the body in the SMC theory framework by letting the agent discovering the visual fields generated by an initially unknown visual sensor, without prior knowledge about the structure nor of the agent body nor of the external world.

Laflaquière et al. [19] have also shown that, beyond the dimension of space, it is also possible to obtain an external space representation by using appropriate partitions of the motor space, resulting in a much more motor oriented than sensor oriented framework. In this contribution, they have proposed to take up the Frolov idea of the robotic arm, but performing an exploration not limited to the agent body. Each end-effector position of the arm is represented by the subset of the motor configurations letting invariant the sensory state, the so-called kernel manifolds. Their objective was to understand how the concept of space can emerge independently of the environment while sensorimotor experiments are dependent on the environment. By noting that, due to the mechanical redundancy of the agent, different motor configurations of the arm can lead to the same position of the end-effector, giving rise to the same sensor signal, the authors have suggested a partitioning of the motor configuration space for which each element of a partition corresponds to a same position of the sensor in its working space. Such a representation becomes then independent of the state of the environment since it is based on a motor coding. Using a CCA-type data analysis and an adequate distance (the Hausdorff distance) defined on the elements of the partition, Laflaquière [20] succeeded to build a motor internal representation of the positions occupied by the end-effector without any external knowledge about its working space.

At this stage it should be noted that there is no mathematical proof that the structure thus constructed is indeed a representation of the external geometric space. Nevertheless, the simulations clearly show a regularity of the built maps suggesting that these are indeed tonotopic maps and that the topology of the external geometric space seems correctly constructed. We have proposed in [21] to formalize this problem by adopting an external point of view from the agent. For that, we have taken at first Frolov's experiment of the action space limited to the body of the agent. By describing mathematically the set of partitions of the motor space as a set of equivalence classes, we have been able to show formally that the motor representation is well isomorphic to the work space of the robotic arm. In [22] we have further shown that the topological properties of the space of the body are well captured. From then on, it becomes possible to plan robotic arm movements directly into the internal representation space.

In this article we propose to extend this proof to the general case. The end-effector arm working space is extended outside the body of the agent, in such a way that all the environment state possibilities have to be considered. By defining the various environment states as state sequences, we show that it is possible to construct a partitioning process of the

motor space (the quotient motor space) which, by successive iterations, allow to obtain a final partition that fit the external working space, unknown to the agent at first. Mathematical proof are given to comfort algorithm convergence. Such a process requires the comparison of successive partitions of the engine space. This is done by defining a partial order relation which is at the origin of the notion of space refinement. Finally, we show that the topological properties of the end-effector working space are identically extracted from the process of refinement.

The first section of this article is devoted to the presentation of the mathematical fundamentals required to formalize the notion of space refinement. The second section presents how the working space can be represented from the quotient motor space. The point of view adopted here is clearly that of the external observer, since the agent himself has no access to this space. The section ends with topological considerations aiming to explain the topological properties of the final obtained structure, in accordance with the topology of the external working space. Finally, we propose a set of simulations as proofs of concepts illustrating the way of achieving the reconstruction of the agent working space.

II. MOTOR SPACE REFINEMENT FROM SENSORY INVARIANCE: DEFINITION AND MATHEMATICAL FORMALIZATION.

This first section is devoted to the mathematical roots required to precisely formalize the notion of space refinement. As a first step, the first subsection is devoted to the mathematical considerations explaining how the motor space can be partitioned using sensory invariance, giving rise to a partial order at the origin of the notion of refinement. On this basis, the theory is generalized to handle successive environments in a second subsection. All along the sections, the same simple example will be used to illustrate the theory and its limits.

A. The notion of space refinement

1) *Notations and definitions:* Let's first consider a naive agent, be it virtual or robotic, which can interact with its environment by generating motor commands lying in the *motor configuration space* \mathcal{M} . This space can be described by latent variables parameterizing the agent actuators states (i.e. joint angles, positions, etc.), thus forming a *motor state* $\mathbf{m} \in \mathcal{M}$. The agent is also endowed with sensors rigidly placed on its own body parts. Those sensors inform the agent about the environment's physical state, thus generating a *sensory state* $\mathbf{s} \in \mathcal{S}$, with \mathcal{S} the *sensory space*. Of course, the sensory state \mathbf{s} relates to the agent motor state \mathbf{m} through the *sensorimotor law* $\Psi_\epsilon(\cdot)$, so that in a given environment state $\epsilon \in \mathcal{E}$ and a motor configuration \mathbf{m} , $\mathbf{s} = \Psi_\epsilon(\mathbf{m})$. This relation also outlines the dependency of the sensory state \mathbf{s} to the environment state ϵ , with \mathcal{E} the set of environment states.

It is obvious that because of the possible redundancies in the agent geometry or in the sensory redundancies for an environment in state ϵ , the function $\Psi_\epsilon(\cdot)$ is surjective. Basically, it means that, at ϵ , two different motor states \mathbf{m}_i and \mathbf{m}_j , $i \neq j$, can lead to the very same sensory state \mathbf{s} .

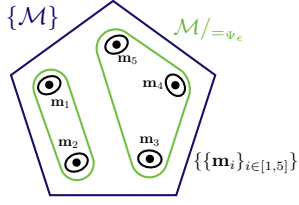


Fig. 1. Representation of three partitions of the motor space \mathcal{M} : the trivial one $\{\mathcal{M}\}$ (blue), the set of equivalence classes at ϵ $\mathcal{M}/=\Psi_\epsilon$ (green), and the set of singletons of motor configuration states $\{\mathbf{m}_i\}$ (black).

As outlined in the previous authors work [22], one can then define an equivalence relation $=_{\Psi_\epsilon}$, such that

$$\mathbf{m}_i =_{\Psi_\epsilon} \mathbf{m}_j \Leftrightarrow \Psi_\epsilon(\mathbf{m}_i) = \Psi_\epsilon(\mathbf{m}_j). \quad (1)$$

Thus, one can regroup all the motor states leading to the same sensory state in their equivalence class $K_{\mathbf{m}}^\epsilon = \{\mathbf{r} \in \mathcal{M} | \mathbf{r} =_{\Psi_\epsilon} \mathbf{m}\}$. It is well known that the set of all equivalence classes forms a *partition*¹ of the set on which the equivalence relation is defined. In other words, every element in \mathcal{M} is included in one and only one equivalence class $K_{\mathbf{m}}^\epsilon$. Additionally, the quotient set $\mathcal{M}/=\Psi_\epsilon = \{K_{\mathbf{m}}^\epsilon | \mathbf{m} \in \mathcal{M}\}$ forms a *refinement* of the trivial partition $\{\mathcal{M}\}$ of \mathcal{M} , i.e. every element in $\mathcal{M}/=\Psi_\epsilon$ is composed of subsets of the trivial equivalence class \mathcal{M} . The refinement relation can then be written as

$$\mathcal{M}/=\Psi_\epsilon \leq \{\mathcal{M}\}, \quad (2)$$

where the symbol \leq refers to the *finer than* relation, which also defines a *partial order*². The partition X is finer than the partition Y if every element in X is a subset of some element of Y , i.e. X can be seen as composed of fragmented parts of Y . Equation (2) can then be completed such that

$$\{\{\mathbf{m}\} | \mathbf{m} \in \mathcal{M}\} \leq \mathcal{M}/=\Psi_\epsilon \leq \{\mathcal{M}\}, \quad (3)$$

which is illustrated in Figure 1 in the particular case of a finite motor configuration space \mathcal{M} , made of 5 different motor configurations $\mathbf{m}_i, i \in [1, 5]$. In this illustration, consider that at environment state ϵ , the subsets surrounded by a green contour are equivalence classes composing the partition $\mathcal{M}/=\Psi_\epsilon$ of \mathcal{M} . Then $\mathcal{M}/=\Psi_\epsilon$ is finer than the trivial partition $\{\mathcal{M}\}$. Similarly the set of singletons $\{\mathbf{m}_i\}$ for $i \in [1, 5]$ is again a finer partition than $\mathcal{M}/=\Psi_\epsilon$ or $\{\mathcal{M}\}$. It is also the finest partition of \mathcal{M} . Then, for a naive agent with absolutely no information about its sensorimotor flow, $\{\mathcal{M}\}$ is the *coarsest* sensorimotor partition it can access to, where each motor configuration is not distinguishable from each other. As such, one can write $\{\mathcal{M}\} = \mathcal{M}/=\Psi_\emptyset$, i.e. $\{\mathcal{M}\}$ is the quotient set obtained with $\epsilon = \emptyset$. Then, the aim of the sensorimotor exploration of the agent could consist in establishing the finest partition that can be obtained exploiting sensory invariances. Such a partition might be coarser than the one obtained from singletons as it captures the structure of sensory invariances

¹A partition of a set X is a set of non-empty, pairwise disjoint, subsets whose union forms the set X itself.

²A partial order is a binary relation satisfying the reflexivity, antisymmetry and transitivity axioms.

of the agent's sensorimotor flow. In that sense, it will be called a *representation* of the agent's interaction in a specific environment state. Moreover, this kind of representation can be extracted for any function Ψ_ϵ , evaluated at any environment state ϵ as relation (3) is always true.

2) *Illustrative example*: All the aforementioned considerations were mainly theoretical. Let's now focus on a more experimental illustration of these points by using a very simple simulated robot agent made of one serial arm composed of two parts of identical length controlled by two revolute joints moving in a plane, see figure 2. The end-effector of the system is endowed with a 1-pixel (punctual) camera which is only sensitive to illumination in such a manner that it can only send two values: $s = 0$ if the illumination is zero and $s = 1$ otherwise. For the sake of simplicity, the system is driven by two motor commands m_1 and m_2 , which are supposed to represent directly the two joint angles, so that by convention $m_1, m_2 \in [-\pi, \pi[$ ($m_1 = m_2 = 0$ makes the arm horizontal on the right in Figure 2).

Suppose now that the environment is made of a black and a white areas separated by a straight line, as depicted in Figure 3. Of course, the agent does not have access to this information and can only rely on its sensorimotor flow, i.e. variations of m_1, m_2 and their sensory consequences. At the very beginning, the set of all motor commands $\mathbf{m} = (m_1, m_2)$ have not been distinguished from each other so that the representation of the sensorimotor interaction is the coarsest partition $\{\mathcal{M}\}$. After having explored this black and white environment, the agent is able to obtain a finer representation. Indeed, two equivalent classes can be easily formed by regrouping all the motor commands \mathbf{m} giving the same sensation, namely, for an environment state ϵ , K_0^ϵ and K_1^ϵ . Then, the set $\{K_0^\epsilon, K_1^\epsilon\}$ forms a partition of the set $\{\mathcal{M}\}$ such that $\{K_0^\epsilon, K_1^\epsilon\} \leq \{\mathcal{M}\}$, which can be represented as a two nodes graph, see Figure 4. This partition is also drawn directly in the motor space in Figure 3b. Of course, for this specific environment state, the representation above, based on sensory invariances, is the finest possible. It is clear the agent will never be able to separate different motor configurations that led to identical sensations in this environment state. Actually, this means that a representation is only valid for *one specific* environment state, thus highlighting the *environment dependency* of the representation, formally captured by the dependency to the environment state ϵ in the equivalence relation $=_{\Psi_\epsilon}$. This dependency has been discussed in many publications [13],

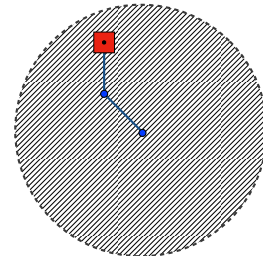


Fig. 2. The 2D serial agent with 2 degrees of freedom and the 1-pixel camera (red) and its working space (dashed line).

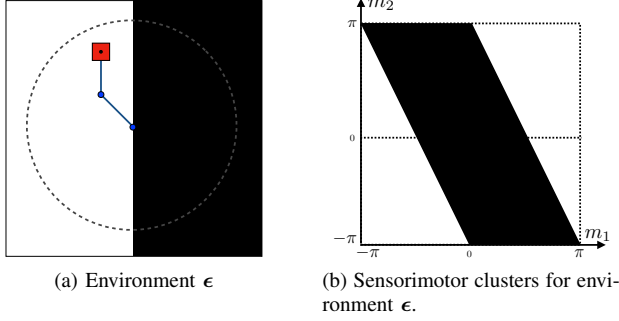


Fig. 3. Representation of the environment state ϵ (left), and its corresponding sensorimotor clusters (right).

[16], [19], [23], and no clear solutions have been proposed so far to deal with the environment variability. In these works, the environment is systematically considered static and often restrict their study to cases where the environment changes do not influence the sensorimotor flow (by working on the agent body, for instance, like in [22]).

However if the environment happens to change, it is possible that previously inseparable motor configurations (thus regrouped in one equivalence class) are now generating different sensations. This would allow the agent to fragment again the previous equivalence class, i.e. to refine the sensorimotor representation. This will be formalized in the following subsection.

B. Generalization to multiple environments

1) *Mathematical formalization:* Following the ideas detailed in §II-A, let's consider a naive agent that has built, for one environment state ϵ , the partition $\mathcal{M}/_{=\Psi_\epsilon}$. If the environment state switches from ϵ to ϵ' , then the agent can build a new representation on the basis of a new equivalence relation $=_{\Psi_{\epsilon'}}$, leading to the new motor space partition $\mathcal{M}/_{=\Psi_{\epsilon'}}$. It should be highlighted that for any $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}$, $\mathbf{m}_1 =_{\Psi_\epsilon} \mathbf{m}_2 \Leftrightarrow \mathbf{m}_1 =_{\Psi_{\epsilon'}} \mathbf{m}_2$ and $\mathbf{m}_1 \neq_{\Psi_\epsilon} \mathbf{m}_2 \Leftrightarrow \mathbf{m}_1 \neq_{\Psi_{\epsilon'}} \mathbf{m}_2$. It clearly means that the two partitions $\mathcal{M}/_{=\Psi_\epsilon}$ and $\mathcal{M}/_{=\Psi_{\epsilon'}}$ can't be directly compared, as they are both representing one specific motor partition dedicated to a given environment state. However, one can define a new *multi-environment* equivalence

relation $=_{\Psi_{(\epsilon, \epsilon')}}$ as

$$\mathbf{m}_1 =_{\Psi_{(\epsilon, \epsilon')}} \mathbf{m}_2 \Leftrightarrow \begin{pmatrix} \mathbf{m}_1 =_{\Psi_\epsilon} \mathbf{m}_2 \\ \text{and} \\ \mathbf{m}_1 =_{\Psi_{\epsilon'}} \mathbf{m}_2 \end{pmatrix}. \quad (4)$$

It is obvious that, according to its definition, this equivalence relation leads to equivalent classes $K_{\mathbf{m}}^{(\epsilon, \epsilon')}$ verifying $K_{\mathbf{m}}^{(\epsilon, \epsilon')} \subseteq K_{\mathbf{m}}^\epsilon$ and $K_{\mathbf{m}}^{(\epsilon, \epsilon')} \subseteq K_{\mathbf{m}}^{\epsilon'}$. In terms of the partial order \leq , this property can also be written

$$\left\{ \begin{array}{l} \{\{\mathbf{m}\} | \mathbf{m} \in \mathcal{M}\} \leq \mathcal{M}/_{=\Psi_{(\epsilon, \epsilon')}} \leq \mathcal{M}/_{=\Psi_\epsilon} \leq \{\mathcal{M}\} \\ \text{and} \\ \{\{\mathbf{m}\} | \mathbf{m} \in \mathcal{M}\} \leq \mathcal{M}/_{=\Psi_{(\epsilon, \epsilon')}} \leq \mathcal{M}/_{=\Psi_{\epsilon'}} \leq \{\mathcal{M}\}, \end{array} \right. \quad (5)$$

which shows that, by definition, the partition $\mathcal{M}/_{=\Psi_{(\epsilon, \epsilon')}}$ is a refinement of both partitions $\mathcal{M}/_{=\Psi_\epsilon}$ and $\mathcal{M}/_{=\Psi_{\epsilon'}}$. Note that this multi-environment equivalence relation $=_{\Psi_{(\epsilon, \epsilon')}}$ does not depend on the order of ϵ and ϵ' . Consequently, the tuple (ϵ, ϵ') can be written as a subset $E = \{\epsilon, \epsilon'\}$.

Based on the idea that intersecting partitions obtained on multiple environments gives a finer representation, one can define the generic multi-environment equivalence relation $=_{\Psi_E}$, for any subset $E \subseteq \mathcal{E}$, as

$$\mathbf{m}_1 =_{\Psi_E} \mathbf{m}_2 \Leftrightarrow \mathbf{m}_1 =_{\Psi_\epsilon} \mathbf{m}_2, \forall \epsilon \in E. \quad (6)$$

Note that the notation $=_{\Psi_E}$ represents the equivalence relation defined in (6), i.e. on all $\epsilon \in E$. The notation Ψ_E will be formally defined as a function in §III-A3, see relation (14).

It is then possible to derive interesting properties of the order relation \leq on subsets of \mathcal{E} . For instance, let's consider two non-empty subsets $E_1, E_2 \subseteq \mathcal{E}$ such that $E_2 \subseteq E_1$. One then have

$$\mathcal{M}/_{=\Psi_{E_1}} \leq \mathcal{M}/_{=\Psi_{E_2}}, \quad (7)$$

i.e. $\mathcal{M}/_{=\Psi_{E_1}}$ is finer than $\mathcal{M}/_{=\Psi_{E_2}}$ since E_1 might contain environment states which are not in E_2 , thus inducing a refinement of the partition obtained on E_2 . Consequently, considering the extreme case where $E = \mathcal{E}$, then it is clear that $\mathcal{M}/_{=\Psi_{\mathcal{E}}}$ is the finer representation the agent can have access to. As such, it constitutes the greatest lower bound for all possible sets of equivalence classes i.e.

$$\mathcal{M}/_{=\Psi_{\mathcal{E}}} \leq \mathcal{M}/_{=\Psi_E} \leq \{\mathcal{M}\}, \forall E \in \mathcal{P}(\mathcal{E})^3. \quad (8)$$

In the same vein, Equation (8) indicates that the set $\{\mathcal{M}\}$ trivially constitutes the least upper bound of all sets of equivalence classes. Among all these partitions, $\mathcal{M}/_{=\Psi_{\mathcal{E}}}$ is of particular importance. Indeed it is made of equivalence classes which can never be further fragmented. In that sense, these equivalent classes constitutes the so-called *sensorimotor refinement points* which are closely related to points in space, see §III-A2.

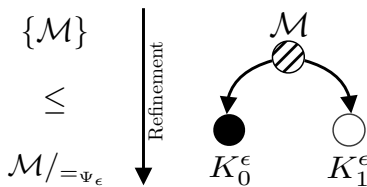


Fig. 4. Clustering of the motor space \mathcal{M} in two equivalence classes K_0^ϵ and K_1^ϵ after the sensorimotor flow analysis. The partition $\{K_0^\epsilon, K_1^\epsilon\} = \mathcal{M}/_{=\Psi_\epsilon}$ is finer than $\{\mathcal{M}\}$.

³The power set of \mathcal{E} is the set of all subsets in \mathcal{E} , including the empty set and the set \mathcal{E} itself.

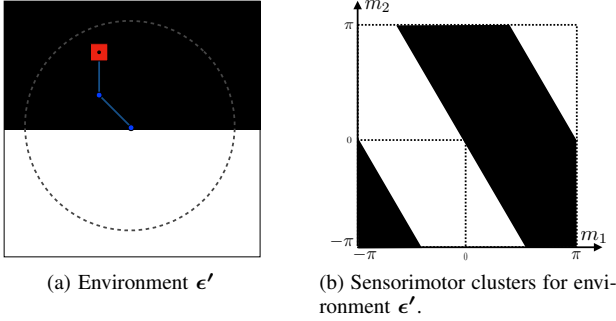


Fig. 5. Representation of the environment state ϵ' (left), and its corresponding sensorimotor clusters (right).

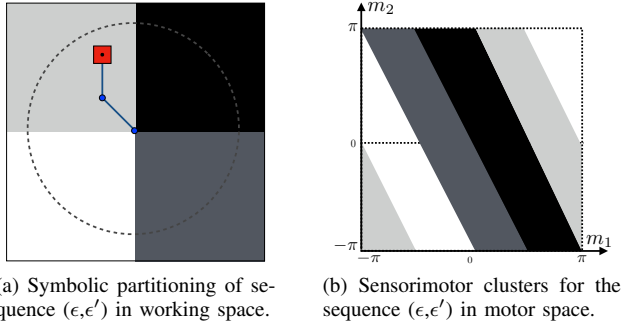


Fig. 6. Sequence $\{\epsilon, \epsilon'\}$: black area is for sequence (black&black), dark gray is for sequence (black&white), light gray is for sequence (white&black) and white for sequence (white&white).

2) *Illustrative example (cont'd)*: Let's come back now to the example used in §II-A2 as an illustration of the theory. Imagine the same agent is now experiencing a new environment state ϵ' which corresponds to a new black and white separation of the robot working space, as shown in Figure 5. Considering this new environment state ϵ' alone, it is clear that the agent can partition its motor space in two equivalence classes K_0^ϵ and K_1^ϵ , thus leading to a new representation which is structurally identical to Figure 4, the variable ϵ being here trade for ϵ' . However, the agent has now the possibility to build a finer representation for having sequentially experienced the environment states ϵ and ϵ' . The resulting *multi-environment* partitioning can be easily deduced in this case, and is shown in Figure 6. In this intuitive example, the agent is now able to separate the equivalence class K_0^ϵ , which relates to all the motor configurations giving the same 0 sensation value for the environment state ϵ , into two new subsets that are denoted K_{00}^E and K_{01}^E , with $E = \{\epsilon, \epsilon'\}$. Of course, the same applies for the initial equivalence class K_1^ϵ , now partitioned in two subsets K_{10}^E and K_{11}^E . Following the conventions used in Figure 4, one can then illustrate this refinement with the binary tree shown in Figure 7. If the experiment is iteratively reproduced, then the multi-environment representation will again be refined, with all the equivalence classes being more and more partitioned into smaller subsets. The tree structure would then of course shows a monotonically growing number of nodes. In this example, there might be an uncountable

number of possible motor configurations as well as (black & white) environments so that the number of nodes will most likely diverges to infinity: all the equivalence classes can always be further partitioned with a new specific environment. However, taking the limit of the refinement in the number of exploited environments states one obtains pointwise clusters: the *sensorimotor refinement points*.

In Figure 7, one can see the tree structure of the refinement: while the number of environment grows, each node is further separated into new nodes. For the moment, at each level of refinement, there is an unconnected graph so that equivalence classes are not linked together by edges which represent a neighborhood relation in the graph. However, the fact that some equivalence classes in one level were inside a common equivalence class in the previous level should imply the existence of a topological relation between them. As an example, looking at Figure 7, the colored nodes at the bottom corresponds to the colored areas in Figure 6 (left). The black and dark gray nodes have a common "ancestor" K_0^ϵ and they are topologically neighbors as they share a frontier in Figure 6 (the same applies for the light gray and white nodes). Furthermore, exchanging the order (ϵ', ϵ) would give a different refinement tree where there would be a common ancestor to dark gray and white nodes but also to light gray and black nodes which are all neighbors in the working space as they share common frontiers. Then, two questions arise:

- what is the interpretation of the obtained finest representation in the physical space ?
- how to compute the neighborhood relations and what is their interpretation in the physical space ?

These two points will be formalized in the next section.

III. SPACE REPRESENTATION: FROM MOTOR SPACE REFINEMENT TO POSE SPACE REPRESENTATION

The previous section was devoted to the introduction of the notion of refinement, exploited here to model how a naive agent can refine its motor space into finer partitions by integrating its sensorimotor experience along successive environment states. But it is not clear how this motor representation can actually captures any information about space, or at least about the agent working space. This link is precisely described in this section by introducing a new intermediary *pose space* \mathcal{X} between the motor configuration space \mathcal{M} and the sensory

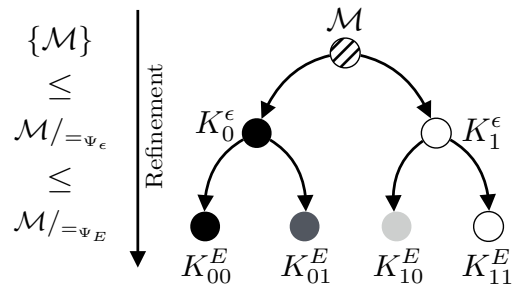


Fig. 7. Clustering of the motor space (cont'd): two equivalence classes (K_0^ϵ and K_1^ϵ) built when experiencing the environment state ϵ can be partitioned again in 4 subsets after experiencing the sequence (ϵ, ϵ') .

space \mathcal{S} . This new set will be shown as fundamental to understand why and how the agent can discover properties about the physical space during its sensorimotor exploration.

This section is organized as follows. In a first subsection, spatial considerations will be introduced through the pose space \mathcal{X} of the sensors as well as a new equivalence relation to form, for any $E \in \mathcal{P}(\mathcal{E})$, a new quotient set $\mathcal{X}/_{=\phi_E}$ which will be shown to be isomorphic to $\mathcal{M}/_{=\Psi_E}$, i.e. the refined partition of \mathcal{M} . Again, the example used all along the paper will be exploited to illustrate the theory. The second subsection introduces some topological considerations into the formalism as well as a minimum number of hypothesis required to prove that it is possible, just from the refinement process, to obtain a representative space with topological properties directly inherited from continuity in the physical space. This will further justify the introduction of technical topological properties in the formalism and the choice of $\mathcal{M}/_{=\Psi_E}$ as a good candidate to represent the agent's working space.

A. From the motor quotient space to the sensor pose

For the sake of clarity, the previous section has highlighted the only two spaces the agent can be aware of: the motor configuration space \mathcal{M} and the sensory space \mathcal{S} , where all its motor and sensory states lie respectively. Both spaces are linked together through the sensorimotor law Ψ . But one have to keep in mind that the sensory state $\mathbf{s} \in \mathcal{S}$ is outputted by rigid sensors whose spatial state in the world is entirely described by their *sensors pose* $\mathbf{x} \in \mathcal{X}$, with \mathcal{X} the *sensors pose space* which has been so far omitted. So let's first focus on this new set and highlight the links between \mathcal{X} , \mathcal{M} and \mathcal{S} .

1) *Definition of the sensor pose space*: It is well known in robotics that the *forward kinematics function* $f(\cdot)$, which accounts for the relative movements allowed at each joint and is dependent on the geometry of the robot, is a function linking the motor state \mathbf{m} to the corresponding sensors pose \mathbf{x} usually in Euclidean space, so that $\mathbf{x} = f(\mathbf{m})$. In all the following, the pose \mathbf{x} –which is the sensors spatial state in the physical world– refers to the sensors positions and their orientations relatively to the frame of the agent's body. The overall state of the sensors in the physical space is composed of an extrinsic, thus not directly accessible, spatial state: their pose \mathbf{x} and an intrinsic sensory state \mathbf{s} corresponding to the response of the transducers to the world physical state ϵ at their specific location and orientation \mathbf{x} in space. Of course, both spatial and sensory states are linked together through the *forward sensory function* ϕ_ϵ , so that $\mathbf{s} = \phi_\epsilon(\mathbf{x})$. In the end, the sensorimotor law Ψ_ϵ can be written as the composition $\Psi_\epsilon = \phi_\epsilon \circ f$, which is summarized by

$$\begin{array}{ccc} & \Psi_\epsilon & \\ & \curvearrowright & \\ \mathcal{M} & \xrightarrow{f} & \mathcal{X} & \xrightarrow{\phi_\epsilon} & \mathcal{S} \end{array} \quad (9)$$

One have to keep in mind that the agent has no way to directly access to the sensor spatial state space \mathcal{X} in any way. Introducing \mathcal{X} is only a convenient theoretical way to understand how the motor refinement strategy outlined in Section II helps the agent to build its own interpretation of space through the analysis of sensorimotor invariants.

In order to generalize, the two functions f and ϕ_ϵ can be both considered surjective. While this is obvious for f (the sensors pose space \mathcal{X} is by definition the image of \mathcal{M} by f), ϕ_ϵ can be rendered surjective by restricting \mathcal{S} to $\mathcal{S}_\epsilon = \phi_\epsilon(\mathcal{X})$, with $\mathcal{S}_\epsilon \subseteq \mathcal{S}$. This means that two different motor configurations can lead to the same sensors pose (i.e. the surjectivity of f captures the agent kinematics redundancy) and in a specific environment state ϵ two different sensors poses can lead to the same sensory state (i.e. the surjectivity in ϕ_ϵ captures the agent environment local physical states redundancies but also poses redundancy such as rotational symmetry). In the vein of Equations (1) and (6), then, for any $\epsilon \in \mathcal{E}$, one can again define an equivalence relation $=_{\phi_\epsilon}$ for two poses with

$$\mathbf{x}_1 =_{\phi_\epsilon} \mathbf{x}_2 \Leftrightarrow \phi_\epsilon(\mathbf{x}_1) = \phi_\epsilon(\mathbf{x}_2), \quad (10)$$

or, the equivalence relation $=_{\phi_E}$ for any subset $E \subseteq \mathcal{E}$, with

$$\mathbf{x}_1 =_{\phi_E} \mathbf{x}_2 \Leftrightarrow \phi_\epsilon(\mathbf{x}_1) = \phi_\epsilon(\mathbf{x}_2), \forall \epsilon \in E. \quad (11)$$

These two equations can be understood as: two poses are said equal after having seen environment $\epsilon \in \mathcal{E}$ (resp: all environments in $E \subseteq \mathcal{E}$) if the sensations they produce are equal (resp: equal for all environment in E). Following the same reasoning as in §II-B1 which has conducted to Equation (8), one can then write immediately

$$\{\{\mathbf{x}\}_{\mathbf{x} \in \mathcal{X}}\} \leq \mathcal{X}/_{=\phi_{\mathcal{E}}} \leq \mathcal{X}/_{=\phi_E} \leq \{\mathcal{X}\}, \forall E \in \mathcal{P}(\mathcal{E}), \quad (12)$$

where $\mathcal{X}/_{=\phi_E}$ represents the pose quotient set refining the trivial partition $\{\mathcal{X}\}$ of \mathcal{X} .

Among all pose partitions, $\mathcal{X}/_{=\phi_{\mathcal{E}}}$ is of particular interest. Exactly like in II-B1 for $\mathcal{M}/_{=\Psi_{\mathcal{E}}}$, this quotient set is made of equivalence classes which can not be further fragmented in subsets. It is interesting to see that the finest equivalence classes can only be coarser than the points in the sensors pose space. This highlights the possible difficulties or even the impossibility for an agent to directly represent the positions and orientations of its sensors, as they may contain a sensory redundancy. As an example, the spatial state of a heat sensor varies with a rotation on itself but its temperature should not. As such, these equivalent classes can also be envisioned as the new *pose refinement points*, which will be shown to be closely related to the sensorimotor refinement points obtained on the finest motor partition.

But how could the agent build $\mathcal{X}/_{=\phi_{\mathcal{E}}}$, since it has no way to access to \mathcal{X} in the first place? This will be illustrated in the next subsection with the same "black and white" example, before the formal demonstration of the existence of a bijection between $\mathcal{M}/_{=\Psi_{\mathcal{E}}}$ and $\mathcal{X}/_{=\phi_{\mathcal{E}}}$.

2) *Illustrative example (cont'd)*: Let's come back to the illustrative example used all along the paper, where a two-DOF robot arm explores a black and white environment. In this simple case:

- the environment state ϵ can be fully described by a straight line delimiting the working space in two areas⁴,

⁴This line can be parameterized by its podal point coordinates in \mathbb{R}^2 , which is the shortest point from the line to the center of the working space

together with a binary value indicating which one is black, so that $\epsilon \in \mathcal{E} \subset \mathbb{R}^2 \times \{0, 1\}$;

- the agent's motor configuration set \mathcal{M} is made of the set of the two joint angles θ_1, θ_2 so that $\mathcal{M} = \{(\theta_1, \theta_2), \theta_1, \theta_2 \in [-\pi, \pi]\}$;
- the forward kinematics function f gives the punctual end-effector position $(x, y) \in \mathcal{X} \subset \mathbb{R}^2$ as a function of θ_1, θ_2 and L , the length of the two body parts, with $(x, y) = (L(\cos \theta_1 + \cos(\theta_1 + \theta_2)), L(\sin \theta_1 + \sin(\theta_1 + \theta_2)))$;
- the pointwise sensor, placed on (x, y) , delivers a sensation⁵ $\mathbf{s} = \phi_\epsilon(x, y) \in \mathcal{S}_\epsilon = \{0, 1\}$.

The agent is there endowed with a pointwise sensor, a pose \mathbf{x} in \mathcal{X} is nothing else but a point in a 2D Euclidean space. Since two distinct points in the 2D euclidean plane can always been separated by a straight line, equivalently for two distinct poses in \mathcal{X} , there always exists an environment state $\epsilon \in \mathcal{E}$ for which the corresponding sensations are distinct. Then, for a given pose \mathbf{x} , the equivalent class $K_{\mathbf{x}}^\mathcal{E}$ regrouping all the poses leading to the same sensation obtained at \mathbf{x} for all $\epsilon \in \mathcal{E}$ is the singleton $\{\mathbf{x}\}$. This means that the finest partition $\mathcal{X}/_{=\phi_\mathcal{E}}$ of the working space is the set of singletons, i.e. $\mathcal{X}/_{=\phi_\mathcal{E}} = \{\{(x, y)\}, (x, y) \in \mathcal{X}\}$. In this particular case, the *pose refinement points* are exactly the *pose space points* in the working space.

Following the same lines, it is clear that the equivalence classes in the motor configurations space \mathcal{M} are the set of motor configurations leading to a same and unique pose in the working space through the forward kinematics function f . Consequently, the finest partition $\mathcal{M}/_{=\Psi_\mathcal{E}}$ is made of equivalence classes $K_{\mathbf{m}}^\mathcal{E}$ individually corresponding to one equivalence class $K_{\mathbf{x}}^\mathcal{E} = K_{f(\mathbf{m})}^\mathcal{E}$. In this particular case, *sensorimotor refinement points* in $\mathcal{M}/_{=\Psi_\mathcal{E}}$ have all one and unique corresponding *pose refinement point* in $\mathcal{X}/_{=\phi_\mathcal{E}}$, which have been shown to represent *pose points* in the working space. Then, without any knowledge on the forward kinematics function f –and through a refinement strategy– the agent can build the set $\mathcal{M}/_{=\Psi_\mathcal{E}}$ which captures all its kinematics redundancies and constitutes a very good candidate for representing its working space.

All these considerations are represented in Figure 8. The working space is represented in the middle, where each pose can be reached by the agent from one or multiple motor configurations due to the kinematics redundancy. For instance, the pose \mathbf{x}_1 (resp. \mathbf{x}_3) can be reached by the 2 different motor configurations \mathbf{m}_1 and \mathbf{m}_2 (resp. \mathbf{m}_3 and \mathbf{m}_4) in \mathcal{M} . The same applies for the pose \mathbf{x}_5 located at the limit of the working space, which can be obtained with a unique motor configuration \mathbf{m}_5 . Another particular case is the pose \mathbf{x}_6 obtained when the sensor is exactly in the center of the working space, which can be reached with all motor configurations $\mathbf{m} = (m_1, m_2)$ such that $m_2 = -\pi$, thus building the set $\mathcal{M}_6 \in \mathcal{M}$. As explained above, each of these poses is linked to an equivalent class in $\mathcal{M}/_{=\Psi_\mathcal{E}}$ once the agent has experienced all the possible environment states ϵ in \mathcal{E} . For instance, the two motor configurations \mathbf{m}_1 and \mathbf{m}_2 (resp. \mathbf{m}_3 and \mathbf{m}_4) can

be regrouped in the equivalent class $K_{\mathbf{m}_1}^\mathcal{E} = \{\mathbf{m}_1, \mathbf{m}_2\}$ (resp. $K_{\mathbf{m}_3}^\mathcal{E} = \{\mathbf{m}_3, \mathbf{m}_4\}$) in $\mathcal{M}/_{=\Psi_\mathcal{E}}$. Then, one can see on this illustration that each undividable equivalent class obtained on the finest partition in $\mathcal{M}/_{=\Psi_\mathcal{E}}$ forms a sensorimotor refinement point, each of them being associated to one and only one pose space point in the working space, i.e. a point in the 2D Euclidean space. Thus, the agent knows for instance that any motor configuration selected in \mathcal{M}_6 would correspond to a sensorimotor refinement point $K_{\mathbf{m}_6 \in \mathcal{M}_6}^\mathcal{E}$ and so to a unique pose space point in the working space. In that sense, along the refinement, one can qualitatively understand that the embedding from \mathcal{M} to the set of equivalence classes pointwise converges to a map which has the same kernel than the forward kinematics function.

This simple example allows to illustrate the link between $\mathcal{M}/_{=\Psi_\mathcal{E}}$ and $\mathcal{X}/_{=\phi_\mathcal{E}}$, but one can prove in the general case the existence of a bijection between these two sets. This will be detailed in the following subsection.

3) *Theoretical generalization*: Before demonstrating the aforementioned bijection, let's introduce the notion of sensory sequence for a motor configuration along with the successive experienced environment states. As an example, in the previous illustration such sequence would be made of a sequence of 0 or 1 obtained for each environment state ϵ in fixed motor configuration. One can then define the set \mathcal{S}_E of all sensory sequences being possibly experienced for any specific subset $E \subseteq \mathcal{E}$ by

$$\mathcal{S}_E = \prod_{\epsilon \in E} \mathcal{S}, \quad (13)$$

where \prod denotes the Cartesian product of sets. On this basis, one can now precisely define the maps ϕ_E and Ψ_E , already used for the notations of the equivalence relations in (6) and (11), as the functions mapping each motor configuration (resp. pose) to their respective sensory sequences indexed by the subset E , with

$$\begin{aligned} \phi_E : \mathcal{X} &\rightarrow \mathcal{S}_E & \text{and } \Psi_E : \mathcal{M} &\rightarrow \mathcal{S}_E \\ \mathbf{x} &\mapsto (\phi_\epsilon(\mathbf{x}))_{\epsilon \in E} & \mathcal{M} &\mapsto (\Psi_\epsilon(\mathbf{m}))_{\epsilon \in E} \end{aligned} \quad (14)$$

Consequently, the equivalence relations $=_{\phi_E}$ and $=_{\Psi_E}$ can now equivalently be respectively defined by, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$,

$$\begin{aligned} \mathbf{x}_1 =_{\phi_E} \mathbf{x}_2 &\Leftrightarrow \forall \epsilon \in E, \phi_\epsilon(\mathbf{x}_1) = \phi_\epsilon(\mathbf{x}_2), \\ &\Leftrightarrow \phi_E(\mathbf{x}_1) = \phi_E(\mathbf{x}_2), \end{aligned} \quad (15)$$

and, for any $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}$,

$$\begin{aligned} \mathbf{m}_1 =_{\Psi_E} \mathbf{m}_2 &\Leftrightarrow \forall \epsilon \in E, \Psi_\epsilon(\mathbf{m}_1) = \Psi_\epsilon(\mathbf{m}_2), \\ &\Leftrightarrow \Psi_E(\mathbf{m}_1) = \Psi_E(\mathbf{m}_2). \end{aligned} \quad (16)$$

The set \mathcal{S}_E does contain sensory sequences that are never obtained from the motor configurations in \mathcal{M} so that both functions ϕ_E and Ψ_E are rendered surjective by restricting the image set to be the sensory sequences that actually occurred for each motor configuration in \mathcal{M} : $\tilde{\mathcal{S}}_E \subseteq \mathcal{S}_E$. Let's now denote the canonic projection that maps points in \mathcal{M} (resp. in \mathcal{X}) to their equivalence classes in $\mathcal{M}/_{=\Psi_E}$ (resp. in $\mathcal{X}/_{=\phi_E}$) by π_{Ψ_E} (resp. π_{ϕ_E}). One can now state the following proposition.

⁵In the particular case where the sensor is placed exactly on the straight line splitting the working space in two areas, it is arbitrary chosen that $\mathbf{s} = 0$.

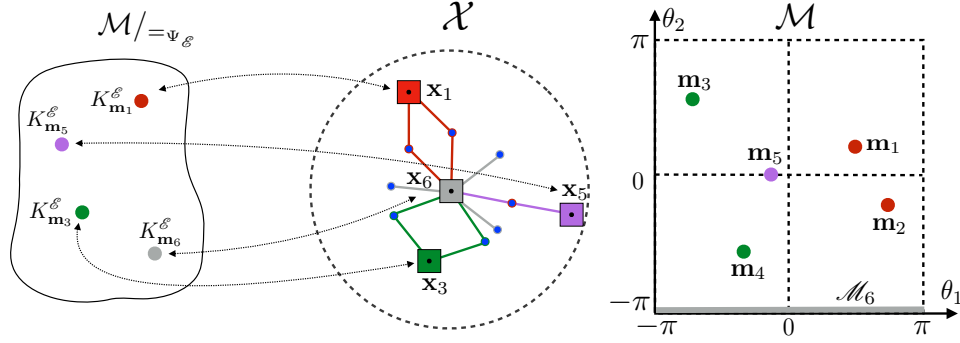


Fig. 8. Illustration of the link between \mathcal{M} , \mathcal{X} and $\mathcal{M}/=\Psi_E$ for the simple example used in the paper. (Left) Sensorimotor representative space, i.e. the finest partition of the motor space. (Middle) Pose space, with the punctual camera represented as a square. (Right) Motor configuration space. In the end, and for the finest motor representation, each *sensorimotor refinement point* $K_{m_i}^E$ in $\mathcal{M}/=\Psi_E$ represents one and only one *point* in the agent working space.

Proposition 1. For any subset $E \subseteq \mathcal{E}$, $\mathcal{X}/=\phi_E$ and $\mathcal{M}/=\Psi_E$ are equinumerous.

Proof. It is clear that the sets $\mathcal{X}/=\phi_E$ and $\tilde{\mathcal{S}}_E$ are equinumerous. Indeed, by definition of $\tilde{\mathcal{S}}_E$, there are as many equivalence classes in \mathcal{X} than there are different sequences of sensations in $\tilde{\mathcal{S}}_E$. Thus there exists a unique bijective map $\check{\phi}_E$ linking them together in $\mathcal{X}/=\phi_E$ and $\tilde{\mathcal{S}}_E$. $\mathcal{M}/=\Psi_E$ and $\tilde{\mathcal{S}}_E$ are also equinumerous for the same reason. Thus, there exists a unique bijective map \check{f}_E , mapping equivalence classes together between $\mathcal{M}/=\Psi_E$ and $\mathcal{X}/=\phi_E$. \square

This property can be summarized with the following commutative diagram:

$$\begin{array}{ccccc}
 & & \Psi_E & & \\
 & & \curvearrowright & & \\
 \mathcal{M} & \xrightarrow{f} & \mathcal{X} & \xrightarrow{\phi_E} & \tilde{\mathcal{S}}_E \\
 \downarrow \pi_{\Psi_E} & & \downarrow \pi_{\phi_E} & & \nearrow \check{\phi}_E \\
 \mathcal{M}/=\Psi_E & \xrightarrow{\check{f}_E} & \mathcal{X}/=\phi_E & &
 \end{array}$$

From the illustration example, the agent is theoretically able to build the motor quotient space $\mathcal{M}/=\Psi_E$ from sensory invariances and by repeating its motor configurations after each environment change. At the end, these motor configurations can theoretically be regrouped into equivalence classes for which sensors always send the same sensations so they are referred as *sensorimotor refinement points*. Each *sensorimotor refinement point* does not always corresponds to a unique sensors pose in space because they might be other poses that always give an identical sensation, and the agent can not distinguish them using only sensory invariants. So the space represented by the sensorimotor representation space $\mathcal{M}/=\Psi_E$ is the quotient pose space $\mathcal{X}/=\phi_E$ composed of *pose refinement points* which include this redundancy.

In order for the motor quotient space to be a representative of the pose quotient space, it has to (i) contain the same number of points (which has been proved here), and (ii) represent the topological properties of the points. It has been postulated in the example that some of these properties can be extracted from the refinement process. The next subsection

deals with the hypothesis and methods needed to build such structure in the set of sensorimotor refinement points.

B. Towards topological considerations

1) *First topological considerations in \mathcal{M} , \mathcal{X} and $\mathcal{X}/=\phi_E$:* Here, one can suppose some topological structures in both space \mathcal{M} and \mathcal{X} . Let's call $\tau_{\mathcal{M}}$ the intrinsic topology in the motor configuration space and let's call $\tau_{\mathcal{X}}$ the extrinsic topology in the pose space. The motor topology is derived from the action and proprioception capabilities of the agent while the extrinsic topology is deduced from possible movements of the sensors in the physical space. As an example for the 2 DOF agent introduced in Figure 2 the motor topology would be the 2-torus because of the two revolute joints, while the topology of the pose space is a surface. From the refinement, one can build a natural topology in the quotient space $\mathcal{X}/=\phi_E$, the quotient topology $\tau_{\mathcal{E}}$, which is coinduced⁶ by the function π_{ϕ_E} on \mathcal{X} . Both topologies $\tau_{\mathcal{X}}$ and $\tau_{\mathcal{E}}$ are extrinsic and are derived from the continuity in the physical world. In order for the agent to build a correct representation of the interaction of the sensors with the physical world, the agent should be able to build, in the representative space $\mathcal{M}/=\Psi_E$, a topological structure which is equivalent to $\tau_{\mathcal{E}}$.

Before exploiting the topological properties of the refinement, there are two hypothesis that the agent motor topology $\tau_{\mathcal{M}}$ should satisfy in order to obtain exploitable results.

(H1) *The forward kinematics map f is continuous.*

Such an hypothesis means that open sets in the topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ have open preimages by f in $(\mathcal{M}, \tau_{\mathcal{M}})$, this intuitively means that small sensors displacements are generated by small motor movements.

For now, \mathcal{M} , \mathcal{X} and $\mathcal{X}/=\phi_E$ can all be considered as topological spaces, and the links between them have been highlighted. In addition to these considerations, this study will be restricted to a motor configuration space verifying the

⁶ $\tau_{\mathcal{E}}$ is the finest topology that makes the canonical projection π_{ϕ_E} continuous

following hypothesis.

(H2) *The motor configuration space \mathcal{M} is compact.*

H2 is a quite technical hypothesis⁷. Basically it implies that if \mathcal{M} is a subset of a Euclidean space, then H2 translates to " \mathcal{M} has to be closed and bounded". For instance, this hypothesis avoids the case where the agent would be able to perform infinite translations of its sensors.

2) *On stochastic properties of the refinement:* It's clear that the refinement process carries some topological information from the successive inclusion of partitions, originating from the interactions with the corresponding environment states. However, the agent can not control these states changes: from its point of view, they can be seen as random events. One then should introduce some stochastic properties to actually exploit the refinement.

(H3) *The refinement follows a Bernoulli process.*

To illustrate H3, let's consider two poses \mathbf{x}_i and \mathbf{x}_j belonging the theoretical pose equivalence classes $K_{\mathbf{x}_i}$ and $K_{\mathbf{x}_j}$ in the finest pose quotient space $\mathcal{X}/_{=\phi_{\mathcal{E}}}$. Then one have, for any $\epsilon \in \mathcal{E}$, $\phi_{\epsilon}(\mathbf{x}_i) = \check{\phi}_{\epsilon}(K_{\mathbf{x}_i})$ and $\phi_{\epsilon}(\mathbf{x}_j) = \check{\phi}_{\epsilon}(K_{\mathbf{x}_j})$. However, there will be environment states for which the sensations for both equivalence classes are identical $\check{\phi}_{\epsilon}(K_{\mathbf{x}_i}) = \check{\phi}_{\epsilon}(K_{\mathbf{x}_j})$ and others for which they are different. Let's consider a successive sequence of environment states the agent can interact with, indexed by the sample number $k \in \mathbb{N}$ so that at sample k the environment is in a state $\epsilon[k] \in \mathcal{E}$. Let's introduce a binary random variable $\delta_k(K_{\mathbf{x}_i}, K_{\mathbf{x}_j})$ such that

$$\delta_k(K_{\mathbf{x}_i}, K_{\mathbf{x}_j}) = \begin{cases} 1 & \text{if } \check{\phi}_{\epsilon[k]}(K_{\mathbf{x}_i}) \neq \check{\phi}_{\epsilon[k]}(K_{\mathbf{x}_j}), \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

H3 states that $\delta_k(K_{\mathbf{x}_i}, K_{\mathbf{x}_j})$ has a Bernoulli distribution, i.e.

$$\Pr(\delta_k(K_{\mathbf{x}_i}, K_{\mathbf{x}_j}) = x) = \begin{cases} \rho(K_{\mathbf{x}_i}, K_{\mathbf{x}_j}) & \text{if } x = 1, \\ 1 - \rho(K_{\mathbf{x}_i}, K_{\mathbf{x}_j}) & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (18)$$

such that

$$\rho : \mathcal{X}/_{=\phi_{\mathcal{E}}} \times \mathcal{X}/_{=\phi_{\mathcal{E}}} \rightarrow [0, 1] \quad (19)$$

One can notice that ρ does not depend on the time step k , so that ρ represents the probability for two equivalence classes $K_{\mathbf{x}_i}$ and $K_{\mathbf{x}_j}$ to be associated to a different sensation in any environment state or equivalently the probability to sample an environment for which the associated sensations are different. Thus for any pair of equivalence classes, and for any $k \in \mathbb{N}$, the random variables δ_k are identically distributed, and supposed mutually independent. Consequently, the sequences of δ_k constitute Bernoulli processes. One important property on ρ follows.

Proposition 2. For any pair of equivalence classes $K_{\mathbf{x}_i}, K_{\mathbf{x}_j} \in \mathcal{X}/_{=\phi_{\mathcal{E}}}$, $K_{\mathbf{x}_i} = K_{\mathbf{x}_j}$ iff $\rho(K_{\mathbf{x}_i}, K_{\mathbf{x}_j}) = 0$

⁷Formally, a topological space X is said compact iff every open cover of X has a finite subcover.

Proof. If $K_{\mathbf{x}_i} = K_{\mathbf{x}_j}$ then it is obvious that $\rho(K_{\mathbf{x}_i}, K_{\mathbf{x}_j}) = 0$. Now let's take equivalence classes $K_{\mathbf{x}_i}$ and $K_{\mathbf{x}_j}$ such that $\rho(K_{\mathbf{x}_i}, K_{\mathbf{x}_j}) = 0$. If there exists some environmental states for which their respective sensations are different, then by definition of ρ , the probability of sampling any number of these environments states is 0, the equivalence classes are identical with probability 1, which will be simplified as pure equality as these states can be removed from the set \mathcal{E} without loss of generality. \square

From the previous statements, one can obtain the following property.

Proposition 3. The map ρ is a metric in $\mathcal{X}/_{=\phi_{\mathcal{E}}}$.

Proof. It is obvious that, according to Equation (19) and proposition 2, ρ is symmetric positive-definite. Let's show that it also satisfies the triangular inequality. For any K_1, K_2 and $K_3 \in \mathcal{X}/_{=\phi_{\mathcal{E}}}$, having $\delta_k(K_1, K_2) = 1$ –which means that K_1 is associated to a different sensation than K_2 at environment sample k –, should lead to $(\delta_k(K_1, K_3), \delta_k(K_2, K_3)) = (0, 1), (1, 0)$ or $(1, 1)$. Therefore

$$\delta_k(K_1, K_2) \leq \delta_k(K_1, K_3) + \delta_k(K_2, K_3), \quad (20)$$

in a deterministic way. Applying the expectation symbol on (20), and recalling that $\mathbb{E}[\delta_k(K_1, K_2)] = \rho(K_1, K_2)$ from Equation (18), then $\rho(K_1, K_2) \leq \rho(K_1, K_3) + \rho(K_3, K_2)$. \square

$\mathcal{X}/_{=\phi_{\mathcal{E}}}$ is now endowed with a metric which is calculable by the naive agent. It is well-known that the metric ρ , which will now be called the *refinement distance*, defines a topology τ_{ρ} on $\mathcal{X}/_{=\phi_{\mathcal{E}}}$, which is generated by open balls $B_r(K)$ of radius $r > 0$ centered at all $K \in \mathcal{X}/_{=\phi_{\mathcal{E}}}$ as the set $B_r(K) = \{K' \in \mathcal{X}/_{=\phi_{\mathcal{E}}}, \rho(K, K') < r\}$.

3) *Link between the finest quotient topologies:* In order to deduce spatial interpretation from this newly defined topology, one has to add one last important hypothesis.

(H4) τ_{ρ} is coarser than the quotient topology $\tau_{\mathcal{E}}$.

This means that continuous actions in the topological space $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\rho})$ are also physically continuous transformation in the space $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\mathcal{E}})$. The hypothesis H4 guarantee the conservation of the physical continuity through the maximum refinement.

Proposition 4. The topological spaces $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\rho})$ and $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\mathcal{E}})$ are equal.

Proof. From H4, as $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\rho})$ is a metric space, it is also Hausdorff, moreover $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\mathcal{E}})$ is also Hausdorff because it has a finer topology. Let's remember that $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\mathcal{E}})$ is compact as the continuous image of the compact space \mathcal{M} (hypothesis H2), therefore $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\rho})$ is compact because of its coarser topology. Both space are compact Hausdorff spaces, therefore their topology are either incomparable or equal. As τ_{ρ} is coarser than $\tau_{\mathcal{E}}$ they are comparable thus equal. More details about this proof in [24]. \square

Therefore, the agent can build a topology in the representative space $\mathcal{M}/_{=\check{\nu}_{\mathcal{E}}}$ as the topology $\check{\tau}_{\rho} = \check{f}_{\mathcal{E}}^{-1}(\tau_{\rho})$, which can be built by computing the metric ρ and applying it to the motor states generating the sensations, then:

Proposition 5. The topological spaces $(\mathcal{M}/_{=\check{\nu}_{\mathcal{E}}}, \check{\tau}_{\rho})$ and $(\mathcal{X}/_{=\phi_{\mathcal{E}}}, \tau_{\mathcal{E}})$ are topologically equivalent.

Proof. From proposition 1, $\check{f}_{\mathcal{E}}$ is a bijective map. From proposition 4, it is obvious that it is continuous, with a continuous inverse, so it is a homeomorphism. \square

The authors have shown that, under technical but natural hypothesis and by definition of a refinement distance ρ , it is theoretically possible to build an intrinsic representative space of the outside physical space. Apart from the hypothesis, this space is built with no a priori information and uniquely from the repetition of motor configurations and the exploitation of sensory invariances. In the next section will be shown some of the results that can be obtained in a simulated context.

IV. SIMULATIONS

This section is devoted to some simulations used as proofs of concept illustrating how a sensorimotor refinement strategy can be numerically applied to actually compute a representation of the quotient pose space unknown to the agent. The results will be obtained by using the agent introduced in Figure 2 endowed with light sensors and environments that satisfy the hypothesis shown in the previous section. The simulation setup is introduced in a first subsection, together with some numerical considerations regarding the refinement distance estimation and a short conclusion. In the next subsection, some first results are proposed concerning the example used all along the paper. It will illustrate how the refinement strategy can be implement and how topological information can be captured in the representative space. Finally, a more complex, environment is used in the third subsection to emphasize the generality of the approach.

A. Simulation setup

Refining a representation is an iterative process *per se* – as formalized in §III-B2– where each iteration is enumerated by the sample number k and corresponds to a supposed static environment $\epsilon[k]$. At each iteration, the agent samples a finite number N of motor configurations \mathbf{m}_i in $\mathcal{M} \subseteq \mathbb{R}^M$ where M is the number of degrees of freedom of the agent’s actuators. In order to obtain meaningful statistical results, the agent will repeat the same movements along the refinement process. Thereby to each environment state $\epsilon[k]$ and motor configuration \mathbf{m}_i correspond a sensation $\mathbf{s}_i[k] \in \mathcal{S}$ where \mathcal{S} is a finite set possibly corresponding to a numerical encoding of all possible values the sensors can return.

1) *Estimation of the metric ρ* : in the vein of Equation (17), let’s define the sensory equivalence between two motor configurations \mathbf{m}_i and \mathbf{m}_j at sample k by $\delta_k(\mathbf{m}_i, \mathbf{m}_j) = (\mathbf{s}_i[k] \neq \mathbf{s}_j[k])$. From the successive iterations along the refinement process, one can define the *refinement*

distance estimator at sample k between two motor configurations \mathbf{m}_i and \mathbf{m}_j as

$$D_{ij}[k] = \frac{1}{k} \sum_{\ell=1}^k \delta_{\ell}(\mathbf{m}_i, \mathbf{m}_j). \quad (21)$$

Imagine now that the two motor configurations \mathbf{m}_i and \mathbf{m}_j belong to two theoretical equivalence classes in the *finest motor quotient space* that the agent is trying to represent, $K_{\mathbf{m}_i} = \pi_{\Psi_{\mathcal{E}}}(\mathbf{m}_i)$ and $K_{\mathbf{m}_j} = \pi_{\Psi_{\mathcal{E}}}(\mathbf{m}_j)$ respectively. Then, according to the law of large numbers,

$$\frac{1}{k} \sum_{\ell=1}^k \delta_{\ell}(\mathbf{m}_i, \mathbf{m}_j) \xrightarrow[k \rightarrow \infty]{\text{a.s.}} \rho(K_{\mathbf{m}_i}, K_{\mathbf{m}_j}). \quad (22)$$

Therefore the refinement distance estimator $D_{ij}[k]$ is a consistent estimator of the refinement distance $\rho(K_{\mathbf{m}_i}, K_{\mathbf{m}_j})$. The refinement distance estimator can be further computed using the *hamming distance* between two sequences of sensations as the percentage of sensations that differs, i.e.

$$D_{ij}[k] = \text{hamming_distance}(\mathbf{s}_i[1:k], \mathbf{s}_j[1:k]). \quad (23)$$

From the N sampled motor configurations used in the simulation, one can then build the symmetric *refinement distance matrix* $\mathbf{D}[k] \in [0, 1]^{N \times N}$ made of the components $D_{ij}[k]$ estimating the distance between \mathbf{m}_i and \mathbf{m}_j . Importantly, as shown in the following, $\mathbf{D}[k]$ contains all the topological structure of the representative space at sample k . In the end, the numerical implementation used in the following is reproduced in Algorithm 1.

2) *Intrinsic performance*: From an *internal* point of view, the only thing an agent can intrinsically evaluate, without any knowledge about the finest pose quotient topology, is the convergence of $\mathbf{D}[k]$ as the number of samples goes to infinity. First, from property 2 each zero in the matrix $\mathbf{D}[k]$ corresponds to an equivalence class. If $D_{ij}[k] = 0$ then the motor configurations \mathbf{m}_i and \mathbf{m}_j have always given the same sensation at all samples before k . If number of zeros in the refinement matrix converges to some value and does not change anymore along iterations, then the agent can suppose that its representative space is not likely to be further refined. The finest representation has been obtained and the agent can focus on the convergence of pairwise distances.

From relation (22), one can say that $\mathbf{D}[k]$ is a consistent representation of the topology in the finest quotient pose space. However, the estimated distances are constantly updated along with the environment sampling. Let’s introduce a stopping rule that tells the agent when its final representation is obtained. One solution is to check whether topological properties are conserved in successive samples. This can be assessed by verifying the existence of a monotonous function between the estimated distance at two samples $\mathbf{D}[k]$ and $\mathbf{D}[k - \Delta]$ where Δ is a given number of samples. Indeed, if both pairwise distance matrices have the same ordering of their distances, then it is obvious that, for both matrices, each points have the same neighbors. Consequently, they will have an identical topological structure, this property is used as an example in non-metric multidimensional scaling [25] and more recently for a similar application in [26]. When trying evaluate

Algorithm 1 Computation of the refinement distance matrix*Inputs:*

- N : number of target motor configurations.
- \mathbf{m}_0 : rest motor configuration.
- `hamming_distance`: percentage of pairwise different values.
- (q, Δ, η) : stopping rule.

Ouputs:

- \mathbf{m}_i : motor configurations for $i \in [1, N]$.
- $\mathbf{s}_i[k]$: sensation at step k and for the motor state \mathbf{m}_i .
- $\mathbf{D}[k]$: refinement distance matrix at step k .

```

1: {Step 1: motor babbling}
2: Go to rest motor configuration  $\mathbf{m}_0$ ,
3: for  $n = 1$  until  $n = N$  do
4:   random action
5:    $\mathbf{m}_n \leftarrow$  current proprioception.
6: end for
7: Go to the rest motor configuration  $\mathbf{m}_0$ ,
8:
9: {Step 2: environment sampling and interaction}
10: for  $k = 1$  do
11:   suppose steady environment at state  $\epsilon[k]$ ,
12:   for  $n = 1$  until  $n = N$  do
13:     action from  $\mathbf{m}_{n-1}$  to  $\mathbf{m}_n$ ,
14:      $\mathbf{s}_n[k] \leftarrow$  current sensation,
15:   end for
16:   go to the rest motor configuration  $\mathbf{m}_0$ .
17:
18:   {Similarity computations}
19:   for all pairs of motor configurations  $i, j$  do
20:      $D_{ij}[k] \leftarrow$  hamming_distance( $\mathbf{s}_i[1 : k], \mathbf{s}_j[1 : k]$ ),
21:   end for
22:   {Stopping rule}
23:   if  $q(\mathbf{D}[k], \mathbf{D}[k - \Delta]) > 1 - \eta$  then
24:     break
25:   end if
26:    $k \leftarrow k + 1$ ,
27: end for

```

the monotonicity between two arrays of values one should consider using the Spearman correlation.

Let $\text{spear_corr}(x, y)$ be the Spearman correlation of two arrays x, y , with

$$\text{spear_corr}(x, y) \triangleq \text{corr}(\text{sort_ind}(x), \text{sort_ind}(y)), \quad (24)$$

where corr is the Pearson correlation and $\text{sort_ind}(x)$ gives the rank of each value in x , for instance $\text{sort_ind}([14, 13, 11, 12]) = [4, 3, 1, 2]$. One interesting property of the Spearman correlation is that $\text{spear_corr}(x, y) = \pm 1$ iff there exists a monotonic function h such that $y = h(x)$. Therefore one can derive, similarly to the Spearman score in [26], a measure q of the topological similarity between two distance matrices \mathbf{D} and \mathbf{D}' :

$$q(\mathbf{D}, \mathbf{D}') \triangleq |\text{spear_corr}(\text{vec}(\mathbf{D}), \text{vec}(\mathbf{D}'))|, \quad (25)$$

where $\text{vec}()$ denotes the flattening operator. When $q(\mathbf{D}, \mathbf{D}')$ is sufficiently close to 1 and for a big enough Δ to avoid any premature stops, then the refinement can end.

3) *Extrinsic evaluation of the algorithm, topological properties conservation:* From an *external* point of view, one can envisage other ways to assess the quality of the representation. Indeed, the programmer has access to data which are not accessible to the agent, like the actual working space for instance. To evaluate extrinsically the algorithm, the quotient pose space $\mathcal{X}/_{=\phi_\epsilon}$ will be considered as a metric subspace of the Euclidean space, i.e. $\mathcal{X}/_{=\phi_\epsilon} \subseteq \mathbb{R}^X$, so that the Euclidean distance, denoted by d , generates its topology. Let then $K_{\mathbf{m}_i}$ and $K_{\mathbf{m}_j}$ be the theoretical finest equivalence classes of motor configurations \mathbf{m}_i and \mathbf{m}_j . One can then define \mathbf{D}^* as the target Euclidean distance matrix, with components $D_{ij}^* = d(K_{\mathbf{m}_i}, K_{\mathbf{m}_j})$. Again, \mathbf{D}^* contains all the theoretically available topological structure from the discrete data. The idea of the extrinsic evaluation consists in checking whether topological properties are conserved by the representative space built by the agent. This can be assessed by verifying the existence of a monotonous function between $\mathbf{D}[k]$ and \mathbf{D}^* : indeed, one can use the measure of topological similarity q defined previously in (25). Let's call $q^*[k] = q(\mathbf{D}[k], \mathbf{D}^*)$ the refinement score. Thus, q^* constitutes an extrinsic evaluation of the refinement, since $q^*[k] = 1$ corresponds to the best topologically correct representation of the extrinsic quotient pose space built by the agent from the available data. However, a bad refinement score does not imply a bad representation as will be shown in subsection IV-C.

B. Illustrative example.

Let's now work again with the example used all along the paper. Recall that the agent is a 2 degrees of freedom serial arm with revolute joints, endowed with a single pixel camera placed at the end-effector and that reacts to the presence of light in the environment, see § III-A2. Each environment state is parameterized by a straight line separating the working space into two black and a white complementary areas. Each line is picked randomly to obtain a uniform distribution of their distances to the center of the working space.

1) *Illustration of the refinement:* A simulated example of the space refinement described in section II-A is shown in Figure 9 with a growing number of environments, i.e. a growing number of straight lines delimiting the working space in two areas. On the top is drawn the agent working space and its partitioning into equivalence classes, shown with multiple colors. The corresponding partitions of the motor equivalence classes are shown on the bottom. One can see in (a, d), that even though the pose equivalence classes are convex subsets of the working space, the motor equivalence classes can possibly be separated subsets (see the yellow subset) or non-convex subsets (see the blue subset). This is a direct consequence of the forward kinematics function, which is a not-trivially a non-linear continuous map. For this specific agent/environment setup, and if a finite number of motor states is used, then from (H4), there exists a finite number of environment states beyond which the equivalence classes cannot be partitioned

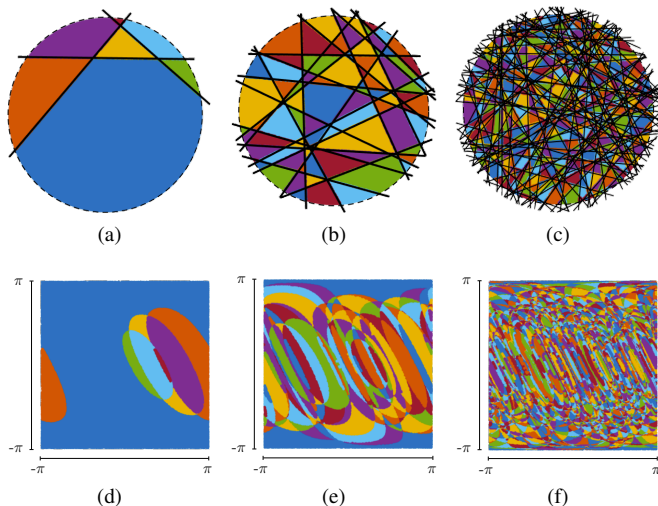


Fig. 9. Illustration of the refinement process. (a), (b) and (c) are refined partitions of the working space, whereas (d), (e) and (f) are refined partitions of the motor configuration space. From left to right are shown the partitions after $k = 3, 20$ and 100 environment states.

anymore. Then, it is obvious that if enough environments are experienced by the agent, all different sensor poses will be separated by at least one straight line, and thus will be associated to different sensations. The finest representation is then actually obtained when the number of equivalence classes does not grow anymore. In this extreme case, each point in the working space is associated to one and only one motor state. While difficult to illustrate, Subfigures (c,f) exhibits a pretty advanced step in the refinement where very small regions (points) in the working space are associated to one and only one subsets in the motor space (sensorimotor points). When this maximum number of equivalence classes is reached, the refinement distance can still evolve with the successive environments, so that the representation should converge to a topologically correct representation. This is illustrated in the next subsection.

2) *Illustration of the topological conservation:* Algorithm 1 is now ran with $N = 400$, and has stopped around $k = 1000$ samples. The N motor states are each associated to their corresponding sensor poses (a point in the working space), which are shown in Figure 9e with a color depending on their distance to the center. After running the algorithm, the agent obtains a refinement distance matrix $\mathbf{D}[1000]$ which can be compared to the euclidean distance matrix \mathbf{D}^* using the refinement score $q^*[1000]$. The comparison is shown in Subfigure 10b which is a scatter plot of the refinement distance estimates compared with pairwise $2D$ euclidean distances between the poses in the working space. In this plot, one can directly verify if the refinement distances are a monotonic transformation of the euclidean distances between the poses. If so then the representative space has a correct topology. Here, both distances are almost linearly correlated with a refinement score of $q^*[1000] = 0.9979$, therefore topological properties are conserved by the refinement. Actually, this is not a surprise, since this simple environment setup –with a straight line delimiting two areas– can be totally formalized,

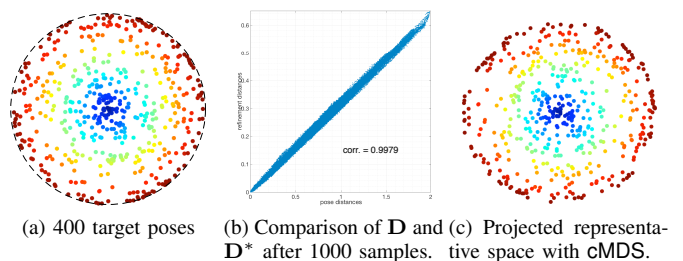


Fig. 10. Results for the illustrative example. The two subfigures on the right show how topological properties are conserved through the refinement distances.

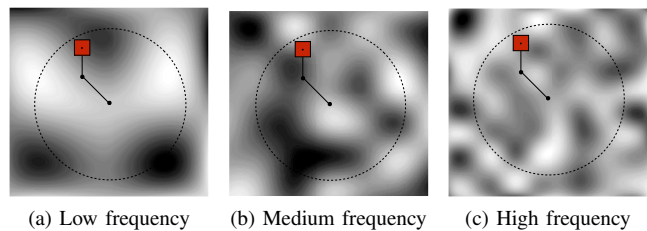


Fig. 11. States from three environments with different illumination distributions.

and it can be proved that the refinement distance matrix $\mathbf{D}[k]$ converges to the matrix $\alpha\mathbf{D}^*$ where α is a positive scalar. Furthermore, to properly visualize the obtained topology, one can project the refinement distance matrix $\mathbf{D}[1000]$ into a $2D$ euclidean space by using a dimensionality reduction techniques. Importantly, this projection is only used here to provide some insight to the reader and to illustrate how the representative space, whose topology is entirely captured in the refinement distance matrix \mathbf{D} , has the same topology than the agent working space. As an example, let's project the refinement distance matrix into the $2D$ euclidean space, by applying the classical MultiDimensional Scaling (cMDS) algorithm [27]. The result of the projection is shown in Subfigure 10c, colored with the same color as the corresponding target poses in Subfigure 10a. As expected, the low dimensional representation is almost identical to the sensor poses in the $2D$ working space, up to a rotation and a scaling factor: points which are close/far from the working space center are also close/far from the low-dimensional projection center. The working space topology has thus been correctly estimated. But what would happen with a more complex environment state, for which one would not be able to easily demonstrate the convergence of the representation? This case is simulated in the next subsection.

C. A more complex 2D environment

In the following simulations, the environment is now made of images of normalized spatially coherent noise whose statistics are invariant through translation or rotation. The implemented noise function is very similar to Perlin noise [28]. Different environments with different spatial frequencies are presented to the agent, as shown in figure 11. The agent is still endowed with a single pixel camera which send a 0

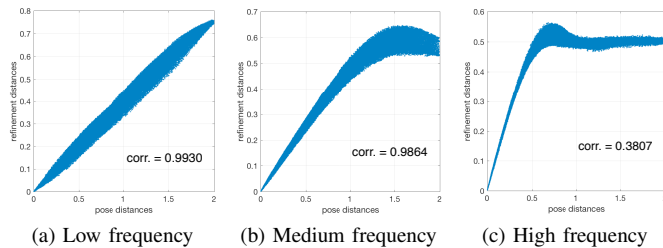


Fig. 12. Comparison of the refinement distance with the pairwise euclidean distance of the sensor poses.

when luminance is < 0.5 and 1 otherwise. For three different environments sets (low, medium and high illumination frequencies), algorithm 1 is ran with $N = 400$. Let's compare the obtained pairwise refinement distances to the pairwise euclidean distances in the working space in Figure 12. From the subfigures, its obvious that lower is the illumination frequency, better is the refinement score. Looking more carefully to the distance curves, one can remark that for the high frequency environments, the refinement distances saturate around 0.5 for high poses distances in the working space. This means that if two sensor poses are farther to each other than a certain distance in the working space, then they have probability 0.5 to generate a different binary sensation, this can be explained by the fact that the physical stimuli, here the illuminations, at two points in space tend to be uncorrelated if these points are far from each other. On the contrary, for smaller distances in the working space the physical stimuli tends to be more and more correlated and the refinement distance tends to 0. This is characterized, in the subfigure 12c, by the monotonous part for pose distances below 0.7.

Since topological information is by definition kept by local structures, the agent has properly extracted the topological information even-though the refinement score is not close to 1.

In order to illustrate how the topology is correctly conserved by the representation, let's consider an artificial target pose space with some highlighted topological structures such as the target pose space in subfigure 13a. One can remark that the working space now excludes points inside 4 different circles, each of them being spotted with different colors. The projected representation from cMDS with the more complex high frequency environment is shown in subfigure 13b: the projection does not appear to conserve the topology of the original working space. Since cMDS does not specifically privilege small distances, it can not keep topological properties by projecting the refinement distance matrix in $2D$. However, to show that the representation is still topologically correct, one can apply an other algorithm that favors small distances such as Isomap [29]. Isomap works by translating the distance matrix into a neighborhood graph where edges are linking nodes that are closer than a distance parameter ϵ . Then a new distance matrix is computed using the shortest path between all pair of nodes and this new dissimilarity matrix is embedded in a low dimension euclidean space using again cMDS. Applying now Isomap with $\epsilon = 0.4$ in $2D$ produces subfigure 13c.

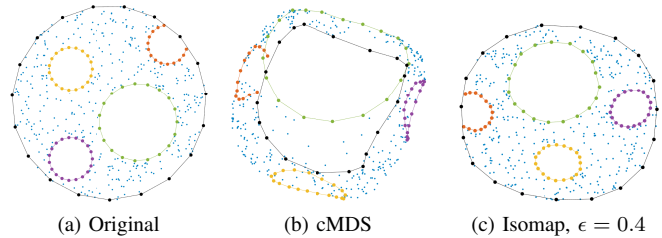


Fig. 13. Refinement $2D$ projections with cMDS and Isomap for high frequency environments.

In this subfigure, its clear that topological information has been kept in the representative space. More generally a good topological representation is always obtained as long as small distances in the working space corresponds to small refinement distances.

D. Conclusion

In this section, the authors have a proposed an algorithm that enable a naive agent to numerically build a topologically correct representation of the interaction of its sensors with the environment. It has been shown that the agent is also intrinsically able to evaluate the convergence of its representation until it has stable topological properties. The approach has been tested in different types of environments where the agent has been able to build a correct topological representation of the pose spaces. Moreover it has been shown the difficulty to extrinsically evaluate the representation in the case of a highly varying environment, however in this case the representation is still carrying good topological properties. It can be noted that the refinement approach is not limited to number of degrees of freedom of the agent.

For the sake of clarity the authors have focused on simple agents in simple environments however the approach is clearly generalizable to more complex sensors and natural environments as will show extended works.

V. CONCLUSION

This paper, rooted in the sensorimotor contingencies theory, has proposed a mathematical formalization to handle changing environments states. While often considered as static in other contributions, these changes have been explicitly exploited here to define a refinement approach to space representation. Some important theoretical results have followed: definition of sensorimotor points, representation of the quotient pose space, etc. These theoretical ideas have been assessed successfully in realistic simulations, and a simple algorithm with guaranteed convergence properties has been proposed. Ongoing theoretical and experimental works are now concerned with continuous changes in the environment states. Since a real implementation of the approach on a real robotic agent is targeted in the near future, some plausible experimental issues also have to be considered, especially regarding noise in the sensory and proprioceptive data. First raw results seem to exhibit a good robustness of the approach, but this has to be confirmed, and more importantly, understood. This might

constitute an important step before exploiting the SMC theory in the real world.

REFERENCES

- [1] J. K. O'Regan and A. Noë, "A sensorimotor account of vision and visual consciousness." *The Behavioral and brain sciences*, vol. 24, no. 5, pp. 939–973; discussion 973–1031, Oct. 2001.
- [2] J. K. O'Regan, "Solving the "real" mysteries of visual perception: The world as an outside memory," *Canadian Journal of Psychology*, vol. 46, no. 3, pp. 461–488, 1992.
- [3] A. V. Terekhov and J. K. O'Regan, "Space as an invention of biological organisms," *available at: arXiv:1308.2124v1 [cs.AI]*, August 2013.
- [4] G. Le Clec'H, J. K. O'Regan, and B. Gas, "Acquisition of a space representation by a naive agent from sensorimotor invariance and proprioceptive compensation," *International Journal of Advanced Robotic Systems*, vol. 13, no. 6, 2016.
- [5] H. Poincaré, "L'espace Et la Géométrie," *Revue de Métaphysique Et de Morale*, vol. 3, no. 6, pp. 631–646, 1895.
- [6] ———, "Science and hypothesis," *London: Walter Scott Publishing*, 1905.
- [7] J. J. Gibson, *The senses considered as perceptual systems*. Boston: Houghton, 1966.
- [8] P. Bach-y Rita, C. Collins, F. Saunders, B. White, and L. Scasden, "Vision substitution by tactile image projection," *Nature*, vol. 221, no. 5184, pp. 963–964, mar 1969.
- [9] J. J. Gibson, *The ecological approach to visual perception*. Boston: Houghton, 1979.
- [10] A. Berthoz, *Le sens du mouvement*. Odile Jacob, 1997.
- [11] A. Noë, *Action in perception*. The MIT Press, 2004.
- [12] J. Bohg, K. Hausman, B. Sankaran, O. Brock, D. Kragic, S. Schaal, and G. S. Sukhatme, "Interactive perception: Leveraging action in perception and perception in action," *IEEE Transactions on Robotics*, vol. PP, no. 99, pp. 1–19, 2017.
- [13] D. Philipona, J. K. O'Regan, and J.-P. Nadal, "Is there something out there?: Inferring space from sensorimotor dependencies," *Neural Comput.*, vol. 15, no. 9, pp. 2029–2049, 2003.
- [14] A. Laflaquière, S. Argentieri, O. Breysse, S. Genet, and B. Gas, "A non-linear approach to space dimension perception by a naive agent," in *Intelligent Robots and Systems (IROS), 2012 IEEE/RSJ International Conference on*, Oct 2012, pp. 3253–3259.
- [15] P. Demartines and J. Hérault, "Curvilinear component analysis: a self-organizing neural network for nonlinear mapping of data sets." *IEEE Transactions on Neural Networks*, vol. 8, no. 1, pp. 148–54, jan 1997.
- [16] V. Y. Roschin and A. Frolov, "A neural network model for the acquisition of a spatial body scheme through sensorimotor interaction," *Neural Computation*, vol. 23, pp. 1821–1834, 2011.
- [17] O. Gapenne, "The co-constitution of the self and the world: action and proprioceptive coupling," *Frontiers in Psychology*, vol. 5, 2014.
- [18] A. Laflaquière, "Grounding the experience of a visual field through sensorimotor contingencies," *Neurocomputing*, 2017.
- [19] A. Laflaquière, J. K. O'Regan, S. Argentieri, B. Gas, and A. Terekhov, "Learning agents spatial configuration from sensorimotor invariants," *Robotics and Autonomous Systems*, vol. 71, pp. 49–59, September 2015.
- [20] A. Laflaquière, "Approche sensorimotrice de la perception de l'espace pour la robotique autonome," 2013.
- [21] V. Marcel, B. Garcia, S. Argentieri, and B. Gas, "Building the representation of an agent body from its sensorimotor invariants," in *Workshop on Sensorimotor Contingencies for Robotics, IEEE/RSJ International Conference on Intelligent Robots and Systems, IROS'15*, 2015.
- [22] V. Marcel, S. Argentieri, and B. Gas, "Building a sensorimotor representation of a naive agent's tactile space," *IEEE Transactions on Cognitive and Developmental Systems*, vol. 9, no. 2, pp. 141–152, June 2017.
- [23] D. Philipona, J. K. O'Regan, J.-P. Nadal, and O. J.-M. Coenen, "Perception of the structure of the physical world using unknown sensors and effectors," *Advances in Neural Information Processing Systems*, vol. 15, 2004.
- [24] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*. New York: Springer, 1978, §3: Compactness Properties and the T_i Axioms.
- [25] J. B. Kruskal, "Multidimensional scaling by optimizing goodness of fit to a nonmetric hypothesis," *Psychometrika*, vol. 29, no. 1, pp. 1–27, 1964.
- [26] A. Censi and D. Scaramuzza, "Calibration by correlation using metric embedding from nonmetric similarities," *IEEE transactions on pattern analysis and machine intelligence*, vol. 35, no. 10, pp. 2357–2370, 2013.
- [27] W. S. Torgerson, "Multidimensional scaling: I. theory and method," *Psychometrika*, vol. 17, no. 4, pp. 401–419, 1952.
- [28] K. Perlin, "An image synthesizer," *ACM Siggraph Computer Graphics*, vol. 19, no. 3, pp. 287–296, 1985.
- [29] J. B. Tenenbaum, V. de Silva, and J. C. Langford, "A global geometric framework for nonlinear dimensionality reduction," *Science*, vol. 290, no. 5500, pp. 2319–2323, 2000.