



**HAL**  
open science

# GENERALIZED CURIE-WEISS MODEL AND QUADRATIC PRESSURE IN ERGODIC THEORY

Renaud Leplaideur, Frédérique Watbled

► **To cite this version:**

Renaud Leplaideur, Frédérique Watbled. GENERALIZED CURIE-WEISS MODEL AND QUADRATIC PRESSURE IN ERGODIC THEORY. Bulletin de la Société Mathématique, 2019, pp.197-219. 10.24033/bsmf.2779 . hal-01678176

**HAL Id: hal-01678176**

**<https://hal.science/hal-01678176>**

Submitted on 9 Jan 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# GENERALIZED CURIE-WEISS MODEL AND QUADRATIC PRESSURE IN ERGODIC THEORY

RENAUD LEPLAIDEUR AND FRÉDÉRIQUE WATBLED

ABSTRACT. We explain the Curie Weiss model in Statistical Mechanics within the Ergodic viewpoint. More precisely, we simultaneously define in  $\{-1, +1\}^{\mathbb{N}}$ , on the one hand a generalized Curie Weiss model within the Statistical Mechanics viewpoint and on the other hand, quadratic free energy and quadratic pressure within the Ergodic Theory viewpoint. We show that there are finitely many invariant measures which maximize the quadratic free energy. They are all Dynamical Gibbs Measures. Moreover, the Probabilistic Gibbs measures for generalized Curie Weiss model converge to a determined combination of the (dynamical) conformal measures associated to these Dynamical Gibbs Measures. The standard Curie Weiss model is a particular case of our generalized Curie Weiss model. An Ergodic viewpoint over the Curie Weiss Potts model is also given.

## 1. INTRODUCTION

**1.1. Background, main motivations and results.** The notion of Gibbs measure comes from Statistical Mechanics. It has been studied a lot from the probabilistic viewpoint (see [12, 5, 8, 9]). This notion was introduced in Ergodic Theory in the 70's by Sinai, Ruelle and Bowen (see [25, 26, 23, 22, 2]). Since that moment, the thermodynamic formalism became in Dynamical Systems a purely mathematical question and has somehow become isolated from the original physical questions.

With years, it turned out that this situation has generated sources of confusions. The first one is that people share the same vocabulary but it is not clear that the same names precisely define the same notions in each viewpoint (ergodic vs physicist). We *e.g.* refer to *phase transition*, *Gibbs measures*, *pressure*. Furthermore, the confusion is also internal to Ergodic Theory. Indeed, the thermodynamic formalism is very differently presented for  $\mathbb{Z}$ -actions (where the Transfer Operator plays a crucial role) or for  $\mathbb{Z}^d$ -actions (with  $d > 1$ ). For this later case, the thermodynamic formalism is much closer to what people in Statistical Mechanics or in Probability do. Now,

---

*Date:* Version of October 6, 2017.

*2010 Mathematics Subject Classification.* 37A35, 37A50, 37A60, 82B20, 82B30, 82C26.

*Key words and phrases.* thermodynamic formalism, equilibrium states, Curie-Weiss model, Curie-Weiss-Potts model, Gibbs measure, phase transition.

F. Watbled thanks the IRMAR, CNRS UMR 6625, University of Rennes 1, for its hospitality.

The authors thank the Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment.

it turns out that several questions arising for 1-dimensional actions ergodic theory have to be exported to the higher dimensional case (see [3, 1]). Therefore, it became important to make clear similitudes and differences in the thermodynamic formalism between physicist and (1-d) ergodic viewpoints.

Our first result (see Theorem 1) states a kind of dictionary between thermodynamic formalism in Statistical Mechanics and Probability on the one hand, and Ergodic Theory on the other hand. More precisely we explain with the ergodic vocabulary the first-order phase transition arising for the Curie-Weiss Model (mean field case) and make precise the link between Gibbs measures within the Physicist/Probabilistic viewpoints and the Ergodic viewpoint. We initially decided to focus on the mean field case for the following reasons. First, there is a large literature dealing with this topic. Second, the mean field model is naturally represented into  $\{-1, +1\}^{\mathbb{N}}$  and exhibits “physical phase transitions” that we wanted to compare with “1-d ergodic phase transitions” in  $\{-1, +1\}^{\mathbb{N}}$ .

From there, a natural question was to get a similar dictionary for the Curie-Weiss-Potts model which is a generalization of the Curie-Weiss model. This is done in Theorem 3.

These two results are then the motivation for our main result (see Theorem 2). The key point is that the Hamiltonian for the Curie-Weiss model is almost equal to the square of a Birkhoff sum. Now, Birkhoff sum is a key object in Dynamical Systems. We thus introduce within the Ergodic viewpoint the notion of *quadratic free energy*. It is equal to the entropy plus the square of an integral. We are naturally led to study a variational principle, that is which invariant measures do maximize the quadratic free energy. This maximum defines the *quadratic pressure*. At the same time, we introduce a generalized Hamiltonian in the Curie Weiss model and show the link between the associated Gibbs measures (within Physicist/Probabilistic viewpoint) and the Gibbs measures within the Ergodic viewpoint. We show how first order phase transitions for this generalized Curie Weiss model are related to a bifurcation into the set of measures which maximize the quadratic free energy. Theorem 1 is thus a particular case of Theorem 2.

We believe that this quadratic pressure generates further possible research questions in Ergodic Theory. Some of them are discussed later (see Subsubsection 1.2.5). Similarly, we believe that our generalized Curie Weiss model may have physical interest.

Finally, we point out that Theorem 2 is not an extension of Theorem 3. There is no obstruction to define and study the quadratic pressure for more general subshift of finite type. Nevertheless, the Hamiltonian for the Curie-Weiss-Potts model does not write itself as a square of a Birkhoff sum, because one considers a vector-valued “potential”. This is work in progress to give an extension of Theorem 3 with the flavour of Theorem 2.

## 1.2. Precise settings and results.

1.2.1. *Ergodic and Dynamical setting.* We consider a finite set  $\Lambda$  with cardinality bigger or equal to 2. It is called the alphabet. Then we consider the one-sided full shift  $\Sigma = \Lambda^{\mathbb{N}}$  over  $\Lambda$ . A point  $x$  in  $\Sigma$  is a sequence  $x_0, x_1, \dots$  (also called an infinite word) where the  $x_i$  are in  $\Lambda$ . Most of the times we shall use the notation  $x = x_0x_1x_2\dots$ . A  $x_i \in \Lambda$  can either be called a letter, or a digit or a symbol.

The shift map  $\sigma$  is defined by

$$\sigma(x_0x_1x_2\dots) = x_1x_2\dots$$

The distance between two points  $x = x_0x_1\dots$  and  $y = y_0y_1\dots$  is given by

$$d(x, y) = \frac{1}{2^{\min\{n, x_n \neq y_n\}}}.$$

A finite string of symbols  $x_0\dots x_{n-1}$  is also called a *word*, of length  $n$ . For a word  $w$ , its length is  $|w|$ . A *cylinder* (of length  $n$ ) is denoted by  $[x_0\dots x_{n-1}]$ . It is the set of points  $y$  such that  $y_i = x_i$  for  $i = 0, \dots, n-1$ . We shall also talk about  $n$ -cylinder instead of cylinder of length  $n$ .

If  $w$  is the word of finite length  $w_0\dots w_{n-1}$  and  $x$  is a word, the concatenation  $wx$  is the new word  $w_0w_1\dots w_{n-1}x_0x_1\dots$ .

For  $\psi : \Sigma \rightarrow \mathbb{R}$  continuous and  $\beta > 0$ , the *pressure function* is defined by

$$(1) \quad \mathcal{P}(\beta\psi) := \sup_{\mu} \left\{ h_{\mu} + \beta \int_{\Sigma} \psi d\mu \right\},$$

where the supremum is taken among the set  $\mathcal{M}_{\sigma}(\Sigma)$  of  $\sigma$ -invariant probabilities on  $\Sigma$  and  $h_{\mu}$  is the Kolmogorov-Sinai entropy of  $\mu$ . The real parameter  $\beta$  is assumed to be positive because it represents the inverse of the temperature in statistical mechanics. It is known that the supremum is actually a maximum and any measure for which the maximum is attained in (1) is called an *equilibrium state for  $\beta\psi$* . We refer the reader to [2, 23] for basic notions on thermodynamic formalism in ergodic theory.

If  $\psi$  is Lipschitz continuous then the Ruelle theorem (see [21]) states that for every  $\beta$ , there is a unique equilibrium state for  $\beta\psi$ , which is denoted by  $\tilde{\mu}_{\beta\psi}$ . It is *ergodic* and it shall be called the *dynamical Gibbs measure* (DGM for short<sup>1</sup>). It is the unique  $\sigma$ -invariant probability measure which satisfies the property that for every  $x = x_0x_1\dots$  and for every  $n$ ,

$$(2) \quad e^{-C_{\beta}} \leq \frac{\tilde{\mu}_{\beta\psi}([x_0\dots x_{n-1}])}{e^{\beta \cdot S_n(\psi)(x) - n\mathcal{P}(\beta\psi)}} \leq e^{C_{\beta}},$$

where  $C_{\beta}$  is a positive real number depending only on  $\beta$  and  $\psi$  (but not on  $x$  or  $n$ ), and  $S_n(\psi)$  stands for  $\psi + \psi \circ \sigma + \dots + \psi \circ \sigma^{n-1}$ .

---

<sup>1</sup>We prefer the adjective ‘‘dynamical’’ instead of ‘‘ergodic’’ to avoid the discussion if an ergodic Gibbs measure is ergodic or not.

In this setting, the  $\beta\psi$ -conformal measure is the unique probability measure such that for every  $x$  and for every  $n$ ,

$$(3) \quad \nu_{\beta\psi}([x_0 \dots x_{n-1}]) = \int e^{\beta S_n(\psi)(x_0 \dots x_{n-1} y) - n\mathcal{P}(\beta\psi)} d\nu_{\beta\psi}(y).$$

A precise (and more technical) definition of conformal measure is given in page 10, where the connection between conformal measures and DGM is stated. We emphasize that in our setting, conformal measures and DGM are equivalent measures and one can obtain one from the other.

If the choice of  $\psi$  is clear we shall drop the  $\psi$  and write  $\tilde{\mu}_\beta$ ,  $\nu_\beta$  and  $\mathcal{P}(\beta)$ .

1.2.2. *The Curie-Weiss model.* We consider the case  $\Lambda = \{-1, +1\}$ ;  $\Sigma$  will be denoted by  $\Sigma_2$ .

If  $\omega_0 \dots \omega_{n-1}$  is a finite word, we set

$$(4) \quad H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \omega_j \omega_i.$$

It is called the *Curie-Weiss Hamiltonian*. The *empirical magnetization* for  $\omega$  is

$$m_n(\omega) := \frac{1}{n} \sum_{j=0}^{n-1} \omega_j. \text{ Then we have}$$

$$(5) \quad H_n(\omega) = -\frac{n}{2} (m_n(\omega))^2.$$

We denote by  $\mathbb{P} := \rho^{\otimes \mathbb{N}}$  the product measure on  $\Sigma_2$ , where  $\rho$  is the uniform measure on  $\{-1, 1\}$ , i.e.  $\rho(\{1\}) = \rho(\{-1\}) = \frac{1}{2}$ , and we define the *probabilistic Gibbs measure* (PGM for short)  $\mu_{n,\beta}$  on  $\Sigma_2$  by

$$(6) \quad \mu_{n,\beta}(d\omega) := \frac{e^{-\beta H_n(\omega)}}{Z_{n,\beta}} \mathbb{P}(d\omega),$$

where  $Z_{n,\beta}$  is the normalization factor

$$Z_{n,\beta} = \frac{1}{2^n} \sum_{\omega', |\omega'|=n} e^{-\beta H_n(\omega')}.$$

Note that  $\mu_{n,\beta}$  can also be viewed as a probability defined on  $\Lambda^n$ .

The measure  $\mathbb{P}$  is a Bernoulli measure and is  $\sigma$ -invariant. In Ergodic Theory it is usually called the Parry-measure (see [19]) and turns out to be the unique measure with maximal entropy. With our previous notations it corresponds to the DGM  $\tilde{\mu}_0$ .

If  $P_n, P$  are probability measures on the Borel sets of a metric space  $S$ , we say that  $P_n$  converges weakly to  $P$  if  $\int_S f dP_n \rightarrow \int_S f dP$  for each  $f$  in the class  $C_b(S)$  of bounded, continuous real functions  $f$  on  $S$ . In this case we write  $P_n \xrightarrow[n \rightarrow +\infty]{w} P$ .

Our first result concerns the weak convergence of the measures  $\mu_{n,\beta}$ .

**Theorem 1. Weak convergence for the CW model**

Let  $\xi_\beta$  be the unique point in  $[0, 1]$  which realizes the maximum for

$$\varphi_I(x) := \log(\cosh(\beta x)) - \frac{\beta}{2}x^2.$$

Let  $\tilde{\mu}_b$  be the dynamical Gibbs measure for  $b(\mathbb{1}_{[+1]} - \mathbb{1}_{[-1]})$ . Then

$$(7) \quad \mu_{n,\beta} \xrightarrow[n \rightarrow +\infty]{w} \begin{cases} \tilde{\mu}_0 & \text{if } \beta \leq 1, \\ \frac{1}{2} [\tilde{\mu}_{\beta\xi_\beta} + \tilde{\mu}_{-\beta\xi_\beta}] & \text{if } \beta > 1. \end{cases}$$

**Remark 1.** Actually  $\mu_{n,\beta}$  converges towards  $\frac{1}{2} [\tilde{\mu}_{\beta\xi_\beta} + \tilde{\mu}_{-\beta\xi_\beta}]$  for every  $\beta > 0$  since we shall see that for  $\beta \leq 1$  we have  $\xi_\beta = 0$ , and it is clear that  $\tilde{\mu}_0 = \rho^{\otimes \mathbb{N}}$ .

We refer to [7], sections IV.4 and V.9, for a discussion of the Curie-Weiss model and historical references (see also [20], section 3.4). We also mention that the weak convergence of  $\mu_{n,\beta}$  was already proved by Orey ([18], Corollary 1.2) by a nice simple probabilistic argument. We remind that our motivation is the dictionary aspect and not the convergence.

We emphasize the equality

$$(8) \quad m_n(\omega) := \frac{1}{n} S_n(\mathbb{1}_{[+1]} - \mathbb{1}_{[-1]})(\omega)$$

which shows that  $m_n$  can be written as a Birkhoff mean of a continuous function.

A consequence of (8) is that (5) can be rewritten under the form

$$H_n(\omega) = -\frac{n}{2} \left( \frac{1}{n} S_n(\psi)(\omega) \right)^2,$$

where  $\psi := \mathbb{1}_{[+1]} - \mathbb{1}_{[-1]}$ .

1.2.3. *The generalized Curie Weiss model.* If  $\psi$  is a Hölder continuous function on  $\Sigma_2$ , we define the *generalized Curie-Weiss Hamiltonian*  $H_n^\psi$  associated to  $\psi$  by setting

$$H_n^\psi(\omega) = -\frac{n}{2} \left( \frac{1}{n} S_n(\psi)(\omega) \right)^2.$$

Then  $\mu_{n,\beta}^\psi$  is the PGM defined by

$$(9) \quad d\mu_{n,\beta}^\psi(d\omega) := \frac{e^{-\beta H_n^\psi(\omega)}}{Z_{n,\beta}^\psi} d\mathbb{P}(\omega), \quad \text{with } Z_{n,\beta}^\psi = \int_{\Sigma_2} e^{-\beta H_n^\psi} d\mathbb{P}.$$

If  $\mu$  is an invariant measure on  $\Sigma_2$ , we define its *quadratic free energy* by

$$h_\mu + \frac{\beta}{2} \left( \int_{\Sigma_2} \psi d\mu \right)^2.$$

Then we define the *quadratic pressure function* (associated to  $\Psi$ ) by

$$(10) \quad \mathcal{P}_2(\beta\psi) := \sup_{\mu} \left\{ h_\mu + \frac{\beta}{2} \left( \int_{\Sigma_2} \psi d\mu \right)^2 \right\}.$$

Upper semi-continuity for the entropy immediately shows that the supremum is a maximum. The function  $\beta \mapsto \mathcal{P}_2(\beta\psi)$  is obviously convex (thus continuous).

### Theorem 2. Weak convergence for the generalized Curie Weiss model

Let  $\psi$  be a Hölder continuous function on  $\Sigma_2$ , let  $\beta$  be a positive real number.

- (1) There are finitely many invariant probabilities  $m_1, \dots, m_J$  (with  $J = J(\beta)$ ) whose quadratic free energy (for  $\beta$ ) is maximal and thus equal to the quadratic pressure  $\mathcal{P}_2(\beta\psi)$ .
- (2) Each  $m_i$  is the unique equilibrium state  $\tilde{\mu}_{\beta t_i \psi}$  for the potential  $\beta t_i \psi$ .
- (3) The numbers  $t_1, \dots, t_J$  are the maxima of the auxiliary function

$$\varphi_{OS}(t) := \mathcal{P}(\beta t \psi) - \frac{\beta}{2} t^2.$$

- (4) As  $n$  goes to  $+\infty$ ,  $\mu_{n,\beta}^\psi$  converges weakly to a convex combination of the conformal measures  $\nu_{\beta t_j}$ 's associated to  $\beta t_j \psi$ :

$$\mu_{n,\beta} \xrightarrow[n \rightarrow +\infty]{w} \sum_{j=1}^J c_j \nu_{\beta t_j}.$$

The  $c_j$ 's are well identified (see formulas (27) and (28)).

We emphasize that Theorem 1 is a particular case of Theorem 2 with  $\psi = \mathbb{1}_{[+]} - \mathbb{1}_{[-]}$ . In that case the pressure is easy to compute and is equal to

$$\mathcal{P}(\beta\psi) = \log 2 + \log(\cosh \beta),$$

and we get  $\varphi_{OS}(x) = \log 2 + \varphi_I(x)$ . Note that for this particular case, the DGM is also the conformal measure.

1.2.4. *Comparison of definitions of phase transition.* Nowadays, a phase transition in ergodic theory means the lack of analyticity for the pressure function (see *e.g.* [4, 24, 17]). It is known that this notion is transversal to the number of equilibrium states. One can have a loss of analyticity with only one equilibrium state (see the Manneville-Pomeau example with good parameters, [28]) or analyticity with several equilibrium states (see [16]).

For the quadratic pressure, things may be different. We remind that  $z \mapsto \mathcal{P}(z\psi)$  is analytic (for Hölder continuous  $\psi$ ). Each  $t_i$  is a maximum for  $\varphi_{\text{OS}}$  and then satisfies  $\mathcal{P}'(\beta t_i) = t_i$ . It is thus highly probable that  $t_i(\beta)$  is locally analytic (and surely locally  $\mathcal{C}^\infty$ ). Then, the quadratic pressure satisfies

$$\mathcal{P}_2(\beta) = h_{\tilde{\mu}_{\beta t_i \psi}} + \frac{\beta}{2} \left( \int_{\Sigma_2} \psi d\tilde{\mu}_{\beta t_i \psi} \right)^2 = \mathcal{P}(\beta t_i \psi) - \beta t_i + \frac{\beta}{2} t_i^2.$$

It is thus reasonable to expect  $\mathcal{P}_2(\beta)$  to be at least piecewise  $\mathcal{C}^\infty$  and even probably piecewise  $\mathcal{C}^\omega$ . Moreover, we expect the borders of intervals of analyticity to be exactly where there is a change in the number of  $t_i$ 's.

It is therefore very likely that the loss of analyticity for the quadratic pressure is equivalent to a bifurcation in the number of “quadratic” equilibrium states. Actually, this is corroborated by Theorem 1, where the quadratic pressure is piecewise analytic (and not analytic) and the number of quadratic equilibrium states change with respect to  $\beta$  exactly where analyticity fails.

1.2.5. *Some consequences of Theorem 2.* Several questions naturally arise from Theorem 2. At that stage, we do not have more precise conjectures or ideas for answers.

- For more geometric dynamical systems, one usually considers or studies the special class of *physical or/and SRB-measures*. These measures are usually considered as the most natural ones with the measures of maximal entropy. It is clear that measures of maximal entropy also maximize  $h_\mu + \left( \int_{\Sigma_2} \psi d\mu \right)^2$  for  $\psi \equiv 0$ .

A natural question is thus to know if for a system admitting one SRB-measure, there exists some potential  $\psi$  such that the SRB measure maximizes the quadratic free energy  $h_\mu + \left( \int_{\Sigma_2} \psi d\mu \right)^2$ .

- More generally, one can ask how big is the set of measures which maximize the quadratic pressure for some potential  $\psi$ ? It is for instance known that any ergodic measure is an equilibrium state for some continuous potential (see [23, Cor. 3.17]). Does it still hold for quadratic pressure?

- Ergodic Optimization studies what happens to DGM  $\tilde{\mu}_{\beta\psi}$  as  $\beta$  goes to  $+\infty$ . It is known that any accumulation point maximizes the integral of  $\psi$  among invariant measures. The goal is to study if there is convergence and how is the limit selected among the simplex of  *$\psi$ -maximizing measures*. The same kind of questions may be studied with the quadratic pressure. We point-out that non-linearity may introduce very new and different phenomena compared to the “usual pressure”.



1.2.6. *The Curie-Weiss-Potts model. Probabilistic settings 2 and result.* The Curie-Weiss-Potts model will be for  $\Lambda = \{\theta^1, \dots, \theta^q\}$  with  $q > 2$ . In that case we shall write  $\Sigma_q$  instead of  $\Sigma$ .

The Curie-Weiss-Potts Hamiltonian is defined for a finite word  $\omega = \omega_0 \cdots \omega_{n-1}$  by

$$(11) \quad H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \mathbb{1}_{\omega_j = \omega_i}.$$

We define the vector  $L_n(\omega) = (L_{n,1}(\omega), \dots, L_{n,q}(\omega))$  where

$$L_{n,k}(\omega) = \sum_{i=0}^{n-1} \mathbb{1}_{\omega_i = \theta^k}$$

is the number of digits of  $\omega$  which take the value  $\theta^k$ , so that we can write

$$\sum_{i,j=0}^{n-1} \mathbb{1}_{\omega_j = \omega_i} = \sum_{k=1}^q \left( \sum_{i=0}^{n-1} \mathbb{1}_{\omega_i = \theta^k} \right)^2 = \|L_n(\omega)\|^2,$$

where  $\|\cdot\|$  stands for the Euclidean norm on  $\mathbb{R}^q$ .

We denote by  $\mathbb{P} := \rho^{\otimes \mathbb{N}}$  the product measure on  $\Sigma_q$ , where  $\rho$  is the uniform measure on  $\Lambda$ , i.e.  $\rho = \frac{1}{q} \sum_{k=1}^q \delta_{\theta^k}$ , and we define the probabilistic Gibbs measure  $\mu_{n,\beta}$  on  $\Sigma_q$  by

$$(12) \quad \mu_{n,\beta}(d\omega) := \frac{e^{-\beta H_n(\omega)}}{Z_{n,\beta}} \mathbb{P}(d\omega) = \frac{e^{\frac{\beta}{2n} \|L_n(\omega)\|^2}}{Z_{n,\beta}} \mathbb{P}(d\omega),$$

where  $Z_{n,\beta}$  is the normalization factor

$$Z_{n,\beta} = \frac{1}{q^n} \sum_{\omega', |\omega'|=n} e^{\frac{\beta}{2n} \|L_n(\omega')\|^2}.$$

Now we can state the analog of Theorem 1.

### Theorem 3. Weak convergence for the CWP model

For  $1 \leq k \leq q$ ,  $b \in \mathbb{R}$ , let  $\tilde{\mu}_b^k$  be the dynamical Gibbs measure for  $b\mathbb{1}_{[\theta^k]}$ . Let  $\beta_c = \frac{2(q-1)\log(q-1)}{q-2}$ . For  $0 < \beta < \beta_c$  set  $s_\beta = 0$  and for  $\beta \geq \beta_c$  let  $s_\beta$  be the largest solution of the equation

$$(13) \quad s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}.$$

Then,

$$(14) \quad \mu_{n,\beta} \xrightarrow[n \rightarrow +\infty]{w} \begin{cases} \rho^{\otimes \mathbb{N}} & \text{if } 0 < \beta < \beta_c, \\ \frac{1}{q} \sum_{k=1}^q \tilde{\mu}_{\beta s_\beta}^k & \text{if } \beta > \beta_c, \\ \frac{A \tilde{\mu}_0^1 + B \sum_{k=1}^q \tilde{\mu}_{\beta_c s_{\beta_c}}^k}{A + qB} & \text{if } \beta = \beta_c, \end{cases}$$

with  $A = \left(1 - \frac{\beta_c}{q(q-1)}\right)^{\frac{q-2}{2}}$  and  $B = \left(1 - \frac{\beta_c}{q}\right)^{\frac{q-2}{2}}$ .

**Remark 2.** Actually  $\mu_{n,\beta}$  converges towards  $\frac{1}{q} \sum_{k=1}^q \tilde{\mu}_{\beta s_\beta}^k$  for every  $\beta \neq \beta_c$  since  $s_\beta = 0$  for  $\beta < \beta_c$ , and it is clear that  $\tilde{\mu}_0^k = \rho^{\otimes \mathbb{N}}$  for each  $1 \leq k \leq q$ .

We refer to [10] for a discussion of the Curie-Weiss-Potts model and historical references. Orey ([18], Theorem 4.4) mentions the weak convergence of  $\mu_{n,\beta}$  towards an explicit atomic measure, but he makes a mistake concerning the case  $\beta = \beta_c$ , as pointed out in [10].

**1.3. Plan of the paper.** The paper is composed as follows.

In Section 2 we prove Theorem 2, in Section 3 we prove Theorem 3. Both proofs are based on the convergence of  $\mu_{n,\beta}(C)$  where  $C$  is a cylinder in  $\Sigma$ .

We point out that in Theorem 2 the proofs of the parts (3)-(4) and of the parts (1)-(2) are independent.

Theorem 1 is a simple consequence of Theorem 2 as said above.

## 2. PROOF OF THEOREM 2

**2.1. Convergence of  $\mu_{n,\beta}^\psi$ .** To lighten the notations we drop the  $\psi$  in  $H_n^\psi$ ,  $\mu_{n,\beta}^\psi$ ,  $Z_{n,\beta}^\psi$ . To prove the weak convergence of  $\mu_{n,\beta}$  towards a measure  $\mu$ , it is enough to show that for every cylinder  $C$ ,

$$(15) \quad \lim_{n \rightarrow \infty} \mu_{n,\beta}(C) = \mu(C).$$

Let  $\omega = \omega_0 \dots \omega_{p-1}$  be a finite word of length  $p$ , let  $n > p$ . We use the equality

$$e^{a^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} + \sqrt{2}ax} dx,$$

sometimes called the Hubbard-Stratonovich transformation ([14],[27]), to compute the following.

$$\begin{aligned}
Z_{n,\beta}\mu_{n,\beta}([\omega]) &= \int_{\Sigma_2} e^{\frac{\beta}{2n}(S_n(\psi)(\alpha))^2} \mathbb{1}_{[\omega]}(\alpha) d\mathbb{P}(\alpha) \\
&= \int_{\Sigma_2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} e^{\sqrt{\frac{\beta}{n}}x S_n(\psi)(\alpha)} \mathbb{1}_{[\omega]}(\alpha) dx d\mathbb{P}(\alpha), \\
(16) \quad &= \sqrt{\frac{\beta n}{2\pi}} \int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} \int_{\Sigma_2} e^{\beta z S_n(\psi)(\alpha)} \mathbb{1}_{[\omega]}(\alpha) d\mathbb{P}(\alpha) dz,
\end{aligned}$$

where we have made the change of variable  $\beta z = \sqrt{\frac{\beta}{n}}x$ .

Let us define the Transfer operator  $\mathcal{L}_\xi$ , depending on a real or complex parameter  $\xi$ , by

$$\mathcal{L}_\xi(T)(x) := \sum_{y, \sigma(y)=x} e^{\xi\psi(y)} T(y).$$

Then for every  $n \in \mathbb{N}$ ,

$$(17) \quad \mathcal{L}_\xi^n(T)(x) = \sum_{y, \sigma^n(y)=x} e^{\xi S_n\psi(y)} T(y).$$

We remind the following properties for  $\mathcal{L}_\xi$  (see [2], [19]):

**Facts .** *The operator  $\mathcal{L}_\xi$  acts on continuous and Hölder continuous functions. On Hölder continuous functions, its spectral radius  $\lambda_\xi$  is a simple dominating eigenvalue. We denote by  $H_\xi$  a positive associated eigenfunction. The rest of the spectrum of  $\mathcal{L}_\xi$  is included into the disk of radius  $\lambda_\xi e^{-\varepsilon(\xi)}$  for some positive  $\varepsilon(\xi)$ . The dual operator (for continuous functions) acts on measures. There exists a unique probability measure  $\nu_\xi$  which is the eigen-measure for the eigen-value  $\lambda_\xi$ . The measure  $\nu_\xi$  is the conformal measure. The DGM is then equal to*

$$d\tilde{\mu}_\xi = H_\xi d\nu_\xi,$$

where  $H_\xi$  is normalized such that  $\tilde{\mu}_\xi$  is a probability measure. The pressure is  $\log \lambda_\xi$ .

Therefore we have

$$(18) \quad \mathcal{L}_\xi^n(\mathbb{1}_{[\omega]})(x) = \lambda_\xi^n \nu_\xi([\omega]) H_\xi(x) + \lambda_\xi^n e^{-n\varepsilon(\xi)} T(n, \xi)(x)$$

with  $\|T(n, \xi)\|_\infty \leq 1$ .

For  $\alpha \in \Sigma_2$  one writes  $\alpha = \bar{\alpha}\theta$  with  $\bar{\alpha}$  equal to the suffix of length  $n$  of  $\alpha$  and  $\theta$  in  $\Sigma_2$ . Using (17) we can rewrite (16) as

$$\begin{aligned}
Z_{n,\beta}\mu_{n,\beta}([\omega]) &= \frac{1}{2^n} \sqrt{\frac{\beta n}{2\pi}} \int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} \int_{\Sigma_2} \sum_{\bar{\alpha}} e^{\beta z S_n(\psi)(\bar{\alpha}\theta)} \mathbb{1}_{[\omega]}(\bar{\alpha}) d\mathbb{P}(\theta) dz, \\
(19) \quad &= \frac{1}{2^n} \sqrt{\frac{\beta n}{2\pi}} \int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} \int_{\Sigma_2} \mathcal{L}_{\beta z}^n(\mathbb{1}_{[\omega]})(\theta) d\mathbb{P}(\theta) dz.
\end{aligned}$$

The normalization factor  $Z_{n,\beta}$  can be computed by replacing  $[\omega]$  by  $\Sigma_2$ , and we get

$$(20) \quad \mu_{n,\beta}([\omega]) = \frac{\int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} \int_{\Sigma_2} \mathcal{L}_{\beta z}^n(\mathbb{1}_{[\omega]})(\theta) d\mathbb{P}(\theta) dz}{\int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2} \int_{\Sigma_2} \mathcal{L}_{\beta z}^n(\mathbb{1})(\theta) d\mathbb{P}(\theta) dz} =: \frac{N_{n,\beta}}{D_{n,\beta}}.$$

Using (18) we get

$$(21) \quad N_{n,\beta} = \int_{-\infty}^{+\infty} e^{-n\frac{\beta}{2}z^2 + n \log \lambda_{\beta z}} \left[ \int_{\Sigma_2} \nu_{\beta z}([\omega]) H_{\beta z}(\theta) + e^{-n\varepsilon(\beta z)} T(n, \beta z)(\theta) d\mathbb{P}(\theta) \right] dz.$$

We want to use the Laplace method but the last term in the inner integral depends on  $n$ . This term converges to zero as  $n$  goes to infinity but the speed of convergence depends on  $z$  and  $|z|$  may go to infinity. Setting  $A := \|\psi\|_\infty$ , we deduce from (17) that for every  $n$ , every  $\xi$  and every  $T$  continuous

$$(22) \quad \|\mathcal{L}_\xi^n(T)\|_\infty \leq 2^n e^{n\xi A} \|T\|_\infty.$$

Therefore the term in the integral defining the numerator  $N_{n,\beta}$  in (20) is bounded from above by  $e^{-n\frac{\beta}{2}z^2 + n \log 2 + n\beta z A}$ . Furthermore, there exists  $Z(\beta)$  such that for  $|z| > Z(\beta)$

$$(23) \quad -\frac{\beta}{2}z^2 + \log 2 + \beta z A \leq -\frac{\beta}{4}z^2$$

holds, from which we deduce that there exists  $\kappa(\beta) > 0$  such that for every  $n > p$ ,

$$(24) \quad \int_{|z| \geq \kappa(\beta)} e^{-n\frac{\beta}{2}z^2} \int_{\Sigma_2} \mathcal{L}_{\beta z}^n(\mathbb{1}_{[\omega]})(\theta) d\mathbb{P}(\theta) dz \leq e^{-n\kappa(\beta)}.$$

From this we claim that the computation of the integral in (21) can be done for  $z$  in the compact set  $[-Z(\beta), Z(\beta)]$  instead of  $\mathbb{R}$ . As the spectral gap  $\xi \mapsto \varepsilon(\xi)$  is lower semi-continuous (see [13]), the map  $z \mapsto \varepsilon(\beta z)$  attains its infimum on  $[-Z(\beta), Z(\beta)]$  so that  $\int_{\Sigma_2} e^{-n\varepsilon(\beta z)} T(n, \beta z)(\theta) d\mathbb{P}(\theta)$  converges uniformly to zero on  $[-Z(\beta), Z(\beta)]$ . This yields that one can use the Laplace method for the convergence in (21), as we now explain.

The Laplace method shows that if  $\varphi : I \rightarrow \mathbb{R}$  is a twice continuously differentiable function, if  $\varphi'$  vanishes on a single point  $\xi$  in the interior of the interval  $I$ , with  $\varphi''(\xi) < 0$ , and if  $f : I \rightarrow \mathbb{R}$  is continuous with  $f(\xi) \neq 0$ , then

$$(25) \quad \int_I e^{n\varphi(y)} f(y) dy \underset{n \rightarrow \infty}{\sim} \frac{\sqrt{2\pi}}{\sqrt{|\varphi''(\xi)|}} e^{n\varphi(\xi)} f(\xi) n^{-1/2},$$

where by  $u_n \underset{n \rightarrow \infty}{\sim} v_n$  we mean that  $u_n = v_n(1 + \epsilon(n))$  with  $\lim_{n \rightarrow +\infty} \epsilon(n) = 0$ . We refer to [11] for a report about the Laplace method, and for a generalization to the case where the least integer  $k$  such that  $\varphi^{(k)}(\xi) \neq 0$  is greater than two. Of course when  $\varphi$  has a finite number of maxima we may break up the integral into a finite

number of integrals so that in each integral  $\varphi$  reaches its maximum at only one interior point.

In our case we claim that the function  $\varphi_{\text{OS}} : z \mapsto -\frac{\beta}{2}z^2 + \log \lambda_{\beta z}$  admits only finitely many maxima. Indeed, for  $z \notin [-Z(\beta), Z(\beta)]$ ,  $\varphi(z) < -\frac{\beta}{4}z^2 < 0$  (this is a consequence of (22) and (23)) and  $\varphi_{\text{OS}}(0) = \log 2 > 0$ . Therefore, the maxima for  $\varphi_{\text{OS}}$  must be in the compact interval  $[-Z(\beta), Z(\beta)]$ . If there are infinitely many, there must be some accumulation point. As  $\varphi_{\text{OS}}$  is analytic in some complex neighborhood of  $[-Z(\beta), Z(\beta)]$ , it must be equal to the constant function, which is clearly not the case. Let  $t_1, \dots, t_J$  the points where  $\varphi_{\text{OS}}$  attains its maximum. We write the integral (21) over the segment  $[-Z(\beta), Z(\beta)]$  as a finite sum of integrals over segments  $[a_j, b_j]$  where each segment  $[a_j, b_j]$  contains exactly one of the points  $t_j$ ,  $1 \leq j \leq J$ .

We state the following lemma, which is an immediate adaptation of the Laplace method.

**Lemma 2.1.** *Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  a function of class  $C^2$ , with  $\varphi'$  vanishing on a single point  $c$  in  $]a, b[$  and  $\varphi''(c) < 0$ . Let  $(f_n)_{n \geq 1}$ ,  $f$  some continuous functions from  $[a, b]$  to  $\mathbb{R}$  such that  $f_n$  converges to  $f$  uniformly on  $[a, b]$ , and  $f(c) \neq 0$ . Then as  $n \rightarrow \infty$*

$$\int_a^b e^{n\varphi(x)} f_n(x) dx \sim \sqrt{\frac{\pi}{2|\varphi''(c)|}} e^{n\varphi(c)} f(c) n^{-1/2}.$$

We apply this lemma on every  $[a_j, b_j]$  to the functions  $f_n$  defined by

$$f_n(z) = \int_{\Sigma_2} \nu_{\beta z}([\omega]) H_{\beta z}(\theta) + e^{-n\varepsilon(\beta z)} T(n, \beta z)(\theta) d\mathbb{P}(\theta).$$

The functions  $f_n$  converge uniformly on  $[a_j, b_j]$  to  $f$  defined by

$$f(z) = \left( \int_{\Sigma_2} H_{\beta z}(\theta) d\mathbb{P}(\theta) \right) \nu_{\beta z}([\omega]).$$

Putting together (24) and the result of Lemma 2.1 applied to every  $[a_j, b_j]$ , assuming for the moment that  $\varphi''_{\text{OS}}(t_j) < 0$  for every  $j = 1, \dots, J$ , we obtain that  $N_{n,\beta}$  is equivalent when  $n$  goes to infinity to

$$\sqrt{\frac{\pi}{2n}} e^{n\varphi_{\text{OS}}(t_1)} \sum_{j=1}^J \frac{\left( \int_{\Sigma_2} H_{\beta t_j}(\theta) d\mathbb{P}(\theta) \right) \nu_{\beta t_j}([\omega])}{\sqrt{|\varphi''_{\text{OS}}(t_j)|}}$$

and  $D_{n,\beta}$  is equivalent to

$$\sqrt{\frac{\pi}{2n}} e^{n\varphi_{\text{OS}}(t_1)} \sum_{j=1}^J \frac{\left( \int_{\Sigma_2} H_{\beta t_j}(\theta) d\mathbb{P}(\theta) \right)}{\sqrt{|\varphi''_{\text{OS}}(t_j)|}}.$$

Recalling (20) we get that  $\mu_{n,\beta}([\omega])$  converges to

$$(26) \quad \sum_{j=1}^J c_j \nu_{\beta t_j}([\omega]),$$

where

$$(27) \quad c_j := \frac{\int_{\Sigma_2} H_{\beta t_j} d\mathbb{P}}{\sqrt{|\varphi''_{\text{OS}}(t_j)|}} \cdot \frac{\sum_{i=1}^J \int_{\Sigma_2} H_{\beta t_i} d\mathbb{P}}{\sum_{i=1}^J \sqrt{|\varphi''_{\text{OS}}(t_i)|}}.$$

If  $\varphi''_{\text{OS}}(t_i) = 0$  then the contribution of the integral over  $[a_i, b_i]$  is of order  $e^{n\varphi_{\text{OS}}(t_i)} n^{-1/k_i}$  where  $k_i$  is the least integer such that  $\varphi_{\text{OS}}^{(k_i)}(t_i) < 0$ . Note that all  $\varphi_{\text{OS}}(t_i)$  are equal and the  $k_i$ 's are all even numbers because  $\varphi_{\text{OS}}$  reaches its maximum at each  $t_i$ .

Let  $K := \max k_i$  and let  $\mathcal{I}$  be the set of indexes  $i$ 's such that  $k_i = K$ . Then we still get the convergence of  $\mu_{n,\beta}([\omega])$  to a convex combination (26) of measures  $\nu_{\beta t_j}$ 's, but with  $c_j = 0$  whenever  $j \notin \mathcal{I}$  and

$$(28) \quad c_j := \frac{\int_{\Sigma_2} H_{\beta t_j} d\mathbb{P}}{|\varphi_{\text{OS}}^{(K)}(t_j)|^{1/K}} \cdot \frac{\sum_{i \in \mathcal{I}} \int_{\Sigma_2} H_{\beta t_i} d\mathbb{P}}{\sum_{i \in \mathcal{I}} |\varphi_{\text{OS}}^{(K)}(t_i)|^{1/K}}$$

for  $j \in \mathcal{I}$ . This finishes the proof of part (4) of Theorem 2.

**2.2. Measures maximizing the quadratic pressure.** We want to determine the invariant measures  $m$  which maximize

$$h_m + \frac{\beta}{2} \left( \int_{\Sigma_2} \psi dm \right)^2.$$

We set  $\bar{A}$  (resp.  $\underline{A}$ ) for  $\max_{m \text{ } \sigma\text{-inv}} \int_{\Sigma_2} \psi dm$  (resp.  $\min_{m \text{ } \sigma\text{-inv}} \int_{\Sigma_2} \psi dm$ ). For  $z \in \mathbb{R}$ , we set

$$\bar{H}(z) := \begin{cases} \max_{m \text{ } \sigma\text{-inv}} \left\{ h_m, \int_{\Sigma_2} \psi dm = z \right\} & \text{if } z \in [\underline{A}, \bar{A}], \\ -\infty & \text{if not.} \end{cases}$$

We point out the equality

$$\mathcal{P}_2(\beta\psi) := \max_m \left\{ h_m + \frac{\beta}{2} \left( \int_{\Sigma_2} \psi \, dm \right)^2 \right\} = \max_{z \in [\underline{A}, \bar{A}]} \left\{ \overline{H}(z) + \frac{\beta}{2} z^2 \right\}.$$

Let us set  $\overline{\varphi}(z) := \overline{H}(z) + \frac{\beta}{2} z^2$ . We claim that the maxima of  $\varphi_{\text{OS}}$  and  $\overline{\varphi}$  are the same. First we observe that

$$\begin{aligned} \mathcal{P}(t\psi) &:= \max_m \left\{ h_m + t \int_{\Sigma_2} \psi \, dm \right\} = \max_{z \in \mathbb{R}} \{ \overline{H}(z) + tz \} \\ &= \max_{z \in \mathbb{R}} \{ tz - (-\overline{H}(z)) \}. \end{aligned}$$

As the function  $-\overline{H}$  is convex lower semi-continuous, we deduce from the duality property of the Fenchel-Legendre transform (see for instance [6], Lemma 4.5.8) that

$$(29) \quad \overline{H}(z) = \inf_{t \in \mathbb{R}} \{ \mathcal{P}(t\psi) - tz \}.$$

**Lemma 2.2.** *For every  $z$  in  $[\underline{A}, \bar{A}]$ ,  $\overline{\varphi}(z) \leq \varphi_{\text{OS}}(z)$ .*

*Proof.* Let  $t = \beta z$ . Using (29) we get

$$\overline{\varphi}(z) \leq \mathcal{P}(t\psi) - tz + \frac{\beta}{2} z^2 = \mathcal{P}(\beta z\psi) - \frac{\beta}{2} z^2 = \varphi_{\text{OS}}(z). \quad \square$$

**Lemma 2.3.**  *$\overline{\varphi}(z)$  is maximal if and only if  $\varphi_{\text{OS}}(z)$  is maximal. In that case,  $\overline{\varphi}(z) = \varphi_{\text{OS}}(z)$ .*

*Proof.* Let  $z$  be a maximum for  $\varphi_{\text{OS}}$ . Then, it is a critical point for  $\varphi_{\text{OS}}$ . This yields

$$\beta \mathcal{P}'(\beta z) = \beta z.$$

In other words,  $\int_{\Sigma_2} \psi \, d\tilde{\mu}_{\beta z} = z$  because  $\mathcal{P}'(t\psi) = \int_{\Sigma_2} \psi \, d\tilde{\mu}_t$  holds for every  $t$ . Then,

$$\begin{aligned} \overline{\varphi}(z) &\geq h_{\tilde{\mu}_{\beta z}} + \frac{\beta}{2} z^2 = h_{\tilde{\mu}_{\beta z}} + \beta z^2 - \frac{\beta}{2} z^2 \\ &= h_{\tilde{\mu}_{\beta z}} + \beta z \int_{\Sigma_2} \psi \, d\tilde{\mu}_{\beta z} - \frac{\beta}{2} z^2 \\ &= \mathcal{P}(\beta z\psi) - \frac{\beta}{2} z^2 = \varphi_{\text{OS}}(z) \geq \overline{\varphi}(z). \end{aligned}$$

This means that  $\overline{\varphi}(z) = \varphi_{\text{OS}}(z)$  holds. On the other hand for any  $z'$ ,

$$\overline{\varphi}(z') \leq \varphi_{\text{OS}}(z') \leq \varphi_{\text{OS}}(z) = \overline{\varphi}(z),$$

which shows that  $z$  is also a maximum for  $\overline{\varphi}$ .

Conversely, if  $z$  is a maximum for  $\bar{\varphi}$ , let  $z'$  be any maximum for  $\varphi_{\text{OS}}$ . We get

$$\bar{\varphi}(z) \geq \bar{\varphi}(z') = \varphi_{\text{OS}}(z') \geq \varphi_{\text{OS}}(z) \geq \bar{\varphi}(z).$$

This shows that  $z$  is also a maximum for  $\varphi_{\text{OS}}$ . □

Now we are ready to finish the proof of Theorem 2. Indeed let  $m$  maximizing

$$h_m + \frac{\beta}{2} \left( \int_{\Sigma_2} \psi \, dm \right)^2.$$

Then  $z := \int_{\Sigma_2} \psi \, dm$  is a maximum for  $\bar{\varphi}$ , hence according to Lemma 2.3  $z$  is a maximum for  $\varphi_{\text{OS}}$  with  $\varphi_{\text{OS}}(z) = \bar{\varphi}(z)$ . Therefore there exists  $i \in \llbracket 1, J \rrbracket$  such that  $z = t_i$ , and

$$h_m + \frac{\beta}{2} t_i^2 = \mathcal{P}(\beta t_i \psi) - \frac{\beta}{2} t_i^2.$$

We deduce that

$$h_m + \beta t_i^2 = \mathcal{P}(\beta t_i \psi) = h_m + \beta t_i \int_{\Sigma_2} \psi \, dm,$$

which implies that  $m = \tilde{\mu}_{\beta t_i \psi}$ . It remains to prove that each  $\tilde{\mu}_{\beta t_i \psi}$  does maximize

$$h_m + \frac{\beta}{2} \left( \int_{\Sigma_2} \psi \, dm \right)^2.$$

But this is immediate since

$$\mathcal{P}'(\beta t_i) = \int_{\Sigma_2} \psi \, d\tilde{\mu}_{\beta t_i \psi} = t_i$$

and  $t_i$  is a maximum for  $\bar{\varphi}$ .

### 3. PROOF OF THEOREM 3

In a first step we use an auxiliary function  $\varphi_P$ . Note that this function was already studied by Ellis and Wang in [10]. Then we deduce that  $\mu_{n,\beta}(C)$  converge for any cylinder  $C$ . In a second step we identify the limit as the relevant convex combination of dynamical measures.

**3.1. Auxiliary function  $\varphi_P$  and convergence for  $\mu_{n,\beta}$ .** We shall need the function  $\varphi_P$  defined on  $\mathbb{R}^q$  by

$$(30) \quad \varphi_P(z) = -\frac{\beta}{2} \|z\|^2 + \log \sum_{k=1}^q e^{\beta z_k}.$$

This function attains its maximum on  $\mathbb{R}^q$  since  $\varphi_P(z) \leq -c\|z\|^2$  as  $\|z\|$  tends to  $\infty$ . We recall Theorem 2.1 of [10], which describes precisely the global maximum points of  $\varphi_P$ .



**Theorem 4.** (Ellis Wang [10])

Let  $\beta_c = \frac{2(q-1)\log(q-1)}{q-2}$ . For  $0 < \beta < \beta_c$  set  $s_\beta = 0$  and for  $\beta \geq \beta_c$  let  $s_\beta$  be the largest solution of the equation

$$(31) \quad s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}.$$

The function  $\beta \mapsto s_\beta$  is strictly increasing on the interval  $[\beta_c, +\infty[$ ,  $s(\beta_c) = \frac{q-2}{q-1}$ , and  $\lim_{\beta \rightarrow \infty} s_\beta = 1$ .

Denote by  $\phi$  the function from  $[0, 1]$  into  $\mathbb{R}^q$  defined by

$$\phi(s) = \left( \frac{1 + (q-1)s}{q}, \frac{1-s}{q}, \dots, \frac{1-s}{q} \right),$$

the last  $(q-1)$  components all equal  $\frac{1-s}{q}$ . Let  $K_\beta$  denote the set of global maximum points of the symmetric function  $\varphi_P$ . Define  $\nu^0 = \phi(0) = \left(\frac{1}{q}, \dots, \frac{1}{q}\right)$ . For  $\beta \geq \beta_c$ , define  $\nu^1(\beta) = \phi(s_\beta)$  and let  $\nu^i(\beta)$ ,  $i = 2, \dots, q$  denote the points in  $\mathbb{R}^q$  obtained by interchanging the first and  $i$ th coordinates of  $\nu^1(\beta)$ . Then

$$K_\beta = \begin{cases} \{\nu^0\} & \text{for } 0 < \beta < \beta_c, \\ \{\nu^1(\beta), \nu^2(\beta), \dots, \nu^q(\beta)\} & \text{for } \beta > \beta_c, \\ \{\nu^0, \nu^1(\beta_c), \nu^2(\beta_c), \dots, \nu^q(\beta_c)\} & \text{for } \beta = \beta_c. \end{cases}$$

For  $\beta \geq \beta_c$  the points in  $K_\beta$  are all distinct.

We fix a finite word  $\omega = \omega_0 \cdots \omega_{p-1}$  of length  $p$  and we compute the limit of  $\mu_{n,\beta}([\omega])$ .

**Lemma 3.1.**

$$\lim_{n \rightarrow \infty} \mu_{n,\beta}([\omega]) = \begin{cases} \frac{1}{q^p} & \text{if } \beta < \beta_c, \\ \frac{1}{q} \frac{1}{(e^{\beta s_\beta} + q - 1)^p} \sum_{k=1}^q e^{\beta s_\beta L_{p,k}(\omega)} & \text{if } \beta > \beta_c, \\ \frac{\frac{A}{q^p} + \frac{B}{(e^{\beta s_\beta} + q - 1)^p} \sum_{k=1}^q e^{\beta s_\beta L_{p,k}(\omega)}}{A + qB} & \text{if } \beta = \beta_c. \end{cases}$$

*Proof.* We want to evaluate the limit of

$$\mu_{n,\beta}([\omega]) = \sum_{\alpha, |\alpha|=n-p} \mu_{n,\beta}([\omega\alpha]) = \frac{\sum_{\alpha, |\alpha|=n-p} e^{\frac{\beta}{2n} \|L_n(\omega\alpha)\|^2}}{\sum_{\alpha, |\alpha|=n} e^{\frac{\beta}{2n} \|L_n(\alpha)\|^2}}.$$

With the help of the identity

$$(32) \quad e^{\|u\|^2} = \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{R}^q} \exp\left(-\frac{1}{2}\|y\|^2 + \sqrt{2}\langle y, u \rangle\right) dy,$$

and noticing that  $L_n(\omega\alpha) = L_p(\omega) + L_{n-p}(\alpha)$ , we write

$$\begin{aligned} \sum_{\alpha, |\alpha|=n-p} e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2} &= \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{R}^q} e^{-\frac{1}{2}\|y\|^2} \sum_{\alpha} e^{\sqrt{\frac{\beta}{n}}\langle y, L_n(\omega\alpha) \rangle} dy \\ &= \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{R}^q} e^{-\frac{1}{2}\|y\|^2 + \sqrt{\frac{\beta}{n}}\langle y, L_p(\omega) \rangle} \sum_{\alpha} e^{\sqrt{\frac{\beta}{n}}\langle y, L_{n-p}(\alpha) \rangle} dy. \end{aligned}$$

It is easily seen that

$$\sum_{\alpha, |\alpha|=n-p} e^{\sqrt{\frac{\beta}{n}}\langle y, L_{n-p}(\alpha) \rangle} = \left( \sum_{k=1}^q e^{\sqrt{\frac{\beta}{n}}y_k} \right)^{n-p},$$

therefore we get

$$\begin{aligned} \sum_{\alpha, |\alpha|=n-p} e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2} &= \\ &= \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{R}^q} \exp\left(-\frac{1}{2}\|y\|^2 + \sqrt{\frac{\beta}{n}}\langle y, L_p(\omega) \rangle + (n-p) \log\left(\sum_{k=1}^q e^{\sqrt{\frac{\beta}{n}}y_k}\right)\right) dy. \end{aligned}$$

Now we make the change of variable  $\beta z = \sqrt{\frac{\beta}{n}}y$ , and we obtain

$$(33) \quad \sum_{\alpha, |\alpha|=n-p} e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2} = \left(\frac{n\beta}{2\pi}\right)^{q/2} \int_{\mathbb{R}^q} e^{n\varphi_P(z)} f(z) dz,$$

where  $\varphi_P$  was defined in (30) and  $f$  is defined on  $\mathbb{R}^q$  by

$$(34) \quad f(z) = \exp\left(\beta\langle z, L_p(\omega) \rangle - p \log\left(\sum_{k=1}^q e^{\beta z_k}\right)\right).$$

Similarly,  $p = 0$  yields

$$\sum_{\alpha, |\alpha|=n} e^{\frac{\beta}{2n}\|L_n(\alpha)\|^2} = \left(\frac{n\beta}{2\pi}\right)^{q/2} \int_{\mathbb{R}^q} e^{n\varphi_P(z)} dz,$$

hence

$$\mu_{n,\beta}([\omega]) = \frac{\sum_{\alpha, |\alpha|=n-p} e^{\frac{\beta}{2n}\|L_n(\omega\alpha)\|^2}}{\sum_{\alpha, |\alpha|=n} e^{\frac{\beta}{2n}\|L_n(\alpha)\|^2}} = \frac{\int_{\mathbb{R}^q} e^{n\varphi_P(z)} f(z) dz}{\int_{\mathbb{R}^q} e^{n\varphi_P(z)} dz}.$$

We denote by  $D\varphi_P(z)$ , respectively  $\mathcal{H}(z)$ , the gradient, respectively the Hessian matrix, of  $\varphi_P$  at  $z$ . It is proved in Proposition 2.2 of [10] that the Hessian matrix of  $\varphi_P$  is negative definite at each global maximum point of  $\varphi_P$ . Now if  $D\varphi_P$  vanishes at

a single point  $z_0$  in an open set  $O$  of  $\mathbb{R}^q$ , if  $\mathcal{H}(z_0)$  is negative definite and if  $f(z_0) \neq 0$ , then we know by Laplace's method that

$$\int_0 e^{n\varphi_P(z)} f(z) dz \sim_{n \rightarrow \infty} \frac{(2\pi)^{q/2} f(z_0) e^{n\varphi_P(z_0)}}{n^{q/2} \sqrt{|\det \mathcal{H}(z_0)|}}.$$

**If  $0 < \beta < \beta_c$  :** according to Theorem 4,  $\varphi_P$  attains its maximum at the unique point  $\nu^0$  so applying Laplace's method yields

$$\mu_{n,\beta}([\omega]) \sim_{n \rightarrow \infty} \frac{f(\nu^0)}{1} = \frac{1}{q^p}.$$

**If  $\beta > \beta_c$  :** Theorem 4 states that  $\varphi_P$  attains its maximum at exactly  $q$  points  $\nu^i(\beta)$ ,  $i = 1, \dots, q$ , where  $\nu^i(\beta)$ ,  $i = 2, \dots, q$  is obtained by interchanging the first and  $i$ th coordinates of  $\nu^1(\beta)$ . Due to the symmetry of the function  $\varphi_P$  it is clear that  $\det \mathcal{H}(\nu^i) = \det \mathcal{H}(\nu^1)$ ,  $i = 2, \dots, q$ . Considering a family of disjoint open sets  $(O_i)_{1 \leq i \leq q}$  such that  $O_i$  contains  $\nu^i$  and  $\mathbb{R}^q = \cup_{i=1}^q O_i \cup N$ , where  $N$  is a set of measure zero, Laplace's method yields

$$\mu_{n,\beta}([\omega]) \sim_{n \rightarrow \infty} \frac{1}{q} \sum_{i=1}^q f(\nu^i).$$

Recall that

$$f(\nu^i) = \frac{e^{\beta \langle \nu^i, L_p(\omega) \rangle}}{(\sum_{k=1}^q e^{\beta \nu_k^i})^p}$$

with

$$\nu_k^i = \begin{cases} \frac{1 - s_\beta}{q} & \text{if } k \neq i, \\ \frac{1 + (q-1)s_\beta}{q} & \text{if } k = i. \end{cases}$$

As  $\sum_{k=1}^q L_{p,k}(\omega) = p$  it is easily seen that

$$(35) \quad e^{\beta \langle \nu^i, L_p(\omega) \rangle} = \exp \left( \frac{\beta p (1 - s_\beta)}{q} + \beta s_\beta L_{p,i}(\omega) \right).$$

As  $\nu^i$  is a critical point of  $\varphi_P$  and  $\frac{\partial \varphi_P}{\partial z_i}(z) = \frac{\beta e^{\beta z_i}}{\sum_{k=1}^q e^{\beta z_k}} - \beta z_i$ , we know that

$$(36) \quad \sum_{k=1}^q e^{\beta \nu_k^i} = \frac{e^{\beta \nu_j^i}}{\nu_j^i} = \frac{q}{1 - s_\beta} e^{\frac{\beta(1-s_\beta)}{q}}.$$

Putting together (35) and (36) we obtain

$$f(\nu^i) = \left( \frac{1 - s_\beta}{q} \right)^p e^{\beta s_\beta L_{p,i}(\omega)},$$

which can also be written

$$(37) \quad f(\nu^i) = \frac{1}{(e^{\beta s_\beta} + q - 1)^p} e^{\beta s_\beta L_{p,i}(\omega)}$$

since  $s_\beta$  is solution of the equation (13). Therefore

$$\mu_{n,\beta}([\omega]) \sim_{n \rightarrow \infty} \frac{1}{q} \frac{1}{(e^{\beta s_\beta} + q - 1)^p} \sum_{i=1}^q e^{\beta s_\beta L_{p,i}(\omega)}.$$

**If  $\beta = \beta_c$  :** the function  $\varphi_P$  admits exactly  $q + 1$  maximum points  $\nu^i(\beta)$ ,  $i = 0, \dots, q$  but  $\det \mathcal{H}(\nu^0) \neq \det \mathcal{H}(\nu^1)$ , therefore Laplace's method yields

$$(38) \quad \mu_{n,\beta}([\omega]) \sim_{n \rightarrow \infty} \frac{|\det \mathcal{H}(\nu^0)|^{-1/2} f(\nu^0) + |\det \mathcal{H}(\nu^1)|^{-1/2} \sum_{i=1}^q f(\nu^i)}{|\det \mathcal{H}(\nu^0)|^{-1/2} + q |\det \mathcal{H}(\nu^1)|^{-1/2}}.$$

In the proof of Proposition 2.2 of [10] it is proved that  $\mathcal{H}(\nu^0)$  has a simple eigenvalue at  $\beta$  and an eigenvalue of multiplicity  $(q-1)$  at  $\beta q^{-1}(q-\beta)$  whereas  $\mathcal{H}(\nu^1)$  has simple eigenvalues at  $\beta$  and  $\beta - \beta^2 q a b$  and an eigenvalue of multiplicity  $(q-2)$  at  $\beta - \beta^2 b$ , where  $a = q^{-1}(1 + (q-1)s_\beta)$  and  $b = q^{-1}(1 - s_\beta)$ . Recalling that  $s(\beta_c) = \frac{q-2}{q-1}$  we deduce that

$$|\det \mathcal{H}(\nu^0)| = \beta_c^q (1 - q^{-1} \beta_c)^{q-1},$$

$$|\det \mathcal{H}(\nu^1)| = \beta_c^q (1 - q^{-1} \beta_c) \left(1 - \frac{\beta_c}{q(q-1)}\right)^{q-2}.$$

Reporting in (38) and recalling (37) we get the result. □

**3.2. Identification of the limit.** We can already deduce from Lemma 3.1 that  $\mu_{n,\beta} \xrightarrow[n \rightarrow +\infty]{w} \tilde{\mu}_0$  if  $\beta < \beta_c$ .

**Lemma 3.2. Computation for  $\tilde{\mu}_b^k$**

For  $k = 1, \dots, q$ ,

$$(39) \quad \tilde{\mu}_b^k([\omega]) = \frac{e^{b L_{p,k}(\omega)}}{(e^b + q - 1)^p}.$$

*Proof.* The function  $b \mathbb{1}_{[\theta^k]}$  depends only on the zero coordinate, therefore the supremum in (1) is attained for the product measure  $(m^k)^{\otimes \mathbb{N}}$ , where the probability vector  $(m_j^k)_{1 \leq j \leq q}$  on  $\Lambda$  maximizes the quantity

$$- \sum_{j=1}^q p_j \log p_j + b p_k$$

over all the probability vectors  $(p_j)_{1 \leq j \leq q}$  on  $\Lambda$ , and is given by  $m_k^k = \frac{e^b}{e^b + q - 1}$ ,  $m_j^k = \frac{1}{e^b + q - 1}$  if  $j \neq k$  (see for instance Example 4.2.2 of [15]). The result is then clear. □

The limit in (14) is now a direct consequence of the lemmas 3.1 and 3.2.

## REFERENCES

- [1] N. Bedaride, P. Hubert, and R. Leplaideur. Thermodynamic formalism and Substitutions. *ArXiv e-prints*, November 2015.
- [2] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics, Vol. 470. Springer-Verlag, Berlin, 1975. 2nd ed. - 2008 by JR Chazottes.
- [3] H. Bruin and R. Leplaideur. Renormalization, freezing phase transitions and Fibonacci quasicrystals. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(3):739–763, 2015.
- [4] D. Coronel and J. Rivera-Letelier. Low-temperature phase transitions in the quadratic family. *Adv. Math.*, 248:453–494, 2013.
- [5] Marius Costeniuc, Richard S. Ellis, and Hugo Touchette. Complete analysis of phase transitions and ensemble equivalence for the Curie-Weiss-Potts model. *J. Math. Phys.*, 46(6):063301, 25, 2005.
- [6] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*. Berlin: Springer, 2nd edition, 2010.
- [7] Richard S. Ellis. *Entropy, large deviations, and statistical mechanics*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1985 original.
- [8] Richard S. Ellis and Charles M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete*, 44(2):117–139, 1978.
- [9] Richard S. Ellis and Charles M. Newman. The statistics of Curie-Weiss models. *J. Statist. Phys.*, 19(2):149–161, 1978.
- [10] Richard S. Ellis and Kongming Wang. Limit theorems for the empirical vector of the Curie-Weiss-Potts model. *Stochastic Processes Appl.*, 35(1):59–79, 1990.
- [11] A. Erdélyi. *Asymptotic expansions*. Dover Publications, Inc., New York, 1956.
- [12] Hans-Otto Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2011.
- [13] Hubert Hennion and Loïc Hervé. *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness*, volume 1766 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
- [14] J. Hubbard. Calculation of partition functions. *Phys. Rev. Lett.*, 3:77–78, Jul 1959.
- [15] Gerhard Keller. *Equilibrium states in ergodic theory*. Cambridge: Cambridge University Press, 1998.
- [16] Renaud Leplaideur. Chaos: butterflies also generate phase transitions. *J. Stat. Phys.*, 161(1):151–170, 2015.
- [17] N. Makarov and S. Smirnov. On thermodynamics of rational maps. II. Non-recurrent maps. *J. London Math. Soc. (2)*, 67(2):417–432, 2003.
- [18] Steven Orey. Large deviations for the empirical field of Curie-Weiss models. *Stochastics*, 25(1):3–14, 1988.
- [19] W. Parry and M. Pollicott. *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, volume 187-188 of *Astérisque*. SMF, 1990.
- [20] Firas Rassoul-Agha and Timo Seppäläinen. *A course on large deviations with an introduction to Gibbs measures*, volume 162 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
- [21] D. Ruelle. Statistical mechanics of a one-dimensional lattice gas. *Commun. Math. Phys.*, 9:267–278, 1968.
- [22] D. Ruelle. *Statistical mechanics*. World Scientific Publishing Co. Inc., River Edge, NJ, 1999. Rigorous results, Reprint of the 1989 edition.
- [23] D. Ruelle. *Thermodynamic formalism*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.

- [24] O. Sarig. Continuous phase transitions for dynamical systems. *Comm. Math. Phys.*, 267(3):631–667, 2006.
- [25] Y. Sinai. Gibbs measures in ergodic theory, *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
- [26] Y. Sinai. *Theory of phase transitions: rigorous results*, volume 108 of *International Series in Natural Philosophy*. Pergamon Press, Oxford, 1982. Translated from the Russian by J. Fritz, A. Krámli, P. Major and D. Szász.
- [27] R. L. Stratonovič. A method for the computation of quantum distribution functions. *Dokl. Akad. Nauk SSSR (N.S.)*, 115:1097–1100, 1957.
- [28] Maximilian Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37(4):303–314, 1980.

ISEA, UNIVERSITÉ DE NOUVELLE CALÉDONIE, 145, AVENUE JAMES COOK - BP R4 98 851 - NOUMÉA CEDEX. NOUVELLE CALÉDONIE

LMBA, UMR6205, UNIVERSITÉ DE BREST.

*E-mail address:* [Renaud.Lepplaideur@univ-nc.nc](mailto:Renaud.Lepplaideur@univ-nc.nc), <http://pagesperso.univ-brest.fr/~leplaide/>

LMBA, UMR 6205, UNIVERSITÉ DE BRETAGNE SUD, CAMPUS DE TOHANNIC, BP 573, 56017 VANNES, FRANCE.

*E-mail address:* [frederique.watbled@univ-ubs.fr](mailto:frederique.watbled@univ-ubs.fr)