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# Modeling Longevity Risk using Consistent Dynamics Affine Mortality Models :

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## Abstract

Longevity Risk becomes an important challenge in the recent Year because of the decreases in the mortality rates and the rising in the life expectancy through the decades. In this article, we propose a consistent multi-factor dynamics affine mortality model to the longevity risk modeling, we show that this model is an appropriate model to fit the historical mortality rates. To our Knowledge this is the first work that uses a consistent Mortality models to model USA Longevity risk. Indeed the multiple risk factors permitting applications not only to the hedge and price of the longevity risk but also in mortality derivatives and the general problems in the risk management .A state space presentation is used to estimate the model parameters through the kalman filter .To capture the effect of the size of the population sample we include a measurement error variance for each age. We evaluate 2-and 3-factor implementation of the model through the use of the USA mortality data, we employ Bootstrapping method to derive parameter estimated and the Consistent models prove the performance and the stability of the model. We show that the 3-factor independent model is the best model that can provide a better fit to our survivals curves and especially for the elderly persons.

**JEL classification** : C02; C14; C65; G13.

**Keywords** :Longevity Risk, Mortality model,multi-factor,consistent, Affine,Kalman filter, arbitrage-free, USA mortality..

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# 1 Introduction

Due to the reduction of the mortality rate around the world among the elderly, social security systems and private annuity providers are facing the problem of imbalances between contribution and benefits significantly and partly because of the retirement of baby boomers in the recent years, as result longevity risk become the major problem to many insurance companies and become more important in the presence of interest rate and mortality . The observed constrant improvements in longevity are bringing new issues challenges at various levels : social, political, economics and regulatory ...

Many previous studies discuss longevity risk , to start with Blake and Burrows (2001) who was the first talked about a Mortality-linked securities when they proposed a survivor bond to hedge this risk and evoke the role of the government to issue this kind of bond based on population mortality. Dowd et al (2003) suggested others hedging methods like survivor options, swaps and futures ,and in 2006 the longevity bonds and gave some construction which can used it to hedge longevity risk the same thing to Blake et al (2006).

To manage this risk many previous research find that the most prevalent way is the securitization as result several types of product have been suggested to hedge this risk like longevity bonds survivor swaps and options-type longevity derivatives. Li and COX (2005) suggested a successful securitization of longevity risk through a longevity bond with payments which were based on the loss experience of the insurer to eliminate the others basis risk.

Brown and Orszag (2006) afforded a discussion of the disadvantages and the advantage of the participation of the government in longevity bonds like the ability to spread the longevity risk across the generations. Longevity risk can be managed through the use of hedging in financial markets or reinsurance blake et al (2006a). D. Blake, A. J. G. Cairns and K. Dowd (2006) discuss some possible ways to manage longevity risk for life companies and pension plans by using Mortality-linked securities. By time there have been many others innovation to remove the longevity risk to the capital markets containing mortality-linked securities (MLSs), and many others proposal to deal with this risk containing the development of longevity-linked securities and derivatives taking the example of the EINB BNP longevity bond to transfer this risk , without forgetting the first capital market derivative transaction in the world : the q-forward contract between J.P Morgan and the UK company Lucida which is occurred in January 2008, Coughlan et al (2008), and in july 2008 it was the first capital market longevity swap which gave the possibility to canadians to hedge their UK-based annuity policies and last but not least Swiss Re launched the called 'Kortis notes' which is a group eight-year longevity-based insurance-linked security notes , like Denuit et al (2007) when they propose a survivor Bond-design directly issued by insures suffering from longevity risk , which based on public Mortality index using the classical Lee Carter Model to price a risky coupon survivor Bond. Other studies talked also about longevity bond and analyse the role of longevity risk on the solvency of some portfolio of pension annuities for example Håri et al (2008) who distinguish between two kind of this risk , a micro longevity risk and a macro longevity risk and precise that longevity bond could be a tool to hedge just the second type of risk ,in other way Hua chen et al (2009) propose a new design of longevity bond through the utilization of a random walk model with drift. Nevertheless Denuit (2009) gave a detailed analysis of a longevity index through out the lee carter model to transfer the longevity risk to the market and bring to bear that choosing reinsurance as a cover from longevity risk could be expensive for many life insurance companies.

The increase of life expectancy causes an increase in the duration of pension payments so this

can abuse a burden on pension providers, many previous studies discuss this problem and propose an alternative tool to manage the longevity risk in pension annuities by introducing many product design. Ralph et al (2009) propose in retirement age a defined benefit pension plan which give the choice between two different type of annuities. Horneff et al (2010) propose a deferred annuities as a pure longevity insurance to hedge this risk however Ngai and sherris (2011) examine the capability of static hedging strategies using longevity bonds and many others retail products-derivatives and proved that variable annuities gives a limited protection against longevity risk in comparison with life annuities and indexed annuities. Joao F. Cocco and Francisco J. Gomes (2011) interpret a life cycle model with longevity risk and propose an endogenous savings and retirement decisions as a hedging for this risk also they give an implications for optimal security design.

Pricing was the center of attraction of others studies for example Johnny Siu and Hang Li (2010) discuss an alternative pricing model by using the parametric boot strap, however Chen, Zhang and Zhao (2010) use to price a longevity bond three popular method first neutral method sharp ratio rule and the wang transform, Kogue et al (2010) propose a Bayesian approach to pricing longevity risk through out the use of the Lee carter methodology.

Some studies focus on modelling this kind of risk as Loisel and Daniel Serant (2007) when they introduce and discuss a new model so called stochastic logit's Deltas Model which is based on the multivariation studies, also we find Tzuling lin et al (2009) who gave a variables method to modelling longevity risk and Andrew et al (2011) who suggested a mortality model with an age shift by using a principal component analysis PCA.

Many others study also discuss a stochastic modeling of mortality to solve the problem of measuring longevity risk, as B. Browne et al (2009), who suggests a simple model to move from a deterministic Mortality scenario to a stochastic model, owing to describe the behavior of the longevity enclosing the projected expected values, this model can be used on future mortality rates estimation. Weny Bal et al (2009) also discuss this kind of models by introducing four stochastic mortality models and bring to light the weakness and the strengths of the four models, in addition this research analyze the possibility to manipulate the input data by adjusting the parameters and show that the stochastic modeling is more advantageous than the deterministic one.

Other articles give a comparison discussion between a deterministic shock and a stochastic value at risk in order to propose an analysis of the Solvency II Standard Model Approach to measure longevity Risk Matthias Börger (2010). Sharron S et al (2010) use the Principal Component Approach to modeling longevity risk and compare the PCA model with the existing stochastic Mortality Models, they use this model to examine the ratio of annuity price for both the deferred life annuity and life annuity products and try to forecast the future mortality rates, moreover they prove that this model can resolve the Lee Carter Model's problem by using the PCA approach. More recently, Vesa Rankainen (2012) uses the famous model so called the Lee Carter Model 1992 to develop a stochastic modeling of financing longevity risk in pension insurance by using the Bayesian MCMC methods and prove that the LC model is not completely adaptable with mortality data.

Søren Fiig Jarner and Thomas Møller (2013) Propose currently a simple Partial internal model for longevity risk inside the framework of Solvency II, which have connection with the Danish longevity benchmark mechanisms, this model comprise a component, which is based on the size of a given portfolio, this component can measure for each insurance portfolio the unsystematic longevity risk.

To solve the problem of measuring and modeling the longevity risk many studies use the affine structure with mortality models application like Dahl (2004) who propose for a cohort live an

affine mortality structure with the same ages. Shragger(2006) introduces an affine mortality intensity model specially for Thiele and Makeham mortality laws and examines all ages in same time , but They do not take into consideration the property of consistent in their Models.

In this work we propose a consistent multifactor dynamics affine mortality model for modeling the longevity risk which is based on affine term structure model ATSM discussed by Duffie and Khan (1996) on modeling interest rates. To our Knowledge this is the first work that uses a consistent Mortality models to model USA Longevity risk.

We present a similar way to term structure of interest rates to model the mortality survival curve age. The affine framework is advantageous than the others econometrics models because we have the ability to use a different mathematical tools used for risk management and in pricing credit risk .

We introduce in our article a Parametric survival curves that are consistent guaranteeing that the set of the predicted survival curves hold an invariable parametric form as the initial fitted parametric curve .

To confirm the model's consistency , we establish different conditions for consistency like it was giving by Bjork and Christensen (1999) also we refer to De Rossi (2004) who gives an application to this conditions in interest rates modeling to derive a 2-factor ATSM which is consistent.

To provide a calibration of the Market price of Risk to an admissible an accessible market data , we used an arbitrage-free formulation. Given our Risk neutral process , we use in our work the Gisanov's theorem to move to a real-world measure in order to model the historical mortality rates. As an alternative to presume a constant price of mortality risk for every factor like it was giving in 'completely affine' model by Duffie and Khan (1996) , we propose in our work an 'Essentially affine ' model as it was giving by Duffie (2002). To estimate our model , we use a state space formulation which contain the measurement and the transition equations. Unlike the Econometrics models as the famous Lee and Carter model (1992), our transition equations which are the time-series dynamics are presented in the model like a latent stochastic factor , and the measurement equation represents the observed survivor curve ,and it is exponentially affine in the factors .Introducing our ATSM model in a state space formulation in this paper,let us estimate the model parameters using the Kalman filter , Kalman (1960) , we include a specific allowance for the age dependent measurement errors , we suppose that the measurement errors are independent between ages to estimates the errors .

We use a USA data to calibrate and evaluate the model fit , we use a state-space re-sampling method to realize a bootstrapped confidence intervals for every paramters estimated.

We present different model assumptions ,these contain 2-and 3-factor independent models. We consider an arbitrage-free type of Nelson Siegel model as presented by Christensen et al (2009),and we compare it with the 3-factor independent model.

This article is organized as follow : in Section 2 we outline the general meaning of the longevity risk and its characteristics through the way how to modelling and to transferring the Risk ,in section 3 we present the life insurance with its different products .Section 4 discuss the the ATSM methodology with the derivation of the 2-and 3-factor model, in Section 5 we introduced the empirical study with the data presentation using the mortality survival curve ,the model calibration using the kalman filter and the bootstrap state space model with the differents estimations results , the sample analysis ,the model robustness and finally we find the conclusion.

## 2 A presentation of Affine Mortality Model:

In this framework we try to use some techniques taked from the credit Risk securities in Mortality modelling.

To start with the probability space  $(\Omega, F, F, P)$  where  $F = G \vee H$  represents the information at time  $t$ , and we noted that  $G_t$  which is the sub-filtration includes all the inforamtion (financial and actuarial) other than the current time of death , the other subfiltration  $H_t$  is the  $\sigma$ -algebra with death information , and the Real world probability space is given by  $P$  .

let's take  $N(t) = 1_{\tau \leq t}$ , if we assume that  $A(t)$  is an anticipated process of  $N(t)$  where:  $A(t) = \int_0^t \mu(s; x) ds$  and:  $dA(t) = \mu(t; x) dt$  so we have:  $dM(t) = dM(t) - dA(t)$   $M(t)$  represents a  $P$ -martingale .

Taking another measure where we considerate  $dM(t)$  as a  $Q$ -martingale process , under which  $dA(t)$  becomes:  $dA(t) = \mu^Q(t; x) dt$  and we can note:  $\mu^Q(t; x) = (1 + \Phi(t))\mu(t; x)$  where  $\Phi(t) \geq -1$  .

We adjust the value of  $\Phi(t)$  at zero  $\Phi(t) = 0$  and we noted that the unsystematic Risk is not priced, this suppose thar we can note  $\mu^Q(t; x) = \mu(t; x)$ .

Under the absence of arbitrage assumption, there exists  $Q$  which is an equivalent measure where  $P(t; T; x)$  is the time  $t$ -value of  $x_t$  (the final payment).  $x_t$  pretend to be a  $G$  adapted process and to the end of time  $t$ ,  $x_t$  is conditional on survival, if not the value is zero .So we can note the time  $t$ -value after the use of the law of 'iterated expectations' as showing by Bielecki and Rutkowski (2002) , as follow :

$$\begin{aligned} P(t; T; x) &= E^Q \left[ e^{-\int_t^T r(s) ds} X_T 1_{\tau > t} | F_t \right] \\ &= 1_{\tau > t} E^Q \left[ e^{-\int_t^T [r(s) + \mu(s; x)] ds} X_T | g_t \right] \end{aligned} \quad (1)$$

by adding the independance assumption between mortality rates  $\mu(x; t)$  and the interest rates  $r(s)$  and suppose that  $X_t = 1$  , we can express the time  $t$ -value as :

$$\begin{aligned} P(t; T; x) &= 1_{\tau > t} E^Q \left[ e^{-\int_t^T r(s) ds} | g_t \right] E^Q \left[ e^{-\int_t^T \mu(s; x) ds} | g_t \right] \\ &= 1_{\tau > t} B(t, T) S(t; T; x) \end{aligned} \quad (2)$$

Where  $S(t; T; x)$  is the survival probabily at time 0 for a person aged  $X$  and  $B(t, T)$  is the time- $t$  bond price , this description is adaptable to forward rate modelling presentedd in the work of Health-Jarrow -Morton (HJM) Health et al (1992).

Because of dealing with the problem of exogeneity of different cohorts and cohorts forecasts, the calibration of affine Models becomes difficult, but Chiarella and Kwon (2001), prove in their farmework that ATSM is a special case of HJM with showing a determenistic volatility function , so we apply this characteristic to solve the problem of calibration using our population data.

We assume in our framework that the parameters of Risk neutral are not varying so we apply the HJM Model to give a better estimation to the cohorts survival curves.

By keeping the filtres space of probability  $(\Omega, F, F, P)$  we consider  $\mu(t)$  , the  $G$ -adapted process, as an instantaneous measure rate , so under an equivalent martingal measure  $Q$  , we can express the survival probability as it was expressed by Black burn and Sherris (2013) by :

$$S(t, T) = E^Q \left[ e^{-\int_t^T \mu(s) ds} | g_t \right] \quad (3)$$

The solution of this equation is given by :

$$S(t, T) = e^{-B(t, T)' Z_t + C(t, T)} \quad (4)$$

Where  $B(t, T)$  and  $C(t, T)$  are the solution of the ordinary differential equation existing in our given short rate mortality process as it was explained by Duffi and Kan (1996).

$$\frac{dB}{dt} = \mathbf{1} - \Delta^Q B(t, T), \quad B(0) = 0 \quad (5)$$

$$\frac{dC}{dt} = -\frac{1}{2} \sum_{j=1}^n \Sigma' B(t, T) B'(t, T) \Sigma, \quad C(0) = 0 \quad (6)$$

So  $\mu(t)$  can be expressed as the sum of latent factors :

$$\mu(t) = \mathbf{1}' Z$$

Given our n-factor model when  $B(T, t)'$  represents the n-factor loadings matrix transpose,  $C(t, T)$  a constant and the latent factors  $Z_t$ . We can show that the curve of the average force of mortality is affine in the factors and it can be presented by :

$$\bar{\mu}(t, T) = -\frac{1}{T-t} \log[S(t, T)] = \frac{B(t, T)' Z_t - C(t, T)}{T-t} \quad (7)$$

Under the risk-neutral measure, the n-latent factors can be defined as a gaussian process by :

$$dZ_t = -\Delta^Q Z_t dt + \Sigma dW_t^Q$$

Where  $\Sigma \in R^{n \times n}$ ,  $\Delta^Q \in R^{n \times n}$  without a condition about  $\Delta^Q$  sign, the process of the risk-neutral factors can be: a non-meaning reverting, a random walk or a mean-reverting to zero.  $\Sigma$ , which is the covariance matrix is diagonal and  $W_t$  are an independant Brownian motions.

### 3 Presentation of longevity risk dynamics:

The aim of this paper is to represent a model that is able to fit the survival curve. We have different structures for the dynamics of the latent risk factors which is able to driving our presented survival curve.

We introduce a 2-and 3-factor model dynamics with independence between factors, and the popular 3 factor version of the Nelson Siegel model which is none as a non-consistent interest rate mode to give as a contrast with our consistent model.

#### 3.1 2-Factor independent model :

we represent here a model of the Brownian motions with no inter action and no factor dependence, the dynamics of this model is given as bellow :

$$\begin{pmatrix} dZ_{1,t} \\ dZ_{2,t} \end{pmatrix} = -\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} dW_{1,t}^Q \\ dW_{2,t}^Q \end{pmatrix} \quad (8)$$

So the instantaneous force of mortality can be written as :

$$\mu_{x,t} = Z_{1,t} + Z_{2,t} \quad (9)$$

The solutions for the factor loadings are given by Dufie and Kan(1996), as we have noted that :

$$\frac{dB(t, T)}{dt} = \mathbf{1} - \Delta^Q B(t, T) \quad B(0) = 0$$

to solve the differential equation for  $B(t, T)$  we denote :

$$\begin{aligned} \frac{d}{dt} [e^{\Delta^Q(T-t)} B(t, T)] &= e^{\Delta^Q(T-t)} \frac{dB(t, T)}{dt} - \Delta^Q e^{\Delta^Q(T-t)} B(t, T) \\ &= e^{\Delta^Q(T-t)} \frac{dB(t, T)}{dt} - \left( \frac{dB(t, T)}{dt} - 1 \right) e^{\Delta^Q(T-t)} \text{ We integrate the equation from the both side :} \\ &= e^{\Delta^Q(T-t)} \end{aligned}$$

$$\int_t^T \frac{d}{ds} [e^{\Delta^Q(T-s)} B(t, T)] = \int_t^T e^{\Delta^Q(T-t)} ds$$

by the use of the boundary conditions we find that:

$$B(t, T) = -e^{-\Delta^Q(T-t)} \int_t^T e^{\Delta^Q(T-s)} ds$$

The matrix exponentials in this equation for our model assumption is given by :

$$\begin{aligned} e^{\Delta^Q(T-t)} &= \begin{pmatrix} e^{\delta_{11}(T-t)} & 0 \\ 0 & e^{\delta_{22}(T-t)} \end{pmatrix} \\ e^{-\Delta^Q(T-t)} &= \begin{pmatrix} e^{-\delta_{11}(T-t)} & 0 \\ 0 & e^{-\delta_{22}(T-t)} \end{pmatrix} \end{aligned}$$

After substituting this in the expression of  $B(t, T)$  we have :

$$B(t, T) = - \begin{pmatrix} e^{-\delta_{11}(T-t)} & 0 \\ 0 & e^{-\delta_{22}(T-t)} \end{pmatrix} \times \int_t^T \begin{pmatrix} e^{\delta_{11}(T-s)} & 0 \\ 0 & e^{\delta_{22}(T-s)} \end{pmatrix} ds$$

After simplifying the integral :

$$\begin{aligned} B(t, T) &= - \begin{pmatrix} e^{-\delta_{11}(T-t)} & 0 \\ 0 & e^{-\delta_{22}(T-t)} \end{pmatrix} \times \begin{pmatrix} -\frac{1-e^{\delta_{11}(T-t)}}{\delta_{11}} \\ -\frac{1-e^{\delta_{22}(T-t)}}{\delta_{22}} \end{pmatrix} \\ B(t, T) &= - \begin{pmatrix} -\frac{1-e^{-\delta_{11}(T-t)}}{\delta_{11}} \\ -\frac{1-e^{-\delta_{22}(T-t)}}{\delta_{22}} \end{pmatrix} \end{aligned}$$

So we have finally:

$$B_1(t, T) = \frac{1 - e^{-\delta_{11}(T-t)}}{\delta_{11}} \quad B_2(t, T) = \frac{1 - e^{-\delta_{22}(T-t)}}{\delta_{22}} \quad (11)$$

and:

for the constant term  $C(t, T)$  is given by:

$$\begin{aligned}
C(t, T) &= -\frac{1}{2} \int_t^T \sum_{j=1}^n (\Sigma' B(s, T) B(s, T)' \Sigma)_{i,j} ds \\
&= -\frac{1}{2} \int_t^T [(\sigma_{11}^2) B_1(s, T)^2 + (\sigma_{22}^2) B_2(s, T)^2] ds \\
&= \frac{1}{2} \left[ \frac{\sigma_{11}^2}{\delta_{11}^3} \left( 2e^{-\delta_{11}(T-t)} - \frac{1}{2} e^{-2\delta_{11}(T-t)} + \delta_{11}(T-t) - \frac{3}{2} \right) \right. \\
&\quad \left. \frac{\sigma_{22}^2}{\delta_{22}^3} \left( -2e^{-\delta_{22}(T-t)} - \frac{1}{2} e^{-2\delta_{22}(T-t)} + \delta_{22}(T-t) - \frac{3}{2} \right) \right]
\end{aligned} \tag{12}$$

### 3.2 3-Factor independent model :

We represent here a model when each of the factors are independent , and we have no correlation of the Brownian movements, the dynamics of this model is given by :

$$\begin{pmatrix} dZ_{1,t} \\ dZ_{2,t} \\ dZ_{3,t} \end{pmatrix} = - \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix} \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \\ Z_{3,t} \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_{1,t}^Q \\ dW_{2,t}^Q \\ dW_{3,t}^Q \end{pmatrix} \tag{13}$$

Where the ordinary differential equations solution is given by :

$$B_i(t, T) = \frac{1 - e^{-\delta_i(T-t)}}{\delta_i} \tag{14}$$

where  $i = 1, 2, 3$  for every factor

and we have

$$C(t, T) = \frac{1}{2} \int_t^T \sum_{j=1}^3 (\Sigma^T B(s, T) B(s, T)^T \Sigma)_{jj} ds \tag{15}$$

### 3.3 The Nelson-Siegel model:

The Nelson siegel model is a model who fits the yield curve at any point in time , (Nelson-Siegel 1987) and in the term structure modelling this model can ease many empirically observed shapes, and this is the reason of being the most popular one in the financial market.

We use the Nelson-Siegel model which is not consistent, to demonstrate the consistency notion in one hand and to give a contrast with our consistent models (2-and 3-factor models) in another hand .

The risk neutral dynamics of the 3-factor Nelson-Siegel model is given as CHRISTENSEN et al (2009) :

$$\begin{pmatrix} dZ_{1,t} \\ dZ_{2,t} \\ dZ_{3,t} \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & -\delta \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \\ Z_{3,t} \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_{1,t}^Q \\ dW_{2,t}^Q \\ dW_{3,t}^Q \end{pmatrix} \quad (16)$$

CHRISTENSEN et al (2009) provide us with the solutions of the factor loadings :

$$\begin{aligned} B_1(t, T) &= (T - t) & B_2(t, T) &= \frac{1 - e^{-\delta(T-t)}}{\delta} \\ B_3(t, T) &= \left[ \frac{1 - e^{-\delta(T-t)}}{\delta} - (T - t)e^{-\delta(T-t)} \right] \end{aligned} \quad (17)$$

By the use of the previous equations we can now present the expression of the average force of mortality for each models ,that it is estimated in our framework .

Under the risk neutral measure ,we can write the average force of moratlity for the n-factor independant model by combining (25),(29),(30),and (32),(32) as:

$$\bar{\mu}(t, T) = \frac{1}{T-t} \sum_{i=1}^n \left[ \frac{1 - e^{-\delta_i(T-t)}}{\delta_i} Z_{i,t} - \frac{1}{2} \frac{\sigma_i^2}{\delta_i^3} \times \left( \frac{1}{2} (1 - e^{-2\delta_i(T-t)}) - 2(1 - e^{-\delta_i(T-t)}) + \delta_i(T-t) \right) \right] \quad (18)$$

And the 3-factor Nelson Siegel model by combining (25) and (35) with C(t,T) given by christensen et al.(2009)

$$\bar{\mu}(t, T) = \frac{1}{T-t} \left[ (T-t)Z_{1,t} + \frac{1 - e^{-\delta(T-t)}}{\delta} Z_{2,t} + \left( \frac{1 - e^{-\delta(T-t)}}{\delta} - (T-t)e^{-\delta(T-t)} \right) Z_{3,t} - C(t, T) \right] \quad (19)$$

### 3.4 Change of measure:

As an alternative of the "Complete affine" model of duffie and kan (1996)in which they assume that the price of Mortality risk is constant for each factor ,we choose the "essentially affine" model given by Duffie(2002) and we use the Gisanov's theorem to move from the Risk neutral Q-measure dynamics to a real world P measur .Under the P measure , there is no interaction between factors and the moratlity risk price detail of the 'essentially affine' remove the relation ship between real world or historical drift and the factor loadings ,so this let us choose various drift terms for the real word or the risk neutral process.

Using Gisanov's theorem the n-factor stochastic process can be written under the real world P-measure as follow :

$$dZ_t = K[\Psi - Z_t]dt + W_t^P \quad (20)$$

where  $K \in R^{n \times n}$ ,  $\Psi \in R^{n \times 1}$ ,  $\Sigma \in R^{n \times n}$  .

This expression is taking from the 'essentially affine' model , where the Market price of mortality is given by :

$$\Lambda_t = \lambda_0 + \lambda_1 Z_t$$

and  $\Lambda_t \in R^{n \times n}$ ,  $\lambda_0 \in R^{n \times 1}$ ,  $\lambda_1 \in R^{n \times n}$  .

The change of measure using Ginasov's theorem from the n-factor real world Brownian movement to the risk-neutral measure is given by :

$$dW_t^Q = dW_t^P + \Lambda_t dt$$

Under the P measure , the stochastic process

$Z_t$  is :  $dZ_t = [\Delta\Theta - \Delta Z_t] dt + \Sigma[\lambda_0 + \lambda_1 Z_t] dt + \Sigma dW_t^P = [\Delta\Theta + \Sigma\lambda_0] dt - [\Delta - \Sigma\lambda_1] Z_t dt + \Sigma dW_t^P$   
this can be written the same as (38). We have to note that we set the  $\Psi$  to 0 , which is the long term mean level of the mortality rates , and this can let each factor increase to zero at an axact rate noted K . We can express the real- word dynamics of our given models as bellow : Starting with the 2-factor independent Model :

$$\begin{pmatrix} dZ_{1,t} \\ dZ_{2,t} \end{pmatrix} = - \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} dW_{1,t}^P \\ dW_{2,t}^P \end{pmatrix} \quad (21)$$

For the 3-factor independent and the Nelson-Siegel model :

$$\begin{pmatrix} dZ_{1,t} \\ dZ_{2,t} \\ dZ_{3,t} \end{pmatrix} = - \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \\ Z_{3,t} \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_{1,t}^P \\ dW_{2,t}^P \\ dW_{3,t}^P \end{pmatrix} \quad (22)$$

Owing to this definition we can estimate K which is the mean reversion real world speed, without affecting the other parameters of the Risk neutral. Duffi(2002) proves that when we add the square root diffusion factors , the detail of risk structure price in the 'essentially affine' model turns to the 'complete affine' model.

So we concentrate on the dynamics of the model which is a gaussian dynamics , and we will estimate the real world drift and the Risk-neutral parameter individually using the Kalmna filter.

### 3.5 The Forward mortality rate:

As Chiarella and Kwon (2001) show that the ATSM model is a special case of the HJM model ,we use this specification to give a forward mortality rate presentation.

Under the Risk neutral measure ,we can model the forward mortality dynamics as follow:

$$\begin{aligned} d\mu(t, T) &= \sum_{i=1}^n \left[ \frac{\sigma_1^2 e^{-\delta_i(T-t)}}{\delta_i} (e^{-\delta_i(T-t)} - 1) dt - \sigma_i e^{-\delta_i(T-t)} dW_{i,t}^Q \right] \\ &= v_\mu(t, T) dt + \sigma_\mu(t, T) dW_t^Q \end{aligned} \quad (23)$$

Proof :

$$\begin{aligned} \mu(t, T) &= \frac{\partial \log[S(t, T)]}{\partial T} \\ &= \frac{\partial}{\partial T} [B(t, T)' Z_t - C(t, T)] \\ &= \sum_{i=1}^n \left[ e^{-\delta_i(T-t)} Z_{i,t} - \frac{1}{2} \frac{\sigma_i^2}{\delta_i^3} (2\delta_i e^{-\delta_i(T-t)} - \delta_i e^{-2\delta_i(T-t)} + \delta_i) \right] \end{aligned} \quad (24)$$

Through the Ito'slemma we can find out the dynamics of  $\mu(t, T)$  :

$$\begin{aligned}
d\mu(t, T) &= \frac{\partial \mu}{\partial t} dt + \sum_{i=1}^n \left[ \frac{\partial \mu}{\partial Z_{i,t}} dZ_{i,t} \right] \\
&= \sum_{i=1}^n \left[ \left( \delta_i e^{-\delta_i(T-t)} Z_{i,t} - \frac{1}{2} \frac{\sigma_i^2}{\delta_i^3} (2\delta_i^2 e^{-\delta_i(T-t)} - 2\delta_i^2 e^{-2\delta_i(T-t)}) \right) dt + e^{-\delta_i(T-t)} dZ_{i,t} \right]
\end{aligned} \tag{25}$$

the expression of  $dZ_t$  of each factors ,is determined as we set in equations.

$$dZ_{i,t} = -\delta_i Z_{i,t} dt + \sigma dW_{i,t}^Q$$

After substitution we have:

$$\begin{aligned}
d\mu(t, T) &= \sum_{i=1}^n \left[ \left( \delta_i e^{-\delta_i(T-t)} Z_{i,t} - \frac{1}{2} \frac{\sigma_i^2}{\delta_i^3} (2\delta_i^2 e^{-\delta_i(T-t)} - 2\delta_i^2 e^{-2\delta_i(T-t)}) \right) dt \right. \\
&\quad \left. - e^{-\delta_i(T-t)} \delta_i Z_{i,t} dt - \sigma_i e^{-\delta_i(T-t)} dW_{i,t}^Q \right] \\
&= \sum_{i=1}^n \left[ \frac{\sigma_i^2 e^{-\delta_i(T-t)}}{\delta_i} (e^{-\delta_i(T-t)} - 1) dt - \sigma_i e^{-\delta_i(T-t)} dW_{i,t}^{hbbQ} \right]
\end{aligned} \tag{26}$$

As we can see, this equation confront the HJM drift given by :

$$\begin{aligned}
v_\mu(t, T) &= \sigma_\mu(t, T) \int_t^T \sigma_\mu(t, s) ds \\
&= \sum_{i=1}^n \left[ \sigma_i e^{-\delta_i(T-t)} \int_t^T \sigma_i e^{-\delta_i(s-t)} ds \right] \\
&= \sum_{i=1}^n \left[ \frac{\sigma_i^2 e^{-\delta_i(T-t)}}{\delta_i} (e^{-\delta_i(T-t)} - 1) \right]
\end{aligned} \tag{27}$$

With  $\sigma_\mu(t, T)$  represent a volatility functions matrix.

### 3.6 The consistent forward mortality curve :

The main aim of this work is to give a a consistent models for fitting the survival probability curve.

These curve have the characteristic of keeping the same form of the survival curves through time. Bjork and Christness (1999) propose a study about the consistent forward curves and we use this study to test if the consistent condition is verified in our 2-and 3-factor independent models and the Nelson siegel model(Christness (2009))

\* Consistency Test:

We had shown that the force of mortality is written by :

$$\mu(t, T) = \frac{\partial \log[S(t, T)]}{\partial T}$$

To determine consistency we can apply the Frechet derivatives to  $\mu(T - t)$  by using **Theorem 4.1** of Christness and Bjork (1999). The derivatives can be used in the verification of volatility and drift conditions for the dynamics of our risk factor that confirm consistency . The consistent drift condition is given by :

$$\mu_x(x, z) + \sigma(x) \left( \int_0^x \sigma(y) dy \right)^T \in \text{Im} [\mu_z(x, z)]$$

Where  $\mu_x(x, z)$  and  $\mu_z(x, z)$  represents respectively the Frechet derivatives with respect to  $x$  and  $z$  . with  $x = T - t$ .

Taking our 2-factor model ,The volatility term in the drift can be given by:

$$\begin{aligned} \sigma(x) \left( \int_0^x \sigma(y) dy \right)^T &= [\sigma_1 e^{-\delta_1 x} + \sigma_2 e^{-\delta_2 x}] \left[ \sigma_1 \left( \frac{1 - e^{-\delta_1 x}}{\delta_1} \right) + \sigma_2 \left( \frac{1 - e^{-\delta_2 x}}{\delta_2} \right) \right]^T \\ &= \sigma_1^2 \left( \frac{e^{-\delta_1 x} - e^{-2\delta_1 x}}{\delta_1} \right) + \sigma_2^2 \left( \frac{e^{-\delta_2 x} - e^{-2\delta_2 x}}{\delta_2} \right) \end{aligned}$$

We can simplify the time varying parameters in  $z$  and the exponential terms by  $x$  so we have :

$$\mu(x, z) = z_1 e^{-\delta_1 x} + z_2 e^{-\delta_2 x} + z_3 e^{-2\delta_1 x} + z_4 e^{-2\delta_2 x}$$

The Frechet derivatives of  $\mu(x, z)$ :

$$\begin{aligned} \mu_z(x, z) &= \begin{bmatrix} e^{-\delta_1 x} & e^{-\delta_2 x} & e^{-2\delta_1 x} & e^{-2\delta_2 x} \end{bmatrix} \\ \mu_x(x, z) &= z_1 e^{-\delta_1 x} + z_2 e^{-\delta_2 x} - z_3 e^{-2\delta_1 x} - z_4 e^{-2\delta_2 x} \end{aligned}$$

The consistent drift condition can be given as :

$$\mu_x(x, z) + \gamma_1 e^{-\delta_1 x} + \gamma_2 e^{-\delta_2 x} + \gamma_3 e^{-2\delta_1 x} + \gamma_4 e^{-2\delta_2 x} \in \text{Im} [\mu_z(x, z)]$$

Where  $\gamma_i$  are constant.

So our 2-factor model is constant since it verify the equation The vectors of volatility for the 2-factor model is given by :

$$\sigma_{2f} = \begin{bmatrix} \sigma_1 e^{-\delta_1 x} & \sigma_2 e^{-\delta_2 x} \end{bmatrix} \quad (28)$$

The same thing for the 3-factor model ,and we can write the consistent drift condition as :

$$\mu_x(x, z) + \gamma_1 e^{-\delta_1 x} + \gamma_2 e^{-\delta_2 x} + \gamma_3 e^{-2\delta_1 x} + \gamma_4 e^{-2\delta_2 x} + \gamma_5 e^{-\delta_3 x} + \gamma_6 e^{-2\delta_3 x} \in \text{Im} [\mu_z(x, z)]$$

The vector of volatilities for the 3-factor model is given by:

$$\sigma_{3f}(x) = \begin{bmatrix} \sigma_1 e^{-\delta_1 x} & \sigma_2 e^{-\delta_2 x} & \sigma_3 e^{-\delta_3 x} \end{bmatrix} \quad (29)$$

Taking now the Nelson siegel model , The volatility term in drift can be shown as :

$$\begin{aligned}
& \sigma(x) \left( \int_0^x \sigma(y) dy \right)^T \\
&= \left[ \sigma_1 \quad \sigma_2 e^{-\delta x} + \sigma_3 e^{-2\delta x} \right] \left[ \sigma_1 x \quad \sigma_2 \frac{(1 - e^{-\delta x})}{\delta} + \sigma_3 \left( \frac{(1 - e^{-\delta x})}{\delta} - x e^{-\delta x} \right) \right]^T \\
&= \sigma_1^2 x + \sigma_2^2 \frac{(e^{-\delta x} - e^{-2\delta x})}{\delta} + \sigma_3^2 \left( x \frac{(e^{-\delta x} - e^{-2\delta x})}{\delta} - x^2 \delta e^{-2\delta x} \right)
\end{aligned}$$

So the force of mortality can be written :

$$\mu(x, z) = z_1 + z_2 e^{-\delta x} + z_3 x e^{-\delta x} + z_4 e^{-2\delta x} + z_5 x e^{-2\delta x} + z_6 x^2 e^{-2\delta x} + z_7 x^2$$

Then the frechet derivatives of  $\mu(x, z)$  is :

$$\begin{aligned}
\mu_z(x, z) &= \begin{bmatrix} 1 & e^{-\delta x} & x e^{-\delta x} & e^{-2\delta x} & x e^{-2\delta x} & x^2 e^{-2\delta x} & x^2 \end{bmatrix} \\
\mu_x(x, z) &= (z_3 - \delta z_2) e^{-\delta x} - \delta z_3 x e^{-\delta x} + (z_5 - 2\delta z_4) e^{-2\delta x} \\
&\quad + (2z_6 - 2\delta z_5) x e^{-2\delta x} - 2\delta z_6 x^2 e^{-2\delta x} + z_7 2x
\end{aligned}$$

So we can write the consistency drift after assembling terms in  $x$ , as:

$$\mu_x(x, z) + \gamma_1 e^{-\delta x} + \gamma_2 e^{-2\delta x} + \gamma_3 e^{-2\delta x} + \gamma_4 e^{-2\delta x} + \gamma_5 x^2 e^{-2\delta x} + \gamma_6 x$$

Since  $\gamma_6 \notin \text{Im}[\mu_z(x, z)]$  the Nelson Siegel model is not consistent.

The vector of volatilities for the 3-factor Nelson Siegel model is given by :

$$\sigma_{NS}(x) = \begin{bmatrix} \sigma_1 e^{-\delta x} & \sigma_2 e^{-\delta x} & \sigma_3 e^{-\delta x} \end{bmatrix} \quad (30)$$

## 4 Empirical study:

### 4.1 Data presentation:

To solve the problem of modeling longevity risk, a consistent multi factor dynamics affine mortality model is suggested through the ATSM model.

We prove that ATSMs model are an adaptable model to fit the historical Mortality rates through the use of USA mortality data. Our sample contains the population exposures and the number of deaths for the years 1933-2010 which are taken from the human Mortality data base when the population exposures by age are generally estimated. This mortality data is used to evaluate 2- and 3-factor implementation of the model when the last one is used to give a good fit to the survival curves which is observed especially for the older age.

The United States is considered as the third most populous country in the world since it accounts for about 4.5% of the whole population. Since its 1950 the US population has more than doubled and it was estimated at 308.7 million persons, its just not about size , the US population has become different after 1950 .As it was mentioned by the Population reference Bureau "the US is getting bigger, older and more diverse" .One of most demographic characteristic of the US is the rapid ageing of the population and an important increasing in the persons aged more than 65 which can increase the Longevity risk in this country, and make it one of the most interestent data that we can used for our model fitting .

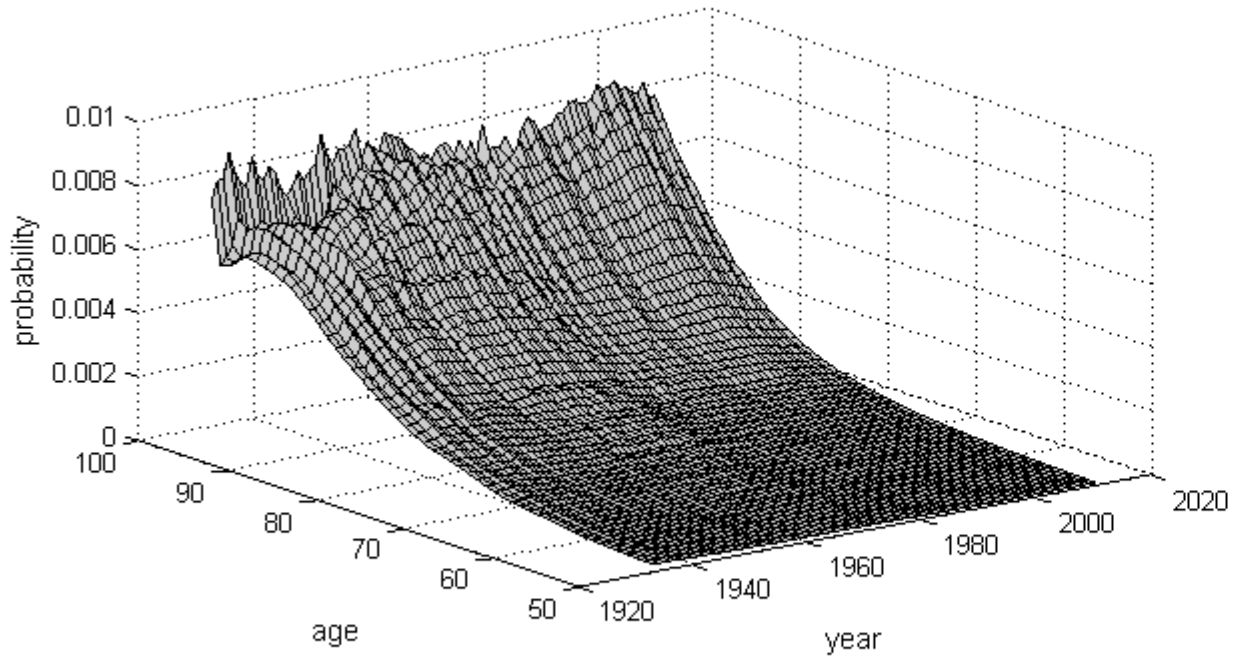


Figure 1: Age-Periode average force of mortality for the USA population aged 50-100 for 1933-2010

#### 4.1.1 Mortality Survival curve :

In our work , we refer to the famous Nelson Siegel model especially the way to fit the yield curves in interest rates. So we introduce our model for the survival curves through a functional form which is a sum of exponentials includes a time varying parameters . In this case , the mortality curve of the survival probabilities average at different survival times , is equivalent to the yield curve of the zero coupon bond yield at variance maturities .

Under the risk neutral measure the survival probability at time  $t$  for a person aged  $x$  to age  $x + T$  (it is mean to survive another  $T - t$  years ),can be represented through the affine form as :

$$S(t, T) = E_t^Q \left[ e^{\int_t^{T-t} \mu_{x+u,t} du} | F_t \right] = e^{-B(t,T)'Z_t + C(t,T)} \quad (31)$$

With  $\mu_{x+u,t}$  correspond to the force of mortality for a person aged  $x + u$  at time  $t$  where  $Z_t$  symbolizes the random factors vectors.

We can presented the maturity curve of the survival probabilities  $S(t, T)$ ,when its log is affine in the random factors and able to drive the mortality changing through time,the maturity curve is correspondent, in the interest rates framework ,to the zero coupon yields to maturity and it is shown by:

$$\bar{\mu}(t, T) = -\frac{1}{T-t} \log[S(t, T)] \quad (32)$$

$\bar{\mu}(t, T)$  is the average force of mortality for survival  $T - t$  years to an individual aged  $x$  at year  $t$  , we use  $\bar{\mu}(t, T)$  to fit our model.

We determine the survival probability from our mortality data and it is given by :

$$S(t, T) = \prod_{s=0}^{T-t} (1 - q(x + s, t)) \quad (33)$$

Where  $q(x, t)$  represents the death probability for one year given for an individual aged at year  $t$  ,and it is given by:

$$q(x, t) = 1 - e^{-m(x,t)} \quad (34)$$

Where  $m(x, t)$  corresponds to the central rate of mortality and we can calculate it from our mortality data as follow:

$$m(x, t) = \frac{D(x, t)}{E(x, t)} = \frac{\#of\ deaths\ aged\ x\ in\ year\ t}{Exposure\ aged\ x\ in\ year\ t} \quad (35)$$

Figure.1 shows the plot of the average force of mortality given by  $\bar{\mu}(t, T)$  for population aged between 50 and 100 for the years 1933-2010. we use this mortality curve data to represent the model fitting.

We can see through this figure the exponential shape of the mortality curve and the improvement in mortality over the periode of the study , this improvements have appeared at various rates for different ages.

Figure.2 shows the mean and the first two principal components of the average of mortality  $\bar{\mu}(t, T)$  for USA data set. The mean and first principal component explains 98.58% of the data variation.This proposes that the 2-and 3-factor mortality models should be able to capture parsimoniosly all changes in our mortality survival curves , in the principal component we assume the consistency of the mean and the orthogonality of the principal components. In our work, we suppose a mean equivalent , it represents a dynamic factor with wich we can provide interaction between the model factors under the dynamics of the risk neutral measure or the real world dynamics.

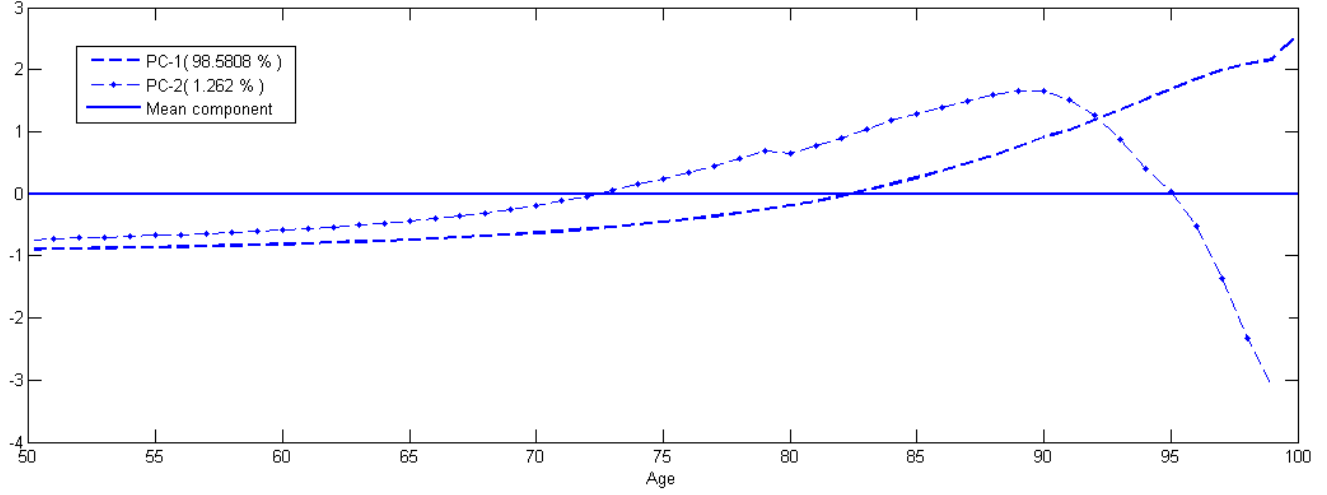


Figure 2: Principal component analysis

## 5 Model calibration :

### 5.1 Kalman filter

The kalman Filter is considered as an optimal linear estimator since it minimises the error covariance (Kalman 1960).The most financial application used this filter , one of this application is the estimaton of the term structure model as it used by Anderson and Land (1997) and Babbs and Nowman (1999) . It bring to us a recursive solution to solve the problem of filtering especially for the linear discreat filtering.

The main advantage of the application of the kalman filter and the state space presentation of the term structure model is that facilitate handling properly the state variable as an unobserved variable by the use the short rate term like a proxy.

The kalman filter is presented by a series of state spaace equation with an n-dimensional unobservable factors, $Z$  of the system which is given by a linear stochastic difference equation.

The transition system  $Z$  can be written in the discrete form as:

$$Z_t = A \times Z_{T-1} + B + \eta_{T-1} \quad (36)$$

We presume that  $A \in \mathbb{R}^{n \times n}$  is a constant matrix and show the relationship between  $Z_t$  and  $Z_{t-1}$  the actual and the previous state factor respectively ,and  $B \in \mathbb{R}^n$  is the controle variable for the factor, $\eta_{t-1} \in \mathbb{R}^{n \times n}$  is a random variable presents the transition noise and it is supposed to be independant and normally distributed.

$$\eta_{T-1} \sim \mathbf{N}(0, \mathbf{Q}) \quad (37)$$

where  $\mathbf{R} \in \mathbb{R}^{m \times m}$  is predicted constant and represents the transition noise covariance of the system .

The measurement equation which is the second equation of the state space illustration , the measurement equation relates the unobserved factors  $Z_t$  and  $M_t$  the observed data which has m-dimensions:

$$M_t = H \times Z_t + C + \epsilon_t \quad (38)$$

Where  $H \in \mathbb{R}^{m \times m}$  is predicted to be a constant matrix and describes the relationship between  $Z_t$  the actual factor and  $M_t$  the measurement of the process with  $C \in \mathbb{R}^m$  a constant matrix which gives the external control of the given measurement equation, the random variable  $\epsilon_t \in \mathbb{R}^{m \times m}$  is the measurement noise with :

$$\epsilon_t \sim \mathbf{N}(0, \mathbf{R})$$

Where  $R \in \mathbb{R}^{m \times m}$  is predicted to be constant and represents the noise covariance matrix . At every interval time the  $Z_t$  which is an unobserved factor is estimated with the kalman filter and it updates estimates by using the feed back from  $M_t$  the measurement equation . There is two kind of update equation in kalman filter , the first one is the time update equation and the second is the measurement update equation.

The time update equation give a projection to the error covariance and to the actual state to get an estimate for the following term.

The measurement update equation integrates an other information to ameliorate the time update equation's estimates.

We can express the time update as bellow:

$$\hat{Z}_t^- = A \times \hat{Z}_{t-1} + B \times \mu_t \quad (39)$$

$$\hat{P}_t^- = A \times \hat{P}_{t-1} \times A' + Q \quad (40)$$

We initialize the filter by giving the unconditional mean  $Z_0$  and  $P_0$  the variance of the transition system ,with which we gonna begin the recursion. We can calculated its from the historival data as follow :

$$\hat{Z}_t^- = A \times Z_0 + B \times \mu_t \quad (41)$$

$$\hat{P}_t^- = A \times P_0 \times A' + Q \quad (42)$$

K which is the kalman gain have the ability to minimize the system's covariance error , it represents the difference between  $M_t$  the actual measurement and the predicted measurement given by  $H\hat{Z}_t^- + C$  .Since the value of the measurement error covariance is closer to zero the actual value measurement is more weighty .

We can express the error term and the residual variance-covariance matrix as :

$$\epsilon_t = M_t - C - H \times \hat{Z}_t^- \quad (43)$$

$$S_t = H \times \hat{P}_t^- \times H' + R \quad (44)$$

From those two expression we can calculate the standardised residuals as:

$$e_t = \frac{\epsilon_t}{\sqrt{S_t}} \quad (45)$$

So the Kalman gain is given by:

$$K_t = \frac{\hat{P}_t^- \times H'}{S_t} \quad (46)$$

We obtain then the measurement update equation :

$$\hat{Z}_t = \hat{Z}_t^- + K_t(M_t - H \times \hat{Z}_t^-) \quad (47)$$

$$\hat{P}_t = (I - K_t \times H) \times \hat{P}_t^- \quad (48)$$

Since we are worked with the state space assumptions , the predictions errors are Gaussian , the kalman filter maximzes then a multi variate normal likelihood ,we can compute the log likelihood for each step of recursion as below :

$$l(\theta) = -\frac{1}{2} \sum_{t=1}^N [\ln(2\pi) + \ln |S_t| + \epsilon_t' S_t^{-1} \epsilon_t] \quad (49)$$

To find the optimal parameter set noted by  $\hat{\theta}$  we need a non-linear optimisation which maximise the likelihood function ,it can be calculated from :

$$\hat{\theta} = \operatorname{argmax}_{\theta} l(\theta)$$

## 5.2 Estimation method:

As we can see from section 4 our model can be clearly represented as a state space form where the measurement equations for the our independent model and the Nelson Siegel model is given by equations (36)and (37) respectively.And it is given as follow :

$$\bar{\mu} = B(t, T)' Z_t - C(t, T) + \Psi, \quad \Psi \rightarrow N(0, R) \quad (50)$$

with  $R \in R^{m \times m}$  is the measurement noise covariance matrix with m represents the number of ages giving in our data set, R is supposed to be time-invariant.

We assume that  $\Psi$  which is the measurement noise is independent between ages as result R is presented by a diagonal matrix.Instead of estimate every measurement error separately for every age, we use an easy parametric curve to estimate the measurement error thanks to this method the number of the parameters are decreased to three parameters.

The diagonal of the covariance matrix can be represented as a parametric form and we can write :

$$R(t, T) = \frac{1}{T-t} \sum_{i=t}^{T-t} [r_c + r_1 e^{r_2 \times i}] \quad (51)$$

And we include the estimation of the different values giving by  $r_c, r_1$  and  $r_2$  with the optimal parameter-set noted by  $\hat{\theta}$ .

The transition systems of equation are presented in our case with the real world dynamics introduced in (4.5) and we can write :

$$Z_{t_i} = e^{-K(t_i-t_{i-1})}Z_{t_{i-1}} + \eta_{t_i}, \quad \eta_{t_i} \rightarrow N(0, Q_{t_i}) \quad (52)$$

$$Q_{t_i} = E \left[ \left( \int_{t_{i-1}}^{t_i} e^{-k(t_i-s)} \Sigma dW_s^p \right)^2 \right] \quad (53)$$

All the errors are Gaussian since we are under the state-space model assumptions , the kalman filter produces a multivariate normal likelihhod , we can calculate the log-likelihhod as:

$$l(\theta) = -\frac{1}{2} \sum_{t=1}^N \left[ \ln(2\pi) + \ln |S_t| + \epsilon_t' S_t^{-1} \epsilon_t \right] \quad (54)$$

$\epsilon_t$  and  $S_t$  are respectively the error term and the residual variance-covariance matrices. And we can conclude  $l(\theta)$  the optimal parameter-set that maximizes the likelihood function like:

$$\hat{\theta} = \operatorname{argmax}_{\theta} l(\theta)$$

### 5.3 Bootstrapping state-space models:

The major advantage of the bootstrap is his simplicity , it is considered as a uncomplicated way to derive estimates of standard errors and confidence intervalles for such a complex paramaters or a complex estimators for any distribution , in addition bootrap is a suitable way to control and confirm the satbility of the results .Despite the fact that most of the problems we can't know or have the exact confidence interval, Bootstrap is asymptotically more exact than the other standard intervals by the use of a sample variance and the normality assumptions.

In our work we noted that the likelihood is flat around the optimal parameter set for this reason we use a bootstrap method to re-sample the observed data, we use the standardised residuals taking from the kalman filter optimal parameters set, so we can have an asymptotically Gaussian distribution for the parameteres estimate.

We use in our framework the re-sample method introduced by Stoffer and Wall (2004). The process of the state-space bootrap method is presented by 4 step as below:

1. We define the standardized innovations from the optimal parameter set :

$$e_t = \frac{\hat{\epsilon}_t}{\sqrt{\hat{S}_t}} \quad (55)$$

2. Re-sample, with replacement, the standardized innovations n times , to create a bootstrap sample of residuals  $e_t^*$  due to the start up irregularities of the kalman filter we exclude the first four innovations .

- Recalculate the measurement and the transition equations using the bootstrap innovations  $e_t^*$

$$Z_t^* = e^{-\hat{K}} Z_{t-1}^* + \frac{e_t^*}{\sqrt{\hat{S}_t}} \quad (56)$$

$$\bar{\mu}_t^* = \hat{B}' Z_t^* - \hat{C} + \hat{K}_t \frac{e_t^*}{\sqrt{\hat{S}_t}} \quad (57)$$

- Employ the set of the bootstrapped data  $\bar{\mu}_t^*$  to have an estimation of the new parameters set which maximise the likelihood function by:

$$\hat{\theta}^* = \operatorname{argmax}_{\theta} l(\theta^*) \quad (58)$$

With  $\theta^*$  is given by the bootstrapped data set  $\bar{\mu}_t^*$ .

- Repeat steps 2 and 4 ,500 times to get for every parameter estimates a sample distribution .

## 6 Results of the estimation:

In this section, we represent the results from the three models estimation and evaluation. Table(1) shows the maximum log likelihood and RMSE: the Root Mean Squared Error for every model.

We presented here a modified Akaike information Criteria noted by AIC test which is used for state space model as it was given by Cavanaugh and Shumway 1997 . The AICb is noticed by his simplistic form and it would be adapted to calculate as part of the bootstrap-based analysis .

We use the result of the bootstrapped log likelihood in the AICb test to obtain an estimation of the penalty term. We can calculate the AICb test as below:

$$AICb = -2 \log L(\hat{\theta}) + 2 \left[ \frac{1}{N} \sum_{i=1}^N -2 \log L(\hat{\theta}_i^*) - (-2 \log L(\hat{\theta})) \right] \quad (59)$$

Where N presents the number of bootstrap distribution, in our case we choose  $N = 500$  , and  $L(\hat{\theta}_i^*)$  presents the bootstrapped maximum likelihood for the  $i$ th distribution .

Table 1: Comparison of log likelihood, RMSE and the number of parameters and factors estimated for every model, and an AICb is also presented :

	3-factor independent	3-factor Nelson Siegel	2-factor independent
Log likelihood	24230	24107	20076
RMSE	0.000251	0.000269	0.000356
No of factors estimated	234	234	156
No of model parameters	15	13	11
$\Delta AIC$	-	$\approx 0$	4404

We note that the 3-factor independent model and the Nelson Siegel model have a comparable loglikelihood and RMSE and both models presented give as a better fit than the two factor model. We noted that the three factor independent model in our framework have 15 parameters and 234 latent factors to be estimated ,(we have 3 factors for every 78 years of our presented data).

We noted that there is no amelioration in the use of the 3-factor independent model in the place of the 3-factor Nelson Siegel model ,when we indicate the rejection of the 2-factor independent model.

## 6.1 Parameter estimation :

Table(2) displays the estimated parameters for the 2-and 3- factor model with 95% confidence intervals from our giving method of state-space bootstrapping introduced in section 5.2.3.

Table 2:the results of 2-and 3-factor model fit with the bootstrap confidence intervals:

ML estimation	2-factor	95%confidence		3-factor	95%confidence	
	independent	Lower	Upper	independent	Lower	Upper
$\delta_1$	-0.0845	-0.0925	-0.0821	-0.09756	-0.10122	-0.08703
$\delta_2$	0.0453	0.0320	0.0567	0.00527	0.00457	0.00752
$\delta_3$	-	-	-	-0.12530	-0.14521	-0.1033
$\kappa_1$	0.0155	0.0025	0.0287	0.00653	0.00320	0.00727
$\kappa_2$	0.0125	0.0108	0.0248	0.00189	0.00127	0.00228
$\kappa_3$	-	-	-	0.00192	0.00103	0.00253
$\sigma_1$	$4.2012e^{-05}$	$3.705e^{-05}$	$4.835e^{-05}$	$3.857e^{-04}$	$3.207e^{-04}$	$4.007e^{-04}$
$\sigma_2$	0.0081	0.0058	0.0092	$3.002e^{-03}$	$2.805e^{-03}$	$3.503e^{-03}$
$\sigma_3$	-	-	-	$6.428e^{-04}$	$3.987e^{-05}$	$2.098e^{-03}$
$r_1$	$6.3169e^{-12}$	$7.501e^{-13}$	$8.233e^{-11}$	$1.657e^{-11}$	$4.002e^{-12}$	$2.430e^{-11}$
$r_2$	0.2589	0.1980	0.3587	0.2908	0.2365	0.3508
$r_c$	$2.4500e^{-08}$	$1.2801e^{-07}$	$2.805e^{-07}$	$3.748e^{-07}$	$2.904e^{-07}$	$4.383e^{-07}$

As we can see for both model that all the parameter  $\delta$  are significant. We can note also that the  $\kappa$  parameters for the real-world process have a large confidence intervals , the value of the lower bound which is close to zero dsignates the random walk whereas the wide values show that the factor returns to its long term mean of zero.

We can note that the measurement variance parameters  $r_1, r_2$  and  $r_c$  ,and the volatilty matrix  $\Sigma$  for both models are significant with a little number of outliers.

Table 3: the results of 3-factor Nelson Siegel model fit with the bootstrap confidence intervals:

ML estimation	3-factor Nelson Siegel model	95%confidence	
		Lower	Upper
$\delta$	-0.08750	-0.09856	-0.07683
$\kappa_1$	0.10082	0.00586	0.18970
$\kappa_2$	0.00369	0.00287	0.00456
$\kappa_3$	0.00115	0.001085	0.00254
$\sigma_1$	4.251e-04	3.658e <sup>-04</sup>	5.257e <sup>-04</sup>
$\sigma_2$	1.328e <sup>-04</sup>	0.982e <sup>-04</sup>	1.698e <sup>-04</sup>
$\sigma_3$	4.921e <sup>-04</sup>	3.984e <sup>-04</sup>	5.327e <sup>-04</sup>
$r_1$	2.425e <sup>-08</sup>	3.205e <sup>-09</sup>	2.967e <sup>-08</sup>
$r_2$	0.253	0.237	0.369
$r_c$	2.845e <sup>-06</sup>	1.914e <sup>-06</sup>	2.968e <sup>-06</sup>

Table(3) displays the estimated parameters of the Nelson Siegel model ,we have similar results of the previous models ,they all have a large confidence intervals and the different parameters of the factor loadings are significant. Even though we have difficulties to estimate and therefore to forecast the real-world drift,we find generally that the risk neutral drift and also the diffusion parameters are stable.

## 6.2 Sample analysis :

The structural changes in our mortality rates for different models can be appeared through the in-sample analysis.

Figure 3 and 4 display the percentage error of the survival curve already estimated for different models taking the years 1950 and 2000. We note that all the models can fit our data up to the age of 85 , wherever over the age of 85 the percentage error of the 2 factor model continue in increasing in both chosen year and we can conclude that the 2 factor model is unable to fit the data at older ages. Whereas the two other models have a similar percentage error variations. Fig.5 summarises for the all periods the fit result using the Mean Absolute Relative Error (MARE), we note that the percentage error is low for all models below the age of 80 , whereas after the age of 80 we notice that only the 3-factor independent model is able to capture the whole survival curve variation this indicates that one of the factors is able to fit the population .

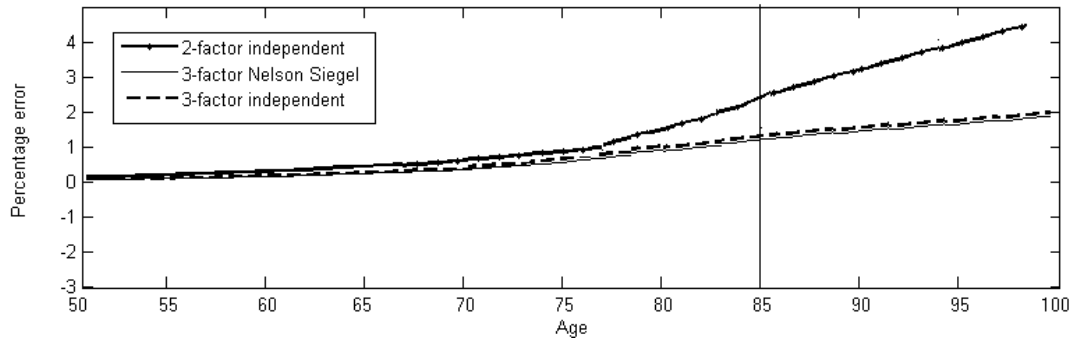


Figure 3: Fit error year 1950

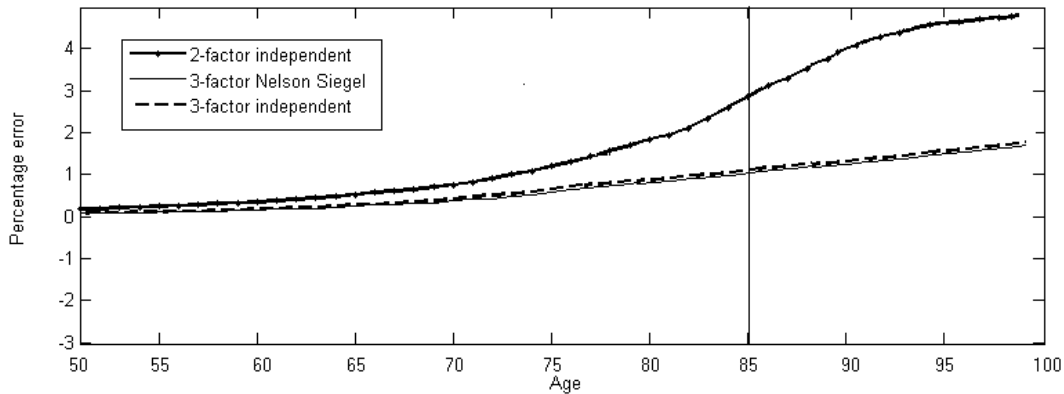


Figure 4: Fit error year 2000

### 6.2.1 Estimated latent factors and factor loadings :

Figure 6 and 7 display respectively the latent factors and the factor loadings for the 2-factor independent model. We note that  $B_1$  the first factor loadings have with age an increasing exponential shape and we present through it the general trend of mortality.  $Z_1$  which is the corresponding factor indicates a heavily improving in mortality rates between 1930 and 1950 with a fairly variation between years. After 1950 we see a little change in the factors trend and the mortality rates are increasing but with a lower volatilities between the years. The second latent factors,  $Z_2$  shows a slowly decreasing trend, and after 1950 the volatility between years became lower, its corresponding factor loadings  $B_2$  has a declining shape which prove the sensitivity to the factor only for the younger ages.

Figure 8 and 9 present respectively the latent factor and the factor loadings for the 3-factor independent model, the first factor loadings have the same shape as the 2-factor independent model, the first latent factor shows a little improving in the mortality rate between 1930 and 1950, since 1950 we note that this trend has changed and the mortality

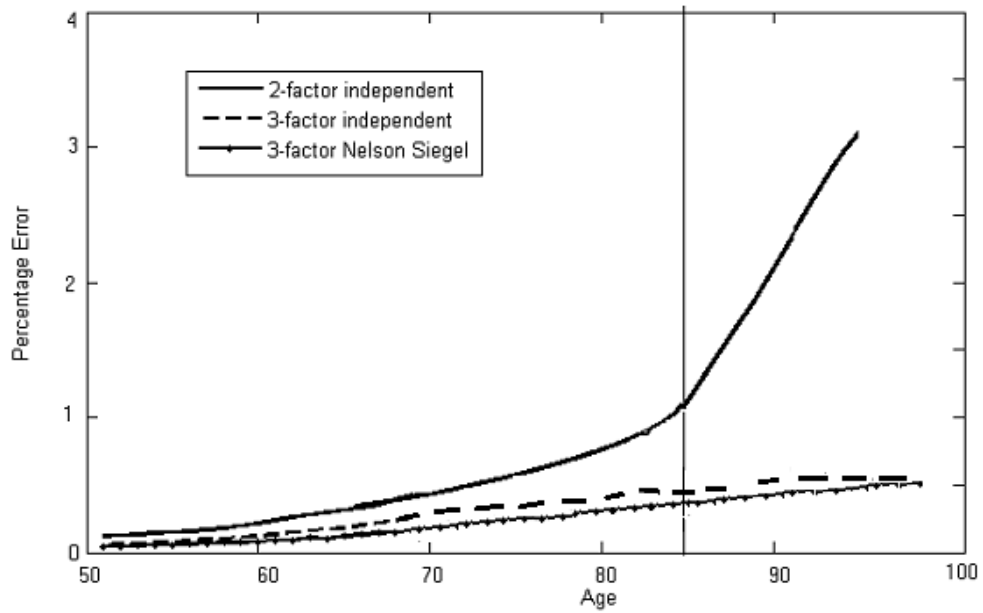


Figure 5: 2-and3-factor survival curve MARE

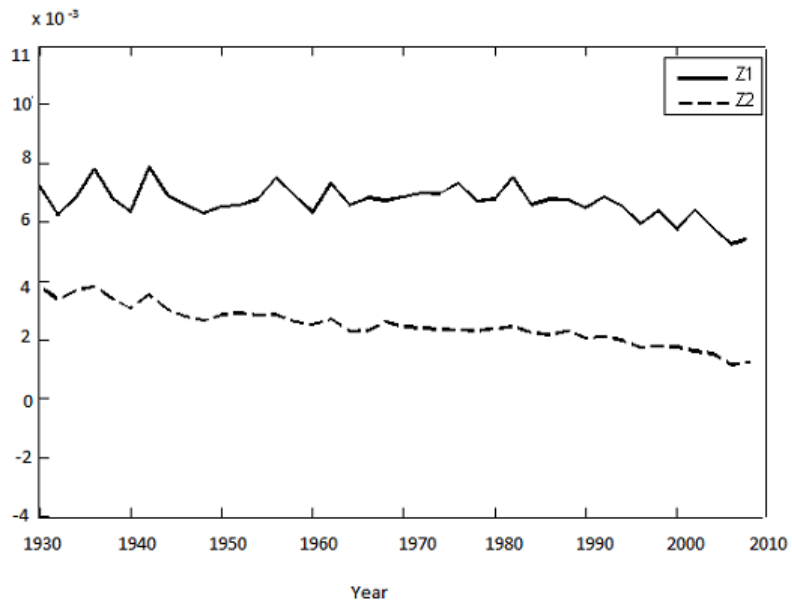


Figure 6: Latent Factor of the 2-factor independent model

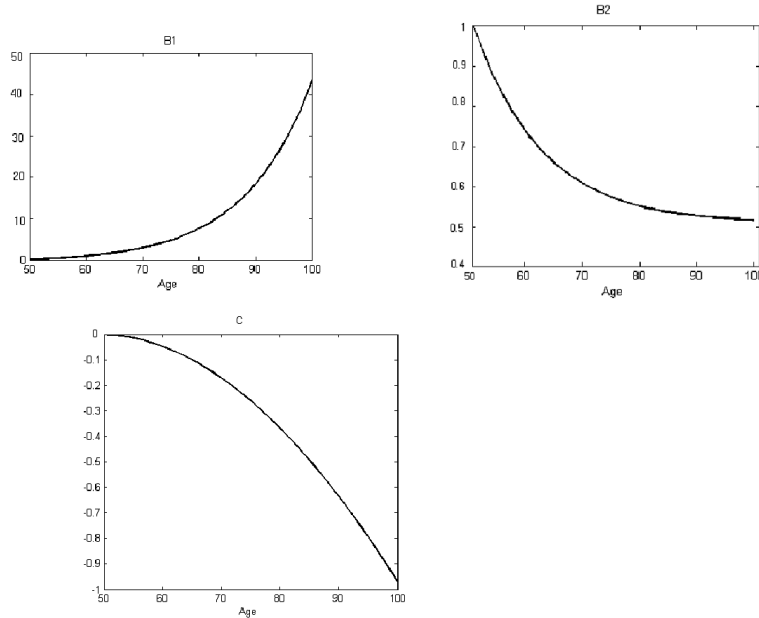


Figure 7: Factor loadings of the 2-factor independent model

rates are decreasing with little volatility between years. The second factor  $Z_2$  shows a slowly increasing trend between 1930 and 1950 and it decreases after 1950, its corresponding factor loadings given by  $B_2$  is declining sloping which shows a greater sensitivity to the factor and indicates that the older age are more sensitive to the factor than the 2-factor model.

We can see that the additional factor  $Z_3$  and the factor loadings  $B_3$  have the ability to capture the mortality trend. We note that the factor  $B_3$  is significantly larger than the first factor  $B_1$  especially in the older ages. The latent factor  $Z_3$  can capture some of the old age volatility and have a slowly increasing trend after 1950, this indicates that the mortality rates of the elderly persons is improving but with heavily trend because of the structural changes in the USA population after 1950.

The Nelson Siegel model is performed in similar way to the consistent models, but the first factor loadings  $B_1$  which is constant forces the latent factor  $Z_1$  to weight all the age in the same way, as a result the Nelson Siegel model loses the flexibility and can not give a good fit in the older ages.

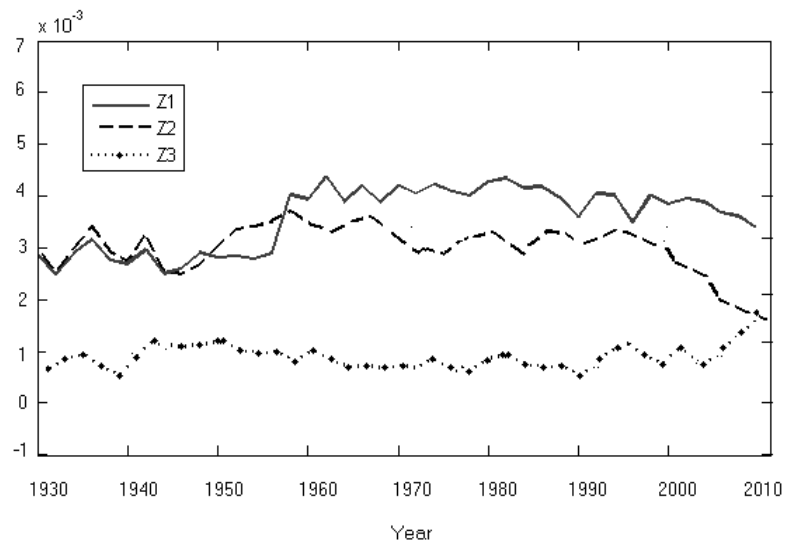


Figure 8: The Latent Factor of the 3-factor independent model

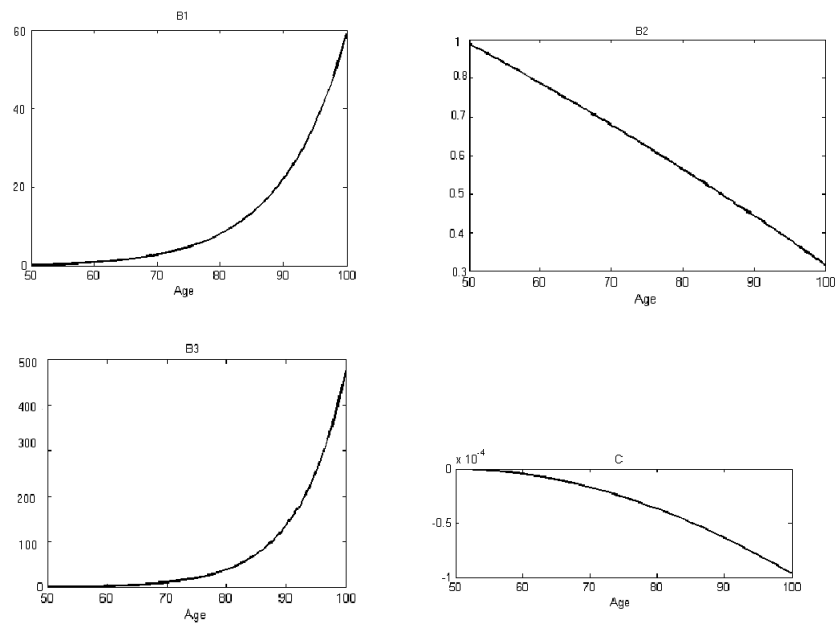
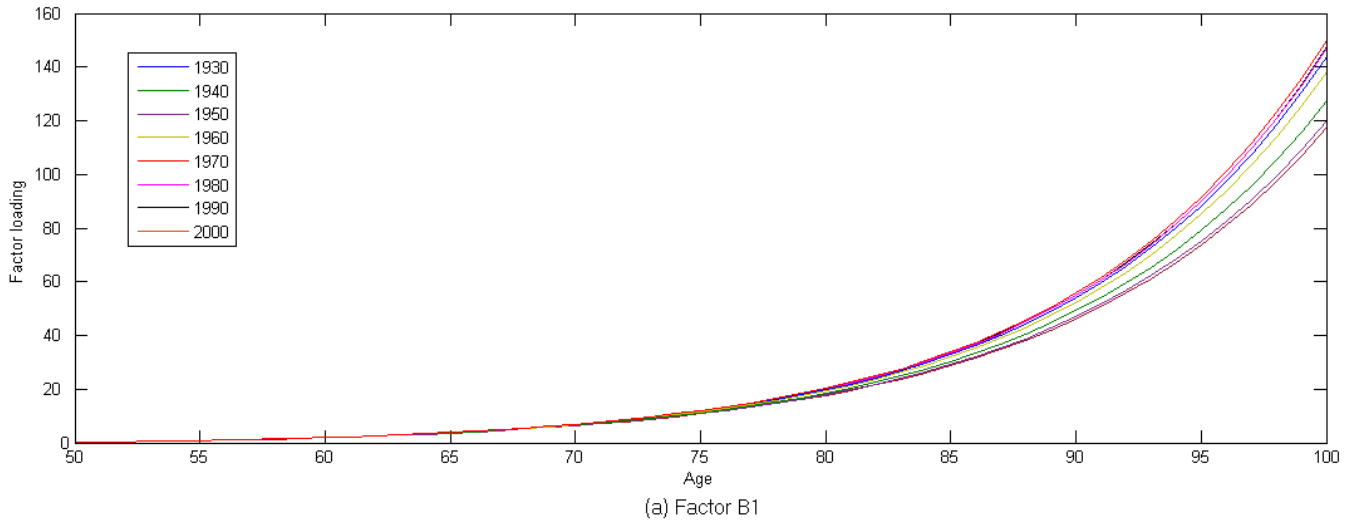
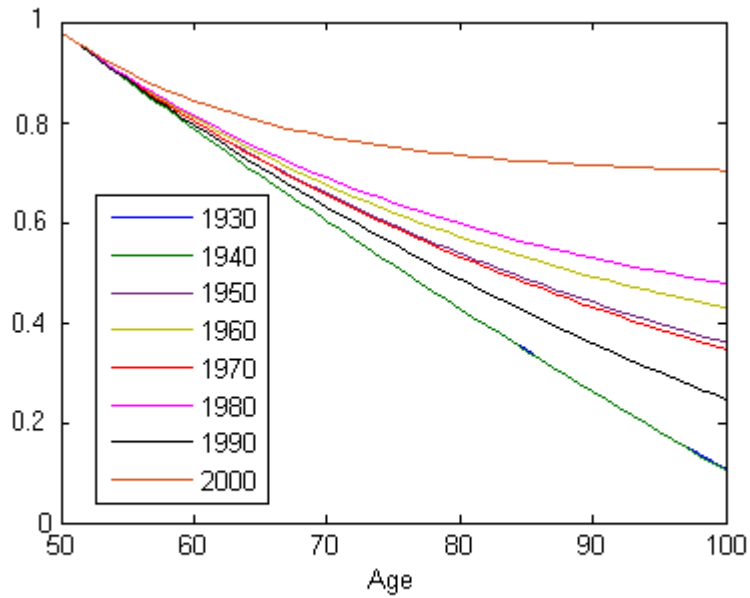


Figure 9: Factor loadings of the 3-factor independent model

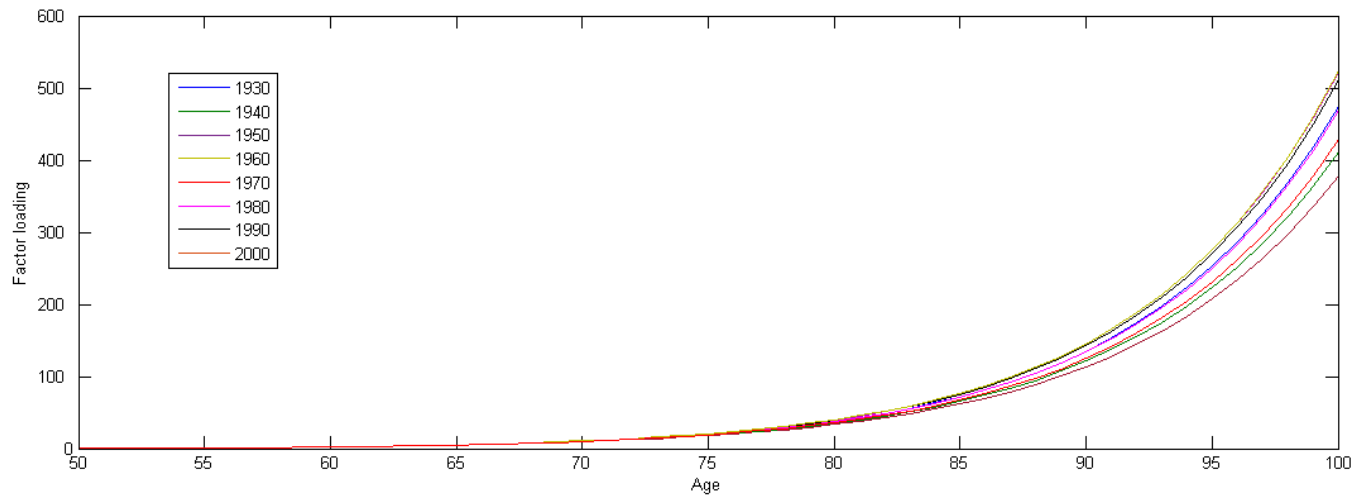
### 6.2.2 Robustness of the model:

Fig.10 prove the robustness of both the estimated factor loadings and latent factors to starting year in our data set. We re-run the 3-factor independent model many times with the different starting year from 1930 to 2000 .Every 10 year we can see a similar results for all the factors. All The latent factors indicate an identical shape for all the starting year. The factor loadings estimated  $B_1$  and  $B_3$  show a stable shape for all the starting year, whereas  $B_2$  indicates some deviation between years, and this prove the robustness of our consistent model.

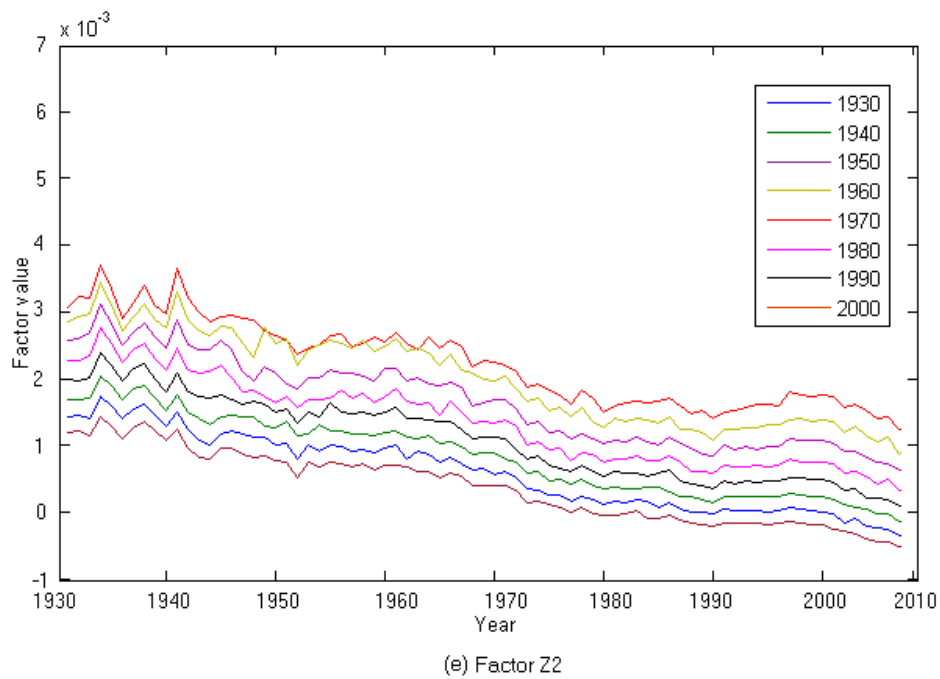
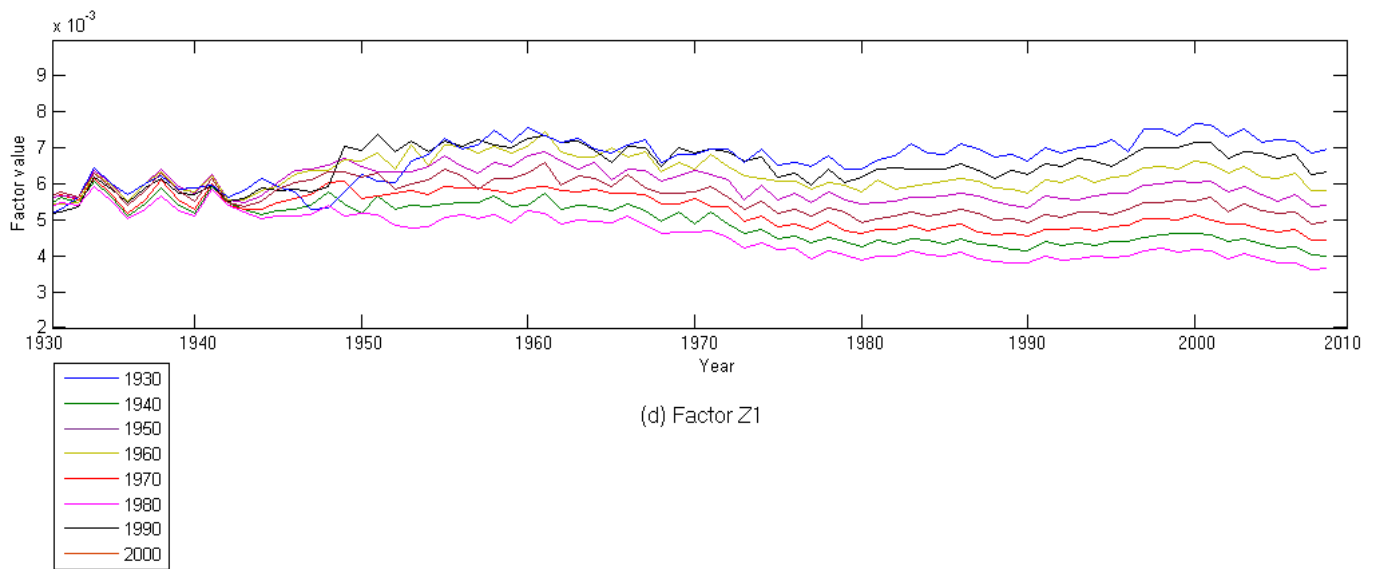




(b) Factor B2



(c) Factor B3



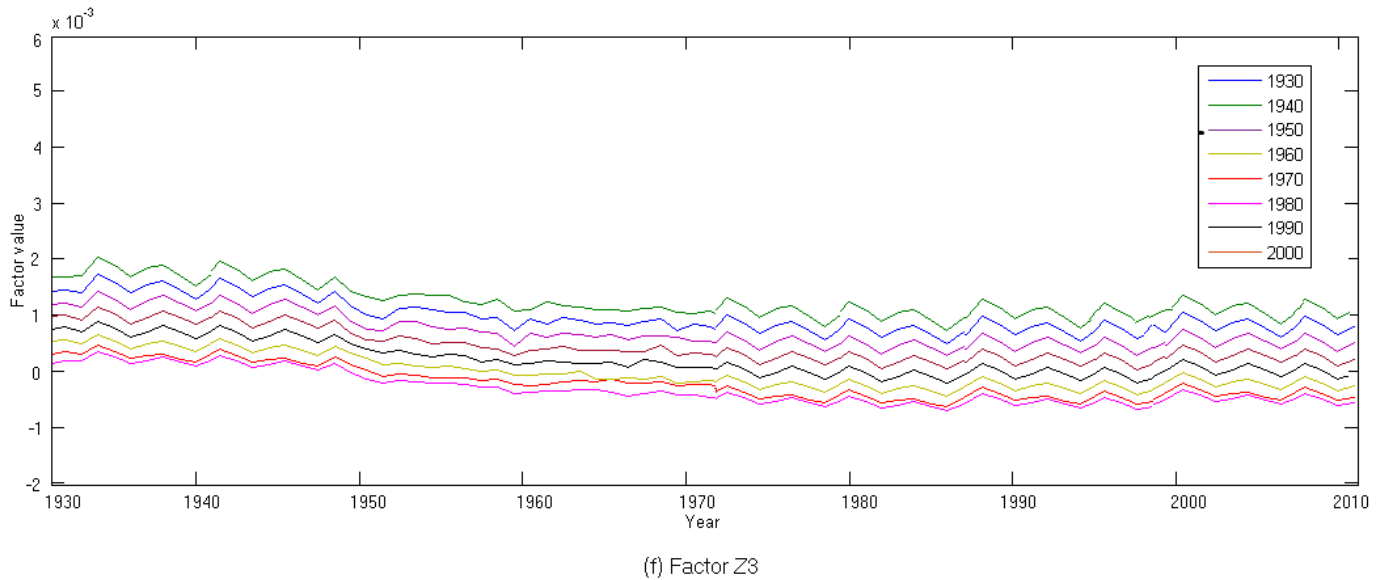


Figure 10: Robustness test:

## 7 Conclusion:

Over the last decades, the human mortality has globally declined and a rising in the life expectancy has occurred. The mortality improvements presents a big challenge especially for the public retirement systems and also for the life annuities business. When we talk about the long term living benefit, we need an adapted mortality projection to calculate the expected present values using in reserving or in pricing in order to avert the underestimation of the future costs. Therefore the actuary have to employ the life tables or the Project tables containing a future forecasting of the mortality trend. Thus, a new Risk appear, is the Longevity Risk.

In our article, we tried to propose a model who can be used in the longevity risk applications. We prove that the affine term structure model (ATSM) which is used in interest rates applications, is a suitable tool to model the mortality data.

To our Knowledge this is the first work that uses the Consistent dynamics affine mortality models to model USA Longevity risk, indeed we show a closed form representation for the risk adjusted survival curve, we give an estimation for the model parameters according to an essentially affine transformation using the kalman filter and we presented a 2-and 3-factor models in a consistent framework.

The aim of this paper is to develop a consistent forecasts that help in pricing and used in risk management applications. We use the USA population between 50 and 100 over 1933 and 2010 to estimate and establish a comparison between models.

Our work shows that the 3-factor independent model is robust during the starting year, and

the mean absolute relative error (MARE) for the fitted survival curves are low for all the age range . We prove that the 3-factor independent model is performed better than the other models and we give an appropriate approach which can be used in the longevity risk application through the use of a consistent framework . The ATSM models is able to be used not only for evaluating capital requirements for risk management but also in pricing and hedging longevity risk for pension funds and also for insurers.

## References

- [1] **Andersen, T., Lund, J., 1997** ,Estimating continuous-time stochastic volatility models of the short-term interest rate. *Journal of Econometrics* 77,343 377
- [2] **Andrew J.G. Cairns 2011**,Modelling and management of longevity risk: Approximations to survivor functions and dynamic hedging.*Insurance: Mathematics and Economics* 49, Issue 3, 438 453.
- [3] **Atsuyuki Kogurea, Yoshiyuki Kurachib 2010**,A Bayesian approach to pricing longevity risk based on risk-neutral predictive distributions.*Insurance: Mathematics and Economics* V 46,162 172.
- [4] **Babbs, S.H., Nowman, K.B., 1999** ,Kalman filtering of generalized vasicek term structure models. *The Journal of Financial and Quantitative Analysis* 34, 115.
- [5] **Barbarin, J., 2008.** , HeathJarrowMorton modelling of longevity bonds and the risk minimization of life insurance portfolios. *Insurance: Mathematics and Economics* 43,41 55.
- [6] **Bauer, D., Börger, M., Ruß, J., 2010.** ,On the pricing of longevity-linked securities. *Insurance: Mathematics and Economics* 46,139 149.
- [7] **Bielecki, T., Rutkowski, M., 2002.** ,*Credit Risk: Modelling, Valuation and Hedging.* Springer-Verlag.
- [8] **Biffis, E., 2005.** ,Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics and Economics* 37,443 468.
- [9] **Bjork, T., Christensen, B.J., 1999** , Interest rate dynamics and consistent forward rate curves. *Mathematical Finance* 9,323 348.
- [10] **Bai, J. Perron, P., (2003)** ,Computation and analysis of multiple structural change models ,*Journal of Applied Econometrics* 18, 1- 22.
- [11] **Bingzheng Chena, , Lihong Zhangb , Lin Zhaoa 2010**,On the robustness of longevity risk pricing.*Insurance: Mathematics and Economics* Volume 47, Issue 3.358 373.
- [12] **Blake, David, Andrew J.G. Cairns, and Kevin Dowd 2006.**,*Living with Mortality: Longevity Bonds and Other Mortality-Linked Securities.* *British Actuarial Journal*12:153- 228.
- [13] **Blake, David and William Burrows. 2001.**,*Survivor Bonds: Helping to Hedge Mortality Risk.* *Journal of Risk and Insurance*68: 339- 348.
- [14] **Brown, JeffreyR.,and Peter Orszag .2006**,*The Political Economy of Government Issued Survivor Bonds .**Journal of Risk and Insurance* 73: 611- 31.

- [15] **Cavanaugh, J.E., Shumway, R.H., 1997** ,A bootstrap variant of AIC for state-space model selection. *Statistica Sinica* 7, 473 496.
- [16] **Chiarella, C., Kwon, O.K., 2001.**,Classes of interest rate models under the HJM framework. *Asia-Pacific Financial Markets* 8, 1 22.
- [17] **Christensen, J.H.E., Diebold, F.X., Rudebusch, G.D., 2009.** ,An arbitrage-free generalized NelsonSiegel term structure model. *Econometrics Journal* 12, C33 C64.
- [18] **Dowd, Kevin. 2003.** ,Survivor Bonds: A Comment on Blake and Burrows. *Journal of Risk and Insurance* 70:339- 348.
- [19] **Cairns, Andrew J.G., David Blake, and Kevin Dowd. 2006.**,A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration. *Journal of Risk and Insurance*73(4):687- 718.
- [20] **Dahl, M., Moller, T., 2006.**, Valuation and hedging of life insurance liabilities with systematic mortality risk. *Insurance: Mathematics and Economics* 39, 193 217.
- [21] **Dai, Q., Singleton, K., 2000.**, Specification analysis of affine term structure models. *The Journal of Finance* 55,1943 1978.
- [22] **De Rossi, G., 2004.**, Kalman filtering of consistent forward rate curves: a tool to estimate and model dynamically the term structure. *Journal of Empirical Finance* 11,277 308.
- [23] **Duffee, G.R., 2002**, Term premia and interest rate forecasts in affine models. *The Journal of Finance* 57,405 443.
- [24] **Duffie, D., Kan, R., 1996.**,A yield-factor model of interest rates. *Mathematical Finance* 6,379406.
- [25] **G. Coughlan, M. Khalaf-Allah, Y. Ye, S. Kumar, A. Cairns, D. Blake, and K. Dowd.2011.** Longevity hedging 101: A framework for longevity basis risk analysis and hedge effectiveness. *North American Actuarial Journal*,15:150 176.
- [26] **Horneff, W., Maurer, R., Mitchell, O., Stamos, M., 2009.** Asset allocation and location over the life cycle with survival contingent payouts.*Journal of Banking and Finance* 33,1688 1699.
- [27] **Hua Chen, J. David Cummins 2010**,Longevity bond premiums: The extreme value approach and risk cubic pricing.*Insurance: Mathematics and Economics* 46.150 161.
- [28] **Jo ao F Cocco,Francisco J. Gomes 2011.**,Longevity risk, retirement savings, and financial innovation.*Journal of Financial Economics* 103 (2012)507 529
- [29] **Johnny Siu-Hang Li 2010**,Pricing longevity risk with the parametric bootstrap: A maximum entropy approach.*Insurance: Mathematics and Economics* 47, Issue 2,176 186.
- [30] **Kalman, R., 1960**,A new approach to linear filtering and prediction problems. *Journal of Basic Engineering* 82,35 45.
- [31] **Lando, D., 1998.**, On cox processes and credit risky securities. *Review of Derivatives Research* 2,99 120.
- [32] **Lee, R., Carter, L., 1992.**, Modeling and forecasting US mortality. *Journal of the American Statistical Association* 87,659 671.

- [33] **Luciano, E., Vigna, E., 2005.**, Non mean reverting affine processes for stochastic mortality. International Centre for Economic Research, 4.2005.
- [34] **M. Borger 2010.** Deterministic shock vs. stochastic value-at-risk an analysis of the solvency II standard model approach to longevity risk. Blatter der DGVFM, 31:225 259.
- [35] **Michel Denuit 2009.**An index for longevity risk transfer.Journal of Computational and Applied Mathematics Volume 230, Issue 2,411 417.
- [36] **Milevsky, M., Promislow, S.D., 2001.**, Mortality derivatives and the option to annuitise. Insurance: Mathematics and Economics 29,299 318.
- [37] **Miltersen, K.R., Persson, S.-A, 2005.**, Is mortality dead Stochastic forward force of mortality rate determined by no arbitrage. Working Paper, University of Bergen.
- [38] **Norbert Hari, Anja De Waegenaere, Bertrand Melenberg, and Theo E. Nijman.,** Longevity risk in portfolios of pension annuities. Insurance: Mathematics and Economics, 42(2):505 519, 2008.
- [39] **Ralph Stevens Anja , De Waegenaere , Joost Driessen , Theo Nijman , Kim Peijnenburg 2009,** Annuity Decisions with Systematic Longevity Risk.The Journal of Risk and Insurance.
- [40] **Schonbucher, P.J., 1998.**, Term structure modelling of defaultable bonds. Review of Derivatives Research 2,161 192.
- [41] **Schrager, D., 2006.**, Affine stochastic mortality. Insurance: Mathematics and Economics 38,81 97.
- [42] **Sharon S. Yanga , Jack C. Yueb, Hong-Chih Huangc 2010,**Modeling longevity risks using a principal component approach: A comparison with existing stochastic mortality models.Insurance: Mathematics and Economics 46, Issue 1,254 270.
- [43] **Stoffer, D.S., Wall, K.D., 2004.**,Resampling in state space models. In: State Space and Unobserved Component Models Theory and Applications.227 258.
- [44] **Stéphane LOISEL,Daniel Serant 2007,**In the core of longevity risk: hidden dependence in stochastic mortality models and cut-offs in prices of longevity swaps,Journal of Economic Behavior and Organization 86.200 220.
- [45] **Søren Fiig Jarner and Thomas Møller 2013,**A partial internal model for longevity risk.Scandinavian Actuarial Journal Volume 2015, Issue 4.
- [46] **Tzuling Lin, , Larry Y. Tzeng 2010,**An additive stochastic model of mortality rates: An application to longevity risk in reserve evaluation.Insurance: Mathematics and Economics 46, Issue 2,423 435.