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# The existence of perfect codes in a family of generalized Fibonacci cubes

Michel Mollard\*

## Abstract

The *Fibonacci cube* of dimension  $n$ , denoted as  $\Gamma_n$ , is the subgraph of the  $n$ -cube  $Q_n$  induced by vertices with no consecutive 1's. In an article of 2016 Ashrafi and his co-authors proved the non-existence of perfect codes in  $\Gamma_n$  for  $n \geq 4$ . As an open problem the authors suggest to consider the existence of perfect codes in generalization of Fibonacci cubes. The most direct generalization is the family  $\Gamma_n(1^s)$  of subgraphs induced by strings without  $1^s$  as a substring where  $s \geq 2$  is a given integer. We prove the existence of a perfect code in  $\Gamma_n(1^s)$  for  $n = 2^p - 1$  and  $s \geq 3 \cdot 2^{p-2}$  for any integer  $p \geq 2$ .

**Keywords:** Error correcting codes, perfect code, Fibonacci cube.

**AMS Subj. Class. :** 94B5,0C69

## 1 Introduction and notations

Let  $G$  be a connected graph. The *open neighbourhood* of a vertex  $u$  is  $N(u)$  the set of vertices adjacent to  $u$ . The *closed neighbourhood* of  $u$  is  $N[u] = N(u) \cup \{u\}$ . The *distance* between two vertices noted  $d_G(x, y)$ , or  $d(x, y)$  when the graph is unambiguous, is the length of the shortest path between  $x$  and  $y$ . We have thus  $N[u] = \{v \in V(G); d(u, v) \leq 1\}$ .

A *dominating set*  $D$  of  $G$  is a set of vertices such that every vertex of  $G$  belongs to the closed neighbourhood of at least one vertex of  $D$ . In [2], Biggs initiated the study of perfect codes in graphs a generalization of classical 1-error perfect correcting codes. A *code*  $C$  in  $G$  is a set of vertices  $C$  such that for all pair of distinct vertices  $c, c'$  of  $C$  we have  $N[c] \cap N[c'] = \emptyset$  or equivalently such that  $d_G(c, c') \geq 3$ .

A *perfect code* of a graph  $G$  is both a dominating set and a code. It is thus a set of vertices  $C$  such that every vertex of  $G$  belongs to the closed neighbourhood of exactly one vertex of  $C$ . A perfect code is some time called an efficient dominating set. The existence or non-existence of perfect codes have been considered for many graphs. See the introduction of [1] for some references.

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The vertex set of the  $n$ -cube  $Q_n$  is the set  $\mathbb{B}_n$  of binary strings of length  $n$ , two vertices being adjacent if they differ in precisely one position. Classical 1-error correcting codes and perfect codes are codes and perfect codes in the graph  $Q_n$ . The concatenation of strings  $\mathbf{x}$  and  $\mathbf{y}$  is noted  $\mathbf{x}||\mathbf{y}$  or just  $\mathbf{xy}$  when there is no ambiguity. A string  $\mathbf{f}$  is a *substring* of a string  $\mathbf{s}$  if there exist strings  $\mathbf{x}$  and  $\mathbf{y}$ , may be empty, such that  $\mathbf{s} = \mathbf{xfy}$ .

A *Fibonacci string* of length  $n$  is a binary string  $\mathbf{b} = b_1 \dots b_n$  with  $b_i \cdot b_{i+1} = 0$  for  $1 \leq i < n$ . In other words a Fibonacci string is a binary string without 11 as substring. The *Fibonacci cube*  $\Gamma_n$  ( $n \geq 1$ ) is the subgraph of  $Q_n$  induced by the Fibonacci strings of length  $n$ . Fibonacci cubes were introduced as a model for interconnection networks [3] and received a lot of attention afterwards. These graphs also found an application in theoretical chemistry. See the survey [4] for more results and applications about Fibonacci cubes.

The sets  $\{00\}$  and  $\{010, 101\}$  are perfect codes in respectively  $\Gamma_2$  and  $\Gamma_3$ . In a recent paper [1] Ashrafi and his co-authors proved the non-existence of perfect codes in  $\Gamma_n$  for  $n \geq 4$ . As an open problem the authors suggest to consider the existence of perfect codes in generalization of Fibonacci cubes. The most complete generalization proposed in [5] is, for a given string  $\mathbf{f}$ , to consider  $\Gamma_n(\mathbf{f})$  the subgraph of  $Q_n$  induced by strings that do not contain  $\mathbf{f}$  as substring. Since Fibonacci cubes are  $\Gamma_n(11)$  the most immediate generalization [6, 7] is to consider  $\Gamma_n(1^s)$  for a given integer  $s$ . We will prove the existence of perfect codes in  $\Gamma_n(1^s)$  for an infinite family of parameters  $(n, s)$ .

It will be convenient to consider the binary strings of length  $n$  as vectors of  $\mathbb{F}^n$  the vector space of dimension  $n$  over the field  $F = \mathbb{Z}_2$  thus to associate to a string  $x_1x_2 \dots x_n$  the vector  $\theta(x_1x_2 \dots x_n) = (x_1, x_2, \dots, x_n)$ . The *Hamming distance* between two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ ,  $d(\mathbf{x}, \mathbf{y})$  is the number of coordinates in which they differ. The *parity function* is the function from  $\mathbb{F}^n$  to  $\mathbb{Z}_2$  defined by  $\pi(\mathbf{x}) = \pi(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$ . By the correspondence  $\theta$  we can define the sum  $\mathbf{x} + \mathbf{y}$ , the Hamming distance  $d(\mathbf{x}, \mathbf{y})$  and the parity  $\pi(\mathbf{x})$  of strings in  $\mathbb{B}_n$ . Note that Hamming distance is the usual graph distance in  $Q_n$ . The complement of a string  $\mathbf{x} \in \mathbb{B}_n$  is the string  $\bar{\mathbf{x}} = \mathbf{x} + \mathbf{1}^n$ .

We will first recall some basic results about perfect codes in  $Q_n$ . Since  $Q_n$  is a regular graph of degree  $n$  the existence of a perfect code of cardinality  $|C|$  implies  $|C|(n+1) = 2^n$  thus a necessary condition of existence is that  $n+1$  is a power of 2 thus that  $n = 2^p - 1$  for some integer  $p$ .

For any integer  $p$  Hamming [8] constructed, a linear subspace of  $\mathbb{F}^{2^p-1}$  which is a perfect code. It is easy to prove that all linear perfect codes are Hamming codes. In 1961 Vasilev [9], and later many authors, see [10, 11] for a survey, constructed perfect codes which are not linear codes. Let us recall Vasilev's construction of perfect codes.

**Theorem 1.1** [9] *Let  $C_r$  be a perfect code of  $Q_r$ . Let  $f$  be a function from  $C_r$  to  $\mathbb{Z}_2$  and  $\pi$  be the parity function. Then the set  $C_{2r+1} = \{\mathbf{x}||\pi(\mathbf{x}) + f(\mathbf{c})||\mathbf{x} + \mathbf{c}; \mathbf{x} \in \mathbb{B}_r, \mathbf{c} \in C_r\}$*

is a perfect code of  $Q_{2r+1}$

We recall also the proof of Theorem 1.1 in such a way our article will be self contained.

**Proof.** First notice that  $|C_{2r+1}| = 2^r |C_r| = 2^r \frac{2^r}{r+1} = \frac{2^{2r+1}}{2r+2}$ . Thus it is sufficient to prove that the distance between two different elements of  $C_{2r+1}$  is at least 3.

Consider  $d(\mathbf{x} || \pi(\mathbf{x}) + f(\mathbf{c}) || \mathbf{x} + \mathbf{c}, \mathbf{x}' || \pi(\mathbf{x}') + f(\mathbf{c}') || \mathbf{x}' + \mathbf{c}') = d_1 + d_2 + d_3$  where  $d_1 = d(\mathbf{x}, \mathbf{x}')$ ,  $d_2 = d(\pi(\mathbf{x}) + f(\mathbf{c}), \pi(\mathbf{x}') + f(\mathbf{c}'))$  and  $d_3 = d(\mathbf{x} + \mathbf{c}, \mathbf{x}' + \mathbf{c}')$ .

If  $d_1 = 0$  then  $\mathbf{x} = \mathbf{x}'$  thus  $d_3 = d(\mathbf{c}, \mathbf{c}') \geq 3$ .

If  $d_1 = 1$  and  $\mathbf{c} = \mathbf{c}'$  then  $d_2 = d_3 = 1$

If  $d_1 = 1$  and  $\mathbf{c} \neq \mathbf{c}'$  then  $d_3 \geq 2$  otherwise  $d(\mathbf{c}, \mathbf{c}') \leq 2$

If  $d_1 = 2$  then  $d_3 \neq 0$  otherwise  $d(\mathbf{c}, \mathbf{c}') = 2$

Thus  $d = d_1 + d_2 + d_3 \geq 3$ . □

If  $f(\mathbf{c}) = 0$  for any  $\mathbf{c} \in C_r$  we obtain the classical inductive construction of Hamming codes with  $C_1 = \{0\}$  as basis.

In the next section we will use this construction starting from the Hamming code in  $Q_r$  as  $C_r$  and a function  $f$  chosen in such a way that the strings of the constructed code  $C_{2r+1}$  has not a too big number of consecutive 1's.

## 2 Main Result

**Lemma 2.1** *Let  $m$  be an integer. Let  $A_0$  be the set of strings  $A_0 = \{0^{m+1}\mathbf{y}; \mathbf{y} \in \mathbb{B}_m\}$ . For  $i \in \{1, \dots, m\}$  let  $A_i = \{z10^{m+1}\mathbf{y}; z \in \mathbb{B}_{i-1}, \mathbf{y} \in \mathbb{B}_{m-i}\}$ . Then the sets  $A_i$  are disjoint and any string of  $B_{2m+1}$  containing  $0^{m+1}$  as substring belongs to a  $A_i$ .*

**Proof.** Let  $\mathbf{x}$  be a string of  $B_{2m+1}$  containing  $0^{m+1}$  as substring and  $i$  be the minimum integer such that  $x_{i+1}x_{i+2}\dots x_{i+m+1} = 0^{m+1}$ . Then  $i = 0$ , and  $\mathbf{x}$  belongs to  $A_0$ , or  $m \geq i \geq 1$ . In this case  $x_i = 1$  thus  $\mathbf{x} \in A_i$ . Assume  $\mathbf{x} \in A_i \cap A_j$  with  $m \geq j > i \geq 0$  then  $x_j = 1$  thus  $j \geq i + m + 2 > m$  a contradiction.

**Theorem 2.2** *Let  $n = 2^p - 1$  where  $p \geq 2$  and let  $s = 3 \cdot 2^{p-2}$ . There exists a perfect code  $C$  in  $Q_n$  such that no elements of  $C$  contains  $1^s$  as substring.*

**Proof.** Let  $m = 2^{p-2} - 1$  thus  $2m + 1 = 2^{p-1} - 1$  and  $s = 3m + 3$ . Let  $C_{2m+1}$  be a perfect code in  $Q_{2m+1}$ . Let  $f$  be the function from  $\mathbb{B}_{2m+1}$  to  $\mathbb{Z}_2$  defined by

- $f(0^{m+1}\mathbf{y}) = 1$  for  $\mathbf{y} \in \mathbb{B}_m$
- $f(10^{m+1}\mathbf{y}) = 0$  for  $\mathbf{y} \in \mathbb{B}_{m-1}$
- $f(z10^{m+1}\mathbf{y}) = \pi(z)$  for  $z \in \mathbb{B}_{i-1}$  and  $\mathbf{y} \in \mathbb{B}_{m-i}$  for  $i = 2$  to  $m$ .
- $f = 0$  otherwise.

Note that from the previous lemma the function is well defined. Let  $C$  be the perfect code obtained from Vasilev's construction from  $C_{2m+1}$  and  $f$ . Assume there exists a string  $\mathbf{d}$  in  $C$  with  $1^{3m+3}$  as substring. Therefore  $\mathbf{d}$  is obtained from  $\mathbf{x} = d_1 d_2 \dots d_{2m+1}$  and  $\mathbf{c} \in C_{2m+1}$ . Since  $n = 4m + 3$  note first that  $d_{m+1} d_{m+2} \dots d_{3m+3} = 1^{2m+2}$ . Let  $i$  be the minimum integer such that  $d_i d_{i+1} \dots d_{3m+i+2} = 1^{3m+3}$ . We consider 3 cases

- $i = 1$  then  $x = d_1 d_2 \dots d_{2m+1} = 1^{2m+1}$  and  $d_{2m+2} d_{2m+3} \dots d_{3m+3} = 1^{m+2}$ . Since  $\mathbf{c} + \mathbf{x} = 1^{m+1} d_{3m+4} d_{3m+5} \dots d_{4m+3}$  we have  $\mathbf{c} = 0^{m+1} \mathbf{y}$  for some  $\mathbf{y} \in \mathbb{B}_m$ . Thus  $f(\mathbf{c}) = 1$  and since  $\pi(\mathbf{x}) = 1$  we obtain  $d_{2m+2} = f(\mathbf{c}) + \pi(\mathbf{x}) = 0$  a contradiction.
- $i = 2$  then  $\mathbf{x} = 01^{2m}$  and  $d_{2m+2} d_{2m+3} \dots d_{3m+4} = 1^{m+3}$ . Since  $\mathbf{c} + \mathbf{x} = 1^{m+2} d_{3m+5} d_{3m+6} \dots d_{4m+3}$  we have  $\mathbf{c} = 10^{m+1} \mathbf{y}$  for some  $\mathbf{y} \in \mathbb{B}_{m-1}$ . Thus  $f(\mathbf{c}) = 0$  and since  $\pi(\mathbf{x}) = 0$  we obtain  $d_{2m+2} = f(\mathbf{c}) + \pi(\mathbf{x}) = 0$  a contradiction.
- $i \geq 3$  then  $\mathbf{x} = \mathbf{z} 01^{2m-i+2}$  for  $\mathbf{z} \in \mathbb{B}_{i-2}$  and  $d_{2m+2} d_{2m+3} \dots d_{3m+i+2} = 1^{m+i+1}$ . Since  $\mathbf{c} + \mathbf{x} = 1^{m+i} d_{3m+i+3} d_{3m+i+4} \dots d_{4m+3}$  we have  $\mathbf{c} = \bar{\mathbf{z}} 10^{m+1} \mathbf{y}$  for some  $\mathbf{y} \in \mathbb{B}_{m-i+1}$ . Thus  $f(\mathbf{c}) = \pi(\bar{\mathbf{z}})$ . Since  $\pi(\mathbf{x}) = \pi(\mathbf{z}) + \pi(1^{2m-i+2})$  and  $\pi(\bar{\mathbf{z}}) + \pi(\mathbf{z}) = \pi(1^{i-2})$  we obtain  $d_{2m+2} = f(\mathbf{c}) + \pi(\mathbf{x}) = \pi(1^{2m}) = 0$  a contradiction.

Therefore there exists no string  $d$  in  $C$  with  $1^{3m+3}$  as substring. □

**Corollary 2.3** *Let  $n = 2^p - 1$  where  $p \geq 2$  and let  $s \geq 3 \cdot 2^{p-2}$ . There exists a perfect code in  $\Gamma_n(1^s)$ .*

**Proof.** Indeed let  $C$  be a perfect code in  $Q_n$  such that no element of  $C$  contains  $1^{3 \cdot 2^{p-2}}$  as substring. The strings of  $C$  are in  $V(\Gamma_n(1^s))$ . Let  $x$  be a vertex of  $V(\Gamma_n(1^s))$ . If  $x \notin C$  then  $x$  is adjacent in  $Q_n$  to a vertex  $c$  in  $C$ . Note that  $x$  and  $c$  are also adjacent in  $\Gamma_n(1^s)$  thus  $C$  is a dominating set of  $\Gamma_n(1^s)$ . If  $c$  and  $c'$  are two strings of  $C$  then  $d_{\Gamma_n(1^s)}(c, c') \geq d_{Q_n}(c, c') \geq 3$ . Therefore  $C$  is a perfect code in  $\Gamma_n(1^s)$ .

### 3 Concluding remark and open problems

Whenever  $n = 2^p - 1$  it will be interesting to determine the minimum  $s$  such that there exists a perfect code in  $\Gamma_n(1^s)$ .

Corollary 2.3 is not always the best result possible. For example for  $n = 7$  the code  $C_7$  obtained in Vasilev's construction starting from  $C_3 = \{000, 111\}$  with  $f(000) = f(111) = 1$  is a perfect code in  $\Gamma_n(1^5)$ . Indeed

- $11111ab$  or  $0011111$  cannot be in  $C_7$  since the  $P(111) + 1 = P(001) + 1 = 0$
- $011111a$  cannot be in  $C_7$  since the possible codewords beginning with  $011$  are  $0111011$  and  $0111100$ .

Note that that all strings of this code are obtained from strings in the Hamming code of length 7 by a translation of 0001000. This simple idea can be generalized but is less efficient than our result in the general case.

We propose also the following conjecture:

**Conjecture 3.1** *For  $n \geq 3$  and  $s \geq 1$  if  $C$  is a perfect code in  $\Gamma_n(1^s)$  then  $n = 2^p - 1$  for some integer  $p$  and furthermore  $C$  is a perfect code in  $Q_n$ .*

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