Simplex–Splines on the Clough–Tocher Element

Tom Lyche, Jean-Louis Merrien

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Tom Lyche [*], Jean-Louis Merrien[†]

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Abstract

We propose a simplex spline basis for a space of C^1-cubics on the Clough-Tocher split on a triangle. The 12 elements of the basis give a nonnegative partition of unity. We derive two Marsden-like identities, three quasi-interpolants with optimal approximation order and prove L_∞ stability of the basis. The conditions for C^1-junction to neighboring triangles are simple and similar to the C^1 conditions for the cubic Bernstein polynomials on a triangulation. The simplex spline basis can also be linked to the Hermite basis to solve the classical interpolation problem on the Clough-Tocher split.

Keywords: Triangle Mesh, Piecewise polynomials, Interpolation, Simplex Splines, Marsden-like Identity.

1 Introduction

Piecewise polynomials over triangles have applications in several branches of the sciences ranging from finite element analysis, surfaces in computer aided design and other engineering problems. For many of these applications, piecewise linear C^0 surfaces do not suffice. In some cases, we need smoother surfaces for modeling, or higher degrees to increase the approximation order. To obtain C^1 smoothness on an arbitrary triangulation, one needs piecewise quintic polynomials, [6]. We can use lower degrees if we are willing to split each triangle into a number of subtriangles. Examples are the Clough-Tocher split (CT), [1] and the Powell-Sabin 6 and 12-splits (PS6, PS12), [11]. The

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number of subtriangles is 3, 6 and 12 for CT, PS6 and PS12, respectively. A B-spline like basis both for C^1 cubics and C^2 quintics has been constructed for PS6, [5, 12] and references therein. Recently a B-spline like basis has also been proposed for a 9 dimensional subspace of C^1 cubics on CT, [5]. The PS12-split can be defined as the complete graph obtained by connecting vertices and edge midpoints of each triangle. A B-spline basis for PS12 and the full space of C^1- cubics on CT seem difficult. An alternative to the B-spline basis is the Hermite basis. Since it uses both values and derivatives it is not as stable as the B-spline basis and it does not form a nonnegative partition of unity.

Here we construct a B-spline basis for one triangle in the coarse triangulation and connect to neighboring triangles using Bernstein-Bézier techniques. This was done for PS12 using C^1 quadratics, [2], and C^2 and C^3 quintics, [7, 8]. These bases, consisting of simplex splines (see for example [10] for a general introduction), all share attractive properties of univariate B-splines such as

- a differentiation formula
- a stable recurrence relation
- a knot insertion formula
- they constitute a nonnegative partition of unity
- simple explicit dual functionals
- L_∞ stability
- simple conditions for C^1 and C^2 joins to neighboring triangles
- well conditioned collocation matrices for Lagrange and Hermite interpolation using certain sites.

In this paper we consider the full 12 dimensional space of C^1 cubics on the CT-split. We will define a simplex spline basis for this split and show that it has all the B-spline and Bernstein-Bézier properties mentioned above.

The CT-split is interesting for many reasons. To obtain a space of C^1 piecewise polynomials of degree at most 3 on an arbitrary triangulation, we only need to divide each triangle into 3 subtriangles, while 6 and 12 subtriangles are needed for PS6 and PS12. Moreover, the approximation order of the space $\mathbb{S}_3$ of piecewise C^1 cubics on CT is 4 and this is at least as good as for the spaces $\mathbb{S}_6$ and $\mathbb{S}_{12}$ of piecewise cubics on PS6 and piecewise quadratics on PS12. The degrees of freedom for $\mathbb{S}_6$ are values and gradients of the vertices of the coarse triangulation while for $\mathbb{S}_3$ and $\mathbb{S}_{12}$ we need in addition cross boundary derivatives at the midpoint of the edges, see Figure 1 (left). For further comparisons of these three spaces see Section 6.6 in [6].

This paper is organized as follows: In the remaining part of the introduction, we review some properties of CT, introduce our notation and recall the main properties of simplex splines. In Section 2, we construct a cubic simplex spline basis for CT, from which, in Section 3, we derive two Marsden identities and then, in Section 4, three quasi-interpolants, and show L_∞ stability of the basis. In Section 5, conditions to ensure C^0 and C^1 continuity through an edge between two triangles are derived. The conversion between the simplex spline basis and the Hermite basis for CT is considered in Section 6. Lagrange and Hermite interpolation on triangulations using C^1 cubics, quartics and higher degrees have also been considered in [3]. We end the paper with numerical examples of interpolation on a triangulation.

1.1 The Clough-Tocher split

To describe this split, let $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$ be a nondegenerate triangle in $\mathbb{R}^2$. Using the barycenter $\boldsymbol{p_T} := (\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)/3$ we can split $\mathcal{T}$ into three subtriangles $\mathcal{T}_1 := \langle \boldsymbol{p_T}, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$, $\mathcal{T}_2 := \langle \boldsymbol{p_T}, \boldsymbol{p}_3, \boldsymbol{p}_1 \rangle$ and $\mathcal{T}_3 := \langle \boldsymbol{p_T}, \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$. On $\mathcal{T}$ we consider the space

$$\mathbb{S}_3^1(\triangle) := \{f \in C^1(\mathcal{T}) : f_{|\mathcal{T}_i} \text{ is a polynomial of degree at most } 3, i = 1, 2, 3\}. \tag{1}$$

This is a linear space of dimension 12, [6]. Indeed, each element in the space can be determined uniquely by specifying values and gradients at the 3 vertices and cross boundary derivatives at the midpoint of the edges, see Figure 1, (right).

We associate the half open edges

$$\langle \boldsymbol{p}_i, \boldsymbol{p}_{\mathcal{T}}) := \{(1-t)\boldsymbol{p}_i + t\boldsymbol{p}_{\mathcal{T}} : 0 \le t < 1\}, \quad i = 1, 2, 3,$$

with subtriangles of $\mathcal{T}$ as follows

$$\langle \boldsymbol{p}_1, \boldsymbol{p_T}) \in \mathcal{T}_2, \quad \langle \boldsymbol{p}_2, \boldsymbol{p_T}) \in \mathcal{T}_3, \quad \langle \boldsymbol{p}_3, \boldsymbol{p_T}) \in \mathcal{T}_1, \tag{2}$$

and we somewhat arbitrarily assume $\boldsymbol{p_T} \in \mathcal{T}_2$.

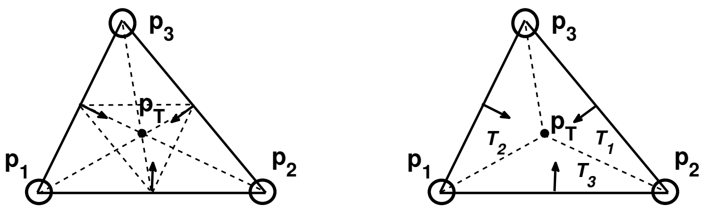


Figure 1: The PS12-split (left) and the CT-split (right). The C^1 quadratics on PS-12 and C^1 cubics on CT have the same degrees of freedom as indicated.

1.2 Notation

We let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of nonnegative integers. For a given degree $d \in \mathbb{N}_0$, the space of polynomials of total degree at most d will be denoted by $\mathbb{P}_d$. The Bernstein polynomials of degree d on $\mathcal{T}$ are given by

$$B^d_{ijk}(\boldsymbol{p}) := B^d_{ijk}(\beta_1, \beta_2, \beta_3) := \frac{d!}{i!j!k!}\beta_1^i \beta_2^j \beta_3^k, \quad i, j, k \in \mathbb{N}_0,\ i+j+k = d, \tag{3}$$

where $\boldsymbol{p} \in \mathbb{R}^2$ and $\beta_1, \beta_2, \beta_3$, given by

$$\boldsymbol{p} = \beta_1 \boldsymbol{p}_1 + \beta_2 \boldsymbol{p}_2 + \beta_3 \boldsymbol{p}_3, \quad \beta_1 + \beta_2 + \beta_3 = 1, \tag{4}$$

are the barycentric coordinates of $\boldsymbol{p}$. The set

$$\mathcal{B}_d := \{B^d_{ijk} : i, j, k \in \mathbb{N}_0,\ i + j + k = d\} \tag{5}$$

is a **partition of unity basis** for $\mathbb{P}_d$. The points

$$\boldsymbol{p}^d_{ijk} := \frac{i\boldsymbol{p}_1 + j\boldsymbol{p}_2 + k\boldsymbol{p}_3}{d},\ i, j, k \in \mathbb{N}_0,\ i + j + k = d, \tag{6}$$

are called the **domain points** of $\mathcal{B}_d$ relative to $\mathcal{T}$. In this paper, we will order the cubic Bernstein polynomials by going counterclockwise around the boundary, starting at $\boldsymbol{p}_1$ with B^3_{300} and ending with B^3_{111}, see Figure 2

$$\{B_1, B_2, \ldots, B_{10}\} := \{B^3_{300}, B^3_{210}, B^3_{120}, B^3_{030}, B^3_{021}, B^3_{012}, B^3_{003}, B^3_{102}, B^3_{201}, B^3_{111}\}. \tag{7}$$

The corresponding ordering of the cubic domain points are

$$\begin{aligned}\{\boldsymbol{p}_1^*,\ldots,\boldsymbol{p}_{10}^*\} := \Big\{&\boldsymbol{p}_1, \frac{2\boldsymbol{p}_1+\boldsymbol{p}_2}{3}, \frac{\boldsymbol{p}_1+2\boldsymbol{p}_2}{3}, \boldsymbol{p}_2, \frac{2\boldsymbol{p}_2+\boldsymbol{p}_3}{3}, \frac{\boldsymbol{p}_2+2\boldsymbol{p}_3}{3}, \\ &\boldsymbol{p}_3, \frac{2\boldsymbol{p}_3+\boldsymbol{p}_1}{3}, \frac{\boldsymbol{p}_3+2\boldsymbol{p}_1}{3}, \boldsymbol{p}_{\boldsymbol{T}}\Big\}.\end{aligned} \tag{8}$$

The partial derivatives of a bivariate function $f = f(x_1, x_2)$ are denoted $\partial_{1,0} f := \frac{\partial f}{\partial x_1}$, $\partial_{0,1} f := \frac{\partial f}{\partial x_2}$, and $\partial_{\boldsymbol{u}} f := (u_1\partial_{1,0} + u_2\partial_{0,1})f$ is the derivative in the direction $\boldsymbol{u} := (u_1, u_2)$. We denote by $\partial_{\beta_j} f$, $j = 1, 2, 3$ the partial derivatives of $f(\beta_1, \beta_2, \beta_3)$ with respect to the barycentric coordinates of f. The symbols $\langle\mathcal{S}\rangle$ and $\langle\mathcal{S}\rangle^o$ are the closed and open convex hull of a set $\mathcal{S} \in \mathbb{R}^m$. For $k \le m$, we let $\mathrm{vol}_k(\mathcal{S})$ be the k dimensional volume of $\mathcal{S}$ and define $\mathbf{1}_{\mathcal{S}} : \mathbb{R}^m \to \mathbb{R}$ by

$$\mathbf{1}_{\mathcal{S}}(\boldsymbol{x}) := \begin{cases} 1, & \text{if } \boldsymbol{x} \in \mathcal{S}, \\ 0, & \text{otherwise.} \end{cases}$$

By the association (2), we note that for any $\boldsymbol{x} \in \mathcal{T}$

$$\mathbf{1}_{\mathcal{T}_1}(\boldsymbol{x}) + \mathbf{1}_{\mathcal{T}_2}(\boldsymbol{x}) + \mathbf{1}_{\mathcal{T}_3}(\boldsymbol{x}) = \mathbf{1}_{\mathcal{T}}(\boldsymbol{x}). \tag{9}$$

We write $\#K$ for the number of elements in a sequence K.

1.3 Bivariate simplex splines

In this section we recall some basic properties of simplex splines.

For $n \in \mathbb{N}, d \in \mathbb{N}_0$, let $m := n + d$ and $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{m+1} \in \mathbb{R}^n$ be a sequence of points called **knots**. The **multiplicity** of a knot is the number of times it occurs in the sequence. Let $\sigma = \langle\overline{\boldsymbol{k}}_1, \ldots, \overline{\boldsymbol{k}}_{m+1}\rangle$ with $\mathrm{vol}_m(\sigma) > 0$ be a simplex in $\mathbb{R}^m$ whose projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$ onto the first n coordinates satisfies $\pi(\overline{\boldsymbol{k}}_i) = \boldsymbol{k}_i$, for $i = 1, \ldots, m + 1$.

With $[K] := [\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{m+1}]$, the **unit integral simplex spline** $M[K]$ can be defined geometrically by

$$M[K] : \mathbb{R}^n \to \mathbb{R}, \quad M[K](\boldsymbol{x}) := \frac{\mathrm{vol}_{m-n}\big(\sigma \cap \pi^{-1}(\boldsymbol{x})\big)}{\mathrm{vol}_m(\sigma)}.$$

For properties of $M[K]$ and proofs see for example [10]. Here, we mention:

- If $n = 1$ then $M[K]$ is the univariate B-spline of degree d with knots K, normalized to have integral one.

- In general $M[K]$ is a nonnegative piecewise polynomial of total degree d and support $\langle K\rangle$.

- For $d = 0$ we have

$$M[K](\boldsymbol{x}) := \begin{cases} 1/\mathrm{vol}_n(\langle K\rangle), & \boldsymbol{x} \in \langle K\rangle^o, \\ 0, & \text{if } \boldsymbol{x} \notin \langle K\rangle. \end{cases} \tag{10}$$

- The value of $M[K]$ on the boundary of $\langle K\rangle$ has to be delt with separately, see below.

- If $\mathrm{vol}_n(\langle K\rangle) = 0$ then $M[K]$ can be defined either as identically zero or as a distribution.

We will deal with the bivariate case $n = 2$, and for our purpose it is convenient to work with **area normalized simplex splines**, [8]. They are defined by $Q[K](\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^2$ if $\mathrm{vol}_2(\langle K\rangle) = 0$, and otherwise

$$Q_{\mathcal{T}}[K] = Q[K] := \frac{\mathrm{vol}_2(\mathcal{T})}{\binom{d+2}{2}} M[K], \tag{11}$$

where $\mathcal{T}$ in general is some subset of $\mathbb{R}^2$, and in our case will be the triangle $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\rangle$. The knot sequence is $[\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_{\mathcal{T}}]$ taken with multiplicities. Using properties of $M[K]$ and (11), we obtain the following for $Q[K]$:

- It is a **piecewise polynomial** of degree $d = \#K - 3$ with support $\langle K\rangle$

- **knot lines** are the lines in the complete graph of K

- **local smoothness**: Across a knot line, $Q[K] \in C^{d+1-\mu}$, where d is the degree and μ is the number of knots on that knot line, including multiplicities

- **differentiation formula**: $\partial_{\boldsymbol{u}} Q[K] = d \sum_{j=1}^{d+3} a_j Q[K \setminus \boldsymbol{k}_j]$,
  for any $\boldsymbol{u} \in \mathbb{R}^2$ and any $a_1, \ldots, a_{d+3}$ such that $\sum_j a_j \boldsymbol{k}_j = \boldsymbol{u}$, $\sum_j a_j = 0$ (A-**recurrence**)

- **recurrence relation**: $Q[K](\boldsymbol{x}) = \sum_{j=1}^{d+3} b_j Q[K \setminus \boldsymbol{k}_j](\boldsymbol{x})$,
  for any $\boldsymbol{x} \in \mathbb{R}^2$ and any $b_1, \ldots, b_{d+3}$ such that $\sum_j b_j \boldsymbol{k}_j = \boldsymbol{x}$, $\sum_j b_j = 1$ (B-**recurrence**)

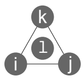
- **knot insertion formula**: $Q[K] = \sum_{j=1}^{d+3} c_j Q[K \cup \boldsymbol{y} \setminus \boldsymbol{k}_j]$,
  for any $\boldsymbol{y} \in \mathbb{R}^2$ and any $c_1, \ldots, c_{d+3}$ such that $\sum_j c_j \boldsymbol{k}_j = \boldsymbol{y}$, $\sum_j c_j = 1$ (*C*-**recurrence**)

- **degree zero**: From (10) and (11) we obtain for $d = 0$

$$Q[K](\boldsymbol{x}) := \begin{cases} \mathrm{vol}_2(\mathcal{T})/\mathrm{vol}_2(\langle K \rangle), & \boldsymbol{x} \in \langle K \rangle^o, \\ 0, & \text{if } \boldsymbol{x} \notin \langle K \rangle. \end{cases} \tag{12}$$

2 A simplex spline basis for the Clough-Tocher split

In this section we determine and study a basis of C^1 cubic simplex splines on the Clough-Tocher split on a triangle. For fixed $\boldsymbol{x} \in \mathcal{T}$ we use the simplified notation

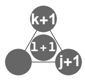


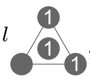
$$:= Q[\boldsymbol{p}_1^{[i]}, \boldsymbol{p}_2^{[j]}, \boldsymbol{p}_3^{[k]}, \boldsymbol{p}_{\boldsymbol{T}}^{[l]}](\boldsymbol{x}), \quad i, j, k, l \in \mathbb{N}_0, \quad i + j + k + l \geq 3,$$

where the notation $\boldsymbol{p}_m^{[n]}$ denotes that $\boldsymbol{p}_m$ is repeated n times.

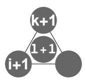
When one of the integers i, j, k, l is zero we have

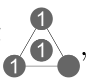
Lemma 1 *For $i, j, k, l \in \mathbb{N}_0$, $i+j+k+l = d \geq 0$ and $\boldsymbol{x} \in \mathcal{T}$ with barycentric coordinates $\beta_1, \beta_2, \beta_3$ we have*

$i = 0$,


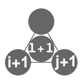
$$= \frac{d!}{j!k!l!}(\beta_2 - \beta_1)^j (\beta_3 - \beta_1)^k (3\beta_1)^l$$


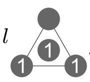
,

$j = 0$,


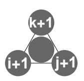
$$= \frac{d!}{i!k!l!}(\beta_1 - \beta_2)^i (\beta_3 - \beta_2)^k (3\beta_2)^l$$


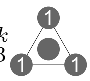
,

$k = 0$,


$$= \frac{d!}{i!j!l!}(\beta_1 - \beta_3)^i (\beta_2 - \beta_3)^j (3\beta_3)^l$$


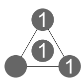
,

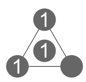
$l = 0$,


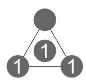
$$= \frac{d!}{i!j!k!}\beta_1^i \beta_2^j \beta_3^k$$


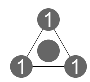
$$= B_{ijk}^d(\boldsymbol{x}), \tag{13}$$

where the constant simplex splines are given by



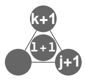
$= 3\,\mathbf{1}_{\mathcal{T}_1}(\boldsymbol{x})$,


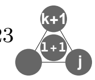
$= 3\,\mathbf{1}_{\mathcal{T}_2}(\boldsymbol{x})$,


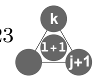
$= 3\,\mathbf{1}_{\mathcal{T}_3}(\boldsymbol{x})$,


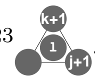
$= \mathbf{1}_{\mathcal{T}}(\boldsymbol{x})$. (14)

Proof: Suppose $i = 0$. The first equation in (13) holds for $d = 0$. Suppose it holds for $d-1$ and let $j+k+l = d$. Let β_j^{023}, $j = 0, 2, 3$ be the barycentric coordinates of $\boldsymbol{x}$ with respect to $\mathcal{T}_1 = \langle \boldsymbol{p}_0, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$, where $\boldsymbol{p}_0 := \boldsymbol{p}_{\mathcal{T}}$. By the B-recurrence



$= \beta_2^{023}$


$+ \beta_3^{023}$


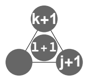
$+ \beta_0^{023}$


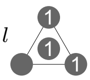
.

It is easily shown that

$$\beta_2^{023} = \beta_2 - \beta_1,\ \beta_3^{023} = \beta_3 - \beta_1,\ \beta_0^{023} = 3\beta_1.$$

Therefore, by the induction hypothesis



$= \dfrac{(d-1)!}{j!k!l!}(j+k+l)(\beta_2^{023})^j(\beta_3^{023})^k(\beta_0^{023})^l$


Since $j + k + l = d$ we obtain the first equation in (13).

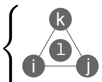
The next two equations in (13) follow similarly using

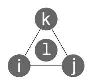
$$\beta_1^{031} = \beta_1 - \beta_2,\ \beta_3^{031} = \beta_3 - \beta_2,\ \beta_0^{031} = 3\beta_2,$$
$$\beta_1^{012} = \beta_1 - \beta_3,\ \beta_2^{012} = \beta_2 - \beta_3,\ \beta_0^{012} = 3\beta_3.$$

Using the B-recurrence repeatedly, we obtain the first equality for $l = 0$. The values of the constant simplex splines are a consequence of (12). $\square$

Remark 2 *For $i = 0$ we note that the expression $\frac{d!}{j!k!l!}(\beta_2 - \beta_1)^j(\beta_3 - \beta_1)^k(3\beta_1)^l$ in (13) is a Bernstein polynomial on $\mathcal{T}_1$. Similar remarks hold for $j, k = 0$.*

The set

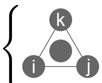
$C1 := \Big\{$


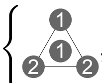
$\in \mathbb{S}_3^1(\triangle) :$


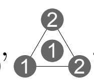
$\neq 0 \Big\}$ (15)

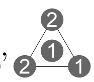
of all nonzero simplex splines that can be used in a basis for $\mathbb{S}_3^1(\triangle)$ contains precisely the following 13 simplex splines.

Lemma 3 *We have*

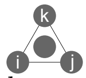
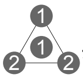
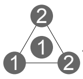
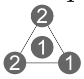
$C1 = \Big\{$


$: i, j, k \in \mathbb{N},\ i + j + k = 6 \Big\} \bigcup \Big\{$


,


,


$\Big\}$.

Proof: For $l = 0$ it follows from Lemma 1 that  $\in \mathbb{S}_3^1(\triangle)$ for all $i + j + k = 6$. Consider next $l = 1$. By the local smoothness property, C^1 smoothness implies that each of i, j, k can be at most 2. But then , ,  are the only possibilities. Now if $l = 2$ then $i + j + k = 4$ implies that one of i, j, k must be at least 2 and we cannot have C^1 smoothness. Similarly $l > 2$ is not feasible. □

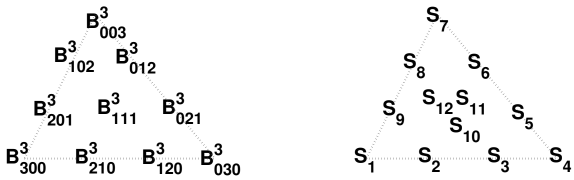
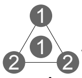
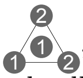
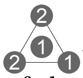
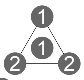
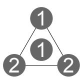
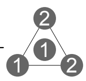
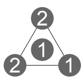
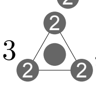
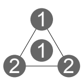
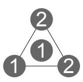
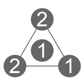


Figure 2: The cubic Bernstein basis (left) and the CTS-basis (right), where B^3_{111} is replaced by S_{10}, S_{11}, S_{12}.

Recall that $\mathbb{S}_3^1(\triangle)$ is a linear space of dimension 12, [1]. Thus, in order to obtain a possible basis for this space, we need to choose 12 of the 13 elements in $C1$. Since $C1$ contains the 10 cubic Bernstein polynomials we have to include at least two of , , . We also want a symmetric basis and therefore, we have to include all of them. But then one of the Bernstein polynomials has to be excluded. To see which one to exclude, we insert the knot $\boldsymbol{p}_3 = -\boldsymbol{p}_1 - \boldsymbol{p}_2 + 3\boldsymbol{p}_{\mathcal{T}}$ into  and use the C-recurrence to obtain  $= -$  $-$  $+ 3$ , or by (13)

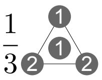
 $+$  $+$  $= 3B^3_{111}(\boldsymbol{x}).$ (16)

Thus, in order to have symmetry and hopefully obtain 12 linearly independent functions, we see that B^3_{111} is the one that should be excluded.

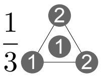
We obtain the following simplex spline basis for $\mathbb{S}_3^1(\triangle)$.

Theorem 4 (CTS-basis) *The 12 simplex splines $S_1, \ldots, S_{12}$, where*

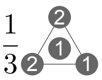
$$S_j(\boldsymbol{x}) := B_j(\boldsymbol{x}), \text{ where } B_j \text{ is given by } (7)\ j = 1, \ldots, 9,$$

$S_{10}(\boldsymbol{x}) := \frac{1}{3}$ 

$$= (B^3_{210} - B^3_{300})\mathbf{1}_{\mathcal{T}_1} + (B^3_{120} - B^3_{030})\mathbf{1}_{\mathcal{T}_2} + (B^3_{111} - B^3_{102} - B^3_{012} + 2B^3_{003})\mathbf{1}_{\mathcal{T}_3}$$

$S_{11}(\boldsymbol{x}) := \frac{1}{3}$ 

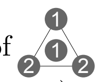
$$= (B^3_{111} - B^3_{210} - B^3_{201} + 2B^3_{300})\mathbf{1}_{\mathcal{T}_1} + (B^3_{021} - B^3_{030})\mathbf{1}_{\mathcal{T}_2} + (B^3_{012} - B^3_{003})\mathbf{1}_{\mathcal{T}_3}$$

$S_{12}(\boldsymbol{x}) := \frac{1}{3}$ 

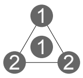
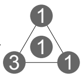
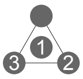
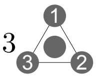
$$= (B^3_{201} - B^3_{300})\mathbf{1}_{\mathcal{T}_1} + (B^3_{111} - B^3_{120} - B^3_{021} + 2B^3_{030})\mathbf{1}_{\mathcal{T}_2} + (B^3_{102} - B^3_{003})\mathbf{1}_{\mathcal{T}_3}. \tag{17}$$

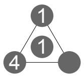
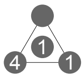
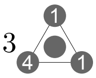
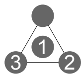
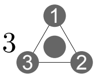
form a partition of unity basis for the space $\mathbb{S}^1_3(\triangle)$ given by (1)*. This basis, which we call the* **CTS-basis***, is the only symmetric simplex spline basis for $\mathbb{S}^1_3(\triangle)$. On the boundary of $\mathcal{T}$ the functions S_{10}, S_{11}, S_{12} have the value zero, while the elements of $\{S_1, S_2, \ldots, S_9\}$ reduce to zero, or to univariate Bernstein polynomials.*

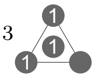
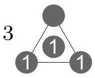
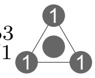
Proof: By Lemma 1, it follows that the Bernstein polynomials $B_1, \ldots, B_9$ are cubic simplex splines, and the previous discussion implies that the functions in (17), apart from scaling, are the only candidates for a symmetric simplex spline basis for $\mathbb{S}^1_3(\triangle)$.

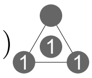
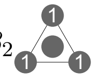
We can find the explicit form of 

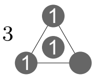
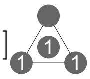
using either the B- or C-recurrence (see definitions at the end of Section 1) . Consider the C-recurrence. Inserting $\boldsymbol{p}_1$ twice and using $\boldsymbol{p}_1 = -\boldsymbol{p}_2 - \boldsymbol{p}_3 + 3\boldsymbol{p}_{\mathcal{T}}$ and (13) we find

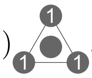
 $= -$  $-$  $+\,3$ 

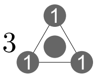
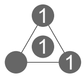
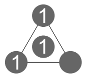
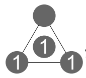
$=$  $+$  $-\,3$  $-$  $+\,3$ 

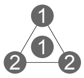
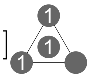
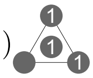
$= (\beta_1 - \beta_2)^3$  $+ (\beta_1 - \beta_3)^3$  $- 3\beta_1^3$ 

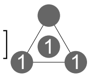
$- 3(\beta_1 - \beta_3)^2(\beta_2 - \beta_3)$  $+ 9\beta_1^2\beta_2$  (18)

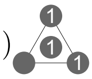
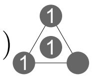
$= (\beta_1 - \beta_2)^3$  $+ [(\beta_1 - \beta_3)^3 - 3(\beta_1 - \beta_3)^2(\beta_2 - \beta_3)]$ 

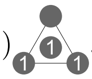
$+ 3\beta_1^2(3\beta_2 - \beta_1)$  .

Using (9) and Lemma 1, we can write 3  $=$  $+$  $+$ , so that

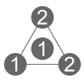
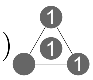
 $= [(\beta_1 - \beta_2)^3 + \beta_1^2(3\beta_2 - \beta_1)]$  $+ \beta_1^2(3\beta_2 - \beta_1)$ 

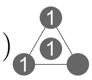
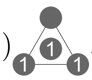
$+ [(\beta_1 - \beta_3)^2(\beta_1 - 3\beta_2 + 2\beta_3) + \beta_1^2(3\beta_2 - \beta_1)]$  (19)

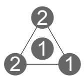
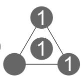
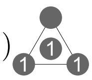
$= (3\beta_1^2\beta_2 - \beta_1^3)$  $+ (3\beta_1\beta_2^2 - \beta_2^3)$ 

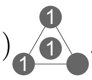
$+ (6\beta_1\beta_2\beta_3 - 3\beta_1\beta_3^2 - 3\beta_2\beta_3^2 + 2\beta_3^3)$  .

By symmetry we obtain

 $= (6\beta_1\beta_2\beta_3 - 3\beta_1^2\beta_2 - 3\beta_1^2\beta_3 + 2\beta_1^3)$ 

$+ (3\beta_2^2\beta_3 - \beta_2^3)$  $+ (3\beta_2\beta_3^2 - \beta_3^3)$ , (20)

 $= (3\beta_1^2\beta_3 - \beta_1^3)$  $+ (3\beta_1\beta_3^2 - \beta_3^3)$ 

$+ (6\beta_1\beta_2\beta_3 - 3\beta_1\beta_2^2 - 3\beta_2^2\beta_3 + 2\beta_2^3)$  .

The formulas for S_{10}, S_{11} and S_{12} in (17) now follows from (19) and (20) using (3) and (14).

By the partition of unity for Bernstein polynomials we find

$$\sum_{j=1}^{12} S_j(\boldsymbol{x}) = \sum_{i+j+k=3} B^3_{ijk}(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \mathcal{T}.$$

It is well known that B^3_{ijk} reduces to univariate Bernstein polynomials or zero on the boundary of $\mathcal{T}$.

Clearly $S_j \in C(\mathbb{R}^2)$, $j = 10, 11, 12$, since no edge contains more than 4 knots. This follows from general properties of simplex splines. By the local support property they must therefore be zero on the boundary. It also follows that $S_j \in C^1(\mathcal{T})$, $j = 10, 11, 12$, since no interior knot line contains more than 3 knots.

It remains to show that the 12 functions S_j, $j = 1, \ldots, 12$ are linearly independent on $\mathcal{T}$. Suppose that $\sum_{j=1}^{12} c_j S_j(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathcal{T}$ and let $(\beta_1, \beta_2, \beta_3)$ be the barycentric coordinates of $\boldsymbol{x}$. On the edge $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$, where $\beta_3 = 0$, the functions S_j, $j = 5, \ldots 12$ vanish, and thus

$$\sum_{j=1}^{12} c_j S_j(\boldsymbol{x}) = c_1 B^3_{300}(\boldsymbol{x}) + c_2 B^3_{210}(\boldsymbol{x}) + c_3 B^3_{120}(\boldsymbol{x}) + c_4 B^3_{030}(\boldsymbol{x}) = 0.$$

On $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$ this is a linear combination of linearly independent univariate Bernstein polynomials and we conclude that $c_1 = c_2 = c_3 = c_4 = 0$. Similarly $c_j = 0$ for $j = 5, \ldots, 9$. It remains to show that S_{10}, S_{11} and S_{12} are linearly independent on $\mathcal{T}$. For $\boldsymbol{x} \in \mathcal{T}_3^o$ and $\beta_3 = 0$ we find

$$\frac{\partial S_{10}}{\partial \beta_3}|_{\beta_3=0} = 6\beta_1\beta_2 \neq 0, \quad \frac{\partial S_j}{\partial \beta_3}|_{\beta_3=0} = 0, \; j = 11, 12.$$

We deduce that $c_{10} = 0$ and similarly $c_{11} = c_{12} = 0$ which concludes the proof. □

In Figure 3 we show graphs of the functions S_{10}, S_{11}, S_{12}.

3 Two Marsden identities and representation of polynomials

We give both a barycentric and a Cartesian Marsden-like identity.

Theorem 5 (Barycentric Marsden-like identity) *For* $\boldsymbol{u} := (u_1, u_2, u_3)$, $\boldsymbol{\beta} := (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ *with* $\beta_i \geq 0$, $i = 1, 2, 3$ *and* $\beta_1 + \beta_2 + \beta_3 = 1$ *we have*

$$\begin{aligned}(\boldsymbol{\beta}^T \boldsymbol{u})^3 = {} & u_1^3 S_1(\boldsymbol{\beta}) + u_1^2 u_2 S_2(\boldsymbol{\beta}) + u_1 u_2^2 S_3(\boldsymbol{\beta}) + u_2^3 S_4(\boldsymbol{\beta}) + u_2^2 u_3 S_5(\boldsymbol{\beta}) \\ & + u_2 u_3^2 S_6(\boldsymbol{\beta}) + u_3^3 S_7(\boldsymbol{\beta}) + u_1 u_3^2 S_8(\boldsymbol{\beta}) + u_1^2 u_3 S_9(\boldsymbol{\beta}) \\ & + u_1 u_2 u_3 \big(S_{10}(\boldsymbol{\beta}) + S_{11}(\boldsymbol{\beta}) + S_{12}(\boldsymbol{\beta})\big).\end{aligned} \tag{21}$$

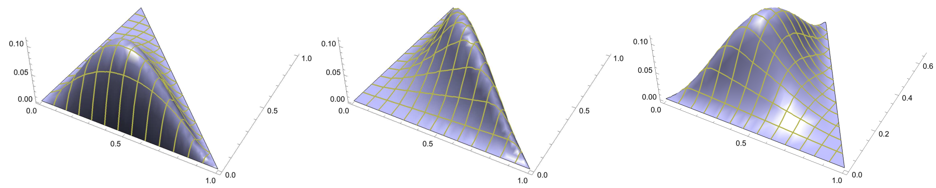


Figure 3: The CTS-basis functions S_{10}, S_{11}, S_{12} on the triangle $\langle(0,0),(1,0),(0,1)\rangle$.

Proof: By the multinomial expansion we obtain

$$\begin{aligned}(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)^3 &= \sum_{i+j+k=3} \frac{3!}{i!j!k!} (\beta_1 u_1)^i (\beta_2 u_2)^j (\beta_3 u_3)^k \\ &= \sum_{i+j+k=3} u_1^i u_2^j u_3^k B_{ijk}^3(\boldsymbol{\beta}).\end{aligned}$$

Using $B_{111}^3 = S_{10} + S_{11} + S_{12}$ and the ordering in Theorem 4 we obtain (21). □

Corollary 6 *For $l, m, n \in \mathbb{N}_0$ with $l + m + n \leq 3$ we have an explicit representation for lower degree Bernstein polynomials in terms of the CTS-basis* (17)*.*

$$\begin{aligned}B_{lmn}^{l+m+n} = \binom{3}{l+m+n}^{-1} \Bigg(&\binom{3}{l}\binom{0}{m}\binom{0}{n} S_1 + \binom{2}{l}\binom{1}{m}\binom{0}{n} S_2 \\ &+ \binom{1}{l}\binom{2}{m}\binom{0}{n} S_3 + \binom{0}{l}\binom{3}{m}\binom{0}{n} S_4 + \binom{0}{l}\binom{2}{m}\binom{1}{n} S_5 \\ &+ \binom{0}{l}\binom{1}{m}\binom{2}{n} S_6 + \binom{0}{l}\binom{0}{m}\binom{3}{n} S_7 + \binom{1}{l}\binom{0}{m}\binom{2}{n} S_8 \\ &+ \binom{2}{l}\binom{0}{m}\binom{1}{n} S_9 + \binom{1}{l}\binom{1}{m}\binom{1}{n} \Big(S_{10} + S_{11} + S_{12} \Big) \Bigg),\end{aligned} \tag{22}$$

where $\binom{0}{0} := 1$ and $\binom{r}{s} := 0$ if $s > r$.

Proof: Differentiating, for any $d \in \mathbb{N}_0$, $(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)^d$ a total of l, m, n times with respect to u_1, u_2, u_3, respectively, and setting $u_1 = u_2 = u_3 = 1$

we find

$$\frac{d!}{(d-l-m-n)!}\beta_1^l\beta_2^m\beta_3^n$$
$$= \sum_{i+j+k=d} i(i-1)\ldots(i-l+1)j\ldots(j-m+1)k\ldots(k-n+1)B_{ijk}^d,$$

and by a rescaling

$$B_{lmn}^{l+m+n} = \binom{d}{l+m+n}^{-1} \sum_{i+j+k=d} \binom{i}{l}\binom{j}{m}\binom{k}{n} B_{ijk}^d, \quad l+m+n \le d. \tag{23}$$

Using (17) with $d=3$, we obtain (22).□

As an example, we find

$$B_{100}^1 = \frac{1}{3}\big(3S_1 + 2S_2 + S_3 + S_8 + 2S_9 + S_{10} + S_{11} + S_{12}\big).$$

Theorem 7 (Cartesian Marsden-like identity) *We have*

$$(1+\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{v})^3 = \sum_{j=1}^{12} \psi_j(\boldsymbol{v})S_j(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{T},\ \boldsymbol{v} \in \mathbb{R}^2, \tag{24}$$

where the **dual polynomials in Cartesian form** *are given by*

$$\psi_j(\boldsymbol{v}) := \prod_{l=1}^{3}(1+\boldsymbol{d}_{j,l}^T\boldsymbol{v}), \quad j=1,\ldots,12, \quad \boldsymbol{v} \in \mathbb{R}^2. \tag{25}$$

Here the **dual points** $\boldsymbol{d}_j := [\boldsymbol{d}_{j,1}, \boldsymbol{d}_{j,2}, \boldsymbol{d}_{j,3}]$, *are given as follows.*

$$\begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \boldsymbol{d}_3 \\ \boldsymbol{d}_4 \\ \boldsymbol{d}_5 \\ \boldsymbol{d}_6 \\ \boldsymbol{d}_7 \\ \boldsymbol{d}_8 \\ \boldsymbol{d}_9 \\ \boldsymbol{d}_{10} \\ \boldsymbol{d}_{11} \\ \boldsymbol{d}_{12} \end{bmatrix} := \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_1 & \boldsymbol{p}_1 \\ \boldsymbol{p}_1 & \boldsymbol{p}_1 & \boldsymbol{p}_2 \\ \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_2 \\ \boldsymbol{p}_2 & \boldsymbol{p}_2 & \boldsymbol{p}_2 \\ \boldsymbol{p}_2 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \\ \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_3 \\ \boldsymbol{p}_3 & \boldsymbol{p}_3 & \boldsymbol{p}_3 \\ \boldsymbol{p}_1 & \boldsymbol{p}_3 & \boldsymbol{p}_3 \\ \boldsymbol{p}_1 & \boldsymbol{p}_1 & \boldsymbol{p}_3 \\ \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \\ \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \\ \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{bmatrix}. \tag{26}$$

The domain points $\boldsymbol{p}_j^*$ *in* (8) *are the coefficients of* $\boldsymbol{x}$ *in terms of the CTS-basis*

$$\boldsymbol{x} = \sum_{j=1}^{12} \boldsymbol{p}_j^* S_j(\boldsymbol{x}), \tag{27}$$

where $\boldsymbol{p}_{10}^* = \boldsymbol{p}_{11}^* = \boldsymbol{p}_{12}^* = \boldsymbol{p}_{\boldsymbol{T}}$.

Proof: We apply (21) with $\beta_1, \beta_2, \beta_3$ the barycentric coordinates of $\boldsymbol{x}$ and $u_i = 1 + \boldsymbol{p}_i^T \boldsymbol{v}$, $i = 1, 2, 3$. Then

$$\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 = \beta_1 + \beta_2 + \beta_3 + \beta_1 \boldsymbol{p}_1^T \boldsymbol{v} + \beta_2 \boldsymbol{p}_2^T \boldsymbol{v} + \beta_3 \boldsymbol{p}_3^T \boldsymbol{v} = 1 + \boldsymbol{x}^T \boldsymbol{v}.$$

and (24), (25), (26) follow from (21). Taking partial derivatives in (24) with respect to $\boldsymbol{v}$,

$$\big(\partial_{v_1}, \partial_{v_2}\big)(1 + \boldsymbol{x}^T \boldsymbol{v})^3 = 3\boldsymbol{x}(1 + \boldsymbol{x}^T \boldsymbol{v})^2 = \sum_{j=1}^{12} \big(\partial_{v_1}, \partial_{v_2}\big)\psi_j(\boldsymbol{v}) S_j(\boldsymbol{x}),$$

where $\big(\partial_{v_1}, \partial_{v_2}\big)\psi_j(\boldsymbol{v}) := \boldsymbol{d}_{j,1}(1+\boldsymbol{d}_{j,2}^T\boldsymbol{v})(1+\boldsymbol{d}_{j,3}^T\boldsymbol{v}) + \boldsymbol{d}_{j,2}(1+\boldsymbol{d}_{j,1}^T\boldsymbol{v})(1+\boldsymbol{d}_{j,3}^T\boldsymbol{v}) + \boldsymbol{d}_{j,3}(1 + \boldsymbol{d}_{j,1}^T\boldsymbol{v})(1 + \boldsymbol{d}_{j,2}^T\boldsymbol{v})$. Setting $\boldsymbol{v} = \boldsymbol{0}$ we obtain (27). □

Note that the domain point $\boldsymbol{p}_{\mathcal{T}}$ for B_{111}^3 has become a triple domain point for the CTS-basis.

Following the proof of (27) we can give explicit representations of all the monomials $x^r y^s$ spanning $\mathbb{P}_3$. We do not give details here.

4 Three quasi-interpolants

We consider three quasi-interpolants on $\mathbb{S}_3^1(\triangle)$. They all use functionals based on point evaluations and the third one will be used to estimate the L_∞ condition number of the CTS-basis.

To start, we consider the following polynomial interpolation problem on $\mathcal{T}$. Find $g \in \mathbb{P}_3$ such that $g(\boldsymbol{p}_i^*) = f_i$, where $\boldsymbol{f} := [f_1, \ldots, f_{10}]^T$ is a vector of given real numbers and the $\boldsymbol{p}_i^*$ given by (8) are the domain points for the cubic Bernstein basis.

Using the ordering (7), we write g in the form $\sum_{j=1}^{10} c_j B_j$ and obtain the linear system

$$\sum_{j=1}^{10} c_j B_j(\boldsymbol{p}_i^*) = f_i, \quad i = 1, \ldots, 10,$$

or in matrix form $\boldsymbol{Ac} = \boldsymbol{f}$ for the unknown coefficient vector $\boldsymbol{c} := [c_1, \ldots, c_{10}]^T$. Since $B_{10}(\boldsymbol{p}_i^*) = B_{111}^3(\boldsymbol{p}_i^*) = 0$ for $i = 1, \ldots, 9$ the coefficient matrix $\boldsymbol{A}$ is block triangular

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{A}_2 & \boldsymbol{A}_3 \end{bmatrix}, \tag{28}$$

and if $\boldsymbol{A}_1$ and $\boldsymbol{A}_3$ are nonsingular then

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{A}_1^{-1} & \boldsymbol{0} \\ -\boldsymbol{A}_3^{-1}\boldsymbol{A}_2\boldsymbol{A}_1^{-1} & \boldsymbol{A}_3^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{B}_2 & \boldsymbol{B}_3 \end{bmatrix}. \tag{29}$$

Using the barycentric form of the domain points in (8) we find $\boldsymbol{A}_2 = [1, 3, 3, 1, 3, 3, 1, 3, 3]/27$, $\boldsymbol{A}_3 = B_{111}^3(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{2}{9}$,

$$\boldsymbol{A}_1 := \frac{1}{27} \begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 12 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 12 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 12 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 12 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 8 & 12 & 6 \\ 8 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 12 \end{bmatrix} \in \mathbb{R}^{9\times 9} \tag{30}$$

and

$$\boldsymbol{B}_1 := \boldsymbol{A}_1^{-1} = \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 18 & -9 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & -9 & 18 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 18 & -9 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -9 & 18 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -5 & 18 & -9 \\ -5 & 0 & 0 & 0 & 0 & 0 & 2 & -9 & 18 \end{bmatrix},$$

$$\boldsymbol{B}_3 = [\frac{9}{2}],\ \boldsymbol{B}_2 := -\boldsymbol{B}_3\boldsymbol{A}_2\boldsymbol{B}_1 = \frac{1}{12}\begin{bmatrix} 4 & -9 & -9 & 4 & -9 & -9 & 4 & -9 & -9 \end{bmatrix}. \tag{31}$$

Define $\tilde{QI}^P : C(\mathcal{T}) \to \mathbb{P}_3$ by

$$\tilde{QI}^P(f) := \sum_{i=1}^{10} \lambda_i^P(f) B_i, \quad \lambda_i^P(f) := \sum_{j=1}^{10} \alpha_{i,j} f(\boldsymbol{p}_j^*), \tag{32}$$

where the matrix $\boldsymbol{\alpha} := \boldsymbol{A}^{-1}$ has elements $\alpha_{i,j}$ in row i and column j, $i,j = 1,\dots,10$. We have

$$\lambda_i^P(B_j) = \sum_{k=1}^{10} \alpha_{i,k} B_j(\boldsymbol{p}_k^*) = \sum_{k=1}^{10} \alpha_{i,k} a_{k,j} = \delta_{i,j}, \quad i,j = 1,\dots,10.$$

It follows that $\tilde{Q}I^P(g) = g$ for all $g \in \mathbb{P}_3$. Since $B_j = S_j$, $j = 1,\dots,9$ and $B_{10} = B_{111}^3 = S_{10} + S_{11} + S_{12}$ the quasi-interpolant

$$QI^P : C(\mathcal{T}) \to \mathbb{S}_3^1(\triangle), \quad QI^P(f) := \sum_{i=1}^{12} \lambda_i^P(f) S_i, \quad \lambda_{11}^P = \lambda_{12}^P = \lambda_{10}^P, \tag{33}$$

where $\lambda_i^P(f)$ is given by (32), $i = 1,\dots,10$, reproduces $\mathbb{P}_3$. Moreover, since for any $f \in C(\mathcal{T})$ and $x \in \mathcal{T}$

$$|QI^P(f)(x)| \leq \max_{1\leq i\leq 12} |\lambda_i^P(f)| \sum_{i=1}^{12} S_i(x) = \max_{1\leq i\leq 10} |\lambda_i^P(f)|,$$

we obtain

$$\|QI^P(f)\|_{L_\infty(\mathcal{T})} \leq \|\boldsymbol{\alpha}\|_\infty \|f\|_{L_\infty(\mathcal{T})} = 10\|f\|_{L_\infty(\mathcal{T})},$$

independently of the geometry of $\mathcal{T}$.

Using the construction in [8], we can derive another quasi-interpolant which also reproduces $\mathbb{P}_3$. It uses more points, but has a slightly smaller norm. Consider the map $\boldsymbol{P} : C(\mathcal{T}) \to \mathbb{S}_3^1(\mathcal{T})$ defined by $\boldsymbol{P}(f) = \sum_{\ell=1}^{12} M_\ell(f) S_\ell$, where

$$\begin{aligned} M_\ell(f) :=& \frac{1}{6}\big(f(\boldsymbol{d}_{\ell,1}) + f(\boldsymbol{d}_{\ell,2}) + f(\boldsymbol{d}_{\ell,3})\big) + \frac{9}{2} f(\boldsymbol{p}_\ell^*) \\ &- \frac{4}{3}\Big(f\Big(\frac{\boldsymbol{d}_{\ell,1} + \boldsymbol{d}_{\ell,2}}{2}\Big) + f\Big(\frac{\boldsymbol{d}_{\ell,1} + \boldsymbol{d}_{\ell,3}}{2}\Big) + f\Big(\frac{\boldsymbol{d}_{\ell,2} + \boldsymbol{d}_{\ell,3}}{2}\Big)\Big). \end{aligned}$$

Here the $\boldsymbol{d}_{\ell,m}$ are the dual points given by (26) and the $\boldsymbol{p}_\ell^*$ are the domain points given by (27). Note that this is an affine combination of function values of f.

We have tested the convergence of the quasi-interpolant, sampling data from the function $f(x,y) = e^{2x+y} + 5x + 7y$ on the triangle $A = [0,0]$, $B = h * [1,0]$, $C = h * [0.2, 1.2]$ for $h \in \{0.05, 0.04, 0.03, 0.02, 0.01\}$. The following array indicates that the error: $\|f - \boldsymbol{P}(f)\|_{L_\infty(\mathcal{T})}$, is $O(h^4)$.

h	0.05	0.04	0.03	0.02	0.01
$error/h^4$	0.0550	0.0547	0.0543	0.0540	0.0537

Using a standard argument the following Proposition shows that the error is indeed $O(h^4)$ for sufficiently smooth functions.

Proposition 8 *The operator $\boldsymbol{P}$ is a quasi-interpolant that reproduces $\mathbb{P}_3$. For any $f \in C(\mathcal{T})$*

$$\|\boldsymbol{P}(f)\|_{L_\infty(\mathcal{T})} \leq 9\|f\|_{L_\infty(\mathcal{T})}, \tag{34}$$

independently of the geometry of $\mathcal{T}$. Moreover,

$$\|f - \boldsymbol{P}(f)\|_{L_\infty(\mathcal{T})} \leq 10 \inf_{g \in \mathbb{P}_3} \|f - g\|_{L_\infty(\mathcal{T})}. \tag{35}$$

Proof: Since $\boldsymbol{d}_{10} = \boldsymbol{d}_{11} = \boldsymbol{d}_{12}$ and $B^3_{111} = S_{10} + S_{11} + S_{12}$, $B^3_{ijk} = S_\ell$ for $(i,j,k) \neq (1,1,1)$ and some ℓ, we obtain $\boldsymbol{P}(f) = \sum_{i+j+k=3} \bar{M}_{ijk}(f) B^3_{ijk}$ where $\bar{M}_{ijk} = M_\ell$ for $(i,j,k) \neq (1,1,1)$ and corresponding ℓ and $\bar{M}_{111} = 3M_{10}$.

To prove that $\boldsymbol{P}$ reproduces polynomials up to degree 3, i.e., $\boldsymbol{P}(B^3_{ijk}) = B^3_{ijk}$, whenever $i + j + k = 3$, it is sufficient to prove the result for B^3_{300}, B^3_{210}, B^3_{111}, using the symmetries. From the following initial values,

$\boldsymbol{p}$	$B^3_{300}(\boldsymbol{p})$	$B^3_{210}(\boldsymbol{p})$	$B^3_{111}(\boldsymbol{p})$
$\boldsymbol{p}_1$	1	0	0
$(2\boldsymbol{p}_1 + \boldsymbol{p}_2)/3$	8/27	4/9	0
$(\boldsymbol{p}_1 + 2\boldsymbol{p}_2)/3$	1/27	2/9	0
$(\boldsymbol{p}_1 + 2\boldsymbol{p}_3)/3$	1/27	0	0
$(2\boldsymbol{p}_1 + \boldsymbol{p}_3)/3$	0	0	0
$(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)/3$	1/27	1/9	2/9
$(\boldsymbol{p}_1 + \boldsymbol{p}_2)/2$	1/8	3/8	0
$(\boldsymbol{p}_1 + \boldsymbol{p}_3)/2$	1/8	0	0

(36)

and the fact that the three polynomials are zero at $\boldsymbol{p}_2$, $\dfrac{2\boldsymbol{p}_2 + \boldsymbol{p}_3}{3}$, $\dfrac{\boldsymbol{p}_2 + 2\boldsymbol{p}_3}{3}$, $\boldsymbol{p}_3$, $\dfrac{\boldsymbol{p}_2 + \boldsymbol{p}_3}{2}$, it is easy to compute that

$$\begin{aligned}
\bar{M}_{300}(B^3_{300}) &= 1,\ \bar{M}_{300}(B^3_{ijk}) = 0 \text{ for } (i,j,k) \neq (3,0,0),\\
\bar{M}_{210}(B^3_{210}) &= 1,\ \bar{M}_{210}(B^3_{ijk}) = 0 \text{ for } (i,j,k) \neq (2,1,0),\\
\bar{M}_{111}(B^3_{111}) &= 1,\ \bar{M}_{111}(B^3_{ijk}) = 0 \text{ for } (i,j,k) \neq (1,1,1).
\end{aligned}$$

Therefore, by a standard argument, $\boldsymbol{P}$ is a quasi-interpolant that reproduces $\mathbb{P}_3$. Since the sum of the absolute values of the coefficients defining $M_\ell(f)$ is equal to 9, another standard argument shows (34) and (35). □

The operators QI^P and $\boldsymbol{P}$ do not reproduce the whole spline space $\mathbb{S}_3^1(\triangle)$. Indeed, since $\lambda_{10}^P(B_{10}) = M_{10}(B_{10}) = 1$, we have $\lambda_{10}^P(S_j) = M_{10}(S_j) = \frac{1}{3}$, $j = 10, 11, 12$.

To give un upper bound for the condition number of the CTS-basis we need a quasi-interpolant which reproduces the whole spline space. We again use the inverse of the coefficient matrix of an interpolation problem to construct such an operator. We need 12 interpolation points and a natural choice is to use the first 9 cubic Bernstein domain points $\boldsymbol{p}_j^*$, $j = 1, \ldots, 9$ and split the barycenter $\boldsymbol{p}_{10}^* = \boldsymbol{p}_{\mathcal{T}}$ into three points. After some experimentation we redefine $\boldsymbol{p}_{10}^*$ and choose $\boldsymbol{p}_{10}^* := (3,3,1)/7$, $\boldsymbol{p}_{11}^* := (3,1,3)/7$ and $\boldsymbol{p}_{12}^* := (1,3,3)/7$. The problem is to find $s = \sum_{j=1}^{12} c_j S_j$ such that $s(\boldsymbol{p}_i^*) = f_i$, $i = 1, \ldots, 12$. The coefficient matrix for this problem has again the block tridiagonal form (28), where $\boldsymbol{A}_1 \in \mathbb{R}^{9\times 9}$ and $\boldsymbol{B}_1 := \boldsymbol{A}_1^{-1}$ are given by (30) and (31) as before. Moreover, using the formulas in Theorem 4 we find

$$\boldsymbol{A}_3 = [S_j(\boldsymbol{p}_i^*)]_{i,j=10}^{12} = \frac{1}{343}\begin{bmatrix} 38 & 8 & 8 \\ 8 & 8 & 38 \\ 8 & 38 & 8 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

This matrix is nonsingular with inverse

$$\boldsymbol{B}_3 := \boldsymbol{A}_3^{-1} = \begin{bmatrix} \frac{7889}{810} & -\frac{686}{405} & -\frac{686}{405} \\ -\frac{686}{405} & -\frac{686}{405} & \frac{7889}{810} \\ -\frac{686}{405} & \frac{7889}{810} & -\frac{686}{405} \end{bmatrix}.$$

With $\boldsymbol{A}_2 = [B_j(\boldsymbol{p}_i^*)]_{i=10,j=1}^{12,9}$ we find

$$\boldsymbol{A}_2 = \frac{1}{343}\begin{bmatrix} 27 & 81 & 81 & 27 & 27 & 9 & 1 & 9 & 27 \\ 27 & 27 & 9 & 1 & 9 & 27 & 27 & 81 & 81 \\ 1 & 9 & 27 & 27 & 81 & 81 & 27 & 27 & 9 \end{bmatrix} \in \mathbb{R}^{3\times 9},$$

and then (29) implies

$$\boldsymbol{\alpha}^S := \boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{B}_2 & \boldsymbol{B}_3 \end{bmatrix},$$

where

$$\boldsymbol{B}_2 = -\boldsymbol{B}_3\boldsymbol{A}_2\boldsymbol{B}_1 = \begin{bmatrix} \frac{643}{810} & -\frac{191}{60} & -\frac{191}{60} & \frac{643}{810} & -\frac{83}{60} & \frac{79}{60} & -\frac{178}{405} & -\frac{83}{60} & \frac{79}{60} \\ -\frac{178}{405} & \frac{79}{60} & -\frac{83}{60} & \frac{643}{810} & -\frac{191}{60} & -\frac{191}{60} & \frac{643}{810} & \frac{79}{60} & -\frac{83}{60} \\ \frac{643}{810} & -\frac{83}{60} & \frac{79}{60} & -\frac{178}{405} & \frac{79}{60} & -\frac{83}{60} & \frac{643}{810} & -\frac{191}{60} & -\frac{191}{60} \end{bmatrix}.$$

It follows that the quasi-interpolant QI given by

$$QI : C(\mathcal{T}) \to \mathbb{S}_3^1(\triangle), \quad QI(f) := \sum_{i=1}^{12} \lambda_i^S(f) S_i, \quad \lambda_i^S(f) = \sum_{j=1}^{12} \alpha_{i,j}^S f(\boldsymbol{p}_j^*), \tag{37}$$

is a projector onto the spline space $\mathbb{S}_3^1(\triangle)$. In particular

$$s := \sum_{i=1}^{12} c_i S_i \implies c_i = \lambda_i^S(s), \quad i = 1, \ldots, 12. \tag{38}$$

The quasi-interpolant (37) can be used to show the L_∞ stability of the CTS-basis. For this we prove that the condition number is independent of the geometry of the triangle.

We define the ∞-norm condition number of the CTS-basis on $\mathcal{T}$ by

$$\kappa_\infty(\mathcal{T}) := \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\|\boldsymbol{b}^T \boldsymbol{c}\|_{L_\infty(\mathcal{T})}}{\|\boldsymbol{c}\|_\infty} \max_{\boldsymbol{c} \neq \boldsymbol{0}} \frac{\|\boldsymbol{c}\|_\infty}{\|\boldsymbol{b}^T \boldsymbol{c}\|_{L_\infty(\mathcal{T})}},$$

where $\boldsymbol{b}^T \boldsymbol{c} := \sum_{j=1}^{12} c_j S_j \in \mathbb{S}_3^1(\triangle)$.

Theorem 9 *For any triangle $\mathcal{T}$ we have $\kappa_\infty(\mathcal{T}) < 27$.*

Proof: Since the S_j form a nonnegative partition of unity it follows that $\max_{\boldsymbol{c} \neq \boldsymbol{0}} \|\boldsymbol{b}^T \boldsymbol{c}\|_{L_\infty(\mathcal{T})} / \|\boldsymbol{c}\|_\infty = 1$.

If $s = \sum_{j=1}^{12} c_j S_j = \boldsymbol{b}^T \boldsymbol{c}$ then by (38) $|c_i| = |\lambda_i^S(\boldsymbol{b}^T \boldsymbol{c})| \leq \|\boldsymbol{\alpha}^S\|_\infty \|\boldsymbol{b}^T \boldsymbol{c}\|_{L_\infty(\mathcal{T})}$. Therefore,

$$\frac{\|\boldsymbol{c}\|_\infty}{\|\boldsymbol{b}^T \boldsymbol{c}\|_{L_\infty(\mathcal{T})}} \leq \|\boldsymbol{\alpha}^S\|_\infty = 27 - \frac{32}{405},$$

and the upper bound $\kappa_\infty < 27$ follows. $\square$

5 C^0 and C^1–continuity

In the following, we derive conditions to ensure C^0 and C^1 continuity through an edge between two triangles. The conditions are very similar to the classical conditions for continuity of Bernstein polynomials.

Theorem 10 *Let $s_1 = \sum_{j=1}^{12} c_j S_j$ and $s_2 = \sum_{j=1}^{12} d_j \tilde{S}_j$ be defined on the triangle $\mathcal{T} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$ and $\tilde{\mathcal{T}} := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \tilde{\boldsymbol{p}}_3 \rangle$, respectively, see Figure 4. The function $s = \begin{cases} s_1 & \text{on } \mathcal{T} \\ s_2 & \text{on } \tilde{\mathcal{T}} \end{cases}$ is continuous on $\mathcal{T} \cup \tilde{\mathcal{T}}$ if*

$$d_1 = c_1, \quad d_2 = c_2, \quad d_3 = c_3, \quad d_4 = c_4. \tag{39}$$

Moreover, $s \in C^1(\mathcal{T} \cup \tilde{\mathcal{T}})$ *if in addition to* (39) *we have*

$$d_5 = \gamma_1 c_3 + \gamma_2 c_4 + \gamma_3 c_5, \;\; d_9 = \gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_9, \;\; d_{10} = \gamma_1 c_2 + \gamma_2 c_3 + \gamma_3 c_{10}. \tag{40}$$

where $\gamma_1, \gamma_2, \gamma_3$ *are the barycentric coordinates of* $\tilde{p}_3$ *with respect to* $\mathcal{T}$*.*

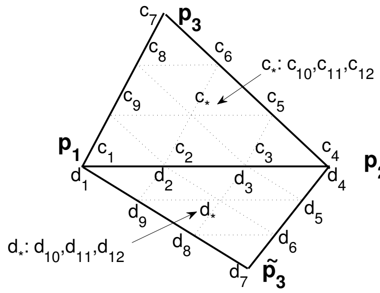
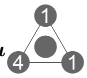
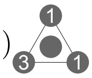


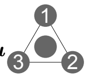
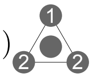
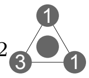
Figure 4: $\mathcal{C}^1$–continuity and splines components

Proof: Consider s_1 on the edge $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$. On that edge only S_1, S_2, S_3, S_4 can be nonzero and they reduce to linearly independent univariate Bernstein polynomials. If $s \in C(\mathcal{T})$ then $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4$ must reduce to the same Bernstein polynomials on $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$. But then (39) follows from linear independence.

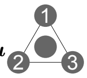
Suppose next (39) holds and $s \in C^1(\mathcal{T} \cup \tilde{\mathcal{T}})$. By the continuity property we see that S_j, $j = 6, 7, 8, 11, 12$ are zero and have zero cross boundary derivatives on $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$ since they have at most 3 knots on that edge. We take derivatives in the direction $\boldsymbol{u} := \tilde{\boldsymbol{p}}_3 - \boldsymbol{p}_1$ using the A-recurrence (defined at the end of Section 1) with $\boldsymbol{a} := (\gamma_1 - 1, \gamma_2, \gamma_3, 0)$ for s_1 and $\boldsymbol{a} := (-1, 0, 1, 0)$

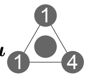
for s_2. We find with $\boldsymbol{x} \in \langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$

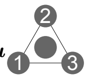
$\frac{1}{3}\partial_{\boldsymbol{u}} S_1(\boldsymbol{x}) := \frac{1}{3}\partial_{\boldsymbol{u}}$  $= (\gamma_1 - 1)$  $= (\gamma_1 - 1) B^2_{200}(\boldsymbol{x}),$

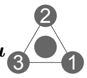
$\frac{1}{3}\partial_{\boldsymbol{u}} S_2(\boldsymbol{x}) := \frac{1}{3}\partial_{\boldsymbol{u}}$  $= (\gamma_1 - 1)$  $+ \gamma_2$ 

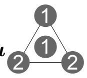
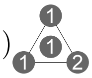
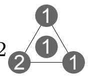
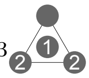
$= (\gamma_1 - 1) B^2_{110}(\boldsymbol{x}) + \gamma_2 B^2_{200}(\boldsymbol{x}),$

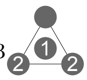
$\frac{1}{3}\partial_{\boldsymbol{u}} S_3(\boldsymbol{x}) := \frac{1}{3}\partial_{\boldsymbol{u}}$  $= (\gamma_1 - 1) B^2_{020}(\boldsymbol{x}) + \gamma_2 B^2_{110}(\boldsymbol{x}),$

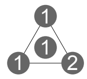
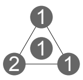
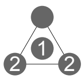
$\frac{1}{3}\partial_{\boldsymbol{u}} S_4(\boldsymbol{x}) := \frac{1}{3}\partial_{\boldsymbol{u}}$  $= \gamma_2 B^2_{020}(\boldsymbol{x}),$ (41)

$\frac{1}{3}\partial_{\boldsymbol{u}} S_5(\boldsymbol{x}) := \frac{1}{3}\partial_{\boldsymbol{u}}$  $= \gamma_2 B^2_{011}(\boldsymbol{x}) + \gamma_3 B^2_{020}(\boldsymbol{x}),$

$\frac{1}{3}\partial_{\boldsymbol{u}} S_9(\boldsymbol{x}) := \frac{1}{3}\partial_{\boldsymbol{u}}$  $= (\gamma_1 - 1) B^2_{101}(\boldsymbol{x}) + \gamma_3 B^2_{200}(\boldsymbol{x}),$

$\frac{1}{3}\partial_{\boldsymbol{u}} S_{10}(\boldsymbol{x}) := \frac{1}{9}\partial_{\boldsymbol{u}}$  $= \frac{1}{3}(\gamma_1 - 1)$  $+ \frac{1}{3}\gamma_2$  $+ \frac{1}{3}\gamma_3$ 

$= \frac{1}{3}\gamma_3$  $= \gamma_3 B^2_{110}(\boldsymbol{x}).$

The last equality follows from (13) since $\beta_3 = 0$ on $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$ so that  $=$  $= 0$ and  $= 3B^2_{110}(\boldsymbol{x})$. Consider next $\tilde{S}_j$. By the same argument as for S_j, we see that $\tilde{S}_j$, $j = 6, 7, 8, 11, 12$ are zero and have zero cross boundary derivatives on $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$. We find for $\boldsymbol{x} \in \langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$

$$\frac{1}{3}\partial_{\boldsymbol{u}} \big[\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4, \tilde{S}_5, \tilde{S}_9, \tilde{S}_{10}\big](\boldsymbol{x}) = \big[-\tilde{B}^2_{200}, -\tilde{B}^2_{110}, -\tilde{B}^2_{020}, 0, \tilde{B}^2_{020}, -\tilde{B}^2_{101} + \tilde{B}^2_{200}, \tilde{B}^2_{110}\big](\boldsymbol{x})$$

We note that on $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$, the polynomials $B^2_{101}, \tilde{B}^2_{101}, B^2_{011}, \tilde{B}^2_{011}$ vanish and $B^2_{ij0} = \tilde{B}^2_{ij0}$. To obtain C^1 smoothness, we need $\partial_{\boldsymbol{u}} \tilde{S}_j = \partial_{\boldsymbol{u}} S_j$ for $j = 1, 2, 3, 4, 5, 9, 10$ on $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$. Using $d_i = c_i$, $i = 1, 2, 3, 4$ we then obtain

$$\begin{aligned} \frac{1}{3}\big(\partial_{\boldsymbol{u}} s_1(\boldsymbol{x}) - \partial_{\boldsymbol{u}} s_2(\boldsymbol{x})\big) = {} & \big(c_1(\gamma_1 - 1) + c_2\gamma_2 + c_9\gamma_3 + c_1 - d_9\big) B^2_{200}(\boldsymbol{x}) \\ & + \big(c_2(\gamma_1 - 1) + c_3\gamma_2 + c_{10}\gamma_3 + c_2 - d_{10}\big) B^2_{110}(\boldsymbol{x}) \\ & + \big(c_3(\gamma_1 - 1) + c_4\gamma_2 + c_5\gamma_3 + c_3 - d_5\big) B^2_{020}(\boldsymbol{x}) = 0. \end{aligned} \tag{42}$$

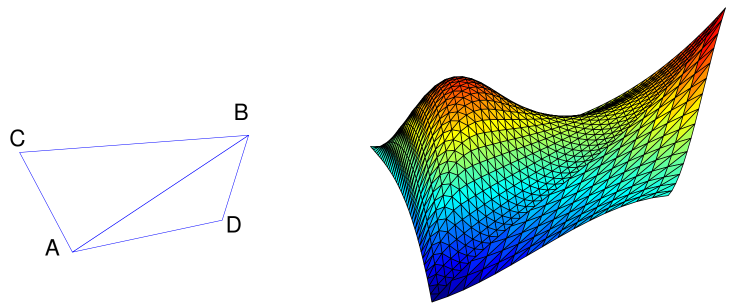


Figure 5: $\mathcal{C}^1$ smoothness

By linear independence we obtain the formulas for $\boldsymbol{d}_5, \boldsymbol{d}_9, \boldsymbol{d}_{10}$. $\square$

In Figure 5, we illustrate C^1 smoothness by connecting two triangles A, B, C and A, B, D with $A = [0, 0]$, $B = [1, 0]$, $C = [0.2, 0.8]$, $D = [0.6, -0.4]$.

6 The Hermite basis

The classical Hermite interpolation problem on the Clough-Tocher split is to interpolate values and gradients at vertices and normal derivatives at the midpoint of edges, see Figure 1.

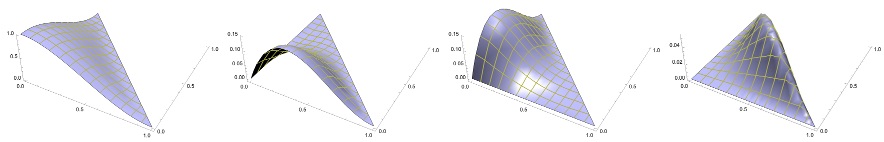


Figure 6: The Hermite basis functions H_1, H_2, H_3, H_{10} on the unit triangle.

These interpolation conditions can be described by the linear functionals

$$\begin{aligned}\boldsymbol{\rho}(f) = [\rho_1(f), \ldots, \rho_{12}(f)]^T := [&f(\boldsymbol{p}_1), \partial_{1,0} f(\boldsymbol{p}_1), \partial_{0,1} f(\boldsymbol{p}_1), f(\boldsymbol{p}_2), \partial_{1,0} f(\boldsymbol{p}_2), \\ &\partial_{0,1} f(\boldsymbol{p}_2), f(\boldsymbol{p}_3), \partial_{1,0} f(\boldsymbol{p}_3), \partial_{0,1} f(\boldsymbol{p}_3), \partial_{\boldsymbol{n}_1} f(\boldsymbol{p}_5), \partial_{\boldsymbol{n}_2} f(\boldsymbol{p}_6), \partial_{\boldsymbol{n}_3} f(\boldsymbol{p}_4)]^T,\end{aligned}$$

where $\boldsymbol{p}_4, \boldsymbol{p}_5, \boldsymbol{p}_6$, are the midpoints on the edges $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle, \langle \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle, \langle \boldsymbol{p}_3, \boldsymbol{p}_1 \rangle$, respectively, and $\partial_{\boldsymbol{n}_j} f$ is the derivative in the direction of the unit normal to that edge in the direction towards $\boldsymbol{p}_j$. We let $\boldsymbol{p}_j = (x_j, y_j)$ be the coordinates of each point. The coefficient vector $\boldsymbol{c} := [c_1, \dots, c_{12}]^T$ of the interpolant $g := \sum_{j=1}^{12} c_j S_j$ is solution of the linear system

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{\rho}(f), \text{ where } \boldsymbol{A} \in \mathbb{R}^{12\times 12} \text{ with } a_{i,j} := \rho_i(S_j). \tag{43}$$

Let $H_1, \dots, H_{12}$ be the **Hermite basis** for $\mathbb{S}_3^1(\triangle)$ defined by $\rho_i(H_j) = \delta_{i,j}$. The matrix $\boldsymbol{A}$ transforms the Hermite basis to the CTS-basis. Since a basis transformation matrix is always nonsingular, we have

$$[S_1, \dots, S_{12}] = [H_1, \dots, H_{12}]\boldsymbol{A}, \quad [H_1, \dots, H_{12}] = [S_1, \dots, S_{12}]\boldsymbol{A}^{-1}. \tag{44}$$

To find the elements $\rho_i(S_j)$ of $\boldsymbol{A}$ we define for $i, j, k = 1, 2, 3$

$$\begin{aligned} &\nu_{ij} := \|\boldsymbol{p}_{ij}\|_2, \ \boldsymbol{p}_{ij} := \boldsymbol{p}_i - \boldsymbol{p}_j, \ x_{ij} := x_i - x_j, \ y_{ij} := y_i - y_j, \\ &\nu_{ijk} := \frac{\boldsymbol{p}_{i,j}^T \boldsymbol{p}_{j,k}}{\nu_{ij}}, \text{ for } i \neq j, \ \delta := \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \end{aligned} \tag{45}$$

We note that ν_{ijk} is the length of the projection of $\boldsymbol{p}_{j,k}$ in the direction of $\boldsymbol{p}_{i,j}$ and that δ is twice the signed area of $\mathcal{T}$.

By the definition of the unit normals and the chain rule for $j = 1, \dots, 12$ we find

$$\begin{aligned}
\partial_{1,0} S_j &= (y_{23}\partial_{\beta_1} S_j + y_{31}\partial_{\beta_2} S_j + y_{12}\partial_{\beta_3} S_j)/\delta, \\
\partial_{0,1} S_j &= (x_{32}\partial_{\beta_1} S_j + x_{13}\partial_{\beta_2} S_j + x_{21}\partial_{\beta_3} S_j)/\delta, \\
\partial_{\boldsymbol{n}_1} S_j &= (y_{23}\partial_{1,0} S_j + x_{32}\partial_{0,1} S_j)/\nu_{32} = (\nu_{32}\partial_{\beta_1} S_j + \nu_{231}\partial_{\beta_2} S_j + \nu_{321}\partial_{\beta_3} S_j)/\delta, \\
\partial_{\boldsymbol{n}_2} S_j &= (y_{31}\partial_{1,0} S_j + x_{13}\partial_{0,1} S_j)/\nu_{31} = (\nu_{132}\partial_{\beta_1} S_j + \nu_{31}\partial_{\beta_2} S_j + \nu_{312}\partial_{\beta_3} S_j)/\delta, \\
\partial_{\boldsymbol{n}_3} S_j &= (y_{12}\partial_{1,0} S_j + x_{21}\partial_{0,1} S_j)/\nu_{21} = (\nu_{123}\partial_{\beta_1} S_j + \nu_{213}\partial_{\beta_2} S_j + \nu_{21}\partial_{\beta_3} S_j)/\delta.
\end{aligned}$$

This leads to

$$\boldsymbol{A} := \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{A}_2 & \boldsymbol{A}_3 \end{bmatrix}, \text{ with } \boldsymbol{A}_1 \in \mathbb{R}^{9\times 9}, \quad \boldsymbol{A}_2 \in \mathbb{R}^{3\times 9}, \quad \boldsymbol{A}_3 \in \mathbb{R}^{3\times 3},$$

where

$$\boldsymbol{A}_1 := \frac{3}{\delta} \begin{bmatrix}
\delta/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{2,3} & y_{3,1} & 0 & 0 & 0 & 0 & 0 & 0 & y_{1,2} \\
x_{3,2} & x_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & x_{2,1} \\
0 & 0 & 0 & \delta/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & y_{2,3} & y_{3,1} & y_{1,2} & 0 & 0 & 0 & 0 \\
0 & 0 & x_{3,2} & x_{1,3} & x_{2,1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y_{3,1} & y_{1,2} & y_{2,3} & 0 \\
0 & 0 & 0 & 0 & 0 & x_{1,3} & x_{2,1} & x_{3,2} & 0
\end{bmatrix},$$

the rows of $\boldsymbol{A}_2$ are given by

$$\boldsymbol{A}_2(1) := \frac{3}{4\delta}\big[0, 0, \nu_{32}, \nu_{231}, \nu_{231} - \nu_{32}, \nu_{321} - \nu_{32}, \nu_{321}, \nu_{32}, 0\big],$$
$$\boldsymbol{A}_2(2) := \frac{3}{4\delta}\big[\nu_{132}, \nu_{31}, 0, 0, 0, \nu_{31}, \nu_{312}, \nu_{312} - \nu_{31}\nu_{132} - \nu_{31}\big],$$
$$\boldsymbol{A}_2(3) := \frac{3}{4\delta}\big[\nu_{123}, \nu_{123} - \nu_{21}, \nu_{213} - \nu_{21}, \nu_{213}, \nu_{21}, 0, 0, 0, \nu_{21}\big],$$

and

$$\boldsymbol{A}_3 := \frac{3}{2\delta}\begin{bmatrix} 0 & \nu_{32} & 0 \\ 0 & 0 & \nu_{31} \\ \nu_{21} & 0 & 0 \end{bmatrix}.$$

We find

$$\boldsymbol{A}^{-1} := \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{B}_2 & \boldsymbol{B}_3 \end{bmatrix} = [b_{i,j}]_{i,j=1}^{12},$$

where

$$\boldsymbol{B}_1 := \boldsymbol{A}_1^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & x_{21} & y_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & x_{12} & y_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & x_{32} & y_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & x_{23} & y_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & x_{13} & y_{13} \\ 3 & x_{31} & y_{31} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{9\times 9},$$

$$\boldsymbol{B}_3 := \boldsymbol{A}_3^{-1} = \frac{2\delta}{3}\begin{bmatrix} 0 & 0 & \nu_{21}^{-1} \\ \nu_{32}^{-1} & 0 & 0 \\ 0 & \nu_{31}^{-1} & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3},$$

and the rows of $\boldsymbol{B}_2 = -\boldsymbol{B}_3\boldsymbol{A}_2\boldsymbol{B}_1 \in \mathbb{R}^{3\times 9}$ are given by

$$\boldsymbol{B}_2(1) := \frac{1}{6\nu_{21}}\big[-6\nu_{123}, x_{12}\nu_{123} + \nu_{21}x_{23}, y_{12}\nu_{123} + \nu_{21}y_{23}, -6\nu_{213},$$
$$x_{21}\nu_{213} + \nu_{21}x_{13}, y_{21}\nu_{213} + \nu_{21}y_{13}, 0, 0, 0\big],$$
$$\boldsymbol{B}_2(2) := \frac{1}{6\nu_{32}}\big[0, 0, 0, -6\nu_{231}, x_{23}\nu_{231} + \nu_{32}x_{31}, y_{23}\nu_{231} + \nu_{32}y_{31}, -6\nu_{321},$$
$$x_{23}\nu_{231} + \nu_{32}x_{21} + \nu_{32}x_{23}, y_{23}\nu_{231} + \nu_{32}y_{21} + \nu_{32}y_{23}\big],$$
$$\boldsymbol{B}_2(3) := \frac{1}{6\nu_{31}}\big[-6\nu_{132}, x_{13}\nu_{132} + \nu_{31}x_{32}, y_{13}\nu_{132} + \nu_{31}y_{32}, 0, 0, 0,$$
$$-6\nu_{312}, x_{31}\nu_{312} + \nu_{31}x_{12}, y_{31}\nu_{312} + \nu_{31}y_{12}\big].$$

As an example, on the unit triangle $(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) = ((0,0),(1,0),(0,1))$ we find

$$\boldsymbol{B}_2 = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{12} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{12} \\ 1 & -\frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 \end{bmatrix}.$$

Some of the Hermite basis functions are shown in Figure 6.

We have also tested the convergence of the Hermite interpolant, sampling again data from the function $f(x,y) = e^{2x+y} + 5x + 7y$ on the triangle $A = [0,0]$, $B = h*[1,0]$, $C = h*[0.2, 1.2]$ for $h \in \{0.05, 0.04, 0.03, 0.02, 0.01\}$. The following array indicates that the error: $\|f - H(f)\|_{L_\infty(\mathcal{T})}$ is $O(h^4)$.

h	0.05	0.04	0.03	0.02	0.01
$error/h^4$	0.1650	0.1640	0.1630	0.1620	0.1610

7 Examples

Several examples have been considered for scattered data on the CT-split, see for example [4, 9]. Here, we consider a triangulation with vertices $\boldsymbol{p}_1 = (0,0)$, $\boldsymbol{p}_2 = (1,0)$, $\boldsymbol{p}_3 = (3/2, 1/2)$, $\boldsymbol{p}_4 = (-1/2, 1)$, $\boldsymbol{p}_5 = (1/4, 3/4)$, $\boldsymbol{p}_6 = (3/2, 3/2)$, $\boldsymbol{p}_7 = (1/2, 2)$ and triangles $T_1 := \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_5 \rangle$, $T_2 := \langle \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_5 \rangle$, $T_3 := \langle \boldsymbol{p}_4, \boldsymbol{p}_1, \boldsymbol{p}_5 \rangle$, $T_4 := \langle \boldsymbol{p}_3, \boldsymbol{p}_6, \boldsymbol{p}_5 \rangle$, $T_5 := \langle \boldsymbol{p}_6, \boldsymbol{p}_4, \boldsymbol{p}_5 \rangle$, $T_6 := \langle \boldsymbol{p}_4, \boldsymbol{p}_6, \boldsymbol{p}_7 \rangle$,. We divide each of the 6 triangles into 3 subtriangles using the Clough-Tocher split. We then obtain a space of C^1 piecewise polynomials of dimension $3V + E = 3 \times 7 + 12 = 33$, where V is the number of vertices and E the number of edges in the triangulation. We can represent a function s in this space by either using the Hermite basis or using CTS-splines on each of the triangles and enforcing the C_1 continuity conditions. The function s on T_1 depends on 12 components, while the C^1–continuity through the edges gives only 5 free components for T_2,T_3 and T_4. Closing the 1-cell at $\boldsymbol{p}_5$ gives only one free component for T_5 and 5 free components for T_6 , Figure 7 left.

In the following graph, Figure 7, right, once the 12 first components on T_1 were chosen, the other free ones are set to zero. Then, in Figure 8, we have plotted the Hermite interpolant of the function $f(x,y) = e^{2x+y} + 5x + 7y$ and gradients using the CTS-splines.

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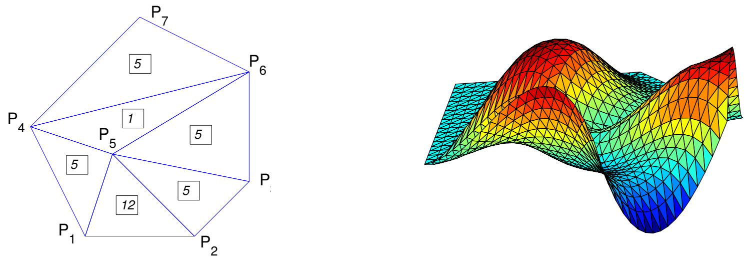

Figure 7: The triangulation and the $\mathcal{C}^1$ surface

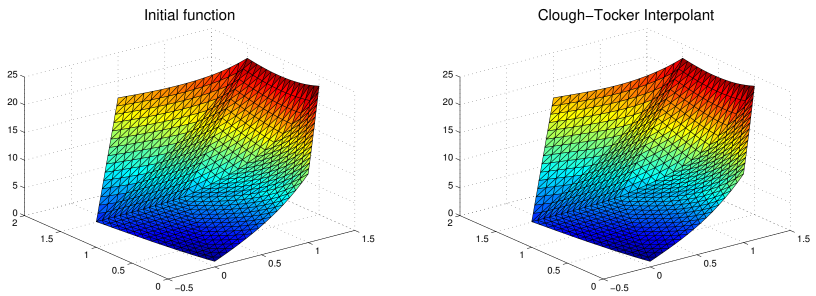

Figure 8: A $\mathcal{C}^1$ Hermite interpolating surface on the triangulation

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