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A SIMPLIFIED MODEL FOR LARGE DEFORMATIONS SOLUTIONS OF MULTILAYERED COMPOSITES

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In order to forecast the phenomenon of delamination in multi-layered beam, it is proposed to extend to large transformations the multiparticle models developed for the linear theory of multi-layered plates : the advantage of such a model is to introduce as variables, stresses and transformations defined at each interface. In the first part, we sum up the steps that lead to different linear models. The second part deals with the extension of the method to finite transformations of one model : a systematic method to build a simple model of composite beam in large rotation is developed.

1. Linear multiparticle models .

The different steps to build multiparticle models for multi-layered plates in linear theory are given .

Step 1 Approximation per layer of the components of the cauchy stress tensor

The thickness of a layer i is noted $e^i = h_i^+ - h_i^- = 2\bar{h}^i$. The components of the Cauchy stress tensor, $\boldsymbol{\sigma}$, are expanded per layer, on a base of Legendre's polynomials depending on z (direction of the stacking) and on the layer i : $\left\{q_l^i(z)\right\}_{i=1}^N$ where the degree relatively to z of $q_l^i(z)$ is l for $z \in \left[h_i^-, h_i^+\right]$:

$$q_i^0 = 1,\ q_i^1(z) = \frac{z-\bar{h}^i}{e^i},\ q_i^2(Z) = -6\left(\frac{z-\bar{h}^i}{e^i}\right)^2 + \frac{1}{2}$$

The only departure hypothesis concerning these expansions is the degree relative to z of the in-plane components of $\boldsymbol{\sigma}$: the three dimensional equilibrium equations then impose the degree of the other components. The key point is that the coefficients of these expansions of $\boldsymbol{\sigma}$, denoted $\boldsymbol{\sigma}^{app}$, are imposed to be dependant of different variables having a maximum physical meaning: the generalized stresses. This approach allows us to introduce standard generalized stresses of the plates' theories, but also to introduce stresses defined on the lower and the upper face of each layer and respecting continuity conditions at the N-1 interfaces. The generalized stresses will be the new unknowns of the simplified two dimensional problem . They are written in the following with the in-plane super- vector $\Sigma^g = \left\{\Sigma_j^i(x,y)\right\}$.

Step2 Conjugate strains and displacements

The former approximations are introduced in the Hellinger-Reissner functional:

$$J(\mathbf{u},\boldsymbol{\sigma}) = \int_\Omega -w^c(\boldsymbol{\sigma})d\Omega + \underbrace{\int_\Omega \boldsymbol{\sigma} : \nabla\mathbf{u}\, d\Omega - \int_{\partial\Omega} \mathbf{t}^d.\mathbf{u}\, ds}_{T(\mathbf{u},\boldsymbol{\sigma})}$$

where $\mathbf{t}^d$ is the prescribed force on the boundary and w^c is the complementary stress potential. The integration along z is operated in $T(\mathbf{u},\boldsymbol{\sigma})$: the cofactors of the generalized stresses are then exhibited and define the generalized strains that derived from generalized displacements : Thus, $T(\mathbf{u},\boldsymbol{\sigma}^{app}) = \tilde{T}(\boldsymbol{\Phi}^g, \boldsymbol{\Sigma}^g)$ where $\boldsymbol{\Phi}^g$ are equivalence classes of displacements defined by $\boldsymbol{\sigma}^{app}$ and J.

Step3 Generalized equilibrium equations

Writing the stationarity of $\tilde{T}$ with respect to $\boldsymbol{\Phi}^g$, equilibrium equations and boundary conditions of the simplified problem (written in $\boldsymbol{\Sigma}^g$) are obtained.

Step4 Generalized constitutive law

The layers studied are made of elastic linear materials whose stiffness tensor is $\boldsymbol{\Lambda}$. The expression of $\boldsymbol{\sigma}^{app}$ is introduce in w^c giving $\tilde{w}^c(\boldsymbol{\Sigma}^g) = \frac{1}{2}\boldsymbol{\sigma}^{app} : \boldsymbol{\Lambda} : \boldsymbol{\sigma}^{app}$. The generalized constitutive equations are then obtained writing the stationarity of $\tilde{J}(\boldsymbol{\Phi}^g, \boldsymbol{\Sigma}^g)$ with respect to $\boldsymbol{\Sigma}^g$.

2. Adaptation to large transformations

A total Lagrangian description of the plane deformations of the beam is used. Variables depend on the two reference coordinates X and Z.

Step1 Mixed Principle

A convenient mixed formulation in $\mathbf{S}$ (2nd Piola-Kirchhoff stress tensor) is used:

$$J = \int_{\Omega_0} -\psi^c(\mathbf{S}) - \frac{1}{2}tr\mathbf{S}\, d\Omega_0 + \int_{\Omega_0} \mathbf{S} : {}^t\nabla\boldsymbol{\varphi}.\nabla\boldsymbol{\varphi}\, d\Omega_0 - \int_{\partial\Omega_0} \mathbf{t}^d.\boldsymbol{\varphi}\, dS$$

The linear models use both the duality between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}\right)$, and the linearity between $\boldsymbol{\varepsilon}$ and $\mathbf{u}$ in order to exhibit kinematics conjugate to a stress approximation. Here, there is duality between the Green-Lagrange strain tensor $\mathbf{e}$ and $\mathbf{S}$

but there is no more linearity between $\mathbf{e}$ and the transformation $\boldsymbol{\varphi}$. The main idea is to both approximate $\boldsymbol{\varphi}$, and $\mathbf{S}$ by polynomial expansions in $q_l^i(Z)$ where Z is the reference coordinate associated to the direction of the stacking in Ω_0.

Step2 approximation of kinematics

We introduce in the polynomial expansion of $\boldsymbol{\varphi}$ on layer i, its values on lower and upper interface: $\boldsymbol{\varphi}^{i\pm} = \boldsymbol{\varphi}\left(X, h_i^{\pm}\right)$

$$\boldsymbol{\varphi}^{app} = \overline{\boldsymbol{\varphi}}^{i}.q_0^i + \left(\boldsymbol{\varphi}^{i+} - \boldsymbol{\varphi}^{i-}\right)q_1^i(Z) + \left(\overline{\boldsymbol{\varphi}}^{i} - \frac{\boldsymbol{\varphi}^{i+} - \boldsymbol{\varphi}^{i-}}{2}\right)q_2^i(Z)$$

where, $\overline{\boldsymbol{\varphi}}^{i}(X) = \frac{1}{e^i}\int_{h_i^-}^{h_i^+} \boldsymbol{\varphi}\, dZ$.

The generalized kinematics is denoted

$$\boldsymbol{\Phi}^g(X) = \left[\overline{\boldsymbol{\varphi}}^{i}, \boldsymbol{\varphi}^{i+}, \boldsymbol{\varphi}^{i-}\right] \text{ and } \boldsymbol{\Phi}^{g'} = \frac{d}{dX}\boldsymbol{\Phi}^g$$

A simple model is obtained by approximating all the fields involved in $J(\boldsymbol{\varphi}, \mathbf{S})$ by polynomials of degree one in Z. The gradient $\mathbf{F} = \nabla\boldsymbol{\varphi}^{app}$ is calculated and then approximated as :

$$\mathbf{F}^{app} = \left(\mathbf{F}^0\right)^i q_0^i + \left(\mathbf{F}^1\right)^i q_1^i(Z)$$

where $\left(\mathbf{F}^k\right)^i_{k=0,1}$ are functions of $\boldsymbol{\Phi}^g$ and $\boldsymbol{\Phi}^{g'}$.

Step3 Approximations of the stresses

Using the same argument, we assume :

$$\mathbf{S}^{app} = \left(\mathbf{S}^0\right)^i q_0^i + \left(\mathbf{S}^1\right)^i q_1^i(Z)$$

Step4 Integration along Z

Using the properties of Legendre's polynomials $\left\{q_l^i(Z)\right\}$, the integration along Z of $J(\boldsymbol{\varphi}, \mathbf{S})$ leads to

$$J\left(\boldsymbol{\varphi}^{app}, \mathbf{S}^{app}\right) = \hat{J}\left(\boldsymbol{\Phi}^g, \boldsymbol{\Phi}^{g'}, \left(\mathbf{S}^0\right)^i, \left(\mathbf{S}^1\right)^i\right)$$

For the study of the beam, the field of unknown kinematical variables is $\boldsymbol{\Phi}^g$: three data per layer. The field of unknown stresses' variables is defined by the six components per layer of $\left(\mathbf{S}^k\right)^i$ with $k = 0$ or 1. The stationarity of $\hat{J}$ can be written to obtain the simplified equations of the problem.

Step5 Definition of the physical generalized forces

The aim is to link those six components of $\left(\mathbf{S}^k\right)^i$ to six generalized forces per layer (grouped in the super-vector $\boldsymbol{\Sigma}^g$) whose physical meanings are clear, in order to rewrite $\hat{J}$ as $\tilde{J}\left(\boldsymbol{\Phi}^g, \boldsymbol{\Sigma}^g\right)$. The generalized forces per layer we define are: normal forces N^i, shearing forces Q^i, shearing and wrenching forces at interfaces $\tau^{i+}, \nu^{i+}, \tau^{i-}, \nu^{i-}$ so that : $\boldsymbol{\Sigma}^g = \left[N^i, Q^i, \tau^{i-}, \tau^{i+}, \nu^{i-}, \nu^{i+}\right]$. They can be defined by both $\mathbf{P}$ (1rst Piola-Kirchhoff stress tensor) and directions given by $\boldsymbol{\varphi}^g$ as follow:

$\mathbf{t}^i$ is the unit vector parallel to $\overline{\boldsymbol{\varphi}}^{i'}$, and $\mathbf{n}^i$ the unit vector orthogonal to $\mathbf{t}^i$. $\mathbf{t}^{i,i+1}\left(\mathbf{t}^{i-1,i}\right)$ is the unit vector parallel to $\boldsymbol{\varphi}^{i+'}\left(\boldsymbol{\varphi}^{i-'}\right)$, and $\mathbf{n}^{i,i+1}\left(\mathbf{n}^{i-1,i}\right)$ the unit vector orthogonal to $\mathbf{t}^{i,i+1}\left(\mathbf{t}^{i-1,i}\right)$. For a section $Z \in \left[h_i^-, h_i^+\right]$ of the beam initially orientated by $\mathbf{E}_1$, we define the generalized forces N^i and Q^i as:

$$\int_{\phi_3\left(X,\left[h_i^-, h_i^+\right]\right)} \boldsymbol{\sigma}\ .\mathbf{t}^i\, dz = \int_{h_i^-}^{h_i^+} \mathbf{P}.\mathbf{E}_1\, dZ = N^i\mathbf{t}^i + Q^i\mathbf{n}^i$$

For an elementary section initially orientated by $\mathbf{E}_3$, we define τ^{i+} and ν^{i+} as follow:

$$\boldsymbol{\sigma}_{\left|\phi\left(X,h_i^+\right)\right.}.\mathbf{n}^{i,i+1} = \tau^{i+}\mathbf{t}^{i,i+1} + \nu^{i+}\mathbf{n}^{i,i+1} = \mathbf{P}_{\left|h_i^+\right.}.\mathbf{E}_3\frac{1}{\left\|\boldsymbol{\varphi}^{i+'}\right\|}$$

(τ^{i-} and ν^{i-} are defined in the same manner).

Step6 Introduction of Σ_g in $J\left(\boldsymbol{\Phi}^g, \boldsymbol{\Phi}^{g'}, \left(\mathbf{S}^0\right)^i, \left(\mathbf{S}^1\right)^i\right)$

The approximation at degree one in Z of $\mathbf{P} = \mathbf{F}^{app}.\mathbf{S}^{app}$, gives a linear dependence between the six components of $\left(\mathbf{S}^k\right)^i$ and six generalized forces per layer : $\hat{J}$ can then be rewritten as $\tilde{J}\left(\boldsymbol{\Phi}^g, \boldsymbol{\Sigma}^g\right)$.

The stationarity of the functional $\tilde{J}\left(\boldsymbol{\Phi}^g, \boldsymbol{\Sigma}^g\right)$ with respect to $\boldsymbol{\Phi}^g$ and $\boldsymbol{\Sigma}^g$ gives equilibrium equations, boundary conditions and constitutive equations written with $\boldsymbol{\Phi}^g$ and $\boldsymbol{\Sigma}^g$. The model of beam with large plane deformations proposed leads to a two dimensional non linear problem where physical stresses at interfaces are introduced as variables .

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