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Abstract

The goal of this article is to provide several sharp results on the minimal time required for observability of several Grushin-type equations. Namely, it is by now well-known that Grushin-type equations are degenerate parabolic equations for which some geometric conditions are needed to get observability properties, contrarily to the usual parabolic equations. Our results concern the Grushin operator $\partial_t - \Delta_x - |x|^2 \Delta_y$ observed from the whole boundary in the multi-dimensional setting (meaning that $x \in \Omega_x$, where Ω_x is a subset of $\mathbb{R}^{d_x}$ with $d_x \geqslant 1$, $y \in \Omega_y$, where Ω_y is a subset of $\mathbb{R}^{d_y}$ with $d_y \geqslant 1$, and the observation is done on $\Gamma = \partial\Omega_x \times \Omega_y$), from one lateral boundary in the one-dimensional setting (i.e. $d_x = 1$), including some generalized version of the form $\partial_t - \partial_x^2 - (q(x))^2 \partial_y^2$ for suitable functions q, and the Heisenberg operator $\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2$ observed from one lateral boundary. In all these cases, our approach strongly relies on the analysis of the family of equations obtained by using the Fourier expansion of the equations in the y (or (y, z)) variables, and in particular the asymptotic of the cost of observability in the Fourier parameters. Combining these estimates with results on the rate of dissipation of each of these equations, we obtain observability estimates in suitably large times. We then show that the times we obtain to get observability are optimal in several cases using Agmon type estimates.

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1 Introduction

The goal of this article is to discuss observability properties of Grushin type equations under various geometric settings. It is a remarkable result known since [3] that observability properties for Grushin type equations, which are degenerate parabolic equations, may require some non-trivial positive time to hold, in strong contrast to what happens for the usual heat equations. Thus, our results will focus on providing precise estimates on the time horizon required for observability estimates for Grushin type equations to hold. In many cases, we will show that our estimates are sharp.

1.1 Scientific context

Before going further, let us start by recalling the scientific context related to our work. To begin with, we shall recall the observability results known in the context of the usual heat equation: let Ω be a smooth bounded domain of $\mathbb{R}^d$ and consider the heat equation

$$\begin{cases} (\partial_t - \Delta_x) u(t,x) = 0\,, & (t,x) \in (0,\infty) \times \Omega\,, \\ u(t,x) = 0\,, & (t,x) \in (0,\infty) \times \partial\Omega\,, \\ u(0,\cdot) = u_0 \in H_0^1(\Omega). \end{cases} \tag{1.1}$$

Given $T > 0$, the observability property for (1.1) at time T through an open subset ω of Ω reads as follows: There exists a constant $C > 0$ such that for all u solution of (1.1),

$$\|u(T)\|_{L^2(\Omega)} \leqslant C \|u\|_{L^2((0,T)\times\omega)}\,. \tag{1.2}$$

When considering the observability property for (1.1) at time T through an open subset Γ of the boundary $\partial\Omega$, the property reads as follows: There exists a constant $C > 0$ such that for all u solution of (1.1),

$$\|u(T)\|_{L^2(\Omega)} \leqslant C \,\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)}\,, \tag{1.3}$$

where ∂_ν denotes the normal derivative of the solution on the boundary of Ω.

Observability is well known to hold for the linear heat equation set in a smooth bounded domain Ω in any arbitrary positive time T for any non-empty observation set, whether it is a distributed domain ω or a non-empty open subset Γ of the boundary. We refer to the works [27] and [21] for the proof of this result (we shall also quote the work [19, Theorem 3.3] when the observation is performed on the boundary of a one-dimensional domain Ω).

More recently, the community investigated this question of observability for degenerate parabolic equations, and several works have shown that they exhibit a wider range of behaviors: In particular, observability may hold true or not depending on the strength of degeneracy of the parabolic operator, the time horizon T, and the geometry.

Strength of the degeneracy It has been shown in the literature that only degenerate parabolic equations with weak enough degeneracies share the same observability properties as the heat equation. We will not detail the case of boundary degeneracy in one space dimension, which is by now rather well understood and for which we refer to the works [12], [13], [1], [28], [10], [9], and [22]. Fewer results are available for multidimensional problems, see [14] and the recent book [15].

For parabolic equations with interior degeneracy, a fairly complete analysis is available for the following Grushin type operators, set in the particular geometry $\Omega := \Omega_x \times \Omega_y$, where Ω_x is a bounded open subset of $\mathbb{R}^{d_x}$ such that $0 \in \Omega_x$, Ω_y is a bounded open subset of $\mathbb{R}^{d_y}$, and $d_x, d_y \in \mathbb{N}^*$:

$$\begin{cases} (\partial_t - \Delta_x - |x|^{2\gamma}\Delta_y)u(t,x,y) = 0\,, & (t,x,y) \in (0,T)\times\Omega\,,\\ u(t,x,y) = 0\,, & (t,x,y) \in (0,T)\times\partial\Omega\,,\\ u(0,.,.) = u_0 \in H_0^1(\Omega)\,. \end{cases} \tag{1.4}$$

where $\gamma > 0$ is a fixed parameter which describes the degeneracy of the parabolic operator.

The observability property at time T for (1.4) through a distributed domain ω (respectively an open subset Γ of the boundary) then reads as follows: There exists a constant $C > 0$ such that all solutions of (1.4) satisfy (1.2) (respectively (1.3)).

It is proved in [3, 4] that the observability inequality holds in any positive time $T > 0$ and with an arbitrary open set $\omega \subset \Omega$ if and only if $\gamma \in (0,1)$. Roughly speaking, this asserts that if the degeneracy is not too strong, i.e. $\gamma < 1$, then the equations (1.4) satisfies the same observability properties as the classical heat equation (1.1), in the sense that observability holds true for any time $T > 0$ and any non-empty open subset ω of Ω. Moreover, [3, 4] show that if $\gamma > 1$ and $\overline{\omega} \cap \{x = 0\} = \emptyset$ with $d_x = 1$, then, whatever $T > 0$ the Grushin equation (1.4) is not observable on $(0,T)\times\omega$. The critical value of γ is then $\gamma = 1$, which is precisely the case that we will handle in this article.

Minimal time For several degenerate parabolic equations, in specific geometric configurations (Ω, ω), a positive minimal time is known to be required for observability to hold. This is in particular the case for the Grushin equation (1.4) with $\gamma = 1$ and $d_x = 1$ when $\omega = \omega_x \times \Omega_y$ and $\overline{\omega_x} \cap \{x = 0\} = \emptyset$, see [3]. To be more precise, given a non-empty subdomain $\omega = \omega_x \times \Omega_y$ of Ω such that $\overline{\omega_x} \cap \{x = 0\} = \emptyset$, it is shown that there exists a critical time $T_* = T_*(\omega, \Omega)$ such that

- The Grushin equation (1.4) (in the case $\gamma = 1$, $d_x = 1$) is not observable through ω in any time $T < T_*$;
- The Grushin equation (1.4) (in the case $\gamma = 1$, $d_x = 1$) is observable through ω in any time $T > T_*$.

The explicit value of this minimal time is obtained in [6] when $\Omega_x = (-1,1)$, $\omega_x = (-1,-a) \cup (a,1)$ and $a \in (0,1)$, for which it is proved that $T_*(\omega,\Omega) = a^2/2$, but there are still many geometric settings for which the precise value of the critical time is not known. Our goal precisely is to give the precise values of the critical times in several geometric settings.

Geometric control condition Let us also mention that, when considering the Grushin equation (1.4) with $d_x = d_y = 1$ and $\gamma = 1$, the work [25] proves that when there exists an horizontal strip which does not intersect ω, then the Grushin equation (1.4) is not observable through ω whatever the time $T > 0$. This emphasizes the requirement of a geometric condition on (Ω, ω) for the Grushin equation with $\gamma = 1$ to be observable on ω. In that setting, the characterization of the sets ω for which observability holds in some time $T > 0$ still seems to be a delicate matter.

This is why our work will focus on cases where the control set is tensorized. Namely, we shall consider the case of boundary observations through sets Γ of the form $\Gamma = \Gamma_x \times \Omega_y$ when Ω takes the form $\Omega = \Omega_x \times \Omega_y$.

Note that, by duality, the observability properties of Grushin equations through Γ are equivalent to the null controllability of Grushin equations with controls acting on Γ. We refer to the textbook [31] for an abstract setting developing these equivalences, and to [3] for more details in the context of Grushin-type equations. This is why we will not investigate the case of distributed observation sets ω, as our results can be extended to cases of tensorized observation sets of the form $\omega = \omega_x \times \Omega_y$ easily by straightforward cut-off and extension arguments on the control problem.

Other related models The above discussion can be extended to Grushin-type operators having singular lower order terms, see e.g. [11] and [30], or for other models, such as Kolmogorov-type equations, see [5]. In fact, we believe that the approach we present here may also allow to investigate the precise value of the critical time of observability for Kolmogorov-type equations in some cases.

Finally note that positive controllability results are also available for hypoelliptic equations on the whole space, with appropriate smoothing properties (in Gevrey or Gelfand-Shilov spaces) and under appropriate geometric assumptions on the control support, see e.g. [26, 8, 7].

1.2 The classical Grushin equations

The multi-dimensional case First, we consider the multi-dimensional classical Grushin equation in a domain $\Omega = \Omega_x \times \Omega_y$, where Ω_x and Ω_y are smooth bounded domains of $\mathbb{R}^{d_x}$ and $\mathbb{R}^{d_y}$ respectively and $d_x, d_y \in \mathbb{N}^*$, which reads as follows:

$$\begin{cases} (\partial_t - \Delta_x - |x|^2 \Delta_y) u(t,x,y) = 0\,, & (t,x,y) \in (0,T) \times \Omega\,, \\ u(t,x,y) = 0\,, & (t,x,y) \in (0,T) \times \partial\Omega\,, \\ u(0,.,.) = u_0 \in H_0^1(\Omega)\,. \end{cases} \tag{1.5}$$

To begin with, we are interested in the boundary observability in time T, when the observation is taken on the part $\Gamma = \partial\Omega_x \times \Omega_y$ of the boundary. In other words, we ask if there exists $C > 0$ such that for any solution u of (1.5) with $u_0 \in H_0^1(\Omega)$,

$$\int_\Omega |u(T,x,y)|^2 dxdy \leqslant C \int_0^T \int_{\partial\Omega_x} \int_{\Omega_y} |\partial_{\nu_x} u(t,x,y)|^2 \, dyds(x)dt, \tag{1.6}$$

where ∂_{ν_x} denotes the normal derivative on $\partial\Omega_x$ and $ds(x)$ is the surface measure on $\partial\Omega_x$.

We will prove the following result.

Theorem 1.1. *Let Ω_x and Ω_y be smooth bounded domains of $\mathbb{R}^{d_x}$ and $\mathbb{R}^{d_y}$ respectively, $d_x, d_y \in \mathbb{N}^*$, and define*

$$L = \sup_{x \in \Omega_x} |x|, \quad T_* = \frac{L^2}{2d_x}. \tag{1.7}$$

Then:

(i) For any time $T > T_*$, there exists a constant C such that for all solutions u of (1.5) *with $u_0 \in H_0^1(\Omega)$, the observability estimate* (1.6) *is satisfied.*

(ii) If $\Omega_x = B(0, L)$, then for any $T \in (0, T_*)$, there is no constant $C > 0$ for which estimate (1.6) *holds for all solutions u of* (1.5) *with $u_0 \in H_0^1(\Omega)$.*

When considering the Grushin equation (1.5) in $\Omega = (-L, L) \times (0, \pi)$, corresponding to $\Omega_x = (-L, L)$ and $\Omega_y = (0, \pi)$, observed from both sides $\Gamma = \{-L, L\} \times (0, \pi)$ ($= \partial\Omega_x \times \Omega_y$), we know from [6][1] that the time $T_* = L^2/2$ is indeed the critical time for observability. Therefore, Theorem 1.1 generalizes the positive result of null-controllability of [6] in large times for the Grushin equation (1.5), and recovers the time known as the sharp time of null-controllability when $\Omega_x = (-L, L)$, $\Omega_y = (0, \pi)$. Note also that [4] derives positive null-controllability results for (1.5) in large times, but with a time T which is not explicitly estimated.

The proof of Theorem 1.1 item (i) is done in Section 2.2 and the proof of Theorem 1.1 item (ii), which closely follows the one of [3, Theorem 5 for $\gamma = 1$], is postponed to Section 5.2.

Theorem 1.1 item (i) is shown by looking at observability properties of the family of equations, indexed by $n \in \mathbb{N}$,

$$\begin{cases} (\partial_t - \Delta_x + \mu_n^2 |x|^2) u_n(t, x) = 0\,, & (t, x) \in (0, T) \times \Omega_x\,, \\ u_n(t, x) = 0\,, & (t, x) \in (0, T) \times \partial\Omega_x\,, \\ u_n(0, .) = u_{0,n} \in H_0^1(\Omega_x), \end{cases} \tag{1.8}$$

which are obtained by expanding the solution u of (1.5) on the basis of eigenfunctions of the operator $-\Delta_y$ with domain $H^2 \cap H_0^1(\Omega_y)$ on $L^2(\Omega_y)$, where μ_n^2 is the n-th eigenvalue of this operator. This allows to reduce the proof of Theorem 1.1 to the proof of observability properties for the family of equations (1.8), uniformly with respect to the parameter n. Following [3], such uniform observability properties for (1.8) are proved by combining the following two ingredients:

- For $T_0 > 0$ arbitrary, an estimate of the cost of observability of each equation (1.8), with precise estimates in the asymptotics $n \to \infty$.
- Dissipation estimates for the semi-group defined by (1.8), with precise estimates in the asymptotics $n \to \infty$.

Concerning the family of equations (1.8), the most delicate part is the analysis of the cost of observability of each equation (1.8). We do it using a global Carleman estimate:

Lemma 1.2. *For every $T_0 > 0$, there exists $C = C(T_0) > 0$ and $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$ and $u_{0,n} \in H_0^1(\Omega_x)$, the solution u_n of* (1.8) *satisfies*

$$\int_{\Omega_x} |u_n(T_0, x)|^2 e^{-\mu_n \coth(2\mu_n T_0)(L^2 - |x|^2)} dx \leqslant C\mu_n \int_0^{T_0} \int_{\partial\Omega_x} |\partial_{\nu_x} u_n(t, x)|^2 ds(x)\, dt\,, \tag{1.9}$$

where L is as in (1.7).

The detailed proof of Theorem 1.1 item (i) is given in Section 2.2, including the proof of Lemma 1.2 in Section 2.2.2.

Let us also point out that the proof of Theorem 1.1 in the one-dimensional case provided by [6] also relies on an estimate on the cost of observability of the equations (1.8) in the asymptotics $n \to \infty$. But the proof of the estimate in [6] is done using precise estimate on the cost of observability of a family of wave type equations and transmutation techniques [29]. Thus, it strongly differs from the approach we propose in Lemma 1.2, which is based on direct Carleman estimates adapted to the equations (1.8).

The case $\Omega_x = (0, L)$ and $\Omega_y = (0, \pi)$. We now focus on the one-dimensional case and discuss the observability properties of the Grushin equations in the case $\Omega = (0, L) \times (0, \pi)$, depending on the boundary conditions imposed at $x = 0$. To be more precise, we shall focus on the equation

$$\begin{cases} (\partial_t - \partial_x^2 - x^2 \partial_y^2) u(t, x, y) = 0, & (t, x, y) \in (0, T) \times \Omega\,, \\ u(t, x, y) = 0\,, & (t, x, y) \in (0, T) \times \partial\Omega\,, \\ u(0, ., .) = u_0 \in H_0^1(\Omega)\,, \end{cases} \tag{1.10}$$

and on the equation

$$\begin{cases} (\partial_t - \partial_x^2 - x^2 \partial_y^2) u(t, x, y) = 0, & (t, x, y) \in (0, T) \times \Omega\,, \\ u(t, x, y) = 0\,, & (t, x, y) \in (0, T) \times (\partial\Omega \setminus (\{0\} \times (0, \pi)))\,, \\ \partial_x u(t, 0, y) = 0\,, & (t, y) \in (0, T) \times (0, \pi)\,, \\ u(0, ., .) = u_0 \in H_N^1(\Omega), \end{cases} \tag{1.11}$$

[1] In fact, most of the references below are concerned with the case of a distributed control. But, as mentioned in Section 1.1, easy cut-off / extension arguments also yield similar results for the Grushin equations controlled from the boundary.

where $H^1_N(\Omega)$ denotes the set of functions of $H^1(\Omega)$ whose trace on $\partial\Omega \setminus (\{0\} \times (0,\pi))$ vanishes.

Here, we are interested in the following observability inequality at time T for (1.10) (respectively (1.11)): There exists a constant $C > 0$ such that for any solution u of (1.10) (respectively (1.11)) with initial datum $u_0 \in H^1_0(\Omega)$ (respectively $H^1_N(\Omega)$), we have

$$\int_\Omega |u(T,x,y)|^2 dxdy \leqslant C \int_0^T \int_0^\pi |\partial_x u(t,L,y)|^2 \, dydt. \tag{1.12}$$

We show the following result:

Theorem 1.3. *Let $\Omega = (0,L) \times (0,\pi)$ and define*

$$T_D = \frac{L^2}{6}, \qquad T_N = \frac{L^2}{2}. \tag{1.13}$$

Then

(i) For any time $T > T_D$ (respectively $T > T_N$), there exists a constant C such that for all solutions u of (1.10)*(respectively* (1.11)*) with $u_0 \in H^1_0(\Omega)$ (respectively $u_0 \in H^1_N(\Omega)$) the observability estimate* (1.12) *is satisfied.*

(ii) For any $T \in (0,T_D)$ (respectively $T \in (0,T_N)$), there is no constant $C > 0$ for which estimate (1.12) *holds for all solutions u of* (1.10) *(respectively* (1.11)*) with $u_0 \in H^1_0(\Omega)$ (respectively $u_0 \in H^1_N(\Omega)$).*

One may be surprised at first that the critical times of observability for (1.10) and (1.11) differ, thus showing that the critical time of observability depends on the boundary conditions at $x = 0$. But one should keep in mind that here the singularity of the Grushin operator precisely lies at $x = 0$, and therefore the change of boundary conditions at $x = 0$ is of paramount importance.

Theorem 1.3 is proved along the same lines as Theorem 1.1, and the main idea is to prove uniform observability results for the following family of one-dimensional heat equations, indexed by the integer $n \in \mathbb{N}$: corresponding to the case of Dirichlet boundary conditions in $x = 0$, we consider

$$\begin{cases} \partial_t u_n - \partial_x^2 u_n + n^2 x^2 u_n = 0, & \text{in } (0,T) \times (0,L), \\ u_n(t,0) = 0, \quad u_n(t,L) = 0, & \text{in } (0,T), \\ u_n(0,x) = u_{0,n}(x), & \text{in } (0,L), \end{cases} \tag{1.14}$$

while, corresponding to the case of Neumann boundary conditions in $x = 0$, we consider instead

$$\begin{cases} \partial_t u_n - \partial_x^2 u_n + n^2 x^2 u_n = 0, & \text{in } (0,T) \times (0,L), \\ \partial_x u_n(t,0) = 0, \quad u_n(t,L) = 0, & \text{in } (0,T), \\ u_n(0,x) = u_{0,n}(x), & \text{in } (0,L). \end{cases} \tag{1.15}$$

As before, we shall proceed in two steps:

- For $T_0 > 0$ arbitrary, an estimate of the cost of observability of each equation (1.14), respectively (1.15), with precise estimates in the asymptotics $n \to \infty$.
- Dissipation estimates for the semi-groups defined by (1.14), respectively (1.15), with precise estimates in the asymptotics $n \to \infty$.

In here, it turns out that the estimates on the cost of observability for both families of equations (1.14) and (1.15) are the same, but the dissipation estimates differ, thus yielding to a difference of the critical times of observability for (1.10) and for (1.11).

The proof of Theorem 1.3 item (i) is given in Section 2.3, while Theorem 1.3 item (ii) is proven in Section 5.3.

1.3 Two-dimensional Grushin equation observed on one vertical side

Let $L_- > 0$ and $L_+ > 0$, and let us consider the Grushin-type equation, in the two-dimensional rectangle domain $\Omega = (-L_-, L_+) \times (0,\pi)$:

$$\begin{cases} (\partial_t - \partial_x^2 - q(x)^2 \partial_y^2) u(t,x,y) = 0\,, & (t,x,y) \in (0,T) \times \Omega\,, \\ u(t,x,y) = 0\,, & (t,x,y) \in (0,T) \times \partial\Omega\,, \\ u(0,.,.) = u_0 \in H^1_0(\Omega)\,. \end{cases} \tag{1.16}$$

Here, we shall assume that q satisfies the following conditions:

$$q(0) = 0, \quad q \in C^3([-L_-, L_+]), \quad \inf_{(-L_-, L_+)} \{\partial_x q\} > 0, \tag{1.17}$$

which encompasses the classical case $q(x) = x$ and slightly generalizes the Grushin type operators that we can handle. We refer to [3] for well posedness results.

Here, we are interested in the boundary observability, when the observation is taken on one vertical side of the rectangle Ω, namely $\Gamma = \{L_+\} \times (0, \pi)$: System (1.16) is observable in time T through Γ if there exists a constant C such that for all solutions u of (1.16) with $u_0 \in H_0^1(\Omega)$,

$$\int_\Omega |u(T, x, y)|^2 dx dy \leqslant C \int_0^T \int_0^\pi |\partial_x u(t, L_+, y)|^2 dy dt. \tag{1.18}$$

We then prove the following result:

Theorem 1.4. *Let $\Omega = (-L_-, L_+) \times (0, \pi)$ with $L_- > 0$ and $L_+ > 0$ and set $\Gamma = \{L_+\} \times (0, \pi)$. Assume* (1.17).

The minimal time for observing system (1.16) *through Γ is*

$$T_* = \frac{1}{q'(0)} \int_0^{L_+} q(s)\, ds. \tag{1.19}$$

More precisely,

(i) for every $T > T_*$, system (1.16) *is observable through Γ ,*

(ii) for every $T \in (0, T_*)$, system (1.16) *is not observable through Γ.*

In fact, it was already known (see [3]) that the Grushin equation (1.16) with $q(x) = x$ and $L_+ = L_-$ is not observable on $(0, T) \times \Gamma$ if the time T is smaller than $T_* = L_+^2/2$. Theorem 1.4 extends this negative result to arbitrary L_+, L_- and q satisfying (1.17), and establishes the positive counterpart for $T > T_*$.

When observing the Grushin equation (1.16) from both sides $\Gamma = \{-L, L\} \times (0, \pi)$, it was shown in [6] that the time $T_* = L^2/2$ was indeed the critical time, in the particular case $q(x) = x$ and $L_+ = L_- = L$. Consequently, Theorem 1.4 proves that $T_* = L^2/2$ is also the critical time for observing this equation from $\Gamma = \{L\} \times (0, \pi)$, i.e. from one side of the domain only.

Let us also recall that if one horizontal strip does not meet the observation set, then the Grushin equation (1.16) with $q(x) = x$ is not observable, whatever the time $T > 0$ is, see [25]. It is therefore natural to restrict ourselves to the case of a tensorized observation set of the form $\Gamma = \{L\} \times (0, \pi)$.

Again, the proof of Theorem 1.4 relies on the analysis of the observability properties of a family of one-dimensional equations obtained after expanding the solution u of (1.16) in Fourier series in the variable y. To be more precise, this allows to look at a family of one-dimensional parabolic equations, indexed by $n \in \mathbb{N}^*$:

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 q(x)^2) u_n = 0\,, & (t, x) \in (0, T) \times (-L_-, L_+)\,, \\ u_n(t, -L_-) = u_n(t, L_+) = 0\,, & t \in (0, T)\,, \\ u_n(0, .) = u_{0,n} \in H_0^1(-L_-, L_+)\,. & \end{cases} \tag{1.20}$$

Here again, we provide a precise estimate on the cost of observability of (1.20) when $n \to \infty$:

Proposition 1.5. *Assume* (1.17). *For every $T_0 > 0$ and $\varepsilon > 0$, there exists $C > 0$ such that, for every $n \in \mathbb{N}$, any solution u_n of* (1.20) *with $u_{0,n} \in H_0^1(-L_-, L_+)$ satisfies*

$$\|u_n(T_0)\|_{L^2(-L_-, L_+)} \leqslant C \exp\left(n \left(\int_0^{L_+} q(s)\, ds + \varepsilon \right) \right) \|\partial_x u_n(., L_+)\|_{L^2(0, T_0)}\,. \tag{1.21}$$

In particular, Proposition 1.5 states observability results for the family of equations (1.20) in small times, but with an explicit dependence of the observability constant with respect to $n \in \mathbb{N}^*$. It can then be suitably combined with dissipation estimates for the semi-groups corresponding to (1.21) (see Lemma 3.7 and Section 4.3) to recover that the family of equations (1.20) are uniformly observable with respect to $n \in \mathbb{N}^*$ provided the time T is larger than T_*.

The proof of Proposition 1.5 is proved by a gluing argument between two appropriate Carleman estimates:

- a dominant one on $(0, L_+)$ in which the weight function roughly behaves like $x \mapsto n \int_x^{L_+} q(s)\, ds$, strongly inspired by the Agmon distance associated to the potential $n^2 q^2(x)$.
- a second one on $(-L_-, 0)$ on which the weight function is essentially constant equal to $n \int_0^{L_+} q(s)\, ds$, up to lower order terms of order $\sqrt{n}$.

The detailed proof of Proposition 1.5 is given in Section 3.2.2. We then show how it implies Theorem 1.4 item (i) in Section 3.2.3. The proof of Theorem 1.4 item (ii) is postponed to Section 5.1.

1.4 Further results

The 3-dimensional Heisenberg equation. The techniques developed to prove Theorem 1.4 apply also to the 3-d Heisenberg equation. More precisely, we consider the heat equation on the Heisenberg group

$$\begin{cases} \big(\partial_t - \partial_x^2 - (x\partial_z + \partial_y)^2\big) u = 0\,, & (t,x,y,z) \in (0,T) \times \Omega\,, \\ u(t, -L_-, y, z) = u(t, L_+, y, z) = 0\,, & (t,y,z) \in (0,T) \times \mathbb{T} \times \mathbb{T}\,, \\ u(0,x,y,z) = u_0(x,y,z)\,, & (x,y,z) \in \Omega\,, \end{cases} \tag{1.22}$$

where $\mathbb{T}$ is the 1-d torus and $\Omega = (-L_-, L_+) \times \mathbb{T} \times \mathbb{T}$. We refer to [4] for well posedness results for system (1.22). We are interested in the observability of equation (1.22) through one side $\Gamma = \{L_+\} \times \mathbb{T} \times \mathbb{T}$ of the cubic domain Ω. More precisely, we will say that system (1.22) is observable in time T from $\Gamma = \{L_+\} \times \mathbb{T} \times \mathbb{T}$ if there exists a constant $C > 0$ such that, for all solution u of (1.22) with $u_0 \in H_0^1(\Omega)$,

$$\int_\Omega |u(T,x,y,z)|^2 dxdydz \leqslant C \int_0^T \int_\mathbb{T} \int_\mathbb{T} |\partial_x u(t, L_+, y, z)|^2 dydzdt\,. \tag{1.23}$$

We prove the following

Theorem 1.6. *Let $\Omega = (-L_-, L_+) \times \mathbb{T} \times \mathbb{T}$ with $L_- > 0$ and $L_+ > 0$.*
The minimal time for observing system (1.22) through $\Gamma = \{L_+\} \times \mathbb{T} \times \mathbb{T}$ is

$$T_* = \frac{(L_+ + L_-)^2}{2}\,. \tag{1.24}$$

More precisely

(i) for every $T > T_*$, system (1.22) is observable in time T through Γ,

(ii) for every $T \in (0, T_*)$, system (1.22) is not observable in time T through Γ.

By giving the precise value of the minimal time T_*, this statement improves [2, Theorem 2], that only establishes the lower bound $T_* \geqslant (L_+ + L_-)^2/8$.

The proof of Theorem 1.6 item (i) is given in Section 3.3, and item (ii) is proven in Section 5.4.

Remark 1.7. *In the case of the 3-d Heisenberg equation (1.22) observed from both sides of the cubic domain Ω, that is $\Gamma = \{-L_-, L_+\} \times \mathbb{T} \times \mathbb{T}$, it is possible to obtain the following result:*

Theorem 1.8. *Let $\Omega = (-L_-, L_+) \times \mathbb{T} \times \mathbb{T}$ with $L_- > 0$ and $L_+ > 0$, and set $T_* = (L_+ + L_-)^2/8$. Then for any $T > T_*$, there exists a constant $C > 0$ such that for any function u solution of (1.22) with $u_0 \in H_0^1(\Omega)$,*

$$\int_\Omega |u(T,x,y,z)|^2 dxdydz \leqslant C \int_0^T \int_\mathbb{T} \int_\mathbb{T} \left(|\partial_x u(t, -L_-, y, z)|^2 + |\partial_x u(t, L_+, y, z)|^2\right) dydzdt\,.$$

In other words, for every $T > T_*$, system (1.22) is observable in time T through Γ.

On the other hand, it is already known, see [2, Theorem 2], that in that configuration, $T_* \geqslant (L_+ + L_-)^2/8$, so that Theorem 1.8 is sharp. A sketch of the proof of Theorem 1.8 is given in Section 3.3.6.

Inverse source problems. We shall also provide, as a corollary of Proposition 1.5 (or of a slightly stronger version of it, see Proposition 3.6), some result on an inverse source problem previously studied in [4].

As before, let $L_- > 0$ and $L_+ > 0$, and consider the Grushin-type equation, in the two-dimensional rectangle domain $\Omega = (-L_-, L_+) \times (0, \pi)$:

$$\begin{cases} (\partial_t - \partial_x^2 - x^2\partial_y^2)u = f\,, & (t,x,y) \in (0,T) \times \Omega\,, \\ u(t,x,y) = 0\,, & (t,x,y) \in (0,T) \times \partial\Omega\,, \\ u(0,.,.) = u_0 \in H_0^1(\Omega)\,, \end{cases} \tag{1.25}$$

with f a source term of the form

$$f(t,x,y) = R(t,x)k(x,y) \quad \text{ for } (t,x,y) \in (0,T) \times \Omega, \tag{1.26}$$

where $R = R(t,x)$ is assumed to be known and to satisfy

$$R \in C^0([0,T] \times [-L_-, L_+])\,, \quad \text{ and } \quad \inf_{[-L_-,L_+]} |R(T_1, x)| > 0, \tag{1.27}$$

for some $T_1 \in [0,T]$, and $k \in L^2(\Omega)$ is an unknown function.

Here, our goal is to recover the unknown function k from informations at time $T_1 \in [0,T]$ in the whole domain Ω and on the time interval (T_0, T) on the boundary $\Gamma = \{L_+\} \times (0, \pi)$, for suitable choices of T_0, T_1.

We will establish in Section 3.4 a Lipschitz stability estimate for this inverse problem when $T_1 - T_0 > T_*$ and $T_1 < T$ in the following sense.

Theorem 1.9. *Let $\Omega = (-L_-, L_+) \times (0,\pi)$ and $\Gamma = \{L_+\} \times (0,\pi)$. Let T_0, T_1 be such that*

$$0 < T_0 < T_1 < T \quad \text{ and } \quad T_1 - T_0 > T_*, \quad \text{ where } T_* = \frac{L_+^2}{2}, \tag{1.28}$$

and assume that R satisfies (1.27).

There exists $C > 0$ such that, for every $k \in L^2(\Omega)$ and $u_0 \in H_0^1(\Omega)$, the solution u of (1.25) *satisfies*

$$\int_\Omega |k(x,y)|^2 dxdy \leqslant C \left(\int_{T_0}^T \int_0^\pi |\partial_x \partial_t u(t, L_+, y)|^2 dydt + \int_\Omega |(\partial_x^2 + x^2\partial_y^2)u(T_1, x, y)|^2 dxdy \right). \tag{1.29}$$

Note that Theorem 1.9 is similar to the one obtained in [4] but yields an explicit estimate on the time interval of observation during which the measurements are done, which can be made of any length $T_1 - T_0 > T_*$. Whether T_* is the minimal time for the Lipschitz stability estimate above is an open problem.

To conclude, we mention that one could prove stability estimates for similar inverse source problems corresponding to the multidimensional Grushin equation (1.5), and to the Heisenberg equation (1.22), using the same arguments as the one developed to prove Theorem 1.9.

1.5 Outline

Theorems 1.1, 1.3, 1.4, and 1.6 are all stating two results: one positive result provided that the time T is large enough, namely $T > T_*$ (the value of T_* varies in each theorem), and one optimality result asserting that if $T < T_*$, then the observability inequality cannot hold. We therefore made the choice to gather all the proofs of the positive results together, and postpone the proof of their optimality to Section 5.

Each of the positive results, i.e. items (i) in Theorems 1.1, 1.3, 1.4, and 1.6 and Theorems 1.8 and 1.9, relies on the same strategy:

- a precise estimate on the cost of observability for a family of heat equations obtained by expanding the solution in Fourier series, in particular with respect to the Fourier parameter;
- a precise estimate on the rate of dissipation for a family of heat equations obtained by expanding the solution in Fourier series.

The second step is more classical. We have therefore chosen to state the dissipation results we need during the proof of each of the positive results (Lemmas 2.3, 2.7, 3.7, and 3.11), and postpone their proof in an independent section, namely Section 4.

The first step, however, deserves more attention, and really corresponds to the main improvements of this article with respect to the literature. We shall therefore focus on this step in most of the article. We thus present in Section 2 the proofs of Theorems 1.1 and 1.3 items (i), while Section 3 gives the proofs of Theorems 1.4 and 1.6 items (i) and of Theorems 1.8 and 1.9.

To sum up, the outline of the article is as follows. The positive results corresponding to Theorems 1.1 and 1.3 items (i) are proved in Section 2. Theorems 1.4 and 1.6 items (i) and Theorems 1.8 and 1.9 are proved in Section 3. Section 4 proves the various dissipation results stated in Sections 2 and 3. Section 5 proves Theorems 1.1, 1.3, 1.4 and 1.6 items (ii). Finally, in Section A, we recall one of the results of [20] on how the observability constant of the heat equation with a potential depends on the norm of the potential, which will be used all along the article.

2 The classical Grushin equation

The goal of this section is to prove items (i) of Theorem 1.1 and of Theorem 1.3 using an appropriate global Carleman estimate proved in the following subsection.

2.1 A global Carleman estimate

Lemma 1.2 is the main step of the proof of Theorem 1.1 and relies on the observability property of the solutions u_n of (1.8). Following the statement of Lemma 1.2, it is interesting to introduce, for each u_n solving (1.8), the function

$$z_n(t,x) = u_n(t,x)\exp\left(-\frac{\mu_n}{2}\coth(2\mu_n t)\left(L^2-|x|^2\right)\right), \quad (t,x)\in(0,T)\times\Omega_x,$$

which solve

$$\begin{cases} \partial_t z_n - \Delta_x z_n + \mu_n\coth(2\mu_n t)\,(2\,x\cdot\nabla_x z_n + d_x\, z_n) - \dfrac{L^2\mu_n^2}{\sinh(2\,\mu_n\,t)^2} z_n = 0, & \text{in } (0,T)\times\Omega_x,\\ z_n(t,x) = 0, & \text{on } (0,T)\times\partial\Omega_x,\\ \lim\limits_{t\to 0}\|z_n(t)\|_{L^2(\Omega_x)} = 0, & \\ \lim\limits_{t\to 0} t\,\|\nabla_x z_n(t)\|_{L^2(\Omega_x)} = 0. & \end{cases}$$

Thus, in this section, for a generic parameter $\mu\in\mathbb{R}_+$, we consider the system

$$\begin{cases} \partial_t z - \Delta_x z + \mu\coth(2\mu t)\,(2\,x\cdot\nabla_x z + d\, z) - \dfrac{L^2\mu^2}{\sinh(2\,\mu\,t)^2} z = 0, & \text{in } (0,T)\times\Omega,\\ z(t,x) = 0, & \text{on } (0,T)\times\partial\Omega,\\ \lim\limits_{t\to 0}\|z(t)\|_{L^2(\Omega)} = 0, & \\ \lim\limits_{t\to 0} t\,\|\nabla_x z(t)\|_{L^2(\Omega)} = 0, & \end{cases} \tag{2.1}$$

where $T,\mu>0$ are fixed, Ω is a bounded domain of $\mathbb{R}^d$, $d\geqslant 1$, and $L=\sup_\Omega|x|$. We then have the following result:

Proposition 2.1. *Any smooth solution z of* (2.1) *verifies the following estimate:*

$$\int_\Omega\left(|\nabla_x z(T)|^2 - \frac{\mu^2}{\sinh(\mu\,T)^2}\frac{L^2}{2}|z(T)|^2\right)dx \leqslant \mu\,L\int_0^T\left(\frac{\sinh(4\,\mu\,t)}{\sinh(2\,\mu\,T)^2}\int_{\Gamma_+}|\nabla_x z(t,x)\cdot\nu|^2\,ds(x)\right)dt \tag{2.2}$$

where $\Gamma_+ = \{x\in\partial\Omega; \langle x,\nu_x\rangle>0\}$ and $L=\sup\{|x|,\, x\in\Omega\}$.

Proof. We denote $\theta(t) = \mu \coth(2\,\mu\, t)$. It is readily seen that

$$\theta'' = -4\theta\theta'', \quad t \in (0,T], \tag{2.3}$$

$$\lim_{t\to 0} 2\,t\,\theta(t) = 1, \quad \text{and} \quad \limsup_{t\to 0} t^2\,|\theta'(t)| < \infty. \tag{2.4}$$

We define the following spatial operators

$$S\,z = -\Delta_x z + \theta'(t)\frac{L^2}{2}z, \quad A\,z = \theta(t)\,(2\,x\cdot\nabla_x z + d\,z)$$

so that z solution of (2.1) verifies

$$\partial_t z + Sz + Az = 0 \text{ in } (0,T)\times\Omega.$$

Note that S and A respectively correspond to the symmetric and skew-symmetric parts of the operator in (2.1). We then consider

$$D(t) = \int_\Omega \left(|\nabla_x z|^2 + \theta'(t)\frac{L^2}{2}|z|^2\right)\,dx = \int_\Omega Sz\,\bar{z}\,dx.$$

A direct calculation shows that

$$D'(t) = \theta(t)''\frac{L^2}{2}\int_\Omega |z|^2\,dx - 2\int_\Omega |S\,z|^2\,dx - 2\Re\left(\int_\Omega S\,z\,\overline{A\,z}\,dx\right).$$

Furthermore, as A is a skew-symmetric operator, we have

$$-2\int_\Omega Sz\,\overline{Az}\,dx = 2\int_\Omega \Delta_x z\,\overline{Az}\,dx = 2\,\theta(t)\int_\Omega \Delta_x z\,(2\,x\cdot\nabla_x\bar{z} + d\,\bar{z})\,dx.$$

On one hand, we obviously have

$$\int_\Omega \Delta_x z\,d\,\bar{z}\,dx = -d\int_\Omega |\nabla_x z|^2\,dx.$$

On the other hand, we note that

$$\begin{aligned}\int_\Omega \Delta_x z\,2\,x\cdot\nabla_x\bar{z}dx &= 2\int_{\partial\Omega}(\nabla_x z\cdot\nu)\,(x\cdot\nabla_x\bar{z})\,ds(x) - 2\int_\Omega \nabla_x z\cdot\nabla_x\,(x\cdot\nabla_x\bar{z})\,dx\\ &= 2\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) - 2\int_\Omega \nabla_x z\cdot\nabla_x\,(x\cdot\nabla_x\bar{z})\,dx.\end{aligned}$$

Here, we have used that as $z = 0$ on $\partial\Omega$, $\nabla_x z = (\nabla_x z\cdot\nu)\nu$ on $\partial\Omega$. As

$$\Re\,(\nabla_x z\cdot\nabla_x\,(x\cdot\nabla_x\bar{z})) = |\nabla_x z|^2 + \frac{x}{2}\cdot\nabla_x\left(|\nabla_x z|^2\right),$$

we have

$$\begin{aligned}\Re\left(\int_\Omega \nabla_x z\cdot\nabla_x\,(x\cdot\nabla_x\bar{z})\,dx\right) &= \int_\Omega |\nabla_x z|^2\,dx + \int_\Omega \frac{x}{2}\cdot\nabla_x\left(|\nabla_x z|^2\right)\,dx\\ &= \int_\Omega |\nabla_x z|^2\,dx + \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z|^2\,ds(x) - \frac{d}{2}\int_\Omega |\nabla_x z|^2\,dx\\ &= \int_\Omega |\nabla_x z|^2\,dx + \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) - \frac{d}{2}\int_\Omega |\nabla_x z|^2\,dx.\end{aligned}$$

Gathering the above computations with (2.3), we get that

$$D'(t) + 2\int_\Omega |Sz|^2\,dx = \theta''(t)\frac{L^2}{2}\int_\Omega |z|^2\,dx - 4\theta(t)\int_\Omega |\nabla_x z|^2\,dx + 2\,\theta(t)\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x).$$

Using (2.3), we finally obtain

$$D'(t) + 4\,\theta(t)\,D(t) + 2\int_\Omega |Sz|^2\,dx = 2\,\theta(t)\int_{\partial\Omega} (x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x).$$

Denoting $\Psi(t) = -4\displaystyle\int_t^T \theta(s)\,ds = \ln\left(\frac{\sinh(2\,\mu\,t)^2}{\sinh(2\,\mu\,T)^2}\right)$, we get in particular,

$$(D(t)e^{\Psi(t)})' \leqslant 2\,e^{\Psi(t)}\,\theta(t)\int_{\Gamma_+} (x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x)\,. \tag{2.5}$$

Using the assumption on z in $(2.1)_{(3,4)}$ and the behavior of θ and θ' as $t\to 0$ (see (2.4)), one easily checks $\lim_{t\to 0} D(t)\exp(\Psi(t)) = 0$, hence we can integrate (2.5) between 0 and T, which gives (2.2), as $|(x\cdot\nu)| \leqslant L$ for all $x\in\overline{\Omega}$. □

2.2 Proof of Theorem 1.1 item (i)

2.2.1 Proof of Theorem 1.1 item (i) up to technical lemmas

This section aims at proving Theorem 1.1 item (i) and we then place ourselves in the setting of this statement. We use the tensorized structure of the equation (1.5) and decompose the solution u on the basis adapted to the Laplace operator $-\Delta_y$ defined on $L^2(\Omega_y)$ with domain $H^2\cap H^1_0(\Omega_y)$, whose eigenvalues will be denoted by $(\mu_n^2)_{n\in\mathbb{N}}$. The observability estimate (1.6) is then equivalent to the existence of a constant $C>0$ such that for all $n\in\mathbb{N}$, any solution u_n of (1.8) satisfies

$$\|u_n(T)\|_{L^2(\Omega_x)} \leqslant C\,\|\partial_{\nu_x} u_n\|_{L^2((0,T)\times\partial\Omega_x)}\,. \tag{2.6}$$

We are thus back to prove a uniform observability estimate (2.6) for the family of heat equations (1.8). We shall mainly focus on the case of large values of $n\geqslant n_0$, for some $n_0\in\mathbb{N}$ to be determined, as the small values of n can be handled using classical observability estimates (see Theorem A.1) for the heat equation as the corresponding potentials $(\mu_n^2|x|^2)_{1\leqslant n\leqslant n_0}$ are uniformly bounded.

We thus restrict ourselves to the proof of uniform observability estimates (2.6) for the Grushin equations (1.8) for $n\geqslant n_0$, for some n_0 to be determined. In order to do this, given $T > L^2/(2d_x)$ with L as in (1.7), we introduce

$$T_0 < T - \frac{L^2}{2d_x}, \tag{2.7}$$

and we will decompose the time interval in $(0,T_0)$, in which the cost of observation and its dependence on n will be of paramount importance, and a time interval (T_0,T) during which we use the dissipation rate of the solutions of (1.8).

More precisely, Theorem 1.1 relies on the following two results, whose proofs are postponed to Section 2.2.2 and Section 4.1 respectively:

Lemma 2.2. *For all $T_0>0$ and $\delta>0$, there exists $C=C(T_0,\delta)>0$ and $n_0=n_0(T_0,\delta)\in\mathbb{N}$ such that, for every $n\geqslant n_0$ and $u_{0,n}\in H^1_0(\Omega_x)$, the solution of (1.8) satisfies*

$$\int_{\Omega_x} |u_n(T_0,x)|^2 dx \leqslant C\,\mu_n\,\exp\left(\mu_n(1+\delta)L^2\right)\int_0^{T_0}\int_{\partial\Omega_x} |\partial_{\nu_x} u_n(t,x)|^2 ds(x)\,dt\,, \tag{2.8}$$

where L is as in (1.7).

Lemma 2.3. *For all $n\in\mathbb{N}$, any solution u_n of (1.8) with initial datum $u_{0,n}\in L^2(\Omega_x)$ satisfies, for all $t\geqslant 0$,*

$$\|u_n(t)\|_{L^2(\Omega_x)} \leqslant \exp(-d_x\mu_n t)\,\|u_{0,n}\|_{L^2(\Omega_x)}\,. \tag{2.9}$$

Let us finish the proof of Theorem 1.1 item (i) assuming Lemma 2.2 and Lemma 2.3. For $T > L^2/(2d_x)$, we set $\varepsilon>0$ so that

$$T - \frac{L^2}{2\,d_x} = \varepsilon\frac{L^2}{2\,d_x},$$

and we choose

$$T_0 = \frac{\varepsilon}{2}\frac{L^2}{2\,d_x} \quad \text{and} \quad \delta = \frac{\varepsilon}{4}.$$

From one hand, we have (2.8), while from the other hand Lemma 2.3 implies

$$\int_{\Omega_x} |u_n(T,x)|^2\,dx \leqslant \exp\left(-2\,d_x\,\mu_n(T-T_0)\right)\int_{\Omega_x}|u_n(T_0,x)|^2dx, \tag{2.10}$$

whose combination easily leads to (2.6) for $n \geqslant n_0(T_0,\delta)$, as

$$\sup_{n\geqslant n_0}\left(\mu_n \exp\left(\mu_n(1+\delta)L^2 - 2d_x\mu_n(T-T_0)\right)\right) = \sup_{n\geqslant n_0}\left(\mu_n\exp\left(-\mu_n\varepsilon\frac{L^2}{4}\right)\right) < \infty.$$

This is enough to prove (2.6) for all $n \in \mathbb{N}$ as for $n \in \{0,\cdots,n_0\}$, the potentials $n^2|x|^2$ are uniformly bounded by $n_0^2L^2$, so that their observability constant can be bounded uniformly for $n \in \{0,\cdots,n_0\}$ from Theorem A.1.

Lemma 2.2 is in fact a straightforward consequence of Lemma 1.2 as μ_n goes to infinity as $n \to \infty$. Therefore, we shall prove Lemma 1.2 and 2.2 together below in Section 2.2.2. This step is in fact the most original part of the proof of Theorem 1.1.

Lemma 2.3 is more classical and can be found in [3]. We nevertheless recall how it can be proved in Section 4.1 for the convenience of the reader.

2.2.2 Proof of Lemma 1.2 and Lemma 2.2

Proof of Lemma 1.2. Let $n \in \mathbb{N}$ and set

$$z_n(t,x) = u_n(t,x)\exp\left(-\frac{\mu_n}{2}\coth(2\mu_n t)(L^2-|x|^2)\right), \quad (t,x)\in(0,T_0)\times\Omega_x. \tag{2.11}$$

Easy computations show that z_n satisfies (2.1) on $\Omega = \Omega_x$ with $\mu = \mu_n$ and $T = T_0$. Hence, applying Proposition 2.1 yields:

$$\begin{aligned}
&\int_{\Omega_x}\left(|\nabla_x z_n(T_0)|^2 - \frac{\mu_n^2}{\sinh(2\mu_nT_0)^2}\frac{L^2}{2}|z_n(T_0)|^2\right)dx \\
\leqslant\ & \mu_n\,L\int_0^{T_0}\left(\frac{\sinh(4\mu_nt)}{\sinh(2\mu_nT_0)^2}\int_{\partial\Omega_x}|\partial_{\nu_x}z_n(t,x)|^2ds(x)\right)dt \\
\leqslant\ & 2\,\mu_n\coth(2\mu_nT_0)\,L\int_0^{T_0}\int_{\partial\Omega_x}|\partial_{\nu_x}z_n(t,x)|^2ds(x)dt.
\end{aligned} \tag{2.12}$$

Now, as Ω_x is bounded, Poincaré inequality holds and there exists a constant $C_{\Omega_x} > 0$ such that for all $g \in H_0^1(\Omega_x)$,

$$\int_{\Omega_x}|g|^2\,dx \leqslant C_{\Omega_x}\int_{\Omega_x}|\nabla_x g|^2\,dx. \tag{2.13}$$

Recalling that $\mu_n \to \infty$ as $n \to \infty$, we now choose $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0, \quad \frac{\mu_n^2L^2}{\sinh(2\mu_nT_0)^2} \leqslant \frac{1}{C_{\Omega_x}}, \quad \text{and} \quad \coth(2\mu_nT_0)\leqslant 2.$$

Combining the estimate (2.12) with the Poincaré inequality (2.13) applied to $z_n(T_0)$, we get a constant $C > 0$ such that for all $n \geqslant n_0$,

$$\int_{\Omega_x}|z_n(T_0,x)|^2\,dx \leqslant C\,\mu_n\int_0^{T_0}\int_{\partial\Omega_x}|\nabla_{\nu_x}z_n(t,x)|^2\,ds(x)dt.$$

Recalling the definition of z_n in (2.11) concludes the proof of Lemma 1.2. □

Proof of Lemma 2.2. Here, we simply remark that for any $\delta > 0$ and $T_0 > 0$, there exists $n_\delta \in \mathbb{N}$ such that for all $n \geqslant n_\delta$,

$$\coth(2\mu_n T_0) < 1 + \delta.$$

Thus, for any $n \geqslant \max\{n_\delta, n_0\}$, where n_0 is the integer given by Lemma 1.2, a straightforward lower bound on (1.9) immediately yields (2.8). $\square$

Remark 2.4. *The weight function*

$$\exp\Big(-\frac{\mu_n}{2}\coth(2\mu_n t)(L^2 - |x|^2)\Big)$$

used in the proof of Lemma 1.2 is closely related to the fundamental solution of the harmonic oscillator in $\mathbb{R}^d$, also known as Melher kernel (see e.g. [18, Proposition 4.3.1] for the one-dimensional case, the d-dimensional kernel being immediately obtained as it is the tensor product of d one-dimensional kernels):

$$K(t; x, y) = \frac{1}{(2\pi \sinh(2t))^{d/2}} \exp\left(-\coth(2t)\left(\frac{|x|^2 + |y|^2}{2}\right) - \frac{2x \cdot y}{\sinh(2t)}\right).$$

More precisely, the change of variable $t \leftrightarrow \mu_n t$, $x \leftrightarrow \sqrt{\mu_n} x$ and $y \leftrightarrow \sqrt{\mu_n} y$ gives the kernel

$$K_n(t; x, y) = \frac{1}{(2\pi \sinh(2\mu_n t))^{d/2}} \exp\left(-\mu_n \coth(2\mu_n t)\left(\frac{|x|^2 + |y|^2}{2}\right) - \frac{2\mu_n\, x \cdot y}{\sinh(2\mu_n t)}\right)$$

which is the fundamental solution associated to the operator defined on $\mathbb{R}$

$$\partial_t - \partial_{xx} + \mu_n^2 |x|^2.$$

Therefore, the weight function used in the Carleman estimate of Lemma 1.2 correspond to the exponential envelop of $K_n(t; x, \imath L)^{-1}$, that is the opposite of the exponential envelop of the fundamental solution after a translation in the complex plane.

This method is closely related to the one developed in [16, 17] to obtain precise estimates for the heat equation: the starting point in both works is the use of the fundamental solution of the heat equation translated in the complex plane as a Carleman weight function.

Remark 2.5. *For the sake of simplicity, we have chosen to state Lemma 1.2, Lemma 2.2, and Theorem 1.1 item (i) with observations on $\partial\Omega_x \times \Omega_y$. However, as all these results derive from Proposition 2.1, all these results can be adapted in a straightforward manner to obtain an observability estimate for* (1.5) *with an observation localized on $\Gamma_{x,+} \times \Omega_y$, where $\Gamma_{x,+} = \{x \in \partial\Omega_x,\ \langle x, \nu_x\rangle > 0\}$.*

2.3 Proof of Theorem 1.3 item (i): the effect of the boundary condition at $x = 0$

We assume that we are in the setting of Theorem 1.3, i.e. that the equations (1.10) (respectively (1.11)) are set on $\Omega = (0, L) \times (0, \pi)$, observed through the flux at $\Gamma = \{L\} \times (0, \pi)$, and have Dirichlet (resp. Neumann) boundary conditions on $\{0\} \times (0, \pi)$.

As before, we can expand the solution u of (1.10) (respectively (1.11)) in Fourier series. Here, it simply means that we write

$$u(t, x, y) = \sum_{n \in \mathbb{N}} u_n(t, x) \sin(ny), \quad (t, x, y) \in (0, T) \times \Omega,$$

where, due to the tensorized structure of the equation (1.10) (resp. (1.11)), the function u solves (1.10) (resp. (1.11)) with initial datum u_0 if and only if for all n, u_n solves the (1.14) (resp. (1.15)) with initial datum $u_{0,n}$, where

$$u_0(x, y) = \sum_{n \in \mathbb{N}} u_{0,n}(x) \sin(ny), \quad (x, y) \in \Omega.$$

Consequently, the observability estimate (1.12) for (1.10) (resp. (1.11)) is equivalent to the observability property

$$\|u_n(T, \cdot)\|_{L^2(0,L)} \leqslant C \|\partial_x u_n(\cdot, L)\|_{L^2(0,T)} \tag{2.14}$$

for all smooth solutions u_n of (1.14) (resp. (1.15)) uniformly with respect to n, i.e. with $C > 0$ independent of n.

In order to prove the uniform observability property (2.14) for (1.14) (resp. (1.15)), we do as before and rely on a precise estimate on the dependence of the cost of observability of (1.14) (resp. (1.15)) in small times and an estimate on the rate of dissipation of the semigroups corresponding to (1.14) (resp. (1.15)).

In fact, a direct application of Lemma 2.2 shows the following estimate on the cost of observability of (1.14) (resp. (1.15)).

Lemma 2.6. *For all $T_0 > 0$ and $\delta > 0$, there exists a constant $C = C(T_0, \delta) > 0$ and $n_0 = n_0(T, \delta) \in \mathbb{N}$ such that for all $n \geqslant n_0$, any solution u of* (1.14) *(resp.* (1.15)*) with initial datum $u_{0,n} \in H_0^1(0, L)$ (resp. $u_{0,n} \in H_N^1(0, L)$) satisfies*

$$\int_0^L |u_n(T_0)|^2 dx \leqslant C n \exp(n(1+\delta)L^2) \int_0^{T_0} |\partial_x u_n(t, L)|^2 dt. \tag{2.15}$$

Proof. The proof relies on a simple symmetrization argument. More precisely, if u_n solves (1.14), for $t \in (0, T_0)$ and $x \in (-L, L)$, we define

$$\tilde{u}_n(t, x) = \begin{cases} u_n(t, x) \text{ if } x \geqslant 0 \\ -u_n(t, -x) \text{ if } x < 0 \end{cases}$$

It is readily seen that $\tilde{u}_n$ satisfies (1.8) with $\mu_n = n$ and $\Omega_x = (-L, L)$. We can thus apply Lemma 2.2 to $\tilde{u}_n$ from which we immediately deduce (2.15).

When considering u_n solving the equation (1.15), a similar argument can be done by introducing the even extension $\tilde{u}_n$ of u_n, namely for $t \in (0, T_0)$ and $x \in (-L, L)$,

$$\tilde{u}_N(t, x) = \begin{cases} u_n(t, x) \text{ if } x \geqslant 0 \\ u_n(t, -x) \text{ if } x < 0 \end{cases} .$$

This easily proves (2.15) for solutions u_n of (1.15). ☐

We have the following dissipation estimate:

Lemma 2.7. *(i) Any function u_n solution of* (1.14) *satisfies for all $t \in (0, T]$*

$$\|u_n(t)\|_{L^2(0,L)} \leqslant e^{-3\,n\,t} \|u_n(0)\|_{L^2(0,L)}. \tag{2.16}$$

(ii) Any function u_n solution of (1.15) *satisfies for all $t \in (0, T]$*

$$\|u_n(t)\|_{L^2(0,L)} \leqslant e^{-n\,t} \|u_n(0)\|_{L^2(0,L)}. \tag{2.17}$$

The proof of Lemma 2.7 is postponed to Section 4.2.

Based on Lemma 2.6 and on Lemma 2.7, we can conclude the proof of Theorem 1.3 as previously.

If we consider the case of Dirichlet boundary conditions, i.e. the case of equation (1.10) and its corresponding family of equations (1.14), for $T > L^2/6$, we set $\varepsilon > 0$ such that

$$T - \frac{L^2}{6} = \varepsilon \frac{L^2}{6},$$

and we choose

$$T_0 = \frac{\varepsilon}{2} \frac{L^2}{6} \quad \text{and} \quad \delta = \frac{\varepsilon}{4}.$$

Applying the Lemma 2.6 on $(0, T_0)$ and the dissipation estimate (2.16) on (T_0, T), we get, for all $n \geqslant n_0$, and all solutions u_n of (1.14) with initial data in $H_0^1(0, L)$,

$$\|u_n(T)\|_{L^2(0,L)}^2 \leqslant C n \exp\left(-\varepsilon n \frac{L^2}{4}\right) \|\partial_x u_n(\cdot, L)\|_{L^2(0,T_0)}^2 .$$

This proves the observability estimate (2.14) for (1.14) uniformly with respect to $n \geqslant n_0$ when $T > L^2/6$. As before, the case of $n \in \{0, \cdots, n_0\}$ follows from classical results on the heat equation with potential, see

Theorem A.1. This shows that the observability estimate (2.14) for (1.14) holds uniformly with respect to $n \in \mathbb{N}$, hence the proof of Theorem 1.3 item (i) in the Dirichlet case.

If we consider the case of Neumann boundary conditions, i.e. the case of equation (1.11) and its corresponding family of equations (1.15), for $T > L^2/2$, we set $\varepsilon > 0$ such that

$$T - \frac{L^2}{2} = \varepsilon \frac{L^2}{2},$$

and we choose

$$T_0 = \frac{\varepsilon}{2}\frac{L^2}{2} \quad \text{and} \quad \delta = \frac{\varepsilon}{4}.$$

The same arguments as in the Dirichlet case allows to prove Theorem 1.3 item (i) in the Neumann case.

3 Observability results for the generalized Grushin equation

The goal of this section is to prove Theorem 1.4 item (i), Theorem 1.6 item (i), Theorem 1.8 and Theorem 1.9. The proof of each of these results strongly rely on Carleman estimates, that we will present separately in a "generic" form in Section 3.1 for later use.

3.1 Carleman estimates: computations

For later use, we will present computations together on a "generic" version of (1.20). Namely, we will consider a generic bounded interval (a, b) with $a < b$, and the following equation, indexed by $n \in \mathbb{N}$:

$$\begin{cases} \partial_t u_n - \partial_{xx} u_n + n^2 q(x)^2 u_n = f_n, & \text{in } (0,\infty) \times (a,b), \\ u_n(t,a) = 0, \quad u_n(t,b) = 0, & \text{in } (0,\infty), \\ u_n(0,x) = u_{0,n}(x), & \text{in } (a,b), \end{cases} \tag{3.1}$$

where f_n is assumed to be in $L^2((0,T) \times (a,b))$ and $u_{0,n} \in H_0^1(a,b)$.

Proposition 3.1. *Let $T > 0$, $a, b \in \mathbb{R}$ with $a < b$, $q \in C^1([a,b],\mathbb{R})$, $n \in \mathbb{N}$ and φ be a weight function such that*

$$\lim_{t\to 0} \inf_{x\in[a,b]} \{\varphi(t,x)\} = \infty, \quad \forall x \in (a,b), \quad \lim_{t\to 0} \partial_x \varphi(t,x) e^{-\varphi(t,x)} = 0, \tag{3.2}$$

$$\lim_{t\to T} \inf_{x\in[a,b]} \{\varphi(t,x)\} = \infty, \quad \forall x \in (a,b), \quad \lim_{t\to T} \partial_x \varphi(t,x) e^{-\varphi(t,x)} = 0, \tag{3.3}$$

$$\varphi \in C^2((0,T); C^4([a,b])). \tag{3.4}$$

Then, for any solution u_n of (3.1) with $u_{0,n} \in H_0^1(a,b)$ and $f_n \in L^2((0,T) \times (a,b))$, the function

$$v_n(t,x) = u_n(t,x)\exp(-\varphi(t,x)), \quad (t,x) \in (0,T) \times (a,b),$$

satisfies

$$2\int_0^T \Big[|\partial_x v_n|^2 \partial_x \varphi\Big]_{x=a}^{x=b} dt + \int_0^T\int_a^b \Big(-4\partial_{xx}\varphi |\partial_x v_n|^2 + |v_n|^2 G_\varphi \Big) dx dt \leqslant \int_0^T\int_a^b |f_n e^{-\varphi}|^2 dx dt \tag{3.5}$$

where we have set

$$G_\varphi(t,x) = 2\,\partial_x\varphi\,\partial_x F_\varphi - \partial_t F_\varphi + \partial_x^4 \varphi, \tag{3.6}$$

in which F_φ is given by

$$F_\varphi(t,x) = \partial_t \varphi - |\partial_x \varphi|^2 + n^2 q(x)^2. \tag{3.7}$$

Proof. In the proof of Proposition 3.1 below, we drop the index n to simplify the notations. Under the assumptions of Proposition 3.1, we compute

$$\begin{aligned} P_\varphi v &= e^{-\varphi}(\partial_t - \partial_{xx} + n^2 q(x)^2)(e^\varphi v) \\ &= \partial_t v - \partial_{xx} v - 2\partial_x v \partial_x \varphi + v\left(\partial_t \varphi - |\partial_x \varphi|^2 - \partial_{xx}\varphi + n^2 q(x)^2\right). \end{aligned} \tag{3.8}$$

In particular, if u denotes a "smooth" (e.g. $L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b))$) solution of (3.1), introducing the functions

$$v = ue^{-\varphi}, \quad g = fe^{-\varphi}, \tag{3.9}$$

v is a "smooth" (up to the regularity of φ in (3.4)) solution to

$$\begin{cases} P_\varphi v = g, & \text{in } (0,T) \times (a,b) \\ v(t,a) = v(t,b) = 0, & \text{in } (0,T), \end{cases} \tag{3.10}$$

with

$$v(0,\cdot) = v(T,\cdot) = 0, \quad \partial_x v(0,\cdot) = \partial_x v(T,\cdot) = 0 \quad \text{in } (a,b). \tag{3.11}$$

We then decompose the operator P_φ as

$$P_\varphi v = P_1 v + P_2 v \quad \text{with} \quad \begin{cases} P_1 v = -\partial_{xx} v + vF_\varphi, \\ P_2 v = \partial_t v - 2\partial_x v \partial_x \varphi - v\partial_{xx}\varphi. \end{cases} \tag{3.12}$$

We therefore have

$$\|P_1 v\|^2_{L^2((0,T)\times(a,b))} + \|P_2 v\|^2_{L^2((0,T)\times(a,b))} + 2\int_0^T \int_a^b P_1 v P_2 v \, dxdt = \|g\|^2_{L^2((0,T)\times(a,b))}. \tag{3.13}$$

This basic identity will be the main point of our argument. We then compute the cross product

$$\int_0^T \int_a^b P_1 v P_2 v \, dtdx = \sum_{i=1}^2 \sum_{j=1}^3 I_{i,j},$$

where $I_{i,j}$ is the cross product between the i-th term of $P_1 v$ and the j-th term of $P_2 v$.

$$\begin{aligned} I_{1,1} &= 0, \\ I_{1,2} &= -\int_0^T \int_a^b |\partial_x v|^2 \partial_{xx}\varphi \, dxdt + \int_0^T |\partial_x v|^2 \partial_x \varphi \Big|_{x=a}^{x=b} dt, \\ I_{1,3} &= -\int_0^T \int_a^b |\partial_x v|^2 \partial_{xx}\varphi \, dxdt + \frac{1}{2}\int_0^T \int_a^b |v|^2 \partial_x^4 \varphi \, dxdt, \\ I_{2,1} &= -\frac{1}{2}\int_0^T \int_a^b |v|^2 \partial_t F_\varphi \, dxdt, \\ I_{2,2} &= \int_0^T \int_a^b |v|^2 \partial_x \left(\partial_x \varphi F_\varphi\right) \, dxdt, \\ I_{2,3} &= -\int_0^T \int_a^b |v|^2 \partial_{xx}\varphi F_\varphi \, dxdt. \end{aligned}$$

Hence we obtain the estimate (3.5). $\square$

Remark 3.2. *Of course, Proposition 3.1 is closely related to Lemma 2.2 and Proposition 2.1. However, the reader will notice that the proofs of Proposition 3.1 differs from the one of Proposition 2.1. This is due to the fact that Lemma 1.2 and Proposition 2.1 rather prove a Carleman estimate for the problem* (1.8)*, for which the fundamental solution on $\mathbb{R}$ is available, see Remark 2.4, and its exponential envelop is used as a Carleman weight, so that many terms cancel in the proof of Lemma 1.2. This is no longer the case when considering Carleman estimates for* (3.1) *for general $q \in C^1([a,b];\mathbb{R})$.*

3.2 The 2D Grushin equation observed from one side: Proof of Theorem 1.4 item (i)

The goal of this section is to prove Theorem 1.4 item (i). In order to do this, as in the previous sections, we use a Fourier expansion to reduce the observability property (1.18) for (1.16) to prove a uniform observability property for solutions of (1.20). As before, the analysis of the observability property of (1.20) will be based on the analysis of the cost of observability in the asymptotics $n \to \infty$ for (1.20), and on the dissipation of the semi-group corresponding to (1.20). Here again, the main difficulty of our result is the asymptotics of the cost of observability of the family of equations (1.20) as $n \to \infty$, which is stated in Proposition 1.5 and is based on suitable Carleman estimates. In particular, we shall do two Carleman estimates, see Section 3.2.1, one on the space interval (a_0, L_+), where a_0 will be a small enough negative number, and the other on the space interval $(-L_-, 0)$, and we will then use a cut-off argument to prove Proposition 1.5 in Section 3.2.2. We finally explain how we conclude Theorem 1.4 item (i) in Section 3.2.3.

3.2.1 Specific choices of weights

From the right end of the domain to the left Here, we prove the following result:

Proposition 3.3. *Let $T \in (0,4)$, $a_0, L_+ \in \mathbb{R}$ be such that $a_0 < L_+$,*

$$q \in C^3([a_0, L_+], \mathbb{R}) \text{ such that } \inf_{[a_0,L_+]}\{q'\} > 0, \tag{3.14}$$

and

$$B \in \mathbb{R}_+^* \text{ such that } q(a_0) + B > 0. \tag{3.15}$$

We define

$$\varphi_{R,n}(t,x) = n\theta(t)\Psi_R(x) + \theta(t), \quad (t,x) \in (0,T) \times (a_0, L_+) \tag{3.16}$$

with θ and Ψ_R as follows

$$\theta \in \mathscr{C}^\infty(0,T), \quad \theta(t) = \begin{cases} 1/t & \text{for } t < T/4, \\ 1 & \text{for } t \in (T/3, 2T/3), \\ 1/(T-t) & \text{for } t > 3T/4, \\ \geqslant 1 & \text{for } t \in (0,T), \end{cases} \tag{3.17}$$

$$\Psi_R(x) = \int_x^{L_+} q(s)\,ds + B(L_+ - x), \quad x \in (a_0, L_+). \tag{3.18}$$

Then there exists $n_0 > 0$ and $C > 0$ such that for all $n \geqslant n_0$, for all u_n satisfying

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 q(x)^2)u_n = f_n\,, & (t,x) \in (0,T) \times (a_0, L_+)\,, \\ u_n(t,a_0) = u_n(t,L_+) = 0\,, & t \in (0,T), \\ u_n(0,.) = u_{0,n} \in H_0^1(a_0, L_+)\,, \end{cases} \tag{3.19}$$

with $f_n \in L^2((0,T) \times (a_0, L_+))$, we have

$$n^3 \left\|\theta^{3/2} u_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T)\times(a_0,L_+))} + n \left\|\theta^{1/2}\partial_x u_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T)\times(a_0,L_+))} \\ \leqslant Cn \left\|\theta^{1/2}\partial_x u_n(t,L_+) e^{-\theta(t)}\right\|^2_{L^2(0,T)} + C\left\|f_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T)\times(a_0,L_+))}. \tag{3.20}$$

Remark 3.4. *In the above statement, the restriction $T \in (0,4)$ is purely technical to guarantee the existence of the C^∞ function θ. Such restriction can be removed with a slightly less explicit construction of the function θ: consider η_1, η_2 and η_3 in $C^\infty([0,1])$ such that for all $s \in [0,1]$,*

$$\forall i,\ 0 \leqslant \eta_i(s) \leqslant 1, \qquad \eta_1(s) + \eta_2(s) + \eta_3(s) = 1,$$

and for all s in $[0,1/5]$, $\eta_1(s) = 1$, for all s in $[2/5,3/5]$, $\eta_2(s) = 1$, for all s in $[4/5,1]$, $\eta_3(s) = 1$. Define on $(0,1)$ the function $\tilde{\theta}$ by

$$\tilde{\theta}(s) = \frac{1}{s}\eta_1(s) + \eta_2(s) + \frac{1}{1-s}\eta_3(s).$$

Then, $\theta(t) = \tilde{\theta}(t/T)$ is an admissible function to use in the construction of the weight function $\varphi_{R,n}$ (and of the weight functions appearing later on), in the sense that all results remain true using it.

Proof. Based on the computations in Section 3.1 with $a = a_0$ and $b = L_+$, we compute the following quantities, where the bounds are obtained by using properties (3.14)–(3.15): for all $(t,x) \in (0,T) \times (a_0, L_+)$,

$$\begin{aligned}
\Psi_R'(x) &= -q(x) - B \leqslant -\Big(q(a_0) + B\Big) < 0, \\
\Psi_R''(x) &= -q'(x) < 0, \\
\partial_x \varphi_{R,n}(t,x) &= n\theta(t)\Psi_R'(x) \leqslant -n\theta(t)\Big(q(a_0) + B\Big) < 0, \\
-\partial_{xx}\varphi_{R,n}(t,x) &= n\theta(t)q'(x) \geqslant n\,\theta(t) \inf_{[a_0,b]}\{q'\} > 0, \\
F_{\varphi_{R,n}}(t,x) &= n\theta'(t)\Psi_R(x) + \theta'(t) - n^2\theta^2(t)(\Psi_R'(x))^2 + n^2 q(x)^2, \\
G_{\varphi_{R,n}}(t,x) &= 2n\theta(t)\Psi_R'(x)\left[2n\theta'(t)\Psi_R'(x) - 2n^2\theta^2(t)\Psi_R''(x)\Psi_R'(x) + 2n^2 q'(x)q(x)\right] \\
&\qquad -n\theta''(t)\Psi_R(x) - \theta''(t) + n\theta(t)\Psi_R^{(4)}(x).
\end{aligned}$$

In the limit $n \to \infty$, the dominant term in $G_{\varphi_{R,n}}$ is the following one and it is positive: for all $(t,x) \in (0,T) \times [a_0, L_+]$, as $\theta(t) \geqslant 1$,

$$\begin{aligned}
2n\theta(t)\Psi_R'(x)\left[-2n^2\theta^2(t)\Psi_R''(x)\Psi_R'(x) + 2n^2 q'(x)q(x)\right] &= 4n^3\theta^3(t)q'(x)(-\Psi_R'(x))\left(-\Psi_R'(x) - \frac{q(x)}{\theta^2(t)}\right) \\
&= 4n^3\theta^3(t)q'(x)(-\Psi_R'(x))\left[\left(1 - \frac{1}{\theta(t)^2}\right)q(x) + B\right] \\
&\geqslant 4n^3\theta(t)^3 q'(x)(-\Psi_R'(x))\left[\left(1 - \frac{1}{\theta^2(t)}\right)q(a_0) + B\right] \\
&\geqslant 4C(B)n^3\theta(t)^3,
\end{aligned}$$

where

$$C(B) = \inf_{[a_0,L_+]}\{q'\}\min\{B, B + q(a_0)\} > 0.$$

Let us note that, with θ as in (3.17), there exists $C > 0$ such that for all $t \in (0,T)$,

$$|\theta'(t)| \leqslant C(\theta(t))^2, \qquad |\theta''(t)| \leqslant C(\theta(t))^3. \tag{3.21}$$

Thus, using furthermore that $\theta \geqslant 1$ on $(0,T)$, there exists $n_0 > 0$ such that for all $n \geqslant n_0$ and $(t,x) \in (0,T) \times (a_0, L_+)$,

$$G_{\varphi_{R,n}}(t,x) \geqslant C(B)n^3\theta(t)^3\,.$$

Using the computations done in Section 3.1, we thus deduce Proposition 3.3. $\square$

From the singularity to the left end of the domain The goal of this section is to prove the following result:

Proposition 3.5. *Assume* (1.17). *Let $T \in (0,4)$, $L_- > 0$, and $A > 0$, and define $\varphi_{L,n}$ for $n \in \mathbb{N}$*

$$\varphi_{L,n}(t,x) = n\theta(t)A + \theta(t) - \sqrt{n}\theta(t)\left(\frac{x^2}{2} + 2L_-x\right), \quad (t,x) \in (0,T) \times (-L_-,0), \tag{3.22}$$

with θ as in (3.17). *Then there exists $n_0 > 0$ and $C > 0$ such that for all $n \geqslant n_0$, for all u_n satisfying*

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 q(x)^2) u_n = f_n\,, & (t,x) \in (0,T) \times (-L_-,0)\,, \\ u_n(t,-L_-) = u_n(t,0) = 0\,, & t \in (0,T), \\ u_n(0,.) = u_{0,n} \in H_0^1(-L_-,0)\,. \end{cases} \tag{3.23}$$

with $f_n \in L^2((0,T) \times (-L_-,0))$, we have

$$\begin{aligned} n^{3/2} \left\| \theta^{3/2} u_n e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T)\times(-L_-,0))} &+ n^{1/2} \left\| \theta^{1/2} \partial_x u_n e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T)\times(-L_-,0))} \\ &\leqslant Cn^{1/2} \left\| \theta^{1/2} \partial_x u_n(t,0) e^{-n\theta(t)A} \right\|^2_{L^2(0,T)} + C \left\| f_n e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T)\times(-L_-,0))}. \end{aligned} \tag{3.24}$$

Proof. Again, we base our proof of Proposition 3.5 on the computations in Section 3.1, this time with $a = -L_-$, $b = 0$, we compute the following quantities: for all $(t,x) \in (0,T) \times (-L_-,0)$,

$$\begin{aligned} \partial_x \varphi_{L,n}(t,x) &= -\sqrt{n}\theta(t)(x + 2L_-) < -\sqrt{n}\theta(t)L_-, \\ -\partial_{xx}\varphi_{L,n}(t,x) &= \sqrt{n}\theta(t) > 0, \\ F_{\varphi_{L,n}}(t,x) &= n\theta'(t)A + \theta'(t) - \sqrt{n}\theta'(t)\left(\frac{x^2}{2} + 2L_-x\right) - n\theta^2(t)(x+2L_-)^2 + n^2 q(x)^2, \\ G_{\varphi_{L,n}}(t,x) &= -n\theta''(t)A - \theta''(t) + \sqrt{n}\theta''(t)\left(\frac{x^2}{2} + 2L_-x\right) + 4n^{3/2}\theta^3(t)(x+2L_-)^2 \\ &\quad -4n^{5/2}\theta(t)(x+2L_-)q'(x)q(x) + 4n\theta'(t)\theta(t)(x+2L_-)^2\,. \end{aligned}$$

Therefore, in order to estimate $G_{\varphi_{L,n}}$ in $(0,T) \times (-L_-,0)$, we use (1.17) and (3.21) to get, for all $(t,x) \in (0,T) \times (-L_-,0)$,

$$\begin{aligned} \left| -n\theta''(t)A - \theta''(t) + \sqrt{n}\theta''(t)\left(\frac{x^2}{2} + 2L_-x\right) + 4n\theta'(t)\theta(t)(x+2L_-)^2 \right| &\leqslant Cn\theta^3(t), \\ 4n^{3/2}\theta^3(t)(x+2L_-)^2 &\geqslant 4n^{3/2}\theta^3(t)L_-^2, \\ -4n^{5/2}\theta(t)(x+2L_-)q'(x)q(x) &\geqslant 0, \end{aligned}$$

because $q \leqslant 0$ on $(-L_-,0)$. Therefore, for n large enough, we have, for all $(t,x) \in (0,T) \times (-L_-,0)$,

$$G_{\varphi_{L,n}}(t,x) \geqslant 2n^{3/2}\theta(t)^3 L_-^2.$$

We finally note that, conditions (3.2)–(3.4) hold, so that we can apply the computations done in Section 3.1. This immediately yields Proposition 3.3. ☐

3.2.2 Proof of Proposition 1.5: a gluing argument

In fact, we will prove a slightly more general result than Proposition 1.5. Namely, we shall consider the equation (1.20) with source terms, that is

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 q(x)^2) u_n = f_n\,, & (t,x) \in (0,T) \times (-L_-,L_+)\,, \\ u_n(t,-L_-) = u_n(t,L_+) = 0\,, & t \in (0,T)\,, \\ u_n(0,.) = u_{0,n} \in H_0^1(-L_-,L_+)\,. \end{cases} \tag{3.25}$$

where $f_n \in L^2((0,T) \times (-L_-,L_+))$, and we prove the following result:

Proposition 3.6. *Assume that q satisfies* (1.17)*. For every $T_0 > 0$ and $\varepsilon > 0$, there exists $C > 0$ such that, for every $n \in \mathbb{N}$, any solution of* (3.25) *with $u_{0,n} \in H_0^1(-L_-, L_+)$ and $f_n \in L^2((0,T_0) \times (-L_-, L_+))$ satisfies*

$$\|u_n(T_0)\|_{L^2(-L_-,L_+)} \leqslant C \exp\left(n\left(\int_0^{L_+} q(s)\,ds + \varepsilon\right)\right)\left[\|\partial_x u_n(.,L_+)\|_{L^2(0,T_0)} + \|f_n\|_{L^2((0,T_0)\times(-L_-,L_+))}\right]. \quad (3.26)$$

Proof. We will prove Proposition 3.6 only in the case $T_0 \in (0,4)$: if $T_0 \geqslant 4$, one can apply the result of Proposition 3.6 on the time interval $(T_0 - 2, T_0)$.

We thus take $T_0 \in (0,4)$ and $\varepsilon > 0$, and we choose $L_0 > 0$ small enough to get

$$\int_{-L_0}^{0} q(s)ds - 2q(-L_0)(L_+ + L_0) \leqslant \varepsilon, \quad (3.27)$$

which is possible from (1.17).

We then define $a_0 = -L_0$, the function Ψ_R as in (3.18) with $B = -2q(-L_0)$ on the interval $(-L_0, L_+)$, and we set

$$A = \Psi_R\left(-\frac{L_0}{2}\right). \quad (3.28)$$

Let $n_0 \in \mathbb{N}$ be large enough so that Propositions 3.3 and 3.5 respectively hold for solutions of (3.19) and (3.23) with $T = T_0$.

In the following argument, we consider a generic $n \geqslant n_0$ and u_n the corresponding solution of (3.25) with initial datum $u_{0,n} \in H_0^1(-L_-, L_+)$ and $f_n \in L^2((0,T_0) \times (-L-, L_+))$.

Step 1: Cut-off argument. We choose two cut-off functions $\eta_L = \eta_L(x)$ and $\eta_R = \eta_R(x)$ such that

$$\eta_L, \eta_R \in \mathscr{C}^\infty(-L_-, L_+), \quad \eta_L(x) = \begin{cases} 1 \text{ if } x \leqslant -L_0/3, \\ 0 \text{ if } x \geqslant 0, \end{cases} \quad \eta_R(x) = \begin{cases} 1 \text{ if } x \geqslant -2L_0/3, \\ 0 \text{ if } x \leqslant -L_0, \end{cases} \quad (3.29)$$

and we set

$$u_{L,n}(t,x) = u_n(t,x)\eta_L(x), \quad u_{R,n}(t,x) = u_n(t,x)\eta_R(x), \quad (t,x) \in (0,T_0) \times (-L_-, L_+). \quad (3.30)$$

According to the construction of the cut-off functions, it is clear that $u_{L,n}$ satisfies (3.23) with source term $f_{L,n} = f_n\eta_L + [\eta_L, \partial_{xx}]u_n$ and that $u_{R,n}$ satisfies (3.19) with source term $f_{R,n} = f_n\eta_R + [\eta_R, \partial_{xx}]u_n$. Therefore, applying Proposition 3.5 to $u_{L,n}$, we obtain a positive constant C such that

$$n^{3/2}\left\|\theta^{3/2}u_{L,n}e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,0))} + n^{1/2}\left\|\theta^{1/2}\partial_x u_{L,n}e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,0))} \leqslant C\left\|f_{L,n}e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,0))}.$$

Using now the properties (3.29) of the cut-off function η_L, we thus obtain

$$n^{3/2}\left\|\theta^{3/2}u_n e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} + n^{1/2}\left\|\theta^{1/2}\partial_x u_n e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} \leqslant C\Big(\left\|(|\partial_x u_n| + |u_n|)e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_0/3,0))} + \left\|f_n e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,0))}\Big).$$

One can similarly apply Proposition 3.3 and, after similar considerations, obtain a positive contant C such that

$$\begin{aligned} n^3\left\|\theta^{3/2}u_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))} &+ n\left\|\theta^{1/2}\partial_x u_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))} \\ \leqslant \quad & C\Big(n\left\|\theta^{1/2}\partial_x u_n(t,L_+)e^{-\theta(t)}\right\|^2_{L^2(0,T_0)} + \left\|(|\partial_x u_n| + |u_n|)e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-L_0,-2L_0/3))} \\ & + \left\|f_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-L_0,L_+))}\Big). \end{aligned}$$

Therefore, summing up the two last estimates, we obtain

$$
\begin{aligned}
n^{3/2} \left\| \theta^{3/2} u_n e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} + n^{1/2} \left\| \theta^{1/2} \partial_x u_n e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} \\
+ n^3 \left\| \theta^{3/2} u_n e^{-\varphi_{R,n}} \right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))} + n \left\| \theta^{1/2} \partial_x u_n e^{-\varphi_{R,n}} \right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))} \\
\leqslant Cn \left\| \theta^{1/2} \partial_x u_n(t,L_+) e^{-\theta(t)} \right\|^2_{L^2(0,T_0)} + C \left\| f_n \right\|^2_{L^2((0,T_0)\times(-L_-,L_+))} \\
+ C \left\| (|\partial_x u_n| + |u_n|) e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T_0\times(-L_0/3,0))} + C \left\| (|\partial_x u_n| + |u_n|) e^{-\varphi_{R,n}} \right\|^2_{L^2((0,T_0)\times(-L_0,-2L_0/3))}.
\end{aligned}
\tag{3.31}
$$

Step 2: Absorption of the last two terms. We prove that, for n large enough

$$
\begin{aligned}
C \left\| (|\partial_x u_n| + |u_n|) e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T_0)\times(-L_0/3,0))} \\
\leqslant \frac{1}{2} \left(n^3 \left\| \theta^{3/2} u_n e^{-\varphi_{R,n}} \right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))} + n \left\| \theta^{1/2} \partial_x u_n e^{-\varphi_{R,n}} \right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))} \right)
\end{aligned}
\tag{3.32}
$$

and

$$
\begin{aligned}
C \left\| (|\partial_x u_n| + |u_n|) e^{-\varphi_{R,n}} \right\|^2_{L^2((0,T_0)\times(-L_0,-2L_0/3))} \\
\leqslant \frac{1}{2} \left(n^{3/2} \left\| \theta^{3/2} u_n e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} + n^{1/2} \left\| \theta^{1/2} \partial_x u_n e^{-\varphi_{L,n}} \right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} \right)
\end{aligned}
\tag{3.33}
$$

so that the last two terms of the right hand side of (3.31) can be absorbed by the left-hand side for large n. To get these estimates, the key points are the following relations: for all $t \in (0, T_0)$,

$$
\sup_{x\in[-L_0/3,0]} \{-\varphi_{L,n}(t,x)\} \leqslant \inf_{x\in[-L_0/3,0]} \{-\varphi_{R,n}(t,x)\}, \tag{3.34}
$$

$$
\sup_{x\in[-L_0,-2L_0/3]} \{-\varphi_{R,n}(t,x)\} \leqslant \inf_{x\in[-L_0,-2L_0/3]} \{-\varphi_{L,n}(t,x)\}. \tag{3.35}
$$

In order to prove (3.34) and (3.35), we first remark that for all $t \in (0,T_0)$, $x \mapsto -\varphi_{L,n}(t,x)$ and $x \mapsto -\varphi_{R,n}(t,x)$ are increasing functions, so that the proof of (3.34)–(3.35) reduces to prove, for all $t \in (0,T_0)$,

$$
-\varphi_{L,n}(t,0) \leqslant -\varphi_{R,n}(t,-L_0/3), \tag{3.36}
$$

$$
-\varphi_{R,n}(t,-2L_0/3) \leqslant -\varphi_{L,n}(t,-L_0). \tag{3.37}
$$

Now, with the choice of A in (3.28), we have

$$
\begin{aligned}
-\varphi_{L,n}(t,0) &= -n\theta(t)A - \theta(t) && \text{by (3.22)}\\
&= -n\theta(t)\Psi_R(-L_0/2) - \theta(t) && \text{by (3.28)}\\
&\leqslant -n\theta(t)\Psi_R(-L_0/3) - \theta(t) && \text{because } (-\Psi_R) \text{ is increasing}\\
&\leqslant -\varphi_{R,n}(t,-L_0/3) && \text{by (3.16)}
\end{aligned}
$$

and

$$
\begin{aligned}
-\varphi_{R,n}(t,-2L_0/3) &= -n\theta(t)\Psi_R(-2L_0/3) - \theta(t) && \text{by (3.16)}\\
&= -n\theta(t)A - \theta(t) + n\theta(t)\left(\Psi_R(-L_0/2) - \Psi_R(-2L_0/3)\right) && \text{by (3.28)}\\
&= -\varphi_{L,n}(t,-L_0) - \sqrt{n}\theta(t)\left(\frac{L_0^2}{2} - 2L_-L_0\right) + n\theta(t)\left(\Psi_R(-L_0/2) - \Psi_R(-2L_0/3)\right) && \\
&\leqslant -\varphi_{L,n}(t,-L_0) && \text{by (3.22)},
\end{aligned}
$$

where the last inequality holds for n large enough, because $\Psi_R(-L_0/2) < \Psi_R(-2L_0/3)$, thus proving estimates (3.34)–(3.35) for n large enough.

Using (3.34)–(3.35) for n large enough, we get, for n large enough,

$$\left\|(|\partial_x u_n|+|u_n|)e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_0/3,0))} \leqslant \left\|(|\partial_x u_n|+|u_n|)e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-L_0/3,0))}$$
$$\leqslant 2\left\|u_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))} + 2\left\|\partial_x u_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))}.$$

and, similarly

$$\left\|(|\partial_x u_n|+|u_n|)e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-L_0,-2L_0/3))} \leqslant \left\|(|\partial_x u_n|+|u_n|)e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_0,-2L_0/3))}$$
$$\leqslant 2\left\|u_n e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} + 2\left\|\partial_x u_n e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))}.$$

These two inequalities imply (3.32) and (3.33) for n large enough, because $\theta \geqslant 1$ on $(0,T_0)$.

Step 3: Conclusion. We thus deduce from (3.31), (3.32) and (3.33) that, for n large enough,

$$n^{1/2}\left\|\theta^{3/2}u_n e^{-\varphi_{L,n}}\right\|^2_{L^2((0,T_0)\times(-L_-,-L_0/3))} + n^2\left\|\theta^{3/2}u_n e^{-\varphi_{R,n}}\right\|^2_{L^2((0,T_0)\times(-2L_0/3,L_+))}$$
$$\leqslant C\left\|\theta^{1/2}\partial_x u_n(t,L_+)e^{-\theta(t)}\right\|^2_{L^2(0,T_0)} + C\left\|f_n\right\|^2_{L^2((0,T_0)\times(-L_-,L_+))}. \quad (3.38)$$

Note that, for every $(t,x)\in(T_0/4,3T_0/4)\times(-L_0,L_+)$,

$$-\varphi_{R,n}(t,x) = -n\Psi_R(x)-1 \geqslant -n\Psi_R(-L_0)-1$$

because $(-\Psi_R)$ is increasing and for every $(t,x)\in(T_0/4,3T_0/4)\times(-L_-,0)$,

$$\begin{aligned}
-\varphi_{L,n}(t,x) &= -nA-1+\sqrt{n}\left(\frac{x^2}{2}+2L_-x\right)\\
&= -n\Psi_R(-L_0)-1+n[\Psi_R(-L_0)-A]+\sqrt{n}\left(\frac{x^2}{2}+2L_-x\right)\\
&\geqslant -n\Psi_R(-L_0)-1+n[\Psi_R(-L_0)-\Psi_R(-L_0/2)]-\frac{3}{2}\sqrt{n}L_-^2 \qquad (\text{ by } (3.28))\\
&\geqslant -n\Psi_R(-L_0)-1
\end{aligned}$$

for n large enough, because $\Psi_R(-L_0)>\Psi_R(-L_0/2)$. Using also that $\sup_{[0,T_0]}\theta(t)^{1/2}e^{-\theta(t)}<\infty$ and $\theta=1$ on $(T_0/4,3T_0/4)$, we obtain, for some constant $C>0$ independent of n,

$$\|u_n\|_{L^2((T_0/4,3T_0/4)\times(-L_-,L_+))} \leqslant Ce^{n\Psi_R(-L_0)}\Big(\|\partial_x u_n(.,L_+)\|_{L^2(0,T_0)}+\|f_n\|_{L^2((0,T_0)\times(-L_-,L_+))}\Big). \quad (3.39)$$

Note that, by (3.27),

$$\Psi_R(-L_0) = \int_{-L_0}^{L_+} q(s)\,ds - 2q(-L_0)(L_++L_0) \leqslant \int_0^{L_+} q(s)ds+\varepsilon. \quad (3.40)$$

To conclude Proposition 3.6, we use rough energy estimates as follows. For $t\in(0,T_0)$, we multiply the equation (3.25) by u_n:

$$\frac{d}{dt}\left(\int_{-L_-}^{L_+}|u_n(t,x)|^2\,dx\right)+\int_{-L_-}^{L_+}|\partial_x u_n(t,x)|^2\,dx \leqslant \|f_n(t)\|_{L^2(-L_-,L_+)}\|u_n(t)\|_{L^2(-L_-,L_+)}.$$

Using Poincaré's inequality, we thus get, for all $t\in(0,T_0)$,

$$\frac{d}{dt}\left(\int_{-L_-}^{L_+}|u_n(t,x)|^2\,dx\right) \leqslant C\|f_n(t)\|^2_{L^2(-L_-,L_+)},$$

from which we easily deduce that

$$\frac{T_0}{2}\|u_n(T_0)\|^2_{L^2(-L_-,L_+)} \leqslant \|u_n\|^2_{L^2((T_0/4,3T_0/4)\times(-L_-,L_+))}+C\|f_n\|^2_{L^2((0,T_0)\times(-L_-,L_+))}.$$

We thus deduce (3.26) from (3.39)–(3.40) and this last estimate. $\square$

3.2.3 Observability in time $T > T_*$: Proof of Theorem 1.4 item (i)

In order to prove Theorem 1.4 item (i), we shall combine the observability estimate of Proposition 1.5 and a dissipation result, that we state below and whose proof is given in Section 4.3:

Lemma 3.7. *There exists $C > 0$ such that, for all $n \in \mathbb{N}$, any solution u_n of* (1.20)*, with initial datum $u_{0,n} \in L^2(-L_-, L_+)$, satisfies, for all $t \geqslant 0$,*

$$\|u_n(t)\|_{L^2(-L_-,L_+)} \leqslant \exp(-(nq'(0) - C\sqrt{n})t) \|u_{0,n}\|_{L^2(-L_-,L_+)} . \tag{3.41}$$

Given $T > T_*$ with T_* as in (1.19), we choose $T_0 > 0$ such that $2T_0 < T - T_*$ and apply Proposition 1.5 with $\varepsilon = q'(0)T_0$: there exists a constant C independent of n such that for all n and u_n solution of (1.20),

$$\begin{aligned} \|u_n(T_0)\|_{L^2(-L_-,L_+)} &\leqslant C \exp\left(n \int_0^{L_+} q(s)ds + nq'(0)T_0\right) \|\partial_x u_n(.,L_+)\|_{L^2(0,T_0)} \\ &\leqslant C \exp\left(nq'(0)(T_* + T_0)\right) \|\partial_x u_n(.,L_+)\|_{L^2(0,T_0)} . \end{aligned}$$

Combined with Lemma 3.7 applied on the time interval (T_0, T_*), we obtain

$$\|u_n(T)\|_{L^2(-L_-,L_+)} \leqslant C \exp\left(-nq'(0)\left(T - T_* - 2T_0\right) + C\sqrt{n}(T - T_0)\right) \|\partial_x u_n(.,L_+)\|_{L^2(0,T_0)}$$

Consequently, the equations (1.20) are uniformly observable from $x = L_+$ in time T, in the sense that there exists $C > 0$ such that for all $n \in \mathbb{N}$, the solutions u_n of (1.20) with $u_{0,n} \in H_0^1(-L_-, L_+)$ satisfy

$$\|u_n(T)\|_{L^2(-L_-,L_+)} \leqslant C \|\partial_x u_n(t, L_+)\|_{L^2(0,T)} .$$

We thus deduce the observability of system (1.16) on $(0,T) \times \Gamma$, by Bessel-Parseval equality.

3.3 Heisenberg equation

The goal of this section is to prove Theorem 1.6 item (i). In order to do this, as before, we take advantage of the tensorized structure of the 3-d Heisenberg equation by developing the solution u of (1.22) in Fourier series with respect to both variables y and z, and therefore consider the following family of one-dimensional heat equations, indexed by n and p in $\mathbb{Z}$:

$$\begin{cases} (\partial_t - \partial_x^2 + (nx+p)^2)u_{n,p}(t,x) = 0 , & (t,x) \in (0,T) \times (-L_-, L_+) , \\ u_{n,p}(t,-L_-) = u_{n,p}(t,L_+) = 0 , & t \in (0,T) , \\ u_{n,p}(0,.) = u_{0,n,p} \in H_0^1(-L_-, L_+) , & \end{cases} \tag{3.42}$$

for which we will prove observability estimates with an observation at $x = L_+$ when $T > T_*$ with T_* as in (1.24). To be more precise, for $T > T_*$, we will show that there exists a constant $C > 0$ such that for all n and p in $\mathbb{Z}$, any solution $u_{n,p}$ of (3.42) with $u_{0,n,p} \in H_0^1(-L_-, L_+)$ satisfies

$$\|u_{n,p}(T)\|_{L^2(-L_-,L_+)} \leqslant C \|\partial_x u_{n,p}(t, L_+)\|_{L^2(0,T)} . \tag{3.43}$$

In order to prove observability properties (3.43) for solutions of (3.42), it will be convenient to write

$$(nx+p)^2 = n^2(x-\alpha)^2, \quad \text{with } \alpha = -\frac{p}{n}, \tag{3.44}$$

to underline the link between the equations (3.42) and the Grushin equation (1.20). But this writing is allowed only for $n \in \mathbb{Z}^*$, and we thus handle separately the case $n = 0$.

In the case $n = 0$, we are considering the family of 1-d heat equation with positive potential p^2 indexed by $p \in \mathbb{Z}$ and given by

$$\begin{cases} (\partial_t - \partial_x^2 + p^2)u_{0,p}(t,x) = 0 , & (t,x) \in (0,T) \times (-L_-, L_+) , \\ u_{0,p}(t,-L_-) = u_{0,p}(t,L_+) = 0 , & t \in (0,T) , \\ u_{0,p}(0,.) = u_{0,0,p} \in H_0^1(-L_-, L_+) , & \end{cases} \tag{3.45}$$

When $p = 0$, the usual observability estimate for the heat equation reads: there exists a constant $C > 0$ such that for all solution $u_{0,0}$ of (3.45) with $p = 0$ and initial datum in $H^1_0(-L_-, L_+)$,

$$\|u_{0,0}(T)\|_{L^2(-L_-,L_+)} \leqslant C \|\partial_x u_{0,0}(t, L)\|_{L^2(0,T)} .$$

It is readily seen that if $u_{0,p}$ solves (3.45) for some $p \in \mathbb{Z}$, then $u_{0,p}e^{p^2 t}$ solves (3.45) with $p = 0$. Thus one can apply the previous estimate to $u_{0,p}e^{p^2 t}$ and straightforwards bounds show that for all $p \in \mathbb{Z}$, any solution $u_{0,p}$ of (3.45) with initial datum in $H^1_0(-L_-, L_+)$ satisfies

$$\|u_{0,p}(T)\|_{L^2(-L_-,L_+)} \leqslant C \|\partial_x u_{0,p}(t, L)\|_{\cdot L^2(0,T)}. \tag{3.46}$$

We then consider the case $n \in \mathbb{Z}^*$ and $p \in \mathbb{Z}$. Based on the writing (3.44), we consider, instead of (3.42), the (larger) family of problems, indexed by $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$,

$$\begin{cases} (\partial_t - \partial_x^2 + n^2(x-\alpha)^2)u_{n,\alpha}(t,x) = 0\,, & (t,x) \in (0,T) \times (-L_-, L_+)\,, \\ u_{n,\alpha}(t,-L_-) = u_{n,\alpha}(t,L_+) = 0\,, & t \in (0,T)\,, \\ u_{n,\alpha}(0,.) = u_{0,n,\alpha} \in H^1_0(-L_-,L_+) \end{cases} \tag{3.47}$$

which we will prove to be observable in time $T > T_*$ with T_* as in (1.24) uniformly with respect to $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$:

Proposition 3.8. *Let T_* be as in (1.24). For every $T > T_*$, there exists $C > 0$ such that, for every $n \in \mathbb{Z}$, $\alpha \in \mathbb{R}$ and $u_{0,n,\alpha} \in L^2(-L_-, L_+)$, the solution of (3.47) satisfies*

$$\int_{-L_-}^{L_+} |u_{n,\alpha}(T,x)|^2 dx \leqslant C \int_0^T |\partial_x u_{n,\alpha}(t, L_+)|^2 dt\,. \tag{3.48}$$

Considering also that equation (3.47) does not depend on the sign of n, from now on we suppose that $n \in \mathbb{N}$. Clearly, equation (3.47) is degenerate only if α belongs to $[-L_-, L_+]$. Therefore, in the arguments afterwards, we shall deal independently with the case $\alpha \in [-L_- - \delta, L_+ + \delta]$ and $\alpha \in \mathbb{R} \setminus [-L_- - \delta, L_+ + \delta]$, where $\delta > 0$ is an arbitrary small parameter.

3.3.1 Cost estimate for α in the interval $[-L_- - \delta, L_+ + \delta]$

In the case $\alpha \in [-L_- - \delta, L_+ + \delta]$, the potential $q(x) = (x - \alpha)$ might cancel anywhere in the interval $(-L_-, L_+)$. Therefore, we shall be cautious and adapt the result we obtained for the Grushin equation, proving first an estimate on the cost of observability in this case, then an estimate on the rate of the dissipation of the semi-group (3.47).

Proposition 3.9. *Let $\delta > 0$ and $T > 0$. There exists $C = C(\delta, T) > 0$ such that, for every $\alpha \in [-L_- - \delta, L_+ + \delta]$, $n \in \mathbb{N}$ and $u_{0,n,\alpha} \in H^1_0(-L_-, L_+)$, the solution $u_{n,\alpha}$ of (3.47) satisfies*

$$\int_{-L_-}^{L_+} |u_{n,\alpha}(T,x)|^2 dx \leqslant C \exp\left(2n\left(\frac{(L_+ + L_-)^2}{2} + 2\delta(L_+ + L_-)\right)\right) \int_0^T |\partial_x u_{n,\alpha}(t, L_+)|^2 dt\,. \tag{3.49}$$

Proof. Let $\delta > 0$ and $\alpha \in [-L_- - \delta, L_+ + \delta]$. As before, we assume, without loss of generality, that $T \in (0, 4)$. The proof of Proposition 3.9 strongly relies on Proposition 3.3 with the choices

$$a_0 = -L_-, \quad L_+ = L_+, \quad q_\alpha(x) = x - \alpha, \quad B_\alpha = L_- + \alpha + 2\delta.$$

As q_α obviously satisfies (3.14) and B_α satisfies

$$B_\alpha \geqslant \delta > 0, \quad q_\alpha(-L_-) + B_\alpha = 2\delta > 0,$$

Proposition 3.3 applies to (3.47), with the weight function

$$\varphi_{n,\alpha}(t,x) = n\theta(t)\Psi_\alpha(x) + \theta(t), \quad (t,x) \in (0,T) \times (-L_-, L_+),$$

with θ as in (3.17) and Ψ_α defined as

$$\begin{aligned}\Psi_\alpha(x) = &\int_x^{L_+} (s-\alpha)\, ds + B_\alpha(L_+ - x)\\ =&\frac{1}{2}\left((L_+-\alpha)^2 - (x-\alpha)^2\right) + (L_- + \alpha + 2\delta)(L_+ - x)\\ =&(L_+ - x)\left(\frac{x+L_+}{2} + L_- + 2\delta\right).\end{aligned}$$

Still, we need to check the uniformity of the constants n_0 and C in Proposition 3.3 for $\alpha \in [-L_- - \delta, L_+ + \delta]$. We thus remark that we have the identities, for $(t,x) \in (0,T) \times (-L_-, L_+)$,

$$\begin{aligned}-\partial_{xx}\varphi_{n,\alpha}(t,x) &= n\theta(t),\\ \partial_x\varphi_{n,\alpha}(t,L_+) &= n\theta(t)(-L_+ + \alpha - B_\alpha) = -n\theta(t)(L_+ + L_- + 2\delta).\end{aligned}$$

It thus remains to bound

$$\begin{aligned}G_{\varphi_{n,\alpha}}(t,x) = 2n\theta(t)\Psi'_\alpha(x)&\left[2n\theta'(t)\Psi'_\alpha(x) - 2n^2\theta^2(t)\Psi''_\alpha(x)\Psi'_\alpha(x) + 2n^2 q'_\alpha(x)q_\alpha(x)\right]\\ &- n\theta''(t)\Psi_\alpha(x) - \theta''(t) + n\theta(t)\Psi_\alpha^{(4)}(x)\end{aligned} \tag{3.50}$$

from below, uniformly with respect to $\alpha \in [-L_- - \delta, L_+ + \delta]$. Arguing as in the proof of Proposition 3.3, we first remark that

$$\begin{aligned}2n\theta(t)\Psi'_\alpha(x)\left[-2n^2\theta^2(t)\Psi''_\alpha(x)\Psi'_{R,\alpha}(x) + 2n^2q'_\alpha(x)q_\alpha(x)\right] =&4n^3\theta(t)^3(-\Psi'_{R,\alpha}(x))\left(-\Psi'_{R,\alpha}(x) - \frac{q_\alpha(x)}{\theta^2(t)}\right)\\ \geqslant&4C(B_\alpha)n^3\theta(t)^3,\end{aligned}$$

where $C(B_\alpha) = \min\{B_\alpha, B_\alpha + (-L_- - \alpha)\} \geqslant \delta > 0$. We thus easily derive that, for all $(t,x) \in (0,T) \times (-L_-, L_+)$,

$$G_{\varphi_{n,\alpha}}(t,x) \geqslant 4n^3\theta(t)^3 + 4n^2\theta(t)\theta'(t)|\Psi'_\alpha(x)|^2 - n\theta''(t)\Psi_\alpha(x) - \theta''(t).$$

Now, it is easy to check that

$$\sup_{\alpha\in[-L_- - \delta, L_+ + \delta]}\ \sup_{x\in[-L_-, L_+]} \left\{|\Psi'_\alpha(x)| + |\Psi_\alpha(x)|\right\} < \infty,$$

so that there exists a constant $C > 0$ independent of $\alpha \in [-L_- - \delta, L_+ + \delta]$ such that for all $(t,x) \in (0,T) \times (-L_-, L_+)$,

$$G_{\varphi_{n,\alpha}}(t,x) \geqslant 4n^3\theta(t)^3 - Cn^2\theta(t)^3.$$

It easily follows that there exists $n_0 \in \mathbb{N}$ and $C > 0$ such that for all $n \geqslant n_0$, $\alpha \in [-L_- - \delta, L_+ + \delta]$, and $u_{0,n,\alpha} \in H_0^1(-L_-, L_+)$, the solution $u_{n,\alpha}$ of (3.47) satisfies:

$$n^3 \left\|\theta^{3/2}u_{n,\alpha}e^{-\varphi_{R,n,\alpha}}\right\|^2_{L^2((0,T)\times(-L_-,L_+))} \leqslant Cn\left\|\theta^{1/2}\partial_x u_{n,\alpha}(t,L_+)e^{-\theta(t)}\right\|^2_{L^2(0,T)}.$$

This leads in particular, with a constant C independent of $n \geqslant n_0$ and $\alpha \in [-L_- - \delta, L_+ + \delta]$, that any solution $u_{n,\alpha}$ of (3.47) satisfies:

$$\begin{aligned}e^{-n\sup_{[-L_-,L_+]}\{\Psi_{R,\alpha}(x)\}-1}\left\|u_{n,\alpha}\right\|_{L^2((T/4,3T/4)\times(-L_-,L_+))} &\leqslant \left\|\theta^{3/2}u_{n,\alpha}e^{-\varphi_{R,n,\alpha}}\right\|_{L^2((T/4,3T/4)\times(-L_-,L_+))}\\ &\leqslant \left\|\theta^{1/2}\partial_x u_{n,\alpha}(t,L_+)e^{-\theta(t)}\right\|_{L^2(0,T)}\\ &\leqslant C\left\|\partial_x u_{n,\alpha}(t,L)\right\|_{L^2(0,T)}.\end{aligned}$$

Straightforward computations then yield

$$\sup_{[-L_-,L_+]}\{\Psi_\alpha(x)\} = \Psi_\alpha(-L_-) = \frac{(L_+ + L_-)^2}{2} + 2\delta(L_+ + L_-).$$

We thus immediately deduce that any solution $u_{n,\alpha}$ of (3.47) satisfies

$$\|u_{n,\alpha}\|_{L^2((T/4,3T/4)\times(-L_-,L_+))} \leqslant C\exp\left(n\left(\frac{(L_++L_-)^2}{2}+2\delta(L_++L_-)\right)\right)\|\partial_x u_{n,\alpha}(t,L)\|_{L^2(0,T)},$$

for some $C>0$ independent of $n\geqslant n_0$ and $\alpha\in[-L_--\delta,-L_++\delta]$. The fact that the observability property (3.49) holds then uniformly for $n\geqslant n_0$ and $\alpha\in[-L_--\delta,-L_++\delta]$ immediately follows from the dissipativity of the equation (3.47).

Now, as n_0 is independent of α,

$$\sup_{n\in\{0,\cdots,n_0\}}\ \sup_{\alpha\in[-L_--\delta,L_++\delta]}\left\|n^2 q_\alpha(x)^2\right\|_{L^\infty(-L_-,L_+)} = n_0^2(L_+-L_-)^2,$$

so that Theorem A.1 easily gives the observability property (3.49) uniformly for $n\in\{0,\cdots,n_0\}$ and $\alpha\in[-L_--\delta,-L_++\delta]$.

Proposition 3.9 immediately follows. □

3.3.2 Cost estimate for $\alpha\in\mathbb{R}\setminus[-L_--\delta,L_++\delta]$

In that case, the potential $q(x)=(x-\alpha)$ is nowhere zero in the interval $(-L_-,L_+)$. For that reason, we will use a rather rough estimate on the cost of observability in this case, which is a consequence of [20]. Namely, Corollary A.2 applied to the family of potentials $V(x)=n^2(x-\alpha)^2$ immediately implies that

Proposition 3.10. *Let $\delta>0$ and $T>0$. There exists $C=C(T)>0$ such that for every $\alpha\in\mathbb{R}\setminus[-L_-,L_+]$, $n\in\mathbb{N}$ and $u_{0,n,p}\in H_0^1(-L_-,L_+)$, the solution $u_{n,\alpha}$ of (3.47) satisfies*

$$\int_{-L_-}^{L_+}|u_{n,\alpha}(T,x)|^2dx \leqslant C\exp\left(Cn^{4/3}\max\{L_+-\alpha,-L_--\alpha\}^{4/3}\right)\int_0^T|\partial_x u_{n,\alpha}(t,L_+)|^2dt\,.$$

3.3.3 Estimate of the rate of dissipation of (3.47)

We claim the following result:

Lemma 3.11. *For all $\alpha\in\mathbb{R}$ and all $n\in\mathbb{N}$, there exists $\lambda_{n,\alpha}>0$ such that any solution $u_{n,\alpha}$ of* (3.47) *with $u_{0,n,\alpha}\in L^2(-L_-,L_+)$ satisfies, for all $t\geqslant0$,*

$$\|u_{n,\alpha}(t)\|_{L^2(-L_-,L_+)} \leqslant \exp(-\lambda_{n,\alpha}t)\|u_{0,n,\alpha}\|_{L^2(-L_-,L_+)}, \tag{3.51}$$

where $\lambda_{n,\alpha}$ satisfies

$$\lambda_{n,\alpha}\geqslant\begin{cases} n\,, & \\ n^2(L_-+\alpha)^2 & \text{when } \alpha\leqslant -L_-,\\ n^2(\alpha-L_+)^2 & \text{when } \alpha\geqslant L_+.\end{cases} \tag{3.52}$$

The proof of Lemma 3.11 is given in Section 4.4.

3.3.4 Proof of Proposition 3.8

We are now in position to prove Proposition 3.8. Let $T>T_*=(L_++L_-)^2/2$. We take

$$T_0>0 \ \text{ such that } 2T_0<T-T_*, \qquad \text{and } \delta:=\frac{T_0}{2(L_++L_-)},$$

and consider three different cases, $\alpha\in[-L_--\delta,L_++\delta]$, $\alpha\leqslant -L_--\delta$ and $\alpha\geqslant L_++\delta$.

First case: $\alpha \in [-L_- - \delta, L_+ + \delta]$**.** We apply Proposition 3.9 with $T = T_0$ and use Lemma 3.11 on the time interval $[T_0, T]$: We obtain a constant $C > 0$ independent of $\alpha \in [-L_- - \delta, L_+ + \delta]$ such that for every $n \in \mathbb{N}$ and $u_{n,\alpha}$ solution of (3.42) with $u_{0,n,\alpha} \in H_0^1(-L_-, L_+)$,

$$
\begin{aligned}
\|u_{n,\alpha}(T)\|_{L^2(-L_-,L_+)} &\leqslant e^{-\lambda_{n,\alpha}(T-T_0)}\|u_{n,\alpha}(T_0)\|_{L^2(-L_-,L_+)} \\
&\leqslant Ce^{-n(T-T_0)} \exp\left(n\left(\frac{(L_+ + L_-)^2}{2} + 2\delta(L_+ + L_-)\right)\right)\|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T)} \\
&\leqslant C\exp\left(-n\left(T - T_* - 2T_0\right)\right)\|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T)} \\
&\leqslant C\|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T)}\,.
\end{aligned}
$$

Second case: $\alpha \leqslant -L_- - \delta$**.** We apply Proposition 3.10 with $T = T_0$ and use Lemma 3.11 on the time interval $[T_0, T]$: We obtain a constant $C > 0$ independent of $\alpha \leqslant -L_- - \delta$ such that for every $n \in \mathbb{N}$ and $u_{n,\alpha}$ solution of (3.42) with $u_{0,n,\alpha} \in H_0^1(-L_-, L_+)$,

$$
\begin{aligned}
\|u_{n,\alpha}(T)\|_{L^2(-L_-,L_+)} &\leqslant e^{-\lambda_{n,\alpha}(T-T_0)}\|u_{n,\alpha}(T_0)\|_{L^2(-L_-,L_+)} \\
&\leqslant Ce^{-n^2(L_-+\alpha)^2(T-T_0)} \exp\left(Cn^{4/3}(L_+ - \alpha)^{4/3}\right)\|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T)}.
\end{aligned}
$$

We now remark that there exists $C = C(\delta)$ such that for all $\alpha \leqslant -L_- - \delta$,

$$
(L_- + \alpha)^2 \geqslant \frac{1}{C}\alpha^2, \quad \text{and} \quad (L_+ - \alpha)^{4/3} \leqslant C|\alpha|^{4/3},
$$

while $T - T_0 > T_*$. We thus deduce that, for all $n \in \mathbb{N}$ and $\alpha \leqslant -L_- - \delta$, any solution $u_{n,\alpha}$ of (3.42) with $u_{0,n,\alpha} \in H_0^1(-L_-, L_+)$ satisfies

$$
\|u_{n,\alpha}(T)\|_{L^2(-L_-,L_+)} \leqslant C\exp\left(-\frac{T_*}{C}n^2\alpha^2 + Cn^{4/3}|\alpha|^{4/3}\right)\|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T)},
$$

where C is independent of $\alpha \leqslant -L_- - \delta$ and $n \in \mathbb{N}$. As

$$
\sup_{n\in\mathbb{N}}\sup_{\alpha\leqslant -L_- -\delta}\left\{-\frac{T_*}{C}n^2\alpha^2 + Cn^{4/3}|\alpha|^{4/3}\right\} \leqslant \sup_{\rho\in\mathbb{R}_+}\left\{-\frac{T_*}{C}\rho^2 + C\rho^{4/3}\right\} < \infty,
$$

we get a constant C independent of $\alpha \leqslant -L_- - \delta$ and $n \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $\alpha \leqslant -L_- - \delta$, any solution $u_{n,\alpha}$ of (3.42) with $u_{0,n,\alpha} \in H_0^1(-L_-, L_+)$ satisfies

$$
\|u_{n,\alpha}(T)\|_{L^2(-L_-,L_+)} \leqslant C\|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T)}.
$$

Third case: $\alpha \geqslant L_+ + \delta$**.** This case can be dealt with as in the second case by applying Proposition 3.10 with $T = T_0$ and Lemma 3.11 on the time interval (T_0, T). The detailed proof is left to the reader as it relies on exactly the same arguments as in the second case.

End of the proof of Proposition 3.8. The proof of the uniform observability inequality (3.48) then easily follows from the fact that if $\alpha \in \mathbb{R}$, then we necessarily are in one of the cases discussed above.

3.3.5 End of the proof of Theorem 1.6 item (i)

Combining the uniform observability estimates (3.46) proved uniformly with respect to $p \in \mathbb{Z}$ and Proposition 3.8, we get the following observability inequality when $T > T_*$: For $T > T_*$, there exists a constant $C > 0$ such that for all $n \in \mathbb{Z}$ and $p \in \mathbb{Z}$, any solution $u_{n,p}$ of (3.42) with $u_{0,n,p} \in H_0^1(-L_-, L_+)$ satisfies (3.43).

Applying Parseval's identity, one then immediately obtains the observability inequality (1.23), which proves Theorem 1.6 item (i).

3.3.6 The case of observations on both sides of the domain: Proof of Theorem 1.8

The goal of this section is to give a sketch of the proof of Theorem 1.8. We consider again system (3.47), and our purpose is to prove that for all $T > T_* = (L_+ + L_-)^2/8$, there exists $C > 0$ such that, for every $n \in \mathbb{Z}$, $\alpha \in \mathbb{R}$ and $u_{0,n,\alpha} \in H_0^1(-L_-, L_+)$, the solution of (3.47) satisfies

$$\int_{-L_-}^{L_+} |u_{n,\alpha}(T,x)|^2 dx \leqslant C \int_0^T (|\partial_x u_{n,\alpha}(t,-L_-)|^2 + |\partial_x u_{n,\alpha}(t,L_+)|^2) dt\,. \tag{3.53}$$

The strategy to prove this result is very similar to the one of the proof of Proposition 3.8, but this time, one should could consider three different cases depending on the location of α (here $\delta > 0$ is an arbitrary small parameter):

(i) the case $\alpha \in \mathbb{R} \setminus [-L_- - \delta, L_+ + \delta]$,

(ii) the case $\alpha \in I_{\mathcal{R}} = \left[\frac{L_+ - L_-}{2}, L_+ + \delta\right]$,

(iii) the case $\alpha \in I_{\mathcal{L}} = \left[-L_- - \delta, \frac{L_+ - L_-}{2}\right]$.

We already considered case (i) in Section 3.3.4, whereas case (iii) reduces to case (ii) by the change of variable $x \leftrightarrow -x + (L_+ - L_-)$. Therefore we only gives a hint on how to prove (3.53) in case (ii). From now on we assume that α belongs to $I_{\mathcal{R}}$.

The key point is again to obtain a precise estimate on the cost of observability of equation (3.47), uniform in α. Doing the change of variable $\tilde{x} = x - \alpha$ in (3.47), we see that $\tilde{u}_{n,\alpha}(\tilde{x}) = u_{n,\alpha}(x)$ verifies the system (3.25) in $(0,T) \times (-\tilde{L}_-, \tilde{L}_+)$, $\tilde{L}_- = L_- + \alpha, \tilde{L}_+ = L_+ - \alpha$, with $q(\tilde{x}) = \tilde{x}$. It is therefore tempting to apply directly Proposition 3.6, which would give the result, but we should guarantee that all the constants appearing in the proof of this proposition can be chosen independent of α. This can be done by a careful reading of sections 3.2.1 and 3.2.2, and using that α belongs to the bounded interval $I_{\mathcal{R}}$ (the proof is left to the reader).

Hence, for any $T_0 > 0$ and $\varepsilon > 0$, there exists a constant C such that for every $n \in \mathbb{N}$ and $\alpha \in I_{\mathcal{R}}$, any $u_{n,\alpha}$ solution of (3.47) with $u_{0,n,\alpha} \in H_0^1(-L_-, L_+)$ satisfies

$$\|u_{n,\alpha}(T_0)\|_{L^2((-L_-,L_+))} \leqslant C e^{n\left(\frac{(L_+-\alpha)^2}{2}+\varepsilon\right)} \|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T_0)}.$$

As for δ small enough,

$$\max_{\alpha \in I_{\mathcal{R}}} \frac{(L_+ - \alpha)^2}{2} = \frac{(L_+ + L_-)^2}{8},$$

we obtain

$$\|u_{n,\alpha}(T_0)\|_{L^2((-L_-,L_+))} \leqslant C e^{n\left(\frac{(L_++L_-)^2}{8}+\varepsilon\right)} \left(\|\partial_x u_{n,\alpha}(.,-L_-)\|_{L^2(0,T_0)} + \|\partial_x u_{n,\alpha}(.,L_+)\|_{L^2(0,T_0)}\right)$$

which combined with Lemma 3.11 gives the desired result.

3.4 Inverse problem for the 2D Grushin equation: Proof of Theorem 1.9

The goal of this section is to prove Theorem 1.9. To that end, we consider

$$\begin{cases} (\partial_t - \partial_x^2 + n^2|x|^2)u_n = f_n\,, & (t,x) \in (0,T) \times (-L_-, L_+)\,, \\ u_n(t,-L_-) = u_n(t,L_+) = 0\,, & t \in (0,T)\,, \\ u_n(0,.) = u_{0,n} \in H_0^1(-L_-, L_+)\,. \end{cases} \tag{3.54}$$

with a source term of the form

$$f_n(t,x) = R(t,x) k_n(x) \quad \text{for } (t,x) \in (0,T) \times (-L_-, L_+), \tag{3.55}$$

where $R = R(t,x)$ is assumed to be known and to satisfy (1.27). Then, Theorem 1.9 is a consequence of Parseval's identity and the following result.

Theorem 3.12. *Let T_* be defined by (1.19), $T > T_*$, T_0, T_1 be such that (1.28) holds, and assume that R satisfies* (1.27). *There exists $C > 0$ such that, for all $n \in \mathbb{N}^*$, for every $k_n \in L^2(-L_-, L_+)$ and $u_{0,n} \in L^2(-L_-, L_+)$, the solution u_n of* (3.54) *with a source term as in* (3.55) *satisfies*

$$\int_{-L}^{L} |k_n(x)|^2 dx \leqslant C \left(\int_{T_0}^{T} |\partial_t \partial_x u_n(t, L_+)|^2 dt + \int_{-L_-}^{L_+} |(-\partial_x^2 + n^2 x^2) u_n(T_1, x)|^2 dx \right). \tag{3.56}$$

Let us emphasize that Theorem 3.12 is relevant for large values of n. Indeed, for a given $n \in \mathbb{N}$, as noticed in [4], the works [24, 32] immediately yields the existence of a constant C_n depending on n such that (3.56) holds for any solution u_n of (3.54) with $u_{0,n} \in L^2(-L_-, L_+)$. We will therefore focus on the proof of Theorem 3.12 for large values of $n \in \mathbb{N}$, i.e. on the existence of $n_0 \in \mathbb{N}$ and a constant $C > 0$ such that for all $n \geqslant n_0$, any solution u_n of (3.54) with $u_{0,n} \in L^2(-L_-, L_+)$ satisfies (3.56).

The proof of Theorem 3.12 relies on the following corollary of Proposition 3.6 and Lemma 3.7.

Proposition 3.13. *Let $T > T_*$. There exists $C > 0$ and a sequence of positive real numbers $(\varepsilon_n)_{n\in\mathbb{N}^*}$ converging to zero as $n \to \infty$, such that for every $n \in \mathbb{N}$, $u_{0,n} \in L^2(-L_-, L_+)$, $f_n \in L^2((0,T) \times (-L_-, L_+))$, the solution of (3.54) with source term $f_n \in L^2((0,T) \times (-L_-, L_+))$ satisfies*

$$\int_{-L_-}^{L_+} |u_n(T, x)|^2 dx \leqslant C \int_0^T |\partial_x u_n(t, L_+)|^2 dt + \varepsilon_n \|f_n\|^2_{L^2((0,T)\times(-L_-,L_+))}.$$

Proof of Proposition 3.13. Let $T > T_*$ and $T_0 > 0$ be such that $2T_0 < T - T_*$. For $n \in \mathbb{N}$, let $S_n(t)$ be the semi-group corresponding to the equation (3.54).

From the Duhamel formula, any solution u_n of (3.54) satisfies:

$$u_n(T) = S_n(T - T_0) u_n(T_0) + \int_{T_0}^{T} S_n(T - t) f_n(t) dt.$$

Therefore, applying Lemma 3.7 and the Cauchy-Schwarz inequality, we get, for any solution u_n of (3.54),

$$\begin{aligned} \|u_n(T)\|_{L^2(-L_-,L_+)} &\leqslant e^{-n(T-T_0)} \|u_n(T_0)\|_{L^2(-L_-,L_+)} + \int_{T_0}^{T} e^{-n(T-t)} \|f_n(t)\|_{L^2(-L_-,L_+)} dt \\ &\leqslant e^{-n(T-T_0)} \|u_n(T_0)\|_{L^2(-L_-,L_+)} + \frac{1}{\sqrt{2n}} \|f_n\|_{L^2((T_0,T)\times(-L_-,L_+))}. \end{aligned}$$

Thus

$$\|u_n(T)\|^2_{L^2(-L_-,L_+)} \leqslant 2e^{-2n(T-T_0)} \|u_n(T_0)\|^2_{L^2(-L_-,L_+)} + \frac{1}{n} \|f_n\|^2_{L^2((0,T)\times(-L_-,L_+))}.$$

Applying Proposition 3.6 with $q(x) = x$ in time T_0 and $\varepsilon = T_0$, we obtain, for any solution u_n of (3.54) with source term $f_n \in L^2((0,T) \times (-L_-, L_+))$,

$$\|u_n(T)\|^2_{L^2(I)} \leqslant 2Ce^{-2n(T-T_*-2T_0)} \int_0^T |\partial_x u_n(t, L_+)|^2 dt + \left(2Ce^{-2n(T-T_*-2T_0)} + \frac{1}{n} \right) \|f_n\|^2_{L^2((0,T)\times(-L_-,L_+))},$$

with a constant C independent of n. From this last estimate, we easily deduce Proposition 3.13 as $T - T_* - 2T_0 > 0$. $\square$

Proof of Theorem 3.12. Let $T > T_*$, T_0, T_1 as in (1.28), and assume that R satisfies (1.27). Let then $n \in \mathbb{N}$ and let u_n be the solution of (3.54) with f_n as in (3.55).

Setting $R_0 = \inf_{(L,L)} |R(T_1, x)|$ (> 0 according to (1.27)), we have

$$R_0 |k_n(x)| \leqslant |R(T_1, x) k_n(x)| = |f_n(T_1, x)| \leqslant |\partial_t u_n(T_1, x)| + |(-\partial_x^2 + n^2 x^2) u_n(T_1, x)|$$

thus

$$\int_{-L_-}^{L_+} |k_n(x)|^2 dx \leqslant \frac{2}{R_0^2} \left(\int_{-L_-}^{L_+} |\partial_t u_n(T_1, x)|^2 dx + \int_{-L_-}^{L_+} |(-\partial_x^2 + n^2 x^2) u_n(T_1, x)|^2 dx \right).$$

We apply Proposition 3.13 to $\partial_t u_n$ between the times T_0 and T_1 (thus corresponding to $T = T_1 - T_0$ in Proposition 3.13, which is larger than T_* in (1.28)), noticing $\partial_t u_n$ solves the Grushin equation (1.20) with source term $\partial_t R(t,x) k_n(x)$:

$$\int_{-L_-}^{L_+} |\partial_t u_n(T_1,x)|^2 dx \leqslant C \int_{T_0}^{T_1} |\partial_x \partial_t u_n(t,L_+)|^2 dt + \varepsilon_n \|\partial_t R\|^2_{L^\infty((0,T)\times(-L_-,L_+))} \|k_n\|^2_{L^2(-L_-,L_+)} ,$$

for a constant $C > 0$ independent of n and ε_n which converges to 0 as $n \to \infty$.

Thus there exists $n_0 \in \mathbb{N}$ such that we can guarantee that for all $n \geqslant n_0$,

$$\frac{2}{R_0^2} \varepsilon_n \|\partial_t R\|^2_{L^\infty((0,T)\times I)} \leqslant \frac{1}{2},$$

and then for all $n \geqslant n_0$,

$$\int_{-L_-}^{L_+} |k_n(x)|^2 dx \leqslant \frac{4C}{R_0^2} \int_{T_0}^{T_1} |\partial_x \partial_t u_n(t,L_+)|^2 dt + \frac{4}{R_0^2} \int_{-L_-}^{L_+} |(-\partial_x^2 + n^2 x^2) u_n(T_1,x)|^2 dx ,$$

which concludes the proof of the estimate (3.56) uniformly for $n \geqslant n_0$.

As said above, the case $n \leqslant n_0$ follows immediately from the works [24, 32], then allowing to conclude Theorem 3.12. ☐

Theorem 1.9 then follows immediately by Parseval's identity from Theorem 3.12.

4 On the rate of dissipation of the semigroups

4.1 In a bounded domain of $\mathbb{R}^d$: Proof of Lemma 2.3

Lemma 2.3 can be proved by writing the equation (1.8) satisfied by u_n using the semigroup formalism under the form $u_n' + \mathscr{G}_{\mu_n} u_n = 0$, where, for $\mu \in \mathbb{R}$, $\mathscr{G}_\mu$ is the operator defined on $L^2(\Omega_x)$ by

$$\mathscr{D}(\mathscr{G}_\mu) = H^2(\Omega_x) \cap H_0^1(\Omega_x) , \qquad \mathscr{G}_\mu \psi := -\Delta_x \psi + \mu^2 |x|^2 \psi. \tag{4.1}$$

It is clear that $\mathscr{G}_\mu$ is a positive self-adjoint operator on $L^2(\Omega_x)$ and has compact resolvent. Therefore, its first eigenvalue λ_μ is characterized by the Rayleigh formula:

$$\begin{aligned} \lambda_\mu &= \inf \left\{ \int_{\Omega_x} \Big(|\nabla \varphi(x)|^2 + \mu^2 |x|^2 \varphi(x)^2 \Big) dx ; \varphi \in H_0^1(\Omega_x) , \ \|\varphi\|_{L^2(\Omega_x)} = 1 \right\} \\ &\geqslant \inf \left\{ \int_{\mathbb{R}^{d_x}} \Big(|\nabla \varphi(x)|^2 + \mu^2 |x|^2 \varphi(x)^2 \Big) dx ; \varphi \in H^1(\mathbb{R}^{d_x}) \cap L^2(\mathbb{R}^{d_x}, |x| dx) , \ \|\varphi\|_{L^2(\mathbb{R}^{d_x})} = 1 \right\} \\ &= \mu \inf \left\{ \int_{\mathbb{R}^{d_x}} \Big(|\nabla \phi(x)|^2 + |x|^2 \phi(x)^2 \Big) dx ; \phi \in H^1(\mathbb{R}^{d_x}) \cap L^2(\mathbb{R}^{d_x}, |x| dx) , \ \|\phi\|_{L^2(\mathbb{R}^{d_x})} = 1 \right\}, \end{aligned}$$

where this last identity is obtained via the transformation $\varphi(x) = |\mu|^{d_x/4} \phi(\sqrt{|\mu|} x)$.

This last expression corresponds, again via Rayleigh formula, to the first eigenvalue of the harmonic oscillator $-\Delta_x + |x|^2$ on $L^2(\mathbb{R}^{d_x})$ with domain $H^2(\mathbb{R}^{d_x}) \cap L^2(\mathbb{R}^{d_x}, |x|^2 \, dx)$, which is known to be equal to d_x, see [23, Section 2.1]. This implies that $\lambda_\mu \geqslant d_x \mu$.

Now, as a solution u_n of the equation (1.8) satisfies $u_n' + \mathscr{G}_{\mu_n} u_n = 0$, and $\mathscr{G}_{\mu_n}$ is a positive self-adjoint operator with compact resolvent whose smallest eigenvalue is larger than $d_x \mu_n$, we readily deduce Lemma 5.1.

4.2 On an interval $(0, L)$: Proof of Lemma 2.7

Similarly as Lemma 2.3, Lemma 2.7 is based on an estimate of the smallest eigenvalue of the operators $\mathscr{G}_{D,n}$ and $\mathscr{G}_{N,n}$ defined for each $n \in \mathbb{N}$ on $L^2(0,L)$ by

$$\mathscr{G}_{D,n}\psi = -\partial_{xx}\psi + n^2x^2\psi, \qquad \mathscr{D}(\mathscr{G}_{D,n}) = H^2(0,L) \cap H_0^1(0,L), \tag{4.2}$$

$$\mathscr{G}_{N,n}\psi = -\partial_{xx}\psi + n^2x^2\psi, \qquad \mathscr{D}(\mathscr{G}_{N,n}) = \{\psi \in H^2(0,L), \ \text{ with } \partial_x\psi(0) = 0, \ \psi(L) = 0\}, \tag{4.3}$$

corresponding respectively to the equations (1.14) and (1.15).

Again, for all $n \in \mathbb{N}$, $\mathscr{G}_{D,n}$ and $\mathscr{G}_{N,n}$ are positive self-adjoint operators on $L^2(0,L)$ with compact resolvent, and the first eigenvalue $\lambda_{D,n}$ of $\mathscr{G}_{D,n}$ as well as the first eigenvalue $\lambda_{N,n}$ of $\mathscr{G}_{N,n}$ can be estimated using Rayleigh formula.

Let us now focus on bounding $\lambda_{D,n}$ from below.

$$\begin{aligned}
\lambda_{D,n} &= \inf\left\{\int_0^L \left(|\varphi'(x)|^2 + n^2|x|^2\varphi(x)^2\right)dx; \varphi \in H_0^1(0,L)\,, \ \|\varphi\|_{L^2(0,L)} = 1\right\} \\
&\geqslant \inf\left\{\int_{\mathbb{R}_+} \left(|\varphi'(x)|^2 + n^2|x|^2\varphi(x)^2\right)dx; \varphi \in H_0^1(\mathbb{R}_+^*) \cap L^2(\mathbb{R}_+^*, |x|dx)\,, \ \|\varphi\|_{L^2(\mathbb{R}_+^*)} = 1\right\} \\
&= n\inf\left\{\int_{\mathbb{R}^{d_x}} \left(|\phi'(x)|^2 + |x|^2\phi(x)^2\right)dx; \phi \in H_0^1(\mathbb{R}_+^*) \cap L^2(\mathbb{R}_+^*, |x|dx)\,, \ \|\phi\|_{L^2(\mathbb{R}_+^*)} = 1\right\},
\end{aligned}$$

where we have used the transformation $\varphi(x) = \sqrt[4]{n}\phi(\sqrt{n}x)$ in the last identity. Now, the Rayleigh formula implies that the quantity

$$\inf\left\{\int_{\mathbb{R}^{d_x}} \left(|\phi'(x)|^2 + |x|^2\phi(x)^2\right)dx; \phi \in H_0^1(\mathbb{R}_+^*) \cap L^2(\mathbb{R}_+^*, |x|dx)\,, \ \|\phi\|_{L^2(\mathbb{R}_+^*)} = 1\right\}$$

coincides with the first eigenvalue of the operator $\mathscr{H}_D$ defined on $L^2(\mathbb{R}_+^*)$ by

$$\mathscr{H}_D\psi = -\partial_{xx}\psi + x^2\psi, \qquad \mathscr{D}(\mathscr{H}_D) = \{\psi \in H^2(\mathbb{R}_+^*) \cap H_0^1(\mathbb{R}_+^*), x^2\psi \in L^2(\mathbb{R}_+^*)\}.$$

(Note that $\mathscr{H}_D$ is a self-adjoint positive definite operator with compact resolvent.) By symmetry arguments, it is clear that any eigenvector ψ_0 of $\mathscr{H}_D$, when extended oddly on $\mathbb{R}$, is an odd eigenvector of the harmonic oscillator $-\partial_{xx} + x^2$ defined on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^2dx)$. As the spectrum of the harmonic oscillator is well-known, see [23, Section 2.1], it follows that the smallest eigenvalue of the operator $\mathscr{H}_D$ equals 3, and actually corresponds to the second eigenvalue of the harmonic operator on $\mathbb{R}$. We have thus proved that $\lambda_{D,n} \geqslant 3n$.

Similarly, one shows that

$$\lambda_{N,n} = n\inf\left\{\int_{\mathbb{R}^{d_x}} \left(|\phi'(x)|^2 + |x|^2\phi(x)^2\right)dx; \phi \in H^1(\mathbb{R}_+^*) \cap L^2(|x|dx)\,, \ \|\phi\|_{L^2(\mathbb{R}_+^*)} = 1\right\}.$$

The quantity

$$\inf\left\{\int_{\mathbb{R}^{d_x}} \left(|\phi'(x)|^2 + |x|^2\phi(x)^2\right)dx; \phi \in H^1(\mathbb{R}_+^*) \cap L^2(\mathbb{R}_+^*, |x|dx)\,, \ \|\phi\|_{L^2(\mathbb{R}_+^*)} = 1\right\}$$

then coincides with the first eigenvalue of the operator $\mathscr{H}_N$ defined on $L^2(\mathbb{R}_+^*)$ by

$$\mathscr{H}_N\psi = -\partial_{xx}\psi + x^2\psi, \qquad \mathscr{D}(\mathscr{H}_N) = \{\psi \in H^2(\mathbb{R}_+^*), \ x^2\psi \in L^2(\mathbb{R}_+^*), \ \text{ with } \partial_x\psi(0) = 0\}.$$

Consequently, the eigenvalues of $\mathscr{H}_N$ coincide with the eigenvalues of the harmonic operator $-\partial_{xx}+x^2$ defined on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^2dx)$ corresponding to even eigenfunctions. From [23, Section 2.1], it follows that the first eigenvalue of H_N equals 1, and thus $\lambda_{N,n} \geqslant n$.

Lemma 2.3 then easily follows, as the equation (1.14), respectively (1.15), can be written under the form $u_n' + \mathscr{G}_{D,n}u_n = 0$, respectively $u_n' + \mathscr{G}_{N,n}u_n = 0$.

4.3 On the rate of dissipation of the generalized Grushin equations: Proof of Lemma 3.7

Let q satisfies (1.17). As in the proofs of Lemma 2.3, 2.7, we will estimate the smallest eigenvalue of the operator $\mathscr{G}_{n,q}$ defined on $L^2(-L_-, L_+)$ by

$$\mathscr{G}_{n,q}\psi = -\partial_{xx}\psi + n^2 q(x)^2\psi, \qquad \mathscr{D}(\mathscr{G}_{n,q}) = H^2 \cap H_0^1(-L_-, L_+).$$

Again, $\mathscr{G}_{n,q}$ is a self-adjoint positive definite operator with compact resolvent, so if we call $\lambda_{n,q}$ its smallest eigenvalue, Lemma 2.7 will follow from an estimate of the form: there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$\lambda_{n,q} \geqslant nq'(0) - C\sqrt{n}. \tag{4.4}$$

Again, we use the Rayleigh formula:

$$\begin{aligned}\lambda_{n,q} &= \inf\left\{\int_{-L_-}^{L_+}\Big(\varphi'(x)^2 + n^2 q(x)^2\varphi(x)^2\Big)dx; \varphi \in H_0^1(-L_-, L_+)\,, \|\varphi\|_{L^2(-L_-,L_+)} = 1\right\} \\ &\geqslant \inf\left\{\int_{\mathbb{R}}\Big(\varphi'(x)^2 + n^2\tilde{q}(x)^2\varphi(x)^2\Big)dx; \varphi \in H^1(\mathbb{R}) \cap L^2(|x|dx)\,, \|\varphi\|_{L^2(\mathbb{R})} = 1\right\},\end{aligned} \tag{4.5}$$

where $\tilde{q}$ denotes any C^2 extension of q over $\mathbb{R}$ such that $\tilde{q}(x)/x$ converges to 1 as $|x| \to \infty$ and vanishes only at $x = 0$. Using Rayleigh formula, the quantity

$$\inf\left\{\int_{\mathbb{R}}\Big(\varphi'(x)^2 + n^2\tilde{q}(x)^2\varphi(x)^2\Big)dx; \varphi \in H^1(\mathbb{R}) \cap L^2(|x|dx)\,, \|\varphi\|_{L^2(\mathbb{R})} = 1\right\}$$

coincides with the first eigenvalue of the operator $\mathscr{H}_{q,n}$ defined on $L^2(\mathbb{R})$ by

$$\mathscr{H}_{q,n}\psi = -\partial_{xx}\psi + n^2(\tilde{q}(x))^2\psi, \qquad \mathscr{D}(\mathscr{H}_{q,n}) = \{\psi \in H^2(\mathbb{R}),\ x^2\psi \in L^2(\mathbb{R})\}$$

With the assumptions (1.17) on q and on the choice of the extension $\tilde{q}$, we are thus in position to apply [23, Proposition 2.2.1 and Remark 2.2.2], which precisely states that, for n large enough, the first eigenvalue of $\mathscr{H}_{q,n}$ is bounded from below by $nq'(0) - C\sqrt{n}$.

We readily deduce (4.4) and then Lemma 2.7.

4.4 Proof of Lemma 3.11

Lemma 3.11 is again based on a bound from below of the first eigenvalue of the operator $\mathscr{G}_{n,\alpha}$ defined for $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ on $L^2(-L_-, L_+)$ by

$$\mathscr{G}_{n,\alpha}\psi = -\partial_{xx}\psi + n^2(x-\alpha)^2\psi, \qquad \mathscr{D}(\mathscr{G}_{n,\alpha}) = H^2 \cap H_0^1(-L_-, L_+). \tag{4.6}$$

These operators are self-adjoint, positive definite, and have compact resolvent. It follows that the dissipation estimate (3.51) in Lemma 3.11 obviously holds with $\lambda_{n,\alpha}$ being the first eigenvalue of $\mathscr{G}_{n,\alpha}$.

We thus estimate the first eigenvalue $\lambda_{n,\alpha}$ of $\mathscr{G}_{n,\alpha}$ for $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned}\lambda_{n,\alpha} &= \inf\left\{\int_{-L_-}^{L_+}\Big(\varphi'(x)^2 + n^2(x-\alpha)^2\varphi(x)^2\Big)dx; \varphi \in H_0^1(-L_-, L_+)\,, \|\varphi\|_{L^2(-L_-,L_+)} = 1\right\} \\ &\geqslant \inf\left\{\int_{\mathbb{R}}\Big(\varphi'(x)^2 + n^2(x-\alpha)^2\varphi(x)^2\Big)dx; \varphi \in H^1(\mathbb{R}) \cap L^2(|x|dx)\,, \|\varphi\|_{L^2(\mathbb{R})} = 1\right\} \\ &= \inf\left\{\int_{\mathbb{R}}\Big(\varphi'(x)^2 + n^2x^2\varphi(x)^2\Big)dx; \varphi \in H^1(\mathbb{R}) \cap L^2(|x|dx)\,, \|\varphi\|_{L^2(\mathbb{R})} = 1\right\} \\ &= n\inf\left\{\int_{\mathbb{R}}\Big(\phi'(x)^2 + x^2\phi(x)^2\Big)dx; \phi \in H^1(\mathbb{R}) \cap L^2(|x|dx)\,, \|\phi\|_{L^2(\mathbb{R})} = 1\right\} = n\,,\end{aligned} \tag{4.7}$$

which proves the first inequality in (3.52).

When $\alpha \notin [-L_-, L_+]$, for every $\varphi \in H_0^1(-L_-, L_+)$,

$$\int_{-L_-}^{L_+} \left(\varphi'(x)^2 + n^2(x-\alpha)^2\varphi(x)^2\right)dx \geqslant n^2 \left(\inf_{[-L_-,L_+]}(x-\alpha)\right)^2 \int_{-L_-}^{L_+} \varphi(x)^2 dx,$$

which immediately proves the second and third inequality in (3.52) by using the variational characterization (4.7).

5 Optimality results

The goal of this section is to prove the optimality results stated in items (ii) of Theorems 1.1, 1.3, 1.4 and 1.6.

In fact, all the proofs of these results are very similar. We shall therefore spend most of this section on the most intricate case, namely the one corresponding to Theorem 1.4 item (ii).

5.1 Proof of Theorem 1.4 item (ii): Non observability in time $T < T_*$ for Grushin equations

We are going to prove that, if system (1.16) is observable on $(0,T)\times\Gamma$, then $T \geqslant T_*$. To that end, we will apply the observability inequality (1.18) to a particular solution of the Grushin equation, with separate variables.

Let $\mathscr{G}_{n,q}$ be the operator defined by

$$D(\mathscr{G}_{n,q}) = H^2 \cap H_0^1(-L_-, L_+)\,, \qquad \mathscr{G}_{n,q} = -\partial_x^2 + n^2 q(x)^2, \tag{5.1}$$

$\lambda_{n,q}$ be its smallest eigenvalue, $\varphi_{n,q}$ be the associated eigenfunction,

$$\left\{\begin{array}{l} -\varphi_{n,q}''(x) + n^2 q(x)^2 \varphi_{n,q}(x) = \lambda_{n,q}\varphi_{n,q}(x)\,, \quad x \in (-L_-, L_+)\,, \\ \varphi_{n,q}(-L_-) = \varphi_{n,q}(L_+) = 0\,, \\ \|\varphi_{n,q}\|_{L^2(-L_-,L_+)} = 1\,. \end{array}\right. \tag{5.2}$$

We then consider the following solutions of system (1.16)

$$u_n(t,x,y) = \varphi_{n,q}(x) e^{-\lambda_{n,q} t} \sin(ny)\,. \tag{5.3}$$

The observability inequality (1.18) for this sequence of specific solution u_n then writes, for all $n \in \mathbb{N}$,

$$e^{-2\lambda_{n,q}T} \leqslant C\frac{1-e^{-2\lambda_{n,q}T}}{2\lambda_{n,q}}\varphi_{n,q}'(L_+)^2 \leqslant \frac{C}{\lambda_{n,q}}\varphi_{n,q}'(L_+)^2\,. \tag{5.4}$$

We shall show that, as C is a constant which does not depend on n, this cannot be satisfied if the time T is too small.

The main points are thus the following ones:

- a precise estimate of $\lambda_{n,q}$, see Proposition 5.1 below,
- an Agmon estimate on $\varphi_{n,q}$, allowing to estimate precisely $\varphi_{n,q}'(L)$, see Proposition 5.2 below.

The precise estimate on $\lambda_{n,q}$ reads as follows:

Proposition 5.1. *Let $L_- > 0$, $L_+ > 0$, $q \in C^3([-L_-, L_+], \mathbb{R})$ satisfying (1.17). Let $\mathscr{G}_{n,q}$ be the operator defined by (5.1) and $\lambda_{n,q}$ be its smallest eigenvalue. Then there exists a constant $C > 0$ such that, for n large enough,*

$$|\lambda_{n,q} - nq'(0)| \leqslant C\sqrt{n}.$$

The proof of Proposition 5.1 is done in Section 5.1.1.

Agmon estimates allow to prove the following result:

Proposition 5.2. *Let L_-, L_+, q, $\mathscr{G}_{n,q}$ and $\lambda_{n,q}$ be as in Proposition 5.1 and $\varphi_{n,q}$ be the eigenfunction of $\mathscr{G}_{n,q}$ associated to the eigenvalue $\lambda_{n,q}$, see (5.2). For every $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that, for n large enough*

$$|\varphi'_{n,q}(L_+)| \leqslant C \exp\left(-n\left(\int_0^{L_+} q(s)ds - \varepsilon\right)\right).$$

The proof of Proposition 5.2 is given in Section 5.1.2.

Let us now explain how Proposition 5.1 and Proposition 5.2 imply Theorem 1.4 item (ii). Indeed assume that the time T is such that system (1.16) is observable in time T through $\{L_+\} \times (0, \pi)$. Then, applying the observability inequality (1.18) to the solutions u_n in (5.3), we get the existence of a constant $C > 0$ such that for all $n \in \mathbb{N}$, (5.4) holds. Now, from Proposition 5.1, for all $n \in \mathbb{N}$ large enough,

$$e^{-2\lambda_{n,q}T} \geqslant e^{-2nq'(0)T - C\sqrt{n}T},$$

while from Proposition 5.1 and Proposition 5.2, for any $\varepsilon > 0$, there exists C such that for all $n \in \mathbb{N}$,

$$\frac{1}{\lambda_{n,q}}|\varphi'_{n,q}(L_+)|^2 \leqslant C(\varepsilon) n \exp\left(-2n\int_0^{L_+} q(s)\,ds + 2n\varepsilon\right).$$

Therefore, the inequality (5.4) implies that for any $\varepsilon > 0$, there exists C such that for all $n \in \mathbb{N}$ large enough,

$$e^{-2nq'(0)T - C\sqrt{n}T} \leqslant CC(\varepsilon) n \exp\left(-2n\int_0^{L_+} q(s)\,ds + 2n\varepsilon\right).$$

Looking at the asymptotics $n \to \infty$, this inequality implies:

$$q'(0)T - \int_0^{L_+} q(s)\,ds + 2\varepsilon \geqslant 0.$$

Now, as $\varepsilon > 0$ is arbitrary, we let it go to zero, and we have thus obtained:

$$T \geqslant \frac{1}{q'(0)}\int_0^{L_+} q(s)\,ds,$$

which concludes the proof of Theorem 1.4 item (ii).

5.1.1 Proof of Proposition 5.1

The proof of the lower bound $\lambda_{n,q} \geqslant nq'(0) - C\sqrt{n}$ for n large enough has been done in Section 4.3.

To prove the upper bound $\lambda_{n,q} \leqslant nq'(0) + \sqrt{n}$ we consider $\varepsilon > 0$ such that $-L_- + \varepsilon < 0 < L_+ - \varepsilon$, $\theta \in C^\infty(\mathbb{R})$ supported on $(-L_- + \varepsilon/2, L_+ - \varepsilon/2)$ such that $0 \leqslant \theta \leqslant 1$, $\theta = 1$ on $(-L_- + \varepsilon, L_+ - \varepsilon)$ and the function

$$\varphi(x) = C_n \theta(x) \exp\left(-n\int_0^x q(s)ds\right) \quad \text{where} \quad \frac{1}{C_n^2} = \int_{-L_-}^{L_+} \theta(x)^2 \exp\left(-2n\int_0^x q(s)ds\right) dx. \tag{5.5}$$

We deduce from the inequality $|q(s)| \leqslant \|q'\|_\infty |s|$ that $C_n^2 = \underset{n\to\infty}{O}(\sqrt{n})$. Indeed,

$$\frac{1}{C_n^2} \geqslant \frac{1}{\sqrt{n}} \int\limits_{(-L_-+\varepsilon)\sqrt{n}}^{(L_+-\varepsilon)\sqrt{n}} e^{-\|q'\|_\infty y^2} dy.$$

We have

$$\begin{cases} -\varphi''(x) + n^2 q(x)^2 \varphi(x) = nq'(x)\varphi(x) + C_n(2nq(x)\theta'(x) - \theta''(x))e^{-n\int_0^x q(s)ds}, & x \in (-L_-, L_+), \\ \varphi(-L_-) = \varphi(L_+) = 0, \\ \|\varphi\|_{L^2(-L_-,L_+)} = 1, \end{cases}$$

thus by multiplying the first identity by φ and integrating by parts, we get

$$\int_{-L_-}^{L_+} \Big(\varphi'(x)^2 + n^2 q(x)^2 \varphi(x)^2\Big) dx = nq'(0) + I_1 + I_2$$

where

$$\begin{aligned}
I_1 &= nC_n^2 \int_{-L_-}^{L_+} [q'(x) - q'(0)]\theta(x)^2 e^{-2n\int_0^x q(s)ds} dx \\
&= C_n^2 \int_{-L_-}^{L_+} \frac{d}{dx}\Big[\frac{[q'(x) - q'(0)]\theta(x)^2}{2q(x)}\Big] e^{-2n\int_0^x q(s)ds} dx \\
&\leqslant \Big\| \frac{d}{dx}\Big[\frac{[q'(x) - q'(0)]}{2q(x)}\Big]\Big\|_\infty + C_n^2 \int_{-L_-}^{L_+} 2\theta'(x)\theta(x) \frac{[q'(x) - q'(0)]}{2q(x)} e^{-2n\int_0^x q(s)ds} dx \\
&\leqslant C\Big(1 + \sqrt{n} e^{-2n\int_0^{L_+-\varepsilon} q(s)ds} + \sqrt{n} e^{-2n\int_{-L_-+\varepsilon}^0 |q(s)|ds}\Big) = \underset{n\to\infty}{O}(1),
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= C_n^2 \int_{-L_-}^{L_+} (2nq(x)\theta'(x) - \theta''(x))\theta(x) e^{-2n\int_0^x q(s)ds} dx \\
&\leqslant Cn^{3/2}\Big(e^{-2n\int_0^{L_+-\varepsilon} q(s)ds} + e^{-2n\int_{-L_-+\varepsilon}^0 |q(s)|ds}\Big) = \underset{n\to\infty}{O}(1)\,,
\end{aligned}$$

in which in both estimates, we used the fact that θ' is supported in $[-L_-, -L_- + \epsilon] \cup [L_+ - \epsilon, L_+]$ and that

$$\int_0^{L_+-\epsilon} q(s)\, ds > 0, \quad \int_0^{-L_-} q(s)\, ds = \int_{-L_-}^0 |q(s)|\, ds > 0,$$

due to the assumptions (1.17).

Now, plugging φ in (4.5), we immediately obtain the upper bound $\lambda_{n,q} \leqslant nq'(0) + C\sqrt{n}$, which concludes the proof of Proposition 5.1 (in fact, we have proved slightly better, namely $\lambda_{n,q} \leqslant nq'(0) + C$).

5.1.2 Proof of Proposition 5.2

To simplify the notations, we drop the subscript q. Let $\varepsilon \in (0, 1)$. We introduce the function

$$g_n(x) := \varphi_n(x) \exp\left(n\sqrt{1-\varepsilon}\int_0^x q(s)ds\right) \tag{5.6}$$

that satisfies

$$\begin{cases} -g_n''(x) + 2n\sqrt{1-\varepsilon} q(x) g_n'(x) + \Big(\varepsilon n^2 q(x)^2 + n\sqrt{1-\varepsilon} q'(x) - \lambda_n\Big) g_n(x) = 0\,, & x \in (-L_-, L_+)\,, \\ g_n(-L_-) = g_n(L_+) = 0\,, \end{cases}$$

and

$$\int\limits_{-L_-}^{L_+} \Big(|g_n'(x)|^2 + (\varepsilon n^2 q(x)^2 - \lambda_n)|g_n(x)|^2\Big) dx = 0\,.$$

Let

$$\delta_n := \frac{2q'(0)}{\varepsilon n}\,.$$

For any $x \in (-L_-, L_+)$ that satisfies $q(x)^2 \geqslant \delta_n$ we have

$$\varepsilon n^2 q(x)^2 - \lambda_n \geqslant \varepsilon n^2 \delta_n - nq'(0) - C\sqrt{n} = nq'(0) - C\sqrt{n} \geqslant 0$$

for n large enough. Therefore, for n large enough,

$$\int_{-L_-}^{L_+} |g_n'(x)|^2 dx \leqslant -\int_{\{q^2<\delta_n\}} (\varepsilon n^2 q(x)^2 - \lambda_n)|g_n(x)|^2 dx$$
$$\leqslant Cn \int_{\{q^2<\delta_n\}} |\varphi_n(x)|^2 e^{2n\sqrt{1-\varepsilon}\int_0^x q} dx.$$

For n large enough, the set $\{q^2 < \delta_n\}$ is close to 0, where $q(x) \sim q'(0)x$. Thus, if $q^2(x) < \delta_n$ then the size of x is almost $\sqrt{\delta_n}/q'(0)$, implying in particular $x \leqslant \sqrt{2\delta_n}/q'(0)$, and

$$\sqrt{1-\varepsilon}\, n \int_0^x |q(s)|ds \leqslant \sqrt{1-\varepsilon} n \left(q'(0)\frac{x^2}{2} + Cx^3\right) \leqslant \sqrt{1-\varepsilon}\frac{n\delta_n}{q'(0)}\left(1 + C\sqrt{\delta_n}\right) \leqslant \frac{2}{\varepsilon}$$

for n large enough.

We get a positive constant $C = C(\varepsilon) > 0$ such that, for n large enough,

$$\int_{-L_-}^{L_+} |g_n'(x)|^2 dx \leqslant Cn\,.$$

We deduce from this H_0^1-estimate and the equation solved by g_n that, for n large enough,

$$\|g_n''\|_{L^2(-L_-,L_+)} \leqslant Cn^{5/2},$$

for some constant $C = C(\varepsilon) > 0$.

We then write

$$\begin{aligned}|g_n'(L_+)|^2 &= \int_0^{L_+} \partial_x(x|g_n'|^2)\, dx \\ &\leqslant \|g_n'\|^2_{L^2(-L_-,L_+)} + L_+\|g_n'\|^2_{L^2(-L_-,L_+)}\|g_n''\|^2_{L^2(-L_-,L_+)} \\ &\leqslant Cn^3.\end{aligned}$$

Using now the identity

$$\varphi_n'(L_+) = g_n'(L_+)\exp\left(-n\sqrt{1-\varepsilon}\int_0^{L_+} q(s)ds\right),$$

and the fact that $\varepsilon > 0$ is arbitrary small, we obtain Proposition 5.2 for ε small enough. The case of large ε is then obvious.

5.2 Proof of Theorem 1.1 item (ii)

First, we shall indicate that when $d_x = 1$, Theorem 1.1 item (ii) is already proved in [3, Theorem 5 for $\gamma = 1$]. Also note that the proof of Theorem 1.4 item (ii) given above immediately yields Theorem 1.1 item (ii) in this case.

In order to show that Theorem 1.1 item (ii) holds when $d_x \geqslant 1$, one should follow the same steps as in Section 5.1 and prove the following two propositions:

Proposition 5.3. *Let $\mathscr{G}_\mu$ be as in (4.1) with $\Omega_x = B(0,L) \subset \mathbb{R}^{d_x}$ for some $L > 0$, and let λ_μ be its smallest eigenvalue. Then there exists a constant $C > 0$ such that, for μ large enough,*

$$|\lambda_\mu - \mu d_x| \leqslant C\sqrt{\mu}.$$

Proposition 5.4. *Within the setting of Proposition 5.3 and φ_μ be the eigenfunction of $\mathscr{G}_\mu$ associated to the eigenvalue λ_μ. For every $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that, for μ large enough*

$$\|\partial_\nu\varphi_\mu(L)\|_{L^2(\partial B(0,L))} \leqslant C\exp\left(-\mu\frac{L^2}{2} + \mu\varepsilon\right).$$

The proofs of Propositions 5.3 and 5.4 closely follow the ones of Propositions 5.3 and 5.4, by working on

$$\varphi(x) = C_\mu \theta(|x|) \exp\left(-\mu \frac{|x|^2}{2}\right)$$

instead of (5.5) for the proof of Proposition 5.3, and on

$$g_\mu(x) = \varphi_\mu(x) \exp\left(-\mu(1-\varepsilon)\frac{|x|^2}{2}\right)$$

instead of (5.6) for the proof of Proposition 5.3. Details are left to the reader.

Based on Propositions 5.3 and 5.4, Theorem 1.1 easily follows from the same considerations as in Section 5.1.

5.3 Proof of Theorem 1.3 item (ii)

Here again, we only sketch the proof of Theorem 1.3 item (ii) as it closely follows the one of Theorem 1.4 presented in Section 5.1.

Proposition 5.5. *For $n \in \mathbb{N}$, let $\mathscr{G}_{D,n}$ be as in (4.2) and $\mathscr{G}_{N,n}$ be as in (4.3), and let $\lambda_{D,n}$, respectively $\lambda_{N,n}$, be the smallest eigenvalue of $\mathscr{G}_{D,n}$, respectively $\mathscr{G}_{N,n}$. Then there exists a constant $C > 0$ such that, for n large enough,*

$$|\lambda_{D,n} - 3n| \leqslant C\sqrt{n}, \qquad |\lambda_{N,n} - n| \leqslant C\sqrt{n}.$$

Proposition 5.6. *Within the setting of Proposition 5.5 and $\varphi_{D,n}$, respectively $\varphi_{N,n}$, be the eigenfunction of $\mathscr{G}_{D,n}$, respectively $\mathscr{G}_{N,n}$ associated to the eigenvalue $\lambda_{D,n}$, respectively $\lambda_{N,n}$. For every $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that, for n large enough*

$$|\varphi'_{N,n}(L)| \leqslant C \exp\left(-n\frac{L^2}{2} + n\varepsilon\right), \qquad |\partial_x \varphi'_{D,n}(L)| \leqslant C \exp\left(-n\frac{L^2}{2} + n\varepsilon\right).$$

The proof of Proposition 5.6 readily follows the one of Proposition 5.2 and is therefore left to the reader.

The proof of Proposition 5.5 has to be slightly modified when considering the Dirichlet case, in which one should take

$$\varphi(x) = C_n \theta(x) x \exp\left(-n\frac{x^2}{2}\right)$$

instead of (5.5) for the proof of Proposition 5.5 in the Dirichlet case. Details are left to the reader.

Again, once Propositions 5.5 and 5.6 are proved, Theorem 1.3 item (ii) easily follows.

5.4 Proof of Theorem 1.6 item (ii): Non observability in time $T < T_*$ for Heisenberg equations

We are going to prove that, if system (1.22) is observable on $(0,T) \times \Gamma$, then $T \geqslant T_*$. To that end, we will apply the observability inequality to a particular solution of the Heisenberg equation, with separate variables.

Let $\varepsilon > 0$, and $\alpha \in \mathbb{Q}$ such that $-L_- < \alpha < -L_- + \varepsilon$, and let $\lambda_{n,\alpha}$ be the smallest eigenvalue and $\varphi_{n,\alpha}$ the corresponding eigenfunction of the operator $\mathscr{G}_{n,\alpha}$ in (4.6).

We write $\alpha = -p_\alpha/n_\alpha$ with $(p_\alpha, n_\alpha) \in \mathbb{N}^2$. For $k \in \mathbb{N}$, we consider the subsequence $(n_k, p_k) = (kn_\alpha, kp_\alpha)$ and define

$$u_{k,\alpha}(t,x,y,z) = \varphi_{n_k,\alpha}(x) e^{-\lambda_{n_k,\alpha} t} e^{-in_k z} e^{-ip_k y}.$$

By construction, for each $k \in \mathbb{N}$, u_k is a solution of (1.22), and the observability inequality (1.23) applied to $u_{k,\alpha}$ implies, for k large,

$$e^{-2\lambda_{n_k,\alpha} T} \leqslant C \frac{1}{2\lambda_{n_k,\alpha}} \varphi'_{n_k,\alpha}(L_+)^2.$$

By Propositions 5.1 and 5.2 applied with $(-L_-, L_+) = (-L_- - \alpha, L_+ - \alpha)$ and $q(x) = x$, following the argument in Section 5.1, we obtain that, for all $\varepsilon > 0$,

$$T > \frac{1}{2}(L_+ - \alpha)^2 - \varepsilon.$$

Now, $\varepsilon > 0$ is arbitrary, and α is any rational number larger than $-L_-$. This leads that T has to be larger than $(L_+ + L_-)^2/2$ as claimed in Theorem 1.6 item (ii).

A On the cost of observability of the heat equation with potential

In this section, we recall the result of [20, Theorem 1.2 and Section 8.6] for the cost of observability of the heat equation with a potential.

Theorem A.1 ([20, Theorem 1.2 and Section 8.6]). *Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \geqslant 1$, and Γ be a non-empty open subset of $\partial\Omega$. Then there exists a constant $C = C(\Omega, \Gamma) > 0$ such that for all $T > 0$, $V \in L^\infty((0,T) \times \Omega)$, $\varphi_0 \in H_0^1(\Omega)$, the solution φ of*

$$\begin{cases} \partial_t \varphi - \Delta \varphi + V\varphi = 0, & \text{in } (0,T) \times \Omega, \\ \varphi = 0, & \text{on } (0,T) \times \partial\Omega, \\ \varphi(0,\cdot) = \varphi_0, & \text{in } \Omega, \end{cases} \tag{A.1}$$

satisfies the following observability property

$$\|\varphi(T)\|_{L^2(\Omega)} \leqslant C \exp\left(C\left(1 + \frac{1}{T} + T\,\|V\|_{L^\infty((0,T)\times\Omega)} + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3}\right)\right) \|\partial_\nu \varphi\|_{L^2((0,T)\times\Gamma)}.$$

One of the main consequence of Theorem A.1 is the fact that, for all $M > 0$, the cost of observability of the heat equation with potential $V \in L^\infty((0,T) \times \Omega)$ with $\|V\|_{L^\infty((0,T)\times\Omega)} \leqslant M$ observed during a time T is bounded by a constant $C = C(T, M)$.

We shall also use the following consequence of Theorem A.1.

Corollary A.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \geqslant 1$, and Γ be a non-empty open subset of $\partial\Omega$. Then there exists a constant $C = C(\Omega, \Gamma) > 0$ such that for all $T > 0$, $V \in L^\infty((0,T) \times \Omega)$ with $V \geqslant 0$, $\varphi_0 \in H_0^1(\Omega)$, the solution φ of* (A.1) *satisfies the following observability property*

$$\|\varphi(T)\|_{L^2(\Omega)} \leqslant C \exp\left(C\left(1 + \frac{1}{T} + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3}\right)\right) \|\partial_\nu \varphi\|_{L^2((0,T)\times\Gamma)}. \tag{A.2}$$

Proof. Let $V \in L^\infty((0,T) \times \Omega)$ with $V \geqslant 0$ and consider the solution φ of (A.1). As $V \geqslant 0$, multiplying (A.1) by $\varphi(t,\cdot)$ and integrating between the times T_0 and T, we easily get that, for all $T_0 \in (0,T)$,

$$\|\varphi(T)\|_{L^2(\Omega)} \leqslant \|\varphi(T_0)\|_{L^2(\Omega)}.$$

Therefore, applying (A.1) to φ on the time interval $(0, T_0)$, there exists a constant $C > 0$ independent of V such that for all $T_0 \in (0,T]$,

$$\begin{aligned} \|\varphi(T)\|_{L^2(\Omega)} &\leqslant C \exp\left(C\left(1 + \frac{1}{T_0} + T_0\,\|V\|_{L^\infty((0,T_0)\times\Omega)} + \|V\|_{L^\infty((0,T_0)\times\Omega)}^{2/3}\right)\right) \|\partial_\nu \varphi\|_{L^2((0,T_0)\times\Gamma)} \\ &\leqslant C \exp\left(C\left(1 + \frac{1}{T_0} + T_0\,\|V\|_{L^\infty((0,T)\times\Omega)} + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3}\right)\right) \|\partial_\nu \varphi\|_{L^2((0,T)\times\Gamma)}. \end{aligned} \tag{A.3}$$

If $T \geqslant \|V\|_{L^\infty((0,T)\times\Omega)}^{-1/3}$, we choose $T_0 = \|V\|_{L^\infty((0,T)\times\Omega)}^{-1/3}$, so that

$$\begin{aligned} 1 + \frac{1}{T_0} + T_0\,\|V\|_{L^\infty((0,T)\times\Omega)} + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3} &= 1 + \|V\|_{L^\infty((0,T)\times\Omega)}^{1/3} + 2\,\|V\|_{L^\infty((0,T)\times\Omega)}^{2/3} \\ &\leqslant 3\left(1 + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3}\right). \end{aligned}$$

If $T \leqslant \|V\|_{L^\infty((0,T)\times\Omega)}^{-1/3}$, we choose $T_0 = T$, so that

$$\begin{aligned} 1 + \frac{1}{T_0} + T_0 \|V\|_{L^\infty((0,T)\times\Omega)} + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3} &= 1 + \frac{1}{T} + T \|V\|_{L^\infty((0,T)\times\Omega)} + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3} \\ &\leqslant 1 + \frac{1}{T} + 2 \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3} \\ &\leqslant 2\left(1 + \frac{1}{T} + \|V\|_{L^\infty((0,T)\times\Omega)}^{2/3}\right). \end{aligned}$$

Therefore, choosing $T_0 \in (0, T]$ appropriately in (A.3), we can always get the observability inequality (A.2), for a constant C independent of T and V. □

References

[1] F. Alabau-Boussouira, P. Cannarsa, and G. Fragnelli. Carleman estimates for degenerate parabolic operators with applications to null controllability. *J. Evol. Equ.*, 6(2):161–204, 2006.

[2] K. Beauchard and P. Cannarsa. Heat equation on the Heisenberg group: Observability and applications. *J. Diff. Eq.*, 262(8):4475–4521, April 2017.

[3] K. Beauchard, P. Cannarsa, and R. Guglielmi. Null controllability of Grushin-type operators in dimension two. *J. Eur. Math. Soc. (JEMS)*, 16(1):67–101, 2014.

[4] K. Beauchard, P. Cannarsa, and M. Yamamoto. Inverse source problem and null controllability for multidimensional parabolic operators of Grushin type. *Inverse Problems*, 30(2):025006, 26, 2014.

[5] K. Beauchard, B. Helffer, L. Henry, and L. Robbiano. Degenerate parabolic operators of Kolmogorov type with a geometric control condition. *ESAIM:COCV*, 21(2):487–512, 2015.

[6] K. Beauchard, L. Miller, and M. Morancey. 2D Grushin-type equations: minimal time and null controllable data. *J. Differential Equations*, 259(11):5813–5845, 2015.

[7] K. Beauchard and K. Pravda-Starov. Null-controllability of non-autonomous Ornstein-Uhlenbeck equations. *J. Math. Anal. Appl.*, 456(1):496–524, 2017.

[8] K. Beauchard and K. Pravda-Starov. Null-controllability of hypoelliptic quadratic differential equations. *J. Éc. polytech. Math.*, 5:1–43, 2018.

[9] P. Cannarsa, G. Fragnelli, and D. Rocchetti. Null controllability of degenerate parabolic operators with drift. *Netw. Heterog. Media*, 2(4):695–715 (electronic), 2007.

[10] P. Cannarsa, G. Fragnelli, and D. Rocchetti. Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form. *J. Evol. Equ.*, 8:583–616, 2008.

[11] P. Cannarsa and R. Guglielmi. Null controllability in large time for the parabolic Grushin operator with singular potential. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 87–102. Springer, Cham, 2014.

[12] P. Cannarsa, P. Martinez, and J. Vancostenoble. Null controllability of degenerate heat equations. *Adv. Differential Equations*, 10(2):153–190, 2005.

[13] P. Cannarsa, P. Martinez, and J. Vancostenoble. Carleman estimates for a class of degenerate parabolic operators. *SIAM J. Control Optim.*, 47(1):1–19, 2008.

[14] P. Cannarsa, P. Martinez, and J. Vancostenoble. Carleman estimates and null controllability for boundary-degenerate parabolic operators. *C. R. Math. Acad. Sci. Paris*, 347(3-4):147–152, 2009.

[15] P. Cannarsa, P. Martinez, and J. Vancostenoble. Global Carleman estimates for degenerate parabolic operators with applications. *Mem. Amer. Math. Soc.*, 239(1133):ix+209, 2016.

[16] J. Dardé and S. Ervedoza. On the reachable set for the one-dimensional heat equation. https://arxiv.org/abs/1609.02692, 2016.

[17] J. Dardé and S. Ervedoza. On the cost of observability in small times for the one-dimensional heat equation. https://hal.archives-ouvertes.fr/hal-01619211/, 2017.

[18] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.

[19] H.O. Fattorini and D. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43:272–292, 1971.

[20] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations: the linear case. *Adv. Differential Equations*, 5(4-6):465–514, 2000.

[21] A.V. Fursikov and O.Y. Imanuvilov. Controllability of evolution equations. *Lecture Notes Series, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul*, 34, 1996.

[22] M. Gueye. Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations. *SIAM J. Control Optim.*, 52(4):2037–2054, 2014.

[23] B. Helffer. *Semi-classical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[24] O. Y. Imanuvilov and M. Yamamoto. Lipschitz stability in inverse parabolic problems by the Carleman estimate. *Inverse Problems*, 14(5):1229–1245, 1998.

[25] A. Koenig. Non null controllability of the Grushin equation in 2d. https://arxiv.org/abs/1701.06467, 2017.

[26] J. Le Rousseau and I. Moyano. Null-controllability of the kolmogorov equation in the whole phase space. *J. Diff. Eq.*, 260:3193–3233, 2016.

[27] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. P.D.E.*, 20:335–356, 1995.

[28] P. Martinez and J. Vancostenoble. Carleman estimates for one-dimensional degenerate heat equations. *J. Evol. Equ.*, 6(2):325–362, 2006.

[29] L. Miller. The control transmutation method and the cost of fast controls. *SIAM J. Control Optim.*, 45(2):762–772 (electronic), 2006.

[30] M. Morancey. Approximate controllability for a 2d Grushin equation with potential having an internal singularity. *Ann. Inst. Fourier*, 65(4), 2015.

[31] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[32] M. Yamamoto. Carleman estimates for parabolic equations and applications. *Inverse Problems*, 25(12):123013, 2009.