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Semi-parametric detection of multiple changes in long-range dependent processes

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Abstract

This paper is devoted to the offline multiple changes detection for long memory processes. The observations are supposed to satisfy a semi-parametric long memory assumption with distinct memory parameters on each stage. A penalized local Whittle contrast is considered for estimating all the parameters. The consistency as well as convergence rates are obtained. Monte-Carlo experiments exhibit the accuracy of the estimators. They also show that the estimation of the number of breaks is improved by using a data-driven slope heuristic procedure of choice of the penalization parameter.

Keywords: 62M10, 62M15, 62F12.

1. Introduction

There exists now a very large literature devoted to long-range dependent processes. The most commonly used definition of long-range dependency requires a second order stationary process $X = (X_n)_{n \in \mathbb{Z}}$ with spectral density f such as:

$$f(\lambda) = |\lambda|^{-2d} L(|\lambda|) \quad \text{for any } \lambda \in [-\pi, \pi], \quad (1.1)$$

where L is a positive slow varying function, satisfying for any $c > 0$, $\lim_{h \rightarrow 0} \frac{L(c|h|)}{L(c|h|)} = 1$, typically L is a function with a positive limit or a logarithm.

From an observed trajectory (X_1, \dots, X_n) of a long-range dependent process, the estimation of the parameter d is an interesting statistical question. The case of a parametric estimator for which the explicit expression of the spectral density f is known, was successively solved in many cases using maximum likelihood estimators (see for instance Dahlhaus, 1989) or Whittle estimators (see for instance Fox and Taqqu, 1987, Giraitis and Surgailis, 1990, or Giraitis and Taqqu, 1999).

However, with numerical applications in view, knowing the explicit form of the spectral density is not a realistic framework. A semi-parametric estimation of d where only the behaviour (1.1) is assumed should be preferred. Thus, numerous semi-parametric estimators of d were defined and studied, the main ones being the log-periodogram (see Geweke and Porter Hudack, 1987, or Robinson, 1995a), the wavelet based (see Bardet *et al.*, 2000) and the local Whittle estimators (see Robinson 1995b).

This last one is a version of the Whittle estimator for which only asymptotically small frequencies are considered. It provides certainly the best trade-off between computation time and accuracy of the estimation (see for instance Bardet *et al.*, 2003b). Its asymptotic normality was extended for numerous kinds of long-memory processes (see Dalla *et al.*, 2005) and also non-stationary processes (see Abadir *et al.*, 2006). However there is still no yet satisfactory adaptive method of choice of the bandwidth parameter even if several interesting attempts have been developed (see for instance Henry and Robinson, 1998, or Henry, 2007). Hence, the usual choice valid for FARIMA or Fractional Gaussian noise is commonly chosen.

In this paper we consider the classical framework of only multiple change detection. It consists on an observed trajectory (X_1, \dots, X_n) of a process X that is the concatenation of $K^* + 1$ independent trajectories of different stationary long memory processes satisfying (1.1) with distinct long-memory parameters d . The aim of this paper is to present a method for estimating the number K^* of abrupt changes, the K^* change-times $(t_1^*, \dots, t_{K^*}^*)$ and the $K^* + 1$ different long-memory parameters $(d_1^*, \dots, d_{K^*+1}^*)$, which are unknown.

The framework of “offline” multiple changes we chose, has to be distinguished from that of the “online” one, for which a monitoring procedure is adopted and test of detection of change is successively applied (such as CUMSUM procedure). The book of Basseville and Nikiforov (1993) is a good reference for an introduction on both online and offline methods. There exists several methods for building a sequential detector of long-range memory, see for instance Giraitis *et al.* (2001), Kokoszka and Leipus (2003) or Lavancier *et al.* (2013).

For our offline framework, following the previous purposes, we chose to build a penalized contrast based on a sum successive local Whittle contrasts and to minimize it. The principle of this method, minimizing a penalized contrast, provides very convincing results in many frameworks: in case of mean changes with least squares contrast (see Bai, 1998), in case of linear models changes with least squares contrast (see Bai and Perron, 1998, generalized by Lavielle, 1999, and Lavielle and Moulines, 2000) or least absolute deviations (see Bai, 1998), in case of spectral densities changes with usual Whittle contrasts (see Lavielle and Ludena, 2000), in case of time series changes with quasi-maximum likelihood (see Bardet *et al.*, 2012),... Clearly, the remarkable paper of Lavielle and Ludena (2000) was the model of this article except that we used a semi-parametric version of their Whittle contrast with the “local” Whittle contrast, and this engenders additional difficulties...

Restricting our paper to long-memory linear processes, we obtained several asymptotic results. First the consistency of the estimator has been established under assumptions on the second order term of the expansion of the spectral density close to 0. A convergence rate of the change times estimators is also provided, but we are not able to reach the usual $\mathcal{O}_P(1)$ converge rate, which is obtained for instance in the parametric case (see Lavielle and Ludena, 2000).

Monte-Carlo experiments illustrate the consistency of the estimators. When the number of changes is known, the theoretical results concerning the consistencies of the estimator is satisfying and $n = 5000$

provides very convincing results while they are still mediocre for $n = 2000$ and bad for $n = 500$. This is not surprising since we considered a semi-parametric statistic framework. When the number of changes is unknown, although we chose an asymptotically consistent choice of penalization sequence, the consistency is not satisfying even for large sample such as $n = 5000$. The accuracy of the number of changes estimator is extremely dependent to the precise choice of the penalization sequence, even if this choice should not be important asymptotically. Then we chose to use a data-driven procedure for computing “optimal” penalty, the so-called “Slope Heuristic” procedure defined in Arlot and Massart (2009). It provides more accurate results than with a fixed penalization sequence and very convincing results when $n = 5000$.

The following Section 2 is devoted to define the framework and the estimator. Its asymptotic properties are studied in Section 3. The concrete estimation procedure and numerical applications are presented in Section 4. Finally, Section 5 contains the main proofs.

2. Definitions and assumptions

2.1. The multiple changes framework

We consider in the sequel the case of log range dependent linear processes defined in Dalla *et al.* (2005). We will say that a process $Y = (Y_k)_{s+1 \leq k \leq t}$ with $(s, t) \in \mathbb{N}^2$, $s < t$ belongs to Class $I(d, \beta)$ with $0 < d < 1/2$ and $\beta \in (0, 2]$ defined by:

Class $I(d, \beta)$: *The process $Y = (Y_k)_{s+1 \leq k \leq t}$ is a second order stationary process satisfying:*

1. *There exist $(a_k)_{k \in \mathbb{N}}$ a sequence of real number and $(\varepsilon_t)_{t \in \mathbb{Z}}$ a sequence of independent and identically distributed random variables (iidrv) with zero mean and unit variance such as*

$$Y_k = \sum_{j=0}^{\infty} a_j \varepsilon_{k-j} \quad \text{for } s+1 \leq k \leq t$$

with $\sum_{j=0}^{\infty} a_j^2 < \infty$ and $\frac{\partial}{\partial \lambda} \alpha(\lambda) = O(|\lambda^{-1} \alpha(\lambda)|)$ when $\lambda \rightarrow 0^+$ with $\alpha(\lambda) = \sum_{j=0}^{\infty} a_j e^{ij\lambda}$.

2. *There exist $c_0 > 0$ and $c_1 \in \mathbb{R}$, such as the spectral density f of Y satisfies*

$$f(\lambda) = |\lambda|^{-2d} (c_0 + c_1 |\lambda|^\beta + o(|\lambda|^\beta)) \quad \text{for all } \lambda \in (-\pi, 0) \cup (0, \pi). \quad (2.1)$$

Note that since $d \in (0, 1/2)$ then Y is a long range dependant process. Using these assumptions, we are going to give the following assumption satisfied by the process X we study the changes:

Assumption A: *The process $X = (X_k)_{1 \leq k \leq n}$ is such that there exist $K^* \in \mathbb{N}$, $\tau_0^* = 0 < \tau_1^* < \dots < \tau_{K^*}^* < 1 = \tau_{K^*+1}^*$ and*

1. *For $i = 0, \dots, K^*$, $X_k = X_k^{(i)}$ for $k \in \{[n\tau_i^*]+1, \dots, [n\tau_{i+1}^*]\}$, where $(X^{(i)})_i$ are independent processes that belong to Class $I(d_i^*, \beta_i^*)$ where $d_i^* \in (0, 1/2)$ and $\beta_i^* \in (0, 2]$;*

2. For $i = 0, \dots, K^*$, $d_{i+1}^* - d_i^* \neq 0$.

The second condition is the key condition insuring that the framework is the one of online multiple long memory change. Moreover we will denote:

$$\Delta_d = \max_{1 \leq i \leq K^*} |d_{i+1}^* - d_i^*| > 0. \quad (2.2)$$

2.2. Definition of the estimator

Now define the following notation: for $K \in \{0, 1, \dots, n-1\}$ and X satisfying Assumption A,

- for $t_0 = 1 < t_1 < t_2 < \dots < t_K < t_{K+1} = n$, denote $T_i = \{t_{i-1} + 1, t_{i-1} + 2, \dots, t_i\}$ and $n_i = t_i - t_{i-1}$ for $i = 1, \dots, K + 1$.
- let $t_i^* = [n\tau_i^*]$, $T_i^* = \{t_{i-1}^* + 1, t_{i-1}^* + 2, \dots, t_i^*\}$ and $n_i^* = t_i^* - t_{i-1}^*$ for $i = 1, \dots, K^* + 1$.
- $T_{ij} = \{t_{i-1} + 1, t_{i-1} + 2, \dots, t_i\} \cap \{t_{j-1}^* + 1, t_{j-1}^* + 2, \dots, t_j^*\}$ and $n_{ij} = \#\{T_{ij}\}$ for $i = 1, \dots, K + 1$ and $j = 1, \dots, K^* + 1$.

For $X = (X_t)_{t \in \{1, \dots, n\}}$ denote by I_T the periodogram of X on the set T where $T \subset \{1, \dots, n\}$ and $|T| = \#\{T\}$:

$$I_T(\lambda) = \frac{1}{2\pi |T|} \left| \sum_{k \in T} X_k e^{-ik\lambda} \right|^2. \quad (2.3)$$

Using the seminal papers of Kunsh (1987), Robinson (1995b) and Robinson and Henry (2003), we define a local Whittle estimator of d . For this, define for $T \subset \{1, \dots, n\}$, $d \in \mathbb{R}$ and $m \in \{1, \dots, n\}$,

$$W_n(T, d, m) = \log(S_n(T, d, m)) - \frac{2d}{m} \sum_{k=1}^m \log(k/m) \quad (2.4)$$

$$\text{with } \lambda_k^{(n)} = 2\pi \frac{k}{n} \quad \text{and} \quad S_n(T, d, m) = \frac{1}{m} \sum_{j=1}^m \left(\frac{j}{m}\right)^{2d} I_T(\lambda_j^{(n)}). \quad (2.5)$$

The local Whittle objective function $d \rightarrow W_n(T, d, m)$ can be minimized for estimating d on the set T providing the local Whittle estimator $\hat{d} = \arg \min_{d \in [0, 0.5]} W_n(T, d, m)$ on T .

Now, for $K \in \{0, \dots, n\}$, we will denote:

- $\mathbf{d} = (d_1, \dots, d_{K+1})$ and $\mathbf{d}^* = (d_1^*, \dots, d_{K^*+1}^*)$,
- $\mathbf{t} = (t_1, \dots, t_K)$ and $\mathbf{t}^* = (t_1^*, \dots, t_{K^*}^*)$,

For $m \in \{1, \dots, n\}$, define the local Whittle contrast of X by

$$L_n(K, \mathbf{t}, \mathbf{d}, m) = \frac{1}{n} \sum_{k=1}^{K+1} n_k W_n(T_k, d_k, m). \quad (2.6)$$

This contrast is therefore a sum of local Whittle objective functions on the $K + 1$ different stages T_k , $k = 1, \dots, K + 1$. Then for any K , we can define

$$(\widehat{\mathbf{t}}, \widehat{\mathbf{d}}) = \arg \min_{\mathbf{d} \in [0, 0.5]^{K+1}, \mathbf{t} \in \mathcal{T}_K(0)} L_n(K, \mathbf{t}, \mathbf{d}, m), \quad (2.7)$$

with $\widehat{\mathbf{d}} = (\widehat{d}_1, \dots, \widehat{d}_{K+1})$ and $\widehat{\mathbf{t}} = (\widehat{t}_1, \dots, \widehat{t}_K)$ and where for $a \geq 0$,

$$\mathcal{T}_K(a) = \left\{ (t_1, \dots, t_K) \in \{2, \dots, n-1\}^K, t_{i+1} > t_i \text{ and } |t_i - t_i^*| \geq a \text{ for all } i = 1, \dots, K \right\}. \quad (2.8)$$

In the sequel we will also use the local Whittle estimators $\widehat{\mathbf{d}}^* = (\widehat{d}_i^*)_{1 \leq i \leq K^*+1}$ obtained when the change dates are known, which are defined by:

$$\widehat{\mathbf{d}}^* = \arg \min_{\mathbf{d} \in [0, 0.5]^{K^*+1}} L_n(K^*, \mathbf{t}^*, \mathbf{d}, m). \quad (2.9)$$

3. Asymptotic behaviors of the estimators

3.1. Case of a known number of changes

We study first the case of a known number K^* of changes.

Theorem 3.1. *For X satisfying Assumption A, and $(\widehat{\tau}, \widehat{\mathbf{d}})$ defined by*

$$(\widehat{\mathbf{t}}, \widehat{\mathbf{d}}) = \arg \min_{\mathbf{d} \in [0, 0.5]^{K^*+1}, \mathbf{t} \in \mathcal{T}_{K^*}(0)} L_n(K^*, \mathbf{t}, \mathbf{d}, m).$$

Then if $m = o(n)$ and with $\widehat{\tau} = (\widehat{\tau}_1, \dots, \widehat{\tau}_K)$ and $\widehat{\tau}^* = (\widehat{\tau}_1^*, \dots, \widehat{\tau}_{K^*}^*)$ where $\widehat{\tau}_i = \frac{\widehat{t}_i}{n}$ for $i = 1, \dots, K^*$,

$$(\widehat{\tau}, \widehat{\mathbf{d}}) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} (\tau^*, \mathbf{d}^*).$$

This first theorem can be improved for specifying the rate of convergence of the estimators:

Theorem 3.2. *For X satisfying Assumption A, if m is such as $m = o(n^{2\underline{\beta}^*/(1+2\underline{\beta}^*)})$ where $\underline{\beta}^* = \min_{1 \leq i \leq K^*+1} \beta_i^*$ and $(\widehat{\mathbf{t}}, \widehat{\mathbf{d}})$ defined as in Theorem 3.1, then*

$$\lim_{\delta \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{\sqrt{m}}{n \log n} |\widehat{\mathbf{t}} - \mathbf{t}^*| \geq \delta \right) = 0. \quad (3.1)$$

This result provides a bound of the “best” convergence rate of $\widehat{\mathbf{t}}$ which is minimized by $\log(n) n^{(1+\underline{\beta}^*)/(1+2\underline{\beta}^*)}$, *i.e.* the “best” convergence rate for $\widehat{\tau}$ is minimized by $\log(n) n^{-\underline{\beta}^*/(1+2\underline{\beta}^*)}$.

Remark 1. This rate of convergence could be compared to the result obtained in the parametric framework of Lavielle and Ludena (2000) where the respective convergence rates of $\widehat{\mathbf{t}}$ and $\widehat{\tau}$ are 1 and n^{-1} . This is the price to pay for going from the parametric to the semi-parametric framework. But also the price to pay to the definition of local Whittle estimator which does not allow some simplifications as in the proof of Theorem 3.4, p. 860. Indeed the random term of their classical used definition of Whittle contrast is $\int_{\pi}^{\pi} I_T(\lambda)/f(\lambda)d\lambda$ while our random term is $\log\left(\frac{1}{m} \sum_{k=1}^m (j/m)^{2d} I_T(\lambda_j^{(n)})\right)$: the logarithm shots the door for simplifications.

Another consequence of this result is that there is asymptotically a small lose on the convergence rates of the long memory parameter local Whittle estimators \widehat{d}_i when the change dates are estimated instead of being known. More formally, using the results of Robinson (1995b) improved by Dalla *et al.* (2005), we know that under conditions of Theorem 3.2, then \widehat{d}_i^* satisfies

$$\sqrt{m}(\widehat{d}_i^* - d_i^*) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{4}\right). \quad (3.2)$$

Unfortunately, the rate of convergence obtained by \widehat{t}_i in Theorem 3.2 does not allow to keep this limit theorem when \widehat{d}_i^* is replaced by \widehat{d}_i . We rather obtain:

Theorem 3.3. Under the assumptions of Theorem 3.2, then for any $i = 1, \dots, K^* + 1$,

$$\frac{\sqrt{m}}{\log n} |\widehat{d}_i - d_i^*| \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (3.3)$$

3.2. Case of an unknown number of changes

Here we consider the case where K^* is unknown. For estimating K^* , we consider now a penalized local Whittle contrast defined by:

$$J_n(K, \mathbf{t}, \mathbf{d}, m) = L_n(K, \mathbf{t}, \mathbf{d}, m) + K z_n = \frac{1}{n} \sum_{k=1}^{K+1} n_k W_n(T_k, d_k, m) + K z_n, \quad (3.4)$$

where (z_n) is a sequence of positive real numbers that will be specify in the sequel.

With $K_{\max} \in \mathbb{N}^*$ a chosen integer number, we define:

$$(\widehat{K}_n, \widehat{\mathbf{t}}, \widehat{\mathbf{d}}) = \arg \min_{K \in \{0, \dots, K_{\max}\}, \mathbf{d} \in [0, 0.5]^{K+1}, \mathbf{t} \in \mathcal{T}_K(0)} J_n(K, \mathbf{t}, \mathbf{d}, m), \quad (3.5)$$

Then we obtain:

Theorem 3.4. Under the assumptions and notations of Theorem 3.1, if $(\widehat{K}, \widehat{\mathbf{t}}, \widehat{\mathbf{d}})$ is defined in (2.7), with m is such as $m = o(n^{2\underline{\beta}^*/(1+2\underline{\beta}^*)})$ where $\underline{\beta}^* = \min_{1 \leq i \leq K^*+1} \beta_i^* z_n$ and $\max\left(z_n, \frac{\log n}{z_n \sqrt{m}}\right) \xrightarrow[n \rightarrow \infty]{} 0$, then

$$(\widehat{K}, \widehat{\tau}, \widehat{\mathbf{d}}) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} (K^*, \tau^*, \mathbf{d}^*),$$

Note that the conditions we obtained on m and z_n imply that $n^{-\underline{\beta}^*/(1+2\underline{\beta}^*)} \log n = o(z_n)$, depending on $\underline{\beta}$ that is generally unknown. However, the choice $z_n = n^{-1/2}$ is a possible choice solving this problem. The provided proof does not allow to establish the consistency of a typical BIC criterion, which should be $z_n = \log n/n$.

Corollary 1. *Under the conditions of Theorem 3.4, the bounds (3.1) and (3.3) hold.*

4. Numerical experiments

In the sequel we first describe the concrete procedure for applying the multiple changes estimator, then we present the numerical results of Monte-Carlo experiments.

4.1. Concrete procedure of estimation

Several details require to be specified to concretely apply the multiple changes estimator. Indeed, we have done:

1. The choice of meta-parameters: 1/ as we mainly studied the cases of FARIMA processes for which $\beta = 2$, we chose $m = n^{0.65}$; 2/ the number $K_{\max} \geq K^*$ is crucial for the heuristic plot procedure (see below) and was chosen such as $K_{\max} = 2([\log(n)] - 1)$, implying $K_{\max} = 10, 12$ and 14 respectively for $n = 500, 2000$ and 5000 .
2. As the choice of the sequence (z_n) of the penalty term is not exactly specified but just has to satisfy $\max\left(z_n, \frac{\log n}{z_n \sqrt{m}}\right) \xrightarrow[n \rightarrow \infty]{} 0$, after many tries, we chose $z_n = 2\sqrt{n}$.
3. A dynamic programming procedure is implemented for notably decreasing the time consuming. Such procedure is very common in the offline multiple change context and has been described with details in Kay (1998).
4. For improving the procedure of selection of the changes number K^* for not too large samples, we implemented a data-driven procedure so-called “the heuristic slop procedure”. This procedure was introduced by Arlot and Massart (2009) in the framework of least squares estimation with fixed design, but can be extended in many statistical fields (see Baudry *et al.*, 2010). Applications in the multiple changes detection problem was already successfully done in Baudry *et al.* (2010) in an i.i.d. context and also for dependent time series in Bardet *et al.* (2012). In a general framework, it consists on computing $-2 \log(\widehat{LIK}(K))$ where $\widehat{LIK}(K)$ is the maximized likelihood for any $K \in \{0, 1, \dots, K_{\max}\}$. Here $-2 \log(\widehat{LIK}(K))$ is replaced by $2 \times L_n(K, \widehat{\mathbf{t}}, \widehat{\mathbf{d}})$. Then for $K > K^*$, the decreasing of this contrast with respect to K is almost linear with a slope s (see Figure 1 where the linearity can be observed when $K > K^* = 4$), which can be estimated for instance by a least-squares estimator \widehat{s} . Then \widehat{K}_H is obtained by minimizing the penalized contrast J_n using $\widehat{z}_n = 2 \times \widehat{s}$, *i.e.*

$$\widehat{K}_H = \arg \min_{0 \leq K \leq K_{\max}} \left\{ L_n(K, \widehat{\mathbf{t}}, \widehat{\mathbf{d}}, m) + 2 \times \widehat{s} \times K \right\}.$$

By construction, the procedure is sensitive to the choice of K_{\max} since a least squares regression is realized for the “largest” values of K and we preferred to chose the largest reasonable value of K_{\max} .

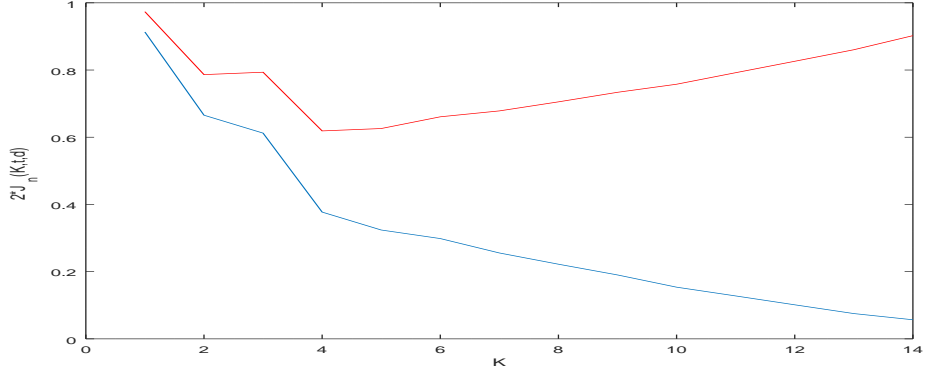


Figure 1: For $n = 5000$, $K^* = 4$ and a FARIMA(0, d , 0) process, the graph of $2 \times J_n(K, \hat{\mathbf{t}}, \hat{\mathbf{d}})$ (in blue), and the one of $2 \times J_n(K, \hat{\mathbf{t}}, \hat{\mathbf{d}}) + 2 \times \hat{s} \times K$

A software was written with Octave software (also executable with Matlab software) and is available on <http://samm.univ-paris1.fr/IMG/zip/detectchange.zip>.

4.2. Monte-Carlo experiments in case of known number of changes

In the sequel we first exhibit the consistency of the multiple breaks estimator when the number of changes is known. Monte-Carlo experiments are realized in the following framework:

1. Three kinds of processes are considered: a FARIMA(0, d , 0) process, a FARIMA(1, d , 1) process with a AR coefficient $\psi = -0.3$ and a MA coefficient $\theta = 0.7$ and a Gaussian stationary process called $X^{(d,1)}$ with the spectral density

$$f(\lambda) = \frac{1}{|\lambda|^{2d}}(1 + |\lambda|^1) \quad \text{for } \lambda \in [-\pi, 0) \cup (0, \pi].$$

Those processes are generated following fast algorithms described in Bardet *et al.* (2003). Note that both the FARIMA processes belongs to Class $I(d, 2)$ while $X^{(d,1)}$ belongs to Class $I(d, 1)$.

Table 1: RMSE of the estimators from 500 independent replications of processes.

	n	FARIMA(0, d , 0)			FARIMA(1, d , 1)			$X^{(d,1)}$		
		500	2000	5000	500	2000	5000	500	2000	5000
$K^* = 1$	$\widehat{\tau}_1$	0.184	0.056	0.013	0.196	0.027	0.015	0.211	0.046	0.012
	\widehat{d}_1	0.094	0.061	0.046	0.100	0.054	0.051	0.201	0.086	0.056
	\widehat{d}_2	0.144	0.060	0.048	0.148	0.064	0.053	0.194	0.093	0.062
$K^* = 3$	$\widehat{\tau}_1$	0.292	0.158	0.014	0.254	0.153	0.013	0.242	0.155	0.012
	$\widehat{\tau}_2$	0.238	0.132	0.013	0.227	0.134	0.011	0.219	0.134	0.013
	$\widehat{\tau}_3$	0.210	0.123	0.016	0.218	0.086	0.013	0.222	0.085	0.014
	\widehat{d}_1	0.120	0.069	0.057	0.254	0.075	0.053	0.233	0.121	0.086
	\widehat{d}_2	0.328	0.179	0.061	0.290	0.119	0.064	0.346	0.171	0.082
	\widehat{d}_3	0.449	0.224	0.048	0.295	0.164	0.052	0.566	0.274	0.080
	\widehat{d}_4	0.234	0.111	0.076	0.264	0.114	0.081	0.255	0.128	0.074

2. For $n = 500, 2000$ and 5000 , two cases are considered:

- One change, $K^* = 1$ and $(d_1^*, d_2^*) = (0.4, 0.1)$ and $\tau_1^* = 0.5$;
- Three changes, $K^* = 3$ and $(d_1^*, d_2^*, d_3^*, d_4^*) = (0.4, 0.1, 0.4, 0.1)$ and $(\tau_1^*, \tau_2^*, \tau_3^*) = (0.25, 0.5, 0.75)$.

3. Each case is independently replicated 500 times, and the RMSE, Root-Mean-Square Error, is computed for each estimator of the parameter.

The results of Monte-Carlo experiments are detailed in Table 1.

4.3. Monte-Carlo experiments in case of unknown number of changes

In this subsection, we consider the result of the model selection using the penalized contrast for estimating the number of changes K^* . We reply exactly the same framework that in the previous subsection and notify the frequencies of the event ' $\widehat{K} = K^*$ ', for:

- $\widehat{K} = \widehat{K}_n$ obtained directly by minimizing J_n with $z_n = 2/\sqrt{n}$;
- $\widehat{K} = \widehat{K}_{BIC}$ obtained directly by minimizing J_n with $z_n = \log n/n$, following the usual BIC procedure;
- $\widehat{K} = \widehat{K}_H$ obtained from the ‘‘Heuristic Slope’’ procedure described previously.

We obtained the results detailed in Table 2:

Table 2: Frequencies of recognition of the true number of changes with several criteria from 500 independent replications of processes.

		FARIMA(0, d , 0)			FARIMA(1, d , 1)			$X^{(d,1)}$		
n		500	2000	5000	500	2000	5000	500	2000	5000
$K^* = 1$	\widehat{K}_n	0.10	0.31	0.54	0.20	0.30	0.57	0.19	0.38	0.63
	\widehat{K}_{BIC}	0	0	0	0	0	0	0	0	0
	\widehat{K}_H	0.49	0.82	0.85	0.49	0.86	0.88	0.32	0.84	0.86
$K^* = 3$	\widehat{K}_n	0.13	0.14	0.40	0.18	0.19	0.43	0.09	0.21	0.55
	\widehat{K}_{BIC}	0	0	0	0	0	0	0	0	0
	\widehat{K}_H	0.02	0.16	0.81	0.03	0.31	0.80	0	0.01	0.90

4.4. Conclusions of Monte-Carlo experiments

From Tables 1 and 2, we may conclude that:

1. If the number of changes is known, the estimators of τ_i and d_i are consistent but their rates of convergence are slightly impacted by the number of changes: as we could imagine, the largest K^* the largest the RMSE of the estimators. However, and it could be counterintuitive, if the kind of process and therefore the roughness of the spectral density, impacts the accuracy of the estimation of d , the estimation of the change instants τ seems not to be sensible to it as we could expect with a semi-parametric method (it is quite surprising to observe that generally a FARIMA(1, d , 1) gives better results than a FARIMA(0, d , 0)!). But finally, the case $n = 5000$ provides extremely convincing results in any cases.
2. The estimators of number of changes \widehat{K}_n and \widehat{K}_H have a satisfying behavior, meaning that they seem to converge to K^* when the sample length increases. Once again, the consistencies are slightly better for small K^* than for large K^* . The results obtained with the ‘‘Slope Heuristic’’ procedure estimator \widehat{K}_H are almost the most accurate and provides very convincing results for $n = 5000$. Note also that the usual BIC penalty is not at all consistent, which can be explained by the use of local Whittle contrast that is not an approximation of the Gaussian likelihood as the usual Whittle contrast is.

Finally we could underline that our detector based on a local Whittle contrast added to a ‘‘slope heuristic’’ data-driven penalization provides very convincing results when $n = 5000$ and not too bad when $n = 2000$ (the case $n = 500$ gives not significant estimation).

5. Proofs

We first provide the statements and the proofs of two useful lemmas:

Lemma 5.1. *Under the assumptions of Theorem 3.1, with $S_n(T, d, m)$ defined in (5.10), for any $i \in \{1, \dots, K^*\}$ and $T \subset T_i$,*

$$\max_{1 \leq |T| \leq n} \sup_{d \in [0, 1/2)} \left(\frac{n}{m} \right)^{2d^*} \left| S_n(T, d, m) \right| - \left(\max \left\{ \frac{n}{m}, |T| \right\} \right)^{2d^*} \frac{c_{0,i}^* (2\pi)^{-2d_i^*}}{1 + 2d - 2d_i^*} \Big|_{m, n/m \rightarrow \infty} \xrightarrow{\mathcal{P}} 0. \quad (5.1)$$

Proof. For T such as $|T| \geq n/m$, i.e. $m \geq n/|T|$ we have

$$\begin{aligned} S_n(T, d, m) &= \frac{1}{m} \sum_{j=1}^m \left(\frac{j}{m} \right)^{2d} I_T(\lambda_j^{(n)}) \\ &= \frac{1}{m} \sum_{j=1}^{n/|T|} \left(\frac{j}{m} \right)^{2d} I_T(\lambda_j^{(n)}) \\ &\quad + \left(\frac{n}{m} \right)^{2d_i^*} c_{0,i}^* (2\pi)^{-2d_i^*} \frac{|T|}{n} \sum_{k=0}^{n/|T|-1} \left(\frac{1}{m|T|/n} \sum_{j=1}^{m|T|/n} \left(\frac{j}{m|T|/n} \right)^{2d-2d_i^*} \eta_j^* \right) \\ &= K_1 + K_2 \end{aligned}$$

with $\eta_j^* = \frac{I_T(\lambda_j^{(n)})}{c_{0,i}^* (\lambda_j^{(n)})^{-2d_i^*}}$ (we follow here the same notation than in Dalla *et al.*, 2005, p.236).

In the one hand, we have $K_1 \geq 0$ and

$$\begin{aligned} \mathbb{E}(K_1) &= \frac{1}{m} \sum_{j=1}^{n/|T|} \left(\frac{j}{m} \right)^{2d} \mathbb{E}[I_T(\lambda_j^{(n)})] \\ &\leq \frac{C}{m} \sum_{j=1}^{n/|T|} \left(\frac{j}{m} \right)^{2d} \sum_{k=-|T|}^{|T|} k^{2d_i^*-1} \\ &\leq \frac{C}{2m d_i^*} \sum_{j=1}^{n/|T|} \left(\frac{j}{m} \right)^{2d} |T|^{2d_i^*} \\ &\leq C' |T|^{2d_i^*} \left(\frac{n}{|T|m} \right)^{1+2d} \end{aligned}$$

with $C > 0$ and $C' > 0$. Therefore, using Markov Inequality we can write

$$\sup_{|T| \geq N} K_1 = O_{\mathbb{P}} \left(|N|^{2d_i^*} \left(\frac{n}{|N|m} \right)^{1+2d} \right). \quad (5.2)$$

In the other hand, using again Dalla *et al.* (2005, Proposition 5, replacing respectively m and n by $\frac{m|T|}{n}$ and $|T|$),

$$\frac{1}{m|T|/n} \sum_{j=1}^{m|T|/n} \left(\frac{j}{m|T|/n} \right)^{2d-2d_i^*} \eta_j^* = \frac{1}{1 + 2d - 2d_i^*} + O_{\mathbb{P}} \left[\left(\frac{m}{n} \right)^{\beta_i^*} + \left(\frac{m|T|}{n} \right)^{-1/2} \log^{1/2} \left(\frac{m|T|}{n} \right) \right].$$

Therefore,

$$\sup_{|T| \geq N} \left| K_2 - \left(\frac{n}{m} \right)^{2d_i^*} \frac{c_{0,i}^* (2\pi)^{-2d_i^*}}{1 + 2d - 2d_i^*} \right| = \left(\frac{n}{m} \right)^{2d_i^*} O_P \left[\left(\frac{m}{n} \right)^{\beta_i^*} + \left(\frac{mN}{n} \right)^{-1/2} \log^{1/2} \left(\frac{mN}{n} \right) \right].$$

As a consequence, if $mN \geq n$,

$$\begin{aligned} \sup_{|T| \geq N} \left| S_n(T, d, m) - \left(\frac{n}{m} \right)^{2d_i^*} \frac{c_{0,i}^* (2\pi)^{-2d_i^*}}{1 + 2d - 2d_i^*} \right| \\ = \left(\frac{n}{m} \right)^{2d_i^*} O_P \left[\left(\frac{m}{n} \right)^{\beta_i^*} + \left(\frac{mN}{n} \right)^{-1/2} \log^{1/2} \left(\frac{mN}{n} \right) + \left(\frac{mN}{n} \right)^{-1+2d_i^*-2d} \right]. \end{aligned} \quad (5.3)$$

Moreover for any $T \subset T_i^*$, we have $E(I_T(\lambda_j^{(n)})) \leq \sigma_i^2 (1 + 2C \sum_{k=1}^{|T|} k^{2d_i^*-1}) \leq C |T|^{2d_i^*}$. As a consequence, there exists $C_i > 0$ such as for any $\delta > 0$,

$$P \left(\sup_{d \in [0, 1/2]} \left| S_n(T, d, m) - \frac{c_{0,i}^* (2\pi)^{-2d_i^*}}{1 + 2d - 2d_i^*} |T|^{2d_i^*} \right| \geq \delta \right) \leq \frac{C_i}{\delta} |T|^{2d_i^*}. \quad (5.4)$$

Thus we deduce (5.1) and this achieves the proof of Lemma 5.1. \square

Lemma 5.2. *Under the assumptions of Theorem 3.1, with $R_n(T, T', d, m)$ defined in (5.12), there exists $C > 0$ such that for any $j \in \{1, \dots, K^*\}$ and $N \in \mathbb{N}^*$,*

$$P \left(\sup_{d \in [0, 1/2]} \max_{n_k \geq N} \min \left\{ n_k, \frac{n}{m} \right\}^{-2 \max(d_j^*, d_{j'}^*)} \frac{1}{n_k} |R_m(T_{kj}, T_{kj'}, d)| \geq \delta \right) \leq \frac{C}{\delta^2} \min \left\{ N, \frac{n}{m} \right\}^{-2|d_{j'}^* - d_j^*| - 1}. \quad (5.5)$$

Proof. Moreover, for $j \neq j'$, and $t \in T_{kj}$, $s \in T_{kj'}$, X_s and X_t are independent from Assumption A. Consequently,

$$R_m(T_{kj}, T_{kj'}, d) = \frac{1}{m} \sum_{p=1}^m \left(\frac{p}{m} \right)^{2d} \frac{1}{2\pi} \left(\sum_{t \in T_{kj}} X_t e^{2\pi i p t / n_k} \right) \left(\sum_{s \in T_{kj'}} X_s e^{-2\pi i p s / n_k} \right)$$

$$\implies E(R_m(T_{kj}, T_{kj'}, d)) = 0$$

$$\begin{aligned} \text{and } \text{Var}(R_m(T_{kj}, T_{kj'}, d)) &= \frac{1}{4\pi^2 m^2} \sum_{p=1}^m \sum_{p'=1}^m \left(\frac{p p'}{m^2} \right)^{2d} \left(\sum_{t \in T_{kj}} \sum_{t' \in T_{kj}} r_{X^{(j)}}(t - t') e^{-2\pi i (p t - p' t') / n_k} \right) \\ &\quad \times \left(\sum_{s \in T_{kj'}} \sum_{s' \in T_{kj'}} r_{X^{(j')}}(s - s') e^{-2\pi i (p s - p' s') / n_k} \right) \end{aligned}$$

$$\implies \text{Var}(R_m(T_{kj}, T_{kj'}, d)) \leq C \sum_{t \in T_{kj}} \sum_{t' \in T_{kj}} \sum_{s \in T_{kj'}} \sum_{s' \in T_{kj'}} |r_{X^{(j)}}(t - t') r_{X^{(j')}}(s - s') b_n(t - s, d, m) b_n(t' - s', d, m)|$$

since $I_{T_{kj}}(\lambda)$ and $I_{T_{kj'}}(\lambda')$ are independent random variables (from Assumption A). Now we use the same arguments than in Lavielle and Ludena (2000). First, from Assumption (2.1) we have $|r_{X^{(j)}}(k)| \leq C |k|^{2d_j^*-1}$

and $|r_{X^{(j')}}(k)| \leq C |k|^{2d_{j'}^* - 1}$. Then,

$$\begin{aligned} \text{Var}(R_m(T_{kj}, T_{kj'}, d)) &\leq C \sum_{t \in T_{kj}} \sum_{t' \in T_{kj'}} |t - t'|^{2d_j^* - 1} \sum_{s \in T_{kj'}} \sum_{s' \in T_{kj'}} |s - s'|^{2d_{j'}^* - 1} b_n(t - s, d, m) b_n(t' - s', d, m) \\ &\leq C n_{kj}^{2d_j^*} \left(\sum_{u=0}^{n_k} |b_n(u, d, m)|^2 + 2 \sum_{u=0}^{n_k} |b_n(u, d, m)| \sum_{v=u+1}^{\infty} |b_n(v, d, m)| |u - v|^{2d_{j'}^* - 1} \right). \end{aligned}$$

But from usual calculations, for any $d \in [-1/2, 1/2)$, there exists $C(d) > 0$ such as we $|b_n(u, d, m)| \leq C(d) \min(1, (n/m)^{1+2d} |u|^{-1-2d})$ for $u \in \mathbb{Z}$. As a consequence,

$$\sum_{u=0}^{n_k} |b_n(u, d, m)|^2 \leq C \left(\sum_{u=0}^{\min(n_k, n/m)} 1 + \sum_{u=\min(n_k, n/m)}^{n_k} (n/m)^{1+2d} |u|^{-1-2d} \right) \leq C(d) \min(n_k, \frac{n}{m}). \quad (5.6)$$

Moreover,

$$\begin{aligned} &\sum_{u=0}^{n_k} |b_n(u, d, m)| \sum_{v=u+1}^{n_k} |b_n(v, d, m)| |u - v|^{2d_{j'}^* - 1} \\ &\leq C \left\{ \sum_{u=0}^{\min(n_k, n/m)} \left(\sum_{v=u+1}^{\min(n_k, n/m)} (v - u)^{2d_{j'}^* - 1} + \left(\frac{n}{m}\right)^{1+2d} \sum_{v=\min(n_k, n/m)}^{n_k} v^{-1-2d} (v - u)^{2d_{j'}^* - 1} \right) \right. \\ &\quad \left. + \left(\frac{n}{m}\right)^{2+4d} \sum_{u=\min(n_k, n/m)}^{n_k} \sum_{v=u+1}^{n_k} (uv)^{-1-2d} (v - u)^{2d_{j'}^* - 1} \right\} \\ &\leq C(d) \left(\min(n_k, \frac{n}{m}) \right)^{1+2d_{j'}^*} \end{aligned} \quad (5.7)$$

after classical computations. From (5.6) and (5.7), we obtain:

$$\text{Var}(R_m(T_{kj}, T_{kj'}, d)) \leq C(d) n_k^{2d_j^*} \left(\min(n_k, \frac{n}{m}) \right)^{1+2d_{j'}^*}. \quad (5.8)$$

Therefore, using Lemma 2.2 in Lavielle and Ludena (2000), we have for any $\delta > 0$, k, j and $j' \neq j$ and $1 \leq N \leq n$,

$$\mathbb{P} \left(\max_{n_k \geq N} \min \left\{ n_k, \frac{n}{m} \right\}^{-2d_j^*} \frac{1}{n_k} |R_m(T_{kj}, T_{kj'}, d)| \geq \delta \right) \leq \frac{C(d)}{\delta^2} \min \left\{ N, \frac{n}{m} \right\}^{2d_{j'}^* - 2d_j^* - 1}.$$

Finally with Lemma 2.4 of Lavielle and Ludena (2000), we obtain for any $\delta > 0$, k, j and $j' \neq j$ and $1 \leq N \leq n$,

$$\mathbb{P} \left(\sup_{d \in (-1/2, 1/2)} \max_{n_k \geq N} \min \left\{ n_k, \frac{n}{m} \right\}^{-2d_j^*} \frac{1}{n_k} |R_m(T_{kj}, T_{kj'}, d)| \geq \delta \right) \leq \frac{C}{\delta^2} \min \left\{ N, \frac{n}{m} \right\}^{2d_{j'}^* - 2d_j^* - 1},$$

with $C > 0$. Using the same method, d_j^* can also be symmetrically replaced by $d_{j'}^*$, and considering the more interesting inequality we obtain (5.5). \square

Now the proof of the consistency of $\hat{\tau}$ can be established:

Proof of Theorem 3.1. Mutatis mutandis, we follow here a similar proof than in Lavielle and Ludena (2000). Denote

$$U_n(\mathbf{t}, \mathbf{d}, m) = J_n(K^*, \mathbf{t}, \mathbf{d}, m) - J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m), \quad (5.9)$$

where J_n is defined in (3.4). Then, we can write that for any \mathbf{d} and \mathbf{t} ,

$$U_n(\mathbf{t}, \mathbf{d}, m) = \frac{1}{n} \left[\sum_{k=1}^{K^*+1} \left(n_k \log (S_n(T_k, d_k, m)) - n_k^* \log (S_n(T_k^*, d_k^*, m)) \right) \right] - \frac{\ell_m}{n} \sum_{k=1}^{K^*+1} 2 (n_k d_k - n_k^* d_k^*)$$

where

$$S_n(T, d, m) = \frac{1}{m} \sum_{j=1}^m \left(\frac{j}{m} \right)^{2d} I_T(\lambda_j^{(n)}) = \frac{1}{2\pi |T|} \sum_{t \in T} \sum_{s \in T} X_s X_t b_n(t - s, d, m) \quad (5.10)$$

$$\text{with } b_n(k, d, m) = \frac{1}{m} \sum_{j=1}^m \left(\frac{j}{m} \right)^{2d} e^{-2\pi i \frac{jk}{n}} \text{ and } \ell_m = \frac{1}{m} \sum_{j=1}^m \log(j/m). \quad (5.11)$$

Now using a decomposition of each S_n on the ‘true’ periods, we can write:

$$S_n(T_k, d_k, m) = \sum_{j=1}^{K^*} \frac{n_{kj}}{n_k} S_n(T_{kj}, d_k, m) + \frac{1}{n_k} \sum_{j=1}^{K^*} \sum_{j'=1, j' \neq j}^{K^*} R_n(T_{kj}, T_{kj'}, d_k, m),$$

$$\text{with } R_n(T_{kj}, T_{kj'}, d, m) = \frac{1}{2\pi} \sum_{t \in T_{kj}} \sum_{s \in T_{kj'}} X_s X_t b_n(t - s, d, m). \quad (5.12)$$

As a consequence,

$$U_n(\mathbf{d}, \mathbf{t}, m) = \frac{1}{n} \sum_{k=1}^{K^*+1} \left[n_k \log \left(\sum_{j=1}^{K^*+1} \frac{n_{kj}}{n_k} S_n(T_{kj}, d_k, m) + \frac{2}{n_k} \sum_{j=1}^{K^*+1} \sum_{j'=1, j' \neq j}^{K^*+1} R_n(T_{kj}, T_{kj'}, d_k, m) \right) \right. \\ \left. - n_k^* \log (S_n(T_k^*, d_k^*, m)) \right] + \frac{\ell(m)}{n} \sum_{k=1}^{K^*+1} 2 (n_k d_k - n_k^* d_k^*). \quad (5.13)$$

Thus, from Lemma 5.1, for any $T_k \subset \{1, \dots, n\}$,

$$\sup_{d_k \in [0, 1/2]} \max_{1 \leq n_k \leq n} \sum_{j=1}^{K^*} \frac{n_{kj}}{n_k} \left(\frac{n}{m} \right)^{-2d_j^*} \left| S_m(T_{kj}, d_k, m) - \left(\frac{n}{m} \right)^{2d_j^*} \frac{c_{0,j}^* (2\pi)^{-2d_j^*}}{1 + 2d - 2d_j^*} \right| \xrightarrow{m, n/m \rightarrow \infty} 0. \quad (5.14)$$

Moreover, using Lemma 5.2, we deduce that for any $j \in \{1, \dots, K^*\}$, there exists $C_j^* > 0$ such as for any $N \in \mathbb{N}^*$,

$$\mathbb{P} \left(\sup_{d \in [0, 1/2]} \max_{n_k \geq N} \left\{ \min \left(n_k, \frac{n}{m} \right)^{-2d_j^*} \frac{1}{n_k} \sum_{j'=1, j' \neq j}^{K^*} |R_m(T_{kj}, T_{kj'}, d, m)| \right\} \geq \delta \right) \leq \frac{C_j^*}{\delta^2} \min \left(N, \frac{n}{m} \right)^{2 \max_{\ell} (d_\ell^*) - 1}.$$

Therefore,

$$\sup_{d_k \in [0, 1/2]} \max_{n_k \geq N} \left| \sum_{j=1}^{K^*+1} n_{kj} S_m(T_{kj}, d_k, m) + \frac{2}{n_k} \sum_{j \neq j'}^{K^*+1} R_m(T_{kj}, T_{kj'}, d, m) - \sum_{j=1}^{K^*+1} n_{kj} \left(\frac{n}{m}\right)^{2d_j^*} \frac{c_{0,j}^* (2\pi)^{-2d_j^*}}{1 + 2d - 2d_j^*} \right| \xrightarrow[n, m, N, n/m \rightarrow \infty]{\mathcal{P}} 0.$$

But from the previous established limits, this consistency is also valid for any $N \in \mathbb{N}^*$ and then:

$$\sup_{d_k \in [0, 1/2]} \sup_{T_k \subset \{1, \dots, n\}} \left| \sum_{j=1}^{K^*+1} n_{kj} S_m(T_{kj}, d_k, m) + \frac{2}{n_k} \sum_{j \neq j'}^{K^*+1} R_m(T_{kj}, T_{kj'}, d, m) - \sum_{j=1}^{K^*+1} n_{kj} \left(\frac{n}{m}\right)^{2d_j^*} \frac{c_{0,j}^* (2\pi)^{-2d_j^*}}{1 + 2d - 2d_j^*} \right| \xrightarrow[n, m, n/m \rightarrow \infty]{\mathcal{P}} 0. \quad (5.15)$$

As a consequence, from (5.13), using the concavity of $x \mapsto \log(x)$ and the same tricks than in Lavielle and Ludena (2000), we could write that there exists a non negative random variable $D(m, n)$ such as $D(m, n) \xrightarrow[n, m, n/m \rightarrow \infty]{\mathcal{P}} 0$ satisfying for any \mathbf{t} and \mathbf{d} ,

$$\begin{aligned} U_n(\mathbf{t}, \mathbf{d}, m) &\geq \frac{1}{n} \left\{ \sum_{k=1}^{K^*+1} \left(\sum_{j=1}^{K^*+1} n_{kj} \log(S_m(T_{kj}, d_k, m)) \right) + D(m, n) - n_k^* \log(S_m(T_k^*, d_k^*, m)) \right. \\ &\quad \left. + 2\ell(m) (n_k d_k - n_k^* d_k^*) \right\} \\ &\geq \frac{1}{n} \sum_{k=1}^{K^*+1} \sum_{j=1}^{K^*+1} n_{kj} \left(\log(S_m(T_{kj}, d_k, m)) + D(m, n) + 2\ell(m) d_k \right) \\ &\quad - n_{jk} \left(\log(S_m(T_j^*, d_j^*, m)) + 2\ell(m) d_j^* \right) \\ &\geq \sum_{k=1}^{K^*+1} \sum_{j=1}^{K^*+1} \frac{n_{kj}}{n} (s(d_j^*, d_k) - s(d_j^*, d_j^*)) - |D(m, n)| \end{aligned}$$

from (5.15) and where for $d \in [0, 1/2)$,

$$s(d_j^*, d) = 2d_j^* \log(n/m) + \log(c_{0,j}^* (2\pi)^{-2d_j^*}) - \log(1 + 2d - 2d_j^*) - 2d. \quad (5.16)$$

Now, simple computations also imply

$$U_n(\mathbf{t}, \mathbf{d}, m) \geq \sum_{k=1}^{K^*+1} \sum_{j=1}^{K^*+1} \frac{n_{kj}}{n} (u(d_j^*, d_k) - u(d_j^*, d_j^*)) - |D(m, n)| \quad (5.17)$$

with $u(d_j^*, d) = -\log(1 + 2d - 2d_j^*) + 2d$. Remark that $u(d_j^*, d_k) - u(d_j^*, d_j^*) > 0$ for any $d_j^* \neq d_k$ and of course $u(d_j^*, d_j^*) - u(d_j^*, d_j^*) = 0$. Now we could use Lemma 2.3 of Lavielle (1999, p.88), adapted in Lemma

3.3 of Lavielle and Ludena (2000, p.858) and we hence obtain that there exists $C^* > 0$ depending only on \mathbf{d}^* such as

$$\sum_{k=1}^{K^*+1} \sum_{j=1}^{K^*+1} \frac{n_{kj}}{n} (u(d_j^*, d_k) - u(d_j^*, d_j^*)) \geq \frac{C}{n} \|\mathbf{t} - \mathbf{t}^*\|_\infty, \quad (5.18)$$

and $\|\mathbf{t} - \mathbf{t}^*\|_\infty = \max_{1 \leq k \leq K^*} \{|t_k - t_k^*|\}$.

Therefore, it is also possible to write that for any $\delta > 0$,

$$\begin{aligned} \mathbb{P}(\|\widehat{\tau} - \tau^*\|_\infty > \delta) &\leq \mathbb{P}\left(\inf_{\mathbf{d} \in [0, 1/2]^{K^*+1}} \min_{\mathbf{t} \in \mathcal{T}_{K^*}(n\delta)} U_n(\mathbf{t}, \mathbf{d}, m) < 0\right) \\ &\leq \mathbb{P}\left(\inf_{\mathbf{d} \in [0, 1/2]^{K^*+1}} \min_{\mathbf{t} \in \mathcal{T}_{K^*}(n\delta)} \sum_{k=1}^{K^*+1} \sum_{j=1}^{K^*+1} \frac{n_{kj}}{n} (u(d_j^*, d_k) - u(d_j^*, d_j^*)) - |D(m, n)| < 0\right) \\ &\leq \mathbb{P}\left(\delta - |D(m, n)| < 0\right) \xrightarrow{n, m, n/m \rightarrow \infty} 0, \end{aligned}$$

since for $\mathbf{t} \in \mathcal{T}_{K^*}(n\delta)$ we have $\|\mathbf{t} - \mathbf{t}^*\|_\infty \geq \delta n$ and for any $k \in \{1, \dots, K^*\}$. This achieves the proof. \square

Proof of Theorem 3.2. Assume with no loss of generality that $K^* = 2$. From Theorem 3.1, there exists $(u_n)_n$ a sequence of real numbers satisfying $u_n \xrightarrow{n \rightarrow \infty} \infty$, $u_n \sqrt{m}/(n \log n) \xrightarrow{n \rightarrow \infty} \infty$, $u_n/n \xrightarrow{n \rightarrow \infty} 0$ and $\mathbb{P}(|\widehat{t}_1 - t_1^*| > u_n) \xrightarrow{n \rightarrow \infty} 0$. For $\delta > 0$, as we have

$$\mathbb{P}\left(|\widehat{t}_1 - t_1^*| > \delta \frac{n \log n}{\sqrt{m}}\right) \leq \mathbb{P}\left(\delta \frac{n \log n}{\sqrt{m}} < |\widehat{t}_1 - t_1^*| \leq u_n\right) + \mathbb{P}(|\widehat{t}_1 - t_1^*| > u_n)$$

As a consequence, it is sufficient to show that $\mathbb{P}(\delta n \log n / \sqrt{m} < |\widehat{t}_1 - t_1^*| \leq u_n) \xrightarrow{n \rightarrow \infty} 0$.

Denote $V_{\delta, n, m} = \{t \in \mathbb{Z} / \delta n \log n / \sqrt{m} < |t_1 - t_1^*| \leq u_n\}$. Then,

$$\mathbb{P}\left(\delta \frac{n \log n}{\sqrt{m}} < |\widehat{t}_1 - t_1^*| \leq u_n\right) \leq \mathbb{P}\left(\min_{t_1 \in V_{\delta, n, m}} (J_n(K^*, t_1, (\widehat{d}_1, \widehat{d}_2), m) - J_n(K^*, t_1^*, (\widehat{d}_1^*, \widehat{d}_2^*), m)) \leq 0\right), \quad (5.19)$$

where

$$\widehat{d}_i^* = \arg \min_{d_i \in [0, 0.5)} \{W_n(T_i^*, d_i, m)\},$$

with $W_n(T, d, m)$ defined in (2.4).

Let $t_1 \in V_{\delta, n, m}$ and with no loss of generality chose $t_1 > t_1^*$. Then $n_1 = t_1$, $n_2 = n - t_1$, $n_{11} = t_1^*$, $n_{12} = t_1 - t_1^*$, $n_{21} = 0$ and $n_{22} = n - t_1$. Then $T_1^* = \{1, \dots, t_1^*\}$, $T_2^* = \{t_1^* + 1, \dots, n\}$, $T_1 = \{1, \dots, t\}$, $T_{11} = T_1^* = \{1, \dots, t_1^*\}$, $T_{12} = \{t_1^* + 1, \dots, t\}$, $T_2 = \{t + 1, \dots, n\} = T_{22}$.

On the one hand, using the same results than in the proof of Theorem 3.1, we have:

$$\begin{aligned}
J_n(K^*, t_1, (\widehat{d}_1, \widehat{d}_2), m) &= \frac{1}{n} \left\{ t_1 \log \left(\frac{t_1^*}{t_1} S_n(T_1^*, \widehat{d}_1, m) + \frac{t_1 - t_1^*}{t_1} S_n(T_{12}, \widehat{d}_1, m) + \frac{2}{t_1} R_n(T_{11}, T_{12}, \widehat{d}_1) \right) \right. \\
&\quad \left. + (n - t_1) \log \left(S_n(T_{22}, \widehat{d}_2, m) \right) + 2 \ell(m) (t_1 \widehat{d}_1 + (n - t_1) \widehat{d}_2) \right\} \\
&\geq \frac{1}{n} \left\{ t_1^* \log \left(S_n(T_1^*, \widehat{d}_1, m) \right) + (t_1 - t_1^*) \log \left(S_n(T_{12}, \widehat{d}_1, m) \right) + O_P \left(t_1 \left(\frac{n}{m} \right)^{-2|d_2^* - d_1^*| - 1} \right) \right. \\
&\quad \left. + (n - t_1) \log \left(S_n(T_{22}, \widehat{d}_2, m) \right) + 2 \ell(m) (t_1 \widehat{d}_1 + (n - t_1) \widehat{d}_2) \right\} \\
&\geq \frac{1}{n} \left\{ t_1^* W_n(T_1^*, \widehat{d}_1, m) + (t_1 - t_1^*) W_n(T_{12}, \widehat{d}_1, m) + (n - t_1) W_n(T_{22}, \widehat{d}_2, m) \right. \\
&\quad \left. + O_P \left(t_1 \left(\frac{n}{m} \right)^{-2|d_2^* - d_1^*| - 1} \right) \right\},
\end{aligned}$$

where $O_P \left(t_1 \left(\frac{n}{m} \right)^{-2|d_2^* - d_1^*| - 1} \right)$ is obtained using Lemma 5.2.

On the other hand, we also have:

$$J_n(K^*, t_1^*, (\widehat{d}_1^*, \widehat{d}_2^*), m) = \frac{1}{n} \left\{ t_1^* W_n(T_1^*, \widehat{d}_1^*, m) + (n - t_1^*) W_n(T_2^*, \widehat{d}_2^*, m) \right\}.$$

First we remark that from the definition of \widehat{d}_1^* ,

$$W_n(T_1^*, \widehat{d}_1^*, m) \leq W_n(T_1^*, \widehat{d}_1, m).$$

Therefore,

$$\begin{aligned}
&\frac{n}{t_1 - t_1^*} \left\{ J_n(K^*, t_1, (\widehat{d}_1, \widehat{d}_2), m) - J_n(K^*, t_1^*, (\widehat{d}_1^*, \widehat{d}_2^*), m) \right\} \\
&\geq \frac{1}{t_1 - t_1^*} \left\{ (t_1 - t_1^*) W_n(T_{12}, \widehat{d}_1, m) + (n - t_1) W_n(T_{22}, \widehat{d}_2, m) \right. \\
&\quad \left. - (n - t_1^*) W_n(T_2^*, \widehat{d}_2^*, m) + O_P \left(t_1 \left(\frac{n}{m} \right)^{-2|d_2^* - d_1^*| - 1} \right) \right\}. \quad (5.20)
\end{aligned}$$

Since $t_1 \in V_{\delta, n, m}$, implying $|T_2^*| \sim |T_{22}| \sim (1 - \tau_1^*)n$ when $n \rightarrow \infty$, Lemma 5.1 and more precisely (5.3) can be applied. Then, conditionally to $\widehat{d}_1, \widehat{d}_2$ and \widehat{d}_2^* , we obtain:

$$\begin{aligned}
W_n(T_{12}, \widehat{d}_1, m) &= 2d_2^* \log(n/m) + \log(c_{0,2}^* (2\pi)^{-2d_2^*}) - \log(1 + 2\widehat{d}_1 - 2d_2^*) - 2\widehat{d}_1 \\
&\quad + O_P \left[\frac{1}{m} + \left(\frac{m}{n} \right)^{\beta_2^*} + \left(\frac{m(t_1 - t_1^*)}{n} \right)^{-1/2} \log^{1/2} \left(\frac{m(t_1 - t_1^*)}{n} \right) + \left(\frac{m(t_1 - t_1^*)}{n} \right)^{-1+2d_2^* - 2\widehat{d}_1} \right] \\
W_n(T_{22}, \widehat{d}_2, m) &= 2d_2^* \log(n/m) + \log(c_{0,2}^* (2\pi)^{-2d_2^*}) - \log(1 + 2\widehat{d}_2 - 2d_2^*) - 2\widehat{d}_2 \\
&\quad + O_P \left[\left(\frac{m}{n} \right)^{\beta_2^*} + m^{-1/2} \log^{1/2}(m) + m^{-1+2d_2^* - 2\widehat{d}_2} \right] \\
W_n(T_2^*, \widehat{d}_2^*, m) &= 2d_2^* \log(n/m) + \log(c_{0,2}^* (2\pi)^{-2d_2^*}) - \log(1 + 2\widehat{d}_2^* - 2d_2^*) - 2\widehat{d}_2^* \\
&\quad + O_P \left[\left(\frac{m}{n} \right)^{\beta_2^*} + m^{-1/2} \log^{1/2}(m) + m^{-1+2d_2^* - 2\widehat{d}_2^*} \right],
\end{aligned}$$

since $\ell(m) = \frac{1}{m} \sum_{j=1}^m \log(j/m) = -1 + O(m^{-1})$ which is negligible with respect to $O_P(m^{-1/2} \log^{1/2}(m))$. Therefore, (5.20) becomes:

$$\begin{aligned} & \frac{n}{t_1 - t_1^*} \left\{ J_n(K^*, t_1, (\widehat{d}_1, \widehat{d}_2), m) - J_n(K^*, t_1^*, (\widehat{d}_1^*, \widehat{d}_2^*), m) \right\} \\ & \geq \frac{1}{t_1 - t_1^*} \left\{ - (t_1 - t_1^*) (\log(1 + 2\widehat{d}_1 + 2d_2^*) - 2\widehat{d}_1) - (n - t_1) (\log(1 + 2\widehat{d}_2 - 2d_2^*) + 2\widehat{d}_2) \right. \\ & \quad \left. + (n - t_1^*) (\log(1 + 2\widehat{d}_2^* - 2d_2^*) + 2\widehat{d}_2^*) + O_P\left(n \left(\frac{m}{n}\right)^{\beta_2^*} + \frac{n\sqrt{\log m}}{\sqrt{m}} + n \left(\frac{n}{m}\right)^{-2|d_2^* - d_1^*| - 1}\right) \right\}, \quad (5.21) \end{aligned}$$

since the other terms (notably those related to $W_n(T_{12}, \widehat{d}_1, m)$) are negligible because t_1 is supposed to belong to $V_{\delta, n, m}$. Now, from Dalla *et al.* (2005, p. 221), when m is such as $m = o(n^{2\beta_2^*/(1+2\beta_2^*)})$, then:

$$\widehat{d}_2 = d_2^* + O_P(m^{-1/2}) \quad \text{and} \quad \widehat{d}_2^* = d_2^* + O_P(m^{-1/2}).$$

Finally, as $|T_{12}| = o(|T_1^*|)$ since $t_1 \in V_{\delta, n, m}$, we also have $\widehat{d}_1 = d_1^* + o_P(1)$. Therefore, from (5.21), we obtain after computations,

$$\begin{aligned} & \frac{n}{t_1 - t_1^*} \left\{ J_n(K^*, t_1, (\widehat{d}_1, \widehat{d}_2), m) - J_n(K^*, t_1^*, (\widehat{d}_1^*, \widehat{d}_2^*), m) \right\} \\ & \geq 2(d_1^* - d_2^*) - \log(1 + 2(d_1^* - d_2^*)) + \frac{n}{t_1 - t_1^*} O_P\left(\frac{1}{\sqrt{m}} + \left(\frac{m}{n}\right)^{\beta_2^*} + \frac{\sqrt{\log m}}{\sqrt{m}} + \left(\frac{n}{m}\right)^{-2|d_2^* - d_1^*| - 1}\right) \\ & \geq 2(d_1^* - d_2^*) - \log(1 + 2(d_1^* - d_2^*)) + O_P\left(\frac{1}{\delta} + \frac{\sqrt{m}}{\delta \log n} \left(\frac{m}{n}\right)^{\beta_2^*}\right). \end{aligned}$$

As $m = o(n^{2\beta_2^*/(1+2\beta_2^*)})$ then $\sqrt{m} \left(\frac{m}{n}\right)^{\beta_2^*} = o(1)$. As a consequence, we finally obtain:

$$\frac{n}{t_1 - t_1^*} \left\{ J_n(K^*, t_1, (\widehat{d}_1, \widehat{d}_2), m) - J_n(K^*, t_1^*, (\widehat{d}_1^*, \widehat{d}_2^*), m) \right\} \geq 2(d_1^* - d_2^*) - \log(1 + 2(d_1^* - d_2^*)) + O_P\left(\frac{1}{\delta}\right). \quad (5.22)$$

As $\log(1 + x) < x$ for any $x \in (-1, 0) \cup (0, 1)$, and since $d_1^* - d_2^* \neq 0$, we obtain that

$$\lim_{\delta \rightarrow \infty} \mathbb{P}\left(\frac{n}{t_1 - t_1^*} \left\{ J_n(K^*, t_1, (\widehat{d}_1, \widehat{d}_2), m) - J_n(K^*, t_1^*, (\widehat{d}_1^*, \widehat{d}_2^*), m) \right\} \leq 0\right) = 0$$

and therefore from (5.19) we deduce (3.1) and therefore the proof of Theorem 3.2 is achieved. \square

Proof of Theorem 3.3. Using Theorem 3.2, we have $\widehat{d}_i = \widehat{d}_i^* + \frac{1}{n} O_P(\widehat{t}_i)$. Using (3.2), we deduce (3.3). \square

Proof of Theorem 3.4. Obviously, the proof is established if for any $K \in \{0, \dots, K^* - 1, K^* + 1, \dots, K_{\max}\}$ the following consistency holds:

$$\mathbb{P}\left(J_n(K, \mathbf{t}, \mathbf{d}, m) - J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m) < 0\right) \xrightarrow[n \rightarrow \infty]{} 0, \quad (5.23)$$

for any \mathbf{t} and \mathbf{d} , with J_n defined as in (3.4). Indeed, as $J_n(K^*, \widehat{\mathbf{t}}^*, \widehat{\mathbf{d}}^*, m) \leq J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m)$ by definition, (5.23) is also satisfied by replacing $J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m)$ by $J_n(K^*, \widehat{\mathbf{t}}^*, \widehat{\mathbf{d}}^*, m)$. We decompose the proof in two

parts, $K < K^*$ and $K > K^*$.

Assume $K < K^*$. Then, for any \mathbf{t} and \mathbf{d} , and using (5.13),

$$\begin{aligned}
& J_n(K, \mathbf{t}, \mathbf{d}, m) - J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m) \\
&= \frac{1}{n} \sum_{k=1}^{K+1} n_k \log \left(\sum_{j=1}^{K^*+1} \frac{n_{kj}}{n_k} S_n(T_{kj}, d_k, m) + \frac{2}{n_k} \sum_{j=1}^{K^*+1} \sum_{j'=1, j \neq j'}^{K^*+1} R_n(T_{kj}, T_{kj'}, d_k, m) \right) \\
&\quad - \frac{1}{n} \sum_{j=1}^{K+1} n_j^* \log (S_n(T_j^*, d_j^*, m)) + 2 \frac{\ell(m)}{n} \left(\sum_{k=1}^{K+1} n_k d_k - \sum_{j=1}^{K^*+1} n_j^* d_j^* \right) + (K - K^*) z_n \\
&\geq \sum_{k=1}^{K+1} \sum_{j=1}^{K^*+1} \frac{n_{kj}}{n} (s(d_j^*, d_k) - s(d_j^*, d_j^*)) - |D(m, n)| + (K - K^*) z_n \\
&\geq \sum_{k=1}^{K+1} \sum_{j=1}^{K^*+1} \frac{n_{kj}}{n} (u(d_j^*, d_k) - u(d_j^*, d_j^*)) - |D(m, n)| + (K - K^*) z_n
\end{aligned}$$

since $\sum_{j=1}^{K^*+1} n_{kj} = n_k$ and $\sum_{k=1}^{K+1} n_{kj} = n_j^*$ and using (5.16) with $D(m, n) \xrightarrow[n, m, n/m \rightarrow \infty]{\mathcal{P}} 0$ and $u(d_j^*, d) = -\log(1 + 2d - 2d_j^*) + 2d \geq 2d_j^*$.

Now, we use again Lemma 2.3 of Lavielle (1999, p. 88). This Lemma was obtained when $K = K^*$ and we obtain that there exist $C_d > 0$ such as

$$\sup_{\mathbf{d} \in \mathbf{t} \in} \sum_{k=1}^{K^*+1} \sum_{j=1}^{K^*+1} \frac{n_{kj}}{n} (u(d_j^*, d_k) - u(d_j^*, d_j^*)) \geq C_d \frac{1}{n} \|\mathbf{t} - \mathbf{t}^*\|_\infty$$

where $\|\mathbf{t} - \mathbf{t}^*\|_\infty = \max_{1 \leq j \leq K^*} |t_j - t_j^*|$. However this result is still valide when K^* is replaced by $K < K^*$ in the first sum, since it is sufficient to add $K^* - K$ fictive times and consider $t_{K+1} = t_{K+2} = \dots = t_{K^*} = t_K$ (and therefore $n_{kj} = 0$ for $k = K + 2, \dots, K^* + 1$. Therefore we obtain:

$$J_n(K, \mathbf{t}, \mathbf{d}, m) - J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m) \geq \frac{1}{3} \min_{1 \leq i \leq K^*} |\tau_{i+1}^* - \tau_i^*| - |D(m, n)| + (K - K^*) z_n \quad (5.24)$$

since $K < K^*$ and therefore $\|\mathbf{t} - \mathbf{t}^*\|_\infty \geq \frac{1}{2} \min_{1 \leq i \leq K^*} |t_{i+1}^* - t_i^*| \geq \frac{n}{3} \min_{1 \leq i \leq K^*} |\tau_{i+1}^* - \tau_i^*|$ when n is large enough. Therefore, if $z_n \xrightarrow[n \rightarrow \infty]{} 0$ then (5.23) is satisfied and therefore $\mathbb{P}(\widehat{K} < K^*) \xrightarrow[n \rightarrow \infty]{} 0$.

Assume $K^* < K \leq K_{\max}$. With $\widehat{\mathbf{t}} = (\widehat{t}_1, \dots, \widehat{t}_K)$, there exists some subset $\{k_j, 1 \leq j \leq K^*\}$ of $\{1, \dots, K\}$ such that for any $j = 1, \dots, K^*$, $|\frac{\widehat{t}_{k_j}}{n} - \tau_j^*| = O_P\left(\frac{\log n}{\sqrt{m}}\right)$. To see this, consider the (\widehat{t}_{k_j}) as the closest times among $(\widehat{t}_1, \dots, \widehat{t}_K)$ to the $(t_1^*, \dots, t_{K^*}^*)$. The other $K - K^*$ change dates \widehat{t}_i could be consider exactly as additional ‘‘false’’ changes (since the parameters d do not change at these times) and therefore the \widehat{t}_{k_j} minimize $J_n(K, \mathbf{t}, \mathbf{d}, m)$ conditionally to those \widehat{t}_i with $i \notin \{k_1, \dots, k_{K^*}\}$ as if the number of changes is known and is K^* . And therefore Theorem 3.2 holds for those \widehat{t}_{k_j} .

Then using the previous expansions detailed in the previous proofs, we obtain

$$\begin{aligned}
& J_n(K, \widehat{\mathbf{t}}, \widehat{\mathbf{d}}, m) - J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m) \\
&= \frac{1}{n} \sum_{j=1}^{K^*+1} \left(\sum_{k=k_j+1}^{k_{j+1}} \widehat{n}_k \log(S_n(\widehat{T}_k, \widehat{d}_k, m)) - n_j^* \log(S_n(T_j^*, d_j^*, m)) \right) \\
&\quad + 2 \frac{\ell(m)}{n} \left(\sum_{i=1}^{K+1} \widehat{n}_i \widehat{d}_i - \sum_{j=1}^{K^*+1} n_j^* d_j^* \right) + (K - K^*) z_n \\
&\geq \frac{1}{n} \sum_{j=1}^{K^*+1} \left(\sum_{k=k_j+1}^{k_{j+1}} \widehat{n}_k s(d_j^*, \widehat{d}_k) - n_j^* s(d_j^*, d_j^*) \right) - |D(m, n)| \\
&\quad + 2 \frac{\ell(m)}{n} \left(\sum_{i=1}^{K+1} \widehat{n}_i \widehat{d}_i - \sum_{j=1}^{K^*+1} n_j^* d_j^* \right) + (K - K^*) z_n \\
&\geq \frac{1}{n} \sum_{j=1}^{K^*+1} \left(\sum_{k=k_j+1}^{k_{j+1}} \widehat{n}_k s(d_j^*, \widehat{d}_k) - n_j^* s(d_j^*, d_j^*) \right) - |D(m, n)| \\
&\quad + 2 \frac{\ell(m)}{n} \left(\sum_{i=1}^{K+1} \widehat{n}_i \widehat{d}_i - \sum_{j=1}^{K^*+1} n_j^* d_j^* \right) + (K - K^*) z_n
\end{aligned}$$

with s defined in (5.16). Now, since $\widehat{T}_k \subset \{\widehat{t}_{k_j+1}, \dots, \widehat{t}_{k_{j+1}}\}$, we have from Theorem 3.4, $\widehat{d}_k = d_j^* + O\left(\frac{\log n}{\sqrt{m}}\right)$. As a consequence, for $k = k_j + 1, \dots, k_{j+1}$ then $s(d_j^*, \widehat{d}_k) = s(d_j^*, d_j^*) + O_P\left(\frac{\log n}{\sqrt{m}}\right)$. Then,

$$\begin{aligned}
& J_n(K, \widehat{\mathbf{t}}, \widehat{\mathbf{d}}, m) - J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m) \\
&\geq \frac{1}{n} \sum_{j=1}^{K^*+1} \left(s(d_j^*, d_j^*) + 2 d_j^* \frac{\ell(m)}{n} \right) \left(\sum_{k=k_j+1}^{k_{j+1}} \widehat{n}_k - n_j^* \right) - |D(m, n)| - |E(m, n)| + (K - K^*) z_n \\
&\geq -|D(m, n)| - |E'(m, n)| + (K - K^*) z_n,
\end{aligned}$$

with $D(m, n) = O_P\left(\frac{\sqrt{\log n}}{\sqrt{m}}\right)$ under condition $m = o(n^{2\beta_-^*/(1+2\beta_-^*)})$ from the proof of Theorem 3.2, $E(m, n) = O_P\left(\frac{\log n}{\sqrt{m}}\right)$, $E'(m, n) = O_P\left(\frac{\log n}{\sqrt{m}}\right)$ since $\left| \sum_{k=k_j+1}^{k_{j+1}} \widehat{n}_k - n_j^* \right| = O_P\left(\frac{n \log n}{\sqrt{m}}\right)$. As a consequence if (z_n) is such that $z_n \frac{\sqrt{m}}{\log n} \xrightarrow[n \rightarrow \infty]{} \infty$ then for any $K > K^*$,

$$\mathbb{P}\left(J_n(K, \widehat{\mathbf{t}}, \widehat{\mathbf{d}}, m) - J_n(K^*, \mathbf{t}^*, \mathbf{d}^*, m) < 0\right) \xrightarrow[n \rightarrow \infty]{} 0.$$

This achieves the proof. □

Proof of Corollary 1. The results are easily obtained by considering conditional probability with respect to the event $\widehat{K} = K^*$. □

- [1] Abadir, K.M., Distaso, W. and Giraitis, L. (2007) Non-stationarity-extended local Whittle estimation. *J. Econometrics*, 141, 1353-1384.
- [2] Arlot, S. and Massart, P. (2009) Data-driven calibration of penalties for least-squares regression. *Journal of Machine Learning Research*, 10, p. 245-279.
- [3] Bai J. (1998) Least squares estimation of a shift in linear processes. *J. of Time Series Anal.*, 5, p. 453-472.
- [4] Bai J. and Perron P. (1998) Estimating and testing linear models with multiple structural changes. *Econometrica*, 66, p. 47-78.
- [5] Bardet, J.-M., Lang, G., Oppenheim, G., Philippe, A. and Taqqu, M.S. (2003a) Generators of long-range dependent processes: a survey. *Theory and applications of long-range dependence*, Birkhauser, Boston,MA, p. 579-623.
- [6] Bardet, J.-M., Lang, G., Oppenheim, G., Philippe, A., Stoev, S. and Taqqu, M.S. (2003b) Semi-parametric estimation of the long-range dependence parameter: a survey *Theory and applications of long-range dependence*, Birkhauser, Boston,MA, p. 5557-577.
- [7] Bardet, J.-M., Kengne, W. and Wintenberger, O. (2012) Detecting multiple change-points in general causal time series using penalized quasi-likelihood. *Electronic Journal of Statistics*, 6, 435-477.
- [8] Basseville, M. and Nikiforov, I. (1993). *Detection of Abrupt Changes: Theory and Applications*. Prentice Hall, Englewood Cliffs, NJ, 1993.
- [9] Beran, J. (1994) *Statistics for Long-Memory Processes*. Chapman and Hall, New York.
- [10] Dalla, V., Giraitis, L. and Hidalgo, J. (2006) Consistent estimation of the memory parameter for nonlinear time series. *Journal of Time Series Analysis*, 27, 211-251.
- [11] Doukhan, P., Oppenheim, G. and Taqqu M.S. (Editors) (2003) *Theory and applications of long-range dependence*, Birkhäuser.
- [12] Geweke, J. and Porter-Hudak, S. (1983), The estimation and application of long-memory time-series models, *J. Time Ser. Anal.*, 4, 221-238.
- [13] Giraitis, L., Kokoszka, P. and Leipus, R. (2001). Testing for Long Memory in the Presence of a General Trend. *Journal of Applied Probability*, 38, 1033-1054.
- [14] Giraitis, L., Koul, H. and Surgailis, D. (2012) *Large-Sample-Inference-Memory-Processes*, Imperial College Press.

- [15] Henry, M. and Robinson, P.M. (1996) Bandwidth choice in Gaussian semiparametric estimation of long range dependence. In: Athens Conference on Applied Probability and Time Series Analysis, Vol. II, 220-232, Springer, New York.
- [16] Henry, M. (2007) Robust automatic bandwidth for long-memory. In: Long Memory in Economics, 157-172, Springer.
- [17] Kay, S.M. (1998). *Fundamentals of Statistical Signal Processing*, 2, Prentice-Hall, Englewood Cliffs.
- [18] Kokoszka, P. and Leipus, R. (2003) Detection and estimation of changes in regime. In P. Doukhan, G. Oppenheim, and M. S. Taqqu, editors, Theory and Applications of Long Range Dependence, pages 325-337.
- [19] Lavancier, F., Leipus, R., Philippe A. and Surgailis, D. (2013). Detection of non-constant long memory parameter. *Econometric Theory*, 29, 1009-1056.
- [20] Lavielle, M. (1999) Detection of multiple change in a sequence of dependent variables. *Stochastic Process. Appl.*, 83, 79–102.
- [21] Lavielle, M. and Ludena, C. (2000). The multiple change-points problem for the spectral distribution. *Bernoulli*, 6, 845–869.
- [22] Robinson, P.M. (1995a). Log-periodogram regression of time series with long range dependence. *Annals of Statistics*, 23, 1048–1072.
- [23] Robinson, P.M. (1995b). Gaussian semiparametric estimation of long range dependence. *Annals of Statistics*, 23, 1630–1661.