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Controllability of the Navier-Stokes equation in a rectangle with a little help of a distributed phantom force

Jean-Michel Coron[*,†] Frédéric Marbach[‡] Franck Sueur[§] Ping Zhang[¶]

February 1, 2018

Abstract

We consider the 2D incompressible Navier-Stokes equation in a rectangle with the usual no-slip boundary condition prescribed on the upper and lower boundaries. We prove that for any positive time, for any finite energy initial data, there exist controls on the left and right boundaries and a distributed force, which can be chosen arbitrarily small in any Sobolev norm in space, such that the corresponding solution is at rest at the given final time.

Our work improves earlier results in [17, 18] where the distributed force is small only in a negative Sobolev space. It is a further step towards an answer to Jacques-Louis Lions' question in [23] about the small-time global exact boundary controllability of the Navier-Stokes equation with the no-slip boundary condition, for which no distributed force is allowed.

Our analysis relies on the well-prepared dissipation method already used in [28] for Burgers and in [10] for Navier-Stokes in the case of the Navier slip-with-friction boundary condition. In order to handle the larger boundary layers associated with the no-slip boundary condition, we perform a preliminary regularization into analytic functions with arbitrarily large analytic radius and prove a long-time nonlinear Cauchy-Kovalevskaya estimate relying only on horizontal analyticity, in the spirit of [3, 35].

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Contents

1 Introduction and statement of the main result **3**
1.1 Historical context . . . 3
1.2 Statement of the main result . . . 3
1.3 Comments and references . . . 4
1.4 Strategy of the proof and plan of the paper . . . 5

2 Regularization enhancement **8**
2.1 Fourier analysis in the tangential direction . . . 8
2.2 Regularization enhancement using the phantom and the control . . . 9
2.3 Proof of the regularization proposition . . . 10

3 Strategy for global approximate controllability **11**
3.1 Small-time to small-viscosity scaling . . . 11
3.2 Return method ansatz . . . 11
3.3 Estimates and proof of approximate controllability . . . 13
3.4 Comments and insights on the proposed expansion . . . 15

4 Well-prepared dissipation method for the boundary layer **16**
4.1 Large time decay of the boundary layer profile . . . 16
4.2 Fast variable scaling and Lebesgue norms . . . 17
4.3 Estimates for the technical profile . . . 18
4.4 Proof of the decay of approximate trajectories . . . 18

5 Estimates on the remainder **19**
5.1 Singular amplification to loss of derivative . . . 20
5.2 A few tools from Littlewood-Paley theory . . . 20
5.3 Long-time weakly nonlinear Cauchy-Kovalevskaya estimate . . . 24
5.4 Proof of Proposition 5.10 . . . 27
5.5 Estimate of $(r \cdot \nabla)u^1 + (u^1 \cdot \nabla)r$. Proof of Lemma 5.14 . . . 29
5.6 Estimate of $(r \cdot \nabla)r$. Proof of Lemma 5.15 . . . 31

6 Analytic estimates for the approximate trajectories **37**
6.1 Preliminary estimates . . . 37
6.2 Estimates for the amplification terms . . . 38
6.3 Estimates for the source terms . . . 39

1 Introduction and statement of the main result

1.1 Historical context

In the late 1980's, Jacques-Louis Lions introduced in [23] (see also [24, 25, 26]) the question of the controllability of fluid flows in the sense of how the Navier-Stokes system can be driven by a control of the flow on a part of the boundary to a wished plausible state, say a vanishing velocity. Jacques-Louis Lions' problem has been solved in [10] by the first three authors in the particular case of the Navier slip-with-friction boundary condition (see also [9] for a gentle introduction to this result). In its original statement with the no-slip Dirichlet boundary condition, it is still an important open problem in fluid controllability.

1.2 Statement of the main result

In this paper we consider the case where the flow occupies a rectangle, where controls are applied to the lateral boundaries and the no-slip condition is prescribed on the upper and lower boundaries. We thus consider a rectangular domain

$$\Omega := (0, L) \times (-1, 1),$$

where $L > 0$ is the length of the domain. We will use (x, y) as coordinates. Inside this domain, a fluid evolves under the Navier-Stokes equation. We will name $u = (u_1, u_2)$ the two components of its velocity. Hence, u satisfies:

$$\begin{cases} \partial_t u + (u \cdot \nabla)\, u + \nabla p - \Delta u = f_g, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

in Ω, where p denotes the fluid pressure and f_g a force term (to be detailed below). We think of this domain as a river or a tube and we assume that we are able to act on the fluid flow at both end boundaries:

$$\Gamma_0 := \{0\} \times (-1, 1) \text{ and } \Gamma_L := \{L\} \times (-1, 1).$$

On the remaining parts of the boundary,

$$\Gamma_\pm := (0, L) \times \{\pm 1\},$$

we assume that we cannot control the fluid flow and that it satisfies null Dirichlet boundary conditions:

$$u = 0 \quad \text{on} \quad \Gamma_\pm. \tag{1.2}$$

We will consider initial data in the space $L^2_{\text{div}}(\Omega)$ of divergence free vector fields, tangent to the boundaries $\Gamma_\pm$. The main result of this paper is the following.

Theorem 1. *Let $T > 0$ and u_* in $L^2_{\text{div}}(\Omega)$. For any $k \in \mathbb{N}$ and for any $\eta > 0$, there exists a force $f_g \in L^1((0, T); H^k(\Omega))$ satisfying*

$$\|f_g\|_{L^1((0,T);H^k(\Omega))} \leq \eta \tag{1.3}$$

and an associated weak Leray solution $u \in C^0([0, T]; L^2_{\text{div}}(\Omega)) \cap L^2((0, T); H^1(\Omega))$ to (1.1) and (1.2) satisfying $u(0) = u_*$ and $u(T) = 0$.

Since the notion of weak Leray solution is classically defined in the case where the null Dirichlet boundary condition is prescribed on the whole boundary, let us detail that we say that $u \in C^0([0,T]; L^2_{\mathrm{div}}(\Omega)) \cap L^2((0,T); H^1(\Omega))$ is a weak Leray solution to (1.1) and (1.2) satisfying $u(0,\cdot) = u_*$ and $u(T,\cdot) = 0$ when it satisfies the weak formulation

$$\begin{aligned} -\int_0^T \int_\Omega u \cdot \partial_t \varphi + \int_0^T \int_\Omega (u \cdot \nabla) u \cdot \varphi + 2 \int_0^T \int_\Omega D(u) : D(\varphi) \\ = \int_\Omega u_* \cdot \varphi(0,\cdot) + \int_0^T \int_\Omega u \cdot f_g, \end{aligned} \tag{1.4}$$

for any test function $\varphi \in C^\infty([0,T] \times \bar{\Omega})$ which is divergence free, tangent to $\Gamma_\pm$, vanishes at $t = T$ and vanishes on the controlled parts of the boundary Γ_0 and Γ_L. Thus, (1.4) encodes the no-slip condition on the upper and lower boundaries only. This under-determination encodes that one has control over the remaining part of the boundary, that is on the lateral boundaries. The controls on the lateral boundaries are therefore not explicit in the statement of Theorem 1. Still the proof below will provide some more insights on the nature of possible controls to the interested reader. Once a trajectory is known, one can indeed deduce that the associated controls are the traces on Γ_0 and Γ_L of the solution.

1.3 Comments and references

Remark 1.1 (Relation to the open problem)**.** *We view Theorem 1 as an intermediate step towards an answer to Jacques-Louis Lions' problem, which requires to prove that the theorem is still true with a vanishing distributed force $f_g = 0$. Here, we need a non-vanishing force but we can choose it very small even in strong topologies. Our result therefore suggests that the answer to Jacques-Louis Lions' question is very likely positive, at least for this geometry. Nevertheless, new ideas are probably necessary to "eliminate" the unwanted distributed force we use.*

Remark 1.2 (Local vs. global null controllability and Reynolds numbers)**.** *The fact that, for any $T > 0$, one can drive to the null equilibrium state $u = 0$ in time T without any distributed force ($f_g = 0$) was already known when the initial data u_* is small enough in $L^2(\Omega)$ (with a maximal size depending on T). In this case, one may think of the bilinear term in Navier-Stokes system as a small perturbation term of the Stokes equation so that the controllability can be obtained by means of Carleman estimates and fixed point theorems. Loosely speaking, such an approach corresponds to low Reynolds controllability.*

More generally, local null controllability is a particular case of local controllability to trajectories. For Dirichlet boundary conditions, the first results have been obtained by Imanuvilov who proved local controllability in 2D and 3D provided that the initial state are close in H^1 norm, with interior controls, first towards steady-states in [21] then towards strong trajectories in [22]. Fursikov and Imanuvilov proved large time global null controllability in 2D for a control supported on the full boundary of the domain in [13]. Still in 2D, they also proved local controllability to strong trajectories for a control acting on a part of the boundary and initial states close in H^1 norm in [14]. Eventually, in [15] they proved in 2D and 3D local controllability to strong trajectories with controls acting on the full boundary, still for initial states close in H^1 norm. More recently, these works have been improved in [11], where the authors proved local controllability towards less regular

trajectories with interior controls and for initial states close in L^2 norm in 2D and L^4 norm in 3D.

In contrast, in Theorem 1, the initial data u_* can be arbitrarily large (and T arbitrarily small). This corresponds to controllability of Navier-Stokes system at large Reynolds numbers. In this regime, the first author and Fursikov proved global null controllability for the Navier-Stokes system in a 2D manifold without boundary in [8].

Remark 1.3 (Comparison with earlier results)**.** *For the large Reynolds regime, let us mention the earlier references [17, 18] where a related result is obtained in a similar setting. In this earlier result, the distributed force can be chosen small in $L^p((0,T);H^{-1}(\Omega))$, where $1 < p < 4/3$. The fact that, in Theorem 1, our phantom force can be chosen arbitrarily small in the space $L^1((0,T),H^k(\Omega))$ for any $k \geq 0$, is the major improvement of this work.*

Remark 1.4 (Geometric setting)**.** *Theorem 1 remains true for any rectangular domain $(0,L_1)\times(0,L_2)$ and any positive viscosity ν, thanks to a straightforward change of variables. On the other hand, we consider the case of a rectangle because it provides many crucial simplifications. This is not for the sake of clarity of the exposition; we suspect that the case of a general domain requires different arguments. A key point is that this geometric setting and the use of well-chosen controls enable us to guarantee that the boundary layer equations we consider will remain linear and well-posed (see* (3.14) *and Remark 3.5).*

Remark 1.5 (Additional properties of the phantom force)**.** *During the proof, we will check that the phantom force f_g we use has C^∞ regularity by parts with respect to time and C^∞ regularity with respect to space for each time. Moreover, we will check that, during the most important step of our strategy (the global approximate control phase, which involves passing through intermediate states of very large size), there exists $\delta > 0$ such that*

$$\operatorname{supp} f_g(t) \subset [0,L] \times [-1+\delta, 1-\delta]. \tag{1.5}$$

1.4 Strategy of the proof and plan of the paper

We explain the strategy of the proof of Theorem 1, which is divided into three steps. First, we prove that the initial data can be regularized into an analytic function with arbitrary analyticity radius. Then, we prove that a large analytic initial data with a sufficient analyticity radius can be driven approximately to the null equilibrium. Last, we know that small enough states can be driven exactly to the rest state. These three steps are implemented in the three propositions below, where we set the Navier-Stokes equations in the horizontal band

$$\mathfrak{B} := \mathbb{R} \times [-1,1], \tag{1.6}$$

with a control supported in the extended region, outside of Ω. Therefore, we look for solutions to

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p - \Delta u = f_c + f_g & \text{in } (0,T)\times\mathfrak{B}, \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\mathfrak{B}, \\ u = 0 & \text{on } (0,T)\times\partial\mathfrak{B}, \end{cases} \tag{1.7}$$

where the force f_c is a control supported in $\mathfrak{B}\setminus\Omega$ and the force f_g is the phantom (ghost) force supported in Ω. Restricting such solutions of Navier-Stokes in the band to the physical

domain Ω will prove Theorem 1. We also introduce a domain

$$G := [-2L, -L] \times [-1, 1] \quad \text{(see Fig. 1).} \tag{1.8}$$

We will denote by e_x and e_y the unit vectors of the canonical basis of $\mathbb{R}^2$.

Proposition 1.6 (Analytic regularization of the initial data)**.** *Let $T > 0$ and $\rho_b > 0$. Let $u_* \in L^2_{\mathrm{div}}(\bar{\Omega})$ with $\int_\Omega u_* \cdot e_x = 0$. For any $k \in \mathbb{N}$ and $\eta_b > 0$, there exists an extension $u_a \in L^2_{\mathrm{div}}(\mathfrak{B})$ of u_* to the band $\mathfrak{B}$, a control force $f_c \in C^\infty([0,T] \times (\mathfrak{B} \setminus \bar{\Omega}))$, a phantom force $f_g \in C^\infty([0,T] \times \bar{\Omega})$ satisfying*

$$\|f_g\|_{L^1((0,T);H^k(\Omega))} \leq \eta_b, \tag{1.9}$$

a weak Leray solution $u \in C^0([0,T]; L^2_{\mathrm{div}}(\mathfrak{B})) \cap L^2((0,T); H^1(\mathfrak{B}))$ to (1.7) associated with the initial data u_a, $C_b > 0$ and $T_b \leq T$ such that $u_b := u(T_b) \in L^2_{\mathrm{div}}(\mathfrak{B})$ satisfies

$$\|u_{b|G}\|_{H^k(G)} + \|u_{b|\mathfrak{B}\setminus\Omega}\|_{L^2(\mathfrak{B}\setminus\Omega)} + \|u_{b|\Omega} - u_*\|_{L^2(\Omega)} \leq \eta_b, \tag{1.10}$$

$$\forall m \geq 0, \quad \|\partial_x^m u_b\|_{H^3(\mathfrak{B})} \leq \frac{m!}{\rho_b^m} C_b, \tag{1.11}$$

$$\sum_{0 \leq \alpha+\beta \leq 3} \|\partial_x^\alpha \partial_y^\beta u_b\|_{L^1_x(L^2_y)} \leq C_b. \tag{1.12}$$

Proposition 1.7 (Global approximate null controllability from any analytic initial data)**.** *Let $T > 0$. There exists $\rho_b > 0$ such that, for every $\sigma > 0$ and each $u_b \in L^2_{\mathrm{div}}(\mathfrak{B})$ for which there exists $C_b > 0$ such that (1.11) and (1.12) hold, for every $k \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$, there exist two forces $f_c \in C^\infty([0,T] \times (\mathfrak{B} \setminus \Omega))$ and $f_g \in C^\infty([0,T] \times \bar{\Omega})$ satisfying*

$$\|f_g\|_{L^1((0,T);H^k(\Omega))} \leq C_{k,\delta} \|u_{b|G}\|_{H^k(G)}, \tag{1.13}$$

$$\operatorname{supp} f_g \subset (0,T) \times [0,L] \times [-1+\delta, 1-\delta], \tag{1.14}$$

$$\operatorname{supp} f_c \subset (0,T) \times \mathfrak{B} \setminus \bar{\Omega} \tag{1.15}$$

and a weak solution $u \in C^0([0,T]; L^2_{\mathrm{loc,div}}(\mathfrak{B})) \cap L^2((0,T); H^1_{\mathrm{loc}}(\mathfrak{B}))$ to (1.7) associated with the initial data u_b, such that there exists $T_c \leq T$ such that $u_c := u(T_c) \in L^2_{\mathrm{loc,div}}(\mathfrak{B})$ satisfies

$$\|u_{c|\Omega}\|_{L^2(\Omega)} \leq \sigma + \|u_{b|\{|y| \geq 1-2\delta\}}\|_{L^2(\mathfrak{B})}. \tag{1.16}$$

Moreover, the constant $C_{k,\delta}$ only depends on k and δ.

Proposition 1.8 (Local null controllability)**.** *Let $T > 0$. There exists $\sigma > 0$ such that, for any $u_c \in L^2_{\mathrm{div}}(\Omega)$ which satisfies*

$$\|u_{c|\Omega}\|_{L^2(\mathfrak{B})} \leq 3\sigma, \tag{1.17}$$

there exists a weak solution $u \in C^0([0,T]; L^2_{\mathrm{div}}(\Omega)) \cap L^2((0,T); H^1(\Omega))$ to (1.1) with the initial data u_c and $f_g = 0$, which satisfies $u(T) = 0$.

Proposition 1.8 is a direct consequence of known results concerning the small-time local null controllability of the Navier-Stokes equation (see Remark 1.2 for references).

Let us prove that the combination of these three propositions implies Theorem 1. First, thanks to standard arguments, it is sufficient to prove Theorem 1 for initial data satisfying $\int_\Omega u_* \cdot e_x = 0$. Indeed, applying vanishing boundary controls on the original system for any positive time guarantees that the state gains this property. Therefore, we assume that the initial data already satisfies this property. We fix the quantities step by step in the following manner.

- Let $T > 0$ and $u_* \in L^2_{\mathrm{div}}(\Omega)$ satisfying $\int_\Omega u_* \cdot e_x = 0$.
- Let $\rho_b > 0$ be given by Proposition 1.7 for a time interval of length $T/3$.
- Let $\sigma > 0$ be given by Proposition 1.8 for a time interval of length $T/3$.
- Let $\delta_1 \in (0, \frac{1}{2})$ small enough such that $\|(u_*)_{|\{|y|\geq 1-2\delta_1\}}\|_{L^2(\Omega)} \leq \sigma$.
- Let $k \in \mathbb{N}$ and $\eta > 0$.
- Let $\eta_b := \min\{\eta/2, \sigma, \eta/(2C_{k,\delta_1})\}$.
- We apply Proposition 1.6 with a time interval of length $T/3$, ρ_b, k and η_b. Hence, there exists $T_b \leq T/3$ and a solution u defined on $[0, T_b]$ with $u(0)_{|\Omega} = u_*$ and such that $u_b := u(T_b)$ satisfies (1.10), (1.11) and (1.12).
- We apply Proposition 1.7 with a time interval of length $T/3$, ρ_b, δ_1, k and σ. This yields a solution u defined on $[T_b, T_c]$ with $T_c \leq T_b + T/3 \leq 2T/3$ such that $u_c := u(T_c)$ satisfies (1.16).
- By triangular inequality, $\|u_{b|\{|y|\geq 1-2\delta_1\}}\|_{L^2(\mathfrak{B})} \leq \|u_{b|\mathfrak{B}\setminus\Omega}\|_{L^2(\mathfrak{B})} + \|u_{b|\Omega} - u_*\|_{L^2(\Omega)} + \|(u_*)_{|\{|y|\geq 1-2\delta_1\}}\|_{L^2(\Omega)}$. Since $\eta_b \leq \sigma$, (1.10) and (1.16) imply that (1.17) holds.
- Finally, we apply Proposition 1.8. This yields a solution u defined on $[T_c, T_3]$ with $T_3 := T_c + T/3 \leq T$ such that $u(T_3) = 0$.
- The concatenated forces f_c and f_g are C^∞ by parts in time with C^∞ regularity in space.
- This concludes the proof of Theorem 1 up to extending the solution and the forces by 0 on $[T_3, T]$.

Remark 1.9. *The fact that, starting from a finite energy initial data, the solution to the Navier-Stokes equation instantly becomes analytic is well-known. However, in the uncontrolled setting, the analytic radius only grows like $\sqrt{t}$. In Proposition 1.6, we use the phantom force to enhance the regularization in short time.*

Remark 1.10. *The small-time global approximate null controllability result of Proposition 1.7 will be proved thanks to a return-method argument (see Section 3). A base flow will shift the whole band $\mathfrak{B}$ of a distance $2L$ towards the right. Roughly speaking, the main part of the initial data $u_{b|\Omega}$ will then be outside of the physical domain and killed by a control force. However, since we need to work in an analytic setting (to establish estimates for a PDE with a derivative loss, see Section 5), this action cannot be exactly localized*

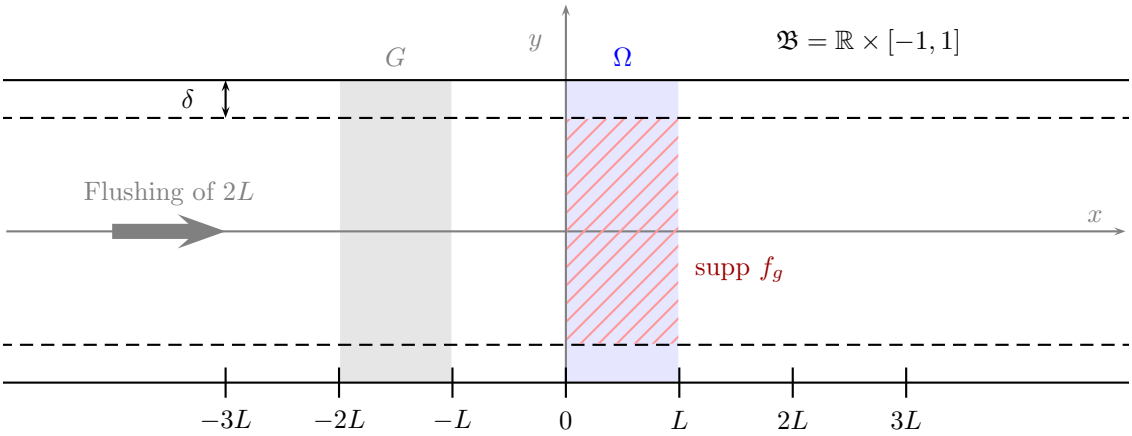

Figure 1: The horizontal band $\mathfrak{B} = \mathbb{R} \times [-1, 1]$, the physical domain $\Omega = [0, L] \times [-1, 1]$ and the domain $G = [-2L, -L] \times [-1, 1]$.

outside of the physical domain. Its leakage inside the physical domain will be related to the values of u_b in G and lead to the unwanted phantom force. This explains estimate (1.13). *Of course, since u_b is analytic in the tangential direction x, it cannot satisfy $u_{b|G} = 0$.*

Proposition 1.6 is proved in Section 2. Proposition 1.7 is the main contribution of this paper. We explain our strategy to prove this small-time global approximate null controllability result in Section 3. It is based on the construction of approximate trajectories and the *well-prepared dissipation method*. We give estimates concerning the boundary layer in Section 4, estimates concerning the remainder in Section 5 and finally analytic estimates on the approximate trajectories themselves as an appendix in Section 6.

2 Regularization enhancement

This section is devoted to the proof of Proposition 1.6.

2.1 Fourier analysis in the tangential direction

We introduce a few notations that will be used throughout this paper.

Tangential Fourier transform. To perform analytic estimates in the tangential direction, we use Fourier analysis in the tangential direction. For $a \in L^2(\mathfrak{B})$, we will denote its Fourier transform with respect to the tangential variable by $\mathcal{F}a$ and define it as

$$\mathcal{F}a(\xi, y) := \int_{\mathbb{R}_x} a(x, y) e^{-ix\xi} \, \mathrm{d}x. \tag{2.1}$$

We define similarly the reciprocal Fourier transform $\mathcal{F}^{-1}$, which obviously also acts only on the tangential variable.

Band-limited functions. Let $N > 0$. We will sometimes need to consider functions in $L^2(\mathfrak{B})$ whose Fourier transform is supported within the set of tangential frequencies ξ satisfying $|\xi| \leq N$. Therefore, we introduce the Fourier multiplier

$$\mathbf{P}_N(\xi) := \mathbf{1}_{[-N,N]}(\xi) \tag{2.2}$$

and the associated functional space

$$L^2_N(\mathfrak{B}) := \left\{a \in L^2(\mathfrak{B});\ \ a = \mathbf{P}_N a\right\}. \tag{2.3}$$

For any $k \in \mathbb{N}$ and $a \in H^k(\mathfrak{B})$, it is clear that

$$\|\mathbf{P}_N a - a\|_{H^k(\mathfrak{B})} \to 0 \quad \text{as} \quad N \to +\infty. \tag{2.4}$$

2.2 Regularization enhancement using the phantom and the control

We start with the following lemma concerning the possibility to remove high tangential frequencies from a smooth initial data. We denote by $\mathbb{P}$ the usual Leray projector on divergence free vector fields, tangent to $\partial\mathfrak{B}$.

Lemma 2.1. *Let $u_a \in L^2_{\text{div}}(\mathfrak{B})$ and $T_b > 0$. We denote by $\widetilde{u}_a$ the solution of the free Navier-Stokes equation at time T_b starting from u_a. There exists a family indexed by $N > 0$ of vector fields $u_N \in C^0([0,T_b]; L^2_{\text{div}}(\mathfrak{B})) \cap L^2((0,T_b); H^1(\mathfrak{B}))$ associated with forces $f_N \in C^0([0,T_b]; H^k(\mathfrak{B}))$, which are weak Leray solutions to*

$$\partial_t u_N - \Delta u_N + \mathbb{P}[(u_N \cdot \nabla) u_N] = \mathbb{P} f_N, \quad u_N(0) = u_a \tag{2.5}$$

and satisfy, for any $k \in \mathbb{N}$ and $\rho_b > 0$,

$$\|(f_N)_{|\Omega}\|_{L^1((0,T_b);H^k(\Omega))} \underset{N\to+\infty}{\longrightarrow} 0, \tag{2.6}$$

$$f_N \in C^\infty([0,T_b] \times \mathfrak{B}), \tag{2.7}$$

$$\|(u_N(T_b) - \widetilde{u}_a)_{|\Omega}\|_{L^2(\Omega)} \underset{N\to+\infty}{\longrightarrow} 0, \tag{2.8}$$

$$\|(u_N(T_b))_{|G}\|_{H^k(G)} \underset{N\to+\infty}{\longrightarrow} 0, \tag{2.9}$$

$$\|(u_N(T_b))_{|\mathfrak{B}\setminus\bar{\Omega}}\|_{L^2(\mathfrak{B}\setminus\bar{\Omega})} \leq \|(\widetilde{u}_a)_{|\mathfrak{B}\setminus\bar{\Omega}}\|_{L^2(\mathfrak{B}\setminus\bar{\Omega})} + \underset{N\to+\infty}{o}(1), \tag{2.10}$$

$$\exists C_N > 0, \quad \sup_{m\geq 0} \|\partial_x^m u_N(T_b)\|_{H^3(\mathfrak{B})} \leq \frac{m!}{\rho_b^m} C_N, \tag{2.11}$$

$$\exists C_N > 0, \quad \sum_{0\leq\alpha+\beta\leq 3} \|\partial_x^\alpha \partial_y^\beta u_N(T_b)\|_{L^1_x(L^2_y)} \leq C_N. \tag{2.12}$$

Proof. Let $u_a \in L^2_{\text{div}}(\mathfrak{B})$. Let $T_b > 0$. Let v be the weak Leray solution to

$$\partial_t v - \Delta v + \mathbb{P}[(v \cdot \nabla) v] = 0, \quad v(0) = u_a. \tag{2.13}$$

Hence, by definition, $\widetilde{u}_a = v(T_b)$. It is classical to prove that $v \in C^\infty((0,T_b] \times \mathfrak{B})$ (see e.g. [34]). Let $\beta \in C^\infty([0,T_b];[0,1])$ with $\beta = 1$ on $[0, T_b/3]$ and $\beta = 0$ on $[2T_b/3, T_b]$. Let

$\theta \in C^\infty(\mathfrak{B};[0,1])$ with $\theta = 1$ for $x \in [0,L]$ and $\theta = 0$ for $x < -L$ or $x > 2L$. We consider $u := \beta v + (1-\beta)\theta v$. Then, u is the weak Leray solution to

$$\partial_t u - \Delta u + \mathbb{P}[(u\cdot\nabla)u] = \mathbb{P}g, \quad u(0) = u_a, \tag{2.14}$$

where we set

$$\begin{aligned} g := \dot\beta(1-\theta)v - 2(1-\beta)(\nabla\theta\cdot\nabla)v - (1-\beta)\Delta\theta v \\ - \beta(1-\beta)(v\cdot\nabla)((1-\theta)v) - (1-\beta)^2(\theta v\cdot\nabla)((1-\theta)v). \end{aligned} \tag{2.15}$$

Hence, $\operatorname{supp} g \subset [T_b/3, T_b]\times(\mathfrak{B}\setminus\bar\Omega)$ and $g \in C^\infty([0,T_b]\times\mathfrak{B})$. We define

$$u_N(t) := \beta(t)u(t) + (1-\beta(t))\mathbf{P}_N u(t), \tag{2.16}$$

where $\mathbf{P}_N$ is defined in (2.2). In particular, (2.16) implies (2.5), provided that one sets

$$\begin{aligned} f_N := \dot\beta(u - \mathbf{P}_N u) + (1-\beta)\Big((\mathbf{P}_N u\cdot\nabla)\mathbf{P}_N u - \mathbf{P}_N((u\cdot\nabla)u)\Big) \\ + \beta(1-\beta)\Big((u\cdot\nabla)(\mathbf{P}_N u - u) - (\mathbf{P}_N u\cdot\nabla)(\mathbf{P}_N u - u)\Big) \\ + \beta g + (1-\beta)\mathbf{P}_N g. \end{aligned} \tag{2.17}$$

Let $k \in \mathbb{N}$. Thanks to definition (2.17), there holds (2.6) and (2.7). Indeed, u belongs to $C^0([T_b/3,T_b];H^{k+1}(\Omega))$ and the family $\mathbf{P}_N u$ converges towards u in this space. From (2.16), at the final time, one has

$$u_N(T_b) = \mathbf{P}_N u(T_b) = \mathbf{P}_N(\theta v(T_b)) = \mathbf{P}_N(\theta\widetilde{u}_a). \tag{2.18}$$

In particular, we deduce from (2.18) that $u_N(T_b)$ is "entire" in x so that, for any $\rho_b > 0$, there exists $C_b > 0$ such that (2.11) holds. We also deduce from (2.18) that $u_N(T_b) \to \theta\widetilde{u}_a$ in $H^k(\mathfrak{B})$, which implies (2.8), (2.9) and (2.10).

To obtain (2.12), we change slightly the definition (2.2) of $\mathbf{P}_N$. Instead of a rectangular window filter, we define $\mathbf{P}_N$ as the Fourier multiplier W_N, where $W_N \in C^\infty(\mathbb{R};[0,1])$ is such that $W_N(\xi) = 1$ for $\xi \in [-N+1, N-1]$ and $W_N(\xi) = 0$ when $|\xi| \geq N$. This preserves the property (2.4) but has a better behavior with respect to L^1 norms in space. Indeed, if $\phi \in \mathcal{S}(\mathbb{R},\mathbb{R})$, one checks that $W_N\phi \in L^1(\mathbb{R})$. This property implies (2.12) because $u(T_b) = \theta v(T_b)$ has a compact support. □

2.3 Proof of the regularization proposition

We turn to the proof of Proposition 1.6.

Let $T, \rho_b, \eta_b > 0$ and $k \in \mathbb{N}$. Let $u_* \in L^2_{\mathrm{div}}(\Omega)$ satisfying $\int_\Omega u_*\cdot e_x = 0$. We start by extending u_* into $u_a \in L^2_{\mathrm{div}}(\mathfrak{B})$ such that $\|(u_a)_{|\mathfrak{B}\setminus\Omega}\|_{L^2(\mathfrak{B}\setminus\Omega)} \leq \eta_b/10$. For $T_b \in (0,T)$ small enough, the free solution starting from u_a at time T_b, say $\widetilde{u}_a$ satisfies $\|(\widetilde{u}_a)_{|\mathfrak{B}\setminus\bar\Omega}\|_{L^2(\mathfrak{B}\setminus\bar\Omega)} \leq \eta_b/5$ and $\|(\widetilde{u}_a)_{|\Omega} - u_*\|_{L^2(\Omega)} \leq \eta_b/5$.

We choose N large enough such that (2.8), (2.9) and (2.10) imply that $u_b := u_N(T_b)$, where the family (u_N, f_N) is given by Lemma 2.1, satisfies (1.10) and such that (2.6) ensures (1.9). Estimates (2.11) and (2.12) prove (1.11) and (1.12). This concludes the proof of Proposition 1.6, provided that we define $f_g := (f_N)_{|\Omega}$ and $f_c := (f_N)_{|\mathfrak{B}\setminus\Omega}$, each being smooth within its support.

3 Strategy for global approximate controllability

We explain our strategy to prove Proposition 1.7. Let $T > 0$, $u_b \in L^2_{\mathrm{div}}(\mathfrak{B})$, $\delta \in (0, \frac{1}{2})$ and $k \in \mathbb{N}$. We intend to construct a family of approximate trajectories depending on a small parameter $0 < \varepsilon \ll 1$ and driving u_b approximately to zero. We detail the construction of this family in the following paragraphs. Then, we prove estimates on boundary layer terms for these approximate trajectories in Section 4. We prove estimates on the remainder in Section 5 and postpone analytic-type estimates for these approximate trajectories to Section 6.

3.1 Small-time to small-viscosity scaling

Let $\kappa \in (0, 1)$. Although it might seem like a further complication, our strategy is based on trying to control the system (1.7) at an even shorter time scale, $\varepsilon^{1-\kappa}T$, passing through intermediate states (velocities) of order $1/\varepsilon$. For $\varepsilon \in (0, 1)$, we introduce the trajectories

$$U^\varepsilon(t, x, y) := u^\varepsilon(t/\varepsilon, x, y)/\varepsilon, \quad P^\varepsilon(t, x, y) := p^\varepsilon(t/\varepsilon, x, y)/\varepsilon^2, \tag{3.1}$$

$$F^\varepsilon_g(t, x, y) := f^\varepsilon_g(t/\varepsilon, x, y)/\varepsilon^2 \quad \text{and} \quad F^\varepsilon_c(t, x, y) := f^\varepsilon_c(t/\varepsilon, x, y)/\varepsilon^2. \tag{3.2}$$

The tuples $(U^\varepsilon, P^\varepsilon, F^\varepsilon_c, F^\varepsilon_g)$ define solutions to (1.7) with initial data u_b if and only if the new unknowns $(u^\varepsilon, p^\varepsilon, f^\varepsilon_c, f^\varepsilon_g)$ are solutions to the rescaled system

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)\, u^\varepsilon + \nabla p^\varepsilon - \varepsilon \Delta u^\varepsilon = f^\varepsilon_g + f^\varepsilon_c & \text{in } (0, T/\varepsilon^\kappa) \times \mathfrak{B}, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } (0, T/\varepsilon^\kappa) \times \mathfrak{B}, \\ u^\varepsilon = 0 & \text{on } (0, T/\varepsilon^\kappa) \times \partial\mathfrak{B}, \\ u^\varepsilon|_{t=0} = \varepsilon u_b & \text{in } \mathfrak{B}. \end{cases} \tag{3.3}$$

Observe the three differences between (3.3) and the original system (1.7):

- the Laplace term has a small factor ε in front of it rather than 1,
- the system is set on the long time interval $(0, T/\varepsilon^\kappa)$ rather than $(0, T)$,
- the initial data is εu_b rather than u_b.

We construct approximate solutions to (3.3) in the following paragraph.

3.2 Return method ansatz

We introduce the following explicit approximate solution to (3.3):

$$u^\varepsilon_{\mathrm{app}}(t, x, y) := u^0(t) + \chi(y) v^0\left(t, \varphi(y)/\sqrt{\varepsilon}\right) + \varepsilon u^1(t, x, y) + \varepsilon^2 w^\varepsilon(t, y), \tag{3.4}$$

$$p^\varepsilon_{\mathrm{app}}(t, x, y) := p^0(t, x), \tag{3.5}$$

$$f^\varepsilon_c(t, x, y) := \varepsilon f^1_{|\mathfrak{B}\setminus\Omega}(t, x, y), \tag{3.6}$$

$$f^\varepsilon_g(t, x, y) := \varepsilon f^1_{|\Omega}(t, x, y). \tag{3.7}$$

In the following lines, we define each of the three terms involved in this approximate solution. We refer to Section 3.4 for comments on the choice of these profiles.

Base Euler flow profile. Let $n \in \mathbb{N}$ satisfying $n \geq 3$ and

$$n > \frac{3}{4}\left(\frac{1}{\kappa} - 1\right). \tag{3.8}$$

Let h in $C^\infty(\mathbb{R}_+, \mathbb{R})$ be such that

$$\text{supp } h \subset (0, T/3] \cup [2T/3, T), \tag{3.9}$$

$$\int_0^{T/3} h(t)\mathrm{d}t = 2L, \tag{3.10}$$

$$\int_0^T t^k h(t)\mathrm{d}t = 0 \quad \text{for} \quad 0 \leq k < n. \tag{3.11}$$

We define

$$u^0(t) := h(t)e_x \quad \text{and} \quad p^0(t,x) := -\dot{h}(t)x. \tag{3.12}$$

For a function $a \in L^2(\mathfrak{B})$, we will denote its translation along the base flow h by

$$(\tau_h a)(t,x,y) := a\left(x - \int_0^t h(s)\,\mathrm{d}s, y\right). \tag{3.13}$$

Boundary layer profile. Let $\varphi \in C^\infty([-1,1],[0,1])$ such that $\varphi(\pm 1) = 0$ and $|\varphi'(y)| = 1$ for $|y| \geq 1/4$. Let $\chi \in C^\infty([-1,1],[0,1])$ such that $\chi(y) = 1$ for $|y| \geq 2/3$, and $\chi(y) = 0$ for $|y| \leq 1/3$. Let $V(t,z)$ be the solution to

$$\begin{cases} \partial_t V - \partial_{zz} V = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ V(t,0) = h(t) & \text{on } \mathbb{R}_+, \\ V(0,z) = 0 & \text{in } \mathbb{R}_+. \end{cases} \tag{3.14}$$

We define

$$v^0(t,z) := -V(t,z)e_x. \tag{3.15}$$

In the sequel, for any function $\mathcal{V}(t,z)$, depending on the fast variable, we will denote its evaluation at $z = \varphi(y)/\sqrt{\varepsilon}$ by

$$\{\mathcal{V}\}(t,y) := \mathcal{V}\left(t, \frac{\varphi(y)}{\sqrt{\varepsilon}}\right). \tag{3.16}$$

Linearized Euler flow profile. Let $\beta \in C^\infty(\mathbb{R}_+, [0,1])$ non-increasing such that $\beta(t) = 1$ for $t \leq T/3$ and $\beta(t) = 0$ for $t \geq 2T/3$. Let $\chi_\delta \in C^\infty([-1,1],[0,1])$ such that $\chi_\delta(y) = 1$ for $|y| \leq 1 - 2\delta$ and $\chi_\delta(y) = 0$ for $|y| \geq 1 - \delta$. We define the stream function associated with u_b, then u^1 and eventually the force f^1.

$$\psi_b(x,y) := -\int_{-1}^y u_b(x,y') \cdot e_x \,\mathrm{d}y', \tag{3.17}$$

$$u^1(t,x,y) := \beta(t)\tau_h \nabla^\perp[\chi_\delta \psi_b] + \tau_h \nabla^\perp[(1-\chi_\delta)\psi_b], \tag{3.18}$$

$$f^1(t,x,y) := \dot{\beta}(t)\left(\chi_\delta(y) u_b(x-2L,y) - \chi_\delta'(y)\psi_b(x-2L,y)e_x\right). \tag{3.19}$$

Technical profile. For $t \in \mathbb{R}_+$ and $y \in [-1,1]$, we define the source

$$f_W^\varepsilon := \frac{\chi''}{\varphi^4}\{z^4 V\} + 2\frac{\chi'}{\varphi^5}\varphi'\{z^5 \partial_z V\}. \tag{3.20}$$

Let $W^\varepsilon(t,y) : \mathbb{R}_+ \times [-1,1] \to \mathbb{R}$ be the solution to

$$\begin{cases} \partial_t W^\varepsilon - \varepsilon \partial_{yy} W^\varepsilon = f_W^\varepsilon & \text{in } \mathbb{R}_+ \times [-1,1], \\ W^\varepsilon(t,\pm 1) = 0 & \text{on } \mathbb{R}_+, \\ W^\varepsilon(0,y) = 0 & \text{in } [-1,1]. \end{cases} \tag{3.21}$$

Finally, we let

$$w^\varepsilon(t,y) := W^\varepsilon(t,y) e_x. \tag{3.22}$$

Equation satisfied by the approximate trajectories. Then $(u_{\text{app}}^\varepsilon, p_{\text{app}}^\varepsilon)$ are solutions to

$$\begin{cases} \partial_t u_{\text{app}}^\varepsilon + \left(u_{\text{app}}^\varepsilon \cdot \nabla\right) u_{\text{app}}^\varepsilon - \varepsilon \Delta u_{\text{app}}^\varepsilon + \nabla p_{\text{app}}^\varepsilon = f_c^\varepsilon + f_g^\varepsilon + \varepsilon f_{\text{app}}^\varepsilon & \text{in } (0, T/\varepsilon^\kappa) \times \mathfrak{B}, \\ \operatorname{div} u_{\text{app}}^\varepsilon = 0 & \text{in } (0, T/\varepsilon^\kappa) \times \mathfrak{B}, \\ u_{\text{app}}^\varepsilon = 0 & \text{on } (0, T/\varepsilon^\kappa) \times \partial\mathfrak{B}, \\ u_{\text{app}}^\varepsilon|_{t=0} = \varepsilon u_b & \text{in } \mathfrak{B}, \end{cases} \tag{3.23}$$

where we define

$$\begin{aligned} f_{\text{app}}^\varepsilon := &- \varepsilon \Delta u^1 + \varepsilon (u^1 \cdot \nabla) u^1 + \varepsilon^2 W^\varepsilon \partial_x u^1 + \varepsilon^2 (u^1 \cdot e_y) \partial_y w^\varepsilon \\ &- \chi\{V\} \partial_x u^1 - \frac{u^1 \cdot e_y}{\varphi}\left(\sqrt{\varepsilon}\chi'\{zV\} + \chi\varphi'\{z\partial_z V\}\right) e_x. \end{aligned} \tag{3.24}$$

3.3 Estimates and proof of approximate controllability

By construction, the approximate trajectory will be small at the final time.

Proposition 3.1. *There exists a constant $C_{\text{app}} > 0$ such that, for $\varepsilon > 0$ small enough,*

$$\begin{aligned} &\frac{1}{\varepsilon}\|u_{\text{app}}^\varepsilon(T/\varepsilon^\kappa)_{|\Omega}\|_{L^2(\mathfrak{B})} \\ &\qquad \le C_{\text{app}}\left(\varepsilon^{\frac{1}{4}} + \varepsilon^{\kappa(n - \frac{3}{4}(\frac{1}{\kappa} - 1))}|\ln \varepsilon|^{n+\frac{3}{4}}\right) + \|u_{b|\{|y| \ge 1-2\delta\}}\|_{L^2(\mathfrak{B})}. \end{aligned} \tag{3.25}$$

Moreover, the approximate trajectory can be arbitrarily close to a true trajectory. Indeed, we can construct a remainder which is small, provided that the initial data u_b is sufficiently regular (its tangential analytic radius is large enough).

Proposition 3.2. *There exists $\rho_b > 0$, depending only on T such that, if u_b satisfies* (1.11) *and* (1.12) *for some $C_b > 0$, there exists $C_r > 0$ such that, for $\varepsilon > 0$ small enough, there*

exists a weak Leray solution $r^\varepsilon \in C^0([0,T/\varepsilon^\kappa], L^2_{\mathrm{div}}(\mathfrak{B})) \cap L^2((0,T/\varepsilon^\kappa), H^1(\mathfrak{B}))$ *to*

$$\begin{cases} \partial_t r^\varepsilon + \left(u^\varepsilon_{\mathrm{app}} \cdot \nabla\right) r^\varepsilon + \varepsilon \left(r^\varepsilon \cdot \nabla\right) r^\varepsilon + \left(r^\varepsilon \cdot \nabla\right) u^\varepsilon_{\mathrm{app}} \\ \qquad\qquad -\varepsilon \Delta r^\varepsilon + \nabla \pi^\varepsilon = -f^\varepsilon_{\mathrm{app}} & \text{in } (0,T/\varepsilon^\kappa) \times \mathfrak{B}, \\ \qquad\qquad \operatorname{div} r^\varepsilon = 0 & \text{in } (0,T/\varepsilon^\kappa) \times \mathfrak{B}, \\ \qquad\qquad r^\varepsilon = 0 & \text{on } (0,T/\varepsilon^\kappa) \times \partial\mathfrak{B}, \\ \qquad\qquad r^\varepsilon|_{t=0} = 0 & \text{in } \mathfrak{B}, \end{cases} \tag{3.26}$$

which moreover satisfies

$$\|r^\varepsilon\|_{L^\infty((0,T/\varepsilon^\kappa);L^2(\mathfrak{B}))} \leq C_r(\varepsilon^{\frac{1}{4}} + \varepsilon^{1-\kappa}). \tag{3.27}$$

It is straightforward to check that Proposition 3.1 and Proposition 3.2 imply Proposition 1.7. Indeed, let ρ_b be given by Proposition 3.2 and assume that u_b satisfies (1.11) and (1.12) for some $C_b > 0$. We choose $\varepsilon > 0$ small enough such that the conclusions of both propositions hold. We construct an exact trajectory by setting

$$u^\varepsilon := u^\varepsilon_{\mathrm{app}} + \varepsilon r^\varepsilon \quad \text{and} \quad p^\varepsilon := p^\varepsilon_{\mathrm{app}} + \varepsilon \pi^\varepsilon. \tag{3.28}$$

Combining (3.28) with the equation (3.23) satisfied by $u^\varepsilon_{\mathrm{app}}$ and the equation (3.26) satisfied by r^ε proves that u^ε is a weak solution to (3.3).

Let $\sigma > 0$. Choosing $\varepsilon > 0$ small enough, summing estimates (3.25) and (3.27) and recalling the definition (3.28) of u^ε, the assumption (3.8) on n and the scaling (3.1) proves that (1.16) holds at the time $T_c := \varepsilon^{1-\kappa} T < T$.

Since u_b satisfies (1.11), $u_b \in C^\infty(\mathfrak{B})$. Thus, thanks to (3.2), (3.6), (3.7) and (3.19), $F_g \in C^\infty([0,T_c] \times \bar{\Omega})$, $F_c \in C^\infty([0,T_c] \times \mathfrak{B} \setminus \bar{\Omega})$ and moreover

$$\operatorname{supp} F_g \subset (0,T_c) \times [0,L] \times [-1+\delta, 1-\delta], \tag{3.29}$$

$$\operatorname{supp} F_c \subset (0,T_c) \times \mathfrak{B} \setminus \bar{\Omega}, \tag{3.30}$$

Moreover, using (3.2), (3.7) and (3.19), one has

$$\begin{aligned} \|F^\varepsilon_g(t)\|_{L^1([0,T_c];H^k(\Omega))} &= \frac{1}{\varepsilon^2} \|f^\varepsilon_g(t/\varepsilon)\|_{L^1([0,\varepsilon^{1-\kappa}T];H^k(\Omega))} \\ &= \frac{1}{\varepsilon} \|f^\varepsilon_g(t)\|_{L^1([0,T/\varepsilon^\kappa];H^k(\Omega))} \\ &= \|f^1_{|\Omega}\|_{L^1([0,T/\varepsilon^\kappa];H^k(\Omega))} \\ &= \|\chi_\delta u_b + \chi'_\delta \psi_b\|_{H^k(G)}, \end{aligned} \tag{3.31}$$

where we recall that the set G is defined in (1.8). This proves the estimate (1.13) concerning the size of the phantom force, for a constant $C_{k,\delta}$ which only depends on the norm of χ_δ in $H^{k+1}(-1,1)$, and thus concludes the proof of the approximate controllability result Proposition 1.7.

We prove Proposition 3.1 in Section 4 (thanks to the well-prepared dissipation method) and Proposition 3.2 in Section 5 (using a long-time nonlinear Cauchy-Kovalevskaya estimate).

3.4 Comments and insights on the proposed expansion

Remark 3.3 (Return method and base Euler flow)**.** *Since system* (3.3) *can be seen as a perturbation of the Euler equations a natural idea is to follow the return method introduced by Coron in [5] (see also [7, Chapter 6]) to prove the controllability of the Euler equations in the 2D case (see also [6], and [16] for the 3D case). Loosely speaking the idea is to overcome that the linearized problem around zero is not controllable by introducing, thanks to the boundary control, a velocity* u^0 *of order* $O(1)$ *(whereas the initial velocity is only of order* $O(\varepsilon)$*) solution to the Euler equation satisfying* $u^0|_{t=0} = u^0|_{t=T} = 0$ *and such that the corresponding flow flushes all the domain out during the time interval* $(0, T)$*. In the present case of a rectangle this step is pretty easy and explicit: it corresponds to the introduction of a flow which flushes out the initial data. From* (3.12)*, we get that* (u^0, p^0) *indeed solves the incompressible Euler equation:*

$$\begin{cases} \partial_t u^0 + \left(u^0 \cdot \nabla\right) u^0 = -\nabla p^0, & \text{in } \mathbb{R}_+ \times \mathfrak{B}, \\ \operatorname{div} u^0 = 0 & \text{in } \mathbb{R}_+ \times \mathfrak{B}, \\ u^0 \cdot e_y = 0 & \text{on } \mathbb{R}_+ \times \partial\mathfrak{B}, \end{cases} \tag{3.32}$$

with initial data $u^0(0) = 0$ *and* $u^0(t) = 0$ *for* $t \geq T$*.*

Remark 3.4 (Transport of the initial data)**.** *The term* u^1 *takes into account the initial data* u_b*, which is transported by the flow* u^0*. Using* (3.12)*,* (3.18) *and* (3.19)*, we obtain that* u^1 *solves*

$$\begin{cases} \partial_t u^1 + h(t)\partial_x u^1 = f^1 & \text{in } \mathbb{R}_+ \times \mathfrak{B}, \\ \operatorname{div} u^1 = 0 & \text{in } \mathbb{R}_+ \times \mathfrak{B}, \\ u^1 = 0 & \text{on } \mathbb{R}_+ \times \partial\mathfrak{B}, \\ u^1(0) = u_b & \text{in } \mathfrak{B}. \end{cases} \tag{3.33}$$

Thanks to assumption (3.10)*, it is clear that the initial data will be flushed outside of the domain at time* $T/3$*. During the time interval* $[T/3, 2T/3]$*, the initial data* u_b *has been shifted towards the right of a distance* $2L$*. This is the time interval during which the force* f^1 *kills most of the initial data (for* $|y| \leq 1 - \delta$*).*

The key point is that, outside of the physical domain, this force is merely a control. However, since we need this force to be analytic, it also acts a little bit within the physical domain. This gives rise to an unwanted phantom force.

Remark 3.5 (Boundary layer correction)**.** *A major difficulty is linked to the discrepancy between the Euler and the Navier-Stokes equations in the vanishing viscosity limit. Indeed, although inertial forces prevail inside the domain, viscous forces play a crucial role near the uncontrolled boundary, and give rise to a boundary layer of order* $O(1)$ *associated with the velocity* u^0 *which does not satisfy the tangential part of the Dirichlet condition on the top and bottom boundaries.*

The purpose of the second term v^0 *is to recover the Dirichlet boundary condition by introducing the boundary layer generated by* u^0*. Thanks to our previous choice of* u^0 *we will avoid the difficulty usually associated with the Prandtl equation. Indeed the boundary layer will also be fully horizontal (tangential) and will not depend on* x *so that the equation for* v^0 *will deplete into a linear heat equation with non-homogeneous Dirichlet data depending*

on u^0. The quantity $\varphi(y)/\sqrt{\varepsilon}$ reflects quick variations within the boundary layer, where $\varphi(y)$ is the distance to the boundary.

4 Well-prepared dissipation method for the boundary layer

The key argument of the well-prepared dissipation method is that the normal dissipation involved in fluid mechanics boundary layer equations can dissipate most of their energy, provided that the created boundary layers are "well-prepared" in some sense. Roughly speaking, this preparation amounts to ensure that they do not contain energy at low frequencies.

4.1 Large time decay of the boundary layer profile

In the work [10] concerning the case of the Navier slip-with-friction boundary condition, we used boundary controls to import enough vanishing moments thanks to the transport by the Euler flow within the boundary layer. In this work, we cannot use this strategy because we do not want the boundary layer profile to depend on the slow tangential variable, see Remark 3.5. Instead we rely on the assumptions (3.11) on the base Euler flow. We prove below that these conditions entail a good decay for the boundary layer profile. This decay will be used both to prove that the source terms generated by v in equation (3.26) for the remainder are integrable with respect to time and that the boundary profile at the final time is small enough to apply a local controllability result. For $s, m \in \mathbb{N}$ and I an interval of $\mathbb{R}$, we introduce the following weighted Sobolev spaces:

$$H^{s,m}(I) := \left\{ f \in H^s(I),\ \sum_{\alpha=0}^{s} \int_I \left(1+z^2\right)^m \left|f^{(\alpha)}(z)\right|^2 \mathrm{d}z < +\infty \right\}, \tag{4.1}$$

which we endow with their natural norm. We will use this definition with $I = \mathbb{R}$ or $I = \mathbb{R}_+$.

Lemma 4.1. *Let $T > 0$, $s, n \in \mathbb{N}$ and $h \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying* (3.9) *and* (3.11)*. We consider V the solution to* (3.14)*. For any $0 \le m \le 2n+1$, there exists a constant C such that the following estimate holds:*

$$|V(t,\cdot)|_{H^{s,m}(\mathbb{R}_+)} \le C \left| \frac{\ln(2+t)}{2+t} \right|^{\frac{1}{4}+\frac{2n+1}{2}-\frac{m}{2}}. \tag{4.2}$$

Proof. Estimate (4.2) is straightforward up to time T because its right-hand side is bounded from below for $t \in [0, T]$. Thus, we focus on large time estimates. We start by explicit computations in the frequency domain using Fourier transform. We consider the auxiliary system

$$\begin{cases} \partial_t f - \partial_{zz} f = (h(t) - \dot{h}(t)) \cdot \mathrm{sgn}(z) e^{-|z|}, & t \ge 0, \quad z \in \mathbb{R}, \\ \quad f(0,z) = 0, & t = 0, \quad z \in \mathbb{R}. \end{cases} \tag{4.3}$$

Since the source term in (4.3) is odd, its unique solution f satisfies $f(t,0) = 0$ for all $t \in \mathbb{R}_+$. Hence, thanks to the uniqueness property for the heat equation on the half-line, there holds $V(t,z) = f(t,z) + h(t)e^{-z}$ for $t, z \ge 0$ because both sides of this equality solve

the same heat equation. Therefore, proving estimates on f will provide estimates on V. After Fourier transform and solving the ODE, we obtain the formula:

$$\hat{f}(t,\zeta) := \int_{\mathbb{R}} f(t,z)e^{-i\zeta z}\mathrm{d}z = -\frac{2i\zeta}{1+\zeta^2}\int_0^t e^{-(t-s)\zeta^2}\Big(h(s)-\dot{h}(s)\Big)\,\mathrm{d}s. \tag{4.4}$$

Since h vanishes after T (see (3.9)), the behavior of f (and thus V) after time T is entirely determined by the "initial" data $f_T(z) := f(T,z)$. Thanks to [10, Lemma 6], to establish (4.2), it suffices to check that, for $0 \le j \le 2n$,

$$\partial_\zeta^j \hat{f}_T(0) = 0. \tag{4.5}$$

Thanks to (4.4) and to the Leibniz rule, for $j \in \mathbb{N}$, one has:

$$\partial_\zeta^j \hat{f}_T(\zeta) = -i\sum_{k=0}^{j}\binom{j}{k}\partial_\zeta^{j-k}\left\{\frac{2\zeta}{1+\zeta^2}\right\}\int_0^T\Big(h(t)-\dot{h}(t)\Big)\,\partial_\zeta^k\left\{e^{-(T-t)\zeta^2}\right\}\mathrm{d}t. \tag{4.6}$$

First, since $\zeta \mapsto 2\zeta/(1+\zeta^2)$ is an odd function, only its odd derivatives don't vanish at zero. Second, thanks to the Arbogast rule for the iterated differentiation of composite functions (also known as Faà di Bruno's formula), one has:

$$\partial_\zeta^k\left\{e^{-(T-t)\zeta^2}\right\} = \sum_{m_1+2m_2=k}\frac{k!}{m_1!m_2!}\left(\frac{-2\zeta(T-t)}{1!}\right)^{m_1}\left(\frac{-2(T-t)}{2!}\right)^{m_2}e^{-(T-t)\zeta^2}. \tag{4.7}$$

Hence, this derivative is non null at zero only if k is even, say $k = 2k'$ and the only non-vanishing term in the right-hand side of (4.7) is the one corresponding to $(m_1, m_2) = (0, k')$ and is proportional to $(T-t)^{k'}$. From (4.6) and (4.7) we deduce that $\partial_\zeta^j \hat{f}_T(0)$ is a linear combination of the moments

$$\int_0^T\Big(h(t)-\dot{h}(t)\Big)(T-t)^{k'}\mathrm{d}t, \tag{4.8}$$

where $0 \le 2k' \le j-1$. Thanks to (3.9) and (3.11), the integrals (4.8) vanish for $0 \le k' < n$. So (4.5) holds for $j \le 2n-1$. Last, (4.5) also holds for $j = 2n$ because, when j is even, all the terms vanish. Indeed, in (4.6), either k is odd or $j-k = 2n-k$ is even. This concludes the proof of the lemma. □

4.2 Fast variable scaling and Lebesgue norms

Let us prove the following lemma, which is a simpler version of [19, Lemma 3, page 150].

Lemma 4.2. *Let $\gamma \in C^0([-1,1])$ with $\gamma \equiv 0$ on $\left(-\frac{1}{3},\frac{1}{3}\right)$. For $\mathcal{V} \in L^2(\mathbb{R}_+)$ and $\varepsilon > 0$:*

$$\|\gamma\{\mathcal{V}\}\|_{L^2(-1,1)} \le 2\varepsilon^{\frac{1}{4}}\|\gamma\|_\infty\|\mathcal{V}\|_{L^2(\mathbb{R}_+)}. \tag{4.9}$$

Proof. For $-1 \le y \le -\frac{1}{4}$, we assumed $\varphi' = 1$. Thus, $\varphi(y) = 1+y$. Recalling the fast variable notation (3.16) and performing an affine change of variables gives

$$\int_{-1}^{-\frac{1}{3}}\gamma^2(y)\mathcal{V}^2\left(\frac{\varphi(y)}{\sqrt{\varepsilon}}\right)\mathrm{d}y = \sqrt{\varepsilon}\int_0^{\frac{2}{3\sqrt{\varepsilon}}}\gamma^2(\sqrt{\varepsilon}z-1)\mathcal{V}^2(z)\,\mathrm{d}z \le \sqrt{\varepsilon}\|\gamma\|_\infty^2\|\mathcal{V}\|_{L^2(\mathbb{R}_+)}^2. \tag{4.10}$$

Proceeding likewise for $\frac{1}{3} \le y \le 1$ and bounding $\sqrt{2}$ by 2 yields (4.9), □

4.3 Estimates for the technical profile

Lemma 4.3. *Assume that* (3.11) *holds for some* $n \geq 3$. *There exists* C_W *such that, for every* $\varepsilon \in (0,1)$, *the solution* W^ε *to* (3.21) *satisfies, for every* $t \geq 0$,

$$\|W^\varepsilon(t)\|_{L^\infty(-1,1)} + \|\partial_y W^\varepsilon(t)\|_{L^\infty(-1,1)} \leq \varepsilon^{-\frac{3}{4}} C_W. \tag{4.11}$$

Proof. Differentiating (3.21) with respect to time, multiplying by $\partial_t W^\varepsilon$ and integrating by parts, we obtain the energy estimate

$$\|\partial_t W^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(-1,1))} \leq 2\|\partial_t f_W^\varepsilon\|_{L^1(\mathbb{R}_+;L^2(-1,1))}. \tag{4.12}$$

Plugging this estimate in the equation (3.21) yields

$$\|\partial_{yy} W^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(-1,1))} \leq \frac{1}{\varepsilon}\left(\|f_W^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(-1,1))} + 2\|\partial_t f_W^\varepsilon\|_{L^1(\mathbb{R}_+;L^2(-1,1))}\right), \tag{4.13}$$

Thanks to estimate (4.9) from Lemma 4.2 applied to the definition (3.20) of f_W^ε, we obtain, for $t \geq 0$,

$$\begin{aligned} \|f_W^\varepsilon(t)\|_{L^2(-1,1)} &\leq 2\varepsilon^{\frac{1}{4}} \|\chi''/\varphi^4\|_\infty \|z^4 V(t,z)\|_{L^2(\mathbb{R}_+)} \\ &\quad + 2\varepsilon^{\frac{1}{4}} \|2\chi'\varphi'/\varphi^5\|_\infty \|z^5 \partial_z V(t,z)\|_{L^2(\mathbb{R}_+)} \\ &\leq C\varepsilon^{\frac{1}{4}} \|V(t)\|_{H^{1,5}(\mathbb{R}_+)}, \end{aligned} \tag{4.14}$$

where C is a finite constant because, by construction, χ' and χ'' vanish for $|y| \geq \frac{2}{3}$, so that the division by φ which vanishes at $y = \pm 1$ is not singular. Proceeding similarly and using the equation (3.14) on V, we obtain

$$\|\partial_t f_W^\varepsilon(t)\|_{L^2(-1,1)} \leq C\varepsilon^{\frac{1}{4}} \|V(t)\|_{H^{3,5}(\mathbb{R}_+)}. \tag{4.15}$$

Combining (4.14) with Lemma 4.1 applied to $m = 5$, $n = 3$, $s = 1$, we obtain

$$\|f_W^\varepsilon(t)\|_{L^2(-1,1)} \leq C\varepsilon^{\frac{1}{4}} \left|\frac{\ln(2+t)}{2+t}\right|^{\frac{5}{4}} \leq C\varepsilon^{\frac{1}{4}} \tag{4.16}$$

Combining (4.15) with Lemma 4.1 applied to $m = 5$, $n = 3$, $s = 3$, we obtain

$$\|\partial_t f_W^\varepsilon\|_{L^1(\mathbb{R}_+;L^2(-1,1))} \leq C\varepsilon^{\frac{1}{4}} \int_0^{+\infty} \left|\frac{\ln(2+t)}{2+t}\right|^{\frac{5}{4}} \mathrm{d}t \leq 26C\varepsilon^{\frac{1}{4}}. \tag{4.17}$$

Eventually, plugging (4.16) and (4.17) into (4.13) proves (4.11) thanks to the boundary conditions $W^\varepsilon(t, \pm 1) = 0$ and the Poincaré-Wirtinger inequality for $\partial_y W^\varepsilon$. □

4.4 Proof of the decay of approximate trajectories

We prove Proposition 3.1. Recalling the definition (3.4) of $u_{\text{app}}^\varepsilon$, we estimate the size of each term at the time T/ε^κ.

- Thanks to (3.9) and (3.12), $u^0(T/\varepsilon^\kappa) = 0$.

- Thanks to (4.9) from Lemma 4.2 and (4.2) from Lemma 4.1, there holds

$$\begin{aligned}\|\chi\{v^0(T/\varepsilon^\kappa)\}\|_{L^2_y} &\leq 2\varepsilon^{\frac{1}{4}}\|\chi\|_\infty\|V(T/\varepsilon^\kappa)\|_{L^2(\mathbb{R}_+)}\\ &\leq 2\varepsilon^{\frac{1}{4}}C\left|\frac{\ln(2+T/\varepsilon^\kappa)}{2+T/\varepsilon^\kappa}\right|^{\frac{3}{4}+n}\\ &\leq \tilde{C}\varepsilon^{1+\kappa(n-\frac{3}{4}(\frac{1}{\kappa}-1))}|\ln\varepsilon|^{n+\frac{3}{4}},\end{aligned} \tag{4.18}$$

  for some constant $\tilde{C} > 0$.

- Thanks to (3.18),

$$u^1(T/\varepsilon^\kappa) = \nabla^\perp[(1-\chi_\delta)\psi_b]. \tag{4.19}$$

  Moreover, since u_b satisfies (1.12), $u_b \in L^1(\mathfrak{B})$. In particular, since u_b is divergence-free, this implies that, for all $x \in \mathbb{R}$,

$$\int_{-1}^{+1} u_b(x,y)\cdot e_x \,\mathrm{d}y = 0, \tag{4.20}$$

  so that ψ_b, which was defined as (3.17) can equivalently be written as

$$\psi_b(x,y) = \int_y^1 u_b \cdot e_x \,\mathrm{d}y. \tag{4.21}$$

  Thanks to (4.19), this implies that there exists a constant $C_\delta > 0$ which only depends on the norm of χ_δ in $H^1(-1,1)$ such that

$$\varepsilon\|u^1(T/\varepsilon^\kappa)\|_{L^2(\mathfrak{B})} \leq \varepsilon C_\delta\|u_{b|\{|y|\geq 1-2\delta\}}\|_{L^2(\mathfrak{B})}. \tag{4.22}$$

- Thanks to estimate (4.11) from Lemma 4.3,

$$\varepsilon^2\|w^\varepsilon(T/\varepsilon^\kappa)\|_{L^\infty_y} \leq \varepsilon^{1+\frac{1}{4}}C_W. \tag{4.23}$$

Gathering these estimates concludes the proof of estimate (3.25) of Proposition 3.1.

5 Estimates on the remainder

This section is devoted to the proof of Proposition 3.2. An important difficulty to obtain some uniform energy estimates of r^ε from system (3.26) is that the term $(r^\varepsilon \cdot \nabla)u^\varepsilon_{\text{app}}$ contains a term with a factor $1/\sqrt{\varepsilon}$ due to the fast variation of the boundary layer term in the normal variable (see the expansion (3.4) of $u^\varepsilon_{\text{app}}$). To deal with this difficulty we use a reformulation of this term where the singular factor is traded against a loss of derivative on r^ε in the tangential direction x (see Section 5.1). Then, we establish a long-time nonlinear Cauchy-Kovalevskaya estimate (see Section 5.3) thanks to some tools from Littlewood-Paley theory which are recalled in Section 5.2.

Remark 5.1. *The well-posedness of the Prandtl equations as well as the convergence of the Navier-Stokes equations to the Prandtl equations in the analytic setting dates back to [32, 33, 27]. The seminal results of Caflisch and Sammartino require analyticity in both spatial directions, and only imply well-posedness of the Prandtl equations on a small time interval. Analytic techniques have been later used in [20, 35] to obtain large-time well-posedness for Prandtl equations by requiring analyticity only in the tangential direction.*

5.1 Singular amplification to loss of derivative

On the one hand, we use the expansion (3.4) of $u^\varepsilon_{\rm app}$ to expand

$$(r^\varepsilon \cdot \nabla)\, u^\varepsilon_{\rm app} = \frac{1}{\sqrt{\varepsilon}}\varphi'\chi r^\varepsilon_2\{\partial_z v^0\} + r^\varepsilon_2(\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon) + \varepsilon\,(r^\varepsilon\cdot\nabla)\,u^1. \tag{5.1}$$

Let M be the operator which associates with any function $a \in L^2(-1,1)$, the function $M[a]$ defined for y in $(-1,1)$ by

$$(M[a])(y) := -\chi(y)\int_0^1 a(\pm 1 \mp s(1\mp y))\,\mathrm{d}s, \tag{5.2}$$

where the signs are chosen depending on whether $\pm y \geq 0$. Using the null boundary condition and the divergence-free condition in (3.26) and the fact that $|\varphi'| = 1$ where $\chi \neq 0$, we obtain that the first term in the right-hand side of (5.1) can be recast as

$$\frac{1}{\sqrt{\varepsilon}}\varphi'\chi r^\varepsilon_2\{\partial_z v^0\} = (M[\partial_x r^\varepsilon_1])\{z\partial_z v^0\}. \tag{5.3}$$

On the other hand we decompose the term $(u^\varepsilon_{\rm app}\cdot\nabla)r^\varepsilon$ of (3.26), thanks to (3.4), into

$$\left(u^\varepsilon_{\rm app}\cdot\nabla\right) r^\varepsilon = (h - \chi\{V\}+\varepsilon^2 W^\varepsilon)\partial_x r^\varepsilon + \varepsilon\left(u^1\cdot\nabla\right) r^\varepsilon, \tag{5.4}$$

Thus, using (5.3) and (5.4), the system (3.26) now reads

$$\begin{cases} \partial_t r^\varepsilon + (h-\chi\{V\}+\varepsilon^2 W^\varepsilon)\partial_x r^\varepsilon - \varepsilon\Delta r^\varepsilon + \nabla\pi^\varepsilon = f^\varepsilon_r & \text{in } (0,T/\varepsilon^\kappa)\times\mathfrak{B},\\ \operatorname{div} r^\varepsilon = 0 & \text{in } (0,T/\varepsilon^\kappa)\times\mathfrak{B},\\ r^\varepsilon = 0 & \text{on } (0,T/\varepsilon^\kappa)\times\partial\mathfrak{B},\\ r^\varepsilon|_{t=0} = 0 & \text{on } \mathfrak{B}, \end{cases} \tag{5.5}$$

where we introduce

$$\begin{aligned} -f^\varepsilon_r := f^\varepsilon_{\rm app} &+ (M[\partial_x r^\varepsilon_1])\{z\partial_z v^0\} + r^\varepsilon_2(\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon)\\ &+ \varepsilon\,(r^\varepsilon\cdot\nabla)\,u^1 + \varepsilon\left(u^1\cdot\nabla\right) r^\varepsilon + \varepsilon\,(r^\varepsilon\cdot\nabla)\,r^\varepsilon. \end{aligned} \tag{5.6}$$

5.2 A few tools from Littlewood-Paley theory

To perform analytic estimates, we use Fourier analysis and Littlewood-Paley decomposition. We refer to [1, Chapter 2] for a detailed course on Littlewood-Paley theory. Although all the functions we consider in this section are defined on the band $\mathfrak{B} = \mathbb{R}_x \times [-1,1]_y$, we only perform Fourier analysis and Littlewood-Paley decomposition in the tangential direction $x \in \mathbb{R}_x$. When a confusion is possible, we will use the subscripts x or y to stress the variable involved in the functional spaces.

Dyadic partition of unity. We recall that, for $a \in L^2(\mathfrak{B})$, we defined its Fourier transform $\mathcal{F}a$ in the tangential direction as (2.1). We fix $\chi_{\rm lp}, \varphi_{\rm lp} \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\operatorname{supp} \varphi_{\rm lp} \subset \left\{ \tau \in \mathbb{R};\ \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\}, \tag{5.7}$$

$$\operatorname{supp} \chi_{\rm lp} \subset \left\{ \tau \in \mathbb{R};\ |\tau| \leq \frac{4}{3} \right\}, \tag{5.8}$$

$$\forall \tau \in \mathbb{R}^*, \quad \sum_{j \in \mathbb{Z}} \varphi_{\rm lp}(2^{-j}\tau) = 1, \tag{5.9}$$

$$\forall \tau \in \mathbb{R}, \quad \chi_{\rm lp}(\tau) + \sum_{j \in \mathbb{N}} \varphi_{\rm lp}(2^{-j}\tau) = 1, \tag{5.10}$$

$$\forall \tau \in \mathbb{R}^*, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi_{\rm lp}^2(2^{-j}\tau) \leq 1, \tag{5.11}$$

The existence of such a dyadic partition of unity is proved in [1, Proposition 2.10]. For $k \in \mathbb{Z}$, we introduce the Fourier multipliers $\dot{\Delta}_k$ and $\dot{S}_k$ by defining, for any $a \in L^2(\mathfrak{B})$,

$$\dot{\Delta}_k a := \mathcal{F}^{-1}\left(\varphi_{\rm lp}(2^{-k}\xi) \mathcal{F}a(\xi, y) \right), \tag{5.12}$$

$$\dot{S}_k a := \mathcal{F}^{-1}\left(\chi_{\rm lp}(2^{-k}\xi) \mathcal{F}a(\xi, y) \right). \tag{5.13}$$

The operators $\dot{\Delta}_k$ and $\dot{S}_k$ are with respect to the horizontal variable only. For $a \in L^2(\mathfrak{B})$, one has, thanks to (5.9) and (5.10),

$$\dot{S}_k a = \sum_{j \leq k-1} \dot{\Delta}_j a. \tag{5.14}$$

Homogeneous Besov spaces. For $a \in L^2(\mathfrak{B})$, we will use for $s = 0$ and $s = \frac{1}{2}$ the following quantity corresponding to a homogeneous Besov norm

$$\|a\|_{\dot{B}^s} := \sum_{k \in \mathbb{Z}} 2^{ks} \|\dot{\Delta}_k a\|_{L^2(\mathfrak{B})}. \tag{5.15}$$

Since we will use such norms for functions whose Fourier transforms in x are compactly supported, we do not provide more details on the definition of the corresponding functional spaces, referring for more to [1].

Classical estimates. We recall the following classical estimates, for which we track the constants. First, we will use the following Bernstein type lemma from [1, Lemma 2.1].

Lemma 5.2. *There exists a universal constant $C_{\rm B} \geq 2$ such that the following properties hold. Let $1 \leq p \leq q \leq +\infty$, $\alpha \in \{0, 1\}$, $k \in \mathbb{Z}$ and $a \in L^2(\mathfrak{B})$.*

- *If the support of $\mathcal{F}a$ is included in $\{(\xi, y);\ 2^{-k}|\xi| \leq 100\}$, then*

$$\|\partial_x^\alpha a\|_{L^q_x(L^2_y)} \leq C_{\rm B} 2^{k\left(\alpha + \left(\frac{1}{p} - \frac{1}{q}\right)\right)} \|a\|_{L^p_x(L^2_y)}. \tag{5.16}$$

- *If the support of $\mathcal{F}a$ is included in $\{(\xi, y);\ \frac{1}{100} \leq 2^{-k}|\xi| \leq 100\}$, then*

$$\|a\|_{L^p_x(L^2_y)} \leq C_{\mathrm{B}} 2^{-k\alpha} \|\partial_x^\alpha a\|_{L^p_x(L^2_y)}. \tag{5.17}$$

Lemma 5.3. *Let $a \in H^1_0([-1,1]_y)$. Then*

$$\|a\|_{L^\infty_y} \leq \|a\|^{\frac{1}{2}}_{L^2_y} \|\partial_y a\|^{\frac{1}{2}}_{L^2_y}. \tag{5.18}$$

Proof. This is a classical Gagliardo-Nirenberg interpolation inequality (see [30]). The fact that (5.18) holds with a unit constant for this particular choice of exponents is proved for example in [29, Corollary 5.12] (which in fact yields a constant $2^{-\frac{1}{2}}$). □

As a consequence of Lemma 5.2, we have the following embedding. Indeed this is the main motivation for considering the ℓ^1 norm rather than the ℓ^2 norm in the definition of the homogeneous Besov norms $\dot{B}^s$.

Lemma 5.4. *Let $a \in H^1_0(\mathfrak{B})$. There holds,*

$$\sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \|\dot{\Delta}_k a\|_{L^2_x(L^\infty_y)} \leq C_{\mathrm{B}} \|\nabla a\|_{\dot{B}^0}, \tag{5.19}$$

$$\sum_{k\in\mathbb{Z}} \|\dot{\Delta}_k a\|_{L^\infty(\mathfrak{B})} \leq C_{\mathrm{B}}^2 \|\nabla a\|_{\dot{B}^0}. \tag{5.20}$$

Proof. Let $a \in H^1_0(\mathfrak{B})$. Hence, for almost every $x \in \mathbb{R}_x$, $a(x,\cdot) \in H^1_0([-1,1]_y)$ and we can apply Lemma 5.3. Using (5.18), Cauchy-Schwarz then (5.17) yields

$$\begin{aligned} 2^{\frac{k}{2}} \|\dot{\Delta}_k a\|_{L^2_x(L^\infty_y)} &\leq 2^{\frac{k}{2}} \|\dot{\Delta}_k a\|^{\frac{1}{2}}_{L^2} \|\dot{\Delta}_k \partial_y a\|^{\frac{1}{2}}_{L^2} \\ &\leq C_{\mathrm{B}}^{\frac{1}{2}} \|\dot{\Delta}_k \partial_x a\|^{\frac{1}{2}}_{L^2} \|\dot{\Delta}_k \partial_y a\|^{\frac{1}{2}}_{L^2} \\ &\leq C_{\mathrm{B}}^{\frac{1}{2}} \|\dot{\Delta}_k \nabla a\|_{L^2}, \end{aligned} \tag{5.21}$$

Hence, since $C_{\mathrm{B}} \geq 1$, (5.21) proves (5.19) by the definition (5.15) of the norm $\dot{B}^0$. Moreover, thanks to (5.16),

$$\|\dot{\Delta}_k a\|_{L^\infty_x(L^\infty_y)} \leq C_{\mathrm{B}} 2^{\frac{k}{2}} \|\dot{\Delta}_k a\|_{L^2_x(L^\infty_y)}. \tag{5.22}$$

Gathering (5.19) and (5.22) proves (5.20). □

Lemma 5.5. *Let $a \in H^1_0(\mathfrak{B})$ such that $\operatorname{div} a = 0$. For each $k \in \mathbb{Z}$,*

$$\|\dot{\Delta}_k a_2\|_{L^2_x(L^\infty_y)} \leq C_{\mathrm{B}} 2^{\frac{k}{2}} \|\dot{\Delta}_k a\|_{L^2(\mathfrak{B})}, \tag{5.23}$$

Proof. Let $a \in H^1_0(\mathfrak{B})$. Hence, for almost every $x \in \mathbb{R}_x$, $a(x,\cdot) \in H^1_0([-1,1]_y)$ and we can apply Lemma 5.3. Using (5.18) and Cauchy-Schwarz, we obtain

$$\|\dot{\Delta}_k a_2\|_{L^2_x(L^\infty_y)} \leq \|\dot{\Delta}_k a_2\|^{\frac{1}{2}}_{L^2} \|\dot{\Delta}_k \partial_y a_2\|^{\frac{1}{2}}_{L^2}. \tag{5.24}$$

Then, using that $\operatorname{div} a = 0$ and Lemma 5.2, we observe that

$$\|\dot{\Delta}_k \partial_y a_2\|_{L^2} = \|\dot{\Delta}_k \partial_x a_1\|_{L^2} \leq C_{\mathrm{B}} 2^k \|\dot{\Delta}_k a_1\|_{L^2}. \tag{5.25}$$

Gathering (5.24) and (5.25) proves (5.23) since $C_{\mathrm{B}} \geq 1$. □

Paraproduct decomposition. We shall use the Bony's decomposition (see [2]) for the horizontal variable:

$$fg = \mathbf{T}_f g + \mathbf{T}_g f + \mathbf{R}(f,g), \tag{5.26}$$

where

$$\mathbf{T}_f g := \sum_k \dot{S}_{k-1} f \dot{\Delta}_k g, \tag{5.27}$$

$$\mathbf{R}(f,g) := \sum_k \dot{\Delta}_k f \widetilde{\dot{\Delta}}_k g \tag{5.28}$$

$$\widetilde{\dot{\Delta}}_k g := \sum_{|k-k'|\leq 1} \dot{\Delta}_{k'} g. \tag{5.29}$$

Thanks to the support properties (5.7) of φ_{lp} and (5.8) of χ_{lp}, the following lemma holds.

Lemma 5.6. *For any f, g and h in $L^2(\mathfrak{B})$,*

$$\langle \mathbf{T}_f g, \dot{\Delta}_k h\rangle = \sum_{k'\in\mathbb{Z}/\ |k'-k|\leq 4} \langle (\dot{S}_{k'-1} f)(\dot{\Delta}_{k'} g), \dot{\Delta}_k h\rangle, \tag{5.30}$$

$$\langle \mathbf{R}(f,g), \dot{\Delta}_k h\rangle = \sum_{k'\in\mathbb{Z}/\ k'\geq k-3} \langle (\dot{\Delta}_{k'} f)(\widetilde{\dot{\Delta}}_{k'} g), \dot{\Delta}_k h\rangle. \tag{5.31}$$

Analyticity by Fourier multipliers. Let $|\partial_x|$ denote the Fourier multiplier with symbol $|\xi|$. We associate with any positive C^1 function of time ρ, the operator $e^{\rho|\partial_x|}$ mapping any reasonable function $f(t,x,y)$ (say such that $f \in L^1_{\mathrm{loc}}(L^2_N(\mathfrak{B}))$, for some $N\in\mathbb{N}$), to

$$(e^{\rho|\partial_x|} f)(t,x,y) := \mathcal{F}^{-1}\big(e^{\rho(t)|\xi|}\mathcal{F}f(t,\xi,y)\big)(x). \tag{5.32}$$

Recall that $\mathcal{F}$ denotes the Fourier transform with respect to the tangential variable x, see (2.1). The function ρ describes the evolution of the radius of analyticity of the considered function. Below we establish a long-time Cauchy-Kovalevskaya estimate, for which the function ρ decays in time but not linearly.

Product estimates for analytic functions. For $a\in L^2(\mathfrak{B})$, we introduce the notation

$$a^+ := \mathcal{F}^{-1}|\mathcal{F}a|. \tag{5.33}$$

Lemma 5.7. *Let $N\in\mathbb{N}^*$ and $a,b,c\in L^2_N(\mathfrak{B})$. There holds*

$$\|a^+\|_{L^2} = \|a\|_{L^2}, \tag{5.34}$$

$$\left|\langle e^{\rho|\partial_x|}\,\mathbf{P}_N(ab), c\rangle\right| \leq \left|\langle (e^{\rho|\partial_x|}\,a^+)(e^{\rho|\partial_x|}\,b^+), c^+\rangle\right|. \tag{5.35}$$

Proof. Equality (5.34) is an immediate consequence of the definition (5.33) and Plancherel's theorem. Moreover, by Plancherel's theorem, the normalization (2.1), the triangle inequality and Plancherel's theorem once more, we have that

$$\begin{aligned}\left|\langle e^{\rho|\partial_x|}\,\mathbf{P}_N(ab), c\rangle\right| &= \frac{1}{2\pi}\left|\int_y\int_{|\xi|\leq N}\mathcal{F}c(\xi)e^{\rho|\xi|}\int_\eta \mathcal{F}a(\xi-\eta)\mathcal{F}b(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}y\right| \\ &\leq \frac{1}{2\pi}\int_y\int_{\xi\in\mathbb{R}}|\mathcal{F}c(\xi)|\int_\eta e^{\rho|\xi-\eta|}|\mathcal{F}a(\xi-\eta)|e^{\rho|\eta|}|\mathcal{F}b(\eta)|\,\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}y \\ &= \langle (e^{\rho|\partial_x|}\,a^+)(e^{\rho|\partial_x|}\,b^+), c^+\rangle.\end{aligned} \tag{5.36}$$

This scalar product is positive and this concludes the proof of (5.35). □

5.3 Long-time weakly nonlinear Cauchy-Kovalevskaya estimate

In this paragraph, we explain how we will prove a long-time weakly nonlinear Cauchy-Kovalevskaya estimate on the remainder. We start by defining quantities that will enable us to define the expected profile of analyticity $\rho(t)$. Then, we close the estimate relying on a Grönwall-type argument. In the following paragraphs, we will prove the required estimates.

Remark 5.8. *The idea of closing an estimate on a nonlinear function of the solution to control the loss of analyticity dates back to Chemin in [3]. It was later used in the context of anisotropic Navier-Stokes equations in [4] and, more recently, for Prandtl equations in [35], using only analyticity in the tangential direction.*

5.3.1 Friedrichs' regularization scheme

In order for our manipulations to make sense, we will restrict (5.5) to a bounded range of frequencies. Then, we establish estimates which are independent on the considered range and we pass to the limit. This process was introduced by Friedrichs in [12] (see also [31] for a recent example of the passage to the limit). Let $N \in \mathbb{N}$. Instead of (5.5), we consider the modified equation

$$\begin{cases} \partial_t r_N^\varepsilon + (h - \chi\{V\} + \varepsilon^2 W^\varepsilon)\partial_x r_N^\varepsilon - \varepsilon \Delta r_N^\varepsilon + \nabla \pi_N^\varepsilon = f_N^\varepsilon & \text{in } (0, T/\varepsilon^\kappa) \times \mathfrak{B}, \\ \operatorname{div} r_N^\varepsilon = 0 & \text{in } (0, T/\varepsilon^\kappa) \times \mathfrak{B}, \\ r_N^\varepsilon = 0 & \text{on } (0, T/\varepsilon^\kappa) \times \partial\mathfrak{B}, \\ r_N^\varepsilon|_{t=0} = 0 & \text{on } \mathfrak{B}, \end{cases} \tag{5.37}$$

where we introduce

$$\begin{aligned} -f_N^\varepsilon := \mathbf{P}_N f_{\text{app}}^\varepsilon + (M[\partial_x r_{N,1}^\varepsilon])\{z\partial_z v^0\} + r_{N,2}^\varepsilon(\chi'\{v^0\} + \varepsilon^2 \partial_y w^\varepsilon) \\ + \varepsilon \mathbf{P}_N \left(r_N^\varepsilon \cdot \nabla\right) u^1 + \varepsilon \mathbf{P}_N \left(u^1 \cdot \nabla\right) r_N^\varepsilon + \varepsilon \mathbf{P}_N \left(r_N^\varepsilon \cdot \nabla\right) r_N^\varepsilon. \end{aligned} \tag{5.38}$$

In the sequel, to lighten the notations, we will write r instead of r_N^ε and we will omit the projections $\mathbf{P}_N$. It will be clear from our proof that we perform *a priori* estimates which are independent of N. Therefore, using usual compactness arguments, our proof will also yield the same energy estimate for the initial equation (5.5). Since this argument is quite classical, we will only detail the *a priori* estimates. Even though this regularization process is transparent in the proof, it is necessary to ensure that all the quantities are well defined.

5.3.2 Definition of the analyticity profile

We start by defining the analyticity radius that we will require on the coefficients and the source terms of the equation for the remainder

$$\rho_0 := 2 + 10^2 C_{\mathrm{B}} \int_0^{+\infty} \|z\partial_z v^0(t,z)\|_{L^\infty(\mathbb{R}_+)} \, \mathrm{d}t. \tag{5.39}$$

Recalling the definition (4.1) of the space $H^{2,2}(\mathbb{R}_+)$, one has, for $t \geq 0$,

$$\|z\partial_z V(t,z)\|_{L^\infty(\mathbb{R}_+)} \leq 2\|V(t)\|_{H^{2,2}(\mathbb{R}_+)}. \tag{5.40}$$

Hence, since $n \geq 2$, thanks to the decay estimate (4.2) from Lemma 4.1, $\rho_0 < +\infty$. Up to a normalization constant due to Bernstein-type estimates, this radius corresponds to the total amount of the loss of derivative that we expect. Then, we set, for $t \geq 0$,

$$\alpha_\varepsilon(t) := \varepsilon^\kappa + (1+t^2)^{-1}, \tag{5.41}$$

$$\ell_1(t) := \sum_{k\in\mathbb{Z}} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k \nabla u^1(t)\|_{L^2(\mathfrak{B})}. \tag{5.42}$$

These quantities will help us to control the (non singular but long-time) amplification terms in the evolution of the remainder. We set

$$\beta(t) := \int_0^t \Big(5\|\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon\|_{L^\infty(\mathfrak{B})} + 10^9 C_{\mathrm{B}}^2 \varepsilon \ell_1^2 + 10\alpha_\varepsilon\Big). \tag{5.43}$$

Proposition 5.9 (Proof in Section 6.2)**.** *If u_b satisfies* (1.11) *for a constant $C_b > 0$ and $\rho_b > \rho_0$, there exists $\beta_\star > 0$ such that, for $\varepsilon, \kappa \in (0,1)$,*

$$\sup_{t\in[0,T/\varepsilon^\kappa]} \beta(t) = \beta(T/\varepsilon^\kappa) \leq \beta_\star. \tag{5.44}$$

We consider the local solution $\rho_N(t)$ to the following nonlinear ODE:

$$\begin{cases} \dot{\rho}_N(t) = -10^2 C_{\mathrm{B}} \|z\partial_z v^0(t,z)\|_{L^\infty(\mathbb{R}_+)} - 10^7 C_{\mathrm{B}}^4 \varepsilon e^{\beta_\star} \|e^{\rho_N(t)|\partial_x|-\beta(t)} \nabla r_N^\varepsilon(t)\|_{\dot{B}^0}, \\ \rho_N(0) = \rho_0. \end{cases} \tag{5.45}$$

Since, for almost every t, $r_N^\varepsilon(t) \in L^2_N(\mathfrak{B})$, the right-hand side is Lipschitz continuous with respect to ρ_N (with constants that may depend on N). Hence, we can apply the Cauchy-Lipschitz theorem and consider the maximal solution of (5.45). We set

$$T_N^* := \sup\ \{t \in [0, T/\varepsilon^\kappa];\ \ \rho_N(t) \geq 1\} \tag{5.46}$$

and consider for $t \leq T_N^*$,

$$\mathfrak{r} := e^{\rho_N|\partial_x|-\beta} r_N^\varepsilon. \tag{5.47}$$

In the sequel, we simply write ρ instead of ρ_N and T^* instead of T_N^* and we prove estimates which are uniform with respect to N.

5.3.3 Grönwall-type energy estimate

We start with deducing from (5.37) that:

$$\begin{cases} \partial_t \mathfrak{r} - \dot{\rho}|\partial_x|\mathfrak{r} + \dot{\beta}\mathfrak{r} + (h - \chi\{V\}+\varepsilon^2 W^\varepsilon)\partial_x \mathfrak{r} \\ \qquad\qquad -\varepsilon\Delta\mathfrak{r} + \nabla e^{\rho|\partial_x|-\beta}\pi = e^{\rho|\partial_x|-\beta} f_N^\varepsilon & \text{on } (0, \dfrac{T}{\varepsilon^\kappa}) \times \mathfrak{B}, \\ \qquad\qquad \operatorname{div}\mathfrak{r} = 0 & \text{on } (0, \dfrac{T}{\varepsilon^\kappa}) \times \mathfrak{B}, \\ \qquad\qquad \mathfrak{r} = 0 & \text{on } (0, \dfrac{T}{\varepsilon^\kappa}) \times \Gamma_\pm, \\ \qquad\qquad \mathfrak{r}|_{t=0} = 0 & \text{on } \mathfrak{B}. \end{cases} \tag{5.48}$$

We apply the dyadic operator $\dot{\Delta}_k$ to (5.48) and take the $L^2(\mathfrak{B})$ inner product of the resulting equation with $\dot{\Delta}_k\mathfrak{r}$. We observe, by integration by parts, that the contributions due to the fourth and sixth terms vanish, so that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_k\mathfrak{r}(t)\|_{L^2}^2 - \dot{\rho}\,\langle|\partial_x|\dot{\Delta}_k\mathfrak{r},\dot{\Delta}_k\mathfrak{r}\rangle + \dot{\beta}\|\dot{\Delta}_k\mathfrak{r}(t)\|_{L^2}^2 + \varepsilon\|\nabla\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2 \\ = \langle\dot{\Delta}_k e^{\rho|\partial_x|-\beta}f_N^\varepsilon,\dot{\Delta}_k\mathfrak{r}\rangle.\end{aligned} \tag{5.49}$$

Above and below we simply denote by L^2 the space $L^2(\mathfrak{B})$. Using the definition (5.12) of $\dot{\Delta}_k$ and the support property (5.7) of φ_{lp}, we know that

$$\langle|\partial_x|\dot{\Delta}_k\mathfrak{r},\dot{\Delta}_k\mathfrak{r}\rangle \geq \frac{1}{2}2^k\|\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2. \tag{5.50}$$

Then, integrating over $[0,t]$, we obtain

$$\begin{aligned}\frac{1}{2}\|\dot{\Delta}_k\mathfrak{r}(t)\|_{L^2}^2 + \frac{1}{2}2^k\int_0^t|\dot{\rho}|\,\|\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2 + \int_0^t\dot{\beta}\|\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2 + \varepsilon\int_0^t\|\nabla\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2 \\ \leq \int_0^t\big|\langle\dot{\Delta}_k e^{\rho|\partial_x|-\beta}f_N^\varepsilon,\dot{\Delta}_k\mathfrak{r}\rangle\big|.\end{aligned} \tag{5.51}$$

We take the square roots and sum the resulting inequalities for $k\in\mathbb{Z}$ to deduce that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k\mathfrak{r}(t)\|_{L^2} + \sum_{k\in\mathbb{Z}}2^{\frac{k}{2}}\left(\int_0^t|\dot{\rho}|\,\|\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}} + \sqrt{2}\sum_{k\in\mathbb{Z}}\left(\int_0^t\dot{\beta}\|\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}} \\ + \sqrt{2\varepsilon}\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k\nabla\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}} \leq 2\sqrt{2}\sum_{k\in\mathbb{Z}}\left(\int_0^t|\langle\dot{\Delta}_k e^{\rho|\partial_x|-\beta}f_N^\varepsilon,\dot{\Delta}_k\mathfrak{r}\rangle|\right)^{\frac{1}{2}}.\end{aligned} \tag{5.52}$$

Proposition 5.10 (Proof in Section 5.4)**.** *For $t\in[0,T^*]$, there holds*

$$\begin{aligned}2\sqrt{2}\sum_{k\in\mathbb{Z}}\left(\int_0^t\big|(\dot{\Delta}_k e^{\rho|\partial_x|-\beta}f_N^\varepsilon,\dot{\Delta}_k\mathfrak{r}\rangle\big|\right)^{\frac{1}{2}} \leq \sum_{k\in\mathbb{Z}}2^{\frac{k}{2}}\left(\int_0^t|\dot{\rho}|\,\|\dot{\Delta}_{k'}\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}} \\ + \sqrt{2}\sum_{k\in\mathbb{Z}}\left(\int_0^t\dot{\beta}\|\dot{\Delta}_k\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}} + \frac{1}{4}\sqrt{\varepsilon}\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k\nabla\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}} \\ + \sqrt{2}\sum_{k\in\mathbb{Z}}\left(\int_0^t\frac{1}{\alpha_\varepsilon}\|e^{\rho_0|\partial_x|}\dot{\Delta}_k f_{\mathrm{app}}^\varepsilon\|_{L^2}^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.53}$$

The proof of Proposition 5.10 is given in Section 5.4. Let us admit Proposition 5.10 for the time being and see how to conclude the proof of Proposition 3.2. Combining (5.52) and (5.53) we deduce that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k\mathfrak{r}(t)\|_{L^2} + \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot{\Delta}_k\nabla\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}} \\ \leq \sqrt{2}\sum_{k\in\mathbb{Z}}\left(\int_0^t\frac{1}{\alpha_\varepsilon}\|e^{\rho_0|\partial_x|}\dot{\Delta}_k f_{\mathrm{app}}^\varepsilon\|_{L^2}^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.54}$$

Proposition 5.11 (Proof in Section 6.3)**.** *If u_b satisfies* (1.11) *and* (1.12) *for a constant $C_b > 0$ and $\rho_b > \rho_0$, there exists $C_f > 0$ such that, for $\varepsilon, \kappa \in (0,1)$,*

$$\sup_{t\in[0,T/\varepsilon^\kappa]} \sum_{k\in\mathbb{Z}} \left(\int_0^t \frac{1}{\alpha_\varepsilon} \|e^{\rho_0|\partial_x|} \dot{\Delta}_k f^\varepsilon_{\text{app}}\|^2_{L^2} \right)^{\frac{1}{2}} \leq C_f(\varepsilon^{\frac{1}{4}} + \varepsilon^{1-\kappa}). \tag{5.55}$$

As long as $t \leq T^*$, (5.54) holds and, thanks to Proposition 5.11, we obtain

$$\sum_{k\in\mathbb{Z}} \|\dot{\Delta}_k \mathfrak{r}(t)\|_{L^2} + \sqrt{\varepsilon} \sum_{k\in\mathbb{Z}} \left(\int_0^t \|\dot{\Delta}_k \nabla \mathfrak{r}\|^2_{L^2} \right)^{\frac{1}{2}} \leq \sqrt{2} C_f(\varepsilon^{\frac{1}{4}} + \varepsilon^{1-\kappa}). \tag{5.56}$$

Moreover,

$$\int_0^t |\dot{\rho}| \leq 10^2 C_{\mathrm{B}} \int_0^{+\infty} \|z\partial_z v^0(t,z)\|_{L^\infty(\mathbb{R}_+)} + 10^7 C_{\mathrm{B}}^4 \varepsilon e^{\beta_\star} \int_0^t \|\nabla \mathfrak{r}\|_{\dot{B}^0} \tag{5.57}$$

and, for $t \leq T/\varepsilon^\kappa$,

$$\begin{aligned} \varepsilon \int_0^t \|\nabla \mathfrak{r}\|_{\dot{B}^0} &= \varepsilon \sum_{k\in\mathbb{Z}} \int_0^t \|\dot{\Delta}_k \nabla \mathfrak{r}\|_{L^2} \\ &\leq \sqrt{\varepsilon t} \cdot \sqrt{\varepsilon} \sum_{k\in\mathbb{Z}} \left(\int_0^t \|\dot{\Delta}_k \nabla \mathfrak{r}\|^2_{L^2} \right)^{\frac{1}{2}} \\ &\leq \sqrt{2T} C_f(\varepsilon^{\frac{1}{4}} + \varepsilon^{1-\kappa}). \end{aligned} \tag{5.58}$$

Combining these estimates yields

$$\rho(T/\varepsilon^\kappa) \geq 2 - 10^7 C_{\mathrm{B}}^4 e^{\beta_\star} \sqrt{2T} C_f(\varepsilon^{\frac{1}{4}} + \varepsilon^{1-\kappa}). \tag{5.59}$$

Thus, for ε small enough, $\rho(T/\varepsilon^\kappa) \geq 1$ and thus $T^*_N = T/\varepsilon^\kappa$ and one has

$$\|r^\varepsilon_N\|_{L^\infty(L^2(\mathfrak{B}))} + \sqrt{\varepsilon} \|\nabla r^\varepsilon_N\|_{L^2(L^2(\mathfrak{B}))} \leq \sqrt{2} e^{\beta_\star} C_f(\varepsilon^{\frac{1}{4}} + \varepsilon^{1-\kappa}). \tag{5.60}$$

This estimate being uniform with respect to N, one can pass to the limit (for fixed ε) towards r^ε, and then take ε small enough to conclude the proof of Proposition 3.2.

5.4 Proof of Proposition 5.10

To prove Proposition 5.10 we estimate separately the terms corresponding to the different terms of the decomposition of the source term f^ε_N in (5.38). Let us start with the term corresponding to a loss of derivative.

Lemma 5.12. *For $t \in [0, T^*]$, there holds*

$$\sum_{k\in\mathbb{Z}} \left(\int_0^t \left| \langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta} (M[\partial_x r_1]) \{z\partial_z v^0\}, \dot{\Delta}_k \mathfrak{r} \rangle \right| \right)^{\frac{1}{2}} \leq$$

$$\sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \left(\int_0^t 2C_{\mathrm{B}} \|z\partial_z v^0\|_{L^\infty_z} \|\dot{\Delta}_k \mathfrak{r}\|^2_{L^2} \right)^{\frac{1}{2}}.$$

Proof. Since $\{z\partial_z v^0\}$ and the operator M do not depend on the x variable,

$$\dot{\Delta}_k e^{\rho|\partial_x|-\beta}(M[\partial_x r_1])\{z\partial_z v^0\} = (M[\dot{\Delta}_k \partial_x \mathfrak{r}_1])\{z\partial_z v^0\}. \tag{5.61}$$

Moreover, using the definition of M in (5.2), Hardy's inequality, and the fact that $|\chi| \leq 1$, we get that, for any $a \in L^2_y(-1,1)$,

$$\|M[a]\|_{L^2_y} \leq 2\|a\|_{L^2_y}. \tag{5.62}$$

Hence, using (5.16) from Lemma 5.2, we obtain

$$\left|\left(\dot{\Delta}_k e^{\rho|\partial_x|-\beta}(M[\partial_x r_1])\{z\partial_z v^0\}, \dot{\Delta}_k \mathfrak{r}\right\rangle\right| \leq 2^{k+1} C_{\mathrm{B}} \|z\partial_z v^0\|_{L^\infty_z} \|\dot{\Delta}_k \mathfrak{r}\|^2_{L^2}. \tag{5.63}$$

The result follows by integration in time and summation over $k \in \mathbb{Z}$ of the square roots. □

Lemma 5.13. *For $t \in [0, T^*]$, there holds*

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \left(\int_0^t \left|\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta} r_2(\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon), \dot{\Delta}_k \mathfrak{r}\rangle\right| \right)^{\frac{1}{2}} \\ &\qquad \leq \sum_{k\in\mathbb{Z}} \left(\int_0^t \|\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon\|_{L^\infty(\mathfrak{B})} \|\dot{\Delta}_k \mathfrak{r}\|^2_{L^2} \right)^{\frac{1}{2}}. \end{aligned} \tag{5.64}$$

Proof. Since $\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon$ does not depend on x,

$$\dot{\Delta}_k e^{\rho|\partial_x|-\beta}(r_2(\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon)) = (\chi'\{v^0\}+\varepsilon^2\partial_y w^\varepsilon)\dot{\Delta}_k \mathfrak{r}_2, \tag{5.65}$$

and therefore the result readily follows by the Cauchy-Schwarz inequality. □

Lemma 5.14 (Proof in Section 5.5)**.** *For $t \in [0, T^*]$, there holds*

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \left(\int_0^t \left|\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta} \left[(r\cdot\nabla)\, u^1 + \left(u^1\cdot\nabla\right) r\right], \dot{\Delta}_k \mathfrak{r}\rangle\right| \right)^{\frac{1}{2}} \\ &\qquad \leq 10^8 C_{\mathrm{B}}^2 \sum_{k\in\mathbb{Z}} \left(\int_0^t \ell_1^2 \|\dot{\Delta}_k \mathfrak{r}\|^2_{L^2} \right)^{\frac{1}{2}} + \frac{1}{20} \sum_{k\in\mathbb{Z}} \left(\int_0^t \|\dot{\Delta}_k \nabla \mathfrak{r}\|^2_{L^2} \right)^{\frac{1}{2}}. \end{aligned} \tag{5.66}$$

Lemma 5.15 (Proof in Section 5.6)**.** *For $t \in [0, T^*]$, there holds*

$$\sum_{k\in\mathbb{Z}} \left(\int_0^t \left|\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta}((r\cdot\nabla)\, r), \dot{\Delta}_k \mathfrak{r}\rangle\right| \right)^{\frac{1}{2}} \leq 600 C_{\mathrm{B}}^2 \sum_{k'\in\mathbb{Z}} 2^{\frac{k'}{2}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_{k'} \mathfrak{r}\|^2_{L^2} \right)^{\frac{1}{2}}.$$

Proposition 5.10 follows from the definition (5.38) of f_N^ε, Lemma 5.12, Lemma 5.13, Lemma 5.14 and Lemma 5.15. Observe in particular that the sum of the right hand sides of Lemma 5.12 and of Lemma 5.15 can be bounded by the first term in the right hand sides of the estimate in Proposition 5.10 thanks to (5.45). On the other hand the sum of the right hand sides of Lemma 5.13 and Lemma 5.14 can be bounded by the other terms in the right hand sides of the estimate in Proposition 5.10 thanks to (5.43).

5.5 Estimate of $(r\cdot\nabla)u^1+(u^1\cdot\nabla)r$. Proof of Lemma 5.14

Due to $\operatorname{div} r=0$, we get, integrating by parts, that

$$\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}\left(r\cdot\nabla\right)u^1,\dot\Delta_k\mathfrak{r}\rangle=-\sum_{i,j=1}^{2}\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}\left(r_iu^1_j\right),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle. \tag{5.67}$$

Due to $\operatorname{div} u^1=0$, we get, integrating by parts, that

$$\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}\left(u^1\cdot\nabla\right)r,\dot\Delta_k\mathfrak{r}\rangle=-\sum_{i,j=1}^{2}\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}\left(u^1_ir_j\right),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle. \tag{5.68}$$

This yields a total of 8 scalar terms, which we estimate separately using the same method for each set of indexes. We explain the proof only for the terms in (5.67) (the terms of (5.68) are handled similarly). By Bony's decomposition (5.26), we expand the products as

$$r_iu^1_j=\mathbf{T}_{r_i}u^1_j+\mathbf{T}_{u^1_j}r_i+\mathbf{R}(r_i,u^1_j). \tag{5.69}$$

We explain the three estimates for each term in the right-hand side of (5.69).

5.5.1 First estimate

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.30), we write

$$\langle \dot\Delta_k e^{\rho|\partial_x|}\left(\mathbf{T}_{r_i}u^1_j\right),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle=\sum_{|k'-k|\le 4}\langle \dot\Delta_k e^{\rho|\partial_x|}\left(\dot S_{k'-1}r_i\right)(\dot\Delta_{k'}u^1_j),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle. \tag{5.70}$$

Thanks to the product estimate (5.35) from Lemma 5.7, the embedding estimate (5.21) from Lemma 5.4, the decomposition (5.14) of $\dot S_{k'-1}$ and estimate (5.16) from Lemma 5.2, and the definition (5.42) of ℓ_1, we obtain

$$\begin{aligned}&|\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}(\dot S_{k'-1}r_i)(\dot\Delta_{k'}u^1_j),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle|\\&\qquad\le \|\dot S_{k'-1}\mathfrak{r}_i^+\|_{L^\infty_x(L^2_y)}\|\dot\Delta_{k'}e^{\rho|\partial_x|}\left(u^1_j\right)^+\|_{L^2_x(L^\infty_y)}\|\dot\Delta_k\partial_i\mathfrak{r}_j\|_{L^2}\\&\qquad\le C_{\mathrm{B}}^{\frac32}\ell_1 2^{-\frac{k'}{2}}\|\dot\Delta_k\nabla\mathfrak{r}\|_{L^2}\sum_{k''\le k'-2}2^{\frac{k''}{2}}\|\dot\Delta_{k''}\mathfrak{r}\|_{L^2}.\end{aligned} \tag{5.71}$$

Thanks to the Peter-Paul inequality, we deduce that

$$\begin{aligned}&|\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}(\dot S_{k'-1}r_i)(\dot\Delta_{k'}u^1_j),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle|\\&\le\frac{1}{(9\cdot8\cdot60)^2}\|\dot\Delta_k\nabla\mathfrak{r}\|^2_{L^2}+\frac14(9\cdot8\cdot60)^2C_{\mathrm{B}}^3 2^{k''-k'}\left(\sum_{k''\le k'-2}\ell_1\|\dot\Delta_{k''}\mathfrak{r}\|_{L^2}\right)^2.\end{aligned} \tag{5.72}$$

Summing these estimates and applying Minkowsky's inequality leads to

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t |\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{r_i}u_j^1),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle|\right)^{\frac12}\\
&\quad\le 2160C_{\mathrm{B}}^2\sum_{k''\in\mathbb{Z}}\left(\sum_{k'\ge k''-2}\sum_{|k-k'|\le 4}2^{\frac{k''-k'}{2}}\right)\left(\int_0^t \ell_1^2\|\dot\Delta_{k''}\mathfrak{r}\|_{L^2}^2\right)^{\frac12}\\
&\qquad+\frac{1}{9\cdot 8\cdot 60}\sum_{k\in\mathbb{Z}}\sum_{|k'-k|\le 4}\left(\int_0^t\|\dot\Delta_k\nabla\mathfrak{r}\|_{L^2}^2\right)^{\frac12}\\
&\quad\le 2\cdot 10^6C_{\mathrm{B}}^2\sum_{k\in\mathbb{Z}}\left(\int_0^t\ell_1^2\|\dot\Delta_k\mathfrak{r}\|_{L^2}^2\right)^{\frac12}+\frac{1}{8\cdot 60}\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot\Delta_k\nabla\mathfrak{r}\|_{L^2}^2\right)^{\frac12}.
\end{aligned}
\tag{5.73}
$$

5.5.2 Second estimate

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.30), we write

$$
\langle \dot\Delta_k e^{\rho|\partial_x|}\,(\mathbf{T}_{u_j^1}r_i),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle=\sum_{|k'-k|\le 4}\langle \dot\Delta_k e^{\rho|\partial_x|}\,(\dot S_{k'-1}u_j^1)(\dot\Delta_{k'}r_i),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle.
$$

It follows from the decomposition (5.14) of $\dot S_{k'-1}$, the estimate (5.20) from Lemma 5.4 and the definition (5.42) of ℓ_1 that

$$
\|e^{\rho|\partial_x|}\,\dot S_{k'-1}(u^1)^+\|_{L^\infty}\le C_{\mathrm{B}}^2\ell_1. \tag{5.74}
$$

Hence, with the product estimate (5.35) from Lemma 5.7, we infer

$$
\begin{aligned}
&|\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}(\dot S_{k'-1}u_j^1)(\dot\Delta_{k'}r_i),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle|\\
&\qquad\le\|e^{\rho|\partial_x|}\,\dot S_{k'-1}(u^1)^+\|_{L^\infty}\|\dot\Delta_{k'}\mathfrak{r}_i^+\|_{L^2}\|\dot\Delta_k\nabla\mathfrak{r}\|_{L^2}\\
&\qquad\le C_{\mathrm{B}}^2\ell_1\|\dot\Delta_{k'}\mathfrak{r}\|_{L^2}\|\dot\Delta_k\nabla\mathfrak{r}\|_{L^2}.
\end{aligned}
\tag{5.75}
$$

Summing these estimates and using the Peter-Paul inequality yields

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t |\langle \dot\Delta_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{u_j^1}r_i),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle|\right)^{\frac12}\\
&\quad\le 2\cdot 10^5C_{\mathrm{B}}^2\sum_{k\in\mathbb{Z}}\left(\int_0^t\ell_1^2\|\dot\Delta_k\mathfrak{r}\|_{L^2}^2\right)^{\frac12}+\frac{1}{8\cdot 60}\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot\Delta_k\nabla\mathfrak{r}\|_{L^2}^2\right)^{\frac12}.
\end{aligned}
\tag{5.76}
$$

5.5.3 Third estimate

By using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.31), and the definition (5.29), we obtain

$$
\begin{aligned}
\langle \dot\Delta_k e^{\rho|\partial_x|}\,\mathbf{R}(r_i,u_j^1),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle&=\sum_{k'\ge k-3}\langle \dot\Delta_k e^{\rho|\partial_x|}\,(\dot\Delta_{k'}r_i)(\widetilde{\dot\Delta}_{k'}u_j^1),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle\\
&=\sum_{k'\ge k-3}\sum_{|k''-k'|\le 1}\langle \dot\Delta_k e^{\rho|\partial_x|}\,(\dot\Delta_{k'}r_i)(\dot\Delta_{k''}u_j^1),\dot\Delta_k\partial_i\mathfrak{r}_j\rangle.
\end{aligned}
$$

Thanks to the product estimate (5.35) from Lemma 5.7, the embedding estimate (5.21) from Lemma 5.4, estimate (5.16) from Lemma 5.2, and the definition (5.42) of ℓ_1, we infer

$$\begin{aligned}
&|\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta}(\dot{\Delta}_{k'} r_i)(\dot{\Delta}_{k''} u_j^1), \dot{\Delta}_k \partial_i \mathfrak{r}_j\rangle| \\
&\qquad \leq \|\dot{\Delta}_{k'} \mathfrak{r}_i^+\|_{L^\infty_x(L^2_y)} \|e^{\rho|\partial_x|}\, \dot{\Delta}_{k''}(u_j^1)^+\|_{L^2_x(L^\infty_y)} \|\dot{\Delta}_k \nabla \mathfrak{r}\|_{L^2} \\
&\qquad \leq 2^{\frac{k-k''}{2}} C_{\mathrm{B}}^{\frac{3}{2}} \ell_1 \|\dot{\Delta}_{k'} \mathfrak{r}\|_{L^2} \|\dot{\Delta}_k \nabla \mathfrak{r}\|_{L^2}.
\end{aligned} \tag{5.77}$$

Using the Peter-Paul inequality and summing the resulting estimates, we find

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left(\int_0^t |\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta} \mathbf{R}(r_i, u_j^1), \dot{\Delta}_k \partial_i \mathfrak{r}_j\rangle| \right)^{\frac{1}{2}} \\
&\leq 10^6 C_{\mathrm{B}}^2 \sum_{k\in\mathbb{Z}} \left(\int_0^t \ell_1^2 \|\dot{\Delta}_k \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}} + \frac{1}{8\cdot 60} \sum_{k\in\mathbb{Z}} \left(\int_0^t \|\dot{\Delta}_k \nabla \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.78}$$

Eventually, summing estimates (5.73), (5.76) and (5.78) for the 8 pairs of indexes concludes the proof of Lemma 5.14 with the claimed constants.

5.6 Estimate of $(r \cdot \nabla) r$. Proof of Lemma 5.15

We prove Lemma 5.15, concerning the estimate of the trilinear term. First, using Bony's paraproduct decomposition (5.26) and the divergence free condition in (3.26), we write the quadratic term as

$$(r \cdot \nabla) r = \mathbf{T}_{r_1} \partial_x r + \mathbf{T}_{\partial_x r} r_1 + \partial_x \mathbf{R}(r_1, r) + \mathbf{T}_{r_2} \partial_y r + \mathbf{T}_{\partial_y r} r_2 + \partial_y \mathbf{R}(r_2, r). \tag{5.79}$$

As a shorthand, we set

$$E := \sum_{k'\in\mathbb{Z}} 2^{\frac{k'}{2}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_{k'} \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}}, \tag{5.80}$$

which appears in the right hand side of the estimate given in Lemma 5.15. The proof of Lemma 5.15 is obtained by summation of the six following estimates:

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t\left|\langle\dot\Delta_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{r_1}\partial_x r),\dot\Delta_k\mathfrak{r}\rangle\right|\right)^{\frac12}\le 10^2C_{\mathrm{B}}^2E, \tag{5.81}$$

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t\left|\langle\dot\Delta_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{\partial_x r}r_1),\dot\Delta_k\mathfrak{r}\rangle\right|\right)^{\frac12}\le 10^2C_{\mathrm{B}}^2E, \tag{5.82}$$

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t\left|\langle\dot\Delta_k e^{\rho|\partial_x|-\beta}\partial_x\mathbf{R}(r_1,r),\dot\Delta_k\mathfrak{r}\rangle\right|\right)^{\frac12}\le 10^2C_{\mathrm{B}}^2E, \tag{5.83}$$

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t\left|\langle\dot\Delta_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{r_2}\partial_y r),\dot\Delta_k\mathfrak{r}\rangle\right|\right)^{\frac12}\le 10^2C_{\mathrm{B}}^2E, \tag{5.84}$$

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t\left|\langle\dot\Delta_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{\partial_y r}r_2),\dot\Delta_k\mathfrak{r}\rangle\right|\right)^{\frac12}\le 10^2C_{\mathrm{B}}^2E, \tag{5.85}$$

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t\left|\langle\dot\Delta_k e^{\rho|\partial_x|-\beta}\partial_y\mathbf{R}(r_2,r),\dot\Delta_k\mathfrak{r}\rangle\right|\right)^{\frac12}\le 10^2C_{\mathrm{B}}^2E. \tag{5.86}$$

5.6.1 Proof of (5.81)

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.30), we expand

$$\langle\dot\Delta_k e^{\rho|\partial_x|}(\mathbf{T}_{r_1}\partial_x r),\dot\Delta_k\mathfrak{r}\rangle=\sum_{|k'-k|\le 4}\langle e^{\rho|\partial_x|}(\dot S_{k'-1}r_1)(\dot\Delta_{k'}\partial_x r),\dot\Delta_k\mathfrak{r}\rangle. \tag{5.87}$$

Thanks to the product estimate (5.35) from Lemma 5.7, we obtain

$$|\langle e^{\rho|\partial_x|-\beta}(\dot S_{k'-1}r_1)(\dot\Delta_{k'}\partial_x r),\dot\Delta_k\mathfrak{r}\rangle|\le e^{\beta_\star}\|\dot S_{k'-1}\mathfrak{r}_1^+\|_{L^\infty}\|\dot\Delta_{k'}\partial_x\mathfrak{r}^+\|_{L^2}\|\dot\Delta_k\mathfrak{r}^+\|_{L^2}. \tag{5.88}$$

Thanks to estimate (5.16) from Lemma 5.2,

$$\|\dot\Delta_{k'}\partial_x\mathfrak{r}\|_{L^2}\le C_{\mathrm{B}}2^{k'}\|\dot\Delta_{k'}\mathfrak{r}\|_{L^2}. \tag{5.89}$$

Using estimate (5.20) from Lemma 5.4, we obtain that

$$\|\dot S_{k'-1}\mathfrak{r}_1^+\|_{L^\infty}\le C_{\mathrm{B}}^2\|\nabla\mathfrak{r}_1^+\|_{\dot B^0}\le C_{\mathrm{B}}^2\|\nabla\mathfrak{r}\|_{\dot B^0}. \tag{5.90}$$

Gathering our estimates, we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^t|\langle\dot\Delta_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{r_1}\partial_x r),\dot\Delta_k\mathfrak{r}\rangle|\right)^{\frac12}\\&\le C_{\mathrm{B}}^{\frac32}\sum_{k\in\mathbb{Z}}\sum_{|k'-k|\le 4}2^{\frac{k'}{2}}\left(\int_0^t e^{\beta_\star}\|\nabla\mathfrak{r}\|_{\dot B^0}\|\dot\Delta_{k'}\mathfrak{r}\|_{L^2}\|\dot\Delta_k\mathfrak{r}\|_{L^2}\right)^{\frac12}\\&\le 10^2C_{\mathrm{B}}^2\sum_{k'\in\mathbb{Z}}2^{\frac{k'}{2}}\left(\int_0^t e^{\beta_\star}\|\nabla\mathfrak{r}\|_{\dot B^0}\|\dot\Delta_{k'}\mathfrak{r}\|_{L^2}^2\right)^{\frac12}.\end{aligned} \tag{5.91}$$

This concludes the proof of (5.81).

5.6.2 Proof of (5.82)

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.30), we expand

$$\langle \dot{\Delta}_k e^{\rho|\partial_x|}(\mathbf{T}_{\partial_x r} r_1), \dot{\Delta}_k \mathfrak{r}\rangle = \sum_{|k'-k|\leq 4} \langle e^{\rho|\partial_x|}(\dot{S}_{k'-1}\partial_x r)(\dot{\Delta}_{k'} r_1), \dot{\Delta}_k \mathfrak{r}\rangle. \tag{5.92}$$

Thanks to the product estimate (5.35) from Lemma 5.7, we obtain

$$\begin{aligned} &|\langle e^{\rho|\partial_x|-\beta}(\dot{S}_{k'-1}\partial_x r)(\dot{\Delta}_{k'} r_1), \dot{\Delta}_k \mathfrak{r}\rangle| \\ &\qquad \leq e^{\beta_\star}\|\dot{S}_{k'-1}\partial_x \mathfrak{r}^+\|_{L^\infty_x(L^2_y)}\|\dot{\Delta}_{k'}\mathfrak{r}_1^+\|_{L^2_x(L^\infty_y)}\|\dot{\Delta}_k \mathfrak{r}^+\|_{L^2}. \end{aligned} \tag{5.93}$$

Thanks to estimate (5.19) from Lemma 5.4, we get

$$\|\dot{\Delta}_{k'}\mathfrak{r}_1^+\|_{L^2_x(L^\infty_y)} \leq C_{\mathrm{B}} 2^{-\frac{k'}{2}}\|\nabla \mathfrak{r}\|_{\dot{B}^0}. \tag{5.94}$$

Gathering our estimates and using the Peter-Paul inequality,

$$\begin{aligned} \left(\int_0^t \|\dot{S}_{k'-1}\partial_x \mathfrak{r}^+\|_{L^\infty_x(L^2_y)}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_k \mathfrak{r}^+\|_{L^2}\right)^{\frac{1}{2}} &\leq 2^{\frac{3k}{4}}\left(\int_0^t \|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_k \mathfrak{r}\|^2_{L^2}\right)^{\frac{1}{2}} \\ &\quad + 2^{-\frac{3k}{4}}\left(\int_0^t \|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{S}_{k'-1}\partial_x \mathfrak{r}^+\|^2_{L^\infty_x(L^2_y)}\right)^{\frac{1}{2}}, \end{aligned} \tag{5.95}$$

we obtain

$$\begin{aligned} &\sum_{k\in\mathbb{Z}}\left(\int_0^t |\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{\partial_x r} r_1), \dot{\Delta}_k \mathfrak{r}\rangle|\right)^{\frac{1}{2}} \\ &\leq C_{\mathrm{B}}^{\frac{1}{2}}\sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \sum_{|k'-k|\leq 4} 2^{\frac{k-k'}{4}}\left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_k \mathfrak{r}\|^2_{L^2}\right)^{\frac{1}{2}} \\ &\quad + C_{\mathrm{B}}^{\frac{1}{2}}\sum_{k\in\mathbb{Z}} 2^{-k} \sum_{|k'-k|\leq 4} 2^{\frac{k-k'}{4}}\left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{S}_{k'-1}\partial_x \mathfrak{r}^+\|^2_{L^\infty_x(L^2_y)}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.96}$$

The first term is bounded by $10C_{\mathrm{B}}E$. Using Minkowski's inequality and (5.14), we estimate the second term as

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} 2^{-k} \sum_{|k'-k|\leq 4} 2^{\frac{k-k'}{4}}\left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{S}_{k'-1}\partial_x \mathfrak{r}^+\|^2_{L^\infty_x(L^2_y)}\right)^{\frac{1}{2}} \\ &\leq \sum_{k\in\mathbb{Z}} 2^{-k} \sum_{|k'-k|\leq 4} 2^{\frac{k-k'}{4}} \sum_{j\leq k'-2}\left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_j\partial_x \mathfrak{r}^+\|^2_{L^\infty_x(L^2_y)}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.97}$$

Now we deduce from Bernstein's Lemma 5.2, estimate (5.16), that

$$\|\dot{\Delta}_j\partial_x \mathfrak{r}^+\|^2_{L^\infty_x(L^2_y)} \leq C_{\mathrm{B}}^2 2^{3j}\|\dot{\Delta}_j \mathfrak{r}\|^2_{L^2}. \tag{5.98}$$

We deduce from these estimates that the second term of (5.96) is bounded by $80C_{\mathrm{B}}^2E$. This concludes the proof of (5.82).

5.6.3 Proof of (5.83)

First, using integration by parts in the horizontal direction, we get

$$\langle \dot{\Delta}_k e^{\rho|\partial_x|}\, \partial_x \mathbf{R}(r_1, r), \dot{\Delta}_k \mathfrak{r}\rangle = -\langle \dot{\Delta}_k e^{\rho|\partial_x|}\, \mathbf{R}(r_1, r), \dot{\Delta}_k \partial_x \mathfrak{r}\rangle. \tag{5.99}$$

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.31), and the definition (5.29) we expand the term as

$$\begin{aligned}\langle \dot{\Delta}_k e^{\rho|\partial_x|}\, \mathbf{R}(r_1, r), \partial_x \dot{\Delta}_k \mathfrak{r}\rangle &= \sum_{k' \geq k-3} \langle e^{\rho|\partial_x|}\, (\dot{\Delta}_{k'} r_1)(\widetilde{\dot{\Delta}}_{k'} r), \dot{\Delta}_k \partial_x \mathfrak{r}\rangle \\ &= \sum_{k' \geq k-3} \sum_{|k''-k'| \leq 1} \langle e^{\rho|\partial_x|}\, (\dot{\Delta}_{k'} r_1)(\dot{\Delta}_{k''} r), \dot{\Delta}_k \partial_x \mathfrak{r}\rangle.\end{aligned} \tag{5.100}$$

Thanks to the product estimate (5.35) from Lemma 5.7, we obtain

$$\begin{aligned}&|\langle e^{\rho|\partial_x|-\beta}(\dot{\Delta}_{k'} r_1)(\dot{\Delta}_{k''} r), \dot{\Delta}_k \partial_x \mathfrak{r}\rangle| \\ &\qquad \leq e^{\beta_\star} \|\dot{\Delta}_{k'} \mathfrak{r}_1^+\|_{L^2_x(L^\infty_y)} \|\dot{\Delta}_{k''} \mathfrak{r}^+\|_{L^2_x(L^2_y)} \|\dot{\Delta}_k \partial_x \mathfrak{r}^+\|_{L^\infty_x(L^2_y)}.\end{aligned} \tag{5.101}$$

Thanks to estimate (5.19) from Lemma 5.4, we have

$$\|\dot{\Delta}_{k'} \mathfrak{r}_1^+\|_{L^2_x(L^\infty_y)} \leq C_{\mathrm{B}} 2^{-\frac{k'}{2}} \|\nabla \mathfrak{r}\|_{\dot{B}^0}. \tag{5.102}$$

Thanks to estimate (5.16) from Lemma 5.2, we have

$$\|\dot{\Delta}_k \partial_x \mathfrak{r}^+\|_{L^\infty_x(L^2_y)} \leq C_{\mathrm{B}} 2^{\frac{3k}{2}} \|\dot{\Delta}_k \mathfrak{r}\|_{L^2}. \tag{5.103}$$

Gathering our estimates, we obtain

$$\begin{aligned}&\sum_{k \in \mathbb{Z}} \left(\int_0^t |\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta} \partial_x \mathbf{R}(r_1, r), \dot{\Delta}_k \mathfrak{r}\rangle| \right)^{\frac{1}{2}} \\ &\leq C_{\mathrm{B}} \sum_{k \in \mathbb{Z}} \sum_{k' \geq k-3} \sum_{|k''-k| \leq 1} 2^{\frac{3k}{4} - \frac{k'}{4}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_k \mathfrak{r}\|_{L^2} \|\dot{\Delta}_{k''} \mathfrak{r}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C_{\mathrm{B}} \sum_{k \in \mathbb{Z}} \sum_{k' \geq k-3} \sum_{|k''-k| \leq 1} 2^{\frac{3k}{4} - \frac{k'}{4}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_k \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\quad + C_{\mathrm{B}} \sum_{k \in \mathbb{Z}} \sum_{k' \geq k-3} \sum_{|k''-k| \leq 1} 2^{\frac{3k}{4} - \frac{k'}{4}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_{k''} \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}}\end{aligned} \tag{5.104}$$

Up to reordering the sums, one gets that the first term is bounded by $32 C_{\mathrm{B}} E$ and that the second term is bounded by $51 C_{\mathrm{B}} E$. This concludes the proof of (5.83).

5.6.4 Proof of (5.84)

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.30), we expand

$$\langle \dot{\Delta}_k e^{\rho|\partial_x|}(\mathbf{T}_{r_2}\partial_y r), \dot{\Delta}_k \mathfrak{r}\rangle = \sum_{|k'-k|\leq 4} \langle e^{\rho|\partial_x|}(\dot{S}_{k'-1}r_2)(\dot{\Delta}_{k'}\partial_y r), \dot{\Delta}_k \mathfrak{r}\rangle. \tag{5.105}$$

Thanks to the product estimate (5.35) from Lemma 5.7, we obtain

$$|\langle e^{\rho|\partial_x|-\beta}(\dot{S}_{k'-1}r_2)(\dot{\Delta}_{k'}\partial_y r), \dot{\Delta}_k \mathfrak{r}\rangle| \leq e^{\beta_\star}\|\dot{S}_{k'-1}\mathfrak{r}_2^+\|_{L^\infty}\|\dot{\Delta}_{k'}\partial_y \mathfrak{r}^+\|_{L^2}\|\dot{\Delta}_k \mathfrak{r}^+\|_{L^2}. \tag{5.106}$$

From the definition (5.15) of $\dot{B}^0$,

$$\|\dot{\Delta}_{k'}\partial_y \mathfrak{r}\|_{L^2} \leq \|\nabla \mathfrak{r}\|_{\dot{B}^0}. \tag{5.107}$$

Summing over $k \in \mathbb{Z}$ and using Young's inequality, we obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left(\int_0^t |\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{r_2}\partial_y r), \dot{\Delta}_k \mathfrak{r}\rangle|\right)^{\frac{1}{2}} \\
&\leq \sum_{k\in\mathbb{Z}}\sum_{|k'-k|\leq 4} 2^{-\frac{k}{2}} \left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{S}_{k'-1}\mathfrak{r}_2^+\|_{L^\infty}^2\right)^{\frac{1}{2}} \\
&\quad + \sum_{k\in\mathbb{Z}}\sum_{|k'-k|\leq 4} 2^{\frac{k}{2}} \left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_k \mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}}
\end{aligned} \tag{5.108}$$

Using Minkowski's inequality and (5.14), the first sum is bounded as

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{|k'-k|\leq 4} 2^{-\frac{k}{2}} \left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{S}_{k'-1}\mathfrak{r}_2^+\|_{L^\infty}^2\right)^{\frac{1}{2}} \\
&\leq \sum_{k\in\mathbb{Z}}\sum_{|k'-k|\leq 4} 2^{-\frac{k}{2}} \sum_{j\leq k'-2} \left(\int_0^t e^{\beta_\star}\|\nabla \mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_j \mathfrak{r}_2^+\|_{L^\infty}^2\right)^{\frac{1}{2}}
\end{aligned} \tag{5.109}$$

Thanks to estimates (5.23) from Lemma 5.5 and (5.16) from Lemma 5.2, we get

$$\|\dot{\Delta}_j \mathfrak{r}_2\|_{L^\infty} \leq C_{\mathrm{B}}^2 2^j \|\dot{\Delta}_j \mathfrak{r}\|_{L^2}. \tag{5.110}$$

Gathering these two estimates and reordering the sums we conclude that the first term in the right hand side of (5.109) is bounded by $36C_{\mathrm{B}}^2 E$. Since the second term is bounded by $9E$ this concludes the proof of (5.84).

5.6.5 Proof of (5.85)

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.30), we expand

$$\langle \dot{\Delta}_k e^{\rho|\partial_x|}(\mathbf{T}_{\partial_y r} r_2), \dot{\Delta}_k \mathfrak{r}\rangle = \sum_{|k'-k|\leq 4} \langle e^{\rho|\partial_x|}(\dot{S}_{k'-1}\partial_y r)(\dot{\Delta}_{k'} r_2), \dot{\Delta}_k \mathfrak{r}\rangle. \tag{5.111}$$

Thanks to the product estimate (5.35) from Lemma 5.7, we obtain

$$|\langle e^{\rho|\partial_x|-\beta}(\dot{S}_{k'-1}\partial_y r)(\dot{\Delta}_{k'}r_2), \dot{\Delta}_k\mathfrak{r}\rangle| \leq e^{\beta_\star}\|\dot{S}_{k'-1}\partial_y\mathfrak{r}^+\|_{L^2}\|\dot{\Delta}_{k'}\mathfrak{r}_2^+\|_{L^\infty}\|\dot{\Delta}_k\mathfrak{r}^+\|_{L^2}. \tag{5.112}$$

On the one hand, thanks to estimate (5.23) from Lemma 5.5 and estimate (5.16) from Lemma 5.2, we have

$$\|\dot{\Delta}_{k'}\mathfrak{r}_2^+\|_{L^\infty} \leq C_{\mathrm{B}}^2 2^{k'}\|\dot{\Delta}_{k'}\mathfrak{r}\|_{L^2}. \tag{5.113}$$

On the other hand, from the definition (5.15) of $\dot{B}^0$,

$$\|\dot{S}_{k'-1}\partial_y\mathfrak{r}\|_{L^2} \leq \|\nabla\mathfrak{r}\|_{\dot{B}^0}. \tag{5.114}$$

Gathering our estimates, we obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t |\langle\dot{\Delta}_k e^{\rho|\partial_x|-\beta}(\mathbf{T}_{\partial_y r}r_2), \dot{\Delta}_k\mathfrak{r}\rangle|\right)^{\frac{1}{2}} \\
&\leq C_{\mathrm{B}}\sum_{k\in\mathbb{Z}}\sum_{|k'-k|\leq 4} 2^{\frac{k'}{2}}\left(\int_0^t e^{\beta_\star}\|\nabla\mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_{k'}\mathfrak{r}\|_{L^2}\|\dot{\Delta}_k\mathfrak{r}\|_{L^2}\right)^{\frac{1}{2}} \\
&\leq 10^2 C_{\mathrm{B}}\sum_{k'\in\mathbb{Z}} 2^{\frac{k'}{2}}\left(\int_0^t e^{\beta_\star}\|\nabla\mathfrak{r}\|_{\dot{B}^0}\|\dot{\Delta}_{k'}\mathfrak{r}\|_{L^2}^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.115}$$

This concludes the proof of (5.85).

5.6.6 Proof of (5.86)

First, using integration by parts in the vertical direction, and the null boundary condition in (5.48), we get

$$\langle\dot{\Delta}_k e^{\rho|\partial_x|}\,\partial_y\mathbf{R}(r_2, r), \dot{\Delta}_k\mathfrak{r}\rangle = -\langle\dot{\Delta}_k e^{\rho|\partial_x|}\,\mathbf{R}(r_2, r), \dot{\Delta}_k\partial_y\mathfrak{r}\rangle \tag{5.116}$$

Using the support properties of the paraproduct decomposition as in Lemma 5.6, equality (5.31), and the definition (5.29) we expand the term as

$$\begin{aligned}
\langle\dot{\Delta}_k e^{\rho|\partial_x|}\,\mathbf{R}(r_2, r), \dot{\Delta}_k\partial_y\mathfrak{r}\rangle &= \sum_{k'\geq k-3}\langle e^{\rho|\partial_x|}\,(\dot{\Delta}_{k'}r_2)(\widetilde{\dot{\Delta}}_{k'}r), \dot{\Delta}_k\partial_y\mathfrak{r}\rangle \\
&= \sum_{k'\geq k-3}\sum_{|k''-k'|\leq 1}\langle e^{\rho|\partial_x|}\,(\dot{\Delta}_{k'}r_2)(\dot{\Delta}_{k''}r), \dot{\Delta}_k\partial_y\mathfrak{r}\rangle
\end{aligned} \tag{5.117}$$

Thanks to the product estimate (5.35) from Lemma 5.7, we obtain

$$\begin{aligned}
&|\langle e^{\rho|\partial_x|-\beta}(\dot{\Delta}_{k'}r_2)(\dot{\Delta}_{k''}r), \dot{\Delta}_k\partial_y\mathfrak{r}\rangle| \\
&\qquad\leq e^{\beta_\star}\|\dot{\Delta}_{k'}\mathfrak{r}_2^+\|_{L^2_x(L^\infty_y)}\|\dot{\Delta}_{k''}\mathfrak{r}^+\|_{L^2_x(L^2_y)}\|\dot{\Delta}_k\partial_y\mathfrak{r}^+\|_{L^\infty_x(L^2_y)}.
\end{aligned} \tag{5.118}$$

Thanks to estimates (5.23) from Lemma 5.5 and (5.16) from Lemma 5.2, we get

$$|\langle e^{\rho|\partial_x|-\beta}(\dot{\Delta}_{k'}r_2)(\dot{\Delta}_{k''}r), \dot{\Delta}_k\partial_y\mathfrak{r}\rangle| \leq e^{\beta_\star}C_{\mathrm{B}}^2 2^{\frac{k}{2}+\frac{k'}{2}}\|\dot{\Delta}_{k'}\mathfrak{r}\|_{L^2}\|\dot{\Delta}_{k''}\mathfrak{r}\|_{L^2}\|\dot{\Delta}_k\partial_y\mathfrak{r}\|_{L^2}. \tag{5.119}$$

We use the following crude estimate, which follows from definition (5.15).

$$\|\dot{\Delta}_k \partial_y \mathfrak{r}\|_{L^2} \leq \|\nabla \mathfrak{r}\|_{\dot{B}^0}. \tag{5.120}$$

Gathering our estimates, we obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left(\int_0^t |\langle \dot{\Delta}_k e^{\rho|\partial_x|-\beta} \partial_y \mathbf{R}(r_2, r), \dot{\Delta}_k \mathfrak{r} \rangle| \right)^{\frac{1}{2}} \\
&\leq C_{\mathrm{B}} \sum_{k\in\mathbb{Z}} \sum_{k'\geq k-3} \sum_{|k''-k'|\leq 1} 2^{\frac{k}{4}+\frac{k'}{4}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_{k'} \mathfrak{r}\|_{L^2} \|\dot{\Delta}_{k''} \mathfrak{r}\|_{L^2} \right)^{\frac{1}{2}} \\
&\leq 6 C_{\mathrm{B}} \sum_{k\in\mathbb{Z}} \sum_{k'\geq k-3} 2^{\frac{k}{4}+\frac{k'}{4}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_{k'} \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\leq 6 C_{\mathrm{B}} \sum_{k'\in\mathbb{Z}} 2^{\frac{k'}{2}} \left(\sum_{k\leq k'+3} 2^{\frac{k-k'}{4}} \right) \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_{k'} \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\leq 10^2 C_{\mathrm{B}} \sum_{k'\in\mathbb{Z}} 2^{\frac{k'}{2}} \left(\int_0^t e^{\beta_\star} \|\nabla \mathfrak{r}\|_{\dot{B}^0} \|\dot{\Delta}_{k'} \mathfrak{r}\|_{L^2}^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.121}$$

This concludes the proof of (5.86).

6 Analytic estimates for the approximate trajectories

6.1 Preliminary estimates

We introduce notations and prove preliminary estimates that will be used in the sequel. In this paragraph, a denotes a function in $L^2(\mathfrak{B})$ for which all the norms and sums that we manipulate are finite.

- For $s \in \mathbb{N}$, $\rho > 0$ and $p \in \{2, +\infty\}$, we introduce the tangential analytic-type norm

$$\mathcal{G}^{\rho}_{s,p}(a) := \sum_{0\leq\alpha+\beta\leq s} \sup_{m\geq 0} \frac{\rho^m}{m!} \|\partial_x^m \partial_x^\alpha \partial_y^\beta a\|_{L^2_x(L^p_y)}. \tag{6.1}$$

  In particular, thanks to condition (1.11), one has $\mathcal{G}^{\rho_b}_{3,2}(u_b) \leq C_b$.

- We introduce the notations $u_i^1 = \nabla^\perp[\chi_\delta \psi_b]$ and $u_o^1 := \nabla^\perp[(1-\chi_\delta)\psi_b]$. Hence, recalling the translation notation (3.13), the definition (3.18) of u^1 can be recast as

$$u^1 = \beta \tau_h u_i^1 + \tau_h u_o^1. \tag{6.2}$$

  From the definition (3.17) of ψ_b, the smoothness of χ_δ and the conditions (1.11) we deduce that there exists C_1 such that,

$$\mathcal{G}^{\rho_b}_{3,2}(u_i^1) + \mathcal{G}^{\rho_b}_{3,2}(u_o^1) \leq C_1. \tag{6.3}$$

  Thus, using the relation (6.2), for any $t \geq 0$,

$$\mathcal{G}^{\rho_b}_{3,2}(u^1(t)) \leq C_1. \tag{6.4}$$

- Let $p \in \{2, +\infty\}$. Thanks to the Peter-Paul inequality, then using the property (5.11) and the support property (5.7) for the operator $\dot{\Delta}_k$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k a\|_{L^2_x(L^p_y)} &\leq \Big(\sum_{k\in\mathbb{Z}} 2^{-\frac{|k|}{2}}\Big)^{\frac{1}{2}}\Big(\sum_{k\in\mathbb{Z}} 2^{\frac{|k|}{2}}\|e^{\rho_0|\partial_x|}\dot{\Delta}_k a\|^2_{L^2_x(L^p_y)}\Big)^{\frac{1}{2}} \\ &\leq \frac{3}{2\pi^2}\Big(\int (|\xi|^{\frac{1}{2}} + |\xi|^{-\frac{1}{2}})e^{2\rho_0|\xi|}\|\mathcal{F}a(\xi)\|^2_{L^p_y}\,\mathrm{d}\xi\Big)^{\frac{1}{2}}\end{aligned} \tag{6.5}$$

For low frequencies, using a uniform bound for the Fourier transform yields

$$\int_{|\xi|\leq 1} (|\xi|^{\frac{1}{2}} + |\xi|^{-\frac{1}{2}})e^{\rho_0|\xi|}\|\mathcal{F}a(\xi)\|^2_{L^p_y}\,\mathrm{d}\xi \leq \frac{16}{3}e^{\rho_0}\|a\|^2_{L^1_x(L^p_y)}. \tag{6.6}$$

For high frequencies, using the elementary inequality $e^x \leq 2\cosh x$,

$$\int_{|\xi|\geq 1} (|\xi|^{\frac{1}{2}} + |\xi|^{-\frac{1}{2}})e^{\rho_0|\xi|}\|\mathcal{F}a(\xi)\|^2_{L^p_y}\,\mathrm{d}\xi \leq 16\pi^2 \sum_{m\geq 0} \frac{(2\rho_0)^{2m}}{(2m)!}\|\partial_x^m \partial_x a\|^2_{L^2_x(L^p_y)}. \tag{6.7}$$

Hence, for any $\rho > \rho_0$, there exists $C_\rho > 0$ such that,

$$\sum_{k\in\mathbb{Z}} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k a\|_{L^2_x(L^p_y)} \leq C_\rho\left(\|a\|_{L^1_x(L^p_y)} + \mathcal{G}^\rho_{1,p}(a)\right). \tag{6.8}$$

6.2 Estimates for the amplification terms

We prove Proposition 5.9. Recalling the definition (5.43) of β, we proceed term by term.

- Recalling the definition (4.1) of the space $H^{1,1}(\mathbb{R}_+)$ and using the decay estimate (4.2) for the boundary layer from Lemma 4.1 with $n = 1$, we get

$$\begin{aligned}\int_0^{+\infty} \|\chi'\{v^0(s)\}\|_{L^\infty(\mathfrak{B})}\,\mathrm{d}s &\leq \|\chi'\|_{L^\infty(-1,1)}\int_0^{+\infty}\|V(s)\|_{L^\infty(\mathbb{R}_+)}\,\mathrm{d}s \\ &\leq 2\|\chi'\|_{L^\infty(-1,1)}\int_0^{+\infty}\|V(s)\|_{H^{1,1}(\mathbb{R}_+)}\,\mathrm{d}s \\ &\leq 2C\|\chi'\|_{L^\infty(-1,1)}\int_0^{+\infty}\left|\frac{\ln(2+s)}{2+s}\right|^{\frac{5}{4}}\,\mathrm{d}s \\ &\leq 52C\|\chi'\|_{L^\infty(-1,1)}.\end{aligned} \tag{6.9}$$

- Recalling the definition (3.22) of w^ε and using estimate (4.11) from Lemma 4.3,

$$\varepsilon^2\int_0^{T/\varepsilon^\kappa}\|\partial_y w^\varepsilon(t)\|_{L^\infty(\mathfrak{B})}\,\mathrm{d}t \leq T\varepsilon^{2-\kappa}\|\partial_y W^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^\infty(\mathfrak{B}))} \leq C_W T\varepsilon^{1-\kappa+\frac{1}{4}}. \tag{6.10}$$

- Recalling the definition (5.42) of ℓ_1 and applying estimate (6.8) to $a = \nabla u^1(t)$ and (6.4), there holds for any $t \geq 0$,

$$\begin{aligned}\ell_1(t) &\leq C_{\rho_b}\left(\|\nabla u^1(t)\|_{L^1_x(L^2_y)} + \mathcal{G}^{\rho_b}_{1,2}(\nabla u^1(t))\right) \\ &\leq C_{\rho_b}\left(\|\nabla u^1_i\|_{L^1_x(L^2_y)} + \|\nabla u^1_o\|_{L^1_x(L^2_y)} + C_1\right) \\ &\leq C_{\rho_b}\left(2C_b + C_1\right),\end{aligned} \tag{6.11}$$

thanks to estimate (1.12).

- Recalling the definition (5.41) of α_ε,

$$\int_0^{T/\varepsilon^\kappa} \alpha_\varepsilon(t)\,\mathrm{d}t \leq T + \frac{\pi}{2}. \tag{6.12}$$

Gathering these four estimates concludes the proof of Proposition 5.9.

6.3 Estimates for the source terms

We prove Proposition 5.11. We will use the two following inequalities. First, for $g \in L^2((0,T/\varepsilon^\kappa)\times\mathfrak{B})$, thanks to the first part of the definition (5.41) of α_ε

$$\sum_{k\in\mathbb{Z}} \left(\int_0^{T/\varepsilon^\kappa} \frac{1}{\alpha_\varepsilon} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k(\varepsilon g)\|^2_{L^2(\mathfrak{B})} \right)^{\frac{1}{2}} \leq \varepsilon^{1-\kappa} \sum_{k\in\mathbb{Z}} \sup_{t\geq 0} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k g(t)\|_{L^2(\mathfrak{B})}. \tag{6.13}$$

For $g \in L^2((0,T/\varepsilon^\kappa)\times\mathfrak{B})$ and $\mathcal{V} \in L^2(\mathbb{R}_+\times\mathbb{R}_+)$, thanks to the second part of the definition (5.41) of α and estimate (4.9) from Lemma 4.2,

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \left(\int_0^{T/\varepsilon^\kappa} \frac{1}{\alpha_\varepsilon} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k(\{\mathcal{V}\}g)\|^2_{L^2(\mathfrak{B})} \right)^{\frac{1}{2}} \\ &\quad \leq 2\varepsilon^{\frac{1}{4}} \left(\int_{\mathbb{R}_+} (1+t^2)\|V(t)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}t \right)^{\frac{1}{2}} \sum_{k\in\mathbb{Z}} \sup_{t\geq 0} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k g(t)\|_{L^2_x(L^\infty_y)}. \end{aligned} \tag{6.14}$$

Recalling the definition (3.24) of $f^\varepsilon_{\mathrm{app}}$, we proceed slightly differently for the first four terms (using (6.13)), and for the last three terms (using (6.14)).

$$\begin{aligned} f^\varepsilon_{\mathrm{app}} := &-\varepsilon\Delta u^1 + \varepsilon(u^1\cdot\nabla)u^1 + \varepsilon^2 W^\varepsilon \partial_x u^1 + \varepsilon^2(u^1\cdot e_y)\partial_y w^\varepsilon \\ &- \chi\{V\}\partial_x u^1 - \frac{u^1\cdot e_y}{\varphi}\left(\sqrt{\varepsilon}\chi'\{zV\} + \chi\varphi'\{z\partial_z V\}\right) e_x. \end{aligned} \tag{6.15}$$

- *First term.* We apply (6.13) to $g := -\Delta u^1$. For $t \geq 0$, thanks to (6.2),

$$\begin{aligned} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta u^1(t)\|_{L^2(\mathfrak{B})} &= \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta(\beta(t)(\tau_h u^1_i)(t) + (\tau_h u^1_o)(t))\|_{L^2(\mathfrak{B})} \\ &= \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta(\beta(t)u^1_i + u^1_o)\|_{L^2(\mathfrak{B})} \\ &\leq \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta u^1_i\|_{L^2(\mathfrak{B})} + \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta u^1_o\|_{L^2(\mathfrak{B})} \end{aligned} \tag{6.16}$$

  Thanks to (6.8), (6.3) and (1.12), we deduce that

$$\sum_{k\in\mathbb{N}} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta u^1_i\|_{L^2(\mathfrak{B})} + \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta u^1_o\|_{L^2(\mathfrak{B})} < +\infty. \tag{6.17}$$

  Hence,

$$\sum_{k\in\mathbb{Z}} \sup_{t\geq 0} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k\Delta^1_u(t)\|_{L^2(\mathfrak{B})} < +\infty. \tag{6.18}$$

- *Second term.* We apply (6.13) to $g := (u^1 \cdot \nabla)u^1$. Similarly, we decompose u^1 thanks to (6.2) to get an estimate which is uniform with respect to time. We get four terms. As an example, let us bound using (6.8), for some $\rho \in (\rho_0, \rho_b)$,

$$\sum_{k\in\mathbb{Z}} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k(u_i^1\cdot\nabla)u_i^1\|_{L^2(\mathfrak{B})} \leq \tilde{C}_\rho \left(\|(u_i^1 \cdot \nabla)u_i^1\|_{L^1_x(L^2_y)} + \mathcal{G}^\rho_{1,2}((u_i^1 \cdot \nabla)u_i^1)\right). \quad (6.19)$$

The first term is finite thanks to (1.12). The second term requires an analytic estimate for this quadratic term. For $m \geq 0$, and $a, b \in L^2(\mathfrak{B})$, thanks the Leibniz differentiation rule,

$$\begin{aligned}
\frac{\rho^m}{m!}\|\partial_x^m[(a \cdot \nabla)b]\|_{H^1(\mathfrak{B})} &\leq \sum_{j=0}^m \binom{m}{j}\frac{\rho^m}{m!}\|((\partial_x^j a) \cdot \nabla)(\partial_x^{m-j} b)\|_{H^1(\mathfrak{B})} \\
&\leq \sum_{j=0}^m \binom{m}{j}\frac{\rho^m}{m!}\frac{j!}{\rho_b^j}\mathcal{G}^{\rho_b}_{1,2}(a)\frac{(m-j)!}{\rho_b^{m-j}}\mathcal{G}^{\rho_b}_{2,2}(b) \\
&\leq (m+1)\left(\frac{\rho}{\rho_b}\right)^m \mathcal{G}^{\rho_b}_{1,2}(a)\mathcal{G}^{\rho_b}_{2,2}(b) \\
&\leq C\mathcal{G}^{\rho_b}_{1,2}(a)\mathcal{G}^{\rho_b}_{2,2}(b),
\end{aligned} \quad (6.20)$$

for some $C > 0$ independent of m because $\rho < \rho_b$. Hence,

$$\sum_{k\in\mathbb{Z}} \sup_{t\geq 0} \|e^{\rho_0|\partial_x|}\dot{\Delta}_k(u^1(t) \cdot \nabla)u^1(t)\|_{L^2(\mathfrak{B})} < +\infty. \quad (6.21)$$

- *Third and fourth terms.* We apply (6.13) to $g := \varepsilon W^\varepsilon \partial_x u^1 + \varepsilon(u^1 \cdot e_y)\partial_y w^\varepsilon$. Since W^ε does not depend on x, we use the estimate

$$\begin{aligned}
\|e^{\rho_0|\partial_x|}\dot{\Delta}_k g(t)\|_{L^2(\mathfrak{B})} &\leq \varepsilon(\|W^\varepsilon\|_\infty + \|\partial_y w^\varepsilon\|_\infty)\times \\
&\quad (\|e^{\rho_0|\partial_x|}\dot{\Delta}_k\partial_x u^1(t)\|_{L^2(\mathfrak{B})} + \|e^{\rho_0|\partial_x|}\dot{\Delta}_k u^1(t)\|_{L^2(\mathfrak{B})}) \\
&\leq \varepsilon^{\frac{1}{4}} C_W(\|e^{\rho_0|\partial_x|}\dot{\Delta}_k\partial_x u^1(t)\|_{L^2(\mathfrak{B})} + \|e^{\rho_0|\partial_x|}\dot{\Delta}_k u^1(t)\|_{L^2(\mathfrak{B})})
\end{aligned} \quad (6.22)$$

thanks to estimate (4.11) from Lemma 4.3. We proceed as above using (6.2) to get uniform bounds on the sums.

- *Fifth term.* We apply (6.14) to $g := \chi\partial_x u^1$ and $\mathcal{V} := V$. The integral in time of the boundary layer is finite thanks to (4.2) from Lemma 4.1 because $n \geq 3$. The sum in k is also finite using the same techniques as above.

- *Sixth term.* We apply (6.14) to $g := \chi'(u^1 \cdot e_y)/\varphi$ and $\mathcal{V} := zV$. The integral in time of the boundary layer is finite thanks to (4.2) from Lemma 4.1 because $n \geq 3$. The sum in k is also finite using the same techniques as above.

- *Seventh term.* We apply (6.14) to $g := \chi\varphi'(u^1 \cdot e_y)/\varphi$ and $\mathcal{V} := z\partial_z V$. The integral in time of the boundary layer is finite thanks to (4.2) from Lemma 4.1 because $n \geq 3$. The sum in k is also finite using the same techniques as above.

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