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# Estimation of Local Anisotropy Based on Level Sets

Corinne Berzin\*

Consider an affine Gaussian field  $X : \mathbb{R}^2 \rightarrow \mathbb{R}$ , that is a process equal in law to  $Z(At)$ , where  $Z$  is isotropic and  $A : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a self-adjoint definite positive matrix. Denote  $0 < \lambda = \frac{\lambda_2}{\lambda_1} \leq 1$  the ratio of the eigenvalues of  $A$ . This paper is aimed at testing the null hypothesis " $X$  is isotropic" versus the alternative " $X$  is affine". Roughly speaking, this amounts to testing " $\lambda = 1$ " versus " $\lambda < 1$ ". By setting level  $u$  in  $\mathbb{R}$ , this is implemented by the partial observations of process  $X$  through some particular level functionals viewed over a square  $T$ , which grows to  $\mathbb{R}^2$ . This leads us to provide estimators for the affinity parameters that are shown to be almost surely consistent. Their asymptotic normality results in confidence intervals for parameters.

This paper offered an important opportunity to study general level functionals near the level  $u$ , part of the difficulties arises from the fact that the topology of level set  $C_{T,X}(u) = \{t \in T : X(t) = u\}$  can be irregular, even if the trajectories of  $X$  are regular. A significant part of the paper is dedicated to show the  $L^2$ -continuity in the level  $u$  of these general functionals.

**Keywords:** Affine processes; isotropic processes; level sets; Rice formulas for random fields; test of isotropy; Gaussian fields..

**AMS MSC 2010:** Primary 62G10; 53C65; 62F12; 60G60, Secondary 60G10; 60G15.

## 1 Introduction

The aim of the present paper is to test the null hypothesis that a given Gaussian process  $X$  indexed in  $\mathbb{R}^2$  and living in the class of affine processes is isotropic. We assume that  $X$  is partially observed through some level functionals of its level curve  $C_{T,X}(u)$  for a fixed level  $u$ , say  $C_{T,X}(u) = \{t \in T : X(t) = u\}$ . The set  $T$  is a bounded square of  $\mathbb{R}^2$ , having the following shape  $] -n, n[^2$  and we are interested in the asymptotic as the square  $T$  tends to  $\mathbb{R}^2$ . Many methods have been provided to test such an isotropy. An overview of methods have been very well summarized in Weller et al. [24]. Also a lot of papers have been devoted to propose estimators of  $\theta$ , the field deformation parameter, modelled as  $X_\theta = Z \circ \theta$ , where  $Z$  is isotropic. Among them, we can cite Allard et al. [2], Anderes et al. [3], Anderes and Stein [4], Fouedjio et al. [13], Perrin and Senoussi [20] and Sampson and Guttorp [22].

Note that in [2], the authors focus on the case where the deformation is linear which will also be our aim. However, as far as we know, very little research are based on the use of level sets of the observed process  $X_\theta$  and those quoted previously are not an

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exception to the rule. In some cases, we may be interested in the use of the information given by some functional of level sets of the process rather than given by a simple realization of the process itself. This is done for example in Fournier [14] where the author chose as particular functional the Euler characteristic of some of excursion sets of the field. In doing so, she proposes an identification of the matrix parameter  $\theta$  which is a determination of this matrix up to an unknown rotation with two alternative angles. Having in mind to estimate the affinity parameters by using the level sets of the process  $X_\theta$  and in the case where the deformation is a linear one, our starting point was an example extract from the Lecture Notes [25].

In this example Wschebor proposes estimators of local anisotropy of a process  $X$  indexed in  $\mathbb{R}^2$  based on the shape of its level curve, corresponding to a given level  $u$ . The author explains as preamble that his work was largely based on Cabaña [8], although his estimators were different. In [8], according to his own words, the author defines the class of affine processes as a reasonable alternative to test the null hypothesis that a given stationary process is isotropic. Estimators for the affinity parameters based on particular functionals of the level set of the observed process are provided and they are shown to be consistent under uniform mixing conditions. These particular functionals being the measure of dimensional area of the level set by unit of volume, say  $J_1^{(n)}(u)$  and also the integrals on the level set of the cosine and of the sine of the angle of the gradient process, these estimates lead to a test of isotropy. Wschebor proposed probability consistent estimators of anisotropy directions  $\theta_o$ , say  $\hat{\theta}_{o,n}$  and also of its value  $\lambda$ , say  $\hat{\lambda}_n$ , based on the observation of the ratio of functionals  $J_{f^*}^{(n)}(u)$  and  $J_1^{(n)}(u)$  where  $J_{f^*}^{(n)}(u)$  is the integral on the level set of a chosen particular function  $f^*$ , taking values in  $\mathbb{R}^2$ , evaluated in the value of the normalized gradient of  $X$  and being its value if this one lives in the same half-plane as that of  $v^*$ , a fixed unitary vector in  $\mathbb{R}^2$ , and minus this value if not. Here a condition of  $\eta$ -dependence was required to obtain consistent estimators.

Our purpose consists in revisiting the example of Wschebor [25] by taking into account the ideas developed in Cabaña [8].

**Main contribution of the paper** In the present work following the way opened by Wschebor, we consider his proposed estimators  $(\hat{\lambda}_n, \hat{\theta}_{o,n})$ . Our main contribution has consisted in the one hand, by using the Birkhoff-Kinchine ergodic theorem of Cramér and Leadbetter [9] and Rice formula (see the seminal work of Rice [21]) to establish the almost surely consistence of those estimators (Theorem 3.12). In the second hand our contribution was to propose a Central Limit Theorem (CLT) (Theorem 5.5) for those estimators and some confidence intervals (Corollary 5.6). This was done by breaking loose of the hypothesis of  $\eta$ -dependence of process  $X$ . This leads us also and as in Section 2.1 of [8] to propose statistical tests for the null hypothesis " $X$  is isotropic" versus the alternative " $X$  is affine", those one suggesting by the properties of the estimators (Theorem 5.16). The tools for proving the asymptotic normality of the estimators is the use of the technique of the CLT for functionals  $J_f^{(n)}(u)$ ,  $u$  being a fixed level in  $\mathbb{R}$  and  $f$  being a general function belonging to  $L^2$ , taking values in  $\mathbb{R}$ , evaluated in the value of the normalized gradient of  $X$ , these functionals belonging to the Wiener chaos. This method has been developed by Nourdin et al. [17], Nualart et al. [18] and Peccati and Tudor [19] among others. The idea for proving such a TCL is inspired by a precursor work of Kratz and León [15]. In this one and in the case where the process  $X$  is a stationary Gaussian isotropic process indexed by  $\mathbb{R}^2$ , the authors propose a way to approximate the length of the level curve  $J_1^{(n)}(u)$  by other functionals  $J_1^{(n)}(u, \sigma)$  ( $\sigma \rightarrow 0$ ) with the help of a kernel  $K$  tending to the delta-Dirac function in  $u$ . This is done in such a way that the

approximating functional can be expressed into the Wiener chaos. It consists in using the coarea formula (see Theorem 3.2.5 p 244 of Federer [12]) for  $J_f^{(n)}(u, \sigma)$ , transforming then this functional initially expressed on the level curve as a temporal functional on the square  $T$  and getting then its Hermite expansion in  $H(X)$  the space of real square integral functionals of the field  $X$ . By using this technic we were enabled by taking  $\sigma \rightarrow 0$  to express in turn the initial functional  $J_f^{(n)}(u)$  into the Wiener chaos (Proposition 4.9) and then to explicit its asymptotic variance (Proposition 4.12) and finally by applying the Peccati-Tudor method [19] to obtain the CLT (Theorem 4.14). The way of proceeding is completely based on the methods developed into the paper Estrade and León [11] itself inspired by the article [15]. In this work the authors show a CLT for the Euler characteristic of the excursions above  $u$  of the field  $X$  on  $T$  as  $T$  grows to  $\mathbb{R}^d$ ,  $X$  being a stationary Gaussian isotropic process indexed in  $\mathbb{R}^d$ .

Our real contribution for proving the CLT, apart from showing the non degeneration of the asymptotic limit matrix variance (Remark 5.3), was to rely on the two functionals  $J_f^{(n)}(u)$  and  $J_f^{(n)}(u, \sigma)$  (Proposition 4.2), that is to show that  $J_f^{(n)}(u, \sigma)$  is an  $L^2$ -convergent approximation of  $J_f^{(n)}(u)$ . It was the opportunity to obtain as a bonus the  $L^2$ -continuity in the level  $u$  of function  $E[J_f^{(n)}(u)]$  (Theorem 4.3), that is a very nice interesting result in itself. We did not find it in the literature and we believe that this result deserves consideration. The proof is far from obvious and implements a number of ideas developed in Berzin et al. [7], from which a local parametrization of the level set  $C_{T,X}(u + \delta)$  near the level  $u$  (see Theorem 3.1.2 of [7]).

**Outline of the paper** Section 2 contains some definitions, assumptions and notations, among others definitions of an affine process, of an isotropic process and explicit the type of general functionals on the level set  $u$ , say  $J_f^{(n)}(u)$ , we are looking for.

Section 3 is devoted to establish a Rice formula for such functionals, in other words to compute  $E[J_f^{(n)}(u)]$  and to show the almost sure convergence of these ones. Also this is to enable us by choosing a particular function  $f^*$  to define the affinity estimators  $\hat{\lambda}_n$  and  $\hat{\theta}_{o,n}$  of the process and to prove their almost sure consistency.

In Section 4, we focus on the convergence in law of  $J_f^{(n)}(u)$  for general function  $f$ . First by using the coarea formula for an approximation of the functional, say  $J_f^{(n)}(u, \sigma)(\sigma \rightarrow 0)$ , we express this last one into the Itô-Wiener chaos from which we deduce an Hermite expansion for the initial functional. It is made by establishing the  $L^2$ -convergence of  $J_f^{(n)}(u, \sigma)$  towards  $J_f^{(n)}(u)$ , thereby showing the  $L^2$ -continuity in the level  $u$  for  $J_f^{(n)}(u)$ . Then the asymptotic variance as the square  $T$  grows to  $\mathbb{R}^2$  is expressed as a series and we give an explicit lower bound. We then proved the asymptotic normality for  $J_f^{(n)}(u)$  through the Peccati-Tudor theorem.

The results obtained in Section 4 are used in Section 5 by considering the particular function  $f^*$ , giving rise to a first result, the convergence in law for the estimators of affinity parameters. Also the coefficients of the asymptotic matrix variance are computed in Appendix A and here a lower bound is given for its determinant. Secondly this law convergence result gives rise to confidence intervals for parameters  $\lambda$  and  $\theta_o$  in the specific special case where the covariance  $r_z$  of the underlying isotropic process is known. Some complementary convergence results for  $\hat{\lambda}_n$  are proposed when the affinity parameters  $\lambda$  and  $\theta_o$  belong to the edges of the parameter space, including the particular case where  $\lambda = 1$ . Finally, supposing that the covariance  $r_z$  is known, we conclude by proposing an isotropy test.

This paper is complemented with Appendix A giving technical proofs of some lemmas.

## 2 Hypothesis and notations

Let us give some definitions, assumptions and notations.

A process  $(Z(t), t \in \mathbb{R}^2)$  is said to be *isotropic* if it is a stationnary process and if for any isometry  $U$  in  $\mathbb{R}^2$ ,  $k \in \mathbb{N}$  and  $t_1, \dots, t_k \in \mathbb{R}^2$ , the joint laws of  $(Z(t_1), \dots, Z(t_k))$  and  $(Z(U(t_1)), \dots, Z(U(t_k)))$  are the same.

In what follows  $(X(t), t \in \mathbb{R}^2)$  is an *affine* process, that is equal in law to  $(Z(A(t)), t \in \mathbb{R}^2)$ , where  $Z$  is isotropic and  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear self-adjoint and positive definite transformation. The eigenvalues of  $A$  are denoted by  $\lambda_1, \lambda_2$ ,  $0 < \lambda_2 \leq \lambda_1$ . Let  $0 < \lambda \leq 1$  be the quotient of the eigenvalues,  $\lambda = \frac{\lambda_2}{\lambda_1}$ . The process  $X$  is isotropic means that  $\lambda = 1$  (Adler and Taylor [1], Section 5.7).

The process  $X : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is a centered Gaussian process twice continuously differentiable in  $\mathbb{R}^2$ , that is  $Z \in \mathbf{C}^2(\mathbb{R}^2)$ .

Denoting by  $r_z$  (resp.  $r_x$ ) the covariance function for  $Z$  (resp. for  $X$ ), the regularity assumption on  $Z$  implies that  $r_z \in C^4(\mathbb{R}^2)$ .

**Assumptions on the covariance** For any multidimensional index  $\mathbf{m} = (i_1, \dots, i_k)$  with  $0 \leq k \leq 2$  and  $1 \leq i_j \leq 2$ , we write  $\frac{\partial^{\mathbf{m}} r_z}{\partial t^{\mathbf{m}}}(t) = \frac{\partial^k r_z}{\partial t_{i_1} \dots \partial t_{i_k}}(t)$ .

Let  $\Psi(t) = \max \left\{ \left| \frac{\partial^{\mathbf{m}} r_z}{\partial t^{\mathbf{m}}}(t) \right|, \mathbf{m} \in \{1, 2\}^k, 0 \leq k \leq 2 \right\}$ . We make the assumption that  $\Psi(t) \rightarrow 0$  when  $\|t\|_2 \rightarrow +\infty$ ,  $\Psi \in L^1(\mathbb{R}^2)$  and  $\int_{\mathbb{R}^2} r_z(t) dt > 0$ , where  $\|\cdot\|_2$  denotes the Euclidean norm in  $\mathbb{R}^2$ . We denote by  $\langle \cdot, \cdot \rangle$  the canonical scalar product in  $\mathbb{R}^2$ .

Note that  $r_z \in L^1(\mathbb{R}^2)$  implies that  $r_z \in L^q(\mathbb{R}^2)$  for all  $q \geq 1$  and hence that  $Z$  (resp.  $X$ ) admits a spectral density  $f_z$  (resp.  $f_x$ ) that is continuous and such that  $f_z(0) > 0$ .

We suppose that  $\int_{\mathbb{R}^2} f_z(t) \|t\|_2^2 dt < +\infty$ .

Let  $T$  be an open bounded square of  $\mathbb{R}^2$ , with the following form  $T = ]-n, n[^2$  with  $n \in \mathbb{N}^* = \{x \in \mathbb{Z}, x > 0\}$ , a positive integer, and let  $n$  tend to infinity. We suppose that  $r_z^2(0) - r_z^2(t) \neq 0$ , for all  $t \neq 0$  and  $t \in A \times (\bar{T} - \bar{T})$ , where  $\bar{T}$  is the closure of  $T$ .

**Level set** For  $u \in \mathbb{R}$  we define the level set at  $u$  as:

$$C_{T,X}(u) = \{t \in T : X(t) = u\}$$

and we denote  $D_X^r$  the following set

$$D_X^r = \{t \in \mathbb{R}^2 : \nabla X(t) \text{ is of rank } 1\} = \{t \in \mathbb{R}^2 : \|\nabla X(t)\|_2 \neq 0\}.$$

Also  $C_{T,X}^{D^r}(u)$  denotes the level set,  $C_{T,X}^{D^r}(u) = C_{T,X}(u) \cap D_X^r$ .

**General level functionals**  $S^1$  is the boundary of the unit ball of  $\mathbb{R}$  and for  $d = 1, 2$ ,  $\sigma_d$  denotes the Lebesgue measure on  $\mathbb{R}^d$ .

For  $f : S^1 \rightarrow \mathbb{R}^d$  a continuous and bounded function, we define the following general functional  $J_f^{(n)}(u)$  of the fixed level  $u$  by:

$$J_f^{(n)}(u) = \frac{1}{\sigma_2(T)} \int_{C_{T,X}(u)} f(\nu_X(t)) d\sigma_1(t),$$

where  $\nu_X(t) = \frac{\nabla X(t)}{\|\nabla X(t)\|_2}$ , while  $\nabla X$  stands for the Jacobian of  $X$ .

**Hermite polynomials** We use the Hermite polynomials  $(H_n)_{n \in \mathbb{N}}$  defined by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2}).$$

They provide an orthogonal basis of  $L^2(\mathbb{R}, \phi(x) dx)$  where  $\phi$  denotes the standard Gaussian density on  $\mathbb{R}$ . We also denote by  $\phi_m$  the standard Gaussian density on  $\mathbb{R}^m$ , for  $m = 2$  or  $3$ .

For  $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{N}^3$  and  $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ , we note  $|\mathbf{k}| = k_1 + k_2 + k_3$ ,  $\mathbf{k}! = k_1! k_2! k_3!$  and  $\tilde{H}_{\mathbf{k}}(y) = \prod_{1 \leq j \leq 3} H_{k_j}(y_j)$ .

$C$  is a generic constant that could change of value in the interior of a proof.

### 3 Rice formula

Let  $u$  a fixed level in  $\mathbb{R}$ .

#### 3.1 Almost sure convergence for $J_f^{(n)}(u)$

For  $f : S^1 \rightarrow \mathbb{R}$  a continuous and bounded function, we show that process  $X$  and  $f(\nu_X)$  verify the assumptions of Theorem 3.3.1 of Berzin et al. [7]. Then the one order Rice formula for the general functional  $J_f^{(n)}(u)$  is valid. More precisely we have the

**Proposition 3.1.**  $\mathbb{E}[J_f^{(n)}(u)] = p_{X(0)}(u) \mathbb{E}\left[f\left(\frac{\nabla X(0)}{\|\nabla X(0)\|_2}\right) \|\nabla X(0)\|_2\right].$

*Proof of Proposition 3.1.* Let us show that process  $X$  verifies assumptions of Remark 3.3.1 of Theorem 3.3.1 of [7].

First  $X/T : \Omega \times T \rightarrow \mathbb{R}$  is a stationary Gaussian field belonging to  $C^2(T)$  and  $T$  is a bounded convex open set of  $\mathbb{R}^2$ .

Furthermore by using results given in Section 4.3 of Azaïs and Wschebor [6], one can show that

$$\mathbb{E}\left[\sup_{\mathbf{x} \in T} \|\nabla^2 X(\mathbf{x})\|_{1,2}^{(s)}\right]^2 < +\infty, \quad (3.1)$$

where  $\nabla^2 X$  stands for the  $2 \times 2$  Hessian matrix of  $X$  and  $\|\cdot\|_{1,2}^{(s)}$  stands for the norm in  $\mathfrak{L}_s^2(\mathbb{R}^2, \mathbb{R})$ , the vectorial space of symmetric linear continuous functions from  $\mathbb{R}^2 \times \mathbb{R}^2$  to  $\mathbb{R}$ .

Now let  $Y : \Omega \times T \rightarrow \mathbb{R}$  defined as  $Y(t) = f(\nu_X(t)) = f\left(\frac{\nabla X(t)}{\|\nabla X(t)\|_2}\right)$ .

Process  $Y$  verifies condition (3.18) of [7] since  $Y(t) = G(\nabla X(t))$ , where

$$G : \mathfrak{L}(\mathbb{R}^2, \mathbb{R}) \longrightarrow \mathbb{R} \\ A \mapsto G(A) = f\left(\frac{A}{\|A\|_{1,2}}\right),$$

is a bounded continuous function on  $\mathfrak{L}(\mathbb{R}^2, \mathbb{R})$ , the vectorial space of linear functions from  $\mathbb{R}^2$  to  $\mathbb{R}$  with the norm  $\|\cdot\|_{1,2}$ .

$Y$  is thus a continuous process on  $T$ . Furthermore, for all  $t \in T$ , the vector  $(X(t), \nabla X(t))$  has a density. Indeed, this vector has the same law as that of  $(Z(0), A \times \nabla Z(0))$ . And since process  $Z$  is an isotropic process, it is such that the random variable  $Z(0)$ ,  $\frac{\partial Z}{\partial t_1}(0)$  and  $\frac{\partial Z}{\partial t_2}(0)$  are mutually independent.

The non-degeneration of processes  $Z(0)$  and  $\nabla Z(0)$  provides from the assumptions made on the covariance  $r_z$ , more precisely from the fact that the spectral density  $f_z$  of process  $Z$  is continuous with  $f_z(0) > 0$  and that  $\int_{\mathbb{R}^2} f_z(t) \|t\|_2^2 dt < +\infty$ . Finally, we check if the hypothesis  $\mathbf{H}_6$  is verified. In this context, let us compute the

following integral:

$$\int_T p_{X(t)}(u) \mathbb{E} [ |Y(t)| \|\nabla X(t)\|_2 | X(t) = u ] dt = \sigma_2(T) p_{X(0)}(u) \mathbb{E} \left[ \left| f \left( \frac{\nabla X(0)}{\|\nabla X(0)\|_2} \right) \right| \|\nabla X(0)\|_2 \right] < +\infty,$$

where for all  $t \in T$ ,  $p_{X(t)}(\cdot)$  stands for the density of  $X(t)$ . All conditions are met to apply the Rice formula, and one gets,

$$\begin{aligned} \mathbb{E} [ J_f^{(n)}(u) ] &= \frac{1}{\sigma_2(T)} \int_T p_{X(t)}(u) \mathbb{E} \left[ f \left( \frac{\nabla X(t)}{\|\nabla X(t)\|_2} \right) \|\nabla X(t)\|_2 | X(t) = u \right] dt \\ &= p_{X(0)}(u) \mathbb{E} \left[ f \left( \frac{\nabla X(0)}{\|\nabla X(0)\|_2} \right) \|\nabla X(0)\|_2 \right]. \end{aligned}$$

The proof is completed.  $\square$

In order to propose estimators of the directions of anisotropy and also to estimate its value, we consider a particular functional of the level set  $u$ , that is  $J_{f^*}^{(n)}(u)$ , where the function  $f^*$  is defined as follows. Let  $v^* \in S^1$  **a fixed vector** and consider

$$\begin{aligned} S^1 &\longrightarrow S^1 \\ \theta &\mapsto f^*(\theta) = \theta \times (\mathbb{1}_{\{\langle \theta, v^* \rangle \geq 0\}} - \mathbb{1}_{\{\langle \theta, v^* \rangle < 0\}}). \end{aligned} \tag{3.2}$$

Applying the Rice formula for this particular function  $f^*$  and for the function  $\mathbf{1}$  taking values in  $\mathbb{R}$  and identically equal to one, we show the following corollary.

**Corollary 3.2.**

$$\frac{\mathbb{E} [ J_{f^*}^{(n)}(u) ]}{\mathbb{E} [ J_{\mathbf{1}}^{(n)}(u) ]} = \frac{2}{\pi} \frac{A^2(v^*)}{\|A(v^*)\|_2} \frac{1}{\int_{S^1} \|A\alpha\|_2 d\alpha},$$

where  $d\alpha$  denotes the normalized area measure on  $S^1$ .

**Remark 3.3.** Note that  $J_{\mathbf{1}}^{(n)}(u) = \frac{\sigma_1(C_{T,X}(u))}{\sigma_2(T)}$ . This functional is nothing but the measurement of the dimensional surface of the level set by unit of volume.

*Proof of Corollary 3.2.* Since  $Z$  is an isotropic process, its density  $p_{\nabla Z(0)}(v)$  depends only on  $\|v\|_2$ . We denote it by  $g(\|v\|_2)$ .

Thus if  $f : S^1 \rightarrow \mathbb{R}^2$  is a continuous and bounded function,

$$\begin{aligned} \mathbb{E} [ f(\nu_X(0)) \|\nabla X(0)\|_2 ] &= \mathbb{E} \left[ f \left( \frac{A\nabla Z(0)}{\|A\nabla Z(0)\|_2} \right) \|A\nabla Z(0)\|_2 \right] \\ &= \int_{\mathbb{R}^2} f \left( \frac{Av}{\|Av\|_2} \right) \|Av\|_2 g(\|v\|_2) dv. \end{aligned}$$

Letting  $v = r\alpha$ ,  $\alpha \in S^1$ , one gets

$$\begin{aligned} &\mathbb{E} [ f(\nu_X(0)) \|\nabla X(0)\|_2 ] \\ &= \sigma_1(S^1) \left( \int_0^\infty g(r) r^2 dr \right) \left( \int_{S^1} f \left( \frac{A\alpha}{\|A\alpha\|_2} \right) \|A\alpha\|_2 d\alpha \right) \\ &= \mathbb{E} [\|\nabla Z(0)\|_2] \left( \int_{S^1} f \left( \frac{A\alpha}{\|A\alpha\|_2} \right) \|A\alpha\|_2 d\alpha \right). \end{aligned}$$

In a similar way taking  $f \equiv \mathbf{1}$ , one gets

$$\mathbb{E} [\|\nabla X(0)\|_2] = \mathbb{E} [\|\nabla Z(0)\|_2] \left( \int_{S^1} \|A\alpha\|_2 d\alpha \right).$$

Finally by using Proposition 3.1, one obtains

$$\frac{\mathbb{E}\left[J_f^{(n)}(u)\right]}{\mathbb{E}\left[J_1^{(n)}(u)\right]} = \frac{\int_{S^1} f\left(\frac{A\alpha}{\|A\alpha\|_2}\right) \|A\alpha\|_2 \, d\alpha}{\int_{S^1} \|A\alpha\|_2 \, d\alpha}.$$

Now choosing  $f = f^*$  defined by (3.2) and since  $A$  is a self-adjoint matrix, one has

$$\begin{aligned} \int_{S^1} f^*\left(\frac{A\alpha}{\|A\alpha\|_2}\right) \|A\alpha\|_2 \, d\alpha &= 2 \int_{S^1} A\alpha \mathbf{1}_{\{\langle A\alpha, v^* \rangle \geq 0\}} \, d\alpha \\ &= 2 \int_{S^1} A\alpha \mathbf{1}_{\{\langle \alpha, Av^* \rangle \geq 0\}} \, d\alpha = 2A \int_{S^1 \cap \{\langle \alpha, w^* \rangle \geq 0\}} \alpha \, d\alpha, \end{aligned}$$

where we have noted  $w^* = \frac{Av^*}{\|Av^*\|_2}$ .

If  $R$  denotes the change of basis matrix from the canonical orthonormal basis  $\mathbf{e} = (\vec{i}, \vec{j})$  to an orthonormal basis choosen as  $\mathbf{w} = (w_1^*, w^*)$  and making the change of variable  $\beta = R^t \alpha$  in the last integral ( $^t$  stands for the transpose symbol), one obtains

$$\begin{aligned} &\left( \int_{S^1 \cap \{\langle \alpha, w^* \rangle_{\mathbf{e}} \geq 0\}} \alpha \, d\alpha \right)_{\mathbf{e}} \\ &= \left( \int_{S^1 \cap \{\langle \alpha, (0,1) \rangle_{\mathbf{w}} \geq 0\}} \alpha \, d\alpha \right)_{\mathbf{w}} = \left( 0, \left( \int_{S^1 \cap \{\alpha_2 \geq 0\}} \alpha_2 \, d\alpha \right) \right)_{\mathbf{w}} \\ &= \left( \int_{S^1 \cap \{\alpha_2 \geq 0\}} \alpha_2 \, d\alpha \right) w^* = \left( \int_0^\pi \sin(\theta) \frac{d\theta}{2\pi} \right) w^* = \frac{1}{\pi} w^*. \end{aligned}$$

Thus we have proved that

$$\int_{S^1} f^*\left(\frac{A\alpha}{\|A\alpha\|_2}\right) \|A\alpha\|_2 \, d\alpha = \frac{2}{\pi} Aw^* = \frac{2}{\pi} \frac{A^2 v^*}{\|Av^*\|_2},$$

that yields Corollary 3.2.  $\square$

Now by applying an ergodic theorem for stationnary processes (Cramér and Leadbetter [9], §7.11), we shall show the following general almost sure convergence theorem.

**Theorem 3.4.** *For  $f : S^1 \rightarrow \mathbb{R}$  a continuous and bounded function,*

$$J_f^{(n)}(u) \xrightarrow[n \rightarrow +\infty]{a.s.} \mathbb{E}\left[J_f^{(1)}(u)\right].$$

*Proof of Theorem 3.4.* Let  $f : S^1 \rightarrow \mathbb{R}$  a bounded and continuous fonction. As a first step we suppose that function  $f$  is positive and that the square  $T$  has the following shape:  $T = ]0, n[ \times ]0, n[$ . Lemma 3.5 is proved in Appendix A.

**Lemma 3.5.**

$$\begin{aligned} &\int_0^{n-1} \int_0^{n-1} \int_{C_{]t, t+1[ \times ]s, s+1[}, X(u)} f(\nu_X(x)) \, d\sigma_1(x) \, dt \, ds \\ &\quad - \int_0^1 \int_0^1 \int_{C_{]0, t[ \times ]0, s[}, X(u)} f(\nu_X(x)) \, d\sigma_1(x) \, dt \, ds \\ &\leq \int_{C_{T, X(u)}} f(\nu_X(x)) \, d\sigma_1(x) \leq \\ &\int_0^{n+1} \int_0^{n+1} \int_{C_{]t-1, t[ \times ]s-1, s[}, X(u)} f(\nu_X(x)) \, d\sigma_1(x) \, dt \, ds \end{aligned}$$

Let us note  $H(t, s) = \int_{C_{]0, t[ \times ]0, s[, X(u)} f(\nu_X(x)) d\sigma_1(x)$ .

On the one hand, since function  $f$  is bounded, we have the convergence that follows

$$\frac{1}{(2n)^2} \int_0^1 \int_0^1 H(t, s) dt ds \leq \frac{C}{(2n)^2} \sigma_1(C_{]0, 1[^2, X(u)}) \xrightarrow[n \rightarrow +\infty]{a.s.} 0,$$

last convergence providing from Proposition 3.1. Indeed, since

$E[\sigma_1(C_{]0, 1[^2, X(u)])] < +\infty$ , we deduce that  $\sigma_1(C_{]0, 1[^2, X(u)})$  is almost surely finite. On the other hand, noting by

$$\xi(t, s) = \int_{C_{]t, t+1[ \times ]s, s+1[, X(u)}} f(\nu_X(x)) d\sigma_1(x),$$

one has

$$\begin{aligned} \frac{1}{(2n)^2} \int_0^{n-1} \int_0^{n-1} \int_{C_{]t, t+1[ \times ]s, s+1[, X(u)}} f(\nu_X(x)) d\sigma_1(x) dt ds = \\ \left( \frac{n-1}{2n} \right)^2 \frac{1}{(n-1)^2} \int_0^{n-1} \int_0^{n-1} \xi(t, s) dt ds. \end{aligned}$$

Since process  $X$  is strictly stationary,  $r_z \in L^1(\mathbb{R}^2)$  and the spectral density  $f_z$  is such that  $\int_{\mathbb{R}^2} f_z(t) \|t\|_2^2 dt < +\infty$ , we deduce from [9], that process  $\xi(t, s)$  is an ergodic process. Furthermore from Proposition 3.1 we know that  $E[|\xi(t, s)|] < +\infty$ . By the Birkhoff-Kintchine ergodic Theorem (Cramér and Leadbetter [9]) and last proposition, we deduce that

$$\begin{aligned} \left( \frac{n-1}{2n} \right)^2 \frac{1}{(n-1)^2} \int_0^{n-1} \int_0^{n-1} \xi(t, s) dt ds \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{1}{4} E[\xi(0, 0)] \\ = \frac{1}{4} E[J_f^{(1)}(u)]. \end{aligned}$$

In the same way, one obtains

$$\begin{aligned} \left( \frac{n+1}{2n} \right)^2 \frac{1}{(n+1)^2} \int_0^{n+1} \int_0^{n+1} \int_{C_{]t-1, t[ \times ]s-1, s[, X(u)}} f(\nu_X(x)) d\sigma_1(x) dt ds \\ \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{1}{4} E[J_f^{(1)}(u)]. \end{aligned}$$

Finally, by using Lemma 3.5, one proved that

$$\frac{1}{(2n)^2} \int_{C_{]0, n[^2, X(u)}} f(\nu_X(x)) d\sigma_1(x) \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{1}{4} E[J_f^{(1)}(u)].$$

Now, working in a similar way succesively with  $T = ]-n, 0[ \times ]0, n[$  or  $T = ]-n, 0[ \times ]-n, 0[$  or still with  $T = ]0, n[ \times ]-n, 0[$ , one should prove the same convergence result for each square  $T$ . Finally, if  $f$  is a positive function, by using the linearity of the interest functional one have proved that

$$J_f^{(n)}(u) \xrightarrow[n \rightarrow +\infty]{a.s.} 4 \times \frac{1}{4} E[J_f^{(1)}(u)] = E[J_f^{(1)}(u)].$$

To conclude the proof of Theorem 3.4, we decompose function  $f$  into its negative and into its positive part, that is under the shape,  $f = f^+ - f^-$ , and we apply the previous result to each of the interest functionals  $J_{f^+}^{(n)}(u)$  and  $J_{f^-}^{(n)}(u)$ . □

This last theorem provides consistent estimators for the directions of anisotropy.

### 3.2 Estimation of anisotropy directions

#### 3.2.1 The affinity parameters $\hat{\lambda}_n$ and $\hat{\theta}_{o,n}$

Theorem 3.4 applied to the particular functions  $f^*$  and  $\mathbf{1}$  and the result of convergence of Corollary 3.2 imply that,

$$\frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)} \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} = \frac{2}{\pi} \frac{A^2 v^*}{\|Av^*\|_2} \frac{1}{\int_{S^1} \|A\alpha\|_2 d\alpha}. \quad (3.3)$$

Let  $P = (v_1, v_2)$  be an orthonormal basis of eigenvectors of matrix  $A$ , with  $\lambda_1$  and  $\lambda_2$  their respective eigenvalues and  $\lambda = \frac{\lambda_2}{\lambda_1}$ ,  $0 < \lambda \leq 1$ . The vector  $v^*$  can always be written in this basis:

$$v^* = \cos(\theta_o)v_1 + \sin(\theta_o)v_2.$$

It is always possible to choose  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ .

Indeed,  $\theta_o$  could be the angle between  $v^*$  and the eigenvector corresponding to the highest eigenvalue, because of the symmetry with respect to the point  $(0, 0)$  and the fact that the mapping transforms  $\lambda$  into  $\frac{1}{\lambda}$  and  $\theta_o$  into  $\frac{\pi}{2} - \theta_o$  has this effect.

Our aim is now to provide some estimators of parameters  $\lambda$  and  $\theta_o$ .

Using last decomposition we show that  $\frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]}$  can be expressed in the direct orthonormal basis  $(v^*, v^{**})$  as follows.

**Corollary 3.6.**

$$\begin{aligned} & \frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} \\ &= \frac{1}{I(\lambda)} \left[ (\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{1}{2}} v^* + \frac{\sin(\theta_o) \cos(\theta_o) (\lambda^2 - 1)}{(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{1}{2}}} v^{**} \right], \end{aligned}$$

where  $I(\lambda)$  is the elliptic integral

$$I(\lambda) = \int_0^{\frac{\pi}{2}} (\cos^2(\theta) + \lambda^2 \sin^2(\theta))^{\frac{1}{2}} d\theta. \quad (3.4)$$

*Proof of Corollary 3.6.* One has:

$$A^2 v^* = \lambda_1^2 [\cos(\theta_o) v_1 + \lambda^2 \sin(\theta_o) v_2],$$

and

$$\|Av^*\|_2 = \lambda_1 [\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o)]^{\frac{1}{2}}.$$

Also

$$\int_{S^1} \|A\alpha\|_2 d\alpha = \lambda_1 \int_0^{2\pi} [\cos^2(\theta) + \lambda^2 \sin^2(\theta)]^{\frac{1}{2}} \frac{d\theta}{2\pi} = \frac{2\lambda_1}{\pi} I(\lambda),$$

where  $I(\lambda)$  is the elliptic integral defined by (3.4).

Thus by using Corollary 3.2 we proved that

$$\frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} = \frac{1}{I(\lambda)} \frac{\cos(\theta_o) v_1 + \lambda^2 \sin(\theta_o) v_2}{[\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o)]^{\frac{1}{2}}}.$$

At this stage of the proof we are going to change the basis, that is expressing last identity in the direct orthonormal basis  $(v^*, v^{**})$ , where  $v^{**} = \cos(\theta_o + \frac{\pi}{2})v_1 + \sin(\theta_o + \frac{\pi}{2})v_2 = -\sin(\theta_o)v_1 + \cos(\theta_o)v_2$ , getting

$$\frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} = F_1(\lambda, \theta_o)v^* + F_2(\lambda, \theta_o)v^{**},$$

$F = (F_1, F_2)$  being defined by:

$$\begin{aligned} F : ]0, 1] \times ]-\frac{\pi}{2}, \frac{\pi}{2}] &\longrightarrow \{(X, Y) \in \mathbb{R}^2, X > 0, X^2 + Y^2 < 1\} \\ (\lambda, \theta_o) &\mapsto F(\lambda, \theta_o) = (F_1(\lambda, \theta_o), F_2(\lambda, \theta_o)), \end{aligned} \quad (3.5)$$

where

$$\begin{cases} F_1(\lambda, \theta_o) &= \frac{1}{I(\lambda)} (\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{1}{2}} \\ F_2(\lambda, \theta_o) &= \frac{1}{I(\lambda)} \frac{\sin(\theta_o) \cos(\theta_o) (\lambda^2 - 1)}{(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{1}{2}}}. \end{cases}$$

That yields Corollary 3.6. □

Writting the **observed ratio** of functionals  $\frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)}$  as

$$\frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)} = X_n v^* + Y_n v^{**}, \quad (3.6)$$

we shall show the following proposition.

**Proposition 3.7.** *Let consider the following system of equations*

$$\begin{cases} X_n &= F_1(\lambda, \theta_o) \\ Y_n &= F_2(\lambda, \theta_o), \end{cases}$$

where  $F = (F_1, F_2)$  has been defined by (3.5).

1. In the case where  $Y_n \neq 0$  or  $X_n \neq \frac{2}{\pi}$ , it admits a unique solution  $(\hat{\lambda}_n, \hat{\theta}_{o,n})$  such that  $0 < \hat{\lambda}_n < 1$  and  $-\frac{\pi}{2} < \hat{\theta}_{o,n} \leq \frac{\pi}{2}$ .

- The estimator  $0 < \hat{\lambda}_n < 1$  is solution of

$$X_n^2 \left( \frac{I(\lambda)}{\lambda} \right)^2 = \frac{X_n^2 I^2(\lambda) - 1}{(X_n^2 + Y_n^2) I^2(\lambda) - 1},$$

- except for  $Y_n = 0$  and  $X_n > \frac{2}{\pi}$ , where  $\hat{\lambda}_n = I^{-1}(\frac{1}{X_n})$ .

- The estimator  $\hat{\theta}_{o,n}$  is defined as

$$\hat{\theta}_{o,n} = \arcsin \left( \sqrt{\frac{1 - X_n^2 I^2(\hat{\lambda}_n)}{1 - \hat{\lambda}_n^2}} \right) (\mathbb{1}_{\{Y_n \leq 0\}} - \mathbb{1}_{\{Y_n > 0\}}).$$

2. In the case where  $Y_n = 0$  and  $X_n = \frac{2}{\pi}$ , it admits a unique solution  $\hat{\lambda}_n = 1, \hat{\theta}_{o,n}$  being any number belonging to  $] -\frac{\pi}{2}, \frac{\pi}{2}]$ .

*Proof of Proposition 3.7.* As a first step we use a lemma proved in Appendix A.

**Lemma 3.8.** *The random variables  $X_n$  and  $Y_n$  defined in (3.6) are such that for  $n \in \mathbb{N}^*$ , a.s.  $X_n > 0$  and  $X_n^2 + Y_n^2 < 1$ .*

Let us consider the following system of equations

$$\begin{cases} X_n &= F_1(\lambda, \theta_o) \\ Y_n &= F_2(\lambda, \theta_o) \end{cases} \quad (3.7)$$

If the system admits a solution  $\hat{\lambda}_n$ , this solution ought to verify the following equation in  $\lambda$ :

$$X_n^2 I^4(\lambda) (X_n^2 + Y_n^2) - X_n^2 I^2(\lambda) (\lambda^2 + 1) + \lambda^2 = 0. \quad (3.8)$$

At this step, two cases appear: the case where  $(X_n^2 + Y_n^2)(\frac{\pi}{2})^2 - 1 < 0$  and the case where  $(X_n^2 + Y_n^2)(\frac{\pi}{2})^2 - 1 \geq 0$ . Let us consider the first one.

1.  $(X_n^2 + Y_n^2)(\frac{\pi}{2})^2 - 1 < 0$ .

Since for all  $\lambda \in [0, 1]$ ,  $I(\lambda) \leq \frac{\pi}{2}$ , the following inequality holds,  $(X_n^2 + Y_n^2) I^2(\lambda) - 1 < 0$ .

Now dividing (3.8) by  $I^2(\lambda) X_n^2$ , ( $\neq 0$  by Lemma 3.8), one gets

$$I^2(\lambda) (X_n^2 + Y_n^2) - (\lambda^2 + 1) + \frac{\lambda^2}{I^2(\lambda) X_n^2} = 0,$$

that is, since  $(X_n^2 + Y_n^2) I^2(\lambda) - 1 < 0$ ,

$$f_1(\lambda) = f_2(\lambda), \quad (3.9)$$

where

$$\begin{aligned} f_1(\lambda) &= X_n^2 \left( \frac{I(\lambda)}{\lambda} \right)^2 \\ f_2(\lambda) &= \frac{X_n^2 I^2(\lambda) - 1}{(X_n^2 + Y_n^2) I^2(\lambda) - 1}. \end{aligned}$$

Equation (3.9) admits a unique solution in the interval  $0 < \lambda < 1$ . Let us argue this last assertion.

Since  $X_n > 0$ , function  $f_1$  is strictly decreasing, while  $f_2$  is such that  $f_2'(\lambda) = \frac{2 I(\lambda) I'(\lambda)}{((X_n^2 + Y_n^2) I^2(\lambda) - 1)^2} Y_n^2 > 0$ , if  $Y_n \neq 0$ .

Let us consider the case where  $Y_n \neq 0$ . In this case function  $f_2$  is strictly increasing.

By summarizing the situation we know that  $f_1 - f_2$  is continuous on  $]0, 1]$ , strictly de-

creasing and such that  $(f_1 - f_2)(0^+) = +\infty$  and  $(f_1 - f_2)(1) = \frac{X_n^2 Y_n^2 (\frac{\pi}{2})^4 + (X_n^2 (\frac{\pi}{2})^2 - 1)^2}{(X_n^2 + Y_n^2) (\frac{\pi}{2})^2 - 1} < 0$ .

Thus there exists  $0 < \lambda < 1$ , such that  $f_1(\lambda) = f_2(\lambda)$ .

Now, in the case where  $Y_n = 0$ ,  $f_2(\lambda) = 1$ . We define  $g$  by  $g(\lambda) = \frac{I(\lambda)}{\lambda}$ . Since  $X_n < \frac{2}{\pi}$ ,  $\lambda = g^{-1}(\frac{1}{X_n})$  is the unique solution in  $]0, 1]$  of (3.9).

Let us consider now the second case.

2.  $(X_n^2 + Y_n^2)(\frac{\pi}{2})^2 - 1 \geq 0$ .

By Lemma 3.8 we know that  $X_n^2 + Y_n^2 > 0$ , and  $h : \lambda \rightarrow (X_n^2 + Y_n^2) I^2(\lambda) - 1$  is strictly increasing and continuous on  $]0, 1]$ .

Moreover, again thanks to Lemma 3.8,  $h(0) = X_n^2 + Y_n^2 - 1 < 0$  and  $h(1) = (X_n^2 + Y_n^2)(\frac{\pi}{2})^2 - 1 \geq 0$ . Thus there exists a unique  $0 < \lambda_0 \leq 1$  such that  $h(\lambda_0) = (X_n^2 + Y_n^2) I^2(\lambda_0) - 1 = 0$  and then for  $\lambda < \lambda_0$ , one has  $(X_n^2 + Y_n^2) I^2(\lambda) - 1 < 0$  and for  $\lambda > \lambda_0$ ,  $(X_n^2 + Y_n^2) I^2(\lambda) - 1 > 0$ .

Arguing as in first part, we can deduce that if (3.7) admits a solution  $\hat{\lambda}_n \neq \lambda_0$ , this solution ought to verify  $f_1(\lambda) = f_2(\lambda)$ .

Function  $f_1$  is strictly decreasing, while function  $f_2$  is strictly increasing if  $Y_n \neq 0$ . In this case  $f_1 - f_2$  is continuous on  $]0, 1]$ , strictly decreasing and such that  $(f_1 - f_2)(0^+) = +\infty$  and  $(f_1 - f_2)(\lambda_0^-) = -\infty$ , since  $Y_n \neq 0$ . Thus there exists  $0 < \lambda < \lambda_0 \leq 1$ , such that  $f_1(\lambda) = f_2(\lambda)$ .

On the other side, still if  $Y_n \neq 0$  and if  $\lambda_0 < 1$ , we have  $(f_1 - f_2)(\lambda_0^+) = +\infty$  and  $(f_1 - f_2)(1^-) > 0$ . Then there is no more solution of (3.9) into interval  $]\lambda_0^+, 1]$ .

Thus we have proved that in the case where  $Y_n \neq 0$ , if (3.7) admits a solution  $\hat{\lambda}_n \neq \lambda_0$ , this solution is the unique solution in  $]0, 1]$  of (3.9) and this solution belongs to the interval  $]0, \lambda_0[$ .

Now we are going to prove that in the case where  $Y_n = 0$ , there is no solution to (3.7) different from  $\lambda_0$ .

In fact, if  $\lambda$  is such a solution then  $f_1(\lambda) = X_n^2 \left( \frac{I(\lambda)}{\lambda} \right)^2 = f_2(\lambda) = 1$ , that is not possible. Indeed,  $X_n^2 \left( \frac{\pi}{2} \right)^2 - 1 > 0$  and then we would have  $1 < X_n^2 \left( \frac{\pi}{2} \right)^2 \leq X_n^2 \left( \frac{I(\lambda)}{\lambda} \right)^2 = 1$ , or  $X_n^2 \left( \frac{\pi}{2} \right)^2 - 1 = 0 = X_n^2 I^2(\lambda_0) - 1$ , and since  $X_n \neq 0$ , this would imply that  $\frac{I(\lambda)}{\lambda} = \frac{\pi}{2}$  and  $\lambda_0 = 1$  and then  $\lambda = \lambda_0 = 1$ .

To finish the proof of this part of proposition, we look for conditions on  $\lambda = \lambda_0$  to be solution of (3.7). If such a solution exists necessarily it has to verify (3.8), and then since  $(X_n^2 + Y_n^2) I^2(\lambda_0) - 1 = 0$ , we would have  $\lambda_0^2(1 - X_n^2 I^2(\lambda_0)) = 0$ , so that  $Y_n = 0$  and  $\lambda_0 = I^{-1}(\frac{1}{X_n})$ .

Thus, or  $X_n^2 \left( \frac{\pi}{2} \right)^2 > 1$  and then  $0 < \lambda = \lambda_0 = I^{-1}(\frac{1}{X_n}) < 1$ , or  $X_n^2 \left( \frac{\pi}{2} \right)^2 = 1$  and then  $0 < \lambda = \lambda_0 = 1$ . If we combine all results we get the following: if (3.7) admits a solution  $\hat{\lambda}_n$  then,

- $(X_n^2 + Y_n^2) \left( \frac{\pi}{2} \right)^2 - 1 < 0$ , and this solution  $0 < \hat{\lambda}_n < 1$  verifies equation  $f_1(\lambda) = f_2(\lambda)$ .  
Furthermore in the case where  $Y_n = 0$  then  $\hat{\lambda}_n = g^{-1}(\frac{1}{X_n})$ .
- Or  $(X_n^2 + Y_n^2) \left( \frac{\pi}{2} \right)^2 - 1 \geq 0$  and  $Y_n \neq 0$ , and this solution  $0 < \hat{\lambda}_n < 1$  verifies equation  $f_1(\lambda) = f_2(\lambda)$ .
- Or  $X_n^2 \left( \frac{\pi}{2} \right)^2 > 1$  and  $Y_n = 0$ , and this solution  $0 < \hat{\lambda}_n < 1$  is  $\hat{\lambda}_n = I^{-1}(\frac{1}{X_n})$ .
- Or  $X_n^2 \left( \frac{\pi}{2} \right)^2 = 1$  and  $Y_n = 0$ , and  $\hat{\lambda}_n = 1$ .

**Remark 3.9.** Note that in all cases  $\hat{\lambda}_n$  verifies (3.8).

**Remark 3.10.** Note that in all cases  $(X_n^2 + Y_n^2) I^2(\hat{\lambda}_n) - 1 \leq 0$ .

Now by continuing the reasoning by necessary condition, if we want that  $\hat{\lambda}_n < 1$  verifies the first equation of (3.7), we have to resolve the following equation:

$$X_n^2 I^2(\hat{\lambda}_n) = (\hat{\lambda}_n^2 - 1) \sin^2(\hat{\theta}_{o,n}) + 1,$$

that is

$$\sin^2(\hat{\theta}_{o,n}) = \frac{1 - X_n^2 I^2(\hat{\lambda}_n)}{1 - \hat{\lambda}_n^2} = \frac{\frac{1}{\hat{\lambda}_n^2} - f_1(\hat{\lambda}_n)}{\frac{1}{\hat{\lambda}_n^2} - 1}.$$

Let us see that  $0 \leq \frac{\frac{1}{\hat{\lambda}_n^2} - f_1(\hat{\lambda}_n)}{\frac{1}{\hat{\lambda}_n^2} - 1} \leq 1$ . We have the following equivalence:

$$\left( \frac{\frac{1}{\hat{\lambda}_n^2} - f_1(\hat{\lambda}_n)}{\frac{1}{\hat{\lambda}_n^2} - 1} \leq 1 \right) \iff (f_1(\hat{\lambda}_n) \geq 1).$$

In the case where  $X_n^2(\frac{\pi}{2})^2 > 1$  and  $Y_n = 0$ ,  $\hat{\lambda}_n = I^{-1}(\frac{1}{X_n})$ , we have  $f_1(\hat{\lambda}_n) = \frac{1}{\hat{\lambda}_n^2} \geq 1$ , since  $0 < \hat{\lambda}_n < 1$ .

In the other cases,  $f_1(\hat{\lambda}_n) = f_2(\hat{\lambda}_n)$  and since  $f_2$  is an increasing function,  $f_1(\hat{\lambda}_n) \geq f_2(0^+) = \frac{X_n^2 - 1}{X_n^2 + Y_n^2 - 1} \geq 1$ , since  $Y_n^2 \geq 0$  and  $X_n^2 + Y_n^2 - 1 < 0$ .

$$\text{Now } \left( \frac{\frac{1}{\hat{\lambda}_n^2} - f_1(\hat{\lambda}_n)}{\frac{1}{\hat{\lambda}_n^2} - 1} \geq 0 \right) \iff \left( X_n^2 I^2(\hat{\lambda}_n) - 1 \leq 0 \right).$$

Remark 3.10 gives the result.

If we look at the sign of expression  $F_2(\lambda, \theta_o)$ , we have to set:

$$\hat{\theta}_{o,n} = \arcsin \left( \sqrt{\frac{1 - X_n^2 I^2(\hat{\lambda}_n)}{1 - \hat{\lambda}_n^2}} \right) (\mathbb{1}_{\{Y_n \leq 0\}} - \mathbb{1}_{\{Y_n > 0\}}).$$

In this way, we always have  $-\frac{\pi}{2} \leq \hat{\theta}_{o,n} \leq \frac{\pi}{2}$  and in fact  $\hat{\theta}_{o,n} > -\frac{\pi}{2}$ . To verify last inequality, remark that

$$\begin{aligned} \left( \hat{\theta}_{o,n} = -\frac{\pi}{2} \right) &\iff \left( Y_n > 0 \text{ and } X_n^2 I^2(\hat{\lambda}_n) = \hat{\lambda}_n^2 \right) \\ &\implies \left( Y_n > 0 \text{ and } f_1(\hat{\lambda}_n) = f_2(\hat{\lambda}_n) = 1 \right) \implies (Y_n > 0 \text{ and } Y_n = 0), \end{aligned}$$

which is impossible.

Let us see now that we can go backward. That is let us verify that these proposed estimators  $\hat{\lambda}_n (< 1)$  and  $\hat{\theta}_{o,n}$  are actually solutions of (3.7).

$$\begin{aligned} \left( \sin^2(\hat{\theta}_{o,n}) = \frac{1 - X_n^2 I^2(\hat{\lambda}_n)}{1 - \hat{\lambda}_n^2} \right) &\iff \left( X_n^2 = \frac{\cos^2(\hat{\theta}_{o,n}) + \hat{\lambda}_n^2 \sin^2(\hat{\theta}_{o,n})}{I^2(\hat{\lambda}_n)} \right) \\ &\iff \left( X_n = \frac{(\cos^2(\hat{\theta}_{o,n}) + \hat{\lambda}_n^2 \sin^2(\hat{\theta}_{o,n}))^{\frac{1}{2}}}{I(\hat{\lambda}_n)} = F_1(\hat{\lambda}_n, \hat{\theta}_{o,n}) \right), \end{aligned}$$

last equivalence provides from Lemma 3.8 which insures that  $X_n > 0$ .

Let us consider the second equation of (3.7). In the case where  $Y_n = 0$  and  $X_n > \frac{2}{\pi}$ , one has  $\hat{\lambda}_n = I^{-1}(\frac{1}{X_n})$  so that  $\hat{\theta}_{o,n} = \arcsin(0) = 0$  and  $Y_n = 0 = F_2(\hat{\lambda}_n, 0)$ . In the other cases, one has  $f_1(\hat{\lambda}_n) = f_2(\hat{\lambda}_n)$ , and since  $X_n^2 I^2(\hat{\lambda}_n) = \cos^2(\hat{\theta}_{o,n}) + \hat{\lambda}_n^2 \sin^2(\hat{\theta}_{o,n})$  one has

$$\begin{aligned} \left( f_1(\hat{\lambda}_n) = f_2(\hat{\lambda}_n) \right) &\iff \left( \frac{X_n^2 I^2(\hat{\lambda}_n)}{\hat{\lambda}_n^2} = \frac{X_n^2 I^2(\hat{\lambda}_n) - 1}{(X_n^2 + Y_n^2) I^2(\hat{\lambda}_n) - 1} \right) \\ &\iff \left( Y_n^2 = \left[ \frac{1}{I(\hat{\lambda}_n)} \frac{\sin(\hat{\theta}_{o,n}) \cos(\hat{\theta}_{o,n}) (\hat{\lambda}_n^2 - 1)}{(\cos^2(\hat{\theta}_{o,n}) + \hat{\lambda}_n^2 \sin^2(\hat{\theta}_{o,n}))^{\frac{1}{2}}} \right]^2 \right) \\ &\iff \left( Y_n = \frac{1}{I(\hat{\lambda}_n)} \frac{\sin(\hat{\theta}_{o,n}) \cos(\hat{\theta}_{o,n}) (\hat{\lambda}_n^2 - 1)}{(\cos^2(\hat{\theta}_{o,n}) + \hat{\lambda}_n^2 \sin^2(\hat{\theta}_{o,n}))^{\frac{1}{2}}} = F_2(\hat{\lambda}_n, \hat{\theta}_{o,n}) \right), \end{aligned}$$

last equivalence providing from the fact that if  $Y_n \leq 0$ ,  $0 \leq \hat{\theta}_{o,n} \leq \frac{\pi}{2}$  and if  $Y_n > 0$ ,  $-\frac{\pi}{2} \leq \hat{\theta}_{o,n} \leq 0$ .

To conclude this proof, we just have to check the case where  $\hat{\lambda}_n = 1$ . In this case  $X_n = \frac{2}{\pi} = F_1(1, \theta_o)$  and  $Y_n = 0 = F_2(1, \theta_o)$ , for all  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ .

This yields the proposition.

**Remark 3.11.** We have proved that  $F$  is a one to one function from  $]0, 1[ \times ] - \frac{\pi}{2}, \frac{\pi}{2}[$  onto  $\{(X, Y) \in \mathbb{R}^2, X > 0, X^2 + Y^2 < 1 \text{ and } (X \neq \frac{2}{\pi} \text{ or } Y \neq 0)\}$ .

Furthermore and since  $(\theta_o = \frac{\pi}{2}) \iff (Y = 0 \text{ and } X < \frac{2}{\pi})$ ,  $F$  is a one to one function from  $]0, 1[ \times ] - \frac{\pi}{2}, \frac{\pi}{2}[$  onto  $\{(X, Y) \in \mathbb{R}^2, X > 0, X^2 + Y^2 < 1 \text{ and } (X > \frac{2}{\pi} \text{ or } Y \neq 0)\}$ .

□

### 3.2.2 Consistency for the parameters $\hat{\lambda}_n$ and $\hat{\theta}_{o,n}$

Now we are ready to state the following results of consistency for the two proposed estimators  $\hat{\lambda}_n$  and  $\hat{\theta}_{o,n}$ .

**Theorem 3.12.** 1. For  $0 < \lambda \leq 1$  and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ , one has

$$\hat{\lambda}_n \xrightarrow[n \rightarrow +\infty]{a.s.} \lambda.$$

2. For  $0 < \lambda < 1$  and  $-\frac{\pi}{2} < \theta_o < \frac{\pi}{2}$ , one has

$$\hat{\theta}_{o,n} \xrightarrow[n \rightarrow +\infty]{a.s.} \theta_o.$$

*Proof of Theorem 3.12.* We prove in Lemma A.1 that the Jacobian of the transformation  $F$  is different from zero when  $0 < \lambda < 1$ .

Thus by Remark 3.11 and the last lemma we now know that  $F$  is a  $C^2$ -diffeomorphism from the open set  $U = ]0, 1[ \times ] - \frac{\pi}{2}, \frac{\pi}{2}[$  onto the open set  $V = \{(X, Y) \in \mathbb{R}^2, X > 0, X^2 + Y^2 < 1, (Y \neq 0 \text{ or } X > \frac{2}{\pi})\}$ .

Furthermore by using the result convergence given in (3.3), decomposition of  $\frac{E[J_{f^*}^{(1)}(u)]}{E[J_1^{(1)}(u)]}$  in the basis  $(v^*, v^{**})$  given in Corollary 3.6 and of that of  $\frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)}$  given in (3.6), we obtain the following convergence result: for all  $0 < \lambda \leq 1$  and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ ,

$$(X_n, Y_n) \xrightarrow[n \rightarrow +\infty]{a.s.} F(\lambda, \theta_o) = (X, Y). \quad (3.10)$$

If we reduce the definition domain of  $(\lambda, \theta_o)$ , that is if we suppose that  $0 < \lambda < 1$  and  $-\frac{\pi}{2} < \theta_o < \frac{\pi}{2}$ , then  $(X, Y)$  belongs to the open set  $V$ . If  $F^{-1}$  denotes the inverse function of  $F$ , thus since  $F^{-1}$  is continuous from  $V$  to  $U$ , one deduces that almost surely  $F^{-1}(X_n, Y_n) = (\hat{\lambda}_n, \hat{\theta}_{o,n})$  converges to  $F^{-1}(X, Y) = (\lambda, \theta_o)$ .

It remains to consider two cases, case where  $\lambda = 1$  and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$  and case where  $0 < \lambda < 1$  and  $\theta_o = \frac{\pi}{2}$ .

First let us consider the isotropic case, that is the case where  $\lambda = 1$  and  $\theta_o$  being any parameter belonging to  $] - \frac{\pi}{2}, \frac{\pi}{2}[$ . By Remark 3.9 stated at the end of Proposition 3.7 proof, the estimator  $\hat{\lambda}_n$  has to verify the following equation

$$X_n^2 I^4(\hat{\lambda}_n) (X_n^2 + Y_n^2) - X_n^2 I^2(\hat{\lambda}_n) (\hat{\lambda}_n^2 + 1) + \hat{\lambda}_n^2 = 0.$$

Thus

$$\begin{aligned} & (X_n^2 - (\frac{2}{\pi})^2) \left[ I^4(\hat{\lambda}_n) (X_n^2 + Y_n^2) + (\frac{2}{\pi})^2 I^4(\hat{\lambda}_n) - I^2(\hat{\lambda}_n) (\hat{\lambda}_n^2 + 1) \right] \\ & + (\frac{2}{\pi})^2 I^4(\hat{\lambda}_n) Y_n^2 - (\frac{2}{\pi})^4 \left( (\frac{\pi}{2})^2 - I^2(\hat{\lambda}_n) \right) \left( I^2(\hat{\lambda}_n) - (\frac{\pi}{2})^2 \hat{\lambda}_n^2 \right) = 0. \end{aligned} \quad (3.11)$$

We define the function  $h$  for  $0 < \lambda \leq 1$ , by

$$h(\lambda) = \left( \frac{\pi}{2} - I(\lambda) \right) \left( I(\lambda) - \frac{\pi}{2} \lambda \right). \quad (3.12)$$

By using convergence given in (3.10), we establish that

$$(X_n, Y_n) \xrightarrow[n \rightarrow +\infty]{a.s.} \left( \frac{2}{\pi}, 0 \right),$$

and since  $0 < \hat{\lambda}_n \leq 1$  and  $1 < I(\hat{\lambda}_n) \leq \frac{\pi}{2}$ , we finally showed that

$$h(\hat{\lambda}_n) \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

Since  $h$  is a strictly decreasing function on  $]0, 1]$  and  $h(1) = 0$ , we obtain that

$$\hat{\lambda}_n \xrightarrow[n \rightarrow +\infty]{a.s.} 1,$$

that is the required convergence.

Let us look now at the case where  $0 < \lambda < 1$  and  $\theta_o = \frac{\pi}{2}$ .

Convergence established in (3.10) now gives that

$$(X_n, Y_n) \xrightarrow[n \rightarrow +\infty]{a.s.} (X, Y) = \left( \frac{\lambda}{I(\lambda)}, 0 \right).$$

In the same way as before and using Remark 3.9, one gets the following equality.

$$\begin{aligned} (X_n^2 - X^2) & \left[ I^4(\hat{\lambda}_n) (X_n^2 + Y_n^2) + X^2 I^4(\hat{\lambda}_n) - I^2(\hat{\lambda}_n) (\hat{\lambda}_n^2 + 1) \right] + X^2 I^4(\hat{\lambda}_n) Y_n^2 \\ & - \left( \frac{1}{I(\lambda)} \right)^4 \left( I^2(\lambda) - \lambda^2 I^2(\hat{\lambda}_n) \right) \left( \lambda I(\hat{\lambda}_n) + I(\lambda) \hat{\lambda}_n \right) \left( \lambda I(\hat{\lambda}_n) - I(\lambda) \hat{\lambda}_n \right) = 0. \end{aligned} \quad (3.13)$$

For fixed  $0 < \lambda < 1$ , let us define  $f$  by

$$f(x) = \lambda I(x) - I(\lambda) x, \text{ for } 0 \leq x \leq 1. \quad (3.14)$$

Previous almost sure convergence and the fact that  $0 < \hat{\lambda}_n \leq 1$ ,  $1 < I(\hat{\lambda}_n) \leq \frac{\pi}{2}$  and for  $\lambda < 1$ ,  $I(\lambda) - \lambda I(\hat{\lambda}_n) > I(\lambda) - \frac{\pi}{2} > 0$ , imply that

$$f(\hat{\lambda}_n) \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

A straightforward calculation shows that function  $f$  is a strictly decreasing function on  $[0, 1]$  such that  $f(\lambda) = 0$ . We deduce that

$$\hat{\lambda}_n \xrightarrow[n \rightarrow +\infty]{a.s.} \lambda,$$

this yields Theorem 3.12. □

We thus set up a first approach to detect if the process  $X$  is isotropic or not.

#### 4 Convergence in law for $J_f^{(n)}(u)$

We built consistent estimators,  $\hat{\lambda}_n$  and  $\hat{\theta}_{o,n}$  for parameters  $\lambda$  and  $\theta_o$ , by using Theorem 3.4 applied in the particular cases where  $f = f^*$  and  $f = 1$ . In other words, we used the almost sure convergence of  $J_{f^*}^{(n)}(u)$  and of  $J_1^{(n)}(u)$ . In order to propose a CLT for these estimators, we believe it would be appropriate to propose a CLT for  $J_{f^*}^{(n)}(u)$ .

Without really having to do extra work, we can establish a CLT for a general functional  $J_f^{(n)}(u)$ , function  $f : S^1 \rightarrow \mathbb{R}$  being any continuous and bounded function. Roughly speaking we decided to give the rate of convergence in Theorem 3.4. In this aim, for  $u$  a fixed level in  $\mathbb{R}$ , we define the random variable  $\xi_f^{(n)}(u)$  by

$$\xi_f^{(n)}(u) = 2n \left( J_f^{(n)}(u) - \mathbb{E} \left[ J_f^{(n)}(u) \right] \right).$$

#### 4.1 Hermite expansion for $J_f^{(n)}(u)$

Our objective is to decompose the random variable  $\xi_f^{(n)}(u)$  as a sum of multiple Itô integrals. In this aim, the idea consists in approaching functionals  $J_f^{(n)}(u)$  by other functionals, say  $J_f^{(n)}(u, \sigma)$  ( $\sigma \rightarrow 0$ ), in such a way that the last ones be expressed into the Itô-Wiener chaos. This is possible through the use of a kernel  $K$  and via the coarea formula.

##### 4.1.1 Coarea formula

For  $\sigma > 0$ , we define an approximation of  $J_f^{(n)}(u)$  given by

$$J_f^{(n)}(u, \sigma) = \frac{1}{\sigma} \int_{-\infty}^{+\infty} K\left(\frac{u-v}{\sigma}\right) J_f^{(n)}(v) dv,$$

where  $K$  is a continuous density function with a compact support in  $[-1, 1]$ .

By applying Corollary 2.1.1. of [7] to the function  $h : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$  defined as :

$$h(x, y) = f(\nu_X(x)) \frac{1}{\sigma} K\left(\frac{u-y}{\sigma}\right),$$

to the  $C^1$ -function  $G = X : D = \mathbb{R}^2 \rightarrow \mathbb{R}$  and to the borel set  $B = T$ , we get:

$$J_f^{(n)}(u, \sigma) = \frac{1}{\sigma_2(T)} \int_T f(\nu_X(t)) \frac{1}{\sigma} K\left(\frac{u-X(t)}{\sigma}\right) \|\nabla X(t)\|_2 dt.$$

Let  $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ , a matrix that factorizes  $A$  such that  $A = P\Lambda P^t$ , where  $P$  is a unitary matrix.

The matrix  $\Gamma^{\mathbf{X}}$  stands for the covariance matrix of the 3-dimensional Gaussian vector

$$\mathbf{X}(t) = (\nabla X(t), X(t))^t.$$

We recall that since  $Z$  is isotropic,  $E[\nabla Z(0)\nabla Z(0)^t] = \mu I_2$ , where  $\mu = -\frac{\partial^2 r_z}{\partial t_i^2}(0)$  for any  $i = 1, 2$  and  $I_2$  stands for the identity matrix in  $\mathbb{R}^2$ .

We denote  $\Delta$  for the  $3 \times 3$  matrix,  $\Delta = \begin{pmatrix} \sqrt{\mu}P\Lambda & 0 \\ 0 & \sqrt{r_z(0)} \end{pmatrix}$ . It is such that  $\Delta\Delta^t = \Gamma^{\mathbf{X}}$ .

We can write, for any fixed  $t \in \mathbb{R}^2$ ,  $\mathbf{X}(t) = \Delta Y(t)$ , where  $Y(t)$  is a 3-dimensional standard Gaussian vector.

With these notations and if  $Y(t) = (Y_i(t))_{1 \leq i \leq 3}$ , one obtains:

$$J_f^{(n)}(u, \sigma) = \frac{\sqrt{\mu}}{\sigma_2(T)} \frac{1}{\sigma} \int_T f \left( \frac{P\Lambda \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix}}{\sqrt{\lambda_1^2 Y_1^2(t) + \lambda_2^2 Y_2^2(t)}} \right) \times K \left( \frac{u - \sqrt{r_z(0)} Y_3(t)}{\sigma} \right) \sqrt{\lambda_1^2 Y_1^2(t) + \lambda_2^2 Y_2^2(t)} dt.$$

We define for  $\sigma > 0$  and  $y = (y_i)_{1 \leq i \leq 3} \in \mathbb{R}^3$ , the map  $g_\sigma$  as:

$$g_\sigma(y) = \frac{\sqrt{\mu}}{\sigma} K \left( \frac{u - \sqrt{r_z(0)} y_3}{\sigma} \right) f \left( \frac{P\Lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}{\sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2}} \right) \sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2}.$$

Since the map belongs to  $L^2(\mathbb{R}^3, \phi_3(y) dy)$ , the following expansion converges in this space:

$$g_\sigma(y) = \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_{f,\sigma}(\mathbf{k}, u) \tilde{H}_{\mathbf{k}}(y),$$

while taking  $\mathbf{k} = (k_i)_{1 \leq i \leq 3} \in \mathbb{N}^3$ ,

$$a_{f,\sigma}(\mathbf{k}, u) = a_f(k_1, k_2) a_\sigma(k_3, u), \quad (4.1)$$

where for  $(k_1, k_2) \in \mathbb{N}^2$ ,

$$a_f(k_1, k_2) = \frac{\sqrt{\mu}}{k_1! k_2!} \int_{\mathbb{R}^2} f \left( \frac{P\Lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}{\sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2}} \right) \sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2} H_{k_1}(y_1) \phi(y_1) \\ H_{k_2}(y_2) \phi(y_2) dy_1 dy_2$$

and

$$a_\sigma(k_3, u) = \frac{1}{k_3!} \int_{-\infty}^{\infty} \frac{1}{\sigma} K \left( \frac{u - \sqrt{r_z(0)} y}{\sigma} \right) H_{k_3}(y) \phi(y) dy.$$

In this way, we obtain the expansion of the functional  $J_f^{(n)}(u, \sigma)$  in  $L^2(\Omega)$ , that is:

$$J_f^{(n)}(u, \sigma) \stackrel{L^2(\Omega)}{=} \frac{1}{\sigma_2(T)} \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_{f,\sigma}(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) dt. \quad (4.2)$$

Now observe that as  $\sigma$  tends to zero,  $a_\sigma(k_3, u) \rightarrow a(k_3, u)$ , where coefficient  $a(k_3, u)$  is defined as

$$a(k_3, u) = \frac{1}{k_3!} H_{k_3} \left( \frac{u}{\sqrt{r_z(0)}} \right) \phi \left( \frac{u}{\sqrt{r_z(0)}} \right) \frac{1}{\sqrt{r_z(0)}}.$$

This remark yields the following expansion in  $L^2(\Omega)$  of  $J_f^{(n)}(u)$  as a sum of multiple Wiener-Itô integrals of order  $q$ , that is

$$J_f^{(n)}(u) = \frac{1}{\sigma_2(T)} \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) dt,$$

where coefficients  $a_f(\mathbf{k}, u)$  are defined by

$$a_f(\mathbf{k}, u) = a_f(k_1, k_2) a(k_3, u). \quad (4.3)$$

**Remark 4.1.** This equality has nothing obvious because we do not know a priori that the right member in the last equality really belongs to  $L^2(\Omega)$ . This fact comes from the way we obtained this expansion, the series

$\sum_{k_3=0}^{\infty} a^2(k_3, u) k_3!$  being equal to  $+\infty$  as the Hermite development in  $L^2(\mathbb{R}, \phi(x) dx)$  of delta's Dirac function in point  $u$ .

So now the idea consists in proving that  $J_f^{(n)}(u, \sigma)$  tends in  $L^2(\Omega)$  towards  $J_f^{(n)}(u)$  as  $\sigma \rightarrow 0$ . We emphasize that this convergence is not an obvious purpose since the difficulty is to prove the continuity of function  $(x, y) \mapsto \mathbb{E} \left[ J_f^{(n)}(x) J_f^{(n)}(y) \right]$ . This fact is far from being trivial and require a number of results expressed and demonstrated in [7], of which the second order Rice formula and a in-depth study of the level curve  $C_{T,X}(u + \delta)$  in a neighborhood of the level  $u$ .

#### 4.1.2 $L^2$ -convergence of $J_f^{(n)}(u, \sigma)$

We demonstrate Proposition 4.2.

**Proposition 4.2.** *One has the following convergence,*

$$J_f^{(n)}(u, \sigma) \xrightarrow[\delta \rightarrow 0]{L^2(\Omega)} J_f^{(n)}(u).$$

*Proof of Proposition 4.2.* One can easily see that the proof of this proposition follows immediately from that of Theorem 4.3. □

**Theorem 4.3.** *The function*

$$(x, y) \mapsto \mathbb{E} \left[ J_f^{(n)}(x) J_f^{(n)}(y) \right],$$

*is continuous. In particular the same holds for function,*

$$y \mapsto \mathbb{E} \left[ J_f^{(n)}(y) \right]^2.$$

*Proof of Theorem 4.3.* Let us note  $I_f^{(n)}(u) = \int_{C_{T, X}(u)} f(\nu_X(t)) d\sigma_1(t)$ .

We are going to need several lemmas among which here is the first one. This one is a little more general than one of part 2. given in theorem.

**Lemma 4.4.** *Let  $Y : T \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  a process such that for all  $t \in T$ ,  $Y(t) = G(t, \nabla X(t))$  where  $G : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is a bounded continuous function of its arguments, then*

$$y \mapsto \mathbb{E} \left[ \int_{C_{T, X}(y)} Y(t) d\sigma_1(t) \right]^2$$

*is a continuous function.*

*Proof of Lemma 4.4.* In order to apply the second order Rice formula to the random variable  $\int_{C_{T, X}(y)} Y(t) d\sigma_1(t)$  for all  $y \in \mathbb{R}$ , let us show that process  $X$  verifies assumptions of Remark 3.3.5 of Theorem 3.3.3 of [7].

As in the proof of Proposition 3.1,  $X/T : \Omega \times T \rightarrow \mathbb{R}$  is a stationary Gaussian field belonging to  $C^2(T)$  and  $T$  is a bounded convex open set of  $\mathbb{R}^2$ .

Furthermore as for inequality (3.1) one can prove that

$\mathbb{E} \left[ \sup_{x \in T} \|\nabla^2(X(x))\|_{1,2}^{(s)} \right]^4 < +\infty$ . Now  $Y$  verifies condition (3.18) of [7] since  $Y(t) = G(t, \nabla X(t))$ , where

$$\begin{aligned} G : \mathbb{R}^2 \times \mathfrak{L}(\mathbb{R}^2, \mathbb{R}) &\rightarrow \mathbb{R} \\ (t, A) &\mapsto G(t, A), \end{aligned}$$

is a bounded continuous function of its arguments.  $Y$  is thus a continuous process on  $T$  and we already saw that for all  $t \in T$ , the vector  $(X(t), \nabla X(t))$  has a density. Furthermore for all  $(t_1, t_2) \in T \times T$ ,  $t_1 \neq t_2$ , the vector  $(X(t_1), X(t_2))$  has a density  $p_{X(t_1), X(t_2)}(\cdot, \cdot)$  since we made the hypothesis that  $r_z^2(0) - r_z^2(t) \neq 0$  for all  $t \neq 0$  such that  $t \in A \times (\overline{T} - \overline{T})$ . To apply Rice formula, we just have to check hypothesis **H<sub>7</sub>** that for all  $y \in \mathbb{R}$ ,

$$\begin{aligned} H(y) = \int_{T \times T} \mathbb{E} [ |Y(t_1)| |Y(t_2)| \|\nabla X(t_1)\|_2 \|\nabla X(t_2)\|_2 |X(t_1) = X(t_2) = y] \\ \times p_{X(t_1), X(t_2)}(y, y) dt_1 dt_2 < +\infty. \end{aligned}$$

Then we have

$$\begin{aligned} & \mathbb{E} \left[ \int_{C_{T, X}(y)} Y(t) d\sigma_1(t) \right]^2 \\ &= \int_{T \times T} \mathbb{E} [Y(t_1) Y(t_2) \|\nabla X(t_1)\|_2 \|\nabla X(t_2)\|_2 | X(t_1) = X(t_2) = y] \times \\ & \quad p_{X(t_1), X(t_2)}(y, y) dt_1 dt_2. \end{aligned}$$

To check hypothesis **H**<sub>7</sub>, we are going to prove that function  $y \mapsto H(y)$  is continuous. Then applying the second order Rice formula and using similar arguments, working with  $Y$  instead of  $|Y|$ , we can prove that  $y \mapsto \mathbb{E} \left[ \int_{C_{T, X}(y)} Y(t) d\sigma_1(t) \right]^2$  is still continuous. This would achieve the proof of Lemma 4.4.

In this aim, we use the decomposition given in ([25], p. 60) that is, for  $\tau \in T - T$

$$\begin{aligned} \nabla X(0) &= \xi + (X(0)\alpha + X(\tau)\beta) \\ \nabla X(\tau) &= \xi^* - (X(\tau)\alpha + X(0)\beta), \end{aligned}$$

where  $\xi$  and  $\xi^*$  are centered Gaussian vectors taking values in  $\mathbb{R}^2$ , with joint Gaussian distribution, each of them independent of  $(X(0), X(\tau))$ , such that if  $r_x$  stands for the  $X$  covariance function,

$$\begin{aligned} \alpha &= \frac{r_x(\tau)\nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)}; \quad \beta = -\frac{r_x(0)\nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} \\ \text{Var}(\xi) &= \text{Var}(\xi^*) = -\nabla^2 r_x(0) - \frac{r_x(0)}{r_x^2(0) - r_x^2(\tau)} \nabla r_x(\tau) (\nabla r_x(\tau))^t \\ \text{Cov}(\xi, \xi^*) &= -\nabla^2 r_x(\tau) - \frac{r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} \nabla r_x(\tau) (\nabla r_x(\tau))^t. \end{aligned}$$

Thus with these notations one has

$$H(y) = \int_T du \int_{T-u} F(y, u, \tau) d\tau,$$

where

$$\begin{aligned} F(y, u, \tau) &= \exp \left( -\frac{y^2}{r_x(0) + r_x(\tau)} \right) \frac{1}{2\pi} \frac{1}{(r_x^2(0) - r_x^2(\tau))^{\frac{1}{2}}} \\ &\times \mathbb{E} \left[ \left\| G(u, \xi - \frac{y}{r_x(0) + r_x(\tau)} \nabla r_x(\tau)) \right\| \left\| \xi - \frac{y}{r_x(0) + r_x(\tau)} \nabla r_x(\tau) \right\|_2 \right. \\ &\times \left. \left\| G(\tau + u, \xi^* + \frac{y}{r_x(0) + r_x(\tau)} \nabla r_x(\tau)) \right\| \left\| \xi^* + \frac{y}{r_x(0) + r_x(\tau)} \nabla r_x(\tau) \right\|_2 \right]. \end{aligned}$$

Since  $r_x^2(0) - r_x^2(t) \neq 0$  for all  $t \neq 0$  such that  $t \in A \times (\bar{T} - \bar{T})$ , this implies in force that  $r_x^2(0) - r_x^2(\tau) \neq 0$  and  $r_x(0) + r_x(\tau) \neq 0$  for all  $\tau \in T - T$ ,  $\tau \neq 0$ , thus

- $y \mapsto F(y, u, \tau)$  is continuous for almost  $(u, \tau) \in T \times (T - T)$ .

Now let us enunciate the second lemma proved in Appendix A.

**Lemma 4.5.**  $\exists A > 0, \forall \tau \in \bar{T} - \bar{T}, \left( \|\tau\|_2 \leq A \implies r_x^2(0) - r_x^2(\tau) \geq A \|\tau\|_2^2 \right)$

Now let us choose  $A > 0$  small enough such that for all  $\tau \in T - T$  with  $\|\tau\|_2 \leq A$ , one has the following inequalities,

- (1)  $r_x(0) + r_x(\tau) \geq \frac{3}{2} r_x(0)$

- (2)  $r_x^2(0) - r_x^2(\tau) \geq A \|\tau\|_2^2$
- (3)  $\|\nabla r_x(\tau)\|_2 \leq C \|\tau\|_2$

Note that these three inequalities are always possible to realize. Indeed the first inequality just translate the fact that covariance function  $r_x$  is continuous in zero, the second one comes from Lemma 4.5 and the third one from a first order Taylor expansion of  $\nabla r_x(\cdot)$  about zero.

These inequalities allow showing the following bound on  $|F(y, u, \tau)|$

- $\exists D > 0$  such that for all  $(u, \tau) \in T \times T - T$ ,  
 $|F(y, u, \tau)| \leq D(1 + y^2) \left\{ \mathbb{1}_{\{\|\tau\|_2 \geq A\}} + \frac{1}{\|\tau\|_2} \mathbb{1}_{\{\|\tau\|_2 \leq A\}} \right\}$

Before explaining the last inequality, let us remark that since  $\int_{T \times T - u} \frac{du d\tau}{\|\tau\|_2^2} < +\infty$ , an application of Lebesgue's convergence theorem yields that  $H$  is continuous. Indeed, two cases occur:  $\|\tau\|_2 \geq A$  and  $\|\tau\|_2 \leq A$ . If  $\|\tau\|_2 \geq A$ , since  $r_x$  is continuous on the compact set  $K = \overline{T} - \overline{T} \cap \{\|\tau\|_2 \geq A\}$  and  $r_x^2(0) - r_x^2(\tau) \neq 0$  for  $\tau \in K$ , thus there exists  $E > 0$  such for all  $\|\tau\|_2 \geq A$ ,  $r_x^2(0) - r_x^2(\tau) \geq E$  and  $r_x(0) + r_x(\tau) \geq E$ , so that  $|F(y, u, \tau)| \leq D(1 + y^2)$ . If  $\|\tau\|_2 \leq A$ , inequalities (2) and (3) give that  $E[\|\xi\|_2] \leq C$  and  $E[\|\xi^*\|_2] \leq C$  and inequalities (1) and (2) give that  $|F(y, u, \tau)| \leq D(1 + y^2) \frac{1}{\|\tau\|_2}$ .

That completes the proof of Lemma 4.4.  $\square$

**Remark 4.6.** Note that for all  $y \in \mathbb{R}$ , a.s.  $C_{T,X}(y) = C_{T,X}^{Dr}(y)$ . To convince oneself of this equality, it suffices to remark that the hypotheses of Theorem 3.3.1 of [7] are satisfied so that one has, for all  $y \in \mathbb{R}$ ,

$$P(\omega \in \Omega, \exists t \in T, X(t)(\omega) = y, \|\nabla X(t)(\omega)\|_2 = 0) = 0.$$

**Remark 4.7.** The approach chosen to prove Lemma 4.4 does not work for showing that

$$(y_1, y_2) \rightarrow E \left[ \left( \int_{C_{T,X}(y_1)} Y(t_1) d\sigma_1(t_1) \right) \times \left( \int_{C_{T,X}(y_2)} Y(t_2) d\sigma_1(t_2) \right) \right],$$

is continuous. In fact if we try to reproduce the previous proof, we realize that the previous hypothesis **H<sub>7</sub>** changes in the following checking. For all  $y_1, y_2 \in \mathbb{R}$ ,

$$\begin{aligned} H(y_1, y_2) = & \int_{T \times T} E[|Y(t_1)| |Y(t_2)| \|\nabla X(t_1)\|_2 \|\nabla X(t_2)\|_2 |X(t_1) = y_1, X(t_2) = y_2] \\ & \times p_{X(t_1), X(t_2)}(y_1, y_2) dt_1 dt_2 < +\infty. \end{aligned}$$

As before, performing a regression, the expectation appearing in the expression of  $H(y)$  is replaced by

$$\begin{aligned} E \left[ \left| G(u, \xi + \frac{r_x(\tau) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_1 - \frac{r_x(0) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_2) \right| \times \right. \\ \left. \left\| \xi + \frac{r_x(\tau) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_1 - \frac{r_x(0) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_2 \right\|_2 \times \right. \\ \left. \left| G(\tau + u, \xi^* + \frac{r_x(0) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_1 - \frac{r_x(\tau) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_2) \right| \times \right. \\ \left. \left\| \xi^* + \frac{r_x(0) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_1 - \frac{r_x(\tau) \nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)} y_2 \right\|_2 \right], \end{aligned}$$

and the principal difficulty results in bounding  $\frac{\nabla r_x(\tau)}{r_x^2(0) - r_x^2(\tau)}$  when  $\|\tau\|_2 \leq A$ . That is why we developed a little more sophisticated approach to prove Lemma 4.8.

**Lemma 4.8.** *Let  $y$  be fixed in  $\mathbb{R}$  and  $(y_k)_{k \in \mathbb{N}}$  be a real sequence tending to  $y$ . Then for all  $n \in \mathbb{N}^*$ ,*

$$\left\| I_f^{(n)}(y_k) - I_f^{(n)}(y) \right\|_{L^2(\Omega)} \xrightarrow{k \rightarrow +\infty} 0,$$

where  $\|\cdot\|_{L^2(\Omega)}$  stands for the  $L^2(\Omega)$ -norm.

Note that Lemma 4.8 imply Theorem 4.3.

*Proof of Lemma 4.8.* Let us give an outline of the proof.

Let  $Y$  be the process defined by  $Y(t) = f(\nu_X(t)) = G(\nabla X(t))$ ,  $t \in T$ . The idea consists, for fixed  $n \in \mathbb{N}^*$ , in approximating the functional  $I_f^{(n)}(y)$  by  $I_{f,m}^{(n)}(y)$  the following one. For  $m \in \mathbb{N}^*$ , let

$$I_{f,m}^{(n)}(y) = \int_{C_{T,X}(y)} Y_m(t) d\sigma_1(t),$$

where  $Y_m$  defined on  $T$  is such that

1.  $Y_m$  verifies assumptions of Lemma 4.4 and then
2.  $Y_m$  is continuous on  $T$
3. Almost surely for all  $t \in T$ ,  $Y_m(t) \xrightarrow{m \rightarrow +\infty} Y(t) \mathbf{1}_{D_X^r}(t)$
4.  $\text{supp}(Y_m) \subset D_X^r$ ,

where for any continuous function  $f$ ,  $\text{supp}(f)$  is its support.

Assertions (2) and (4) imply (see [7], Theorem 3.1.2) that almost surely

$$x \rightarrow \int_{C_{T,X}^{D_X^r}(x)} Y_m(t) d\sigma_1(t)$$

is continuous and applying Remark 4.6 we have, for all  $m \in \mathbb{N}^*$  that, almost surely

$$I_{f,m}^{(n)}(y_k) \xrightarrow{k \rightarrow +\infty} I_{f,m}^{(n)}(y).$$

Note that this almost sure convergence is what we need to move forward. Furthermore, assertion (1) implies that for all  $m \in \mathbb{N}^*$ ,

$$\left\| I_{f,m}^{(n)}(y_k) \right\|_{L^2(\Omega)} \xrightarrow{k \rightarrow +\infty} \left\| I_{f,m}^{(n)}(y) \right\|_{L^2(\Omega)},$$

so that Scheffé's lemma allows to conclude that for all  $m \in \mathbb{N}^*$ ,

$$\lim_{k \rightarrow +\infty} \left\| I_{f,m}^{(n)}(y_k) - I_{f,m}^{(n)}(y) \right\|_{L^2(\Omega)} = 0. \quad (4.4)$$

An upper bound is

$$\begin{aligned} \left\| I_f^{(n)}(y_k) - I_f^{(n)}(y) \right\|_{L^2(\Omega)} &\leq \left\| I_f^{(n)}(y_k) - I_{f,m}^{(n)}(y_k) \right\|_{L^2(\Omega)} + \\ &\quad \left\| I_{f,m}^{(n)}(y_k) - I_{f,m}^{(n)}(y) \right\|_{L^2(\Omega)} + \left\| I_{f,m}^{(n)}(y) - I_f^{(n)}(y) \right\|_{L^2(\Omega)}. \end{aligned} \quad (4.5)$$

Applying Lemma 4.4 to  $Y - Y_m$  and using the convergence assumption given in point (3), we show that

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{k \rightarrow +\infty} \left\| I_f^{(n)}(y_k) - I_{f,m}^{(n)}(y_k) \right\|_{L^2(\Omega)} &= \\ \lim_{m \rightarrow +\infty} \left\| I_f^{(n)}(y) - I_{f,m}^{(n)}(y) \right\|_{L^2(\Omega)} &= 0, \end{aligned}$$

ending the proof of Lemma 4.8.

Thus let us build such a random variable  $Y_m$ , by defining for  $m \in \mathbb{N}^*$  and  $t \in T$ :

$$Y_m(t) = g_m(t) \varphi \left( \frac{1}{m \|\nabla X(t)\|_2} \mathbb{1}_{\{\nabla X(t) \neq 0\}} + 2 \mathbb{1}_{\{\nabla X(t) = 0\}} \right) Y(t),$$

where  $\varphi$  is an even continuous function on  $\mathbb{R}$ , decreasing on  $\mathbb{R}^+$  such that

$$\varphi(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{if } 2 \leq t \end{cases}$$

and  $(g_m)_{m \in \mathbb{N}^*}$  is a sequence of functions defined on  $\mathbb{R}^2$  to  $[0, 1]$  in the following manner

$$g_m(x) = \frac{d(x, T^{2m})}{d(x, T^{2m}) + d(x, T^{(m)})},$$

where  $d(x, A)$  stands for the distance between the point  $x$  and the set  $A \subset \mathbb{R}^2$ . The closed sets  $T^{2m}$  and  $T^{(m)}$  are defined by

$$T^{2m} = \{x \in \mathbb{R}^2, d(x, T^c) \leq \frac{1}{2m}\} \text{ and } T^{(m)} = \{x \in \mathbb{R}^2, d(x, T^c) \geq \frac{1}{m}\},$$

$T^c$  denoting the complement of  $T$  on  $\mathbb{R}^2$ .

We have shown in ([7], Lemmas 3.2.1 and 3.2.2) that the functions  $(g_m)_{m \in \mathbb{N}^*}$  are well defined, continuous and such that the support of  $g_m/T$  is contained in  $T$  for each  $m \in \mathbb{N}^*$ . Furthermore this sequence is bounded by one and tends to one when  $m$  goes to infinity. Thus to check that  $Y_m$  verifies assertions (1) to (4) we proceed as in [7].

Indeed, one can write  $Y_m(t) = G_m(t, \nabla X(t))$ , where for  $t \in T$  and  $z \in \mathbb{R}^2$ ,  $G_m(t, z) = g_m(t) \varphi \left( \frac{1}{m \|z\|_2} \mathbb{1}_{\{z \neq 0\}} + 2 \mathbb{1}_{\{z = 0\}} \right) G(z)$ . Using the crucial central term in  $\varphi$ , the fact that the support of  $g_m/T$  is contained in  $T$  and that the sequence  $(g_m)_m$  tends to one, one establishes assertions (1) to (4). As explained before convergence (4.4) ensues. Now going back to the upper bound given in (4.5), let us look closer at its first and third terms.

By using assertion (1) one proves that  $Z_m = Y - Y_m$  verifies assumptions of Lemma 4.4 and then, one has

$$\lim_{k \rightarrow +\infty} \left\| I_f^{(n)}(y_k) - I_{f,m}^{(n)}(y_k) \right\|_{L^2(\Omega)} = \left\| I_f^{(n)}(y) - I_{f,m}^{(n)}(y) \right\|_{L^2(\Omega)}. \quad (4.6)$$

By using Remark 4.6 almost surely for all  $m \in \mathbb{N}^*$ , one has

$$I_f^{(n)}(y) - I_{f,m}^{(n)}(y) = \int_{C_{T,X}^{D^r}(y)} Z_m(t) d\sigma_1(t).$$

Now by using convergence given in point (3), we obtain that almost surely for all  $t \in D_X^r$ ,  $\lim_{m \rightarrow \infty} Z_m(t) = 0$ .

Furthermore for all  $t \in D_X^r$ , one has  $|Z_m(t)| \leq C$ . Moreover by applying Lemma 4.4 to function  $G \equiv 1$ , one gets that

$$\mathbb{E} \left[ \sigma_1(C_{T,X}^{D^r}(y)) \right]^2 < +\infty, \quad (4.7)$$

and then almost surely  $\sigma_1(C_{T,X}^{D^r}(y)) < +\infty$ . The Lebesgue convergence theorem induces that

- Almost surely,  $\lim_{m \rightarrow \infty} I_f^{(n)}(y) - I_{f,m}^{(n)}(y) = 0$ . Furthermore

- Almost surely,  $\left| I_f^{(n)}(y) - I_{f,m}^{(n)}(y) \right| \leq C \sigma_1(C_{T,X}^{\text{Dr}}(y)) \in L^2(dP)$ ,  
last belonging providing from (4.7).

Applying once again the Lebesgue convergence theorem, one has finally proved that  $\lim_{m \rightarrow +\infty} \left\| I_{f,m}^{(n)}(y) - I_f^{(n)}(y) \right\|_{L^2(\Omega)} = 0$ . Thus by using inequality (4.5) and convergences obtained in (4.4) and (4.6), one concludes that

$$\overline{\lim}_{k \rightarrow +\infty} \left\| I_f^{(n)}(y_k) - I_f^{(n)}(y) \right\|_{L^2(\Omega)} = 0,$$

yielding Lemma 4.8. □

Theorem 4.3 follows from last lemma. □

We are now ready to propose a  $L^2(\Omega)$ -expansion for  $\xi_f^{(n)}(u)$ .

#### 4.1.3 The functional $J_f^{(n)}(u)$ viewed into the Wiener chaos

**Proposition 4.9.** *One has the following expansion in  $L^2(\Omega)$ ,*

$$\xi_f^{(n)}(u) = \frac{1}{\sqrt{\sigma_2(T)}} \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) dt,$$

where coefficients  $a_f(\mathbf{k}, u)$  have been defined by (4.3).

*Proof of Proposition 4.9.* The proof of this proposition strongly leans on that given in [11] and extensively uses the orthogonality of the various chaos.

As in the proof of ([11], Proposition 1.3) let us define formally

$$\eta(T) = \frac{1}{\sigma_2(T)} \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) dt,$$

and prove the following lemma.

**Lemma 4.10.**  $\eta(T) \in L^2(\Omega)$ .

*Proof of Lemma 4.10.* First, remark that if  $\mathbf{k}, \mathbf{m} \in \mathbb{N}^3$  are such that  $|\mathbf{k}| \neq |\mathbf{m}|$ , then for all  $s, t \in T \times T$ ,  $E[\tilde{H}_{\mathbf{k}}(Y(t))\tilde{H}_{\mathbf{m}}(Y(s))] = 0$  and the above expression,  $\eta(T)$  turns out to be a sum of orthogonal terms in  $L^2(\Omega)$ .

Indeed, to prove this we need a generalization of Mehler's formula given in Azaïs and Wschebor ([6], Lemma 10.7, part b, page 269) via the following lemma for proved in Appendix A.

**Lemma 4.11.** *Let  $X = (X_i)_{i=1,2,3}$  and  $Y = (Y_j)_{j=1,2,3}$  be two centered standard Gaussian vectors in  $\mathbb{R}^3$  such that for  $1 \leq i, j \leq 3$ ,  $E[X_i Y_j] = \rho_{ij}$ , then for  $\mathbf{k}, \mathbf{m} \in \mathbb{N}^3$ , one has*

$$E[\tilde{H}_{\mathbf{k}}(X)\tilde{H}_{\mathbf{m}}(Y)] = \left( \sum_{\substack{d_{ij} \geq 0 \\ \sum_j d_{ij} = k_i \\ \sum_i d_{ij} = m_j}} \mathbf{k}! \mathbf{m}! \prod_{1 \leq i, j \leq 3} \frac{\rho_{ij}^{d_{ij}}}{d_{ij}!} \right) \mathbb{1}_{|\mathbf{k}|=|\mathbf{m}|}.$$

As in [11], let us fix  $Q \in \mathbb{N}$  and let us denote by  $\pi^Q$  the projection onto the first  $Q$  chaos in  $L^2(\Omega)$  and by  $\pi_Q$  the projection onto the remaining one. With these notations

one has

$$\pi^Q(\eta(T)) = \frac{1}{\sigma_2(T)} \sum_{q=0}^Q \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) \, dt,$$

and  $\pi_Q(\eta(T)) = \eta(T) - \pi^Q(\eta(T))$ .

By the precedent remark, we have

$$\mathbb{E}[\pi^Q(\eta(T))]^2 = \sum_{q=0}^Q \mathbb{E} \left[ \frac{1}{\sigma_2(T)} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) \, dt \right]^2.$$

Remember that coefficients  $a_{f,\sigma}(\mathbf{k}, u)$  have been defined by (4.1).

Applying the Fatou's lemma, since  $\lim_{\sigma \rightarrow 0} a_{f,\sigma}(\mathbf{k}, u) = a_f(\mathbf{k}, u)$ , we obtain

$$\begin{aligned} \mathbb{E}[\pi^Q(\eta(T))]^2 &\leq \liminf_{\sigma \rightarrow 0} \sum_{q=0}^Q \mathbb{E} \left[ \frac{1}{\sigma_2(T)} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_{f,\sigma}(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) \, dt \right]^2 \\ &\leq \liminf_{\sigma \rightarrow 0} \sum_{q=0}^{\infty} \mathbb{E} \left[ \frac{1}{\sigma_2(T)} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_{f,\sigma}(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) \, dt \right]^2 \\ &= \lim_{\sigma \rightarrow 0} \mathbb{E} \left[ \frac{1}{\sigma_2(T)} \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_{f,\sigma}(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) \, dt \right]^2 \\ &= \lim_{\sigma \rightarrow 0} \mathbb{E} \left[ J_f^{(n)}(u, \sigma) \right]^2 = \mathbb{E} \left[ J_f^{(n)}(u) \right]^2 < +\infty, \end{aligned}$$

the last equality providing from equality (4.2) and from Proposition 4.2 and the third last equality from the expansion in  $L^2(\Omega)$  of  $J_f^{(n)}(u, \sigma)$  (see (4.2)) and precedent remark.

Thus the random variable  $\eta(T) \in L^2(\Omega)$  and

$$\mathbb{E}[\eta(T)]^2 = \sum_{q=0}^{\infty} \mathbb{E} \left[ \frac{1}{\sigma_2(T)} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) \, dt \right]^2. \quad (4.8)$$

This achieves proof of Lemma 4.10. □

It remains to prove that  $J_f^{(n)}(u) = \eta(T)$  in  $L^2(\Omega)$ .

As in the proof of Theorem 4.3, we write  $\|\cdot\|_{L^2(\Omega)}$  for the  $L^2(\Omega)$ -norm.

For fixed  $Q \in \mathbb{N}$  and  $\sigma > 0$ , one has the following inequalities

$$\begin{aligned} \|J_f^{(n)}(u) - \eta(T)\|_{L^2(\Omega)} &\leq \|\pi_Q(J_f^{(n)}(u) - \eta(T))\|_{L^2(\Omega)} \\ &\quad + \|\pi^Q(J_f^{(n)}(u) - J_f^{(n)}(u, \sigma))\|_{L^2(\Omega)} + \|\pi^Q(J_f^{(n)}(u, \sigma) - \eta(T))\|_{L^2(\Omega)} \\ &\leq \|\pi_Q(J_f^{(n)}(u))\|_{L^2(\Omega)} + \|\pi_Q(\eta(T))\|_{L^2(\Omega)} + \|J_f^{(n)}(u) - J_f^{(n)}(u, \sigma)\|_{L^2(\Omega)} \\ &\quad + \|\pi^Q(J_f^{(n)}(u, \sigma) - \eta(T))\|_{L^2(\Omega)}. \end{aligned}$$

Now by Lemma 4.4 and Lemma 4.10,  $J_f^{(n)}(u)$  and  $\eta(T)$  belong to  $L^2(\Omega)$ , thus  $\lim_{Q \rightarrow +\infty} \left\| \pi_Q(J_f^{(n)}(u)) \right\|_{L^2(\Omega)} = \lim_{Q \rightarrow +\infty} \left\| \pi_Q(\eta(T)) \right\|_{L^2(\Omega)} = 0$ .

Furthermore due to Proposition 4.2,  $\lim_{\sigma \rightarrow 0} \left\| J_f^{(n)}(u) - J_f^{(n)}(u, \sigma) \right\|_{L^2(\Omega)} = 0$  and for fixed

$Q \in \mathbb{N}$  and since  $\lim_{\sigma \rightarrow 0} a_{f,\sigma}(\mathbf{k}, u) = a_f(\mathbf{k}, u)$ ,

$$\lim_{\sigma \rightarrow 0} \left\| \pi_Q(J_f^{(n)}(u, \sigma) - \eta(T)) \right\|_{L^2(\Omega)} = 0.$$

Hence, for fixed  $Q \in \mathbb{N}$  and taking limit as  $\sigma$  tends to zero one obtains,

$$\left\| J_f^{(n)}(u) - \eta(T) \right\|_{L^2(\Omega)} \leq \left\| \pi_Q(J_f^{(n)}(u)) \right\|_{L^2(\Omega)} + \left\| \pi_Q(\eta(T)) \right\|_{L^2(\Omega)}.$$

Then, taking limit as  $Q$  tends to infinity one finally gets

$$\left\| J_f^{(n)}(u) - \eta(T) \right\|_{L^2(\Omega)} = 0.$$

Now to complete the proof of Proposition 4.9 it remains to establish that  $E[J_f^{(n)}(u)] = a_f(\mathbf{0}, u)$ , where  $\mathbf{0} = (0, 0, 0) \in \mathbb{N}^3$ . Indeed it is enough to remark that  $a(0, u) = \phi\left(\frac{u}{\sqrt{r_z(0)}}\right) \frac{1}{\sqrt{r_z(0)}} = p_{X(0)}(u)$  and that  $a_f(0, 0) = E\left[f\left(\frac{\nabla X(0)}{\|\nabla X(0)\|_2}\right) \|\nabla X(0)\|_2\right]$ , since

$$\|\nabla X(0)\|_2 = \sqrt{\mu} \left\| P\Lambda \begin{pmatrix} Y_1(0) \\ Y_2(0) \end{pmatrix} \right\|_2 = \sqrt{\mu} \left\| \Lambda \begin{pmatrix} Y_1(0) \\ Y_2(0) \end{pmatrix} \right\|_2.$$

Proposition 3.1 gives the result.  $\square$

Now we are ready to compute the asymptotic variance of  $\xi_f^{(n)}(u)$  as  $n$  goes to infinity, which is the object of the following proposition. The proof very closely follows the one given in ([11], Proposition 2.1).

#### 4.2 Asymptotic variance for $\xi_f^{(n)}(u)$

The functionals  $\xi_f^{(n)}(u)$  are also orthogonal in  $L^2(\Omega)$ . This is a crucial fact for computing its variance. Using the Arcones inequality (see [5], Lemma 1, p. 2245), we deduce the asymptotic variance of  $\xi_f^{(n)}(u)$  as  $T$  grows to  $\mathbb{R}^2$ , this variance depending on the level  $u$  as follows.

**Proposition 4.12.** *We have the following convergence,*

$$\lim_{n \rightarrow +\infty} \text{Var}[\xi_f^{(n)}(u)] = \Sigma_{f,f}(u),$$

$\Sigma_{f,f}(u)$  being defined by

$$\Sigma_{f,f}(u) = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{k}, \mathbf{m} \in \mathbb{N}^3 \\ |\mathbf{k}|=|\mathbf{m}|=q}} a_f(\mathbf{k}, u) a_f(\mathbf{m}, u) R(\mathbf{k}, \mathbf{m}), \quad (4.9)$$

while  $R(\mathbf{k}, \mathbf{m})$  is defined as

$$R(\mathbf{k}, \mathbf{m}) = \int_{\mathbb{R}^2} E\left[\tilde{H}_{\mathbf{k}}(Y(0)) \tilde{H}_{\mathbf{m}}(Y(v))\right] dv. \quad (4.10)$$

*Proof of Proposition 4.12.*  $\xi_f^{(n)}(u) = 2n \left( J_f^{(n)}(u) - \mathbb{E} \left[ J_f^{(n)}(u) \right] \right)$ .

Since the random variable  $\xi_f^{(n)}(u)$  is a centered one, using equality given in (4.8), we obtain

$$\text{Var} \left[ \xi_f^{(n)}(u) \right] = \mathbb{E} \left[ \xi_f^{(n)}(u) \right]^2 = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{k}, \mathbf{m} \in \mathbb{N}^3 \\ |\mathbf{k}|=|\mathbf{m}|=q}} a_f(\mathbf{k}, u) a_f(\mathbf{m}, u) R_n(\mathbf{k}, \mathbf{m}) \quad (4.11)$$

with

$$R_n(\mathbf{k}, \mathbf{m}) = \frac{1}{(2n)^2} \int_{[-n, n]^2} \int_{[-n, n]^2} \mathbb{E} \left[ \tilde{H}_{\mathbf{k}}(Y(s)) \tilde{H}_{\mathbf{m}}(Y(t)) \right] ds dt.$$

Thus and since  $Y$  is a stationary process, we have

$$R_n(\mathbf{k}, \mathbf{m}) = \int_{[-2n, 2n]^2} \mathbb{E} \left[ \tilde{H}_{\mathbf{k}}(Y(0)) \tilde{H}_{\mathbf{m}}(Y(v)) \right] \left( 1 - \frac{|v_1|}{2n} \right) \left( 1 - \frac{|v_2|}{2n} \right) dv.$$

Now by applying Lemma 4.11 of Section 4.1.3 to  $X = Y(0)$  and  $Y = Y(v)$ , one has for  $|\mathbf{k}| = |\mathbf{m}|$ ,

$$\mathbb{E} \left[ \tilde{H}_{\mathbf{k}}(Y(0)) \tilde{H}_{\mathbf{m}}(Y(v)) \right] = \sum_{\substack{d_{ij} \geq 0 \\ \sum_j d_{ij} = k_i \\ \sum_i d_{ij} = m_j}} \mathbf{k}! \mathbf{m}! \prod_{1 \leq i, j \leq 3} \frac{(\Gamma_{ij}^Y(v))^{d_{ij}}}{d_{ij}!},$$

where

$$\Gamma_{ij}^Y(v) = \mathbb{E}[Y_i(0)Y_j(v)]. \quad (4.12)$$

Since

$$\begin{aligned} \Gamma^Y(v) &= (\Gamma_{ij}^Y(v))_{1 \leq i, j \leq 3} \\ &= \begin{pmatrix} -\frac{1}{\mu} P^t \left( \frac{\partial^2 r_z}{\partial v_i \partial v_j} (Av) \right)_{1 \leq i, j \leq 2} P & -\frac{1}{\sqrt{\mu r_z(0)}} P^t \left( \frac{\partial r_z}{\partial v_i} (Av) \right)_{1 \leq i \leq 2} \\ -\frac{1}{\sqrt{\mu r_z(0)}} \left( \frac{\partial r_z}{\partial v_i} (Av) \right)_{1 \leq i \leq 2}^t P & \frac{r_z(Av)}{r_z(0)} \end{pmatrix}, \end{aligned}$$

we have for any  $v \in \mathbb{R}^2$ ,

$$\sup_{1 \leq i, j \leq 3} |\Gamma_{ij}^Y(v)| \leq \mathbf{L} \Psi(Av), \quad (4.13)$$

where the function  $\Psi$  has been introduced in Section 2 and  $\mathbf{L}$  is some positive constant. Hence, for  $|\mathbf{k}| = |\mathbf{m}| = q$ , with  $q \in \mathbb{N}^*$ ,

$$\left| \mathbb{E} \left[ \tilde{H}_{\mathbf{k}}(Y(0)) \tilde{H}_{\mathbf{m}}(Y(v)) \right] \right| \leq \mathbf{L}' \Psi^q(Av),$$

where  $\mathbf{L}'$  is some constant depending on  $\mathbf{k}$  and  $\mathbf{m}$ .

By the covariance assumptions previously stated,  $\Psi \in L^1(\mathbb{R}^2)$  and then  $\Psi^q(A \cdot) \in L^1(\mathbb{R}^2)$ . We can apply the Lebesgue convergence theorem and obtain, for  $\mathbf{k}, \mathbf{m} \in (\mathbb{N}^3)^*$ ,

$$\lim_{n \rightarrow +\infty} R_n(\mathbf{k}, \mathbf{m}) = R(\mathbf{k}, \mathbf{m}) = \int_{\mathbb{R}^2} \mathbb{E} \left[ \tilde{H}_{\mathbf{k}}(Y(0)) \tilde{H}_{\mathbf{m}}(Y(v)) \right] dv.$$

Now turning back to (4.11), we write

$$\text{Var} \left[ \xi_f^{(n)}(u) \right] = \sum_{q=1}^{\infty} V_q^{(n)}(u),$$

and according to what we have just seen, for all  $q \in \mathbb{N}^*$ ,

$$\lim_{n \rightarrow +\infty} V_q^{(n)}(u) = V_q(u) = \sum_{\substack{\mathbf{k}, \mathbf{m} \in \mathbb{N}^3 \\ |\mathbf{k}|=|\mathbf{m}|=q}} a_f(\mathbf{k}, u) a_f(\mathbf{m}, u) R(\mathbf{k}, \mathbf{m}). \quad (4.14)$$

Note that for any  $q$ ,  $V_q^{(n)}(u) \geq 0$  and so  $V_q(u)$ .

Thus, if we prove that  $\lim_{Q \rightarrow +\infty} \sup_n \sum_{q=Q+1}^{\infty} V_q^{(n)}(u) = 0$ , Fatou's lemma implies that  $\lim_{Q \rightarrow +\infty} \sum_{q=Q+1}^{\infty} V_q(u) = 0$ . Thus  $(\sum_{q=1}^Q V_q(u))_Q$  is an upper bounded increasing sequence and consequently a converging sequence, that is the series  $\sum_{q=1}^{\infty} V_q(u) = \Sigma_{f,f}(u)$  will be convergent. Also the first convergence will imply that  $\text{Var} \left[ \xi_f^{(n)}(u) \right]$  tends to  $\Sigma_{f,f}(u)$ . The proof of Proposition 4.12 will be completed. Thus let us prove this convergence via a lemma.

**Lemma 4.13.**

$$\lim_{Q \rightarrow +\infty} \sup_{n \geq 1} \sum_{q=Q+1}^{\infty} V_q^{(n)}(u) = 0. \quad (4.15)$$

*Proof of Lemma 4.13.* First, let us remark that the convergence in (4.15) is equivalent to the following one:

$$\lim_{Q \rightarrow +\infty} \text{Var} \left[ \pi_Q(\xi_f^{(n)}(u)) \right] = 0,$$

uniformly with respect to  $n$ , where  $\pi_Q$  stands for the projection onto the chaos of strictly upper order in  $Q$ .

For the sake of simplicity of writing, let us note  $V_{n,Q} = \text{Var} \left[ \pi_Q(\xi_f^{(n)}(u)) \right]$ .

Let  $s \in \mathbb{R}^2$  and  $\theta_s$  be the shift operator associated with the field  $X$ , that is,  $\theta_s X = X_{s+\cdot}$ . Let us also introduce the set of indices  $I_n = [-n, n]^2 \cap \mathbb{Z}^2$ , clearly we have

$$\pi_Q(\xi_f^{(n)}(u)) = \frac{1}{2n} \sum_{s \in I_n} \theta_s(\pi_Q(\xi_{f,1}(u))),$$

where the random variable  $\xi_{f,1}(u)$  is

$$\xi_{f,1}(u) = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_{[0,1]^2} \tilde{H}_{\mathbf{k}}(Y(t)) \, dt.$$

The stationarity of  $X$  leads to

$$V_{n,Q} = \left( \frac{1}{2n} \right)^2 \sum_{s \in I_{2n}} \alpha_s(n) \mathbb{E}[\pi_Q(\xi_{f,1}(u)) \theta_s(\pi_Q(\xi_{f,1}(u)))],$$

where  $\alpha_s(n) = \text{card}\{t \in I_n, t - s \in I_n\}$ .

Obviously, one has  $\alpha_s(n) \leq (2n)^2$ .

Now, on the one hand, by the assumptions made of Section 2,  $\lim_{\|x\|_2 \rightarrow +\infty} \Psi(x) = 0$ , and since the eigenvalues of  $A$  are strictly positive one also has

$\lim_{\|x\|_2 \rightarrow +\infty} \Psi(Ax) = 0$ .

On the other hand, let  $0 < \rho < 1$  such that

$$\rho \mathbf{L}(u) < 1 \text{ where } \mathbf{L}(u) = 2 \frac{u^2}{r_z(0)} + 1, \quad (4.16)$$

and  $a > 0$  such that  $\|x\|_2 \geq a$  implies

$$3 \mathbf{L}\Psi(Ax) \leq \rho < 1, \quad (4.17)$$

where  $\mathbf{L}$  defined by (4.13).

We split  $V_{n,Q}$  into  $V_{n,Q} = V_{n,Q}^{(1)} + V_{n,Q}^{(2)}$ , where in  $V_{n,Q}^{(1)}$  the sum runs over the indices  $s$  in  $\{s \in I_{2n}, \|s\|_\infty < a + 3\}$  and in  $V_{n,Q}^{(2)}$  over  $s$  in  $\{s \in I_{2n}, \|s\|_\infty \geq a + 3\}$ ,  $\|\cdot\|_\infty$  standing for the supremum norm. By Schwarz inequality and since  $\alpha_s(n) \leq (2n)^2$ , using the stationarity of  $X$  one has the following upper bound,

$$|V_{n,Q}^{(1)}| \leq (2(a+3))^2 \mathbb{E}[\pi_Q(\xi_{f,1}(u))]^2,$$

which goes to zero as  $Q$  goes to infinity uniformly with respect to  $n$ , since as in Proposition 4.9 one can prove that  $\lim_{Q \rightarrow +\infty} \mathbb{E}[\pi_Q(\xi_{f,1}(u))]^2 = 0$ .

We proved that  $\lim_{Q \rightarrow +\infty} \sup_n V_{n,Q}^{(1)} = 0$ .

Now, let us consider the term  $V_{n,Q}^{(2)}$ .

$$V_{n,Q}^{(2)} = \left(\frac{1}{2n}\right)^2 \sum_{\substack{s \in I_{2n} \\ \|s\|_\infty \geq a+3}} \alpha_s(n) \mathbb{E}[\pi_Q(\xi_{f,1}(u)) \theta_s(\pi_Q(\xi_{f,1}(u)))].$$

For  $q \in \mathbb{N}^*$ , let us define function  $F_q$  by

$$F_q(x) = \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \tilde{H}_{\mathbf{k}}(x), \quad x \in \mathbb{R}^3.$$

For  $s \in I_{2n}$  such that  $\|s\|_\infty \geq a + 3$ ,

$$\mathbb{E}[\pi_Q(\xi_{f,1}(u)) \theta_s(\pi_Q(\xi_{f,1}(u)))] = \sum_{q=Q+1}^{\infty} \int_{[0,1]^2} \int_{[0,1]^2} \mathbb{E}[F_q(Y(t)) F_q(Y(s+v))] \, dt \, dv.$$

At this step of the proof we want to propose a bound for  $\mathbb{E}[F_q(Y(t)) F_q(Y(s+v))]$ ,  $t, v \in [0, 1]^2$ .

Note that if  $\sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f^2(\mathbf{k}, u) \mathbf{k}! = 0$ , then  $F_q(x) = 0$  for all  $x \in \mathbb{R}^3$  and a trivial bound is zero. So let us suppose that  $\sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f^2(\mathbf{k}, u) \mathbf{k}! \neq 0$ .

We are going to apply Arcones inequality (see [5], Lemma 1 p. 2245). By using notations of this lemma, we apply it to  $f = F_q$  and to  $X = (X^{(j)})_{1 \leq j \leq 3} = Y(t)$  and  $Y = (Y^{(k)})_{1 \leq k \leq 3} = Y(s+v)$ , with  $d = 3$ , such that  $r^{(j,k)} = \mathbb{E}[X^{(j)} Y^{(k)}] = \Gamma_{jk}^Y(s-t+v)$ ,  $\Gamma^Y$  having defined in (4.12).

Now by using inequalities given in (4.13) and (4.17),

$$\begin{aligned} \psi &= \left( \sup_{1 \leq j \leq 3} \sum_{k=1}^3 |r^{(j,k)}| \right) \vee \left( \sup_{1 \leq k \leq 3} \sum_{j=1}^3 |r^{(j,k)}| \right) \\ &\leq 3 \mathbf{L} \Psi(A(s-t+v)) \leq \rho < 1. \end{aligned} \quad (4.18)$$

It remains to verify that  $F_q$  function on  $\mathbb{R}^3$  has finite second moment and rank  $q$ . In the first place by Lemma 4.11 given in Section 4.1.3 one has

$$\mathbb{E}[F_q(X)]^2 = \mathbb{E}[F_q(Y(t))]^2 = \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f^2(\mathbf{k}, u) \mathbf{k}! < +\infty.$$

In the second place and since  $\sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f^2(\mathbf{k}, u) \mathbf{k}! \neq 0$ , this last equality ensures that  $\text{rank } F_q \leq q$ . Furthermore let  $\mathbf{m} \in \mathbb{N}^3$  such that  $\mathbb{E}[F_q(X) \tilde{H}_{\mathbf{m}}(X)] \neq 0$ . By Lemma 4.11,  $\mathbb{E}[F_q(X) \tilde{H}_{\mathbf{m}}(X)] = \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \mathbf{k}! \mathbb{1}_{|\mathbf{k}|=|\mathbf{m}|}$ , which implies  $|\mathbf{m}| = q$  and  $\text{rank } F_q = q$ . Thus we have all the ingredients to apply Arcones inequality. For  $q \geq 2$ , using inequality given in (4.18) we get the bound

$$\begin{aligned} \mathbb{E}[F_q(Y(t)) F_q(Y(s+v))] &\leq \psi^q \mathbb{E}[F_q(Y(t))]^2 \\ &\leq \rho^{q-2} (3L)^2 \Psi^2(A(s-t+v)) \left( \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f^2(\mathbf{k}, u) \mathbf{k}! \right). \end{aligned}$$

As already pointed out in Remark 4.1, the series  $\sum_{\mathbf{k} \in \mathbb{N}^3} a_f^2(\mathbf{k}, u) \mathbf{k}! = +\infty$ , so that we have to tread carefully in what follows.

Finally and since  $\alpha_s(n) \leq (2n)^2$ , one has

$$\left| V_{n,Q}^{(2)} \right| \leq C \sum_{q=Q+1}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} \rho^{q-2} a_f^2(\mathbf{k}, u) \mathbf{k}! \sum_{s \in I_{2n}} \int_{[0,1]^2} \int_{[0,1]^2} \Psi^2(A(s-t+v)) \, dt \, dv.$$

Using that

$$\begin{aligned} \sum_{s \in I_{2n}} \int_{[0,1]^2} \int_{[0,1]^2} \Psi^2(A(s-t+v)) \, dt \, dv &\leq \int_{\mathbb{R}^2} \Psi^2(Av) \, dv \\ &\leq C \int_{\mathbb{R}^2} \Psi^2(v) \, dv < +\infty, \end{aligned}$$

last finiteness providing from hypotheses given in Section 2, one has

$$\left| V_{n,Q}^{(2)} \right| \leq C \sum_{q=Q+1}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} \rho^{q-2} a_f^2(\mathbf{k}, u) \mathbf{k}!.$$

To conclude the proof of this lemma we just have to check that

$$\sum_{q=2}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} \rho^{q-2} a_f^2(\mathbf{k}, u) \mathbf{k}! < +\infty.$$

Now remember that for  $\mathbf{k} = (k_i)_{1 \leq i \leq 3} \in \mathbb{N}^3$ ,

$$a_f(\mathbf{k}, u) = a_f(k_1, k_2) a(k_3, u),$$

with

$$a(k_3, u) = \frac{1}{k_3!} H_{k_3} \left( \frac{u}{\sqrt{r_z(0)}} \right) \phi \left( \frac{u}{\sqrt{r_z(0)}} \right) \frac{1}{\sqrt{r_z(0)}}.$$

On the one hand, since the function

$$h : (y_1, y_2) \mapsto f\left(\frac{P\Lambda\left(\frac{y_1}{y_2}\right)}{\sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2}}\right) \sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2}$$

is such that  $h \in L(\mathbb{R}^2, \phi_2(y) dy)$ , we deduce that,

$$\sum_{k_1, k_2 \in \mathbb{N}} a_f^2(k_1, k_2) k_1! k_2! < +\infty. \quad (4.19)$$

On the other hand by using the expression of Hermite's polynomials given in Appendix A by (A.4) and (A.5), for all  $k \in \mathbb{N}$  and for all  $x \in \mathbb{R}$  we get the bound

$$H_k^2(x) \leq (k+1)! (2x^2 + 1)^k,$$

so that for all  $\mathbf{k} \in \mathbb{N}^3$  such that  $|\mathbf{k}| = q$  and remembering that  $\mathbf{L}(u)$  has been defined in (4.16), one has

$$k_3! a^2(k_3, u) \leq \mathbf{C} (k_3 + 1) \mathbf{L}^{k_3}(u) \leq \mathbf{C} (q + 1) \mathbf{L}^q(u).$$

Thus by inequality (4.19), finally one obtains

$$\sum_{q=2}^{\infty} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} \rho^{q-2} a_f^2(\mathbf{k}, u) \mathbf{k}! \leq \mathbf{C} \sum_{q=2}^{\infty} \rho^{q-2} (q+1)^2 \mathbf{L}^q(u) < +\infty,$$

last finiteness providing from inequality (4.16).

This yields Lemma 4.13. □

Proposition 4.12 ensues. □

Now, we have got all the tools to prove that the random variable  $\xi_f^{(n)}(u)$  converges in law as  $n$  tends to infinity to a centered Gaussian variable with finite variance  $\Sigma_{f,f}(u)$  given by (4.9), see Theorem 4.14. The proof we give for this theorem and the following remark, are inspired by the one presented in ([11], Proposition 2.4).

### 4.3 Convergence in law of $\xi_f^{(n)}(u)$

Using the Peccati and Tudor theorem (see [19]), we obtain the following theorem.

**Theorem 4.14.**

$$\xi_f^{(n)}(u) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma_{f,f}(u)).$$

**Remark 4.15.** If  $f$  is a function with constant sign, then  $\Sigma_{f,f}(u) > 0$ .

For example if  $f \equiv 1$ , we find that the curve length converges in law to a non degenerate Gaussian random variable.

Note that for all real numbers  $a$  and  $b$  and for all continuous and bounded functions,  $f_1$  and  $f_2 : S^1 \rightarrow \mathbb{R}$ , one has  $\xi_{af_1 + bf_2}^{(n)}(u) = a\xi_{f_1}^{(n)}(u) + b\xi_{f_2}^{(n)}(u)$ .

Thus if we define

$$\Sigma_{f_1, f_2}(u) = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{k}, \mathbf{m} \in \mathbb{N}^3 \\ |\mathbf{k}|=|\mathbf{m}|=q}} a_{f_1}(\mathbf{k}, u) a_{f_2}(\mathbf{m}, u) R(\mathbf{k}, \mathbf{m}), \quad (4.20)$$

where  $R(\mathbf{k}, \mathbf{m})$  is defined by (4.10), we readily get Corollary 4.16.

**Corollary 4.16.** *For all  $f = (f_1, f_2) : S^1 \rightarrow \mathbb{R}^2$  continuous and bounded function, one has*

$$\xi_f^{(n)}(u) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma_{(f_1, f_2)}(u)),$$

where  $\Sigma_{(f_1, f_2)}(u) = (\Sigma_{f_i, f_j}(u))_{1 \leq i, j \leq 2}$ .

*Proofs of Theorem 4.14 and of Remark 4.15.* First, let  $Q$  a fixed integer in  $\mathbb{N}^*$  and let us consider the random variable

$$\pi^Q(\xi_f^{(n)}(u)) = \frac{1}{2n} \sum_{q=1}^Q \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) dt.$$

We will show the asymptotic normality of this sequence as  $n$  tends to infinity. For this purpose and in order to apply the Peccati and Tudor theorem (see [19], Theorem 1), we will give an expansion of this random variable into the Wiener-Itô chaos of order less or equal to  $Q$ .

To this end, remember that for any  $t \in \mathbb{R}^2$  one has defined

$$\mathbf{X}(t) = (\nabla X(t), X(t))^t,$$

and  $Y(t) = \Delta^{-1} \mathbf{X}(t) = (Y_i(t))_{i=1, 2, 3}$ .

We have seen in Section 2 that  $Z$  admits a spectral density  $f_z$  and then the same fact occurs for  $X$ . We have noted  $f_x$  the spectral density for  $X$ .

We obtain the following spectral representations: for  $t \in \mathbb{R}^2$ ,

$$X(t) = \int_{\mathbb{R}^2} e^{i \langle t, \lambda \rangle} \sqrt{f_x(\lambda)} dW(\lambda),$$

where  $W$  stands for the standard Brownian motion.

Thus, for any  $\lambda = (\lambda_i)_{1 \leq i \leq 2}$  in  $\mathbb{R}^2$ , we let

$$\nu(\lambda) = (i\lambda_1, i\lambda_2, 1)^t,$$

so that for any  $t \in \mathbb{R}^2$ ,

$$Y(t) = \Delta^{-1} \mathbf{X}(t) = \int_{\mathbb{R}^2} e^{i \langle t, \lambda \rangle} \sqrt{f_x(\lambda)} \Delta^{-1} \nu(\lambda) dW(\lambda).$$

In what follows, for any  $t \in \mathbb{R}^2$  and  $j = 1, 2, 3$ , we denote by  $\varphi_{t,j}$  the square integrable map on  $\mathbb{R}^2$  such that,

$$Y_j(t) = \int_{\mathbb{R}^2} \varphi_{t,j}(\lambda) dW(\lambda).$$

Now, since  $(\varphi_{t,j})_{1 \leq j \leq 3}$  is an orthonormal system in  $L^2(\mathbb{R}^2)$ , using Itô's formula, see ([16], Theorem 4.2 p. 30), for fixed  $\mathbf{k} = (k_i)_{1 \leq i \leq 3} \in \mathbb{N}^3$  such that  $|\mathbf{k}| = q$ ,

$$\begin{aligned} \tilde{H}_{\mathbf{k}}(Y(t)) &= \prod_{j=1}^3 H_{k_j}(Y(t)) \\ &= \int_{\mathbb{R}^{2q}} \varphi_{t,1}^{\otimes k_1} \otimes \varphi_{t,2}^{\otimes k_2} \otimes \varphi_{t,3}^{\otimes k_3}(\lambda_1, \dots, \lambda_q) dW(\lambda_1) \cdots dW(\lambda_q) \\ &= I_q(\varphi_{t,1}^{\otimes k_1} \otimes \varphi_{t,2}^{\otimes k_2} \otimes \varphi_{t,3}^{\otimes k_3}), \end{aligned}$$

where  $I_q$  stands for the Itô-Wiener integral of order  $q$ .

We shall use notations introduced in Slud (see [23]).

For each  $q \geq 1$ , let consider  $L^2_{\text{sym}}((\mathbb{R}^2)^q)$  the complex Hilbert-space

$$L^2_{\text{sym}}((\mathbb{R}^2)^q) = \{f_q \in L^2((\mathbb{R}^2)^q), \text{ such that for all } x \in (\mathbb{R}^2)^q, \\ f_q(x) = \overline{f_q(-x)}, f_q(x_1, \dots, x_q) = f_q(x_{\pi(1)}, \dots, x_{\pi(q)}), \forall \pi \in S_q\},$$

where  $S_q$  denotes the symmetric group of permutations of  $\{1, \dots, q\}$ .

For  $f_q \in L^2((\mathbb{R}^2)^q)$ , we denote by  $\text{sym}(f_q)$  the symmetrization of  $f_q$ , that is for  $x_1, \dots, x_q \in \mathbb{R}^2$ ,

$$\text{sym}(f_q)(x_1, \dots, x_q) = \frac{1}{q!} \sum_{\pi \in S_q} f_q(x_{\pi(1)}, \dots, x_{\pi(q)}).$$

Observe that for  $f_q \in L^2((\mathbb{R}^2)^q)$  such that for all  $x \in (\mathbb{R}^2)^q$ ,  $f_q(x) = \overline{f_q(-x)}$ , one has

$$I_q(f_q) = I_q(\text{sym}(f_q)). \quad (4.21)$$

For  $q \geq 1$ ,  $f_q \in L^2_{\text{sym}}((\mathbb{R}^2)^q)$  and  $p = 1, \dots, q$ , we will write  $f_q \otimes_p f_q$  for the  $p$ -th contraction of  $f_q$  defined as

$$f_q \otimes_p f_q(\lambda_1, \dots, \lambda_{2q-2p}) = \int_{(\mathbb{R}^2)^p} f_q(\lambda_1, \dots, \lambda_{q-p}, x_1, \dots, x_p) \\ f_q(\lambda_{q-p+1}, \dots, \lambda_{2q-2p}, -x_1, \dots, -x_p) \, dx_1 \cdots dx_p.$$

Using the property of  $I_q$  given in (4.21), the random variable of interest can be written as

$$\pi^Q(\xi_f^{(n)}(u)) = \sum_{q=1}^Q I_q(f_q^{(n)}),$$

where

$$f_q^{(n)} = \frac{1}{2n} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \text{sym}(\varphi_{t,1}^{\otimes k_1} \otimes \varphi_{t,2}^{\otimes k_2} \otimes \varphi_{t,3}^{\otimes k_3}) \, dt.$$

We symmetrized the function  $f_q^{(n)}$  with the aim of applying ([19], Theorem 1).

Symmetrizing the function complicates a lot the calculations in the study of the contractions. So we are going to write function  $f_q^{(n)}$  in another way.

For  $\mathbf{k} = (k_i)_{1 \leq i \leq 3}$  such that  $|\mathbf{k}| = q$ , we define

$$\mathcal{A}_{\mathbf{k}} = \{m = (m_1, \dots, m_q) \in \{1, 2, 3\}^q, \forall i = 1, 2, 3, \sum_{j=1}^q \mathbb{1}_{\{i\}}(m_j) = k_i\},$$

one has  $\text{card}\{\mathcal{A}_{\mathbf{k}}\} = \mathbf{k}!/q!$ . Let us remark that the sets  $(\mathcal{A}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}^3, |\mathbf{k}|=q}$  provide a partition of  $\{1, 2, 3\}^q$ .

With these notations one has

$$\sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \text{sym}(\varphi_{t,1}^{\otimes k_1} \otimes \varphi_{t,2}^{\otimes k_2} \otimes \varphi_{t,3}^{\otimes k_3}) = \\ \text{sym} \left( \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} \sum_{m \in \mathcal{A}_{\mathbf{k}}} \frac{a_f(\mathbf{k}, u)}{\text{card}(\mathcal{A}_{\mathbf{k}})} \varphi_{t,m_1} \cdots \varphi_{t,m_q} \right).$$

Since  $(\mathcal{A}_{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}^3, |\mathbf{k}|=q}$  provides a partition of  $\{1, 2, 3\}^q$ , then for all  $m \in \{1, 2, 3\}^q, \exists! \mathbf{k} \in \mathbb{N}^3$  such that  $|\mathbf{k}| = q$  and  $m \in \mathcal{A}_{\mathbf{k}}$ . So for fixed  $m \in \{1, 2, 3\}^q$ , let  $b_f(m, u) = \frac{a_f(\mathbf{k}, u)}{\text{card}(\mathcal{A}_{\mathbf{k}})}$ . Thus

$$\begin{aligned} & \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \text{sym}(\varphi_{t,1}^{\otimes k_1} \otimes \varphi_{t,2}^{\otimes k_2} \otimes \varphi_{t,3}^{\otimes k_3}) \\ &= \text{sym} \left( \sum_{m \in \{1,2,3\}^q} b_f(m, u) \varphi_{t,m_1} \cdots \varphi_{t,m_q} \right) \\ &= \sum_{m \in \{1,2,3\}^q} b_f(m, u) \varphi_{t,m_1} \cdots \varphi_{t,m_q} \end{aligned}$$

since  $m \mapsto b_f(m, u)$  is symmetric on  $\{1, 2, 3\}^q$ .

Finally the random variable  $\pi^Q(\xi_f^{(n)}(u))$  can be written as

$$\pi^Q(\xi_f^{(n)}(u)) = \sum_{q=1}^Q I_q(f_q^{(n)}),$$

where

$$f_q^{(n)} = \frac{1}{2n} \int_{]-n, n[^2} \sum_{m \in \{1,2,3\}^q} b_f(m, u) \varphi_{t,m_1} \cdots \varphi_{t,m_q} dt,$$

that ends our first objective.

To obtain convergence of  $\pi^Q(\xi_f^{(n)}(u))$ , we use ([19], Theorem 1). Convergence in Proposition 4.12 gives the required conditions appearing at the beginning of this latter theorem. So we just verify condition (i) in proving the following lemma.

**Lemma 4.17.** *For fixed integers  $q$  and  $p$  such that  $q \geq 2$  and  $p = 1, \dots, q-1$ ,*

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{(\mathbb{R}^2)^{2(q-p)}} \left| f_q^{(n)} \otimes_p f_q^{(n)}(\lambda_1, \dots, \lambda_{q-p}, \mu_1, \dots, \mu_{q-p}) \right|^2 \\ & d\lambda_1 \dots d\lambda_{q-p} d\mu_1 \dots d\mu_{q-p} = 0. \end{aligned}$$

*Proof of Lemma 4.17.* Let

$$\begin{aligned} C_n &= \int_{(\mathbb{R}^2)^{2(q-p)}} \left| f_q^{(n)} \otimes_p f_q^{(n)}(\lambda_1, \dots, \lambda_{q-p}, \mu_1, \dots, \mu_{q-p}) \right|^2 \\ & d\lambda_1 \dots d\lambda_{q-p} d\mu_1 \dots d\mu_{q-p}. \end{aligned}$$

Straightforwards calculations show that

$$\begin{aligned} C_n &= \left( \frac{1}{2n} \right)^4 \int_{]-n, n[^2} \sum_{k, m \in \{1,2,3\}^q} \sum_{K, M \in \{1,2,3\}^q} b_f(k, u) b_f(m, u) \\ & b_f(K, u) b_f(M, u) \Gamma_{k_1, K_1}^Y(t_1 - t_2) \dots \Gamma_{k_{q-p}, K_{q-p}}^Y(t_1 - t_2) \times \\ & \Gamma_{m_1, M_1}^Y(s_1 - s_2) \dots \Gamma_{m_{q-p}, M_{q-p}}^Y(s_1 - s_2) \Gamma_{k_{q-p+1}, m_{q-p+1}}^Y(t_1 - s_1) \dots \times \\ & \Gamma_{k_q, m_q}^Y(t_1 - s_1) \Gamma_{K_{q-p+1}, M_{q-p+1}}^Y(t_2 - s_2) \dots \Gamma_{K_q, M_q}^Y(t_2 - s_2) dt_1 ds_1 dt_2 ds_2, \end{aligned}$$

where  $\Gamma^Y$  has been defined in (4.12).

Using inequality (4.13), we get the bound

$$\begin{aligned} C_n &\leq \left( \frac{1}{2n} \right)^4 \mathbf{L}^{2q} \left( \sum_{m \in \{1,2,3\}^q} |b_f(m, u)| \right)^4 \int_{]-n, n[^2} \Psi^{q-p}(A(t_1 - t_2)) \times \\ & \Psi^{q-p}(A(s_1 - s_2)) \Psi^p(A(t_1 - s_1)) \Psi^p(A(t_2 - s_2)) dt_1 ds_1 dt_2 ds_2. \end{aligned}$$

Moreover, we have

$$\Psi^{q-p}(A(t_1 - t_2))\Psi^p(A(t_2 - s_2)) \leq \Psi^q(A(t_1 - t_2)) + \Psi^q(A(t_2 - s_2)).$$

Furthermore for  $r \in \mathbb{N}^*$ , one has

$$\int_{]-n, n[^2} \Psi^r(A(u - v)) \, du \, dv \leq \mathbf{C} \int_{\mathbb{R}^2} \Psi^r(u) \, du < +\infty.$$

Applying the penultimate and last inequalities to  $p \geq 1$ ,  $q \geq 1$  and  $q - p \geq 1$ , one obtains

$$C_n \leq \mathbf{C}_q \left( \frac{1}{2n} \right)^2 \left( \int_{\mathbb{R}^2} \Psi^q(u) \, du \right) \left( \int_{\mathbb{R}^2} \Psi^{q-p}(u) \, du \right) \left( \int_{\mathbb{R}^2} \Psi^p(u) \, du \right),$$

thus we proved that  $\lim_{n \rightarrow +\infty} C_n = 0$ , this achieves proof of Lemma 4.17.  $\square$

Hence we proved that for all  $Q \geq 1$ ,

- $\pi^Q(\xi_f^{(n)}(u)) \xrightarrow[n \rightarrow +\infty]{Law} N(0; \sum_{q=1}^Q V_q(u))$ , where  $V_q(u)$  has been defined by (4.14).

On the other hand we proved in Lemma 4.13 that

- for all  $n \geq 1$ ,  $\pi^Q(\xi_f^{(n)}(u)) \xrightarrow[Q \rightarrow +\infty]{L^2(\Omega)} \xi_f^{(n)}(u)$ ,

and that

- $\lim_{Q \rightarrow +\infty} \sum_{q=1}^Q V_q(u) = \Sigma_{f,f}(u)$ .

Finally and by Proposition 4.12 we also have

- $\lim_{Q \rightarrow +\infty} \lim_{n \rightarrow +\infty} \left\| \pi^Q(\xi_f^{(n)}(u)) - \xi_f^{(n)}(u) \right\|_{L^2(\Omega)} = 0$ .

Applying ([10], Lemma 1.1), we can conclude that  $\xi_f^{(n)}(u) \xrightarrow[n \rightarrow +\infty]{Law} N(0; \Sigma_{f,f}(u))$ , that achieves proof of Theorem 4.14.

*Proof of Remark 4.15.* Remark ensues from the following argumentation.

We have seen in the proof of Proposition 4.12 that  $\Sigma_{f,f}(u) = \sum_{q=1}^{\infty} V_q(u)$ , with

$$V_q(u) = \sum_{\substack{\mathbf{k}, \mathbf{m} \in \mathbb{N}^3 \\ |\mathbf{k}| = |\mathbf{m}| = q}} a_f(\mathbf{k}, u) a_f(\mathbf{m}, u) R(\mathbf{k}, \mathbf{m}) \geq 0,$$

for all  $q \geq 1$ . Thus

$$\Sigma_{f,f}(u) \geq V_1(u) + V_2(u).$$

By using Lemma 4.11 and the inversion formula, a computation gives that for  $|\mathbf{k}| = |\mathbf{m}| = 1$ ,  $R(\mathbf{k}, \mathbf{m}) = 0$  except when  $\mathbf{k} = \mathbf{m} = (0, 0, 1)$  and in this case one has

$$R((0, 0, 1), (0, 0, 1)) = \frac{1}{r_x(0)} \int_{\mathbb{R}^2} r_x(v) \, dv = (2\pi)^2 \frac{f_z(0)}{\lambda_1 \lambda_2 r_z(0)}.$$

Thus

$$V_1(u) = a_f^2(0, 0) \frac{u^2}{r_z^2(0)} \phi^2\left(\frac{u}{\sqrt{r_z(0)}}\right) (2\pi)^2 \frac{f_z(0)}{\lambda_1 \lambda_2 r_z(0)} > 0,$$

if  $u \neq 0$ , since  $f$  is supposed to have constant sign and  $f_z(0) > 0$  (see observation given in Section 2).

Using arguments similar to the previous ones, the fact that  $\int_{\mathbb{R}^2} f_z(t) \|t\|_2^2 dt < +\infty$  and Parseval equality, straightforward calculations provide that

$$\begin{aligned} V_2(u) = 2 \times (2\pi)^2 \int_{\mathbb{R}^2} f_x^2(t) \times & \left[ a_f((1, 1, 0), u)(d_{11}d_{21}t_1^2 \right. \\ & + (d_{11}d_{22} + d_{12}d_{21})t_1t_2 + d_{12}d_{22}t_2^2) + a_f((1, 0, 1), u) \frac{1}{\sqrt{r_z(0)}}(d_{11}t_1 + d_{12}t_2) \\ & + a_f((0, 1, 1), u) \frac{1}{\sqrt{r_z(0)}}(d_{21}t_1 + d_{22}t_2) + a_f((2, 0, 0), u)(d_{11}t_1 + d_{12}t_2)^2 \\ & \left. + a_f((0, 2, 0), u)(d_{21}t_1 + d_{22}t_2)^2 + a_f((0, 0, 2), u) \frac{1}{r_z(0)} \right]^2 dt \geq 0, \end{aligned}$$

where  $(d_{ij})_{1 \leq i, j \leq 2} = D = \frac{1}{\sqrt{\mu}} \Lambda^{-1} P^t$ .

**Remark 4.18.** Note that in the case where the process  $X$  is isotropic our result contains that of ([15], Theorem 3).

Since  $\det(D) \neq 0$ , one gets the following equivalence:

$$(V_2(u) = 0) \iff (a_f(\mathbf{k}, u) = 0, \text{ for all } \mathbf{k} \in \mathbb{N}^3 \text{ such that } |\mathbf{k}| = 2)$$

In particular, since  $f$  has a constant sign,  $a_f(0, 0) \neq 0$  so that  $a_f((0, 0, 2), 0) \neq 0$  and  $V_2(0) > 0$ .

Finally we proved that for  $u \neq 0$ ,  $\Sigma_{f,f}(u) \geq V_1(u) + V_2(u) \geq V_1(u) > 0$  and for  $u = 0$ ,  $\Sigma_{f,f}(0) \geq V_1(0) + V_2(0) \geq V_2(0) > 0$ .

The proof of Remark 4.15 is completed.  $\square$

$\square$

Now we have all the elements to prove a limit theorem about the distributions of estimators  $\hat{\lambda}_n$  and  $\hat{\theta}_{o,n}$ . Consequently, confidence intervals for the estimated parameters  $\lambda$  and  $\theta_o$  can be proposed and an isotropy test will follow.

## 5 Towards a test of isotropy

Let  $u$  be a fixed level in  $\mathbb{R}$ .

### 5.1 Convergence in law for the affinity parameters

First, we apply results of Section 4.3.

Recalling that  $f^* = (f_1^*, f_2^*)$  has been defined in (3.2), by Theorems 3.4 and 4.14 one can show the following proposition.

**Proposition 5.1.**

$$2n \left( \frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)} - \frac{\mathbb{E}[J_{f^*}^{(n)}(u)]}{\mathbb{E}[J_1^{(n)}(u)]} \right) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma^*(u)),$$

where  $\Sigma^*(u) = \Sigma^*(u, \lambda_1, \lambda_2, P) = B \Sigma_{(f_1^*, f_2^*, 1)}(u) B^t$  and

$$B = \frac{1}{a_1(\mathbf{0}, u)} \begin{pmatrix} 1 & 0 & -\frac{a_{f_1^*}(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} \\ 0 & 1 & -\frac{a_{f_2^*}(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} \end{pmatrix},$$

where  $\Sigma_{(f_1^*, f_2^*, 1)}(u) = \left( \Sigma_{f_i^*, f_j^*}(u) \right)_{1 \leq i, j \leq 3}$  with  $f_3^* = \mathbf{1}$ , the covariance  $\Sigma_{f_i^*, f_j^*}$  being defined by (4.20).

**Remark 5.2.** The expression of coefficients  $a_{f_i^*}(\cdot, u)$  for  $i = 1, 2$  and of  $a_1(\cdot, u)$  are respectively given in Lemma A.3 and Lemma A.5 of Appendix A.

**Remark 5.3.** The asymptotic variance matrix  $\Sigma^*(u)$  is a non-degenerate matrix.

*Proof of Proposition 5.1.* Let  $f : S^1 \rightarrow \mathbb{R}$  be a continuous and bounded function. Since  $\mathbb{E}[J_f^{(n)}(u)] = a_f(\mathbf{0}, u)$  the following decomposition ensues

$$\begin{aligned} 2n \left( \frac{J_f^{(n)}(u)}{J_1^{(n)}(u)} - \frac{\mathbb{E}[J_f^{(n)}(u)]}{\mathbb{E}[J_1^{(n)}(u)]} \right) &= \frac{1}{a_1(\mathbf{0}, u)} \left( \xi_f^{(n)}(u) - \frac{a_f(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} \xi_1^{(n)}(u) \right) \\ &\quad + \left( \frac{1}{J_1^{(n)}(u)} - \frac{1}{a_1(\mathbf{0}, u)} \right) \left( \xi_f^{(n)}(u) - \frac{a_f(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} \xi_1^{(n)}(u) \right) \\ &\equiv \frac{1}{a_1(\mathbf{0}, u)} \left( \xi_f^{(n)}(u) - \frac{a_f(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} \xi_1^{(n)}(u) \right), \end{aligned}$$

the last law equivalence providing from Theorem 3.4 and from Theorem 4.14.

Applying this reasoning again successively to  $f = f_1^*$  and  $f = f_2^*$ , and using Corollary 4.16 by taking  $f_1 = f_1^* - \frac{a_{f_1^*}(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} \mathbf{1}$ ,  $f_2 = f_2^* - \frac{a_{f_2^*}(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} \mathbf{1}$ , we get Proposition 5.1.

The calculus of coefficients  $a_{f_i^*}(\cdot, u)$  for  $i = 1, 2$  and of  $a_1(\cdot, u)$  are respectively given in Lemma A.3 and Lemma A.5 of Appendix A.  $\square$

*Proof of Remark 5.3.* It remains to prove that the matrix  $\Sigma^*(u)$  is positive definite. In this aim, for  $f_1, f_2 : S^1 \rightarrow \mathbb{R}$  continuous and bounded functions, let us note for  $q \in \mathbb{N}^*$ ,

$$(\Sigma_{f_1, f_2}(u))_q = \sum_{\substack{\mathbf{k}, \mathbf{m} \in \mathbb{N}^3 \\ |\mathbf{k}| = |\mathbf{m}| = q}} a_{f_1}(\mathbf{k}, u) a_{f_2}(\mathbf{m}, u) R(\mathbf{k}, \mathbf{m}),$$

and

$$(\Sigma_{(f_1, f_2)}(u))_q = \left( (\Sigma_{f_i, f_j}(u))_q \right)_{1 \leq i, j \leq 2}.$$

The proof of Lemma 5.4 is given in Appendix A.

**Lemma 5.4.** For all  $q \in \mathbb{N}^*$ , one has

$$\det(\Sigma_{(f_1, f_2)}(u)) \geq \sum_{q=1}^{\infty} \det(\Sigma_{(f_1, f_2)}(u))_q \geq \det(\Sigma_{(f_1, f_2)}(u))_q.$$

Thus applying this lemma to functions  $f_1 = \frac{f_1^*}{a_1(\mathbf{0}, u)} - \frac{a_{f_1^*}(\mathbf{0}, u)}{a_1^2(\mathbf{0}, u)} \mathbf{1}$  and  $f_2 = \frac{f_2^*}{a_1(\mathbf{0}, u)} - \frac{a_{f_2^*}(\mathbf{0}, u)}{a_1^2(\mathbf{0}, u)} \mathbf{1}$ , we obtain

$$\det(\Sigma^*(u)) \geq \det(\Sigma_{(f_1, f_2)}(u))_2.$$

Note that  $\det(\Sigma_{(f_1, f_2)}(u))_1 = 0$  and this is why we let  $q = 2$  in Lemma 5.4. Using arguments similar to those given in the proof of Remark 4.15, simple calculations give

$$\det(\Sigma^*(u)) \geq 4 \times (2\pi)^2 \int_{\mathbb{R}^4} f_x^2(s) f_x^2(t) \{g^2(s, t) - g(s, t)g(t, s)\} ds dt \geq 0,$$

where

$$g(s, t) = s_1^2 t_2^2 a + s_1^2 t_1 t_2 b - s_1 s_2 t_2^2 c,$$

with

$$\begin{aligned} a &= [(d_{11}d_{22} + d_{12}d_{21}) A + d_{11}d_{12} B - d_{21}d_{22} C] \det(D) \\ b &= [2d_{11}d_{21}A + d_{11}^2 B - d_{21}^2 C] \det(D) \\ c &= [-2d_{12}d_{22} A - d_{12}^2 B + d_{22}^2 C] \det(D), \end{aligned}$$

while

$$\begin{aligned} A &= a_{f_1}((2, 0, 0), u) a_{f_2}((0, 2, 0), u) - a_{f_1}((0, 2, 0), u) a_{f_2}((2, 0, 0), u) \\ B &= a_{f_1}((2, 0, 0), u) a_{f_2}((1, 1, 0), u) - a_{f_1}((1, 1, 0), u) a_{f_2}((2, 0, 0), u) \\ C &= a_{f_1}((0, 2, 0), u) a_{f_2}((1, 1, 0), u) - a_{f_1}((1, 1, 0), u) a_{f_2}((0, 2, 0), u), \end{aligned}$$

and  $D$  is defined as in the proof of Remark 4.15 as  $D = (d_{ij})_{1 \leq i, j \leq 2} = \frac{1}{\sqrt{\mu}} \Lambda^{-1} P^t$ . The positivity of the last integral provides from an application of Hölder inequality.

Let us remark on the one hand that the nullity of the integral is equivalent to the equality in Hölder inequality and thus to  $a = b = c = 0$ . On the other hand since  $\det(D) \neq 0$ ,  $A, B, C$  is solution of a Cramer linear system. We deduce that  $\Sigma^*(u)$  will be strictly positive if  $(A, B, C) \neq (0, 0, 0)$ .

Let us see that  $C \neq 0$ .

By using Lemmas A.3 and A.5, straightforward computations show that,

$$C = -\frac{\det(P)}{a_1^2(0, 0)} \times \frac{\lambda_1 \lambda_2 \mu}{\pi(\lambda_1^2(\omega_1^*)^2 + \lambda_2^2(\omega_2^*)^2)} \times [\lambda_2^2(\omega_2^*)^2 + 2(\lambda_1^2(\omega_1^*)^2 - \lambda_2^2(\omega_2^*)^2)W],$$

where

$$W = \frac{\sum_{n=0}^{\infty} \frac{1}{4(n+1)} V_n}{\sum_{n=0}^{\infty} V_n}, \text{ while for } n \in \mathbb{N}, V_n = \frac{(2n)!(2n+1)!}{(n!)^4 2^{4n+1}} (1 - \lambda^2)^n.$$

By using that  $0 < W \leq \frac{1}{4}$ , one easily gets that  $\lambda_2^2(\omega_2^*)^2 + 2(\lambda_1^2(\omega_1^*)^2 - \lambda_2^2(\omega_2^*)^2)W > 0$ , thus  $C \neq 0$ .

This yields proof of remark.  $\square$

If we translate the convergence result expressed in the above Proposition 5.1 in the basis  $(v^*, v^{**})$ , recalling that function  $F$  has been defined in (3.5), we get for  $0 < \lambda \leq 1$  and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$

$$2n \left( F(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F(\lambda, \theta_o) \right) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma^{(*)}(u)),$$

where  $\Sigma^{(*)}(u) = \Sigma^{(*)}(u, \lambda_1, \lambda_2, P) = Q \times \Sigma^*(u) \times Q^t$  and  $Q$  is the change of basis matrix from the canonical basis  $(\vec{i}, \vec{j})$  to the basis  $(v^*, v^{**})$ .

By using last proposition and the fact that  $F$  is a  $C^2$ -diffeomorphism, we get the following theorem.

**Theorem 5.5.** For  $0 < \lambda < 1$  and  $-\frac{\pi}{2} < \theta_o < \frac{\pi}{2}$ , one has

$$2n \left( \hat{\lambda}_n - \lambda; \hat{\theta}_{o,n} - \theta_o \right) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma_{\lambda, \theta_o}(u)),$$

where  $\Sigma_{\lambda, \theta_o}(u) = C(\lambda, \theta_o) \times \Sigma^{(*)}(u) \times C^t(\lambda, \theta_o)$  and

$$C(\lambda, \theta_o) = \frac{1}{J_F(\lambda, \theta_o)} \begin{pmatrix} \frac{\partial F_2}{\partial \theta_o}(\lambda, \theta_o) & -\frac{\partial F_1}{\partial \theta_o}(\lambda, \theta_o) \\ -\frac{\partial F_2}{\partial \lambda}(\lambda, \theta_o) & \frac{\partial F_1}{\partial \lambda}(\lambda, \theta_o) \end{pmatrix}$$

where the Jacobian  $J_F$  has been defined in (A.1).

*Proof of Theorem 5.5.* The decomposition given in (3.6), Proposition 3.7, Corollary 3.6 and Proposition 5.1 imply that

$$\begin{aligned} 2n \left( \frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)} - \frac{\mathbb{E}[J_{f^*}^{(n)}(u)]}{\mathbb{E}[J_1^{(n)}(u)]} \right) &= 2n \left( F_1(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F_1(\lambda, \theta_o) \right) v^* \\ &\quad + 2n \left( F_2(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F_2(\lambda, \theta_o) \right) v^{**} \\ &\xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma^*(u)). \end{aligned}$$

Thus if we express last convergence in the canonical basis  $(\vec{i}, \vec{j})$  and if  $Q$  is the change of basis matrix from the canonical basis  $(\vec{i}, \vec{j})$  to the basis  $(v^*, v^{**})$ , one has proved that

$$\begin{aligned} 2n \left( F(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F(\lambda, \theta_o) \right) &= 2n (X_n - X, Y_n - Y) \\ &\xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma^{(*)}(u)), \quad (5.1) \end{aligned}$$

where  $\Sigma^{(*)}(u) = Q \times \Sigma^*(u) \times Q^t$ .

Now we suppose that  $(\lambda, \theta_o) \in U = ]0, 1[ \times ] -\frac{\pi}{2}, \frac{\pi}{2}[$ . On the one hand, by Remark 3.11, we know that  $F$  is a  $C^2$ -diffeomorphism from the open set  $U$  onto the open set  $V = f(U)$ . On the other hand, by Theorem 3.12, we know that

$$(\hat{\lambda}_n, \hat{\theta}_{o,n}) \xrightarrow[n \rightarrow +\infty]{a.s.} (\lambda, \theta_o).$$

Thus for  $n$  large enough, almost surely  $F(\hat{\lambda}_n, \hat{\theta}_{o,n})$  and  $F(\lambda, \theta_o)$  belong to  $V$ . Using a second order Taylor-Young expansion of  $F^{-1}$  about  $F(\lambda, \theta_o)$ , we get

$$\begin{aligned} 2n \left( \hat{\lambda}_n - \lambda; \hat{\theta}_{o,n} - \theta_o \right) &= \\ &\sum_{j=1}^2 \frac{\partial F^{-1}}{\partial x_j}(F(\lambda, \theta_o)) \times 2n(F_j(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F_j(\lambda, \theta_o)) + \\ &\frac{1}{2} \sum_{j,k=1}^2 \frac{\partial^2 F^{-1}}{\partial x_j \partial x_k}(F(\lambda, \theta_o)) \times 2n(F_j(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F_j(\lambda, \theta_o)) \times \\ &\quad (F_k(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F_k(\lambda, \theta_o)) \\ &\quad + o(2n \left\| F(\hat{\lambda}_n, \hat{\theta}_{o,n}) - F(\lambda, \theta_o) \right\|_2^2). \end{aligned}$$

Law convergence result expressed in (5.1) gives Theorem 5.5.  $\square$

Now we are in position to propose confidence intervals for  $(\lambda, \theta_o)$  when the covariance  $r_z$  of  $Z$  is known.

## 5.2 Confidence intervals for $(\lambda, \theta_o)$

In this section, we suppose that parameters  $\lambda$  and  $\theta_o$  are such that  $0 < \lambda < 1$ ,  $-\frac{\pi}{2} < \theta_o < \frac{\pi}{2}$ . We will also assume that the covariance function  $r_z$  is a **known function**.

One can build confidence intervals for parameters  $(\lambda, \theta_o)$ . We will show that

$$\Sigma^*(u) = \Sigma^*(u, \lambda_1, \lambda_2, P) = \frac{1}{\lambda_1^2} \Sigma_*(u, \lambda, P), \quad (5.2)$$

where  $\Sigma_*$  is a continuous matrix as function of  $(\lambda, P)$  and is computable provided that  $(\lambda, P)$  are given.

We then consider  $\hat{\lambda}_{1,n}$  and  $\hat{P}_n$  two estimators of respectively  $\lambda_1$  and matrix  $P$  obtained as follows. We propose  $\hat{P}_n = (\hat{v}_{1,n}, \hat{v}_{2,n})$  as estimator of  $P = (v_1, v_2)$ , the orthonormal basis of eigenvectors of matrix  $A$ , with:

$$\begin{cases} \hat{v}_{1,n} = \cos(\hat{\theta}_{o,n})v^* - \sin(\hat{\theta}_{o,n})v^{**} \\ \hat{v}_{2,n} = \sin(\hat{\theta}_{o,n})v^* + \cos(\hat{\theta}_{o,n})v^{**} \end{cases}$$

By Theorem 3.12-(2),  $\hat{P}_n$  is a consistent estimator of  $P$ .

Now for  $\hat{\lambda}_{1,n}$ , first, we apply Theorem 3.4 to the particular function  $f \equiv 1$  and then, we use the result of Proposition 3.1.

We deduce that if  $\tilde{\Lambda}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ , one has

$$J_1^{(n)}(u) \xrightarrow[n \rightarrow +\infty]{a.s.} \lambda_1 p_{Z(0)}(u) \mathbb{E} \left[ \left\| \tilde{\Lambda}(\lambda) P^t \nabla Z(0) \right\|_2 \right] = \lambda_1 \Phi(u, \lambda, P), \quad (5.3)$$

where  $\Phi$  is a continuous function.

A consistent estimator for  $\lambda_1$  is then obtained as:

$$\hat{\lambda}_{1,n} = \frac{J_1^{(n)}(u)}{\Phi(u, \hat{\lambda}_n, \hat{P}_n)},$$

and finally a consistent estimator of  $\Sigma^*(u)$  is given by

$$\hat{\Sigma}_n^*(u) = \frac{1}{\hat{\lambda}_{1,n}^2} \Sigma_*(u, \hat{\lambda}_n, \hat{P}_n).$$

The matrix  $\hat{\Sigma}_n^*(u)$  is computable and can be factorized as:  $\hat{\Sigma}_n^*(u) = R_n \Gamma_n^* R_n^t$ , where  $R_n$  is an unitary matrix and  $\Gamma_n^*$  is a diagonal matrix.

Remark that  $\Gamma_n^*$  is invertible since  $\Sigma^*(u)$  is non-degenerate (see Remark 5.3). Thus let:

$$D_n(u) = (C^{-1}(\hat{\lambda}_n, \hat{\theta}_{o,n}))^t Q R_n (\Gamma_n^*)^{-\frac{1}{2}} R_n^t.$$

Theorem 5.5 implies Corollary 5.6.

**Corollary 5.6.** For  $0 < \lambda < 1$  and  $-\frac{\pi}{2} < \theta_o < \frac{\pi}{2}$ , one has

$$2n \left( \hat{\lambda}_n - \lambda; \hat{\theta}_{o,n} - \theta_o \right) \times D_n(u) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; I_2).$$

*Proof of Corollary 5.6.* First, let us prove that  $\Sigma^*(u) = \Sigma^*(u, \lambda_1, \lambda_2, P) = \frac{1}{\lambda_1^2} \Sigma_*(u, \lambda, P)$ ,

where  $\Sigma_*$  is a computable and a continuous matrix as function of  $(\lambda, P)$ .

On the one hand, this fact provides from the form of the ratio of coefficients  $a_{f_i^*}(\mathbf{k}, u)/a_1(\mathbf{0}, u)$ ,  $i = 1, 2, 3$ , these latter coefficients being defined in Lemmas A.3 and A.5 of Appendix A. Indeed these ratios only depend on the ratio  $\lambda = \lambda_2/\lambda_1$  and  $P$  and they are computable as function of  $\lambda$  and  $P$ . They do not depend on  $\mu$ . On the other hand by Lemma 4.11 given in Section 4.1.3 one can see that the term  $R(\mathbf{k}, \mathbf{m})$  only depends on the covariance function of  $Y$  through the following form. By defining  $W(v)$  as the 3-dimensional vector defined as  $W(v) = \left( \frac{P^t}{\sqrt{\mu}} \nabla Z(v), \frac{Z(v)}{\sqrt{r_z(0)}} \right)^t$ , one has

$$\begin{aligned} R(\mathbf{k}, \mathbf{m}) &= \int_{\mathbb{R}^2} \mathbb{E} \left[ \tilde{H}_{\mathbf{k}}(Y(0)) \tilde{H}_{\mathbf{m}}(Y(v)) \right] dv \\ &= \frac{1}{\lambda_1^2 \lambda} \int_{\mathbb{R}^2} \mathbb{E} \left[ \tilde{H}_{\mathbf{k}}(W(0)) \tilde{H}_{\mathbf{m}}(W(v)) \right] dv = \frac{1}{\lambda_1^2} G(\lambda, P, \mathbf{k}, \mathbf{m}), \end{aligned}$$

and since  $r_z$  is supposed to be known,  $G$  is computable as function of  $\lambda$  and  $P$ . Thus  $\Sigma^*(u) = \frac{1}{\lambda_1^2} \Sigma_*(u, \lambda, P)$ , and  $\Sigma_*$  is computable.

It remains to prove that  $\Sigma_*$  is a continuous matrix as function of  $\lambda$  and  $P$ . In this aim, let us compute  $\Sigma_{f_1, f_1}$ , similar arguments would be raised for  $\Sigma_{f_2, f_2}$  and for  $\Sigma_{f_1, f_2}$ , where  $f_i = \frac{f_i^*}{a_1(\mathbf{0}, u)} - \frac{a_{f_i^*}(\mathbf{0}, u)}{a_1^2(\mathbf{0}, u)} \mathbf{1}$ ,  $i = 1, 2$ .

Applying Proposition 4.12 to  $f = f_1$  and the second order Rice formula for  $\mathbb{E} \left[ \xi_{f_1}^{(n)}(u) \right]^2$ , it is easy to see that another expression for  $\Sigma_{f_1, f_1}$  is

$$\begin{aligned} \Sigma_{f_1, f_1} = \int_{\mathbb{R}^2} \mathbb{E}[f_1(\nu_X(t)) f_1(\nu_X(0)) \|\nabla X(t)\|_2 \|\nabla X(0)\|_2 \mid X(0) = X(t) = u] \\ \times p_{X(0), X(t)}(u, u) - (\mathbb{E}[f_1(\nu_X(0)) \|\nabla X(0)\|_2])^2 p_{X(0)}^2(u) dt. \end{aligned}$$

Denoting by  $B(\lambda, P) = P \tilde{\Lambda}(\lambda) P^t$  and letting  $At = v$ , one gets

$$\begin{aligned} \Sigma_{f_1, f_1} = \frac{1}{a_1^2(\mathbf{0}, u)} \frac{1}{\lambda} \int_{\mathbb{R}^2} \left\{ E \left[ g_1 \left( \frac{B(\lambda, P) \nabla Z(v)}{\|B(\lambda, P) \nabla Z(v)\|_2} \right) g_1 \left( \frac{B(\lambda, P) \nabla Z(0)}{\|B(\lambda, P) \nabla Z(0)\|_2} \right) \right. \right. \\ \left. \left. \|B(\lambda, P) \nabla Z(v)\|_2 \|B(\lambda, P) \nabla Z(0)\|_2 / Z(0) = Z(v) = u \right] p_{Z(0), Z(v)}(u, u) \right. \\ \left. - \left( \mathbb{E} \left[ g_1 \left( \frac{B(\lambda, P) \nabla Z(0)}{\|B(\lambda, P) \nabla Z(0)\|_2} \right) \right] \|B(\lambda, P) \nabla Z(0)\|_2 \right)^2 p_{Z(0)}^2(u) \right\} dv, \end{aligned}$$

where  $g_1 = f_1^* - \frac{a_{f_1^*}(\mathbf{0}, u)}{a_1(\mathbf{0}, u)} = f_1^* - b(u, \lambda, P)$ , while  $b$  is a continuous function of  $\lambda$  and  $P$ . We conclude the argument noticing that  $a_1(\mathbf{0}, u) = \lambda_1 c(u, \lambda)$  with  $c$  a strictly positive continuous function of  $\lambda$ .

Now to prove Corollary 5.6, we have Lemma 5.7 proved in Appendix A.

**Lemma 5.7.** *Let  $A = R\Gamma R^t$  a definite positive matrix such that  $R$  is an unitary matrix, while  $\Gamma$  is a diagonal one. Let also  $(A_n)_n$  be an approximation of matrix  $A$ , i.e.  $\lim_n A_n = A$ , such that  $A_n = R_n \Gamma_n R_n^t$  with  $R_n$  an unitary matrix and  $\Gamma_n$  a diagonal one. Consider  $B_n$  a square root of  $A_n$ , that is  $B_n = R_n \Gamma_n^{\frac{1}{2}} R_n^t$ . Then  $\lim_{n \rightarrow +\infty} B_n = B = R\Gamma^{\frac{1}{2}} R^t$ .*

Turning back to the proof of Corollary 5.6, we apply Lemma 5.7 to  $A = \Sigma^*(u) := R\Gamma^* R^t$ ,  $A_n = \hat{\Sigma}_n^*(u) = R_n \Gamma_n^* R_n^t$ . Since  $\hat{\Sigma}_n^*(u)$  is a consistent estimator of  $\Sigma^*(u)$ , we deduce that  $\lim_{n \rightarrow +\infty} A_n = A$ . Using Lemma 5.7 and Theorem 3.12, we obtain that almost surely

$$\lim_{n \rightarrow +\infty} D_n(u) = (C^{-1}(\lambda, \theta_0))^t Q R (\Gamma^*)^{-\frac{1}{2}} R^t = D(u).$$

Using Theorem 5.5 and the fact that  $D^t(u) \Sigma_{\lambda, \theta_o}(u) D(u) = I_2$ , one finally proved corollary.  $\square$

### 5.3 Complementary results for estimating the parameter $\lambda$

We emphasize that convergence result in Theorem 5.5 is valid under the assumption that  $0 < \lambda < 1$  and  $-\frac{\pi}{2} < \theta_o < \frac{\pi}{2}$ . However, we can better elaborate what is happening to  $\hat{\lambda}_n$  in the isotropic case, when  $\lambda = 1$  (and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ ) and also when  $0 < \lambda < 1$  and  $\theta_o = \frac{\pi}{2}$ , via the following theorem.

**Theorem 5.8.** *For  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ , one has*

1. For  $0 < \lambda < 1$ ,

$$2n \left( \hat{\lambda}_n - \lambda \right) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; (\Sigma_{\lambda, \theta_o}(u))_{11}).$$

2. For  $\lambda = 1$ ,

$$2n(1 - \hat{\lambda}_n) \xrightarrow[n \rightarrow +\infty]{Law} \sqrt{U},$$

where the density  $f_U(t)$  of the positive random variable  $U$  is given by:

$$f_U(t) = \left( \frac{1}{4\pi} \int_0^{2\pi} e^{-t \frac{(\cos(\theta) - a \sin(\theta))^2}{2\sigma_{11}^2(1+a^2)}} e^{-t \frac{\sin^2(\theta)(1+a^2)}{2\tilde{\sigma}_{22}^2}} d\theta \right) \mathbb{1}_{\{t>0\}},$$

where the coefficients  $a$  and  $\tilde{\sigma}_{22}$  are defined by:

$$a = \frac{\sigma_{12}}{\sigma_{11}^2}, \tilde{\sigma}_{22}^2 = \frac{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2}{\sigma_{11}^2},$$

while coefficients  $\sigma_{11}$ ,  $\sigma_{22}$  and  $\sigma_{12}$  are

$$\sigma_{11}^2 = 4 \left( \frac{\pi}{2} \right)^2 \Sigma_{11}^{(\star\star)}(u), \sigma_{22}^2 = \left( \frac{\pi}{2} \right)^2 \Sigma_{22}^{(\star\star)}(u), \sigma_{12} = 2 \left( \frac{\pi}{2} \right)^2 \Sigma_{12}^{(\star\star)}(u),$$

with

$$\Sigma^{(\star\star)}(u) = \left( \Sigma_{ij}^{(\star\star)}(u) \right)_{1 \leq i, j \leq 2} = \Sigma^{(\star)}(u, \tau, \tau, I_2),$$

where  $\tau$  is the common value of the eigenvalues of matrix  $A$  under the isotropic hypothesis.

**Remark 5.9.** If  $\lambda \neq 1$ , by part one of Theorem 5.8, we readily get that  $2n(1 - \hat{\lambda}_n)$  converges in law to Gaussian random variable with positive infinite mean when  $n$  tends to infinity. Thus one gets a one more way to detect if the process is isotropic or not.

**Remark 5.10.** By Remark 5.3, the matrix  $\Sigma^{(\star\star)}(u)$  is non-degenerate, ensuring that the coefficient  $\tilde{\sigma}_{22}$  appearing in the expression of the density  $f_U$  does not vanish.

**Remark 5.11.** We suppose that the covariance function  $r_z$  of  $Z$  is **known**. We can estimate the density  $f_U$  by  $\hat{f}_U(t) = f_V(\hat{\tau}_n t)$ , where  $\hat{\tau}_n$  is a consistent estimator of  $\tau$ , the common value of the eigenvalues of matrix  $A$  under the isotropic case and where  $V$  has the same law as that of  $U$ , where we replaced  $\Sigma^{(\star\star)}(u)$  by  $Q \Sigma_\star(u, 1, I_2) Q^t$ , where  $\Sigma_\star$  being defined in equality (5.2) is computable. By using convergence in (5.3) we can take for  $\hat{\tau}_n$ ,

$$\hat{\tau}_n = 2 J_1^{(n)}(u) \sqrt{\frac{r_z(0)}{\mu}} \exp\left(\frac{1}{2} \frac{u^2}{r_z(0)}\right). \quad (5.4)$$

*Proof of Theorem 5.8.* By Theorem 5.5 we already know that if  $0 < \lambda < 1$  and  $-\frac{\pi}{2} < \theta_o < \frac{\pi}{2}$ ,

$$2n(\hat{\lambda}_n - \lambda; \hat{\theta}_{o,n} - \theta_o) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma_{\lambda, \theta_o}(u)).$$

So let us look first at the case where  $0 < \lambda < 1$  and  $\theta_o = \frac{\pi}{2}$ , then we will be treat the case where  $\lambda = 1$  (and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ ).

In the first case, by decomposition obtained in (3.13) in the proof of Theorem 3.12, we have

$$\begin{aligned} 2n(X_n - X)(X_n + X) & \left[ I^4(\hat{\lambda}_n)(X_n^2 + Y_n^2) + X^2 I^4(\hat{\lambda}_n) - I^2(\hat{\lambda}_n)(\hat{\lambda}_n^2 + 1) \right] \\ & + 2nY_n^2 I^4(\hat{\lambda}_n) X^2 - 2n \left( \lambda I(\hat{\lambda}_n) - I(\lambda) \hat{\lambda}_n \right) \times \\ & \frac{1}{I^4(\lambda)} \left( I^2(\lambda) - \lambda^2 I^2(\hat{\lambda}_n) \right) \left( \lambda I(\hat{\lambda}_n) + I(\lambda) \hat{\lambda}_n \right) = 0, \end{aligned}$$

where

$$(X_n, Y_n) \xrightarrow[n \rightarrow +\infty]{a.s.} (X, Y) = (\lambda/I(\lambda), 0).$$

Now by using this almost sure convergence result and those given in (5.1) and part 1 of Theorem 3.12, one obtains the following probability equivalence

$$2n(X_n - X) \frac{2\lambda I(\lambda)(\lambda^2 - 1)}{I(\lambda)} \equiv 2n \left( \lambda I(\hat{\lambda}_n) - I(\lambda) \hat{\lambda}_n \right) \frac{2\lambda}{I(\lambda)} (1 - \lambda^2),$$

and since  $0 < \lambda < 1$ , one finally gets

$$2n \left( \lambda I(\hat{\lambda}_n) - I(\lambda) \hat{\lambda}_n \right) \equiv -2n(X_n - X) I^2(\lambda).$$

Now referring to function  $f$  defined by (3.14) in the proof of Theorem 3.12, let

$$f(x) = \lambda I(x) - I(\lambda)x, \text{ for } 0 \leq x \leq 1.$$

One has  $f(\lambda) = 0$  and  $f'(\lambda) = \lambda I'(\lambda) - I(\lambda) < 0$ .

So with the first order Taylor expansion of  $f$  about  $\lambda$  evaluated at  $\hat{\lambda}_n$ , one gets

$$\begin{aligned} 2n \left( \lambda I(\hat{\lambda}_n) - I(\lambda) \hat{\lambda}_n \right) &= 2nf(\hat{\lambda}_n) \\ &\equiv 2nf'(\lambda)(\hat{\lambda}_n - \lambda) \\ &\equiv -2n(X_n - X) I^2(\lambda), \end{aligned}$$

and then one proved that

$$2n(\hat{\lambda}_n - \lambda) \equiv \frac{2n(X_n - X) I^2(\lambda)}{I(\lambda) - \lambda I'(\lambda)},$$

and by convergence given in (5.1),

$$2n(\hat{\lambda}_n - \lambda) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma),$$

where  $\Sigma = \frac{I^4(\lambda)}{(I(\lambda) - \lambda I'(\lambda))^2} (\Sigma^{(*)}(u))_{11}$ .

To end the first part of this proof we just have to check that  $(\Sigma_{\lambda, \frac{\pi}{2}}(u))_{11} = \Sigma$ . In this aim, remember that  $\Sigma_{\lambda, \frac{\pi}{2}}(u) = C(\lambda, \frac{\pi}{2}) \times \Sigma^{(*)}(u) \times C^t(\lambda, \frac{\pi}{2})$  with

$$C(\lambda, \frac{\pi}{2}) = \frac{1}{J_F(\lambda, \frac{\pi}{2})} \begin{pmatrix} \frac{\partial F_2}{\partial \theta_o}(\lambda, \frac{\pi}{2}) & -\frac{\partial F_1}{\partial \theta_o}(\lambda, \frac{\pi}{2}) \\ - & - \end{pmatrix}.$$

Using Lemma A.1, one obtains that  $J_F(\lambda, \frac{\pi}{2}) = \frac{(\lambda^2 - 1)}{\lambda I(\lambda)} \times \left( \frac{\lambda I'(\lambda) - I(\lambda)}{I^2(\lambda)} \right)$  and

$$C(\lambda, \frac{\pi}{2}) = \frac{1}{J_F(\lambda, \frac{\pi}{2})} \begin{pmatrix} \frac{1 - \lambda^2}{\lambda I(\lambda)} & 0 \\ - & - \end{pmatrix}.$$

In this way, one proved that  $(\Sigma_{\lambda, \frac{\pi}{2}}(u))_{11} = \Sigma$ , yielding the first part of theorem. Now we consider the second part taking  $\lambda = 1$  (and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ ).

The decomposition obtained in (3.11) gives

$$\begin{aligned}
 & 2n \left( X_n - \frac{2}{\pi} \right) \left( X_n + \frac{2}{\pi} \right) \times \\
 & \quad 2n \left[ I^4(\hat{\lambda}_n) (X_n^2 + Y_n^2) + \left( \frac{2}{\pi} \right)^2 I^4(\hat{\lambda}_n) - I^2(\hat{\lambda}_n) (\hat{\lambda}_n^2 + 1) \right] \\
 & \quad + \left( \frac{2}{\pi} \right)^2 I^4(\hat{\lambda}_n) (2nY_n)^2 \\
 & = (2n)^2 \left( \frac{\pi}{2} - I(\hat{\lambda}_n) \right) \left( I(\hat{\lambda}_n) - \frac{\pi}{2} \hat{\lambda}_n \right) \left( \frac{2}{\pi} \right)^4 \left( \frac{\pi}{2} + I(\hat{\lambda}_n) \right) \left( I(\hat{\lambda}_n) + \frac{\pi}{2} \hat{\lambda}_n \right) \quad (5.5)
 \end{aligned}$$

where

$$(X_n, Y_n) \xrightarrow[n \rightarrow +\infty]{a.s.} (X, Y) = \left( \frac{2}{\pi}, 0 \right). \quad (5.6)$$

Note that we can not hold the same reasoning as in the first part of the proof. Indeed, by the first part of Theorem 3.12 and the latter almost sure convergence result, we deduce that

$$Z_n = I^4(\hat{\lambda}_n) (X_n^2 + Y_n^2) + \left( \frac{2}{\pi} \right)^2 I^4(\hat{\lambda}_n) - I^2(\hat{\lambda}_n) (\hat{\lambda}_n^2 + 1) \xrightarrow[n \rightarrow +\infty]{a.s.} 0,$$

thus we have to renormalize this expression by  $(2n)^2$ .

However, if in a first time we do not normalize this expression by  $(2n)^2$  but rather by  $(2n)$ , we get

$$\begin{aligned}
 & 2n \left( X_n - \frac{2}{\pi} \right) \left( X_n + \frac{2}{\pi} \right) Z_n + \left( \frac{2}{\pi} \right)^2 I^4(\hat{\lambda}_n) 2nY_n^2 \\
 & = 2n h(\hat{\lambda}_n) \left( \frac{2}{\pi} \right)^4 \left( \frac{\pi}{2} + I(\hat{\lambda}_n) \right) \left( I(\hat{\lambda}_n) + \frac{\pi}{2} \hat{\lambda}_n \right),
 \end{aligned}$$

remembering that the function  $h$  (see (3.12)) has been defined in the proof of Theorem 3.12 by

$$h(\lambda) = \left( \frac{\pi}{2} - I(\lambda) \right) \left( I(\lambda) - \frac{\pi}{2} \lambda \right),$$

for  $0 < \lambda \leq 1$ .

Now by using the almost sure convergence to 0 and those given in (5.1) and (5.6) and in the first part of Theorem 3.12, we get

$$2n h(\hat{\lambda}_n) \xrightarrow[n \rightarrow +\infty]{Pr} 0.$$

Since  $h(1) = h'(1) = 0$ , one obtains by using a second order Taylor expansion of  $h$  about 1 evaluated at point  $\hat{\lambda}_n$ ,

$$h(\hat{\lambda}_n) = \frac{\pi^2}{16} (\hat{\lambda}_n - 1)^2 + o((\hat{\lambda}_n - 1)^2). \quad (5.7)$$

Thus as a bonus we proved that in probability

$$2n (\hat{\lambda}_n - 1)^2 = o(1). \quad (5.8)$$

This last equality will help us to study the normalized expression  $2n Z_n$  and show that in probability

$$2n Z_n \equiv 2n \left( X_n - \frac{2}{\pi} \right) \frac{\pi^3}{4}.$$

Indeed

$$2n Z_n = I^2(\hat{\lambda}_n) \left[ 2n \left( X_n^2 - \left( \frac{2}{\pi} \right)^2 \right) I^2(\hat{\lambda}_n) + 2n I^2(\hat{\lambda}_n) Y_n^2 + 2n \left( 2 I^2(\hat{\lambda}_n) \left( \frac{2}{\pi} \right)^2 - \left( \hat{\lambda}_n^2 + 1 \right) \right) \right].$$

Using an order two Taylor expansion of the elliptic integral  $I$  (see (3.4) for definition) in  $\hat{\lambda}_n$  about  $\lambda = 1$ , one gets

$$I(\hat{\lambda}_n) = \frac{\pi}{2} + \frac{\pi}{4}(\hat{\lambda}_n - 1) + O((\hat{\lambda}_n - 1)^2),$$

in such a way that

$$2n \left( 2 I^2(\hat{\lambda}_n) \left( \frac{2}{\pi} \right)^2 - \left( \hat{\lambda}_n^2 + 1 \right) \right) = -n(\hat{\lambda}_n - 1)^2 + O(2n(\hat{\lambda}_n - 1)^2).$$

By equality (5.8) this expression converges in probability to zero.

Part 1 of Theorem 3.12, convergences in (5.1) and (5.6) give the required result

$$2n Z_n \equiv 2n \left( X_n - \frac{2}{\pi} \right) \frac{\pi^3}{4}.$$

Back to expression (5.5), using once again the convergence obtained in part 1 of Theorem 3.12 and equality (5.7), one finally proved that in probability

$$\pi^2 \left( 2n \left( X_n - \frac{2}{\pi} \right) \right)^2 + (2n Y_n)^2 \left( \frac{\pi}{2} \right)^2 \equiv (2n)^2 h(\hat{\lambda}_n) \frac{16}{\pi^2} \equiv (2n(\hat{\lambda}_n - 1))^2.$$

By using convergence given in (5.1), we get the required convergence. This ends the proof of this theorem.  $\square$

*Proof of Remark 5.11.* The density  $f_U$  of the positive random variable  $U$  can be expressed as  $f_U(t) = f_V(t\tau^2)$ , where  $\tau$  is the common eigenvalue of matrix  $A$  under the isotropic case and  $V$  is defined as  $U$ , substituting  $Q\Sigma_*(u, 1, I_2)Q^t$  to  $\Sigma^*(u, \tau, \tau, I_2)$ . The matrix  $\Sigma_*$  is computable and is defined by (5.2). So, it is enough to estimate  $\tau$  by a consistent estimator, say  $\hat{\tau}_n$ . Using convergence given in (5.3) we propose a consistent estimator of  $\lambda_1 = \tau$  in the isotropic case by taking

$$\hat{\tau}_n = \frac{J_1^{(n)}(u)}{p_{Z(0)}(u)E[\|\nabla Z(0)\|_2]} = 2 J_1^{(n)}(u) \sqrt{\frac{r_z(0)}{\mu}} \exp\left(\frac{1}{2} \frac{u^2}{r_z(0)}\right),$$

that is computable since  $\mu$  and  $r_z(0)$  are supposed to be known.  $\square$

In the next section, statistical tests are proposed for the null hypothesis “ $X$  is isotropic” against “ $X$  is affine”.

#### 5.4 Testing the isotropy

We test

$$H_0 : \lambda = 1 \quad \text{against} \quad H_1 : \lambda < 1.$$

We still obtain a way to detect the possible isotropy of the process via the following corollaries.

**Corollary 5.12.** For  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ , under the hypothesis  $H_0$ , the following convergence holds:

$$\frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)} \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{2}{\pi} v^*$$

**Corollary 5.13.** For  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ , under the hypothesis  $H_0$ , we have the following convergence:

$$T_{f^*}^{(n)}(u) = 2n \left( \frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)} - \frac{2}{\pi} v^* \right) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; \Sigma^*(u, \tau, \tau, I_2)),$$

where  $\tau$  is the common value of the eigenvalues of matrix  $A$ .

**Remark 5.14.** We can observe that the matrix  $\Sigma^*(u, \tau, \tau, I_2)$  is non-degenerate, as in Remark 5.10.

**Remark 5.15.** Under the alternative hypothesis  $H_1$ , the test statistic  $T_{f^*}^{(n)}(u)$  converges in law toward a Gaussian random variable, the mean of which will tend to infinity. Indeed by using Corollary 3.6 one can easily show that

$$\left( \frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} - \frac{2}{\pi} v^* \right) \iff (\lambda = 1).$$

*Proof of Corollaries 5.12 and 5.13.* Using the almost sure convergence (3.3), convergence in law of Proposition 5.1, and taking  $\lambda = 1$  in Corollary 3.6 one obviously gets Corollaries 5.12 and 5.13.  $\square$

*Proof of Remark 5.15.* Applying Proposition 5.1, under  $H_1$ , the test statistic  $T_{f^*}^{(n)}(u)$  converges in law to a Gaussian random variable with asymptotically mean equivalent to  $2n \left( \frac{\mathbb{E}[J_{f^*}^{(n)}(u)]}{\mathbb{E}[J_1^{(n)}(u)]} - \frac{2}{\pi} v^* \right)$ . Since  $\left( \frac{\mathbb{E}[J_{f^*}^{(n)}(u)]}{\mathbb{E}[J_1^{(n)}(u)]} - \frac{2}{\pi} v^* \right)$  does not depend on  $n$  and is equal to zero if and if  $\lambda = 1$ , this argument ends the proof of the remark.  $\square$

In the case where covariance **function**  $r_z$  **is known**, Corollary 5.13 suggests another test statistic. More precisely, let  $R$  an unitary matrix obtained by diagonalizing the computable matrix  $\Sigma_*(u, 1, I_2)$  defined in (5.2). That is  $\Sigma_*(u, 1, I_2) = R \Gamma_* R^t$ . We consider the following test statistic  $S_{f^*}^{(n)}(u)$ :

$$S_{f^*}^{(n)}(u) = \hat{\tau}_n \Gamma_*^{-\frac{1}{2}} R^t T_{f^*}^{(n)}(u),$$

where  $\hat{\tau}_n$  is given by (5.4).

We can state the following theorem.

**Theorem 5.16.** For  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ , under the hypothesis  $H_0$ , we have the following convergence:

$$\Xi_{f^*}^{(n)}(u) = (S_{f^*}^{(n)}(u))^t S_{f^*}^{(n)}(u) \xrightarrow[n \rightarrow +\infty]{Law} \chi_2^2.$$

**Remark 5.17.** The rejection region is then  $\Xi_{f^*}^{(n)}(u) > \gamma$ . This critical region provides a consistent test for any positive constant  $\gamma$ , because  $\Xi_{f^*}^{(n)}(u)$  is stochastically unbounded, for  $n \rightarrow +\infty$ , except under the null hypothesis. In fact when  $\lambda < 1$ ,  $\frac{1}{(2n)^2} \Xi_{f^*}^{(n)}(u)$  converges in probability to  $b > 0$ , and this implies that  $\Xi_{f^*}^{(n)}(u)$  converges in probability to  $+\infty$ .

*Proof of Theorem 5.16 and Remark 5.17.* As we have already pointed out in Remark 5.14, the matrix  $\Sigma^*(u, \tau, \tau, I_2)$  is invertible, so that we can factorize  $\Sigma_*(u, I_2, 1)$  into  $\Sigma_*(u, 1, I_2) = R \Gamma_* R^t$ , where  $R$  is an unitary matrix. Let

$$S_{f^*}^{(n)}(u) = \hat{\tau}_n \Gamma_*^{-\frac{1}{2}} R^t T_{f^*}^{(n)}(u).$$

Since  $\hat{\tau}_n$  is a consistent estimator of  $\tau$  under hypothesis  $H_0$ , by Corollary 5.13, asymptotically this random vector is a standard Gaussian one and Theorem 5.16 ensues.

To achieve the proof of Remark 5.17, let us check that  $\frac{1}{(2n)^2} \Xi_{f^*}^{(n)}(u)$  converges in probability to  $b > 0$  under the alternative hypothesis  $H_1$  that is when  $\lambda < 1$ .

In this aim, let us note

$$\tilde{S}_{f^*}^{(n)}(u) = \hat{\tau}_n \Gamma_*^{-\frac{1}{2}} R^t \tilde{T}_{f^*}^{(n)}(u),$$

where

$$\tilde{T}_{f^*}^{(n)}(u) = 2n \left( \frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)} - \frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} \right).$$

We have the following decomposition

$$\begin{aligned} \frac{1}{(2n)^2} \Xi_{f^*}^{(n)}(u) &= \frac{1}{(2n)^2} (S_{f^*}^{(n)}(u))^t S_{f^*}^{(n)}(u) \\ &= \frac{1}{(2n)^2} \left( S_{f^*}^{(n)}(u) - \tilde{S}_{f^*}^{(n)}(u) \right)^t \left( S_{f^*}^{(n)}(u) - \tilde{S}_{f^*}^{(n)}(u) \right) \\ &\quad + \frac{2}{(2n)^2} (\tilde{S}_{f^*}^{(n)}(u))^t \left( S_{f^*}^{(n)}(u) - \tilde{S}_{f^*}^{(n)}(u) \right) \\ &\quad + \frac{1}{(2n)^2} (\tilde{S}_{f^*}^{(n)}(u))^t (\tilde{S}_{f^*}^{(n)}(u)) \\ &= (1) + (2) + (3). \end{aligned}$$

Let us look at the third term (3).

$$(3) = \left( \frac{1}{2n} (\tilde{T}_{f^*}^{(n)}(u))^t \right) (\hat{\tau}_n^2 R \Gamma_*^{-1} R^t) \left( \frac{1}{2n} \tilde{T}_{f^*}^{(n)}(u) \right) \xrightarrow[n \rightarrow +\infty]{Pr} 0,$$

by using Proposition 5.1 and the fact that  $\hat{\tau}_n$  almost surely converges to  $\sqrt{\frac{2}{\pi\mu}} \mathbb{E}[\|\nabla X(0)\|_2]$  (see Theorem 3.4 and Proposition 3.1).

For the second term (2).

$$(2) = \left( \frac{1}{n} (\tilde{T}_{f^*}^{(n)}(u))^t \right) (\hat{\tau}_n^2 R \Gamma_*^{-1} R^t) \left( \frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} - \frac{2}{\pi} v^* \right),$$

and this term tends in probability toward zero as for the third term (3).

Finally, the first term (1) gives

$$\begin{aligned} (1) &= \left( \hat{\tau}_n \Gamma_*^{-\frac{1}{2}} R^t \left( \frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} - \frac{2}{\pi} v^* \right) \right)^t \times \\ &\quad \left( \hat{\tau}_n \Gamma_*^{-\frac{1}{2}} R^t \left( \frac{\mathbb{E}[J_{f^*}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} - \frac{2}{\pi} v^* \right) \right) = b_n, \end{aligned}$$

and

$$\lim_{n \rightarrow +\infty} b_n = \frac{2}{\pi\mu} (\mathbb{E}[\|\nabla X(0)\|_2])^2 \left( \Gamma_\star^{-\frac{1}{2}} R^t \left( \frac{\mathbb{E}[J_{f^\star}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} - \frac{2}{\pi} v^\star \right) \right)^t \times \\ \left( \Gamma_\star^{-\frac{1}{2}} R^t \left( \frac{\mathbb{E}[J_{f^\star}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} - \frac{2}{\pi} v^\star \right) \right) = b \geq 0.$$

Since  $\Gamma_\star^{-\frac{1}{2}} R$  is invertible, one has

$$(b > 0) \iff \left( \left( \frac{\mathbb{E}[J_{f^\star}^{(1)}(u)]}{\mathbb{E}[J_1^{(1)}(u)]} - \frac{2}{\pi} v^\star \right) \neq 0 \right) \iff (\lambda < 1).$$

That ends the proof of remark.  $\square$

## A Appendix

*Proof of Lemma 3.5.* We have

$$(1) = \int_0^{n-1} \int_0^{n-1} \int_{C_{\lfloor t, t+1[\times]s, s+1[, X(u)} f(\nu_X(x)) d\sigma_1(x) dt ds \\ - \int_0^1 \int_0^1 \int_{C_{\lfloor 0, t[\times]0, s[, X(u)} f(\nu_X(x)) d\sigma_1(x) dt ds \\ = \int_0^{n-1} \int_0^{n-1} (H(t+1, s+1) - H(t, s+1) - H(t+1, s) + H(t, s)) dt ds \\ - \int_0^1 \int_0^1 H(t, s) dt ds.$$

Thus by making change of variable in the first integral, one obtains

$$(1) = \int_{n-1}^n \int_{n-1}^n H(t, s) dt ds - \int_0^1 \int_{n-1}^n H(t, s) dt ds - \int_{n-1}^n \int_0^1 H(t, s) dt ds.$$

Since  $f$  is a positive function then  $H(t, s) \geq 0$  and we get the following upper bound

$$(1) \leq \int_{n-1}^n \int_{n-1}^n H(t, s) dt ds \\ \leq \int_{n-1}^n \int_{n-1}^n \int_{C_{\lfloor 0, n[\times]0, n[, X(u)} f(\nu_X(x)) d\sigma_1(x) dt ds = \int_{C_{T, X(u)}} f(\nu_X(x)) dx.$$

The first inequality of lemma is then achieved. To obtain the second inequality, arguing as previously we obtain the following lower bound

$$(2) = \int_0^{n+1} \int_0^{n+1} \int_{C_{\lfloor t-1, t[\times]s-1, s[, X(u)} f(\nu_X(x)) d\sigma_1(x) dt ds \\ = \int_n^{n+1} \int_n^{n+1} H(t, s) dt ds - \int_{-1}^0 \int_n^{n+1} H(t, s) dt ds - \int_n^{n+1} \int_{-1}^0 H(t, s) dt ds \\ \geq \int_n^{n+1} \int_n^{n+1} \int_{C_{\lfloor 0, t[\times]0, s[, X(u)} f(\nu_X(x)) d\sigma_1(x) dt ds \geq \int_{C_{T, X(u)}} f(\nu_X(x)) dx,$$

this yields Lemma 3.5.  $\square$

*Proof of Lemma 3.8.* First let us see that for  $n \in \mathbb{N}^*$ , a.s.  $X_n^2 + Y_n^2 < 1$ .

$$\begin{aligned} X_n^2 + Y_n^2 &= \frac{\left\| \int_{C_{T,X}(u)} f^*(\nu_X(t)) d\sigma_1(t) \right\|_2^2}{\sigma_1^2(C_{T,X}(u))} \\ &\leq \frac{\int_{C_{T,X}(u)} \|f^*(\nu_X(t))\|_2^2 d\sigma_1(t)}{\sigma_1(C_{T,X}(u))} = 1, \end{aligned}$$

last equality coming from the fact that  $f^*$  takes its values into  $S^1$ . But the strict inequality is true otherwise, we would have equality in Hölder inequality that remains impossible.

Now let us see that, a.s.  $X_n > 0$ .

$$X_n = \left\langle \frac{J_{f^*}^{(n)}(u)}{J_1^{(n)}(u)}, v^* \right\rangle = \frac{1}{J_1^{(n)}(u)} \int_{C_{T,X}(u)} |\langle \nu_X(t), v^* \rangle| d\sigma_1(t) \geq 0,$$

and in a manner similar to that previously used it is proved that, a.s.  $X_n > 0$ . This yields lemma.  $\square$

The Jacobian  $J_F$  of transformation  $F$  (see (3.5)) is given by Lemma A.1.

**Lemma A.1.**

$$\begin{aligned} J_F(\lambda, \theta_o) &= \frac{\lambda(1 - \lambda^2)}{I^3(\lambda)(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))} \times \\ &\quad \left[ \sin^2(\theta_o) \int_0^1 \frac{\sqrt{1 - u^2}}{\sqrt{1 - (1 - \lambda^2)u^2}} du \right. \\ &\quad \left. + \cos^2(\theta_o) \int_0^1 \frac{u^2}{\sqrt{1 - (1 - \lambda^2)u^2}} \frac{du}{\sqrt{1 - u^2}} \right] \neq 0, \quad (\text{A.1}) \end{aligned}$$

for  $0 < \lambda < 1$ .

*Proof of Lemma A.1.* For  $0 < \lambda \leq 1$  and  $-\frac{\pi}{2} < \theta_o \leq \frac{\pi}{2}$ , one has

$$\begin{cases} \frac{\partial F_1}{\partial \lambda}(\lambda, \theta_o) = \frac{1}{I^2(\lambda)} \frac{[\lambda \sin^2(\theta_o) I(\lambda) - (\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o)) I'(\lambda)]}{(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{1}{2}}} \\ \frac{\partial F_1}{\partial \theta_o}(\lambda, \theta_o) = \frac{1}{I(\lambda)} \frac{(\lambda^2 - 1) \sin(\theta_o) \cos(\theta_o)}{(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{1}{2}}}, \end{cases}$$

and

$$\begin{cases} \frac{\partial F_2}{\partial \lambda}(\lambda, \theta_o) = \frac{\sin(\theta_o) \cos(\theta_o)}{I^2(\lambda)} \frac{\left( \begin{aligned} &(\lambda I(\lambda) - (\lambda^2 - 1) I'(\lambda)) \\ &\times (\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o)) \\ &+ \lambda I(\lambda) \end{aligned} \right)}{(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{3}{2}}} \\ \frac{\partial F_2}{\partial \theta_o}(\lambda, \theta_o) = \frac{(\lambda^2 - 1)}{I(\lambda)} \frac{[\cos^4(\theta_o) - \lambda^2 \sin^4(\theta_o)]}{(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))^{\frac{3}{2}}}. \end{cases}$$

From straightforward calculations we deduce that the Jacobian  $J_F$  of the transformation  $F$  can be written as

$$J_F(\lambda, \theta_o) = \frac{(1 - \lambda^2)}{I^3(\lambda)} \frac{[\lambda \sin^2(\theta_o) I(\lambda) + (\cos^2(\theta_o) - \lambda^2 \sin^2(\theta_o)) I'(\lambda)]}{(\cos^2(\theta_o) + \lambda^2 \sin^2(\theta_o))}$$

Now  $I(\lambda) = \int_0^{\frac{\pi}{2}} (\cos^2(\theta) + \lambda^2 \sin^2(\theta))^{\frac{1}{2}} d\theta$ , and the change of variable:

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \sqrt{1 - u^2}}{u} \text{ gives } I(\lambda) = \int_0^1 \frac{(1 - (1 - \lambda^2)u^2)^{\frac{1}{2}}}{(1 - u^2)^{\frac{1}{2}}} du, \text{ and } I'(\lambda) = \lambda \int_0^1 \frac{u^2}{(1 - (1 - \lambda^2)u^2)^{\frac{1}{2}}} \frac{du}{\sqrt{1 - u^2}}.$$

This yields

$$\begin{aligned} \lambda \sin^2(\theta_o) I(\lambda) + (\cos^2(\theta_o) - \lambda^2 \sin^2(\theta_o)) I'(\lambda) = \\ \lambda \left[ \sin^2(\theta_o) \int_0^1 \frac{\sqrt{1 - u^2}}{\sqrt{1 - (1 - \lambda^2)u^2}} du \right. \\ \left. + \cos^2(\theta_o) \int_0^1 \frac{u^2}{\sqrt{1 - (1 - \lambda^2)u^2}} \frac{du}{\sqrt{1 - u^2}} \right], \end{aligned}$$

and lemma ensues from the fact that  $(J_F(\lambda, \theta_o) = 0) \iff (\lambda = 1)$ .  $\square$

*Proof of Lemma 4.5.* Note that since the process  $Z$  admits a spectral density  $f_z$  so does the process  $X$ . Its spectral density is denoted by  $f_x$ .

To prove Lemma 4.5 we need to prove another lemma.

**Lemma A.2.**  $\exists C > 0, \exists B > 0, \inf_{v \in S^1} \int_{\|t\|_2 < B} \langle t, v \rangle^2 f_x(t) dt \geq C$ .

*Proof of Lemma A.2.* Let us prove it by contradiction supposing that for all  $B > 0$ ,  $\inf_{v \in S^1} \int_{\|t\|_2 < B} \langle t, v \rangle^2 f_x(t) dt = 0$ . Thus, for all  $n \in \mathbb{N}^*$ , there exists  $v_n \in S^1$  such that  $\int_{\|t\|_2 < n} \langle t, v_n \rangle^2 f_x(t) dt \leq \frac{1}{n}$ .

Since  $S^1$  is a compact set, there exists  $(v_{n_k})_{k \in \mathbb{N}^*}$  a subsequence of  $(v_n)_{n \in \mathbb{N}^*}$  such that  $\lim_{k \rightarrow +\infty} v_{n_k} = v_o$ , with  $v_o \in S^1$ .

Now the hypothesis  $\int_{\mathbb{R}^2} f_z(t) \|t\|_2^2 dt < +\infty$  implies that  $\int_{\mathbb{R}^2} f_x(t) \|t\|_2^2 dt < +\infty$  and the Lebesgue convergence theorem gives that  $\int_{\mathbb{R}^2} \langle t, v_o \rangle^2 f_x(t) dt = 0$ , this leads to a contradiction since the fact that  $f_z$  is a continuous function and  $f_z(0) > 0$  implies that  $f_x$  is also a continuous function and that  $f_x(0) > 0$ . Lemma A.2 is proved.  $\square$

Back to the proof of Lemma 4.5, by using Lemma A.2, let us choose  $B > 0$  and  $C > 0$  such that

$$\inf_{v \in S^1} \int_{\|t\|_2 < B} \langle t, v \rangle^2 f_x(t) dt \geq C, \quad (\text{A.2})$$

and let  $\tau \in \overline{T} - \overline{T}$ , such that  $\|\tau\|_2 \leq \frac{1}{B}$ . Now by using the following inequality,  $\exists D > 0$ , such that if  $|y| \leq 1$ , then  $1 - \cos(y) \geq Dy^2$  and (A.2), we get the serie of inequalities

$$\begin{aligned} r_x(0) - r_x(\tau) &= \int_{\mathbb{R}^2} (1 - \cos\langle t, \tau \rangle) f_x(t) dt \geq \int_{\|t\|_2 < B} (1 - \cos\langle t, \tau \rangle) f_x(t) dt \\ &\geq D \int_{\|t\|_2 < B} \langle t, \tau \rangle^2 f_x(t) dt \geq DC \|\tau\|_2^2. \end{aligned}$$

To end the proof of this lemma, just notice that the covariance  $r_x$  is continuous in zero.  $\square$

*Proof of Lemma 4.11.* Let  $X = (X_i)_{i=1,2,3}$  and  $Y = (Y_j)_{j=1,2,3}$  be two centered standard Gaussian vectors in  $\mathbb{R}^3$  such that for  $1 \leq i, j \leq 3$ ,  $E[X_i Y_j] = \rho_{ij}$ .

Let  $\mathbf{k} = (k_1, k_2, k_3)$  and  $\mathbf{m} = (m_1, m_2, m_3)$  be two vectors of  $\mathbb{N}^3$ . We want to give an explicit formula for  $E[\tilde{H}_{\mathbf{k}}(X) \tilde{H}_{\mathbf{m}}(Y)]$ .

For ease of notations let us set  $Y_j = X_{j+1}$  for  $j = 1, 2, 3$ .

As in ([6], p. 269) a straightforward calculation on Gaussian characteristic functions gives

$$\begin{aligned} \mathbb{E} \left[ \prod_{i=1}^6 \exp(t_i X_i - \frac{1}{2} t_i^2) \right] &= \exp \left( \sum_{i=1}^3 \sum_{j=1}^3 \rho_{ij} t_i t_{j+3} \right) \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \left( \sum_{i=1}^3 \sum_{j=1}^3 \rho_{ij} t_i t_{j+3} \right)^r = \sum_{r=0}^{\infty} \sum_{\sum_{i,j} d_{ij}=r} \prod_{i,j=1}^3 \frac{1}{d_{ij}!} (\rho_{ij} t_i t_{j+3})^{d_{ij}} \\ &= \sum_{r=0}^{\infty} \sum_{\sum_{i,j} d_{ij}=r} \left( \prod_{i,j=1}^3 \frac{\rho_{ij}^{d_{ij}}}{d_{ij}!} \right) \left( \prod_{i=1}^3 t_i^{\sum_{j=1}^3 d_{ij}} \right) \left( \prod_{j=1}^3 t_{j+3}^{\sum_{i=1}^3 d_{ij}} \right) \quad (\text{A.3}) \end{aligned}$$

Now, by definition

$$\exp(tx - \frac{1}{2} t^2) = \sum_{q=0}^{\infty} t^q \frac{H_q(x)}{q!},$$

thus

$$\begin{aligned} \mathbb{E} \left[ \prod_{i=1}^6 \exp(t_i X_i - \frac{1}{2} t_i^2) \right] \\ = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \frac{t_1^{k_1}}{k_1!} \frac{t_2^{k_2}}{k_2!} \frac{t_3^{k_3}}{k_3!} \frac{t_4^{m_1}}{m_1!} \frac{t_5^{m_2}}{m_2!} \frac{t_6^{m_3}}{m_3!} \mathbb{E} [\tilde{H}_{\mathbf{k}}(X) \tilde{H}_{\mathbf{m}}(Y)]. \end{aligned}$$

Identifying this last equality with (A.3), it follows that  $\mathbb{E} [\tilde{H}_{\mathbf{k}}(X) \tilde{H}_{\mathbf{m}}(Y)] = 0$  if  $|\mathbf{k}| \neq |\mathbf{m}|$ . In the case where  $|\mathbf{k}| = |\mathbf{m}|$ , one gets

$$\mathbb{E} [\tilde{H}_{\mathbf{k}}(X) \tilde{H}_{\mathbf{m}}(Y)] = \sum_{\substack{d_{ij} \geq 0 \\ \sum_j d_{ij} = k_i \\ \sum_i d_{ij} = m_j}} \mathbf{k}! \mathbf{m}! \prod_{1 \leq i, j \leq 3} \frac{\rho_{ij}^{d_{ij}}}{d_{ij}!},$$

yielding the lemma.  $\square$

Recall that for  $f : S^1 \rightarrow \mathbb{R}$  a continuous and bounded functions and for all  $u$  fixed level in  $\mathbb{R}$  the coefficients  $a_f(\cdot, u)$  are defined by  $a_f(\mathbf{k}, u) = a_f(k_1, k_2) a(k_3, u)$ , for  $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{N}^3$ , with

$$a(k_3, u) = \frac{1}{k_3!} H_{k_3} \left( \frac{u}{\sqrt{r_z(0)}} \right) \phi \left( \frac{u}{\sqrt{r_z(0)}} \right) \frac{1}{\sqrt{r_z(0)}}.$$

So for giving an expression of the coefficients  $a_{f_i^*}(\cdot, u)$  for  $i = 1, 2$  and of  $a_1(\cdot, u)$  just give one of  $a_{f_i^*}(k_1, k_2)$  and of that of  $a_1(k_1, k_2)$ .

**Lemma A.3.** *The coefficients  $a_{f_i^*}$  are given by: for  $m, \ell \in \mathbb{N}$*

$$\begin{aligned} &\begin{pmatrix} a_{f_1^*}(2m, 2\ell) & a_{f_1^*}(2m+1, 2\ell+1) & a_{f_1^*}(2m, 2\ell+1) & a_{f_1^*}(2m+1, 2\ell) \\ a_{f_2^*}(2m, 2\ell) & a_{f_2^*}(2m+1, 2\ell+1) & a_{f_2^*}(2m, 2\ell+1) & a_{f_2^*}(2m+1, 2\ell) \end{pmatrix} = \\ &P \times \begin{pmatrix} A(\lambda_1, \lambda_2, \omega_1^*, \omega_2^*, m, \ell, \mu) & B(\lambda_1, \lambda_2, \omega_1^*, \omega_2^*, m, \ell, \mu) & 0 & 0 \\ A(\lambda_2, \lambda_1, \omega_2^*, \omega_1^*, \ell, m, \mu) & B(\lambda_2, \lambda_1, \omega_2^*, \omega_1^*, \ell, m, \mu) & 0 & 0 \end{pmatrix}, \end{aligned}$$

where  $\omega^* = (\omega_1^*, \omega_2^*) = v^* \times P$  and

$$\begin{aligned} A(\lambda_1, \lambda_2, \omega_1^*, \omega_2^*, m, \ell, \mu) &= \sqrt{\frac{2\mu}{\pi}} \lambda_1 \sum_{p=0}^m \frac{(-2)^{p-m}}{(2p)!(m-p)!} \sum_{k=0}^p \frac{2^k p!}{(p-k)!} \left( \frac{\lambda_2 \omega_2^*}{\lambda_1 \omega_1^*} \right)^{2(p-k)} \times \\ &\quad \sum_{n=0}^{\ell} \frac{(-2)^{n-\ell}}{(2n)!(\ell-n)!} \left( \frac{\lambda_1 \omega_1^*}{[(\lambda_1 \omega_1^*)^2 + (\lambda_2 \omega_2^*)^2]^{\frac{1}{2}}} \right)^{2(n+p-k)+1} \times \\ &\quad \frac{(2(n+p-k))!}{2^{n+p-k}(n+p-k)!}, \end{aligned}$$

and

$$\begin{aligned} B(\lambda_1, \lambda_2, \omega_1^*, \omega_2^*, m, \ell, \mu) &= \sqrt{\frac{2\mu}{\pi}} \lambda_1 \sum_{p=0}^{\ell} \frac{(-2)^{p-\ell}}{(2p+1)!(\ell-p)!} \sum_{k=0}^p \frac{2^k p!}{(p-k)!} \left( \frac{\lambda_1 \omega_1^*}{\lambda_2 \omega_2^*} \right)^{2(p-k)} \times \\ &\quad \sum_{n=0}^m \frac{(-2)^{n-m}}{(2n+1)!(m-n)!} \left( \frac{\lambda_2 \omega_2^*}{[(\lambda_1 \omega_1^*)^2 + (\lambda_2 \omega_2^*)^2]^{\frac{1}{2}}} \right)^{2(p-k+n+1)+1} \times \\ &\quad \frac{(2(p-k+n+1))!}{2^{p-k+n+1}(p-k+n+1)!}. \end{aligned}$$

*Proof of Lemma A.3.* First, let us compute  $a_{f_1^*}(k_1, k_2)$ , for  $k_1, k_2 \in \mathbb{N}$ . Let  $P = (a_{i,j})_{1 \leq i,j \leq 2}$ . By definition of coefficient  $a_{f_1^*}(k_1, k_2)$ , we have

$$\begin{aligned} a_{f_1^*}(k_1, k_2) &= \frac{\sqrt{\mu}}{k_1! k_2!} \int_{\mathbb{R}^2} (a_{11} \lambda_1 y_1 + a_{12} \lambda_2 y_2) \times \\ &\quad \left( \mathbb{1}_{\left\{ \left\langle \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}, (\omega^*)^t \right\rangle \geq 0 \right\}} - \mathbb{1}_{\left\{ \left\langle \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}, (\omega^*)^t \right\rangle < 0 \right\}} \right) \times \\ &\quad H_{k_1}(y_1) \phi(y_1) H_{k_2}(y_2) \phi(y_2) dy_1 dy_2. \end{aligned}$$

By using the fact that  $H_n$  is even when  $n$  is even and odd if not (see (A.4) and (A.5)), the coefficients  $k_1$  and  $k_2$  ought to be of the same parity, otherwise the coefficients  $a_{f_1^*}(k_1, k_2)$  would be null. Therefore let us suppose first that  $k_1 = 2m$  and  $k_2 = 2\ell$ ,  $m, \ell \in \mathbb{N}$ . In that way and using that polynomial  $H_{2n}$  is even, one has

$$\begin{aligned} a_{f_1^*}(2m, 2\ell) &= \frac{2\sqrt{\mu}}{(2m)!(2\ell)!} \int_{\mathbb{R}^2} (a_{11} \lambda_1 y_1 + a_{12} \lambda_2 y_2) \times \\ &\quad \mathbb{1}_{\left\{ \left\langle \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}, (\omega^*)^t \right\rangle \geq 0 \right\}} \times H_{2m}(y_1) \phi(y_1) H_{2\ell}(y_2) \phi(y_2) dy_1 dy_2. \end{aligned}$$

Let us compute

$$A = \int_{\mathbb{R}^2} y_1 \times \mathbb{1}_{\left\{ \left\langle \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}, (\omega^*)^t \right\rangle \geq 0 \right\}} \times H_{2m}(y_1) \phi(y_1) H_{2\ell}(y_2) \phi(y_2) dy_1 dy_2.$$

(similar computations would be done for the second integral on  $y_2$ , the arguments  $\omega_2^*, \omega_1^*, \lambda_2, \lambda_1, \ell, m$ , in this order, playing the role of  $\omega_1^*, \omega_2^*, \lambda_1, \lambda_2, m, \ell$  in last integral). At this step of the proof Lemma A.4 is required.

**Lemma A.4.** Let  $p \in \mathbb{N}$ ,  $a \in \mathbb{R}$  and  $F_p(a) = \int_a^{+\infty} z^{2p+1} \phi(z) dz$ . Then

$$F_p(a) = \phi(a) \sum_{k=0}^p \frac{2^k p!}{(p-k)!} a^{2(p-k)}.$$

*Proof of Lemma A.4.* Integration by parts and straightforward calculations give the lemma.  $\square$

Three cases occur:  $\omega_1^* > 0$ ,  $\omega_1^* < 0$  and  $\omega_1^* = 0$ .

Let us consider the first one. If  $\omega_1^* > 0$ ,  $A$  can be written as

$$A = \int_{\mathbb{R}} H_{2\ell}(y_2) \phi(y_2) \left[ \int_{-\frac{\lambda_2 \omega_2^*}{\lambda_1 \omega_1^*} y_2}^{+\infty} y_1 H_{2m}(y_1) \phi(y_1) dy_1 \right] dy_2.$$

For real  $x$  and  $m \in \mathbb{N}$ , the polynomial form of  $H_{2m}(x)$  is

$$H_{2m}(x) = (2m)! \sum_{p=0}^m \frac{(-2)^{p-m}}{(2p)!(m-p)!} x^{2p}. \quad (\text{A.4})$$

Using Lemma A.4, one gets

$$A = (2m)! \sum_{p=0}^m \frac{(-2)^{p-m}}{(2p)!(m-p)!} \sum_{k=0}^p \frac{2^k p!}{(p-k)!} \left( \frac{\lambda_2 \omega_2^*}{\lambda_1 \omega_1^*} \right)^{2(p-k)} \times \\ \sum_{n=0}^{\ell} (2\ell)! \frac{(-2)^{n-\ell}}{(2n)!(\ell-n)!} G_{n+p-k} \left( \frac{\lambda_2 \omega_2^*}{\lambda_1 \omega_1^*} \right),$$

where for  $q \in \mathbb{N}$  and  $x \in \mathbb{R}$ , we defined

$$G_q(x) = \int_{-\infty}^{+\infty} y^{2q} \phi(y) \phi(xy) dy = \frac{1}{\sqrt{2\pi}} \frac{1}{(1+x^2)^{q+\frac{1}{2}}} \frac{(2q)!}{2^q q!}.$$

If  $\omega_1^* < 0$ , using that for  $m \in \mathbb{N}$ , polynomial  $H_{2m}$  is even, one obtains that

$$A = \int_{\mathbb{R}} H_{2\ell}(y_2) \phi(y_2) \left[ \int_{-\infty}^{-\frac{\lambda_2 \omega_2^*}{\lambda_1 \omega_1^*} y_2} y_1 H_{2m}(y_1) \phi(y_1) dy_1 \right] dy_2 \\ = - \int_{\mathbb{R}} H_{2\ell}(y_2) \phi(y_2) \left[ \int_{-\frac{\lambda_2 \omega_2^*}{\lambda_1 \omega_1^*} y_2}^{+\infty} y_1 H_{2m}(y_1) \phi(y_1) dy_1 \right] dy_2,$$

and in a same way, when  $\omega_1^* = 0$ ,

$$A = \left( \int_{\mathbb{R}} H_{2\ell}(y_2) \phi(y_2) \mathbf{1}_{\{y_2 \omega_2^* \geq 0\}} dy_2 \right) \left( \int_{\mathbb{R}} y_1 H_{2m}(y_1) \phi(y_1) dy_1 \right) = 0.$$

Finally, knowing that  $|\omega_1^*| \times \text{sign}(\omega_1^*) = \omega_1^*$ , one gets the expression of coefficient  $a_{f_1^*}(2m, 2\ell)$ .

For coefficient  $a_{f_1^*}(2m+1, 2\ell+1)$ , similar arguments would be developed using the previous way and the polynomial form of  $H_{2\ell+1}(x)$ , that is

$$H_{2\ell+1}(x) = (2\ell+1)! \sum_{p=0}^{\ell} \frac{(-2)^{p-\ell}}{(2p+1)!(\ell-p)!} x^{2p+1}. \quad (\text{A.5})$$

To conclude the proof of Lemma A.3, just remark that  $a_{f_2^*}(2m+1, 2\ell) = a_{f_2^*}(2m, 2\ell+1) = 0$  and that  $a_{f_2^*}(2m, 2\ell)$  (resp.  $a_{f_2^*}(2m+1, 2\ell+1)$ ) would be computed replacing  $a_{11}$  by  $a_{21}$  and  $a_{12}$  by  $a_{22}$  in the expression of  $a_{f_1^*}(2m, 2\ell)$  (resp.  $a_{f_1^*}(2m+1, 2\ell+1)$ ).  $\square$

In order to give the coefficients  $a_1$  we introduce the following functions. The  $\beta$  function is defined by:

$$\beta(x; y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0,$$

while the  $\Gamma$  function is defined by:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

**Lemma A.5.** The coefficients  $a_1$  are given by: for  $m, \ell \in \mathbb{N}$

$$\begin{pmatrix} a_1(2m, 2\ell) & a_1(2m+1, 2\ell+1) & a_1(2m, 2\ell+1) & a_1(2m+1, 2\ell) \end{pmatrix} = \begin{pmatrix} c(\lambda_1, \lambda_2, m, \ell, \mu) & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} c(\lambda_1, \lambda_2, m, \ell, \mu) &= \sqrt{2\pi\mu} \frac{(-2)^{-(m+\ell)}}{m!\ell!} \sum_{p=0}^{\ell} \sum_{q=0}^m \binom{\ell}{p} \binom{m}{q} (-1)^{p+q} \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{2q+1} \times \\ &\quad \sum_{n=0}^{\infty} \frac{\binom{q+n}{q} \binom{2q+2n}{q+n}}{\binom{2q}{q}} \left(\frac{1}{2}\right)^{2n} \frac{1}{\beta(p+q+n+1; \frac{1}{2})} \left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^2\right)^n. \end{aligned}$$

**Remark A.6.** Note that in the case where  $\lambda = 1$ , that is when the process is isotropic our result contains that of the authors expressed in Theorem 2 of [15].

*Proof of Lemma A.5.* Let us compute  $a_1(k_1, k_2)$ , for  $k_1, k_2 \in \mathbb{N}$ . We have

$$a_1(k_1, k_2) = \frac{\sqrt{\mu}}{k_1!k_2!} \int_{\mathbb{R}^2} \sqrt{\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2} H_{k_1}(y_1) H_{k_2}(y_2) \phi(y_1) \phi(y_2) dy_1 dy_2.$$

Similar arguments as those given in Lemma A.3 show that  $a_1(k_1, k_2) = 0$  except when  $k_1$  and  $k_2$  are even.

So let us compute  $a_1(2m, 2\ell)$  for  $m, \ell \in \mathbb{N}$ .

The change of variable  $y_1 = \lambda r \sin(\theta)$  and  $y_2 = r \cos(\theta)$  in last integral and expression of  $H_{2\ell}$  given in (A.4) yield

$$\begin{aligned} a_1(2m, 2\ell) &= \frac{2\sqrt{\mu}}{\pi} \frac{\lambda_2^2}{\lambda_1} \sum_{p=0}^m \sum_{q=0}^{\ell} \frac{(-2)^{p-m} (-2)^{q-\ell}}{(2p)!(2q)!(m-p)!(\ell-q)!} \lambda^{2p} \times \\ &\quad \int_0^{\frac{\pi}{2}} \int_0^{+\infty} r^{2p+2q+2} \cos^{2q}(\theta) \sin^{2p}(\theta) e^{-\frac{1}{2}r^2[\cos^2(\theta)+\lambda^2\sin^2(\theta)]} dr d\theta. \end{aligned}$$

Now making the change of variable  $r^2 = \frac{2v}{\cos^2(\theta)+\lambda^2\sin^2(\theta)}$  and  $\tan(\frac{\theta}{2}) = \frac{1-\sqrt{1-w}}{\sqrt{w}}$ , one gets

$$\begin{aligned} a_1(2m, 2\ell) &= \frac{\sqrt{\mu}}{\pi} \frac{\lambda_2^2}{\lambda_1} \sum_{p=0}^m \sum_{q=0}^{\ell} \frac{(-2)^{p-m} (-2)^{q-\ell}}{(2p)!(2q)!(m-p)!(\ell-q)!} \lambda^{2p} 2^{p+q+\frac{1}{2}} \times \\ &\quad \Gamma(p+q+\frac{3}{2}) F(p+q+\frac{3}{2}; p+\frac{1}{2}; p+q+1; 1-\lambda^2), \end{aligned}$$

where the hypergeometric function  $F$  is defined by

$$F(a; b; c; z) = \int_0^1 u^{b-1} (1-u)^{c-b-1} (1-uz)^{-a} du.$$

for  $|z| < 1$ ,  $0 < b < c$  and  $a > 0$ .

The proof of lemma ensues from the following one.

**Lemma A.7.** For  $|z| < 1$ ,  $0 < b < c$  and  $a > 0$ , one has

$$F(a; b; c; z) = \frac{\Gamma(c-b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)} z^n.$$

*Proof of Lemma A.7.* The proof consists in a serial development of function  $f(z) = (1 - uz)^{-a}$ . □

□

*Proof of Lemma 5.4.* We have the following decomposition of  $\det(\Sigma_{(f_1, f_2)}(u))$ .

$$\begin{aligned} & \det(\Sigma_{(f_1, f_2)}(u)) \\ &= \Sigma_{f_1, f_1}(u) \Sigma_{f_2, f_2}(u) - (\Sigma_{f_1, f_2}(u))^2 \\ &= \sum_{q=1}^{\infty} \sum_{q'=1}^{\infty} (\Sigma_{f_1, f_1}(u))_q (\Sigma_{f_2, f_2}(u))_{q'} - \left( \sum_{q=1}^{\infty} (\Sigma_{f_1, f_2}(u))_q \right)^2 \\ &= \sum_{q=1}^{\infty} \det(\Sigma_{(f_1, f_2)}(u))_q \\ &+ \sum_{q < q'} \left( \sqrt{(\Sigma_{f_1, f_1}(u))_q} \sqrt{(\Sigma_{f_2, f_2}(u))_{q'}} - \sqrt{(\Sigma_{f_1, f_2}(u))_{q'}} \sqrt{(\Sigma_{f_2, f_2}(u))_q} \right)^2 \\ &+ 2 \sum_{q < q'} \left[ \sqrt{(\Sigma_{f_1, f_1}(u))_q} \sqrt{(\Sigma_{f_2, f_2}(u))_{q'}} \sqrt{(\Sigma_{f_1, f_2}(u))_{q'}} \sqrt{(\Sigma_{f_2, f_2}(u))_q} \right. \\ &\quad \left. - (\Sigma_{f_1, f_2}(u))_q (\Sigma_{f_1, f_2}(u))_{q'} \right]. \end{aligned}$$

To conclude the proof of lemma, we just have to verify that for all  $q \in \mathbb{N}^*$ ,

$$\left| (\Sigma_{f_1, f_2}(u))_q \right| \leq \sqrt{(\Sigma_{f_1, f_1}(u))_q} \sqrt{(\Sigma_{f_2, f_2}(u))_q}. \quad (\text{A.6})$$

So, let  $q \in \mathbb{N}^*$ ,

$$\left( \xi_f^{(n)}(u) \right)_q = \frac{1}{\sqrt{\sigma_2(T)}} \sum_{\substack{\mathbf{k} \in \mathbb{N}^3 \\ |\mathbf{k}|=q}} a_f(\mathbf{k}, u) \int_T \tilde{H}_{\mathbf{k}}(Y(t)) dt.$$

Corollary 4.16 implies that,

$$\left( \left( \xi_{f_1}^{(n)}(u) \right)_q ; \left( \xi_{f_2}^{(n)}(u) \right)_q \right) \xrightarrow[n \rightarrow +\infty]{Law} \mathcal{N}(0; (\Sigma_{(f_1, f_2)}(u))_q),$$

and in force  $(\Sigma_{(f_1, f_2)}(u))_q$  is a semi-definite positive matrix. This argument yields inequality (A.6) and Lemma 5.4. □

*Proof of Lemma 5.7.* Using that  $B_n^2 = A_n$ ,  $\lim_n A_n = A = B^2$  and that  $\lim_n \text{tr}(B_n) = \text{tr}(B) > 0$ , we obviously deduce that  $\lim_n B_n = B$ . □

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