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► **To cite this version:**

Andrea Seppi. On the maximal dilatation of quasiconformal minimal Lagrangian extensions. *Geometriae Dedicata*, 2019, 203, pp.25-52. 10.1007/s10711-019-00422-8 . hal-01675476v2

HAL Id: hal-01675476

<https://hal.science/hal-01675476v2>

Submitted on 31 Jan 2019

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ON THE MAXIMAL DILATATION OF QUASICONFORMAL MINIMAL LAGRANGIAN EXTENSIONS

ANDREA SEPPI

ABSTRACT. Given a quasisymmetric homeomorphism φ of the circle, Bonsante and Schlenker proved the existence and uniqueness of the minimal Lagrangian extension $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ to the hyperbolic plane. By previous work of the author, its maximal dilatation satisfies $\log K(f_\varphi) \leq C \|\varphi\|_{cr}$, where $\|\varphi\|_{cr}$ denotes the cross-ratio norm. We give constraints on the value of an optimal such constant C , and discuss possible lower inequalities, by studying two one-parameter families of minimal Lagrangian extensions in terms of maximal dilatation and cross-ratio norm.

1. INTRODUCTION

A classical problem in quasiconformal Teichmüller theory consists in finding particular quasiconformal extensions $f : \mathbb{D} \rightarrow \mathbb{D}$, where \mathbb{D} is the unit disc in \mathbb{C} , of a quasisymmetric homeomorphism $\varphi : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ of the circle. Classical extensions are for instance the *Beurling-Ahlfors* extension ([BA56, Leh83]) and the *Douady-Earle* extension ([DE86]). More recently Markovic ([Mar17]), and independently Benoist and Hulin ([BH17]), proved the existence of the *harmonic* extension, hence solving the so-called *Schoen conjecture*. As the notion of harmonicity depends on the choice of a Riemannian metric on the target, the Poincaré metric is considered on \mathbb{D} , thus identifying the target \mathbb{D} with the hyperbolic plane \mathbb{H}^2 . Uniqueness of the harmonic extension was previously known (see [LT93]).

Minimal Lagrangian extensions. In [BS10], Bonsante and Schlenker proved the existence and uniqueness of the *minimal Lagrangian* extension of φ — where the notion of minimal Lagrangian diffeomorphism depends on the choice of the Riemannian metric (here, the Poincaré metric) both on the source and on the target. Indeed, an orientation-preserving diffeomorphism $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is called minimal Lagrangian if it is area-preserving and its graph is a minimal surface in the Riemannian product $\mathbb{H}^2 \times \mathbb{H}^2$. Minimal Lagrangian diffeomorphisms $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ are also closely related to harmonic maps, as they are characterized by the condition of being the composition $f = h_1 \circ (h_2)^{-1}$, where h_1, h_2 are harmonic diffeomorphisms to \mathbb{H}^2 with opposite Hopf differential — see for instance [Lab92, BS10]. This paper concerns the study of the minimal Lagrangian extension of a quasisymmetric homeomorphism $\varphi : \partial\mathbb{H}^2 \rightarrow \partial\mathbb{H}^2$, which we will simply denote by f_φ .

More concretely, let us consider \mathbb{H}^2 in the upper half-plane model, so that its boundary is identified to $\mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\}$. Given a quasisymmetric homeomorphism $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$, its *cross-ratio norm* is defined as:

$$\|\varphi\|_{cr} := \sup_{cr(Q)=-1} |\log |cr(\varphi(Q))|| ,$$

where $cr(Q)$ denotes the cross-ratio of a quadruple of points Q in \mathbb{RP}^1 , and our definition of cross-ratio is such that a quadruple Q is symmetric if and only if $cr(Q) = -1$.

Then, if $K(f_\varphi)$ denotes the maximal dilatation of the minimal Lagrangian extension $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ of φ , the main result of [Sep19] is the following inequality:

$$\log K(f_\varphi) \leq C \|\varphi\|_{cr} , \quad (1)$$

where $C > 0$ is a universal constant. We remark that similar results were obtained for the Beurling-Ahlfors extension:

$$\log K(f_\varphi^{BA}) \leq 2 \|\varphi\|_{cr} ,$$

as proved in [BA56], while [Leh83] showed:

$$\log K(f_\varphi^{BA}) \leq \|\varphi\|_{cr} + \log 2 .$$

Concerning the Douady-Earle extension, in [HM12] Hu and Muzician proved the inequality:

$$\log K(f_\varphi^{DE}) \leq C \|\varphi\|_{cr} ,$$

thus improving previous results of [DE86].

It seems therefore a natural question to ask what is the best possible value of the constants, for the different extensions taken into account. Unfortunately, the methods used in [Sep19] do not give any control on the value of the constant C appearing in (1). On the other hand, a partial lower bound on the constant C is provided by the following estimate, again proved in [Sep19]: for every $\epsilon > 0$ there exists $\delta > 0$ such that:

$$\log K(f_\varphi) \geq \left(\frac{1}{2} - \epsilon \right) \|\varphi\|_{cr} , \quad (2)$$

provided $\|\varphi\|_{cr} \leq \delta$.

Two families of minimal Lagrangian diffeomorphisms. In this paper, we give a rather explicit description of two families of minimal Lagrangian diffeomorphisms, each family depending on a real parameter, for which $\|\varphi\|_{cr}$ varies between 0 and $+\infty$. We therefore study the best possible value of the constant C for each of these families, as $\|\varphi\|_{cr}$ approaches 0 and $+\infty$. This will provide more precise conditions on the limit inferior and limit superior of the ratio $\log K(f_\varphi)/\|\varphi\|_{cr}$ as $\|\varphi\|_{cr} \rightarrow 0$ and $\|\varphi\|_{cr} \rightarrow +\infty$.

The first family of examples consists in the extensions f_{φ_λ} of a *simple (left) earthquake map* on $\partial\mathbb{H}^2$. That is, we consider for every $\lambda \geq 0$ the orientation-preserving homeomorphisms:

$$\varphi_\lambda(x) = \begin{cases} x & \text{if } x \leq 0 \\ e^\lambda x & \text{if } x > 0 \end{cases} .$$

It turns out that φ_λ is quasymmetric and $\|\varphi_\lambda\|_{cr} = \lambda$. Observe that φ_0 is the identity. Then we prove the following:

Theorem A. *Let $f_{\varphi_\lambda} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the minimal Lagrangian diffeomorphisms which extend a simple left earthquake φ_λ of weight λ . Then*

$$\lim_{\lambda \rightarrow 0^+} \frac{\log K(f_{\varphi_\lambda})}{\|\varphi_\lambda\|_{cr}} = \frac{2}{\pi} \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} \frac{\log K(f_{\varphi_\lambda})}{\|\varphi_\lambda\|_{cr}} = \frac{\sqrt{2}}{2} .$$

Numerically, we have

$$\frac{2}{\pi} \approx 0.64, \quad \frac{\sqrt{2}}{2} \approx 0.71.$$

We remark that the Douady-Earle extension $f_{\varphi_\lambda}^{DE}$ of simple earthquakes was studied in [HM12], where the following inequality was obtained:

$$\lim_{\lambda \rightarrow +\infty} \frac{\log K(f_{\varphi_\lambda}^{DE})}{\|\varphi_\lambda\|_{cr}} \geq \frac{1}{4}.$$

The second family considered consists in the extensions of *power functions* $\psi_\alpha : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$. Namely, we consider the orientation-preserving homeomorphisms:

$$\psi_\alpha(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ -|x|^\alpha & \text{if } x < 0 \end{cases}.$$

When $\alpha = 1$, ψ_1 is the identity. Since $(\psi_\alpha)^{-1} = \psi^{1/\alpha}$, and the inverse of a minimal Lagrangian map is minimal Lagrangian with the same maximal dilatation, we will only consider the case $\alpha \in [1, +\infty)$. We will prove the following:

Theorem B. *Let $f_{\psi_\alpha} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the minimal Lagrangian diffeomorphisms which extend the power function ψ_α . Then*

$$\limsup_{\alpha \rightarrow 1^+} \frac{\log K(f_{\psi_\alpha})}{\|\psi_\alpha\|_{cr}} \leq \frac{\sqrt{2}}{\log(3 + \sqrt{2})} \quad \text{and} \quad \lim_{\alpha \rightarrow +\infty} \frac{\log K(f_{\psi_\alpha})}{\|\psi_\alpha\|_{cr}} = 0.$$

Numerically, the first upper bound is $\sqrt{2}/\log(3 + \sqrt{2}) \approx 0.80$.

Since Theorem A exhibits a family of minimal Lagrangian extensions for which the ratio $\log K(f_\varphi)/\|\varphi\|_{cr}$ converges to a positive value as $\|\varphi\|_{cr} \rightarrow +\infty$, we obtain as a consequence that the inequality (1) of [Sep19] is *sharp*;

Corollary C. *There exists no function $F : [0, +\infty) \rightarrow [0, +\infty)$ which grows less than linearly such that $\log K(f_\varphi) \leq F(\|\varphi\|_{cr})$ for every minimal Lagrangian extension $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ of a quasimetric homeomorphism $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$.*

On the other hand, Theorem B shows an inequality of the form (2), as obtained in [Sep19], cannot hold globally, but only holds for bounded values of $\|\varphi\|_{cr}$.

Corollary D. *There exists no lower bound of the form $\log K(f_\varphi) \geq C\|\varphi\|_{cr}$, that holds for every minimal Lagrangian extension $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ of a quasimetric homeomorphism $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$.*

Applications to the study of optimal constants. As already mentioned, Theorem A and Theorem B, as an application, provide explicit bounds on the quantities:

$$\liminf \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \quad \text{and} \quad \limsup \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}},$$

as $\|\varphi\|_{cr} \rightarrow 0$ and $\|\varphi\|_{cr} \rightarrow +\infty$, which can be compared with the known estimates for the more classical extensions. More precisely, we obtain:

Corollary E. *If $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ denotes the minimal Lagrangian extension of a quasimetric homeomorphism $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$, then:*

$$\liminf_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \in \left[\frac{1}{2}, \frac{2}{\pi} \right] \quad \text{and} \quad \limsup_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \in \left[\frac{2}{\pi}, +\infty \right).$$

Such result can be compared for instance with the Beurling-Ahlfors extension f_φ^{BA} , for which the limit superior of $\log K(f_\varphi^{BA})/\|\varphi\|_{cr}$ is at most 2 as $\|\varphi\|_{cr} \rightarrow 0^+$, as a

consequence of the inequalities mentioned above. The limit superior is finite also for the Douady-Earle extension, although an explicit value seems not be available in the literature.

It is also worth mentioning that Corollary E highlights that the lower bound (2), obtained in [Sep19] by means of geometric methods using Anti-de Sitter geometry, is at least not far from being optimal, since $2/\pi \approx 0.64$.

On the other hand, when $\|\varphi\|_{cr} \rightarrow +\infty$, we obtain the following corollary:

Corollary F. *If $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ denotes the minimal Lagrangian extension of a quasisymmetric homeomorphism $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$, then:*

$$\liminf_{\|\varphi\|_{cr} \rightarrow +\infty} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} = 0 \quad \text{and} \quad \limsup_{\|\varphi\|_{cr} \rightarrow +\infty} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \in \left[\frac{\sqrt{2}}{2}, +\infty \right).$$

Again, the reader might want to compare the aforementioned inequality for the Beurling-Ahlfors extension, which shows that the limit superior of $\log K(f_\varphi^{BA})/\|\varphi\|_{cr}$ is at most 1 as $\|\varphi\|_{cr} \rightarrow +\infty$. Finally, let us observe that from the work of Strebel ([Str57, Str62]), it is known that the extremal extension $f_{\varphi_\lambda}^e$ (namely, the extension which minimizes the maximal dilatation) of the simple earthquake φ_λ satisfies $K(f_{\varphi_\lambda}^e) = O(\lambda^2)$ as $\lambda \rightarrow +\infty$. Hence a consequence of Theorem A is the following:

Corollary G. *The minimal Lagrangian extension of the simple earthquake φ_λ stays exponentially far from being extremal as $\lambda \rightarrow +\infty$.*

Compare also [HM12, Theorem 5] for the Douady-Earle extension.

Discussion of the techniques. The proof of the inequality (1), as well as the proof of the existence and uniqueness result of Bonsante and Schlenker, involves the use of *Anti-de Sitter* three-dimensional geometry, and in particular the study of *maximal surfaces*. See also [BS18a] which provides an alternative proof of the existence, [BBZ07, KS07, Tam18, BS18b, Tou16, Sep16, Bar16] for related results on (maximal) surfaces, [Mes07] as a standard reference on three-dimensional Anti-de Sitter geometry, and [BBD⁺12] for a survey on related questions. On the other hand, we remark that the results of this paper do not rely on Anti-de Sitter geometry, and their setting might also be compared with [Tou15a].

We will not enter into technical details here, but we mention the idea behind the construction of the two families of minimal Lagrangian extensions which appear in Theorem A and Theorem B. Instead of extending a given quasisymmetric homeomorphism, we rather start from (two) *ansatz* which lead to (two) remarkable families of minimal Lagrangian diffeomorphisms. These *ansatz* basically consist in looking for diffeomorphisms $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ with a certain invariance with respect to 1-parameter families of isometries of \mathbb{H}^2 . Such invariance enables to reduce the condition that f is minimal Lagrangian to ordinary differential equations. Solutions of such differential equations are then studied analytically, and the corresponding diffeomorphisms are interpreted from the point of view of hyperbolic geometry. This enables to estimate the maximal dilatation of f on the one hand, and on the other the cross-ratio distortion of the boundary values of f (which are recognized as the simple left earthquakes in one case, and the power functions in the other).

Finally, we remark that the two one-parameter families f_{φ_λ} and f_{ψ_α} are subfamilies of a promising two-parameter family, whose boundary value is the quasisymmetric

homeomorphism

$$x \mapsto \begin{cases} -|x|^\alpha & \text{if } x \leq 0 \\ e^\lambda x^\alpha & \text{if } x > 0 \end{cases},$$

for parameters $\lambda \in [0, +\infty)$ and $\alpha \in (0, +\infty)$. Clearly, this reduces to φ_λ when $\alpha = 1$ and to ψ_α when $\lambda = 0$. The understanding of the minimal Lagrangian extensions for such more general two-parameter family might lead to more precise estimates in the spirit of Corollary E and Corollary F. However, a description of this two-parameter family of minimal Lagrangian diffeomorphisms seems to be difficult to achieve with the techniques used in this paper, and is therefore not tackled here.

Organization of the paper. Section 2 contains preliminary results on quasiconformal mappings, minimal Lagrangian diffeomorphisms of \mathbb{H}^2 , quasisymmetric homeomorphisms, and state the necessary results known from the literature. Section 3 studies the first one-parameter family of minimal Lagrangian diffeomorphisms, leading to Theorem A, whereas Section 4 considers the second family and leads to Theorem B. Finally, Section 5 contains the Corollary E, F and G and compares them with results, known from the literature, for the more classical extensions.

Acknowledgements. I am extremely grateful to an anonymous referee for several comments which highly improved the present article. I would like to thank Jean-Marc Schlenker for motivating me towards questions of this type since several years, and Francesco Bonsante and Jun Hu for many interesting discussions on related topics.

2. PRELIMINARIES

In this paper, we consider the hyperbolic plane in the upper half-plane model, namely:

$$\mathbb{H}^2 = \left(\{z = x + iy \in \mathbb{C} : y > 0\}, \frac{dx^2 + dy^2}{y^2} \right),$$

and we identify its visual boundary $\partial_\infty \mathbb{H}^2$ with $\mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\}$, which in this model is the projective line $\{y = 0\} \cup \{\infty\}$. Recall that, in this model, the group of orientation-preserving isometries of \mathbb{H}^2 is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, acting on $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$ by Möbius transformations which preserve the real line \mathbb{RP}^1 .

Quasisymmetric homeomorphisms. We will adopt the following definition of cross-ratio:

$$cr(x_1, x_2, x_3, x_4) = \frac{(x_4 - x_1)(x_3 - x_2)}{(x_2 - x_1)(x_3 - x_4)}, \quad (3)$$

for a quadruple $x_1, x_2, x_3, x_4 \in \mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\}$. We say that a quadruple $Q = (x_1, x_2, x_3, x_4)$ of points of $\mathbb{R} \cup \{\infty\}$ is *symmetric* if the geodesics ℓ, ℓ' of \mathbb{H}^2 connecting x_1 to x_3 and x_2 to x_4 intersect orthogonally. Composing with an element of $\mathrm{PSL}_2(\mathbb{R})$, such a quadruple is equivalent to $(-1, 0, 1, \infty)$ and therefore has cross-ratio equal to -1 .

Definition 2.1. Let $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ be an orientation-preserving homeomorphism. Then φ is *quasisymmetric* if

$$\|\varphi\|_{cr} := \sup_{cr(Q)=-1} |\log |cr(\varphi(Q))|| < +\infty.$$

In this case, the quantity $\|\varphi\|_{cr}$ is called *cross-ratio norm* of φ .

It can be shown that $\|\varphi\|_{cr} = 0$ if and only if $\varphi \in \mathrm{PSL}_2(\mathbb{R})$. In this case, φ admits a conformal extension to the upper half-plane \mathbb{H}^2 , which is defined by the action of the same element of $\mathrm{PSL}_2(\mathbb{R})$ on \mathbb{H}^2 by a Möbius transformation.

Quasiconformal diffeomorphisms. Ahlfors and Beurling showed that every quasisymmetric homeomorphism φ admits a *quasiconformal* extension $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ and conversely, every quasiconformal map $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ extends continuously to a quasisymmetric homeomorphism $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$.

It turns out, for instance from Theorem 2.6 below, that a quasisymmetric homeomorphism always admits a *smooth* quasiconformal extension $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$. (That is, f is a diffeomorphism of \mathbb{H}^2 which extends continuously to \mathbb{RP}^1 .) We thus provide the definition of quasiconformality only for diffeomorphisms, although there is a more general definition with weaker regularity. (See for instance [Ahl06, GL00, FM07].)

Definition 2.2. Let $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be an orientation-preserving diffeomorphism. Then f is *quasiconformal* if

$$K(f) := \sup_{z \in \mathbb{C}} \frac{\text{major axis of } df_z(\mathbb{S}^1)}{\text{minor axis of } df_z(\mathbb{S}^1)} < +\infty ,$$

where we denote $\mathbb{S}^1 = \{|z| = 1\} \subset T_z\mathbb{H}^2 = \mathbb{C}$, and df_z is the differential of f at the point z . In this case, the quantity $K(f)$ is called *maximal dilatation* of f .

Roughly speaking, a quasiconformal diffeomorphism maps, at first order, circles to ellipses of bounded eccentricity. Clearly, $K(f) \geq 1$. It turns out that $K(f) = 1$ if and only if $f \in \mathrm{PSL}_2(\mathbb{R})$.

Minimal Lagrangian extensions. In this paper, we will study the properties of the *minimal Lagrangian diffeomorphism* extending φ , which is defined as follows:

Definition 2.3. Let $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be an orientation-preserving diffeomorphism. Then f is *minimal Lagrangian* if it is area-preserving and its graph is a minimal surface in the Riemannian product $\mathbb{H}^2 \times \mathbb{H}^2$.

An important characterization of minimal Lagrangian diffeomorphisms is the following. Let us denote by $g_{\mathbb{H}^2}$ the hyperbolic metric of \mathbb{H}^2 .

Proposition 2.4 ([Lab92],[Tou15b, Proposition 1.2.6]). *An orientation-preserving diffeomorphism $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is minimal Lagrangian if and only if $f^*g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b, b)$, where $b \in \Gamma(\mathrm{End}(T\mathbb{H}^2))$ is a bundle morphism such that:*

- b is self-adjoint for $g_{\mathbb{H}^2}$;
- $\det b = 1$;
- $d^\nabla b = 0$.

Here ∇ is the Levi-Civita connection of $g_{\mathbb{H}^2}$, and $d^\nabla b$ is the exterior derivative defined by:

$$(d^\nabla b)(v, w) = \nabla_v(b(w)) - \nabla_w(b(v)) - b[v, w] ,$$

where v and w are any local smooth vector fields.

Remark 2.5. Given an orientation-preserving diffeomorphism $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$, there is a unique section $b \in \Gamma(\mathrm{End}(T\mathbb{H}^2))$, satisfying $f^*g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b, b)$, which is self-adjoint for $g_{\mathbb{H}^2}$ and positive definite. In fact, if b is self-adjoint, then $f^*g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b, b) = g_{\mathbb{H}^2}(\cdot, b^2 \cdot)$ and therefore (in matrix notation),

$$b^2 = g_{\mathbb{H}^2}^{-1} \cdot f^*g_{\mathbb{H}^2} . \tag{4}$$

Such tensor b^2 then has two self-adjoint square roots, one positive definite and one negative definite, say b and $-b$. The condition of being minimal Lagrangian thus consist in imposing that one of these sections (and thus also the other) is Codazzi and has unit determinant.

We can finally state the main results which motivate this paper. First, the existence and uniqueness of the minimal Lagrangian extension, which is due to Bonsante and Schlenker. See also [BS18a] for an alternative proof.

Theorem 2.6 ([BS10]). *Let $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ be a quasimetric homeomorphism. Then there exists a unique minimal Lagrangian extension $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$, which is quasiconformal.*

We observe that, if φ is in $\mathrm{PSL}_2(\mathbb{R})$, then the minimal Lagrangian extension f_φ is the natural extension, defined by the action of the same element of $\mathrm{PSL}_2(\mathbb{R})$ on \mathbb{H}^2 . (In fact, in this case f_φ is an isometry for $g_{\mathbb{H}^2}$, and therefore one can take b as the identity section in Definition 2.3.) The maximal dilatation of the minimal Lagrangian extension was studied in [Sep19].

Theorem 2.7 ([Sep19]). *There exists a universal constant $C > 0$ such that*

$$\log K(f_\varphi) \leq C \|\varphi\|_{cr} ,$$

for any quasimetric homeomorphism $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$.

In other words, Theorem 2.7 can be stated as:

$$\sup_{\|\varphi\|_{cr} \neq 0} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} < +\infty .$$

The constant C obtained in [Sep19] is not explicit, though. Moreover, in the same paper, the following (partial) converse inequality was obtained:

Theorem 2.8 ([Sep19]). *For every $\epsilon > 0$ there exists $\delta > 0$ such that, if $\varphi : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ is quasimetric with $\|\varphi\|_{cr} \leq \delta$, then*

$$\log K(f_\varphi) \geq \left(\frac{1}{2} - \epsilon \right) \|\varphi\|_{cr} .$$

Theorem 2.8 can be reformulated as

$$\liminf_{\|\varphi\|_{cr} \rightarrow 0} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \geq \frac{1}{2} . \tag{5}$$

3. EXTENDING SIMPLE EARTHQUAKE MAPS

In this section we will construct a first one-parameter family of minimal Lagrangian maps $f_c : \mathbb{H}^2 \rightarrow \mathbb{H}^2$, depending on a parameter $c \in [0, 1)$, and study its behavior.

The band model. It will be convenient to introduce a new coordinate system on \mathbb{H}^2 , which is the *band model* of \mathbb{H}^2 . We use $z = x + iy$ to denote points of \mathbb{H}^2 , and introduce the notation $w = u + iv$. Then consider the global coordinate $\varsigma : \Omega \rightarrow \mathbb{H}^2$, where

$$\Omega = \left\{ w = u + iv : |u| < \frac{\pi}{2} \right\} ,$$

defined by

$$\varsigma : w \mapsto z = i \exp(-iw) .$$

Observe that ς extends to each component $\{u = -\pi/2\}$ and $\{u = \pi/2\}$ of $\partial\Omega$, which are mapped to the two portions $\{x > 0\}$ and $\{x < 0\}$ of $\mathbb{R}P^1$ by means of

$$\varsigma\left(-\frac{\pi}{2} + iv\right) = -e^v \quad \varsigma\left(\frac{\pi}{2} + iv\right) = e^v .$$

In the w -coordinates, the metric of \mathbb{H}^2 takes the form:

$$\varsigma^* g_{\mathbb{H}^2} = \frac{du^2 + dv^2}{\cos^2(u)} .$$

Let us thus remark that this is a conformal model of \mathbb{H}^2 , for which the vertical line $\ell = \{u = 0\}$ is a geodesic with arclength parameter v . The endpoints of ℓ are at infinity and correspond to 0 and ∞ in the upper half-plane model. In fact, the hyperbolic translations along ℓ , with (signed) translation distance δ , have the form

$$T_\delta(w) = w + i\delta .$$

See also [Hub06, Chapter 2].

Ansatz for minimal Lagrangian diffeomorphisms. For convenience, let us use the notation

$$w = (u, v) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}$$

for the band model of \mathbb{H}^2 , instead of the complex notation $w = u + iv$. In this section we will consider diffeomorphisms $\mathfrak{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ of the following form:

$$\mathfrak{f}(u, v) = (u, v + h(u)) . \tag{6}$$

We will derive conditions on h which correspond to the fact that \mathfrak{f} is minimal Lagrangian, using the characterization of Proposition 2.4, and study the solutions of the corresponding ODE. As observed in Remark 2.5, the section $b \in \Gamma(\text{End}(T\mathbb{H}^2))$ is essentially determined by the map \mathfrak{f} , and \mathfrak{f} being minimal Lagrangian corresponds to imposing $\det b = 1$ and $d^\nabla b = 0$. We will see below that the condition $\det b = 1$ will be easily satisfied, while the most complicated condition is the Codazzi condition $d^\nabla b = 0$.

Remark 3.1. The diffeomorphism of the form of Equation (6) is invariant for the 1-parameter group of isometries $\{T_\delta : \delta \in \mathbb{R}\}$, that is,

$$\mathfrak{f} \circ T_\delta = T_\delta \circ \mathfrak{f} ,$$

for every $\delta \in \mathbb{R}$. A diffeomorphism with this invariance can have the more general form

$$\mathfrak{f}(u, v) = (g(u), v + h(u)) . \tag{7}$$

Such maps were considered, in a slightly different context, in [BS18a, Section 6]. There it was showed that, when taking \mathfrak{f} of the form (7), if b is the unique positive definite self-adjoint bundle morphism with $f^* g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b, b)$ (see Remark 2.5), the condition $\det b = 1$ is equivalent to the following ODE on g :

$$\frac{g'(u)}{\cos^2(g(u))} = \pm \frac{1}{\cos^2(u)} ,$$

which is solved by $\tan(g(u)) = \pm \tan u + C$. Since in this work we are looking for global diffeomorphisms, we impose $C = 0$, so that g is defined on the entire interval $(-\pi/2, \pi/2)$, and we take the positive sign since we require \mathfrak{f} to be orientation-preserving. For this reason, we only consider $g(u) = u$ here, as in Equation (6).

Let us now compute the self-adjoint, positive definite section $b \in \Gamma(\text{End}(T\mathbb{H}^2))$, as explained in Remark 2.5.

Lemma 3.2. *Let $\mathfrak{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be an orientation-preserving diffeomorphism of the form (in the band model)*

$$\mathfrak{f}(u, v) = (u, v + h(u)) .$$

Then $\mathfrak{f}^*g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b, b)$ where

$$b = \frac{1}{\sqrt{4 + h'(u)^2}} \begin{pmatrix} 2 + h'(u)^2 & h'(u) \\ h'(u) & 2 \end{pmatrix}$$

is self-adjoint, positive definite, and $\det b = 1$.

Proof. Let us first compute the pull-back metric, in the band model:

$$\begin{aligned} \mathfrak{f}^*g_{\mathbb{H}^2} &= \frac{1}{\cos^2(u)} (du^2 + (dv + h'(u)du)^2) \\ &= \frac{(1 + h'(u)^2)du^2 + 2h'(u)dudv + dv^2}{\cos^2(u)} . \end{aligned}$$

Using Equation (4), and recalling that in these coordinates $g_{\mathbb{H}^2} = (1/\cos^2(u))(du^2 + dv^2)$, we then have:

$$b^2 = \begin{pmatrix} 1 + h'(u)^2 & h'(u) \\ h'(u) & 1 \end{pmatrix} .$$

Hence $\det b^2 = 1$. To compute b , self-adjoint for $g_{\mathbb{H}^2}$ and positive definite, we observe from the Cayley-Hamilton theorem that $b^2 - (\text{tr } b)b + (\det b)\mathbb{1} = 0$. Since we are looking for b with $\det b = 1$, we deduce $(\text{tr } b)b = b^2 + \mathbb{1}$. By taking the trace we obtain $(\text{tr } b)^2 = \text{tr } b^2 + 2$ and therefore $\text{tr } b = \sqrt{2 + \text{tr } b^2} = \sqrt{4 + h'(u)^2}$. (Indeed b is supposed positive definite and thus $\text{tr } b > 0$.) This enables us to conclude

$$b = \frac{1}{\sqrt{4 + h'(u)^2}} (b^2 + \mathbb{1}) = \frac{1}{\sqrt{4 + h'(u)^2}} \begin{pmatrix} 2 + h'(u)^2 & h'(u) \\ h'(u) & 2 \end{pmatrix} ,$$

as claimed. \square

Imposing the Codazzi condition. We are thus left to impose the condition $d^\nabla b = 0$, which will result in an ordinary differential equation for h .

Lemma 3.3. *Let $\mathfrak{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be an orientation-preserving diffeomorphism of the form (in the band model)*

$$\mathfrak{f}(u, v) = (u, v + h(u)) ,$$

and let $\mathfrak{f}^*g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b, b)$ where b is self-adjoint, positive definite and $\det b = 1$, as in Lemma 3.2. Then $d^\nabla b = 0$ if and only if h satisfies

$$\tan(u) = -\frac{2h''(u)}{(4 + h'(u)^2)h'(u)} . \quad (8)$$

Proof. First, let us remark that $d^\nabla b$ is a skew-symmetric 2-form, hence in (u, v) -coordinates, $d^\nabla b(\partial_u, \partial_u) = d^\nabla b(\partial_v, \partial_v) = 0$ and $d^\nabla b(\partial_u, \partial_v) = -d^\nabla b(\partial_v, \partial_u)$. Moreover $[\partial_u, \partial_v] = 0$. It thus suffices to check

$$d^\nabla b(\partial_u, \partial_v) = \nabla_{\partial_u} b(\partial_v) - \nabla_{\partial_v} b(\partial_u) = 0 .$$

Let us now compute:

$$\begin{aligned}
\nabla_{\partial_v} b(\partial_u) &= \nabla_{\partial_v} \left(\frac{1}{\sqrt{4+h'(u)^2}} ((2+h'(u)^2)\partial_u) + h'(u)\partial_v \right) \\
&= \frac{1}{\sqrt{4+h'(u)^2}} ((2+h'(u)^2)\nabla_{\partial_v}\partial_u + h'(u)\nabla_{\partial_v}\partial_v) \\
&= \frac{1}{\sqrt{4+h'(u)^2}} ((2+h'(u)^2)\tan(u)\partial_v - h'(u)\tan(u)\partial_u) \\
&= \frac{\tan(u)}{\sqrt{4+h'(u)^2}} (-h'(u)\partial_u + (2+h'(u)^2)\partial_v) .
\end{aligned}$$

From the second to the third line we have used the expressions

$$\nabla_{\partial_v}\partial_u = \Gamma_{vu}^u\partial_u + \Gamma_{vu}^v\partial_v = \tan(u)\partial_v$$

and

$$\nabla_{\partial_v}\partial_v = \Gamma_{vv}^u\partial_u + \Gamma_{vv}^v\partial_v = -\tan(u)\partial_u ,$$

which arise from a direct computation of the Christoffel symbols: it turns out indeed that $\Gamma_{vu}^u = \Gamma_{vv}^v = 0$, $\Gamma_{vu}^v = \tan(u)$ and $\Gamma_{vv}^u = -\tan(u)$.

On the other hand, let us compute:

$$\begin{aligned}
\nabla_{\partial_u} b(\partial_v) &= \nabla_{\partial_u} \left(\frac{1}{\sqrt{4+h'(u)^2}} (h'(u)\partial_u + 2\partial_v) \right) \\
&= -\frac{h'(u)h''(u)}{(4+h'(u)^2)^{3/2}} (h'(u)\partial_u + 2\partial_v) \\
&\quad + \frac{1}{\sqrt{4+h'(u)^2}} (h''(u)\partial_u + h'(u)\nabla_{\partial_u}\partial_u + 2\nabla_{\partial_u}\partial_v) \\
&= \left(-\frac{h'(u)^2h''(u)}{(4+h'(u)^2)^{3/2}} + \frac{h'(u)\tan(u) + h''(u)}{\sqrt{4+h'(u)^2}} \right) \partial_u \\
&\quad + \left(-\frac{2h'(u)h''(u)}{(4+h'(u)^2)^{3/2}} + \frac{2\tan(u)}{\sqrt{4+h'(u)^2}} \right) \partial_v ,
\end{aligned}$$

where here we have used

$$\nabla_{\partial_u}\partial_u = \Gamma_{uu}^u\partial_u + \Gamma_{uu}^v\partial_v = \tan(u)\partial_u$$

since $\Gamma_{uu}^u = \tan(u)$ and $\Gamma_{uu}^v = 0$, and

$$\nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \tan(u)\partial_v .$$

Let us now equate $\nabla_{\partial_v} b(\partial_u)$ and $\nabla_{\partial_u} b(\partial_v)$. From the ∂_u component, we obtain:

$$\frac{h'(u)^2h''(u)}{(4+h'(u)^2)^{3/2}} = \frac{2h'(u)\tan(u) + h''(u)}{\sqrt{4+h'(u)^2}} ,$$

which leads after some manipulation to:

$$\tan(u) = -\frac{2h''(u)}{(4+h'(u)^2)h'(u)} ,$$

which is precisely Equation (8). Equating the ∂_v components instead, one gets to:

$$\frac{2h'(u)h''(u)}{(4+h'(u)^2)^{3/2}} = -\frac{h'(u)\tan(u)}{\sqrt{4+h'(u)^2}} ,$$

which leads again to Equation (8). Hence $d^\nabla b = 0$ if and only if Equation (8) is satisfied. \square

Solutions of ODE and definition of the diffeomorphisms f_k . We thus need to study solutions of Equation (8). This is not difficult to integrate, since

$$\int \tan(u) du = -\log \cos(u) + c_1$$

and

$$\int \frac{2h''(u)}{(4+h'(u)^2)h'(u)} du = \frac{1}{2} \log h'(u) - \frac{1}{4} \log(4+h'(u)^2) + c_2,$$

and therefore we obtain (put $c = -c_1 - c_2$):

$$e^{4c} \cos^4(u) = \frac{h'(u)^2}{4+h'(u)^2}.$$

After some manipulation, and replacing the constant e^c by k , we obtain

$$h'(u) = \pm \frac{2k^2 \cos^2(u)}{\sqrt{1-k^4 \cos^4(u)}}. \quad (9)$$

Composing with a hyperbolic translation T_δ , which has the form $T_\delta(u, v) = (u, v + \delta)$, we can assume $h(-\pi/2) = 0$. Hence we will consider the following family of minimal Lagrangian diffeomorphisms (corresponding to the choice of the positive sign in Equation (9)):

Definition 3.4. For every $k \in [0, 1)$, let

$$h_k(u) = \int_{-\pi/2}^u \frac{2k^2 \cos^2(\sigma)}{\sqrt{1-k^4 \cos^4(\sigma)}} d\sigma.$$

Then define, in the band model of \mathbb{H}^2 ,

$$f_k(u, v) = (u, v + h_k(u)).$$

The diffeomorphisms f_k obtained in this way (recall the ansatz of Equation (6)) are minimal Lagrangian, since h_k is a solution of Equation (8) in Lemma 3.3. They depend on the parameter k . We only consider $k < 1$ so that h_k is defined for all $u \in (-\pi/2, \pi/2)$. Moreover, we also allowed $k = 0$, which corresponds to $h'_0 \equiv 0$ and thus $h_0 \equiv 0$: the diffeomorphism f_0 is the identity.

Computing the cross-ratio norm. Let us first express the boundary value of the diffeomorphisms f_k we have just defined, and find its cross-ratio norm.

Lemma 3.5. *The minimal Lagrangian diffeomorphisms $f_k : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ extend to \mathbb{RP}^1 , with boundary value (in the upper half-plane model) the homeomorphism:*

$$\phi_k(x) = \begin{cases} x & \text{if } x \leq 0 \\ e^{\lambda_k x} & \text{if } x > 0 \end{cases},$$

where

$$\lambda_k = \int_{-\pi/2}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1-k^4 \cos^4(\sigma)}} d\sigma. \quad (10)$$

Proof. Since we assumed (in the band model coordinates) $h_k(-\pi/2) = 0$, the diffeomorphism f_k extends to the boundary, with

$$f_k(-\pi/2, v) = (-\pi/2, v) \quad f_k(\pi/2, v) = (\pi/2, v + h_k(\pi/2)).$$

In other words, in the upper half-plane model, f_k is the identity on the portion $\{x \leq 0\}$ of $\mathbb{R} \cup \{\infty\}$, whereas it coincides on the portion $\{x > 0\}$ with the restriction on $\mathbb{R} \cup \{\infty\}$ of the hyperbolic transformation whose axis is the geodesic of the upper half-plane with endpoints 0 and ∞ , and having (signed) translation distance

$$\lambda_k = h_k(\pi/2) = \int_{-\pi/2}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^4 \cos^4(\sigma)}} d\sigma . \quad (11)$$

Hence, in the upper half-plane model, the boundary value of f_k is:

$$\phi_k(x) = \begin{cases} x & \text{if } x \leq 0 \\ e^{\lambda_k x} & \text{if } x > 0 \end{cases} .$$

This concludes the proof. \square

Remark 3.6. In the language of hyperbolic geometry, ϕ_k is the boundary value of a *left earthquake map*, whose earthquake lamination is the geodesic line $\{x = 0\}$ with weight λ_k . To simplify the notation, let us denote the restriction to $\partial\mathbb{H}^2$ of a simple left earthquake map with weight λ by φ_λ – namely, we put $\lambda = \lambda_k$ and we denote $\varphi_\lambda = \phi_k$. Using Definition 2.1, it can be easily verified that

$$\|\varphi_\lambda\|_{cr} = \lambda .$$

In fact, picking the quadruple $Q = (x_1, x_2, x_3, x_4)$ with:

$$x_1 = -a \quad x_2 = 0 \quad x_3 = a \quad x_4 = \infty$$

for some $a > 0$, by a direct computation $cr(Q) = -1$. On the other hand,

$$\phi_k(x_1) = -a \quad \phi_k(x_2) = 0 \quad \phi_k(x_3) = e^\lambda a \quad \phi_k(x_4) = \infty ,$$

hence by another direct computation $cr(\phi_k(Q)) = -e^\lambda$. Therefore

$$\|\varphi_\lambda\|_{cr} := \sup_{cr(Q)=-1} |\log |cr(\varphi(Q))|| \geq \lambda .$$

It is not difficult to check that $|\log |cr(\varphi(Q))||$ is actually maximized for quadruples of this form, and therefore $\|\varphi_\lambda\|_{cr} = \lambda$. See also [HM12]. Let us also remark that

$$\lim_{k \rightarrow 0^+} \|\phi_k\|_{cr} = \lim_{\lambda \rightarrow 0^+} \|\varphi_\lambda\|_{cr} = 0 .$$

We will study more precisely the behavior of $\|\phi_k\|_{cr}$ as $k \rightarrow 0$ below, where we will also see (with a precise asymptotic behavior) that

$$\lim_{k \rightarrow 1^-} \|\phi_k\|_{cr} = +\infty .$$

Computing the maximal dilatation. Let us moreover compute the maximal dilatation of the minimal Lagrangian diffeomorphism $f_k : \mathbb{H}^2 \rightarrow \mathbb{H}^2$.

Lemma 3.7. *The maximal dilatation of the minimal Lagrangian diffeomorphisms $f_k : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is:*

$$K(f_k) = \frac{1 + k^2}{1 - k^2} .$$

Proof. Recalling Definition 2.2 and the expression

$$f^* g_{\mathbb{H}^2} = g_{\mathbb{H}^2}(b, b) ,$$

for any point $z \in \mathbb{H}^2$ the ratio between the major axis and the minor axis of the ellipse $df_z(\mathbb{S}^1)$ equals $\eta_+(z)/\eta_-(z)$, where $\eta_+(z)$ and $\eta_-(z)$ are the largest and smallest

eigenvalues of b at z . Since $\det b = 1$, $\eta_+ \eta_- = 1$ and therefore the maximal dilatation of f is:

$$K(f) = \sup_{z \in \mathbb{H}^2} \eta_+(z)^2 .$$

In the case of the minimal Lagrangian diffeomorphisms \mathbf{f}_k , since $\det b = 1$ and $\operatorname{tr} b = \sqrt{4 + h'_k(u)}$, the eigenvalues of b are:

$$\eta_{\pm}(u, v) = \frac{\sqrt{4 + h'_k(u)^2} \pm h'_k(u)}{2} .$$

From Equation (9),

$$h'_k(u) = \frac{2k^2 \cos^2(u)}{\sqrt{1 - k^4 \cos^4(u)}}$$

achieves its maximum at $u = 0$, namely for:

$$h'_k(0) = \frac{2k^2}{\sqrt{1 - k^4}} .$$

Hence we finally have

$$\begin{aligned} K(\mathbf{f}_k) &= \left(\frac{\sqrt{4 + h'_k(0)^2} + h'_k(0)}{2} \right)^2 \\ &= \frac{1}{4} \left(\sqrt{4 + \frac{4k^4}{1 - k^4}} + \frac{2k^2}{\sqrt{1 - k^4}} \right)^2 \\ &= \frac{1}{4} \left(\frac{2(1 + k^2)}{\sqrt{1 - k^4}} \right)^2 = \left(\frac{1 + k^2}{\sqrt{1 - k^4}} \right)^2 = \frac{1 + k^2}{1 - k^2} , \end{aligned}$$

as claimed. \square

Comparison when $k \rightarrow 0$. Let us now study the value of the ratio $\log K(\mathbf{f}_k)/\|\phi_k\|_{cr}$, in particular when $k \rightarrow 0$ and $k \rightarrow 1$.

Theorem A (Case $\lambda \rightarrow 0$). *Let $f_{\varphi_\lambda} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the minimal Lagrangian diffeomorphisms which extend a simple left earthquake φ_λ of weight λ . Then*

$$\lim_{\lambda \rightarrow 0} \frac{\log K(f_{\varphi_\lambda})}{\|\varphi_\lambda\|_{cr}} = \frac{2}{\pi} \approx 0.64 .$$

Proof. First, recall that the cross-ratio norm of φ_λ is $\|\varphi_\lambda\|_{cr} = \lambda$, from Remark 3.6. Since $\phi_k = \varphi_{\lambda_k}$, it follows that $f_{\varphi_\lambda} = \mathbf{f}_k$ is the minimal Lagrangian homeomorphism extending a simple left earthquake of weight λ_k . As $\|\phi_k\|_{cr} = \lambda_k \rightarrow 0$ when $k \rightarrow 0$, we have

$$\lim_{\lambda \rightarrow 0^+} \frac{\log K(f_{\varphi_\lambda})}{\|\varphi_\lambda\|_{cr}} = \lim_{k \rightarrow 0^+} \frac{\log K(\mathbf{f}_k)}{\|\phi_k\|_{cr}} .$$

We first need to estimate $\|\phi_k\|_{cr} = \lambda_k$, which is defined by the integral of Equation (11). Since $1 - k^4 \cos^4(u) = (1 - k^2 \cos^2(u))(1 + k^2 \cos^2(u))$ and

$$1 \leq \sqrt{1 + k^2 \cos^2(u)} \leq \sqrt{1 + k^2} ,$$

we can estimate

$$\frac{1}{\sqrt{1 + k^2}} \int_{-\pi/2}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^2 \cos^2(\sigma)}} d\sigma \leq \lambda_k \leq \int_{-\pi/2}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^2 \cos^2(\sigma)}} d\sigma . \quad (12)$$

From some algebraic steps (in the second line we change variable $\theta = \pi/2 - \sigma$),

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^2 \cos^2(\sigma)}} d\sigma &= 4 \int_0^{\pi/2} \frac{k^2 \cos^2(\sigma)}{\sqrt{1 - k^2 \cos^2(\sigma)}} d\sigma \\ &= 4 \int_0^{\pi/2} \frac{k^2 \sin^2(\theta)}{\sqrt{1 - k^2 \sin^2(\theta)}} d\theta \\ &= 4 \left(\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}} - \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2(\theta)} d\theta \right) \\ &= 4(K(k) - E(k)) , \end{aligned}$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}$$

denotes the complete elliptic integral of the first kind (there is a small abuse of notation, since K also denotes the maximal dilatation, but this is the standard notation in both cases), and

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2(\theta)} d\theta$$

is the complete elliptic integral of the second kind.

When $k \rightarrow 0$, $K(k)$ has the expansion:

$$K(k) = \frac{\pi}{2} + \frac{\pi}{8}k^2 + O(k^4) ,$$

whereas $E(k)$ has the expansion:

$$E(k) = \frac{\pi}{2} - \frac{\pi}{8}k^2 + O(k^4) ,$$

This shows that

$$4(K(k) - E(k)) = \pi k^2 + O(k^4) .$$

On the other hand,

$$\log K(\mathbf{f}_k) = \log \left(\frac{1 + k^2}{1 - k^2} \right) = 2k^2 + O(k^6) .$$

Therefore, using (12),

$$\liminf_{k \rightarrow 0^+} \frac{\log K(\mathbf{f}_k)}{\|\phi_k\|_{cr}} \geq \lim_{k \rightarrow 0^+} \frac{2k^2 + O(k^6)}{\pi k^2 + O(k^4)} = \frac{2}{\pi} .$$

Similarly, using the other inequality in (12), one shows:

$$\limsup_{k \rightarrow 0^+} \frac{\log K(\mathbf{f}_k)}{\|\phi_k\|_{cr}} \leq \lim_{k \rightarrow 0^+} \frac{2k^2 + O(k^6)}{\pi k^2 / \sqrt{1 + k^2} + O(k^4)} = \lim_{k \rightarrow 0^+} \frac{2k^2 + O(k^6)}{\pi k^2 + O(k^4)} = \frac{2}{\pi} ,$$

which concludes the proof. \square

Comparison when $k \rightarrow 1$. Employing the same computations, we can study the limit of the ratio when k approaches 1.

Theorem A (Case $\lambda \rightarrow +\infty$). *Let $f_{\varphi_\lambda} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the minimal Lagrangian diffeomorphisms which extend a simple left earthquake φ_λ of weight λ . Then*

$$\lim_{\lambda \rightarrow +\infty} \frac{\log K(f_{\varphi_\lambda})}{\|\varphi_\lambda\|_{cr}} = \frac{\sqrt{2}}{2} \approx 0.71 .$$

Proof. To simplify the notation, we will prove the equivalent statement

$$\lim_{k \rightarrow 1^-} \frac{\|\phi_k\|_{cr}}{\log K(\mathfrak{f}_k)} = \sqrt{2} ,$$

where $\|\phi_k\|_{cr} = \lambda_k$ is defined by the integral of Equation (11).

As a preliminary remark, from the proof of the previous case let us define

$$F(k) := \int_{-\pi/2}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^2 \cos^2(\sigma)}} d\sigma = 4(K(k) - E(k)) .$$

When $k \rightarrow 1$, $E(k) \rightarrow 1$, while $K(k)$ goes to infinity with the asymptotic expression (see [GLLZ02, Lemma 4]):

$$K(k) = \log 4 - \frac{1}{2} \log(1 - k^2) + O(|(1 - k^2) \log(1 - k^2)|) .$$

In particular, $\lambda_k \rightarrow +\infty$ as $k \rightarrow 1^-$. For the maximal dilatation, from Lemma 3.7,

$$\log K(\mathfrak{f}_k) = \log \left(\frac{1 + k^2}{1 - k^2} \right) = \log(1 + k^2) - \log(1 - k^2) .$$

Hence, when taking the limit for $k \rightarrow 1$, we obtain:

$$\lim_{k \rightarrow 1^-} \frac{F(k)}{\log K(\mathfrak{f}_k)} = \lim_{k \rightarrow 1^-} \frac{4(\log 4 - 1 - \frac{1}{2} \log(1 - k^2))}{\log(1 + k^2) - \log(1 - k^2)} = 2 . \quad (13)$$

From Equation (11), writing $1 - k^4 \cos^4(u) = (1 - k^2 \cos^2(u))(1 + k^2 \cos^2(u))$ and estimating $\sqrt{1 + k^2 \cos^2(u)} \leq \sqrt{2}$, we get (using (13)):

$$\liminf_{k \rightarrow 1^-} \frac{\|\phi_k\|_{cr}}{\log K(\mathfrak{f}_k)} \geq \frac{1}{\sqrt{2}} \lim_{k \rightarrow 1^-} \frac{F(k)}{\log K(\mathfrak{f}_k)} = \frac{2}{\sqrt{2}} = \sqrt{2} .$$

To show the other inequality, fix $\epsilon > 0$. Observe that $1 + k^2 \cos^2(u) \geq 2 - \epsilon$ if and only if $k^2 \cos^2(u) \geq 1 - \epsilon$, which is equivalent to $|\sigma| \leq \sigma_0$ for

$$\sigma_0 := \arccos \left(\frac{\sqrt{1 - \epsilon}}{k} \right) .$$

Hence we have

$$\int_{-\sigma_0}^{\sigma_0} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^4 \cos^4(\sigma)}} d\sigma \leq \frac{1}{\sqrt{2 - \epsilon}} \int_{-\sigma_0}^{\sigma_0} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^2 \cos^2(\sigma)}} d\sigma \leq \frac{1}{\sqrt{2 - \epsilon}} F(k) ,$$

whereas for $|\sigma| \geq \sigma_0$, using $1 + k^2 \cos^2(\sigma) \geq 1$ and $k^2 \cos^2(u) \leq 1 - \epsilon$, we have

$$\int_{\sigma_0}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^4 \cos^4(\sigma)}} d\sigma \leq \frac{2}{\sqrt{\epsilon}} \int_{\sigma_0}^{\pi/2} d\sigma \leq \frac{\pi}{\sqrt{\epsilon}} .$$

Thus we can estimate:

$$\frac{1}{\log K(\mathfrak{f}_k)} \int_{-\pi/2}^{\pi/2} \frac{2k^2 \cos^2(\sigma)}{\sqrt{1 - k^4 \cos^4(\sigma)}} d\sigma \leq \frac{1}{\sqrt{2 - \epsilon}} \frac{F(k)}{\log K(\mathfrak{f}_k)} + \frac{2\pi}{\sqrt{\epsilon} \log K(\mathfrak{f}_k)} .$$

Choosing k large enough, the first term can be made smaller than $(2 + \epsilon)/\sqrt{2 - \epsilon}$ as a consequence of (13), while the second term can be made smaller than ϵ (say) since $\log K(\mathfrak{f}_k)$ goes to infinity. This shows that

$$\limsup_{k \rightarrow 1^-} \frac{\|\phi_k\|_{cr}}{\log K(\mathfrak{f}_k)} \leq \sqrt{2}$$

and therefore concludes the proof. \square

4. EXTENDING POWER FUNCTIONS

In this section, we will construct another one-parameter family of minimal Lagrangian maps \mathfrak{g}_α , depending on a parameter $\alpha \in [1, +\infty)$.

Coordinates from a geodesic. It will be convenient here to use a different coordinate system for \mathbb{H}^2 . Let ℓ be a geodesic of \mathbb{H}^2 as in the previous section, and suppose that in the upper half-plane model $\ell = \{x = 0\}$, so that $T_\delta(z) = e^\delta \cdot z$ is the hyperbolic translation preserving ℓ . Let $\gamma : \mathbb{R} \rightarrow \mathbb{H}^2$ a unit speed parameterization of the geodesic orthogonal to ℓ through the point i , and consider the global coordinate $\varsigma : \mathbb{R}^2 \rightarrow \mathbb{H}^2$ defined by:

$$\varsigma(s, t) = T_t \circ \gamma(s) .$$

It can be checked that in these coordinates the metric takes the form:

$$\varsigma^* g_{\mathbb{H}^2} = ds^2 + \cosh^2(s) dt^2 .$$

The hyperbolic translation takes the simple form

$$T_\delta(s, t) = (s, t + \delta) .$$

Ansatz for self-adjoint Codazzi tensors. In this section we will determine a bundle morphism $b \in \Gamma(\text{End}(T\mathbb{H}^2))$ satisfying the conditions of Proposition 2.4, and we will later study the behavior of a minimal Lagrangian map associated. For this purpose, we will use the following property.

Lemma 4.1 ([OS83],[BS16, Lemma 2.1]). *Given any smooth function $\rho : \mathbb{H}^2 \rightarrow \mathbb{R}$, the bundle morphism*

$$b = \text{Hess}\rho - \rho \mathbb{1}$$

is self-adjoint for $g_{\mathbb{H}^2}$ and satisfies the Codazzi equation $d^\nabla b = 0$, where $\text{Hess}\rho$ denotes the Hessian of ρ for the metric $g_{\mathbb{H}^2}$, as a $(1,1)$ -tensor, and $\mathbb{1}$ is the identity operator.

In [OS83] the converse statement was also proved, that is, any self-adjoint Codazzi tensor on a simply connected hyperbolic surface has the form $\text{Hess}\rho - \rho \mathbb{1}$ for some function ρ .

In this section, using this approach, we will look for a function $\rho : \mathbb{H}^2 \rightarrow \mathbb{R}$ so that $b = \text{Hess}\rho - \rho \mathbb{1}$ is a tensor as in Proposition 2.4. Conversely to the previous section, here the condition $d^\nabla b = 0$ will be automatically satisfied by Lemma 4.1, and the condition to impose is $\det b = 1$.

The ansatz here is to consider a function which only depends on s , that is,

$$\rho(s, t) = r(s) . \tag{14}$$

Imposing the unit determinant condition. Let us now compute the condition that $\det b = 1$ for a tensor $b = \text{Hess}\rho - \rho \mathbb{1}$.

Lemma 4.2. *Let $\rho : \mathbb{H}^2 \rightarrow \mathbb{R}$ be a smooth function of the form*

$$\rho(s, t) = r(s)$$

and let $b = \text{Hess}\rho - \rho \mathbb{1}$. Then $\det b = 1$ if and only

$$g'(s) = \tanh(s) \left(\frac{1}{g(s)} - g(s) \right) , \tag{15}$$

where $g(s) = r'(s) \tanh(s) - r(s)$ and $1/g(s)$ are the eigenvalues of b .

Proof. We need to consider the condition

$$\det(\text{Hess}\rho - \rho\mathbb{1}) = 1$$

for the function $\rho(s, t) = r(s)$. Recall that the hyperbolic Hessian is defined as

$$\text{Hess}\rho(v) = \nabla_v \text{grad } \rho .$$

To compute the Hessian, consider the orthonormal frame

$$v_s = \partial_s \quad v_t = \frac{1}{\cosh(s)} \partial_t .$$

Observe that, by an explicit computation for the bracket,

$$\nabla_{v_s} v_t - \nabla_{v_t} v_s = [v_s, v_t] = -\tanh(s)v_t ,$$

hence $\nabla_{v_t} v_s = \tanh(s)v_t$ and $\nabla_{v_s} v_t = 0$.

Since $\text{grad } \rho(s, t) = r'(s)\partial_s = r'(s)v_s$, we have:

$$\text{Hess}\rho(v_s) = \nabla_{v_s}(r'(s)v_s) = r''(s)v_s ,$$

whereas

$$\text{Hess}\rho(v_t) = \nabla_{v_t}(r'(s)v_s) = r'(s)\nabla_{v_t} v_s = r'(s)\tanh(s)v_t .$$

Hence in matrix form,

$$\text{Hess}\rho - \rho\mathbb{1} = \begin{pmatrix} r''(s) - r(s) & 0 \\ 0 & r'(s)\tanh(s) - r(s) \end{pmatrix} .$$

Let us define $g(s) = r'(s)\tanh(s) - r(s)$, so that $g(s)$ is an eigenvalue of b at the point with coordinates (s, t) . Observe that

$$g'(s) = r''(s)\tanh(s) + r'(s)(1 - \tanh^2(s) - 1) = \tanh(s)(r''(s) - r'(s)\tanh(s)) ,$$

hence

$$r''(s) - r(s) = \frac{1}{\tanh(s)}g'(s) + g(s) .$$

This shows that $\text{Hess}\rho - \rho\mathbb{1}$ takes the form:

$$\text{Hess}\rho - \rho\mathbb{1} = \begin{pmatrix} \frac{1}{\tanh(s)}g'(s) + g(s) & 0 \\ 0 & g(s) \end{pmatrix} ,$$

and therefore we obtain the following ODE on g :

$$\det(\text{Hess}\rho - \rho\mathbb{1}) = g(s) \left(\frac{1}{\tanh(s)}g'(s) + g(s) \right) = 1 .$$

By algebraic manipulation (since, from the form of the equation, any solution g cannot vanish at any point) this equation is equivalent to:

$$g'(s) = \tanh(s) \left(\frac{1}{g(s)} - g(s) \right) ,$$

as claimed. □

Solutions of ODE and definition of the diffeomorphisms \mathfrak{g}_α . Let us now study the solutions of Equation (15). We remark that $g(s) \equiv 1$ is a solution. Assuming $g(s) \neq 1$, we obtain the following form:

$$\frac{g'(s)g(s)}{1 - g(s)^2} = \tanh(s) .$$

Hence we can distinguish three cases:

- $g(s) = 1$ is a solution for which $b = \mathbb{1}$.
- If $g(s) < 1$, then $1 - g(s)^2 > 0$ and we obtain

$$-\frac{1}{2} \log(1 - g(s)^2) = \log \cosh(s) + c ,$$

which leads to

$$1 - g(s)^2 = \frac{e^c}{\cosh^2(s)} = e^c(1 - \tanh^2(s)) .$$

- If $g(s) > 1$, then $1 - g(s)^2 < 0$ and we obtain

$$\frac{1}{2} \log(g(s)^2 - 1) = -\log \cosh(s) + c ,$$

which leads to

$$g(s)^2 - 1 = \frac{e^c}{\cosh^2(s)} = e^c(1 - \tanh^2(s)) .$$

In conclusion, we have the solutions

$$g_a(s) = \pm \sqrt{1 + a(1 - \tanh^2(s))} ,$$

where we can allow any value of $a \in \mathbb{R}$ and include all the three cases. We will only consider the positive root, since we can assume that b is positive definite (and the other choice of sign corresponds to changing b with $-b$).

Moreover, observe that g_a is defined on \mathbb{R} only if $a > -1$. Here we will only consider $a \geq 0$, so that g_a is defined on \mathbb{R} and $g(s) \geq 1$ — namely, recalling Lemma 4.2, $g(s)$ is the largest eigenvalue of b . To study the minimal Lagrangian diffeomorphisms associated to the tensors b obtained in this way, we first need to analyze the metric $g_{\mathbb{H}^2}(b, b)$.

Lemma 4.3. *Let $g_a : \mathbb{R} \rightarrow \mathbb{R}$, for $a \geq 0$, be defined by*

$$g_a(s) = \sqrt{1 + a(1 - \tanh^2(s))} ,$$

and let b_a be the self-adjoint, positive definite, Codazzi tensor

$$b_a = \begin{pmatrix} \frac{1}{g_a(s)} & 0 \\ 0 & g_a(s) \end{pmatrix}$$

in the (s, t) -coordinates of \mathbb{H}^2 . Then $g_{\mathbb{H}^2}(b_a, b_a)$ has constant curvature -1 and is complete, hence it is isometric to \mathbb{H}^2 . Moreover, $\ell_0 = \{s = 0\}$ is a geodesic, while the $\{t = t_0\}$ are geodesics orthogonal to ℓ_0 for every $t_0 \in \mathbb{R}$.

Proof. First, we claim that $g_{\mathbb{H}^2}(b_a, b_a)$ is a hyperbolic metric. In fact, since $d^\nabla b = 0$, we have the following formula for the curvatures κ (see [Lab92] or [KS07]):

$$\kappa_{g_{\mathbb{H}^2}(b_a, b_a)} = \frac{\kappa_{g_{\mathbb{H}^2}}}{\det b_a}$$

and therefore $g_{\mathbb{H}^2}(b_a, b_a)$ has curvature -1 since $g_{\mathbb{H}^2}$ does, and $\det b_a = 1$.

Now, in the (s, t) -coordinates, the metric $g_{\mathbb{H}^2}(b_a, b_a)$ takes the form:

$$g_{\mathbb{H}^2}(b_a, b_a) = \frac{1}{g_a(s)^2} ds^2 + g_a(s)^2 \cosh^2(s) dt^2 .$$

From the form of $g_a(s)$, it turns out that $g_a(-s) = g_a(s)$ and therefore $(s, t) \mapsto (-s, t)$ is an isometry for $g_{\mathbb{H}^2}(b_a, b_a)$. This implies that $\ell_0 = \{s = 0\}$ is a geodesic. It is also complete, since the arclength parameter is $t' = t/g_a(0)$, which is defined in \mathbb{R} .

Also the reflections $(s, t) \mapsto (s, t_0 - t)$ are isometries for every $t_0 \in \mathbb{R}$. Therefore the lines $\{t = t_0\}$ are geodesics for every t_0 , and are clearly orthogonal to ℓ_0 . Finally, all such geodesics are complete, as a consequence of the fact that $g_a(s)$ is bounded from below by 1 (recall that $a \geq 0$). This shows that $g_{\mathbb{H}^2}(b_a \cdot, b_a \cdot)$ is isometric to \mathbb{H}^2 , and the isometries $(s, t) \mapsto (s, t + t_0)$ are hyperbolic isometries preserving ℓ_0 . \square

In the following definition, we use the same notation as in Lemma 4.3.

Definition 4.4. For every $a \geq 0$, we define $\mathfrak{g}_a : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ to be the isometry for the metrics $g_{\mathbb{H}^2}(b_a \cdot, b_a \cdot)$ on the source and $g_{\mathbb{H}^2}$ on the target, which (in the (s, t) -coordinates) maps the geodesic $\ell_0 = \{s = 0\}$ of $g_{\mathbb{H}^2}(b_a \cdot, b_a \cdot)$ to the geodesic $\ell = \{s = 0\}$ of $g_{\mathbb{H}^2}$.

Computing the cross-ratio norm. Clearly, \mathfrak{g}_a is not an isometry, when one takes the metric $g_{\mathbb{H}^2}$ both on the source and on the target. Up to composing with a hyperbolic translation preserving ℓ , we can assume

$$\mathfrak{g}_a(0, 0) = (0, 0) .$$

In other words, \mathfrak{g}_a can be considered as the identity map of \mathbb{H}^2 , with respect to the metric $g_{\mathbb{H}^2}$ on the source and the metric $g_{\mathbb{H}^2}(b_a \cdot, b_a \cdot)$ on the target. With this convention, \mathfrak{g}_a takes the following boundary value:

Lemma 4.5. *The minimal Lagrangian diffeomorphisms $\mathfrak{g}_a : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ extend to $\mathbb{R}P^1$, with boundary value (in the upper half-plane model) the homeomorphism:*

$$\psi_\alpha(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ -|x|^\alpha & \text{if } x < 0 \end{cases} ,$$

where $\alpha = \sqrt{1+a}$.

Proof. In the upper half-plane model, from Lemma 4.3, \mathfrak{g}_a preserves (setwise) the geodesic $\ell = \{x = 0\}$, and maps geodesics orthogonal to ℓ to geodesics orthogonal to ℓ . Moreover, observe that the coordinate t is the arclength parameter of ℓ (which corresponds to points with coordinates $(0, t)$) for the metric $g_{\mathbb{H}^2}$, while the arclength parameter for the metric $g_{\mathbb{H}^2}(b_a \cdot, b_a \cdot)$ is $t' = t/g_a(0)$. That is, \mathfrak{g}_a stretches the geodesic ℓ by a uniform factor

$$g_a(0) = \sqrt{1+a} .$$

Since moreover \mathfrak{g}_a maps the point i (which has coordinates $(s, t) = (0, 0)$) to itself, \mathfrak{g}_a sends a geodesic orthogonal to ℓ , at (signed along ℓ) distance t from i , to the geodesic orthogonal to ℓ at signed distance $g_a(0)t$ from i .

Since the arclength parameter of ℓ , in the upper half-plane model, is $t = \log y$, \mathfrak{g}_a maps $yi = e^t i$ to $e^{g_a(0)t} i = y^{\sqrt{1+a}} i$. But the endpoints at infinity of a geodesic orthogonal to ℓ through $yi \in \mathbb{H}^2$ are y and $-y$, and therefore \mathfrak{g}_a extends to $\mathbb{R}P^1$ with boundary value

$$\psi_a(x) = x^{\sqrt{1+a}}$$

if $x \geq 0$, and

$$\psi_a(-x) = -(|x|^{\sqrt{1+a}})$$

if $x < 0$. This concludes the proof. \square

We conclude this subsection by giving an estimate on the cross-ratio norm of ψ_α .

Lemma 4.6. *Let $\psi_\alpha : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ be the orientation-preserving homeomorphism:*

$$\psi_\alpha(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ -|x|^\alpha & \text{if } x < 0 \end{cases} .$$

Then:

- $\|\psi_\alpha\|_{cr} \geq \log(2^\alpha - 1)$,
- As $\alpha \rightarrow 1^+$, $\|\psi_\alpha\|_{cr} \geq C(\alpha - 1) + O(\alpha - 1)^2$,
where $C = \sqrt{2} (\log(\sqrt{2} + 1) - \log(\sqrt{2} - 1)) \approx 2.49$.

Proof. For the first inequality, consider the quadruple $Q = (x_1, x_2, x_3, x_4)$ with:

$$x_1 = 0 \quad x_2 = a \quad x_3 = 2a \quad x_4 = \infty ,$$

(for some $a > 0$), which is easily seen to be symmetric. Then

$$cr(\psi_\alpha(Q)) = -\frac{(2a)^\alpha - a^\alpha}{a^\alpha} = -(2^\alpha - 1) ,$$

hence $|\log |cr(\psi_\alpha(Q))|| = \log(2^\alpha - 1)$. Since $\|\psi_\alpha\|_{cr} \geq |\log |cr(\psi_\alpha(Q))||$ by Definition 2.1, this concludes the first inequality.

To prove the second inequality, consider the following quadruple of points Q' :

$$x_1 = -\sqrt{2} - 1 \quad x_2 = 1 - \sqrt{2} \quad x_3 = \sqrt{2} - 1 \quad x_4 = \sqrt{2} + 1 .$$

By a direct computation using Equation (3), we have $cr(Q') = -1$. On the other hand,

$$\begin{aligned} cr(\psi_\alpha(Q')) &= \frac{((\sqrt{2} + 1)^\alpha + (\sqrt{2} + 1)^\alpha)((\sqrt{2} - 1)^\alpha + (\sqrt{2} - 1)^\alpha)}{-((\sqrt{2} + 1)^\alpha - (\sqrt{2} - 1)^\alpha)^2} \\ &= -\frac{4(\sqrt{2} + 1)^\alpha(\sqrt{2} - 1)^\alpha}{((\sqrt{2} + 1)^\alpha - (\sqrt{2} - 1)^\alpha)^2} = -\frac{4}{((\sqrt{2} + 1)^\alpha - (\sqrt{2} - 1)^\alpha)^2} . \end{aligned}$$

Now, observe that $(\sqrt{2} + 1)^\alpha - (\sqrt{2} - 1)^\alpha \geq 2$. Hence

$$\begin{aligned} |\log |cr(\psi_\alpha(Q'))|| &= -\log \left(\frac{4}{((\sqrt{2} + 1)^\alpha - (\sqrt{2} - 1)^\alpha)^2} \right) \\ &= 2 \log \left((\sqrt{2} + 1)^\alpha - (\sqrt{2} - 1)^\alpha \right) - \log 4 \\ &= \left((\sqrt{2} + 1) \log(\sqrt{2} + 1) - (\sqrt{2} - 1) \log(\sqrt{2} - 1) \right) (\alpha - 1) + O(\alpha - 1)^2 \\ &= \sqrt{2} \left(\log(\sqrt{2} + 1) - \log(\sqrt{2} - 1) \right) (\alpha - 1) + O(\alpha - 1)^2 . \end{aligned}$$

Since again $\|\psi_\alpha\|_{cr} \geq |\log |cr(\psi_\alpha(Q'))||$, this concludes the proof. \square

Remark 4.7. An asymptotic expansion of the function $\log(2^\alpha - 1)$ which appears in the first inequality gives:

$$\|\psi_\alpha\|_{cr} \geq C(\alpha - 1) + O(\alpha - 1)^2 ,$$

where one obtains $C = 2 \log 2 \approx 1.39$. Hence the second choice of quadruple Q in the proof of Lemma 4.6 is necessary to refine the value of the constant C .

The second inequality could be achieved by taking any quadruple Q' , satisfying $cr(Q') = -1$, which is symmetric with respect to reflection in $x = 0$. In fact, the homeomorphism ψ_α conjugates the action of the Möbius transformation $x \mapsto \lambda x$ to the Möbius transformation $x \mapsto \lambda^\alpha x$. Therefore all such quadruples Q' are equivalent in terms of cross-ratio distortion of ψ_α . Picking a quadruple of the form

$Q' = (-b, -a, a, b)$ for some $b > a > 0$, a direct computation shows that $cr(Q') = -1$ if and only if $b = (3 + 2\sqrt{2})a = (\sqrt{2} + 1)^2 a$. This motivates the choice of the quadruple Q' in the proof, which corresponds to the choice $a = \sqrt{2} - 1$ and thus $b = (\sqrt{2} + 1)^2(\sqrt{2} - 1) = \sqrt{2} + 1$.

Computing the maximal dilatation. On the other hand, we have the following value for the maximal dilatation of \mathfrak{g}_a :

Lemma 4.8. *The maximal dilatation of the minimal Lagrangian diffeomorphisms $\mathfrak{g}_a : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is:*

$$K(\mathfrak{g}_a) = 1 + a .$$

Proof. As for the proof of Lemma 3.7, the eccentricity of the ellipses $d\mathfrak{g}_a(\mathbb{S}^1)$ at every point can be computed as the ratio between the two eigenvalues of b_a , which are $g_a(s)$ and $1/g_a(s)$. Hence the eccentricity is:

$$g_a(s)^2 = 1 + a(1 - \tanh^2(s)) ,$$

which is maximized for $s = 0$. Hence we have

$$K(\mathfrak{g}_a) = g_a(0)^2 = 1 + a ,$$

as claimed. □

Comparison when $a \rightarrow 0$. We have showed that the minimal Lagrangian diffeomorphism \mathfrak{g}_a coincides with the unique minimal Lagrangian extension $\mathfrak{g}_a = f_{\psi_\alpha}$ of the power function ψ_α , for $\alpha = \sqrt{1+a}$. We are now ready to conclude the study of the ratio $\log K(f_{\psi_\alpha})/||\psi_\alpha||_{cr}$. Let us start by the case $\alpha \rightarrow 1$.

Theorem B (Case $\alpha \rightarrow 1$). *Let $f_{\psi_\alpha} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the minimal Lagrangian diffeomorphisms which extend the power function ψ_α . Then*

$$\limsup_{\alpha \rightarrow 1^+} \frac{\log K(f_{\psi_\alpha})}{||\psi_\alpha||_{cr}} \leq \frac{\sqrt{2}}{\log(3 + \sqrt{2})} .$$

Proof. We shall study the limit of the ratio $\log K(f_{\psi_\alpha})/||\psi_\alpha||_{cr}$ as $\alpha \rightarrow 1^+$ — that is, as f_{ψ_α} approaches the identity. On the one hand, we have from Lemma 4.6:

$$||\psi_\alpha||_{cr} \geq \sqrt{2} \left(\log(\sqrt{2} + 1) - \log(\sqrt{2} - 1) \right) (\alpha - 1) + O(\alpha - 1)^2 ,$$

while on the other, recalling that $\alpha = \sqrt{1+a}$:

$$\log K(f_{\psi_\alpha}) = \log(1+a) = \log \alpha^2 = 2 \log \alpha = 2(\alpha - 1) + O((\alpha - 1)^2) .$$

Therefore one has:

$$\limsup_{\alpha \rightarrow 1^+} \frac{\log K(f_{\psi_\alpha})}{||\psi_\alpha||_{cr}} \leq \frac{2}{\sqrt{2}(\log(\sqrt{2} + 1) - \log(\sqrt{2} - 1))} = \frac{\sqrt{2}}{\log(3 + \sqrt{2})} \approx 0.80 ,$$

as claimed. □

Remark 4.9. Numerical computation seems to show that $||\psi_\alpha||_{cr}$ is actually achieved at the quadruple Q' introduced in the proof of Lemma 4.6. The value $\sqrt{2}/\log(3 + \sqrt{2}) \approx 0.80$ is thus expected to be the actual value of the limit of $\log K(f_{\psi_\alpha})/||\psi_\alpha||_{cr}$ as $\alpha \rightarrow 1^+$. This would have as a consequence an improvement on the lower bound for the limit superior in Corollary E.

However, computational difficulties are encountered when trying to prove that the second estimate of Lemma 4.6 is sharp, namely, that the quadruple Q' achieves the

cross-ratio norm (see also Remark 4.7), which is also an expected result given the symmetries of the problem.

A nice way to produce an upper bound on the cross-ratio norm $\|\psi_\alpha\|_{cr}$ uses an argument similar to [HM12, Proposition 2]. Namely, one can estimate $\|\psi_\alpha\|_{cr}$ in terms of the quantity $K_q(\psi_\alpha)$, which is defined as

$$K_q(\psi_\alpha) = \sup_Q \frac{M(\psi_\alpha(Q))}{M(Q)},$$

where M denotes the conformal modulus of the quadrilateral with vertices in the quadruple Q . Following [LV73], one can then estimate K_q by the maximal dilatation of the extremal extension, which by [Str57, Str62] is known to be equal to α . However, performing the precise computations (we omit the details here) leads to inequality:

$$\liminf_{\alpha \rightarrow 1^+} \frac{\log K(f_{\psi_\alpha})}{\|\psi_\alpha\|_{cr}} \geq 2 \left(\frac{E(1/\sqrt{2})}{K(1/\sqrt{2})} \right) - 1 \approx 0.46,$$

which (unfortunately) does not improve the lower bound $1/2$ which we already know from Theorem 2.8 (see Equation (5)).

Comparison when $a \rightarrow +\infty$. Let us finally consider the case $a \rightarrow +\infty$.

Theorem B (Case $\alpha \rightarrow +\infty$). *Let $f_{\psi_\alpha} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the minimal Lagrangian diffeomorphisms which extend the power function ψ_α . Then*

$$\lim_{\alpha \rightarrow +\infty} \frac{\log K(f_{\psi_\alpha})}{\|\psi_\alpha\|_{cr}} = 0.$$

Proof. First, observe that $\alpha = \sqrt{1+a}$ tends to $+\infty$ as $a \rightarrow +\infty$, and $f_{\psi_\alpha} = \mathfrak{g}_a$. From Lemma 4.6, we have

$$\|\psi_\alpha\|_{cr} \geq \log(2^\alpha - 1).$$

As in the previous proof, from Lemma 4.8 we have

$$\log K(f_{\psi_\alpha}) = 2 \log \alpha.$$

Hence we get

$$\lim_{\alpha \rightarrow +\infty} \frac{\log K(f_{\psi_\alpha})}{\|\psi_\alpha\|_{cr}} \leq \lim_{\alpha \rightarrow +\infty} \frac{2 \log \alpha}{\log(2^\alpha - 1)} = 0.$$

This concludes the proof. □

5. CONCLUSIONS

As already observed in Section 2, see Theorem 2.7 and Theorem 2.8, there exists a universal constant $C > 0$ such that, for every quasimetric homeomorphism φ , the minimal Lagrangian extension f_φ satisfies:

$$\frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \leq C.$$

In this section we apply Theorem A and Theorem B to give some conditions on the best possible constant C , in particular when $\|\varphi\|_{cr} \rightarrow 0$ and $\|\varphi\|_{cr} \rightarrow +\infty$.

Let us start by the case $\|\varphi\|_{cr} \rightarrow 0$.

Corollary E. *If $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ denotes the minimal Lagrangian extension of a quasimetric homeomorphism $\varphi : \mathbb{R}P^1 \rightarrow \mathbb{R}P^1$, then:*

$$\liminf_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \in \left[\frac{1}{2}, \frac{2}{\pi} \right] \quad \text{and} \quad \limsup_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \in \left[\frac{2}{\pi}, +\infty \right).$$

Proof. The inequality

$$\liminf_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \geq \frac{1}{2}$$

follows from Theorem 2.8, see Equation (5).

On the other hand, from Theorem A the extensions f_{φ_λ} of the simple earthquake φ_λ satisfy:

$$\lim_{\alpha \rightarrow 1^+} \frac{\log K(f_{\psi_\alpha})}{\|\psi_\alpha\|_{cr}} = \frac{2}{\pi}.$$

Hence this implies

$$\liminf_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \leq \frac{2}{\pi}.$$

For the second claim, we know from Theorem 2.7 that

$$\limsup_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}}$$

is finite. Finally, again from Theorem A,

$$\limsup_{\|\varphi\|_{cr} \rightarrow 0^+} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \geq \frac{2}{\pi}.$$

This concludes the proof of the corollary. \square

When $\|\varphi\|_{cr} \rightarrow +\infty$ we can conclude the following corollary.

Corollary F. *If $f_\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ denotes the minimal Lagrangian extension of a quasymmetric homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, then:*

$$\liminf_{\|\varphi\|_{cr} \rightarrow +\infty} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} = 0 \quad \text{and} \quad \limsup_{\|\varphi\|_{cr} \rightarrow +\infty} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \in \left[\frac{\sqrt{2}}{2}, +\infty \right).$$

Proof. From Theorem B, for the extension f_{ψ_α} of the power function ψ_α we have

$$\lim_{\alpha \rightarrow +\infty} \frac{\log K(f_{\psi_\alpha})}{\|\psi_\alpha\|_{cr}} = 0,$$

and therefore necessarily

$$\liminf_{\|\varphi\|_{cr} \rightarrow +\infty} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} = 0.$$

On the other hand, Theorem A shows

$$\lim_{\lambda \rightarrow +\infty} \frac{\log K(f_{\varphi_\lambda})}{\|\varphi_\lambda\|_{cr}} = \frac{\sqrt{2}}{2}$$

for the extension f_{φ_λ} of the simple earthquake φ_λ . Hence

$$\limsup_{\|\varphi\|_{cr} \rightarrow +\infty} \frac{\log K(f_\varphi)}{\|\varphi\|_{cr}} \geq \frac{\sqrt{2}}{2}.$$

Finally, finiteness of the limit superior is again consequence of Theorem 2.7. \square

Let us also remark that, for the extension of the simple earthquake φ_λ , Strebel in [Str57] and [Str62] proved that the extremal extension $f_{\varphi_\lambda}^e$ of φ_λ satisfies $K(f_{\varphi_\lambda}^e) = O(\lambda^2)$ as $\lambda \rightarrow +\infty$. On the other hand, from Theorem A we showed that

$$K(f_{\varphi_\lambda}) \geq e^{C\|\varphi_\lambda\|_{cr}}$$

for large λ and for some $C > 0$. Hence we have:

Corollary G. *The minimal Lagrangian extension of the simple earthquake φ_λ stays exponentially far from being extremal as $\lambda \rightarrow +\infty$.*

Such result is the analogue for minimal Lagrangian extensions of [HM12, Theorem 5], which proves that the Douady-Earle extension of φ_λ stays exponentially far from being extremal as $\lambda \rightarrow +\infty$. More precisely, Hu and Muzician showed (see [HM12, Theorem 3]) that, if $f_{\varphi_\lambda}^{DE}$ denotes the Douady-Earle extension of φ_λ , then

$$\lim_{\lambda \rightarrow +\infty} \frac{\log K(f_{\varphi_\lambda}^{DE})}{\|\varphi_\lambda\|_{cr}} \geq \frac{1}{4},$$

which should be compared with Theorem A of the present paper.

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