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Ergodic BSDE with an unbounded and multiplicative underlying diffusion and application to large time behavior of viscosity solution of HJB equation

Ying Hu* Florian Lemonnier†

Abstract. In this paper, we study ergodic backward stochastic differential equations (EBSDEs for short), for which the underlying diffusion is assumed to be multiplicative and of at most linear growth. The fact that the forward process has an unbounded diffusion is balanced with an assumption of weak dissipativity for its drift. Moreover, the forward equation is assumed to be non-degenerate. Like in [HMR15], we show that the solution of a BSDE in finite horizon T behaves basically as a linear function of T, with a shift depending on the solution of the associated EBSDE, with an explicit rate of convergence. Finally, we apply our results to an ergodic optimal control problem. In particular, we show the large time behaviour of viscosity solution of Hamilton-Jacobi-Bellman equation with an exponential rate of convergence when the undelrying diffusion is multiplicative and unbounded.

Key words. multiplicative and unbounded diffusion, ergodic backward stochastic differential equation, HJB equation, large time behavior, rate of convergence

AMS subject classifications. 60H10, 35B40, 93E20

1 Introduction

We study the following EBSDE in finite dimension and infinite horizon: for all $t, T \in \mathbb{R}_+$ such that $0 \leqslant t \leqslant T < \infty$,

$$Y_t^x = Y_T^x + \int_t^T \left\{ \psi\left(X_s^x, Z_s^x\right) - \lambda \right\} \mathrm{d}s - \int_t^T Z_s^x \, \mathrm{d}W_s, \tag{1}$$

where the unknown is the triplet (Y^x, Z^x, λ), with:

- Y^x a real-valued and progressively measurable process;
- Z^x a $\left(\mathbb{R}^d\right)^*$-valued and progressively measurable process;
- λ a real number.

The given data of our equation consists in:

- W, a $\mathbb{R}^d$-valued standard Brownian motion;
- $x \in \mathbb{R}^d$;
- X^x, a $\mathbb{R}^d$-valued process, starting from x, and solution of the SDE: for all $t \in \mathbb{R}_+$,

$$X_t^x = x + \int_0^t \Xi\left(X_s^x\right) \mathrm{d}s + \int_0^t \sigma\left(X_s^x\right) \mathrm{d}W_s; \tag{2}$$

- $\psi : \mathbb{R}^d \times \left(\mathbb{R}^d\right)^* \to \mathbb{R}$ a measurable function.

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This class of ergodic BSDEs was first introduced by Fuhrman, Hu and Tessitore in [FHT09], in order to study optimal ergodic control problem. In that paper, the main assumption is the strong dissipativity of Ξ, that is to say:

$$\exists \eta > 0, \forall x, x' \in \mathbb{R}^d, \langle \Xi(x) - \Xi(x'), x - x' \rangle \leqslant -\eta |x - x'|^2.$$

The strong dissipativity assumption was then dropped off in [DHT11] and replaced by a weak dissipativity assumption: in other words, Ξ can be written as the sum of a dissipative function and of a bounded function.

A few years later, in [HMR15], under similar assumptions, the large time behaviour of BSDEs in finite horizon T: for all $t \in [0, T]$,

$$Y_t^{T,x} = g\left(X_T^x\right) + \int_t^T \psi\left(X_s^x, Z_s^{T,x}\right) \mathrm{d}s - \int_t^T Z_s^{T,x} \mathrm{d}W_s, \tag{3}$$

was studied and linked with ergodic BSDEs. The authors prove the existence of a constant $L \in \mathbb{R}$, such that for all $x \in \mathbb{R}^d$,

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T \to \infty}{\longrightarrow} L;$$

moreover, they obtain an exponential rate on convergence.

Those BSDEs are linked with Hamilton-Jacobi-Bellman equations. Indeed, the solution of equation (3) can be written as $Y_t^{T,x} = u\left(T - t, X_t^x\right)$, where u is the solution of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) &= \mathcal{L}u(t,x) + \psi\left(x, \nabla u(t,x)\sigma(x)\right) & \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) &= g(x) & \forall x \in \mathbb{R}^d, \end{cases} \tag{4}$$

and where $\mathcal{L}$ is the generator of the Kolmogorov semigroup of X^x, solution of (2). Also, the ergodic BSDE (1) admits a solution such that $Y_t^x = v\left(X_t^x\right)$ satisfies

$$\mathcal{L}v(x) + \psi\left(x, \nabla v(x)\sigma(x)\right) - \lambda = 0 \qquad \forall x \in \mathbb{R}^d. \tag{5}$$

Because the functions u and v built by solving BSDEs or EBSDEs are not, in general, of class $\mathcal{C}^2$, we only solve these equations in a weak sense. In [HMR15], the authors prove that, under their assumptions, the function v is of class $\mathcal{C}^1$ and they are able to work with mild solutions. But in this article, we only prove the continuity of v, so we link the solution of the EBSDE (1) with the viscosity solution of the ergodic PDE (5). Viscosity solutions of PDEs have already been widely studied (see [FIL06]).

The large time behaviour of such PDEs has already been widely studied, for example in [CGMT15] and the references therein: but we do not make here the same assumptions. The authors work on the torus $\mathbb{R}^n/\mathbb{Z}^n$, and they need the Hamiltonian ψ to be uniformly convex; on the other side, they are less restrictive about the matrix $\sigma(x)\sigma(x)^*$ which only needs to be nonnegative definite. They prove the convergence of $u(t, \bullet) - v - \lambda t$ as $t \to \infty$, but they do not have any rate of convergence. The same method has been used in [MT15] to study the large time behaviour of the solution to the obstacle problem for degenerate viscous Hamilton-Jacobi equations.

But in [FHT09], [DHT11] or [HMR15], the diffusion σ of the forward process X^x is always supposed to be constant. The main contribution in this article is that the function σ is here assumed to be Lipschitz continuous, invertible, and such that σ^{-1} is bounded. Moreover, we will need the linear growth of σ to be small enough, regarding the weak dissipativity of the drift Ξ: consequently, in average, the forward process X^x is attracted to the origin. The key point is a coupling estimate for a multiplicative noise obtained by the irreducibility of X^x (see [DZ96], [CF15] or [EM01]): the proof (see Theorem 7) is different from [Mad15], where a bridge was used in equation (B.6), which required σ to fulfill $\langle y, \sigma(x+y) - \sigma(x) \rangle \leqslant \Lambda |y|$ with $\Lambda \geqslant 0$ not too large. Then, we apply this result to auxiliary monotone BSDEs in infinite horizon:

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T \left\{\psi\left(X_s^x, Z_s^{\alpha,x}\right) - \alpha Y_s^{\alpha,x}\right\} \mathrm{d}s - \int_t^T Z_s^{\alpha,x} \mathrm{d}W_s,$$

where $\alpha \in \mathbb{R}_+^*$ (see [BH98] or [Roy04]); we get that $Y_0^{\alpha,x}$ is locally Lipschitz with respect to x, and it allows us to prove the convergence of $\left(Y^{\alpha,x} - Y_0^{\alpha,0}, Z^{\alpha,x}, \alpha Y_0^{\alpha,0}\right)$ to a solution (Y^x, Z^x, λ) of the EBSDE, for every $x \in \mathbb{R}^d$. We can prove uniqueness for λ, but we can only expect uniqueness of (Y^x, Z^x) as measurable functions of X^x.

In [HMR15], the large time behaviour is obtained when ψ is assumed to be Lipschitz continuous with respect to z only. But we could not extend this result when σ is unbounded: we need here ψ to be Lipschitz continuous with respect to $z\sigma(x)^{-1}$ (it is not equivalent to being Lipschitz continuous with respect to z when σ is unbounded). The price to pay is that we also require the existence of a constant K_x such that:

$$|\psi(x,z) - \psi(x',z')| \leqslant K_x|x-x'| + K_z \left|z\sigma(x)^{-1} - z'\sigma(x')^{-1}\right|,$$

in order to keep the implication "$Y_0^{\alpha,x}$ has quadratic growth, so its increments have also quadratic growth". This part of our work is somewhat technical and is presented in the appendix.

The paper is organised as follows. Some notations are introduced in section 2. In section 3, we study a SDE slightly more general than the one satisfied by X^x: indeed, it will be the SDE satisfied by X^x after a change of probability space, due to Girsanov's theorem. In section 4, we prove that, in a way, our EBSDE admits a unique solution, which allows us also to solve an ergodic PDE. The link between this solution and the solutions of some finite horizon BSDEs is presented in section 5. The section 6 is devoted to an application of our results to an optimal ergodic control problem. Finally, the appendix presents how we keep the estimates of the increments of $Y_0^{\alpha,x}$ with respect to x, despite our twisted Lipschitz assumption on ψ.

2 Notations

Throughout this paper, $(W_t)_{t\geqslant 0}$ will denote a d-dimensional Brownian motion, defined on a parobability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geqslant 0$, set $\mathcal{F}_t$ the σ-algebra generated by W_s, $0 \leqslant s \leqslant t$, and augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$. We write elements of $\mathbb{R}^d$ as vectors, and the star * stands for transposition. The Euclidean norm on $\mathbb{R}^d$ and $\left(\mathbb{R}^d\right)^*$ will be denoted by $|\bullet|$. For a matrix of $\mathcal{M}_d(\mathbb{R})$, we denote by $|\bullet|_F$ its Frobenius norm, that is to say the square root of the sum of the square of its coefficients. The Lipschitz constant of a function is written $\|\bullet\|_{\text{lip}}$ and we also write $\|f\|_\infty := \sup_x |f(x)|$ for a bounded function f. Finally, for any process, $|\bullet|_{t,T}^{*,p}$ stands for $\sup\limits_{s\in[t,T]} |\bullet_s|^p$, and the exponent p is omitted when equal to 1.

3 The SDE

We consider the SDE:

$$\left\{\begin{array}{rcl} \mathrm{d}X_t^x &=& \Xi\left(t, X_t^x\right)\mathrm{d}t + \sigma\left(X_t^x\right)\mathrm{d}W_t, \\ X_0^x &=& x, \end{array}\right. \tag{6}$$

Assumption 1

- $\Xi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous w.r.t. x uniformly in t: precisely, there exists some constants ξ_1 and ξ_2 such that for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, $|\Xi(t,x)| \leqslant \xi_1 + \xi_2|x|$;
- Ξ is weakly dissipative, *i.e.* $\Xi(t,x) = \xi(x) + b(t,x)$, where ξ is dissipative (that is to say $\langle \xi(x), x\rangle \leqslant -\eta|x|^2$, with $\eta > 0$) and locally Lipschitz and b is bounded; this way, $\exists \eta_1, \eta_2 > 0$, $\forall t \in \mathbb{R}_+$, $\forall x \in \mathbb{R}^d$, $\langle \Xi(t,x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$;
- $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ is Lipschitz continuous, the function $x \mapsto \sigma(x)^{-1}$ is bounded and $\forall x \in \mathbb{R}^d$, $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$.

Remark 2. If we know that σ has a sublinear growth, *i.e.* $|\sigma(x)|_F \leqslant C\left(1+|x|^{\alpha}\right)$ with $\alpha \in (0,1)$, then, by Young's inequality, we have $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$, and r_2 can be chosen as close to 0 as possible.

Theorem 3

Under Assumption 1, for all $p \in [2,+\infty)$ and for all $T \in (0,+\infty)$, there exists a unique process $X^x \in \mathrm{L}^p_{\mathcal{P}}\left(\Omega, \mathcal{C}\left([0,T],\mathbb{R}^d\right)\right)$ strong solution to (6).

Proof :
See Theorem 7.4 of [DZ92]. The main idea for the existence is to use a fix point theorem. ■

Proposition 4

Under Assumption 1, for all $p \in (0,+\infty)$, and $T > 0$, we have:

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\left[|X_t^x|^p\right] \leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |X_t^x|^p\right] \leqslant C\left(1+|x|^p\right),$$

where C only depends on p, T, r_1, r_2, ξ_1 and ξ_2.

Proof :
This is a straightforward consequence of Burkholder-Davis-Gundy's inequality and Gronwall's lemma. ■

Proposition 5

Suppose that Assumption 1 holds true. Let $p \in (0,+\infty)$, $T > 0$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ a bounded function. Thanks to Girsanov theorem, the process $\widetilde{W}_t^x = W_t - \int_0^t \gamma\left(s, X_s^x\right) \mathrm{d}s$ is a Brownian motion under the probability $\widetilde{\mathbb{P}}^{x,T}$, on the interval $[0,T]$. When $\sqrt{r_2}\|\gamma\|_\infty + \dfrac{(p\vee 2)-1}{2} r_2 < \eta_2$, we get:

$$\sup_{T\geqslant 0} \widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right] \leqslant C\left(1+|x|^p\right),$$

where C only depends on p, η_1, η_2, r_1, r_2 and $\|\gamma\|_\infty$.

Proof : The case $p = 2$ is easier, we will treat the case $p > 2$.
We will suppose that we have $\sqrt{r_2}\|\gamma\|_\infty + (p-1)\dfrac{r_2}{2} < \eta_2$.
By Itô's formula, we get (where $0 \leqslant t \leqslant T$):

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] \\
&\quad = p\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left\langle X_t^x, \Xi\left(t,X_t^x\right) + \sigma\left(X_t^x\right)\gamma\left(t,X_t^x\right)\right\rangle\right] + \frac{p}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left|\sigma\left(X_t^x\right)\right|_F^2\right] \\
&\qquad\qquad + \frac{p(p-2)}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-4}\sum_{i=1}^d\left(\sum_{j=1}^d \left(X_t^x\right)_j \sigma\left(X_t^x\right)_{i,j}\right)^2\right] \\
&\quad \leqslant p\widetilde{\mathbb{E}}^{x,T}\left[\eta_1|X_t^x|^{p-2} - \eta_2|X_t^x|^p + |X_t^x|^{p-1}\left|\sigma\left(X_t^x\right)\right|_F\left|\gamma\left(t,X_t^x\right)\right|\right] + \frac{p}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left|\sigma\left(X_t^x\right)\right|_F^2\right] \\
&\qquad\qquad + \frac{p(p-2)}{2}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\left|\sigma\left(X_t^x\right)\right|_F^2\right] \\
&\quad \leqslant \left(-p\eta_2 + p\sqrt{r_2}\|\gamma\|_\infty + \frac{p(p-1)}{2}r_2\right)\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] + p\sqrt{r_1}\|\gamma\|_\infty\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-1}\right] \\
&\qquad\qquad + \left(p\eta_1 + \frac{p(p-1)}{2}r_1\right)\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^{p-2}\right].
\end{aligned}$$

But, using Young's inequality, we can show, for every $\varepsilon > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] \leqslant \lambda_\varepsilon\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right] + \sqrt{r_1}\|\gamma\|_\infty\varepsilon^{-p} + \left(2\eta_1 + (p-1)r_1\right)\varepsilon^{-\frac{p}{2}}.$$

We set $\lambda_\varepsilon := -p\eta_2 + p\sqrt{r_2}\|\gamma\|_\infty + \frac{p(p-1)}{2}r_2 + (p-1)\sqrt{r_1}\|\gamma\|_\infty \varepsilon^{\frac{p}{p-1}} + \left((p-2)\eta_1 + \frac{(p-2)(p-1)}{2}r_1\right)\varepsilon^{\frac{p}{p-2}}$ (for ε small enough) and this quantity is negative. Hence, for ε small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda_\varepsilon t}\widetilde{\mathbb{E}}^{x,T}\left[|X_t^x|^p\right]\right) \leqslant \mathrm{e}^{-\lambda_\varepsilon t}\left(\sqrt{r_1}\|\gamma\|_\infty \varepsilon^{-p} + (2\eta_1 + (p-1)r_1)\,\varepsilon^{-\frac{p}{2}}\right)$$
$$\widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right] \leqslant |x|^p + \frac{1}{|\lambda_\varepsilon|}\left(\sqrt{r_1}\|\gamma\|_\infty \varepsilon^{-p} + (2\eta_1 + (p-1)r_1)\,\varepsilon^{-\frac{p}{2}}\right).$$

We have been able to conclude the case $p > 2$ because $\lambda_\varepsilon < 0$. When $p \in (0,2)$, and under the Assumption $\sqrt{r_2}\|\gamma\|_\infty + \frac{r_2}{2} < \eta_2$, we have $\sup_{T\geqslant 0}\widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right] \leqslant C\,(1+|x|^p)$. We use Jensen's inequality (with the concavity of $x \mapsto |x|^{\frac{p}{2}}$) and the inequality $(1+z)^\alpha \leqslant 1 + z^\alpha$ for $z \geqslant 0$ and $\alpha \in (0,1)$. ■

Proposition 6

Under Assumption 1, the process X^x is irreducible, that is to say:

$$\forall t > 0,\ \forall x, z \in \mathbb{R}^d,\ \forall r > 0,\ \mathbb{P}\left(|X_t^x - z| < r\right) > 0.$$

Proof :

The proof is very similar to Theorem 7.3.1 of [DZ96]. ■

Theorem 7

Suppose Assumption 1 holds true. Let $\mu \geqslant 2$ be such that $\frac{\mu-1}{2}r_2 < \eta_2$ and $\phi : \mathbb{R}^d \to \mathbb{R}^d$ a measurable function, satisfying $|\phi(x)| \leqslant c_\phi\,(1+|x|^\mu)$. Then we have,

$$\forall x, y \in \mathbb{R}^d,\ \forall t \geqslant 0,\ |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leqslant \widehat{c}c_\phi\,(1+|x|^\mu+|y|^\mu)\,\mathrm{e}^{-\widehat{\nu}t},$$

where $\mathcal{P}_t[\phi](x) = \mathbb{E}\left[\phi\left(X_t^x\right)\right]$ (Kolmogorov semigroup of X) and $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2 and μ.

Proof :

This proof is based on the application of the Theorem A.2 of [EM01]. Hypoellipticity is a consequence of the boundedness of σ^{-1}. In order to show that the monomial function $V : x \mapsto |x|^\mu$ is a Lyapunov function, it suffices to show that it satisfies for all $x \in \mathbb{R}^d$: $LV(x) < -aV(x) + b\mathbb{1}_C(x)$, where a and b are positive constants, C a compact set and L the generator of X. We have:

$$\begin{aligned} LV(x) &= \mu|x|^{\mu-2}\langle x, \Xi(0,x)\rangle + \frac{\mu}{2}|x|^{\mu-2}|\sigma(x)|_F^2 + \frac{\mu(\mu-2)}{2}|x|^{\mu-4}\sum_{i=1}^d\left(\sum_{j=1}^d x_j\sigma(x)_{i,j}\right)^2 \\ &\leqslant \left(-\mu\eta_2 + \frac{\mu(\mu-1)}{2}r_2\right)|x|^\mu + \left(\mu\eta_1 + \frac{\mu(\mu-1)}{2}r_1\right)|x|^{\mu-2} \end{aligned}$$

One can check that the required inequality can be obtained by setting $C = \overline{\mathcal{B}}(0,R)$, $R = \sqrt{\dfrac{\eta_1 + \frac{\mu-1}{2}r_1}{\eta_2 - \frac{\mu-1}{2}r_2}} + 1$, $a = \mu\eta_2 - \dfrac{\mu(\mu-1)}{2}r_2 - \dfrac{b}{R^2}$ and $b = \mu\eta_1 + \dfrac{\mu(\mu-1)}{2}r_1$.

The last thing we need to apply Theorem A.2 of [EM01] is a consequence of irreducibility of X^x. Then, we get:

$$|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leqslant \widehat{c}c_\phi\,(1+|x|^\mu+|y|^\mu)\,\mathrm{e}^{-\widehat{\nu}t}.$$

See also [MT92] and [MT93] to prove that $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2 and μ. ■

Assumption 8

We can write $\Xi(t,x) = \xi(x) + \rho(t,x)$ where ξ and ρ satisfy the following:

- ρ is a bounded function;
- ξ is Lipschitz continuous and satisfies $\langle \xi(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$, with η_1 and η_2 two positive constants;
- ρ is the pointwise limit of a sequence (ρ_n) of $\mathcal{C}^1$ functions, with bounded derivatives w.r.t. x and uniformly bounded by $\|\rho\|_\infty$.

As before, we still require on σ the following:

- $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ is Lipschitz continuous;
- the function $x \mapsto \sigma(x)^{-1}$ is bounded;
- $\forall x \in \mathbb{R}^d$, $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$.

Corollary 9

Suppose Assumption 8 holds true. Let $\mu \geqslant 2$ be such that $\sqrt{r_2}\|\rho\|_\infty + \dfrac{\mu-1}{2} r_2 < \eta_2$ and $\phi : \mathbb{R}^d \to \mathbb{R}^d$ measurable with $|\phi(x)| \leqslant c_\phi\,(1+|x|^\mu)$. Then we have,

$$\forall x, y \in \mathbb{R}^d,\ \forall t \geqslant 0,\ |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leqslant \widehat{c} c_\phi\,(1+|x|^\mu+|y|^\mu)\,\mathrm{e}^{-\widehat{\nu} t},$$

where $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2, μ and $\|\rho\|_\infty$.

Proof :

We set σ_n a function close to σ on the centered ball of $\mathbb{R}^d$ of radius n, equal to I_d outside the centered ball of radius $n+1$ and of class $\mathcal{C}^1$ with bounded derivatives on $\mathbb{R}^d$; on the ring between the radius n and $n+1$, σ_n is chosen in such a way that $\sigma^{-1}\sigma_n$ is bounded, independently from n. This way, the function $\Xi_n : (t,x) \mapsto \xi(x) + \sigma_n(x)\rho_n(t,x)$ is Lipschitz continuous w.r.t. x. We denote $X^{n,x}$ the solution of the SDE:

$$\begin{cases} \mathrm{d}X_t^{n,x} = \Xi_n\left(t, X_t^{n,x}\right)\,\mathrm{d}t + \sigma\left(X_t^{n,x}\right)\,\mathrm{d}W_t, \\ X_0^{n,x} = x. \end{cases}$$

We can write, for every $\varepsilon > 0$:

$$\langle \Xi_n(t,x), x\rangle \leqslant \eta_1 - \eta_2|x|^2 + (\sqrt{r_1} + \sqrt{r_2}|x|)\,\|\rho\|_\infty|x| \leqslant \left[\eta_1 + \frac{\|\rho\|_\infty^2 r_1}{2\varepsilon^2}\right] - \left[\eta_2 - \sqrt{r_2}\|\rho\|_\infty - \frac{\varepsilon^2}{2}\right]|x|^2.$$

Then, when ε is small enough, for any ϕ with polynomial growth of degree μ, Theorem 7 tells us that $\widehat{c}_n$ and $\widehat{\nu}_n$ are independent of n, and we have:

$$\exists \widehat{c}, \widehat{\nu} > 0,\ \forall n \in \mathbb{N}^*,\ \forall x, y \in \mathbb{R}^d,\ \forall t \geqslant 0,\ \left|\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] - \mathbb{E}\left[\phi\left(X_t^{n,y}\right)\right]\right| \leqslant \widehat{c} c_\phi\,(1+|x|^\mu+|y|^\mu)\,\mathrm{e}^{-\widehat{\nu} t}.$$

Our goal is to take the limit ; let us show that $\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] \underset{n\to\infty}{\longrightarrow} \widetilde{\mathbb{E}}\left[\phi\left(X_t^x\right)\right]$. Let U be the solution of the SDE

$$U_t^x = x + \int_0^t \xi\left(U_s^x\right)\,\mathrm{d}s + \int_0^t \sigma\left(U_s^x\right)\,\mathrm{d}W_s.$$

We can write: $X_t^{n,x} = x + \int_0^t \xi\left(X_s^{n,x}\right)\,\mathrm{d}s + \int_0^t \sigma\left(X_s^{n,x}\right)\,\mathrm{d}W_s^{(n)}$, where $W_t^{(n)} = W_t + \int_0^t \sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s, X_s^{n,x}\right)\,\mathrm{d}s$ is a Brownian motion under the probability $\mathbb{P}^{(n)} = p_T^n\left(X^{n,x}\right)\mathbb{P}$ on $[0,T]$ and where

$$p_t^n\left(X^{n,x}\right) = \exp\left(-\int_0^t \left\langle \sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s, X_s^{n,x}\right), \mathrm{d}W_s\right\rangle - \frac{1}{2}\int_0^t \left|\sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s, X_s^{n,x}\right)\right|^2\,\mathrm{d}s\right).$$

Similarly, $X_t^x = x + \int_0^t \xi\left(X_s^x\right)\,\mathrm{d}s + \int_0^t \sigma\left(X_s^x\right)\,\mathrm{d}W_s^{(\infty)}$, where $W_t^{(\infty)} = \widetilde{W}_t + \int_0^t \rho\left(s, X_s^x\right)\,\mathrm{d}s$ is a Brownian motion under the probability $\mathbb{P}^{(\infty)} = p_T^\infty\left(X^x\right)\widetilde{\mathbb{P}}$ on $[0,T]$ and where

$$p_t^\infty\left(X^x\right) = \exp\left(-\int_0^t \left\langle \rho\left(s, X_s^x\right), \mathrm{d}\widetilde{W}_s\right\rangle - \frac{1}{2}\int_0^t \left|\rho\left(s, X_s^x\right)\right|^2\,\mathrm{d}s\right).$$

By uniqueness in law of the solutions of the SDEs, we get the equalities:

$$\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] = \mathbb{E}^{(n)}\left[p_t^n\left(X^{n,x}\right)^{-1}\phi\left(X_t^{n,x}\right)\right] = \mathbb{E}\left[p_t^n\left(U^x\right)^{-1}\phi\left(U_t^x\right)\right];$$

$$\widetilde{\mathbb{E}}\left[\phi\left(X_t^x\right)\right] = \mathbb{E}^{(\infty)}\left[p_t^\infty\left(X^x\right)^{-1}\phi\left(X_t^x\right)\right] = \mathbb{E}\left[p_t^\infty\left(U^x\right)^{-1}\phi\left(U_t^x\right)\right].$$

But $\rho_n(t,x) \underset{n\to\infty}{\longrightarrow} \rho(t,x)$, so we have $p_t^n\left(U^x\right)^{-1} \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} p_t^\infty\left(U^x\right)^{-1}$. We just have to show that the sequence $\left(p_t^n\left(U^x\right)^{-1}\right)_{n\in\mathbb{N}^*}$ is uniformly integrable, in order to show that $\mathbb{E}\left[\phi\left(X_t^{n,x}\right)\right] \underset{n\to\infty}{\longrightarrow} \widetilde{\mathbb{E}}\left[\phi\left(X_t^x\right)\right]$. We have:

$$\begin{aligned}
&\mathbb{E}\left[\left(p_t^n\left(U^x\right)^{-1}\right)^2\right] \\
&= \mathbb{E}\left[\exp\left(2\int_0^t \left\langle\sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s,X_s^{n,x}\right),\mathrm{d}W_s\right\rangle + \int_0^t\left|\sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s,X_s^{n,x}\right)\right|^2\mathrm{d}s\right)\right] \\
&\leqslant \mathbb{E}\left[\exp\left(\int_0^t\left\langle 4\sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s,X_s^{n,x}\right),\mathrm{d}W_s\right\rangle - \frac{1}{2}\int_0^t\left|4\sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s,X_s^{n,x}\right)\right|^2\mathrm{d}s\right)\right]^{\frac{1}{2}} \\
&\qquad\qquad \mathbb{E}\left[\exp\left(10\int_0^t\left|\sigma\left(X_s^{n,x}\right)^{-1}\sigma_n\left(X_s^{n,x}\right)\rho_n\left(s,X_s^{n,x}\right)\right|^2\mathrm{d}s\right)\right]^{\frac{1}{2}} \\
&\leqslant \exp\left(5t\left(d^2+\left\|\sigma^{-1}\right\|_\infty^2\right)\|\rho\|_\infty\right) < \infty.
\end{aligned}$$

■

4 The EBSDE

We consider the following EBSDE:

$$\forall 0 \leqslant t \leqslant T < \infty,\ Y_t^x = Y_T^x + \int_t^T\left[\psi\left(X_s^x, Z_s^x\right) - \lambda\right]\mathrm{d}s - \int_t^T Z_s^x\,\mathrm{d}W_s \tag{7}$$

and we make on $\psi : \mathbb{R}^d \times \left(\mathbb{R}^d\right)^* \to \mathbb{R}$ and the SDE (6) satisfied by X^x the following assumptions.

Assumption 10

- $\forall x \in \mathbb{R}^d$, $|\psi(x,0)| \leqslant M_\psi(1+|x|)$;
- $\forall x, x' \in \mathbb{R}^d$, $\forall z, z' \in \mathbb{R}^d$, $|\psi(x,z) - \psi(x',z')| \leqslant K_x|x-x'| + K_z\left|z\sigma(x)^{-1} - z'\sigma(x')^{-1}\right|$;
- $\Xi : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous, $|\Xi(x)| \leqslant \xi_1 + \xi_2|x|$;
- $\Xi(x) = \xi(x) + b(x)$, with ξ dissipative and locally Lipschitz and b bounded, and $\langle\Xi(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$ for two positive constants η_1, η_2;
- $\sigma : \mathbb{R}^d \to \mathrm{GL}_d(\mathbb{R})$ is Lipschitz continuous;
- $x \mapsto \sigma(x)^{-1}$ is bounded and $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$;
- $\sqrt{r_2}K_z\left\|\sigma^{-1}\right\|_\infty + \dfrac{r_2}{2} < \eta_2$.

Remark 11. In most papers, the function ψ is assumed to be Lipschitz continuous. We make a slight modification of this assumption in order to have some information about the second and third behaviour (see [HMR15]). But ψ is still Lipschitz continuous w.r.t. z, with a constant equal to $K_z\left\|\sigma^{-1}\right\|_\infty$. Moreover, ψ is still continuous w.r.t. x.

4.1 Existence of a solution

Theorem 3 ensures that the process X^x is well defined. We introduce a new parameter $\alpha > 0$ and we consider a new BSDE of infinite horizon:

$$\forall 0 \leqslant t \leqslant T < \infty,\ Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T \left[\psi\left(X_s^x, Z_s^{\alpha,x}\right) - \alpha Y_s^{\alpha,x}\right] \mathrm{d}s - \int_t^T Z_s^{\alpha,x}\, \mathrm{d}W_s. \tag{8}$$

Lemma 12

Under Assumption 10, for every $x \in \mathbb{R}^d$ and $\alpha > 0$, there exists a unique solution $(Y^{\alpha,x}, Z^{\alpha,x})$ to the BSDE (8), such that $Y^{\alpha,x}$ is a continuous process bounded in L^1 and $Z^{\alpha,x} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty; \left(\mathbb{R}^d\right)^*\right)\right)$. Also, for every $t \geqslant 0$, $|Y_t^{\alpha,x}| \leqslant \dfrac{C}{\alpha}\left(1 + |X_t^x|\right)$, $\mathbb{P}$-a.s., where C only depends on M_ψ, η_1, η_2, r_1, r_2, K_z and $\left\|\sigma^{-1}\right\|_\infty$. The function $v^\alpha : x \mapsto Y_0^{\alpha,x}$ is continuous, and for every $t \geqslant 0$, $Y_t^{\alpha,x} = v^\alpha\left(X_t^x\right)$ $\mathbb{P}$-a.s.

Proof :

For the upperbound for $Y^{\alpha,x}$, see Theorem 2.1 of [Roy04]. The main difference is that we do not require $\psi(\bullet, 0)$ to be bounded. Uniqueness only needs boundedness in L^1 and not almost surely boundedness. Also, in the proof of existence, the constant C is harder to write, because we do the same as in the end of the proof of proposition 5. The continuity of v^α is a consequence of a straightforward adaptation of the Theorem 2.1 of [Roy04] when $\psi(\bullet, 0)$ is no more assumed to be bounded and proposition 2.1 of [EPQ97]. The representation of $Y^{\alpha,x}$ by X^x comes from the Lemma 4.4 of [Roy04]. ■

Lemma 13

Under Assumption 10, for every $\alpha \in (0, 1]$, we have:

$$\forall x, x' \in \mathbb{R}^d,\ \left|v^\alpha(x) - v^\alpha(x')\right| \leqslant C\left(1 + |x| + |x'|\right)|x - x'|, \tag{9}$$

where C only depends on K_x, K_z, M_ψ, α, η_1, η_2, r_1, r_2, $\left\|\sigma^{-1}\right\|_\infty$, ξ_1, $\|\Xi\|_{\mathrm{lip}}$ and $\|\sigma\|_{\mathrm{lip}}$.

Proof :

See Theorem 35, given in the appendix, with $f : (x, y, z) \mapsto \psi(x, z) - \alpha y$ and $g = v^\alpha$. It uses the estimate of Lemma 12. ■

Remark 14.

1. If σ and $\psi(\bullet, 0)$ are bounded and Ξ and σ are of class $\mathcal{C}^1$, then, the Theorem 3.2 of [HT07] tells us that v^α is of class $\mathcal{C}^1$. Then, we can apply the Theorem 3.1 of [MZ02] and write $Z_t^{\alpha,x} = \partial_x v^\alpha\left(X_t^x\right)\sigma\left(X_t^x\right)$.
2. The Lipschitz constant given by the last lemma goes to infinity as α appoximates 0.

Lemma 15

Let $\psi : \mathbb{R}^d \times \left(\mathbb{R}^d\right)^* \to \mathbb{R}$ continuous w.r.t. the first variable and Lipschitz continuous w.r.t. the second one. Let $\zeta, \zeta' : \mathbb{R}_+ \times \mathbb{R}^d \to \left(\mathbb{R}^d\right)^*$ be such that for every $t \geqslant 0$, $\zeta(t, \bullet)$ and $\zeta'(t, \bullet)$ are continuous, and set:

$$\widetilde{\Gamma}(t,x) = \begin{cases} \dfrac{\psi(x,\zeta(t,x)) - \psi(x,\zeta'(t,x))}{|\zeta(t,x) - \zeta'(t,x)|^2}(\zeta(t,x) - \zeta'(t,x)), & \text{if } \zeta(t,x) \neq \zeta'(t,x), \\ 0, & \text{if } \zeta(t,x) = \zeta'(t,x). \end{cases}$$

Then, there exists a uniformly bounded sequence of $\mathcal{C}^1$ functions w.r.t. x with bounded derivatives $\left(\widetilde{\Gamma}_n\right)_{n\geqslant 1}$ (*i.e.* for all n, $\widetilde{\Gamma}_n$ has bounded derivatives w.r.t. x – the bound of derivatives can depend on n – and $\sup_{n\geqslant 1}\left\|\widetilde{\Gamma}_n(t, \bullet)\right\|_\infty < \infty$ for every $t \geqslant 0$), such that $\widetilde{\Gamma}_n \underset{n\to\infty}{\longrightarrow} \widetilde{\Gamma}$ pointwise.

Proof :

See the Lemma 3.7 of [HMR15]: we can approximate the Lipschitz functions by $\mathcal{C}^1$ functions with bounded derivatives and construct a new sequence having the required regularity. ■

Proposition 16

Under Assumption 10, there exists a constant C, such that for every $\alpha \in (0,1]$, we have:

$$\forall x, x' \in \mathbb{R}^d,\ \left|v^\alpha(x) - v^\alpha(x')\right| \leqslant C\left(1 + |x|^2 + |x'|^2\right).$$

The constant C only depends on η_1, η_2, r_1, r_2, K_z, $\left\|\sigma^{-1}\right\|_\infty$ and M_ψ.

Proof :

1. We approximate σ by a sequence $(\sigma^\varepsilon)_{\varepsilon>0}$ of functions satisfying:
   - σ^ε converges pointwise towards σ over $\mathbb{R}^d$;
   - σ^ε is bounded (the bound can depend on ε) and $|\sigma^\varepsilon(x)|_F^2 \leqslant r_1 + r_2|x|^2$;
   - σ^ε is of class $\mathcal{C}^1$ and $\|\sigma^\varepsilon\|_{\text{lip}} \leqslant \|\sigma\|_{\text{lip}}$;
   - $x \mapsto \sigma^\varepsilon(x)^{-1}$ is bounded, the bound is independent of ε.

   We also approximate Ξ by a sequence $(\Xi^\varepsilon)_{\varepsilon>0}$ of $\mathcal{C}^1$ functions which converges uniformly. One can check that the functions Ξ^ε are "uniformly weakly dissipative", because the functions $\Xi^\varepsilon - \xi$ are uniformly bounded. Moreover, ψ is approximated by a sequence (ψ^ε) of functions satisfying:
   - $|\psi^\varepsilon(x,z) - \psi^\varepsilon(x',z')| \leqslant K_x|x - x'| + K_z\left|z\sigma^\varepsilon(x)^{-1} - z'\sigma^\varepsilon(x')^{-1}\right|$;
   - $\psi^\varepsilon(\bullet, 0)$ is bounded;
   - $|\psi^\varepsilon(x,0)| \leqslant M_\psi(1 + |x|)$;
   - (ψ^ε) converges pointwise towards ψ.

   We consider the BSDE:

   $$Y_t^{\varepsilon,\alpha,x} = Y_T^{\varepsilon,\alpha,x} + \int_t^T \left\{\psi^\varepsilon\left(X_s^{\varepsilon,x}, Z_s^{\varepsilon,\alpha,x}\right) - \alpha Y_s^{\varepsilon,\alpha,x}\right\} \mathrm{d}s - \int_t^T Z_s^{\varepsilon,\alpha,x}\,\mathrm{d}W_s,$$

   where the process $X^{\varepsilon,x}$ satisfies the following equation:

   $$X_t^{\varepsilon,x} = x + \int_0^t \Xi^\varepsilon\left(X_s^{\varepsilon,x}\right)\mathrm{d}s + \int_0^t \sigma^\varepsilon\left(X_s^{\varepsilon,x}\right)\mathrm{d}W_s.$$

   This BSDE has a unique solution (see Lemma 12), and $\left|Y_t^{\varepsilon,\alpha,x}\right| \leqslant \dfrac{C}{\alpha}\left(1 + |X_t^x|\right)$ $\mathbb{P}$-a.s. and for every $t \geqslant 0$. Thanks to the previous remark, we can write $Z_t^{\varepsilon,\alpha,x} = \zeta^{\varepsilon,\alpha}\left(X_t^{\varepsilon,x}\right)$ $\mathbb{P}$-a.s. and for a.e. $t \geqslant 0$, and $\zeta^{\varepsilon,\alpha}$ is continuous. We define:

   $$\Gamma^{\varepsilon,\alpha}(x) = \begin{cases} \dfrac{\psi^\varepsilon\left(x, \zeta^{\varepsilon,\alpha}(x)\right) - \psi^\varepsilon(x,0)}{\left|\zeta^{\varepsilon,\alpha}(x)\right|^2}\zeta^{\varepsilon,\alpha}(x), & \text{if } \zeta^{\varepsilon,\alpha}(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

   Then, thanks to Lemma 15, we can approximate $\Gamma^{\varepsilon,\alpha}$ in such a way that we can use Corollary 9. We can rewrite:

   $$-\mathrm{d}Y_t^{\varepsilon,\alpha,x} = \left\{\psi^\varepsilon\left(X_t^{\varepsilon,x}, 0\right) + Z_t^{\varepsilon,\alpha,x}\Gamma^{\varepsilon,\alpha}\left(X_t^{\varepsilon,x}\right)^* - \alpha Y_t^{\varepsilon,\alpha,x}\right\}\mathrm{d}t - Z_t^{\varepsilon,\alpha,x}\,\mathrm{d}W_t.$$

   But $\Gamma^{\varepsilon,\alpha}$ is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$, and there exists a probability $\widehat{\mathbb{P}}^{\varepsilon,\alpha,x,T}$ under which $\widehat{W}_t^{\varepsilon,\alpha,x} = W_t - \int_0^t \Gamma^{\varepsilon,\alpha}\left(X_s^{\varepsilon,x}\right)^*\mathrm{d}s$ is a Brownian motion on $[0,T]$. Finally, we get the equality: $v^{\varepsilon,\alpha}(x) = \widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\mathrm{e}^{-\alpha T}v^{\varepsilon,\alpha}\left(X_T^{\varepsilon,x}\right) + \int_0^T \mathrm{e}^{-\alpha s}\psi^\varepsilon\left(X_s^{\varepsilon,x}, 0\right)\mathrm{d}s\right]$.

   On the one hand, using proposition 5: $\left|\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\mathrm{e}^{-\alpha T}v^{\varepsilon,\alpha}\left(X_T^{\varepsilon,x}\right)\right]\right| \leqslant \mathrm{e}^{-\alpha T}\dfrac{C}{\alpha}\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\left|X_T^{\varepsilon,x}\right|\right] \underset{T\to\infty}{\longrightarrow} 0.$

On the other hand, X^ε satisfies the following SDE under $\widehat{\mathbb{P}}^{\varepsilon,\alpha,x,T}$:

$$\mathrm{d}X_t^{\varepsilon,x} = \left[\Xi^\varepsilon\left(X_t^{\varepsilon,x}\right) + \sigma^\varepsilon\left(X_t^{\varepsilon,x}\right)\Gamma^{\varepsilon,\alpha}\left(X_t^{\varepsilon,x}\right)^*\right]\mathrm{d}t + \sigma^\varepsilon\left(X_t^{\varepsilon,x}\right)\mathrm{d}\widehat{W}_t^{\varepsilon,\alpha,x}.$$

Thanks to corollary 9, we get:

$$\left|\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,T}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x},0\right)\right] - \widehat{\mathbb{E}}^{\varepsilon,\alpha,x',T}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x'},0\right)\right]\right| \leqslant 2M_\psi\widehat{c}\mathrm{e}^{-\widehat{\nu}t}\left(1+|x|^2+|x'|^2\right),$$

where $\widehat{c}$ and $\widehat{\nu}$ only depend on η_1, η_2, r_1, r_2, K_z and $\left\|\sigma^{-1}\right\|_\infty$. As a consequence, we get:

$$\begin{aligned}\left|v^{\varepsilon,\alpha}(x) - v^{\varepsilon,\alpha}(x')\right| &= \lim_{T\to\infty}\left|\int_0^T \mathrm{e}^{-\alpha t}\left(\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x},0\right)\right] - \widehat{\mathbb{E}}^{\varepsilon,\alpha,x',t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x'},0\right)\right]\right)\mathrm{d}t\right| \\ &\leqslant \int_0^\infty \mathrm{e}^{-\alpha t}\left|\widehat{\mathbb{E}}^{\varepsilon,\alpha,x,t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x},0\right)\right] - \widehat{\mathbb{E}}^{\varepsilon,\alpha,x',t}\left[\psi^\varepsilon\left(X_t^{\varepsilon,x'},0\right)\right]\right|\mathrm{d}t \\ &\leqslant 2\int_0^\infty \mathrm{e}^{-\alpha t}\widehat{c}M_\psi\mathrm{e}^{-\widehat{\nu}t}\left(1+|x|^2+|x'|^2\right)\mathrm{d}t \leqslant 2\frac{\widehat{c}M_\psi}{\widehat{\nu}}\left(1+|x|^2+|x'|^2\right). \qquad (10)\end{aligned}$$

2. Now, our goal is to take the limit when $\varepsilon \to 0$. Let D be a dense and countable subset of $\mathbb{R}^d$. By a diagonal argument, there exists a positive sequence $(\varepsilon_n)_n$ such that $(v^{\varepsilon_n,\alpha})_n$ converges pointwise over D to a function $\overline{v}^\alpha$. Because the constant C in equation (9) does not depend on ε, $\overline{v}^\alpha$ satisfies the same inequality. Let (K_n) be a sequence of compact sets whose diameter goes to infinity. The function $\overline{v}^\alpha$ is uniformly continuous on $K_n \cap D$, so it has an extension $\overline{v}$ which is continuous on K_n. Passing to the limit as n goes to infinity, we get a continuous function on $\mathbb{R}^d$, and it is the pointwise limit of the sequence $(v^{\varepsilon_n,\alpha})_n$. We denote $\overline{Y}_t^{\alpha,x} = \overline{v}^\alpha\left(X_t^x\right)$; we have $\left|\overline{Y}_t^{\alpha,x}\right| \leqslant \dfrac{C}{\alpha}\left(1+|X_t^x|\right)$. Using the Lemma 2.1 of [MZ02], we have, for every $T > 0$:

$$\mathbb{E}\left[\left|X^{\varepsilon_n,x} - X^x\right|_{0,T}^{*,2}\right] \leqslant C\left\{\|\Xi - \Xi^{\varepsilon_n}\|_\infty^2 T + \mathbb{E}\left[\int_0^T \left|\sigma^{\varepsilon_n}\left(X_t^x\right) - \sigma\left(X_t^x\right)\right|^2 \mathrm{d}t\right]\right\}.$$

By dominated convergence, this quantity goes to 0 as n goes to infinity. Also by dominated convergence, we get:

$$\mathbb{E}\left[\int_0^T \left|Y_t^{\varepsilon_n,\alpha,x} - \overline{Y}_t^{\alpha,x}\right|^2 \mathrm{d}t\right] \underset{n\to\infty}{\longrightarrow} 0 \text{ and } \mathbb{E}\left[\left|Y_T^{\varepsilon_n,\alpha,x} - \overline{Y}_T^{\alpha,x}\right|^2\right] \underset{n\to\infty}{\longrightarrow} 0.$$

3. We will show that there exists a process denoted $\overline{Z}^{\alpha,x}$ belonging to $\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty;\left(\mathbb{R}^d\right)^*\right)\right)$ which satisfies $\mathbb{E}\left[\int_0^T \left|Z_t^{\varepsilon_n,\alpha,x} - \overline{Z}_t^{\alpha,x}\right|^2 \mathrm{d}t\right] \underset{n\to\infty}{\longrightarrow} 0$, for every $T > 0$. Indeed, $\left(\overline{Y}^{\alpha,x}, \overline{Z}^{\alpha,x}\right)$ is solution of the BSDE (8) ; by uniqueness of the solution, $v^\alpha \equiv \overline{v}^\alpha$ and taking the limit in the equation (10) gives the result.
Let $n \leqslant m \in \mathbb{N}$, we define $\widetilde{Y} = Y^{\varepsilon_n,\alpha,x} - Y^{\varepsilon_m,\alpha,x}$ and $\widetilde{Z} = Z^{\varepsilon_n,\alpha,x} - Z^{\varepsilon_m,\alpha,x}$. We have:

$$\mathrm{d}\widetilde{Y}_t = \alpha\widetilde{Y}_t\,\mathrm{d}t + \underbrace{\left(\psi^{\varepsilon_m}\left(X_t^{\varepsilon_m,x}, Z_t^{\varepsilon_m,\alpha,x}\right) - \psi^{\varepsilon_n}\left(X_t^{\varepsilon_n,x}, Z_t^{\varepsilon_n,\alpha,x}\right)\right)}_{=:\psi_t}\mathrm{d}t + \widetilde{Z}_t\,\mathrm{d}W_t.$$

Thanks to Itô's formula, we obtain: $\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] \leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] - 2\mathbb{E}\left[\int_0^T \widetilde{Y}_t\psi_t\,\mathrm{d}t\right]$.
But, we have: $|\psi_t| \leqslant M_\psi\left(2 + \left|X_t^{\varepsilon_m,x}\right| + \left|X_t^{\varepsilon_n,x}\right|\right) + K_z\left(\left|Z_t^{\varepsilon_m,x}\sigma^{\varepsilon_m}\left(X_t^{\varepsilon_m,x}\right)^{-1}\right| + \left|Z_t^{\varepsilon_n,x}\sigma^{\varepsilon_n}\left(X_t^{\varepsilon_n,x}\right)^{-1}\right|\right)$.
So:

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] \leqslant 2\mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] &+ 2TM_\psi\mathbb{E}\left[\left|\widetilde{Y}\right|_{0,T}^*\left(2 + \left|X^{\varepsilon_m,x}\right|_{0,T}^* + \left|X^{\varepsilon_n,x}\right|_{0,T}^*\right)\right] \\ &+ 2K_z\left\|\sigma^{-1}\right\|_\infty\mathbb{E}\left[\left|\widetilde{Y}\right|_{0,T}^*\int_0^T\left(\left|Z_t^{\varepsilon_m,x}\right| + \left|Z_t^{\varepsilon_n,x}\right|\right)\mathrm{d}t\right].\end{aligned}$$

Now, we need to bound $\mathbb{E}\left[|X^{\varepsilon_n,x}|^{*,2}_{0,T}\right]$ and $\mathbb{E}\left[\int_0^T \left|Z_t^{\varepsilon_n,\alpha,x}\right|^2 \mathrm{d}t\right]$ independently of n. Thanks to proposition 4, we have $\mathbb{E}\left[|X^{\varepsilon_n,x}|^{*,2}_{0,T}\right] \leqslant C\left(1+|x|^2\right)$, where C only depends on T, r_1, r_2, ξ_1 and ξ_2. Also, we have:

$$\mathbb{E}\left[\int_0^T \left|Z_t^{\varepsilon_n,\alpha,x}\right|^2 \mathrm{d}t\right] \leqslant \mathbb{E}\left[\left|Y_T^{\varepsilon_n,\alpha,x}\right|^2\right] + 2\mathbb{E}\left[\int_0^T \left|Y_t^{\varepsilon_n,\alpha,x}\psi^{\varepsilon_n}\left(X_t^{\varepsilon_n,x}, Z_t^{\varepsilon_n,\alpha,x}\right)\right| \mathrm{d}t\right].$$

But, using the estimate of Lemma 12,

$$\begin{aligned}\left|Y_t^{\varepsilon_n,\alpha,x}\psi\left(X_t^{\varepsilon_n,x}, Z_t^{\varepsilon_n,\alpha,x}\right)\right| &\leqslant \frac{C}{\alpha}\left(1+\left|X_t^{\varepsilon_n,x}\right|\right)\left(M_\psi\left(1+\left|X_t^{\varepsilon_n,x}\right|\right) + K_z\left\|\sigma^{-1}\right\|_\infty \left|Z_t^{\varepsilon_n,\alpha,x}\right|\right)\\ &\leqslant \mathcal{C}\left(1+\left|X_t^{\varepsilon_n,x}\right|^2\right) + \frac{1}{4}\left|Z_t^{\varepsilon_n,\alpha,x}\right|^2.\end{aligned}$$

Finally, $\mathbb{E}\left[\int_0^T \left|Z_t^{\varepsilon_n,\alpha,x}\right|^2 \mathrm{d}t\right] \leqslant \mathcal{C}'\left(1+|x|^2\right)$, with $\mathcal{C}'$ depending only on M_ψ, η_1, η_2, r_1, r_2, K_z, $\left\|\sigma^{-1}\right\|_\infty$, α, T, ξ_1 and ξ_2, and $\left(Z^{\varepsilon_n,\alpha,x}\right)_n$ is a Cauchy sequence in $\mathrm{L}^2_{\mathcal{P}}\left(\Omega, \mathrm{L}^2\left([0,T], \left(\mathbb{R}^d\right)^*\right)\right)$; we can define its limit process $\overline{Z}^{\alpha,x} \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty; \left(\mathbb{R}^d\right)^*\right)\right)$, and it satisfies the convergence we claimed. ■

Proposition 17

Under Assumption 10, there exists a constant C, such that for every $\alpha \in (0,1]$, we have:

$$\forall x, x' \in \mathbb{R}^d,\ \left|v^\alpha(x) - v^\alpha(x')\right| \leqslant C\left(1+|x|^2+|x'|^2\right)|x-x'|.$$

The constant C only depends on η_1, η_2, r_1, r_2, K_z, $\left\|\sigma^{-1}\right\|_\infty$, M_ψ, K_x, $\|\Xi\|_{\mathrm{lip}}$ and $\|\sigma\|_{\mathrm{lip}}$.

Proof : See Theorem 35 given in the appendix, with $f : (x,y,z) \mapsto \psi(x,z) - \alpha y$ and $g = v^\alpha$. ■

Theorem 18 (*Existence of solutions to the EBSDE*)

Under Assumption 10, there exists a real number $\overline{\lambda}$, a locally Lipschitz function $\overline{v}$, which satisfies $\overline{v}(0) = 0$, and a process $\overline{Z}^x \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty; \left(\mathbb{R}^d\right)^*\right)\right)$ such that if we define $\overline{Y}_t^x = \overline{v}\left(X_t^x\right)$, then the EBSDE (7) is satisfied by $\left(\overline{Y}^x, \overline{Z}^x, \overline{\lambda}\right)$ $\mathbb{P}$-a.s. and for all $0 \leqslant t \leqslant T < \infty$. Moreover, there exists $C > 0$ such that for all $x \in \mathbb{R}^d$, $|\overline{v}(x)| \leqslant C\left(1+|x|^2\right)$, and there exists $\overline{\zeta}$ measurable such that $\overline{Z}_t^x = \overline{\zeta}\left(X_t^x\right)$ $\mathbb{P}$-a.s. and for a.e. $t \geqslant 0$.

Proof :

The strategy is the same as in the Theorem 4.4 of [FHT09]. We give a sketch of the proof here for completeness.

Step 1: Construction of $\overline{v}$ by a diagonal procedure.

For every $\alpha > 0$, we define $\overline{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0)$; we recall that $|\overline{v}^\alpha(x)| \leqslant C\left(1+|x|^2\right)$ and $|\alpha v^\alpha(0)| \leqslant C$, with C independent of α. Let D be a countable dense set in $\mathbb{R}^d$; by a diagonal argument, we can construct a sequence (α_n), such that $\left(\overline{v}^{\alpha_n}\right)_n$ converges pointwise over D to a function $\overline{v}$ and $\alpha_n v^{\alpha_n}(0) \underset{n\to\infty}{\longrightarrow} \overline{\lambda}$, for a convenient real number $\overline{\lambda}$. Moreover, thanks to the previous proposition:

$$\exists C > 0,\ \forall \alpha \in (0,1],\ \forall x, x' \in \mathbb{R}^d,\ \left|\overline{v}^\alpha(x) - \overline{v}^\alpha(x')\right| \leqslant C\left(1+|x|^2+|x'|^2\right)|x-x'|.$$

Because it is uniformly continuous on every compact subset of D, $\overline{v}$ has an extension which is continuous on $\mathbb{R}^d$. Then, we can show that $\overline{v}$ is the pointwise limit of the functions $\overline{v}^{\alpha_n}$ on $\mathbb{R}^d$, and then $\overline{v}$ is locally Lipschitz continuous and has quadratic growth.

Step 2: Construction of the process $\overline{Z}^x$.

We will show that $(Z^{\alpha_n,x})_n$ is Cauchy in $\mathrm{L}^2_{\mathcal{P}}\left(\Omega;\mathrm{L}^2\left([0,T];\left(\mathbb{R}^d\right)^*\right)\right)$ for every $T>0$. Then, we will be able to define $\overline{Z}^x \in \mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega;\mathrm{L}^2\left(0,\infty;\left(\mathbb{R}^d\right)^*\right)\right)$. When $n \leqslant m \in \mathbb{N}$, we set $\widetilde{Y} = Y^{\alpha_n,x} - Y^{\alpha_m,x}$ and $\widetilde{Z} = Z^{\alpha_n,x} - Z^{\alpha_m,x}$; we have:

$$\mathrm{d}\widetilde{Y}_t = -\widetilde{\psi}_t\,\mathrm{d}t + \left(\alpha_n Y_t^{\alpha_n,x} - \alpha_m Y_t^{\alpha_m,x}\right)\,\mathrm{d}t + \widetilde{Z}_t\,\mathrm{d}W_t,$$

where $\widetilde{\psi}_t = \psi\left(X_t^x, Z_t^{\alpha_n,x}\right) - \psi\left(X_t^x, Z_t^{\alpha_m,x}\right)$. Thanks to Itô's formula:

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] &= \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] - |Y_0|^2 + 2\mathbb{E}\left[\int_0^T \widetilde{\psi}_t\widetilde{Y}_t\,\mathrm{d}t\right] - 2\mathbb{E}\left[\int_0^T \left(\alpha_n Y_t^{\alpha_n,x} - \alpha_m Y_t^{\alpha_m,x}\right)\widetilde{Y}_t\,\mathrm{d}t\right]\\ &\leqslant \mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] + 2K_z\left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right|\left|\widetilde{Z}_t\right|\,\mathrm{d}t\right] + 4M_\psi\mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right|\,\mathrm{d}t\right].\end{aligned}$$

By Cauchy-Schwarz, and noting that $\left|\widetilde{Y}_t\right|\left|\widetilde{Z}_t\right| \leqslant K_z\left\|\sigma^{-1}\right\|_\infty\left|\widetilde{Y}_t\right|^2 + \dfrac{1}{4K_z\left\|\sigma^{-1}\right\|_\infty}\left|\widetilde{Z}_t\right|^2$, we obtain:

$$\mathbb{E}\left[\int_0^T \left|\widetilde{Z}_t\right|^2 \mathrm{d}t\right] \leqslant 2\mathbb{E}\left[\left|\widetilde{Y}_T\right|^2\right] + 4K_z^2\left\|\sigma^{-1}\right\|_\infty^2 \mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right|^2 \mathrm{d}t\right] + 8M_\psi\sqrt{T}\mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t\right|^2 \mathrm{d}t\right]^{\frac{1}{2}}.$$

Dominated convergences (using propositions 16 and 4) give us the result claimed.

Step 3: $\left(\overline{Y}^x, \overline{Z}^x, \overline{\lambda}\right)$ is a solution to the EBSDE (7).

Taking the limit in the BSDE satisfied by $(Y^{\alpha_n,x}, Z^{\alpha_n,x})$ gives us:

$$\overline{Y}_t^x = \overline{Y}_T^x + \int_t^T \left[\psi\left(X_s^x, \overline{Z}_s^x\right) - \overline{\lambda}\right]\mathrm{d}s - \int_t^T \overline{Z}_s^x\,\mathrm{d}W_s.$$

Step 4: $\overline{Z}^x$ can be represented as a measurable function of X^x. We fix $T>0$. We denote $\Delta\bullet := \bullet^{\alpha,x} - \bullet^{\alpha,x'}$. By standard calculations, for every $\alpha \in (0,1]$, $x, x' \in \mathbb{R}^d$:

$$\begin{aligned}\mathbb{E}\left[\int_0^T |\Delta Z_t|^2\,\mathrm{d}t\right] &\leqslant \mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right] + 2K_xT\mathbb{E}\left[|\Delta Y|_{0,T}^*|\Delta X|_{0,T}^*\right] + 2K_z\left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^*|\Delta Z_t|\,\mathrm{d}t\right]\\ &\qquad + 4K_z\left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^*\left|Z_t^{\alpha,x'}\right|\,\mathrm{d}t\right]\\ &\leqslant 2\mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right] + 4K_xT\mathbb{E}\left[|\Delta Y|_{0,T}^*|\Delta X|_{0,T}^*\right] + 2K_z^2\left\|\sigma^{-1}\right\|_\infty^2 T\mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right]\\ &\qquad + 8K_z\left\|\sigma^{-1}\right\|_\infty \mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^*\left|Z_t^{\alpha,x'}\right|\,\mathrm{d}t\right]\end{aligned}$$

But, using Cauchy-Schwarz inequality and the estimate of the end of the proof of proposition 16, we get:

$$\mathbb{E}\left[\int_0^T |\Delta Y|_{0,T}^*\left|Z_t^{\alpha,x'}\right|\,\mathrm{d}t\right] \leqslant C\mathbb{E}\left[|\Delta Y|_{0,T}^{*,2}\right]^{\frac{1}{2}}(1+|x|).$$

Using Lemma 2.1 of [MZ02] and propositions 4 and 17, we finally get:

$$\mathbb{E}\left[\int_0^T \left|Z_t^{\alpha,x} - Z_t^{\alpha,x'}\right|^2 \mathrm{d}t\right] \leqslant \mathcal{C}\left(1 + |x|^4 + |x'|^4\right)|x-x'|^2, \tag{11}$$

where $\mathcal{C}$ is independent of x and x', but depends on α and T. For every $x \in \mathbb{R}^d$, the sequence $\left(\mathbb{E}\left[\int_0^T \left|Z_t^{\alpha_n,x} - Z_t^{\alpha_m,x}\right|^2 \mathrm{d}t\right]\right)_{n\leqslant m\in\mathbb{N}}$ is bounded (it converges). By a diagonal procedure, there exists a subsequence $(\alpha_n') \subset (\alpha_n)$ such that:

$$\forall x \in D,\ \forall n \leqslant m \in \mathbb{N},\ \mathbb{E}\left[\int_0^T \left|Z_t^{\alpha_n',x} - Z_t^{\alpha_m',x}\right|^2 \mathrm{d}t\right] \leqslant 2^{-n}.$$

Equation (11) extends this inequality to $\mathbb{R}^d$. Borel-Cantelli theorem gives for a.e. $t \in [0,T]$, $Z_t^{\alpha'_n,x} \underset{n\to\infty}{\longrightarrow} \overline{Z}_t^x$ $\mathbb{P}$-a.s. Set

$$\overline{\zeta}(x) = \begin{cases} \lim_n \zeta^{\alpha'_n}(x), & \text{if the limit exists,} \\ 0, & \text{elsewhere.} \end{cases}$$

For a.e. $t \in [0,T]$, X_t^x belongs $\mathbb{P}$-a.s. to the set where $\lim_n \zeta^{\alpha'_n}(x)$ exists ; $\overline{Z}_t^x = \overline{\zeta}\left(X_t^x\right)$ $\mathbb{P}$-a.s. and for a.e. $t \in [0,T]$. ■

4.2 Uniqueness of the solution

Theorem 19 (*Uniqueness of the parameter λ*)

Let $p > 0$; we suppose that $\sqrt{r_2}K_z \left\|\sigma^{-1}\right\|_\infty + [(p \vee 2) - 1]\dfrac{r_2}{2} < \eta_2$ and that Assumption 10 holds true. We suppose that, for some $x \in \mathbb{R}^d$, (Y', Z', λ') verifies the EBSDE (7) $\mathbb{P}$-a.s. and for all $0 \leqslant t \leqslant T < \infty$, where Y' is a progressively measurable continuous process, Z' is a process in $\mathrm{L}^2_{\mathcal{P},\mathrm{loc}}\left(\Omega, \mathrm{L}^2\left(0,\infty; \left(\mathbb{R}^d\right)^*\right)\right)$ and $\lambda' \in \mathbb{R}$. Finally, we assume that there exists $c_x > 0$ (that may depend on x) such that

$$\forall t \geqslant 0, |Y'_t| \leqslant c_x\left(1 + |X_t^x|^p\right).$$

Then $\lambda' = \overline{\lambda}$.

Proof : We define $\widetilde{\lambda} = \lambda' - \overline{\lambda}$, $\widetilde{Y} = Y' - \overline{Y}$ and $\widetilde{Z} = Z' - \overline{Z}$. We have:

$$\widetilde{\lambda} = \frac{\widetilde{Y}_T - \widetilde{Y}_0}{T} + \frac{1}{T}\int_0^T \left[\psi\left(X_t^x, Z'_t\right) - \psi\left(X_t^x, \overline{Z}_t^x\right)\right] \mathrm{d}t - \frac{1}{T}\int_0^T \widetilde{Z}_t \,\mathrm{d}W_t.$$

We denote:

$$\gamma_t^x = \begin{cases} \dfrac{\psi\left(X_t^x, Z'_t\right) - \psi\left(X_t^x, \overline{Z}_t^x\right)}{\left|\widetilde{Z}_t\right|^2}\widetilde{Z}_t, & \text{if } \widetilde{Z}_t \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

There exists a probability $\widetilde{\mathbb{P}}^{x,T}$ under which $\widetilde{W}_t^x = W_t - \int_0^t \gamma_s^{x*}\,\mathrm{d}s$ is a Brownian motion on $[0,T]$. For every $\delta > 0$:

$$\widetilde{\lambda} = \frac{1}{T}\widetilde{\mathbb{E}}^{x,T}\left[\widetilde{Y}_T - \widetilde{Y}_0\right] \leqslant \frac{1}{T}\left\{c_x\left(2 + |x|^p + \widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^p\right]\right) + C\left(2 + |x|^2 + \widetilde{\mathbb{E}}^{x,T}\left[|X_T^x|^2\right]\right)\right\}.$$

We conclude by taking the limit $T \to \infty$ and using Proposition 5. ■

Theorem 20 (*Uniqueness of the functions v and ζ*)

Let $p > 0$; we suppose that $\sqrt{r_2}K_z \left\|\sigma^{-1}\right\|_\infty + [(p \vee 2) - 1]\dfrac{r_2}{2} < \eta_2$ and that Assumption 10 holds true.
Let (v, ζ) and $(\widetilde{v}, \widetilde{\zeta})$ be two couples of functions with:

- $v, \widetilde{v} : \mathbb{R}^d \to \mathbb{R}$ are continuous, $|v(x)| \leqslant C\left(1 + |x|^p\right)$, $|\widetilde{v}(x)| \leqslant C\left(1 + |x|^p\right)$ and $v(0) = \widetilde{v}(0) = 0$;
- $\zeta, \widetilde{\zeta} : \mathbb{R}^d \to \left(\mathbb{R}^d\right)^*$ are measurable.

We also assume that for some constants λ, $\widetilde{\lambda}$, and for all $x \in \mathbb{R}^d$, the triplets $\left(v\left(X_t^x\right), \zeta\left(X_t^x\right), \lambda\right)$ and $\left(\widetilde{v}\left(X_t^x\right), \widetilde{\zeta}\left(X_t^x\right), \widetilde{\lambda}\right)$ verify the EBSDE (7).
Then $\lambda = \widetilde{\lambda}$, $v = \widetilde{v}$ and $\zeta\left(X_t^x\right) = \widetilde{\zeta}\left(X_t^x\right)$ $\mathbb{P}$-a.s. and for a.e. $t \geqslant 0$ and for all $x \in \mathbb{R}^d$.

Proof :

By Theorem 19, we already know that $\lambda = \widetilde{\lambda}$. We denote $Y_t^x = v(X_t^x)$, $Z_t^x = \zeta(X_t^x)$, $\widetilde{Y}_t^x = \widetilde{v}(X_t^x)$ and $\widetilde{Z}_t^x = \widetilde{\zeta}(X_t^x)$. We approximate the functions ψ, v and $\widetilde{v}$ by sequences (ψ^ε), (v^ε) and $(\widetilde{v}^\varepsilon)$ of $\mathcal{C}^1$ functions with bounded derivatives that converge uniformly. We can say that for all $\varepsilon > 0$, the bound on the derivatives of ψ^ε is independent of ε and v^ε, $\widetilde{v}^\varepsilon$ have polynomial growth (with constant C and exponent p, all independent of ε). In the following, $x \in \mathbb{R}^d$, $T > 0$ and $\varepsilon > 0$ are fixed. Let us consider the following BSDEs in finite time horizon:

$$\begin{cases} \mathrm{d}Y_t^{\varepsilon,x} = -\left[\psi^\varepsilon\left(X_t^x, Z_t^{\varepsilon,x}\right) - \lambda\right]\mathrm{d}t + Z_t^{\varepsilon,x}\,\mathrm{d}W_t, \\ Y_T^{\varepsilon,x} = v^\varepsilon\left(X_T^x\right), \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{d}\widetilde{Y}_t^{\varepsilon,x} = -\left[\psi^\varepsilon\left(X_t^x, \widetilde{Z}_t^{\varepsilon,x}\right) - \lambda\right]\mathrm{d}t + \widetilde{Z}_t^{\varepsilon,x}\,\mathrm{d}W_t, \\ \widetilde{Y}_T^{\varepsilon,x} = \widetilde{v}^\varepsilon\left(X_T^x\right). \end{cases}$$

We denote $\Delta Y_t^x = Y_t^x - \widetilde{Y}_t^x$, $\Delta Z_t^x = Z_t^x - \widetilde{Z}_t^x$, $\Delta Y_t^{\varepsilon,x} = Y_t^{\varepsilon,x} - \widetilde{Y}_t^{\varepsilon,x}$, $\Delta Z_t^{\varepsilon,x} = Z_t^{\varepsilon,x} - \widetilde{Z}_t^{\varepsilon,x}$. This way, we get:

$$\Delta Y_0^x - \mathbb{E}\left[\Delta Y_T^x\right] = \mathbb{E}\left[\int_0^T \left[\psi\left(X_t^x, Z_t^x\right) - \psi\left(X_t^x, \widetilde{Z}_t^x\right)\right]\mathrm{d}t\right] \text{ and}$$

$$\Delta Y_0^{\varepsilon,x} - \mathbb{E}\left[\Delta Y_T^{\varepsilon,x}\right] = \mathbb{E}\left[\int_0^T \left[\psi^\varepsilon\left(X_t^x, Z_t^{\varepsilon,x}\right) - \psi^\varepsilon\left(X_t^x, \widetilde{Z}_t^{\varepsilon,x}\right)\right]\mathrm{d}t\right].$$

By substraction, it leads us to:

$$\left|\Delta Y_0^x - \Delta Y_0^{\varepsilon,x}\right| \leqslant \mathbb{E}\left[\int_0^T \left|\psi\left(X_t^x, Z_t^x\right) - \psi^\varepsilon\left(X_t^x, Z_t^{\varepsilon,x}\right)\right|\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T \left|\psi\left(X_t^x, \widetilde{Z}_t^x\right) - \psi^\varepsilon\left(X_t^x, \widetilde{Z}_t^{\varepsilon,x}\right)\right|\mathrm{d}t\right] + \left\|v - v^\varepsilon\right\|_\infty + \left\|\widetilde{v} - \widetilde{v}^\varepsilon\right\|_\infty.$$

Set $\delta Y_t^{\varepsilon,x} = Y_t^x - Y_t^{\varepsilon,x}$ and $\delta Z_t^{\varepsilon,x} = Z_t^x - Z_t^{\varepsilon,x}$. We have: $\left|\psi\left(X_t^x, Z_t^x\right) - \psi^\varepsilon\left(X_t^x, Z_t^{\varepsilon,x}\right)\right| \leqslant K_z \left\|\sigma^{-1}\right\|_\infty \left|\delta Z_t^{\varepsilon,x}\right| + \left\|\psi - \psi^\varepsilon\right\|_\infty$.

Our next goal is to estimate $\mathbb{E}\left[\int_0^T \left|\delta Z_t^{\varepsilon,x}\right|\mathrm{d}t\right]$. We see that $(\delta Y_t^{\varepsilon,x}, \delta Z_t^{\varepsilon,x})$ is solution of the BSDE:

$$\begin{cases} \mathrm{d}\delta Y_t^{\varepsilon,x} = -\left[\psi\left(X_t^x, Z_t^x\right) - \psi^\varepsilon\left(X_t^x, Z_t^{\varepsilon,x}\right)\right]\mathrm{d}t + \delta Z_t^{\varepsilon,x}\,\mathrm{d}W_t, \\ \delta Y_T^{\varepsilon,x} = \left(v - v^\varepsilon\right)\left(X_T^x\right). \end{cases}$$

Using Lemma 2.1 of [MZ02], there exists $C > 0$ only depending on T and the Lipschitz constant of ψ, such that:

$$\mathbb{E}\left[\int_0^T \left|\delta Z_t^{\varepsilon,x}\right|^2 \mathrm{d}t\right] \leqslant C\left(T\left\|\psi - \psi^\varepsilon\right\|_\infty^2 + \left\|v - v^\varepsilon\right\|_\infty^2\right).$$

By Cauchy-Schwarz, we finally get: $\mathbb{E}\left[\int_0^T \left|\delta Z_t^{\varepsilon,x}\right|\mathrm{d}t\right] \leqslant \sqrt{CT}\sqrt{T\left\|\psi - \psi^\varepsilon\right\|_\infty^2 + \left\|v - v^\varepsilon\right\|_\infty^2}$. We can do the same for $\mathbb{E}\left[\int_0^T \left|\psi\left(X_t^x, \widetilde{Z}_t^x\right) - \psi^\varepsilon\left(X_t^x, \widetilde{Z}_t^{\varepsilon,x}\right)\right|\mathrm{d}t\right]$, and we have: $\left|\Delta Y_0^x - \Delta Y_0^{\varepsilon,x}\right| \underset{\varepsilon\to 0}{\longrightarrow} 0$. By Theorem 3.1 of [MZ02], because ψ^ε, v^ε and $\widetilde{v}^\varepsilon$ have bounded derivatives, there exists continuous functions ζ_T^ε and $\widetilde{\zeta}_T^\varepsilon$ such that: $Z_t^{\varepsilon,x} = \zeta_T^\varepsilon(t, X_t^x)$ and $\widetilde{Z}_t^{\varepsilon,x} = \widetilde{\zeta}_T^\varepsilon(t, X_t^x)$ $\mathbb{P}$-a.s. and for a.e. $t \in [0,T]$. As usual, we linearise our BSDE; so we set:

$$\Gamma_T^\varepsilon(t,x) = \begin{cases} \dfrac{\psi^\varepsilon\left(x, \zeta_T^\varepsilon(t,x)\right) - \psi^\varepsilon\left(x, \widetilde{\zeta}_T^\varepsilon(t,x)\right)}{\left|\zeta_T^\varepsilon(t,x) - \widetilde{\zeta}_T^\varepsilon(t,x)\right|^2}\left(\zeta_T^\varepsilon(t,x) - \widetilde{\zeta}_T^\varepsilon(t,x)\right), & \text{if } \zeta_T^\varepsilon(t,x) \neq \widetilde{\zeta}_T^\varepsilon(t,x), \\ 0, & \text{otherwise.} \end{cases}$$

The process $\Gamma_T^\varepsilon(t, X_t^x)^*$ is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$; by Girsanov, $W_t^{\varepsilon,x,T} = W_t - \int_0^t \Gamma_T^\varepsilon(s, X_s^x)^*\,\mathrm{d}s$ is a Brownian motion on $[0,T]$ under the probability $\mathbb{Q}_T^{\varepsilon,x}$. $\Delta Y^{\varepsilon,x}$ is a $\mathbb{Q}_T^{\varepsilon,x}$-martingale and we get:

$$\Delta Y_0^{\varepsilon,x} = \mathbb{E}^{\mathbb{Q}_T^{\varepsilon,x}}\left[\Delta Y_T^{\varepsilon,x}\right] = \mathbb{E}^{\mathbb{Q}_T^{\varepsilon,x}}\left[\left(v^\varepsilon - \widetilde{v}^\varepsilon\right)\left(X_T^x\right)\right] = \mathcal{P}_T^\varepsilon\left[v^\varepsilon - \widetilde{v}^\varepsilon\right](x),$$

where $\mathcal{P}_T^\varepsilon$ is the Kolmogorov semigroup of the SDE:

$$\mathrm{d}U_t^x = \left[\Xi\left(U_t^x\right) + \sigma\left(U_t^x\right)\Gamma_T^\varepsilon\left(t, U_t^x\right)^*\right]\mathrm{d}t + \sigma\left(U_t^x\right)\mathrm{d}W_t.$$

Using Lemma 15 and Corollary 9, we get: $\left|\mathcal{P}_T^\varepsilon\left[v^\varepsilon-\widetilde{v}^\varepsilon\right](x)-\mathcal{P}_T^\varepsilon\left[v^\varepsilon-\widetilde{v}^\varepsilon\right](0)\right| \leqslant C\left(1+|x|^{p\vee 2}\right)\mathrm{e}^{-\nu T}$, where ν and C are independent of ε (because the polynomial growths of v^ε and $\widetilde{v}^\varepsilon$ do not depend on ε). Finally,

$$\forall\varepsilon>0,\ \forall T>0,\ \forall x\in\mathbb{R}^d,\ \left|\Delta Y_0^{\varepsilon,x}-\Delta Y_0^{\varepsilon,0}\right|\leqslant C\left(1+|x|^{p\vee 2}\right)\mathrm{e}^{-\nu T}.$$

By taking the limit as ε goes to 0:

$$\forall T>0,\ \forall x\in\mathbb{R}^d,\ \left|\left(v-\widetilde{v}\right)(x)\right|=\left|\Delta Y_0^{x}-\Delta Y_0^{0}\right|\leqslant C\left(1+|x|^{p\vee 2}\right)\mathrm{e}^{-\nu T}.$$

Taking the limit as T goes to infinity leads us to $v=\widetilde{v}$. Then, uniqueness of ζ is the consequence of Itô's formula. ■

5 Large time behaviour

In this section, we always suppose that $\sqrt{r_2}K_z\left\|\sigma^{-1}\right\|_\infty+\dfrac{\mu-1}{2}r_2<\eta_2$, with $\mu\geqslant 2$, and we keep working under Assumption 10. Indeed, we have seen in the previous section that, under this assumption, there exists a unique triplet (v,ζ,λ) such that $\left(v\left(X_t^x\right),\zeta\left(X_t^x\right),\lambda\right)$ is a solution to the EBSDE (7), v is continuous with quadratic growth, $v(0)=0$ and ζ is measurable. Let ξ^T be a real random variable $\mathcal{F}_T$-measurable and such that $\left|\xi^T\right|\leqslant C\left(1+|X_T^x|^\mu\right)$. It will allow us to use proposition 5. We denote by $\left(Y_t^{T,x},Z_t^{T,x}\right)$ the solution of the BSDE in finite horizon:

$$Y_t^{T,x}=\xi^T+\int_t^T\psi\left(X_s^x,Z_s^{T,x}\right)\mathrm{d}s-\int_t^T Z_s^{T,x}\,\mathrm{d}W_s.$$

Theorem 21

We have the following inequality:

$$\left|\frac{Y_0^{T,x}}{T}-\lambda\right|\leqslant\frac{C\left(1+|x|^\mu\right)}{T},$$

where the constant C is independent of x and T; and in particular: $\dfrac{Y_0^{T,x}}{T}\underset{T\to\infty}{\longrightarrow}\lambda$, uniformly in any bounded subset of $\mathbb{R}^d$.

Proof :

For all $x\in\mathbb{R}^d$ and $T>0$, we write:

$$\left|\frac{Y_0^{T,x}}{T}-\lambda\right|\leqslant\left|\frac{Y_0^{T,x}-Y_0^x-\lambda T}{T}\right|+\left|\frac{Y_0^x}{T}\right|.$$

First of all, $\left|Y_0^x\right|=|v(x)|\leqslant C\left(1+|x|^2\right)$. Also, by the usual linearisation technique, we have:

$$Y_0^{T,x}-Y_0^x-\lambda T=\xi^T-v\left(X_T^x\right)+\int_0^T\left(Z_s^{T,x}-Z_s^x\right)\beta_s^{T*}\,\mathrm{d}s-\int_0^T\left(Z_s^{T,x}-Z_s^x\right)\mathrm{d}W_s,$$

where

$$\beta_t^T=\begin{cases}\dfrac{\psi\left(X_t^x,Z_t^{T,x}\right)-\psi\left(X_t^x,Z_t^x\right)}{\left|Z_t^{T,x}-Z_t^x\right|^2}\left(Z_t^{T,x}-Z_t^x\right), & \text{if } Z_t^{T,x}\neq Z_t^x,\\ 0, & \text{otherwise.}\end{cases}$$

The process β^T is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$ and by Girsanov's theorem, there exists a probability measure $\mathbb{Q}^T$ under which $\widetilde{W}_t^T=W_t-\int_0^t\beta_s^{T*}\,\mathrm{d}s$ is a Brownian motion on $[0,T]$. This way, we can see that:

$$\left|Y_0^{T,x}-Y_0^x-\lambda T\right|=\left|\mathbb{E}^{\mathbb{Q}^T}\left[\xi^T-v\left(X_T^x\right)\right]\right|\leqslant\mathbb{E}^{\mathbb{Q}^T}\left[\left|\xi^T\right|\right]+\mathbb{E}^{\mathbb{Q}^T}\left[\left|v\left(X_T^x\right)\right|\right]\leqslant C\left(1+\mathbb{E}^{\mathbb{Q}^T}\left[|X_T^x|^\mu\right]\right).$$

Thanks to Proposition 5, we get: $\sup_{T\geqslant 0} \mathbb{E}^{\mathbb{Q}^T}\left[|X_T^x|^\mu\right] \leqslant \kappa\left(1+|x|^\mu\right)$, where κ is independent of x. ■

Theorem 22

We suppose that $\xi^T = g\left(X_T^x\right)$, where $g : \mathbb{R}^d \to \mathbb{R}$ has polynomial growth: $\forall x \in \mathbb{R}^d$, $|g(x)| \leqslant C\left(1+|x|^\mu\right)$ and satisfies: $\forall x, x' \in \mathbb{R}^d$, $|g(x)-g(x')| \leqslant C\left(1+|x|^\mu+|x'|^\mu\right)|x-x'|$.
Then, there exists $L \in \mathbb{R}$, such that: $\forall x \in \mathbb{R}^d$, $Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to\infty}{\longrightarrow} L$. Furthermore,

$$\forall x \in \mathbb{R}^d,\ \forall T > 0,\ \left|Y_0^{T,x} - \lambda T - Y_0^x - L\right| \leqslant C\left(1+|x|^\mu\right)\mathrm{e}^{-\nu T}.$$

Proof :

We will consider the following equations:

$$\left\{\begin{array}{rcl} \mathrm{d}Y_s^{T,t,x} &=& -\psi\left(X_s^{t,x}, Z_s^{T,t,x}\right)\mathrm{d}s + Z_s^{T,t,x}\mathrm{d}W_s \\ Y_T^{T,t,x} &=& g\left(X_T^{t,x}\right) \end{array}\right. \longrightarrow Y_s^{T,t,x} = u_T\left(s, X_s^{t,x}\right), \text{ see Theorem 4.1 of [EPQ97];}$$

$$\left\{\begin{array}{rcl} \mathrm{d}Y_s^{t,x} &=& -\left\{\psi\left(X_s^{t,x}, Z_s^{t,x}\right) - \lambda\right\}\mathrm{d}s + Z_s^{t,x}\mathrm{d}W_s \\ Y_T^{t,x} &=& v\left(X_T^{t,x}\right) \end{array}\right. \longrightarrow Y_s^{t,x} = v\left(X_s^{t,x}\right), \text{ solution of the EBSDE;}$$

$$\left\{\begin{array}{rcl} \mathrm{d}Y_s^{n,t,x} &=& -\psi^n\left(X_s^{n,t,x}, Z_s^{n,t,x}\right)\mathrm{d}s + Z_s^{n,t,x}\mathrm{d}W_s \\ Y_T^{n,t,x} &=& g^n\left(X_T^{n,t,x}\right) \end{array}\right. \longrightarrow Y_s^{n,t,x} = u_T^n\left(s, X_s^{n,t,x}\right), \text{ where } u_T^n \text{ is } \mathcal{C}^1 \text{ (see appendix);}$$

$$\left\{\begin{array}{rcl} \mathrm{d}\widetilde{Y}_s^{n,t,x} &=& -\psi^n\left(X_s^{n,t,x}, \widetilde{Z}_s^{n,t,x}\right)\mathrm{d}s + \widetilde{Z}_s^{n,t,x}\mathrm{d}W_s \\ \widetilde{Y}_T^{n,t,x} &=& v^n\left(X_T^{n,t,x}\right) \end{array}\right. \longrightarrow \widetilde{Y}_s^{n,t,x} = \widetilde{u}_T^n\left(s, X_s^{n,t,x}\right), \text{ where } \widetilde{u}_T^n \text{ is } \mathcal{C}^1 \text{ (see appendix);}$$

where $g^n, v^n \in \mathcal{C}_b^1$ converge uniformly on every compact towards g and v; also, we take $(\Xi^n)_n$ a sequence of $\mathcal{C}_b^1$ functions that converges uniformly towards Ξ and define $X^{n,t,x}$ as the solution of the SDE

$$\left\{\begin{array}{rcl} \mathrm{d}X_s^{n,t,x} &=& \Xi^n\left(X_s^{n,t,x}\right)\mathrm{d}s + \sigma\left(X_s^{n,t,x}\right)\mathrm{d}W_s, \\ X_t^{n,t,x} &=& x. \end{array}\right.$$

Step 1: Approximation of the function ψ.

We set $\rho(x,p) = \psi(x, p\sigma(x))$; ρ is Lipschitz, and we approximate it by a sequence of $\mathcal{C}_b^1$ functions (ρ^n) which converges uniformly on $\mathbb{R}^d \times \left(\mathbb{R}^d\right)^*$. Then, we define a function β^n satisfying:

$$\beta^n(x,p) = \left\{\begin{array}{ll} \rho^n(x,p), & \text{if } |p\sigma(x)| \leqslant n, \\ 0, & \text{if } |p\sigma(x)| \geqslant f(n), \end{array}\right.$$

where $f(n)$ is chosen such that β^n and ρ^n have the same Lipschitz constant. This way, $\beta^n \underset{n\to\infty}{\longrightarrow} \rho$, uniformly on every compact, and $\beta^n \in \mathcal{C}_b^1$. We set $\psi^n(x,z) = \beta^n\left(x, z\sigma(x)^{-1}\right)$ and $\psi^n \in \mathcal{C}_b^1$. Moreover, we have the following:

$$|(\psi - \psi^n)(x,z)| \leqslant \|\rho - \rho^n\|_\infty + \left(2M_\psi(1+|x|) + 2K_z\left\|\sigma^{-1}\right\|_\infty |z|\right)\mathbb{1}_{|z|>n}.$$

Step 2: Growth of w_T^n.

We set $w_T^n(t,x) = u_T^n(t,x) - \widetilde{u}_T^n(t,x)$ and $w_T(t,x) = u_T(t,x) - \lambda(T-t) - v(x)$. By uniqueness of viscosity solutions (see [EPQ97]), we have $u_T^n(0,x) = u_{T+S}^n(S,x)$ and $\widetilde{u}_T^n(0,x) = \widetilde{u}_{T+S}^n(S,x)$.

Lemma 23

We have: $\forall \varepsilon > 0$, $\forall T > 0$, $\exists \widetilde{C}_T > 0$, $\forall x \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$,

$$\begin{aligned} &|v(x) - \widetilde{u}_T^n(0,x) + \lambda T|^2 \\ &\qquad \leqslant \widetilde{C}_T\left\{\mathbb{E}\left[\left|v\left(X_T^x\right) - v^n\left(X_T^{n,x}\right)\right|^2\right] + \|\rho - \rho^n\|_\infty^2 + \left(\|\Xi - \Xi^n\|_\infty^2 + \frac{1}{n^\varepsilon}\right)\left(1+|x|^{2(2+\varepsilon)}\right)\right\}. \end{aligned}$$

Proof: We set $\delta\widetilde{Y}_s^n = Y_s^x - \widetilde{Y}_s^{n,x} + \lambda(T-s)$ and $\delta\widetilde{Z}_s^n = Z_s^x - \widetilde{Z}_s^{n,x}$; then,

$$\mathrm{d}\delta\widetilde{Y}_s^n = \mathrm{d}Y_s^x - \mathrm{d}\widetilde{Y}_s^{n,x} - \lambda\mathrm{d}s = \left\{\psi^n\left(X_s^{n,x}, \widetilde{Z}_s^{n,x}\right) - \psi\left(X_s^x, Z_s^x\right)\right\}\mathrm{d}s + \delta\widetilde{Z}_s^{n,x}\,\mathrm{d}W_s.$$

We will use [BDH$^+$03]; to use the same notations, we set $f(s,y,z) = \psi\left(X_s^x, Z_s^x\right) - \psi^n\left(X_s^x, Z_s^x - z\right)$. We have:

$$\begin{aligned}|f(s,y,z)| \leqslant \|\rho-\rho^n\|_\infty + \left(2M_\psi\left(1+|X_s^x|\right) + 2K_z\left\|\sigma^{-1}\right\|_\infty |Z_s^x|\right)\mathbb{1}_{|Z_s^x|>n} + K_x\left|X_s^x - X_s^{n,x}\right| \\ + K_z\left|Z_s^x\right|\|\sigma\|_{\mathrm{lip}}\left\|\sigma^{-1}\right\|_\infty^2\left|X_s^x - X_s^{n,x}\right| + K_z\left\|\sigma^{-1}\right\|_\infty|z| \quad =: f_s + \lambda|z|.\end{aligned}$$

We define $F = \int_0^T f_s\,\mathrm{d}s$; because $F \in \mathrm{L}^2$ and $\delta\widetilde{Y}^n \in \mathcal{S}^2$, we get, $\mathbb{E}\left[\left|\delta\widetilde{Y}^n\right|_{0,T}^{*,2}\right] \leqslant Ce^{2\lambda^2 T}\mathbb{E}\left[\left|\delta\widetilde{Y}_T^n\right|^2 + F^2\right]$.
We just need an upper bound for $\mathbb{E}\left[F^2\right]$:

$$\begin{aligned}\mathbb{E}\left[F^2\right] \leqslant 5T^2K_x^2\mathbb{E}\left[\left|X^x - X^{n,x}\right|_{0,T}^{*,2}\right] + 5K_z^2\|\sigma\|_{\mathrm{lip}}^2\left\|\sigma^{-1}\right\|_\infty^4\mathbb{E}\left[\left|X^x - X^{n,x}\right|_{0,T}^{*,2}\left(\int_0^T|Z_s^x|\,\mathrm{d}s\right)^2\right] + 5T^2\|\rho-\rho^n\|_\infty^2 \\ + 40M_\psi^2\mathbb{E}\left[\left(1+\left|X^x\right|_{0,T}^{*,2}\right)\left(\int_0^T\mathbb{1}_{|Z_s^x|>n}\,\mathrm{d}s\right)^2\right] + 20K_z^2\left\|\sigma^{-1}\right\|_\infty^2\mathbb{E}\left[\left(\int_0^T|Z_s^x|\,\mathbb{1}_{|Z_s^x|>n}\,\mathrm{d}s\right)^2\right].\end{aligned}$$

Thanks to the Lemma 2.1 of [MZ02], we have: $\forall p \geqslant 2$, $\exists C_{T,p} > 0$, $\mathbb{E}\left[\left|X^x - X^{n,x}\right|_{0,T}^{*,p}\right] \leqslant C_{T,p}\|\Xi - \Xi^n\|_\infty^p$.
Moreover, we have, for every $\alpha \in (0,2]$:

$$\mathbb{E}\left[\left(\int_0^T\mathbb{1}_{|Z_s^x|>n}\,\mathrm{d}s\right)^2\right] \leqslant T\int_0^T\mathbb{P}\left(|Z_s^x|>n\right)\mathrm{d}s \leqslant \frac{T}{n^\alpha}\int_0^T\mathbb{E}\left[|Z_s^x|^\alpha\right]\mathrm{d}s \leqslant \frac{T^{2-\frac{\alpha}{2}}}{n^\alpha}\mathbb{E}\left[\int_0^T|Z_s^x|^2\,\mathrm{d}s\right]^{\frac{\alpha}{2}}.$$

We also recall – see [BDH$^+$03] – for $p > 1$: $\mathbb{E}\left[\left(\int_0^T|Z_s^x|^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \leqslant C_{p,T}\left(1+|x|^{2p}\right)$, because v has quadratic growth. Using Cauchy-Schwarz inequality and the above remarks, we get:

$$\mathbb{E}\left[F^2\right] \leqslant C_T\left(\|\rho-\rho^n\|_\infty^2 + \left(\|\Xi-\Xi^n\|_\infty^2 + \frac{1}{n^\varepsilon}\right)\left(1+|x|^{2(2+\varepsilon)}\right)\right),$$

for every $\varepsilon > 0$. ■

Lemma 24

We have: $\forall\varepsilon > 0$, $\forall T > 0$, $\exists C_T > 0$, $\forall x \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$,

$$\begin{aligned}&|u_T(0,x) - u_T^n(0,x)|^2 \\ &\quad\leqslant C_T\left\{\mathbb{E}\left[\left|g\left(X_T^x\right) - g^n\left(X_T^{n,x}\right)\right|^2\right] + \|\rho-\rho^n\|_\infty^2 + \left(\|\Xi-\Xi^n\|_\infty^2 + \frac{1}{n^\varepsilon}\right)\left(1+|x|^{(2+\varepsilon)\mu}\right)\right\}.\end{aligned}$$

Proof:
The proof is essentially the same; the main difference is that the exponent μ appears in the following inequality (see [BDH$^+$03]): $\mathbb{E}\left[\left(\int_0^T\left|Z_s^{T,0,x}\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \leqslant C_{p,T}\left(1+\mathbb{E}\left[\left|Y_T^{T,0,x}\right|^p\right]\right)$. ■

We keep in mind the following results:

$$\begin{aligned}|w_T^n(0,x) - w_T(0,x)|^2 \leqslant C_T\left\{\mathbb{E}\left[\left|g\left(X_T^x\right) - g^n\left(X_T^{n,x}\right)\right|^2 + \left|v\left(X_T^x\right) - v^n\left(X_T^{n,x}\right)\right|^2\right]\right. \\ \left. + \|\rho-\rho^n\|_\infty^2 + \left(\|\Xi-\Xi^n\|_\infty^2 + \frac{1}{n^\varepsilon}\right)\left(1+|x|^{(2+\varepsilon)\mu}\right)\right\}, \\ |w_T(0,x)| \leqslant C\left(1+|x|^\mu\right).\end{aligned}$$

Step 3: Variation of $w_T^n(0,\bullet)$.
We write:

$$Y_0^{n,x} - \widetilde{Y}_0^{n,x} = Y_T^{n,x} - \widetilde{Y}_T^{n,x} + \int_0^T \left\{\psi^n\left(X_s^{n,x}, Z_s^{n,x}\right) - \psi^n\left(X_s^{n,x}, \widetilde{Z}_s^{n,x}\right)\right\} \mathrm{d}s - \int_0^T \left\{Z_s^{n,x} - \widetilde{Z}_s^{n,x}\right\} \mathrm{d}W_s.$$

We have $Z_s^{n,x} = \partial_x u_T^n\left(s, X_s^{n,x}\right)\sigma\left(X_s^{n,x}\right)$ and $\widetilde{Z}_s^{n,x} = \partial_x \widetilde{u}_T^n\left(s, X_s^{n,x}\right)\sigma\left(X_s^{n,x}\right)$ (see the Theorem 3.1 of [MZ02]). Consequently, $Z_s^{n,x} - \widetilde{Z}_s^{n,x} = \partial_x w_T^n\left(s, X_s^{n,x}\right)\sigma\left(X_s^{n,x}\right)$. We define the following function:

$$\beta_T^n(t,x) = \begin{cases} \dfrac{\psi^n\left(x, \partial_x u_T^n(t,x)\sigma(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}_T^n(t,x)\sigma(x)\right)}{\left|\partial_x w_T^n(t,x)\sigma(x)\right|^2}\left(\partial_x w_T^n(t,x)\sigma(x)\right)^*, & \text{if } t < T \text{ and } \partial_x w_T^n(t,x) \neq 0, \\ \dfrac{\psi^n\left(x, \partial_x u_T^n(T,x)\sigma(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}_T^n(T,x)\sigma(x)\right)}{\left|\partial_x w_T^n(T,x)\sigma(x)\right|^2}\left(\partial_x w_T^n(T,x)\sigma(x)\right)^*, & \text{if } t \geqslant T \text{ and } \partial_x w_T^n(T,x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The function β_T^n is bounded by $K_z\left\|\sigma^{-1}\right\|_\infty$; $\widetilde{W}_t = W_t - \int_0^t \beta_T^n\left(s, X_s^{n,x}\right)\mathrm{d}s$ is a Brownian motion on $[0,T]$ under the probability $\mathbb{Q}_T^n$. Because $\psi^n\left(X_s^{n,x}, Z_s^{n,x}\right) - \psi^n\left(X_s^{n,x}, \widetilde{Z}_s^{n,x}\right) = \left(Z_s^{n,x} - \widetilde{Z}_s^{n,x}\right)\beta_T^n\left(s, X_s^{n,x}\right)$, we have:

$$w_T^n(0,x) = Y_0^{n,x} - \widetilde{Y}_0^{n,x} = \mathbb{E}^{\mathbb{Q}_T^n}\left[Y_T^{n,x} - \widetilde{Y}_T^{n,x}\right] = \mathbb{E}^{\mathbb{Q}_T^n}\left[g^n\left(X_T^{n,x}\right) - v^n\left(X_T^{n,x}\right)\right] = \mathcal{P}_T^n\left[g^n - v^n\right](x),$$

where $\mathcal{P}^n$ is the Kolmogorov semigroup of the SDE: $\mathrm{d}U_t = \Xi^n\left(U_t\right)\mathrm{d}t + \sigma\left(U_t\right)\left(\mathrm{d}W_t + \beta_T^n\left(t, U_t\right)\mathrm{d}t\right)$. By Lemma 15 and corollary 9, we get:

$$\left|w_T^n(0,x) - w_T^n(0,y)\right| = \left|\mathcal{P}_T^n\left[g^n - v^n\right](x) - \mathcal{P}_T^n\left[g^n - v^n\right](y)\right| \leqslant \widehat{c}_n c_{g^n - v^n}\left(1 + |x|^\mu + |y|^\mu\right)\mathrm{e}^{-\widehat{\nu}_n T},$$

where $\widehat{c}_n$ and $\widehat{\nu}_n$ depend only on η_1, η_2, r_1, r_2, K_z and $\left\|\sigma^{-1}\right\|_\infty$; also $c_{g^n - v^n}$ is the constant appearing in the μ-polynomial growth of $g^n - v^n$: it only depends on $g - v$ and is independent of n. So:

$$\exists \widehat{c}, \widehat{\nu} > 0,\ \forall T > 0,\ \forall x, y \in \mathbb{R}^d,\ \forall n \in \mathbb{N}^*,\ \left|w_T^n(0,x) - w_T^n(0,y)\right| \leqslant \widehat{c}\left(1 + |x|^\mu + |y|^\mu\right)\mathrm{e}^{-\widehat{\nu}T}. \tag{12}$$

Step 4: Upperbound for $\partial_x w_T^n(0,\bullet)$.
We use the Theorem 4.2 of [MZ02]; for every $T' \in (t,T]$, we have:

$$\partial_x u_T^n(t,x) = \mathbb{E}\left[u_T^n\left(T', X_{T'}^{n,t,x}\right)N_{T'}^{n,t,x} + \int_t^{T'} \psi^n\left(X_s^{n,t,x}, Z_s^{n,t,x}\right)N_s^{n,t,x}\,\mathrm{d}s\right],$$

$$\partial_x \widetilde{u}_T^n(t,x) = \mathbb{E}\left[\widetilde{u}_T^n\left(T', X_{T'}^{n,t,x}\right)N_{T'}^{n,t,x} + \int_t^{T'} \psi^n\left(X_s^{n,t,x}, \widetilde{Z}_s^{n,t,x}\right)N_s^{n,t,x}\,\mathrm{d}s\right],$$

where $N_s^{n,t,x} = \dfrac{1}{s-t}\left(\int_t^s \left(\sigma\left(X_r^{n,t,x}\right)^{-1}\nabla X_r^{n,t,x}\right)^*\mathrm{d}W_r\right)^*$. By substraction:

$$\partial_x w_T^n(t,x) = \mathbb{E}\left[w_{T-T'}^n\left(0, X_{T'}^{n,t,x}\right)N_{T'}^{n,t,x} + \int_t^{T'}\left\{\psi^n\left(X_s^{n,t,x}, Z_s^{n,t,x}\right) - \psi^n\left(X_s^{n,t,x}, \widetilde{Z}_s^{n,t,x}\right)\right\}N_s^{n,t,x}\,\mathrm{d}s\right].$$

By Itô's formula, we prove that: $\mathrm{d}\left|\nabla X_s^{n,t,x}\right|_F^2 \leqslant \left(2\|\Xi\|_{\mathrm{lip}} + \|\sigma\|_{\mathrm{lip}}^2\right)\left|\nabla X_s^{n,t,x}\right|_F^2\mathrm{d}s + \mathrm{d}M_s$, with M a martingale starting at 0. We set $\lambda = 2\|\Xi\|_{\mathrm{lip}} + \|\sigma\|_{\mathrm{lip}}^2$ and we get: $\mathbb{E}\left[\left|\nabla X_s^{n,t,x}\right|_F^2\right] \leqslant d^2\mathrm{e}^{\lambda(s-t)}$. Then, for any $s \in [t,T']$,

$$\mathbb{E}\left[\left|N_s^{n,t,x}\right|^2\right] \leqslant \left(\frac{1}{s-t}\right)^2\mathbb{E}\left[\int_t^s \left|\sigma\left(X_r^{n,t,x}\right)^{-1}\nabla X_r^{n,t,x}\right|_F^2\mathrm{d}r\right] \leqslant \frac{d^2\left\|\sigma^{-1}\right\|_\infty^2}{s-t}\mathrm{e}^{\lambda(T'-t)}.$$

Recalling that $(x,p) \mapsto \psi^n(x, p\sigma(x))$ is Lipschitz, we obtain:

$$\begin{aligned}
\left|\partial_x w_T^n(t,x)\right| &\leqslant \mathbb{E}\left[\left|w_{T-T'}^n\left(0, X_{T'}^{n,t,x}\right)\right|\left|N_{T'}^{n,t,x}\right|\right] + \int_t^{T'}\mathbb{E}\left[\left|\psi^n\left(X_s^{n,t,x}, Z_s^{n,t,x}\right) - \psi^n\left(X_s^{n,t,x}, \widetilde{Z}_s^{n,t,x}\right)\right|\left|N_s^{n,t,x}\right|\right]\mathrm{d}s \\
&\leqslant \mathbb{E}\left[\left|w_{T-T'}^n\left(0, X_{T'}^{n,t,x}\right)\right|^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left|N_{T'}^{n,t,x}\right|^2\right]^{\frac{1}{2}} + K_z\int_t^{T'}\mathbb{E}\left[\left|\partial_x w_T^n\left(s, X_s^{n,t,x}\right)\right|^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left|N_s^{n,t,x}\right|^2\right]^{\frac{1}{2}}\mathrm{d}s \\
&\leqslant \mathbb{E}\left[\left|w_{T-T'}^n\left(0, X_{T'}^{n,t,x}\right)\right|^2\right]^{\frac{1}{2}}\frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}}\mathrm{e}^{\frac{\lambda}{2}(T'-t)} + K_z d\left\|\sigma^{-1}\right\|_\infty \mathrm{e}^{\frac{\lambda T'}{2}}\int_t^{T'}\mathbb{E}\left[\left|\partial_x w_T^n\left(s, X_s^{n,t,x}\right)\right|^2\right]^{\frac{1}{2}}\frac{\mathrm{d}s}{\sqrt{s-t}}.
\end{aligned}$$

Thanks to step 2 and proposition 4, we have the following upperbound (for any $\varepsilon > 0$):

$$\mathbb{E}\left[\left|w^n_{T-T'}\left(0, X^{n,t,x}_{T'}\right)\right|^2\right] \leqslant C_{T'}\left(1+|x|^{2\mu}\right) + C_{T-T'}\left(\Delta^{n,x}_t + \|\rho-\rho^n\|^2_\infty + C_{T'}\left(\|\Xi-\Xi^n\|^2_\infty + \frac{1}{n^\varepsilon}\right)\left(1+|x|^{(2+\varepsilon)\mu}\right)\right),$$

where $\Delta^{n,x}_t = \mathbb{E}\left[\left|v\left(X^{0,X^{n,t,x}_{T'}}_{T-T'}\right) - v^n\left(X^{n,0,X^{n,t,x}_{T'}}_{T-T'}\right)\right|^2 + \left|g\left(X^{0,X^{n,t,x}_{T'}}_{T-T'}\right) - g^n\left(X^{n,0,X^{n,t,x}_{T'}}_{T-T'}\right)\right|^2\right]$. So, for every $\varepsilon > 0$,

$$|\partial_x w^n_T(t,x)| \leqslant \left\{C_{T'}\left(1+|x|^\mu\right) + C_{T-T'}\left(\sqrt{\Delta^{n,x}_t} + \|\rho-\rho^n\|_\infty + C_{T'}\left(\|\Xi-\Xi^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)\left(1+|x|^{(1+\varepsilon)\mu}\right)\right)\right\}$$
$$\times \frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}} \mathrm{e}^{\frac{\lambda}{2}(T'-t)} + K_z d\left\|\sigma^{-1}\right\|_\infty \mathrm{e}^{\frac{\lambda T'}{2}} \int_t^{T'} \mathbb{E}\left[\left|\partial_x w^n_T\left(s, X^{t,x}_s\right)\right|^2\right]^{\frac{1}{2}} \frac{\mathrm{d}s}{\sqrt{s-t}}.$$

Let us take $\zeta > \mu$; we set $\varphi^n_T(t) = \sup_{x\in\mathbb{R}^d} \frac{\left|\partial_x w^n_T(t,x)\right|}{1+|x|^\zeta}$. Using the appendix, we see that:

$$|\partial_x w^n_T(t,x)| \leqslant |\partial_x u^n_T(t,x)| + |\partial_x \widetilde{u}^n_T(t,x)| \leqslant C\left(1+|x|^\mu\right).$$

This proves that φ^n_T is well defined on $[0,T)$; moreover, it is bounded on every set $[0,T']$, for every $T' < T$. Also, for every $s \in [t,T']$, $\mathbb{E}\left[\left|\partial_x w^n_T\left(s, X^{t,x}_s\right)\right|^2\right]^{\frac{1}{2}} \leqslant \varphi^n_T(s) C_{T'}\left(1+|x|^\zeta\right)$, where $C_{T'}$ is independent of T and x. We get:

$$\frac{\left|\partial_x w^n_T(t,x)\right|}{1+|x|^\zeta} \leqslant \left\{C_{T'} + C_{T-T'}\left(\frac{\sqrt{\Delta^{n,x}_t}}{1+|x|^\zeta} + \|\rho-\rho^n\|_\infty + C_{T'}\left(\|\Xi-\Xi^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)\right)\right\} \frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{T'-t}} \mathrm{e}^{\frac{\lambda}{2}(T'-t)}$$
$$+ C_{T'} \int_t^{T'} \frac{\varphi^n_T(s)}{\sqrt{s-t}}\,\mathrm{d}s.$$

For ε small enough, we can take the supremum when $x \in \mathbb{R}^d$, and by a change of variable, it can be rewritten as:

$$\varphi^n_T(T'-t) \leqslant \underbrace{\left\{C_{T'} + C_{T-T'}\left(\sup_{x\in\mathbb{R}^d} \frac{\sqrt{\Delta^{n,x}_{T'-t}}}{1+|x|^\zeta} + \|\rho-\rho^n\|_\infty + C_{T'}\left(\|\Xi-\Xi^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)\right)\right\} \frac{d\left\|\sigma^{-1}\right\|_\infty}{\sqrt{t}} \mathrm{e}^{\frac{\lambda}{2}t}}_{=a(t)}$$
$$+ C_{T'} \int_0^t \frac{\varphi^n_T(T'-u)}{\sqrt{t-u}}\,\mathrm{d}u,$$

where $0 < t \leqslant T' < T$. We use the Lemma 7.1.1 of [Hen81]; indeed, the function a is locally integrable on $(0,T')$, since $\sup_{x\in\mathbb{R}^d} \frac{\sqrt{\Delta^{n,x}_{T'-t}}}{1+|x|^\zeta}$ is bounded independently of t. We get $\varphi^n_T(T'-t) \leqslant a(t) + C_{T'} \int_0^t E'\left(C_{T'}(t-u)\right) a(u)\,\mathrm{d}u$, and the integral is well defined, since $E(z) = \sum_{n=0}^{\infty} \frac{z^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ is $\mathcal{C}^1$ and $E'(z) \underset{z\to 0^+}{\sim} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{z}}$. Choose $t = T'$ and get:

$$|\partial_x w^n_T(0,x)| \leqslant \left[C_{T'} + C_{T-T',T'}\gamma^n_{T'}(0) + C_{T'} \int_0^{T'} E'\left(C_{T'}(T'-u)\right)\left\{C_{T'} + C_{T-T'}\gamma^n_{T'}(T'-u)\right\} \frac{\mathrm{d}u}{\sqrt{u}}\right]\left(1+|x|^\zeta\right),$$

where we denoted $\gamma^n_{T'}(t) = \sup_{x\in\mathbb{R}^d} \frac{\sqrt{\Delta^{n,x}_t}}{1+|x|^\zeta} + \|\rho-\rho^n\|_\infty + C_{T'}\left(\|\Xi-\Xi^n\|_\infty + n^{-\frac{\varepsilon}{2}}\right)$.

Step 5: Taking the limit when $n \to \infty$. From Theorem 21 and equation (12), we get easily:

$$\exists C > 0,\ \forall T > 0,\ \forall x \in \mathbb{R}^d,\ |w_T(0,x)| \leqslant C\left(1+|x|^\mu\right), \tag{13}$$

$$\exists \widehat{c}, \widehat{\nu} > 0,\ \forall T > 0,\ \forall x, y \in \mathbb{R}^d,\ |w_T(0,x) - w_T(0,y)| \leqslant \widehat{c}\,(1 + |x|^\mu + |y|^\mu)\,\mathrm{e}^{-\widehat{\nu}T}. \tag{14}$$

The function w_T has no reason to be $\mathcal{C}^1$, so we use the mean value theorem:

$$\frac{\left|w_T^n(0,x) - w_T^n(0,y)\right|}{\left(1 + |x|^\zeta + |y|^\zeta\right)|x-y|} \leqslant \left[C_{T'} \int_0^{T'} E'\left(C_{T'}(T'-u)\right)\left\{C_{T'} + C_{T-T'}\gamma_{T'}^n(T'-u)\right\}\frac{\mathrm{d}u}{\sqrt{u}} + C_{T'} + C_{T-T',T'}\gamma_{T'}^n(0)\right].$$

Our next goal will be to get an upperbound for $\gamma_{T'}^n$. Let $R > 0$; we denote $\alpha = \zeta - \mu$;

$$\begin{aligned}
\mathbb{E}\left[\left|(g-g^n)\left(X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right)\right|^2 \mathbb{1}_{\left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right| > R}\right] &\leqslant \mathbb{E}\left[\left|(g-g^n)\left(X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right)\right|^4\right]^{\frac{1}{2}} \mathbb{P}\left(\left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right| > R\right)^{\frac{1}{2}} \\
&\leqslant C\mathbb{E}\left[\left(1 + \left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right|^\mu\right)^4\right]^{\frac{1}{2}} \frac{\mathbb{E}\left[\left|X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right|^{2\alpha}\right]^{\frac{1}{2}}}{R^\alpha} \\
&\leqslant C_{T-T',T'}\frac{1 + |x|^{2\mu+\alpha}}{R^\alpha}.
\end{aligned}$$

By our proposition 4 and the Lemma 2.1 of [MZ02], we get:

$$\mathbb{E}\left[\left|g\left(X_{T-T'}^{0,X_{T'}^{n,t,x}}\right) - g\left(X_{T-T'}^{n,0,X_{T'}^{n,t,x}}\right)\right|^2\right] \leqslant C_{T-T',T'}\left(1 + |x|^{2\mu}\right)\|\Xi - \Xi^n\|_\infty^2.$$

We can do exactly the same with g and g^n replaced by v and v^n. This way, for any $R > 0$:

$$\sup_{x\in\mathbb{R}^d} \frac{\sqrt{\Delta_{T'-t}^{n,x}}}{1 + |x|^\zeta} \leqslant C_{T-T',T'}\left(R^{-\frac{\alpha}{2}} + \|\Xi - \Xi^n\|_\infty\right) + \|g - g^n\|_{\infty,\mathcal{B}(0,R)} + \|v - v^n\|_{\infty,\mathcal{B}(0,R)}.$$

To sum up, we have: $\forall \alpha > 0$, $\forall T' > 0$, $\exists C_{T'} > 0$, $\forall T > T'$, $\exists C_{T-T',T'} > 0$, $\forall x, y \in \mathbb{R}^d$, $\forall R > 0$, $\forall n \in \mathbb{N}^*$,

$$\begin{aligned}
&|w_T^n(0,x) - w_T^n(0,y)|\left(1 + |x|^{\mu+\alpha} + |y|^{\mu+\alpha}\right)^{-1}|x-y|^{-1} \\
&\leqslant \Big[C_{T'} + C_{T-T',T'}\left(R^{-\frac{\alpha}{2}} + \|\Xi - \Xi^n\|_\infty + \|g - g^n\|_{\infty,\mathcal{B}(0,R)} + \|v - v^n\|_{\infty,\mathcal{B}(0,R)}\right) \\
&+ C_{T'}\int_0^{T'} E'\left(C_{T'}(T'-u)\right)\left\{C_{T'} + C_{T-T',T'}\left(R^{-\frac{\alpha}{2}} + \|\Xi - \Xi^n\|_\infty + \|g - g^n\|_{\infty,\mathcal{B}(0,R)} + \|v - v^n\|_{\infty,\mathcal{B}(0,R)}\right)\right\}\frac{\mathrm{d}u}{\sqrt{u}}\Big]
\end{aligned}$$

We take the limit as n and then R go to the infinity: for every $\alpha > 0$,

$$\forall T' > 0,\ \exists \mathcal{C}_{T'} > 0,\ \forall T > T',\ \forall x, y \in \mathbb{R}^d,\ |w_T(0,x) - w_T(0,y)| \leqslant \mathcal{C}_{T'}\left(1 + |x|^{\mu+\alpha} + |y|^{\mu+\alpha}\right)|x-y|.$$

Step 6: Convergence of $w_T(0,\bullet)$ when $T \to \infty$.

Thanks to equation (13), by a diagonal argument, there exists an increasing sequence $(T_i)_{i\in\mathbb{N}}$, with limit $+\infty$; a function $w : D \to \mathbb{R}$, with $D \subset \mathbb{R}^d$ countable and dense, such that $\left(w_{T_i}\right)_i$ converges pointwise over D to w. The equation (14), with $x, y \in D$, tells us that w is equal to a constant L on D. The same equation, with $x \in \mathbb{R}^d$ and $y \in D$, gives us that $w = L$ on the whole $\mathbb{R}^d$. Let K be a compact in $\mathbb{R}^d$; we define $\mathcal{A} = \{w_T(0,\bullet)|_K | T > 1\} \subset \mathcal{C}(K,\mathbb{R})$; it is equicontinuous, because $w_T(0,\bullet)$ is uniformly Lipschitz. For all $x \in K$, the set $\mathcal{A}(x) = \{w_T(0,x)|T > 1\}$ is bounded (see equation (13). By Ascoli's theorem, $\mathcal{A}$ is relatively compact in $\mathcal{C}(K,\mathbb{R})$. Let $x \in K$,

$$\left|w_{T_i}(0,x) - L\right| \leqslant \left|w_{T_i}(0,x) - w_{T_i}(0,0)\right| + \left|w_{T_i}(0,0) - L\right| \leqslant \widehat{c}\,(1 + |x|^\mu)\,\mathrm{e}^{-\widehat{\nu}T_i} + \left|w_{T_i}(0,0) - L\right|,$$

which proves that the functions $w_{T_i}(0,\bullet)$ converge uniformly to L on K. So L is an accumulation point of $\mathcal{A}$. Since $\mathcal{A}$ is relatively compact, if we prove that L is the unique accumulation point of $\mathcal{A}$, then, we get the uniform convergence of $w_T(0,\bullet)$ to L on K. It works for any compact $K \subset \mathbb{R}^d$, and L does not depend on K. We conclude:

$$\forall x \in \mathbb{R}^d,\ w_T(0,x) \underset{T\to\infty}{\longrightarrow} L.$$

Step 7: L is the only accumulation point of $\mathcal{A}$.

Let $w_{\infty,K}$ an accumulation point of $\mathcal{A}$, *i.e.* $\left\| w_{T_i'}(0,\bullet)\big|_K - w_{\infty,K} \right\|_\infty \underset{i\to\infty}{\longrightarrow} 0$, for a sequence $\left(T_i'\right)$. Again, the equation (14) tells us that $w_{\infty,K}$ is equal to a constant L_K on K. Our goal is now to show that $L = L_K$. Be careful: now, Y^n and $\widetilde{Y}^n$ will follow the same equations as before, but their terminal conditions will be given at time $T+S$. So,

$$w^n_{T+S}(0,x) = Y^n_0 - \widetilde{Y}^n_0 = Y^n_S - \widetilde{Y}^n_S + \int_0^S \left\{ \psi^n\left(X^x_r, Z^n_r\right) - \psi^n\left(X^x_r, \widetilde{Z}^n_r\right) \right\} \mathrm{d}r - \int_0^S \left\{ Z^n_r - \widetilde{Z}^n_r \right\} \mathrm{d}W_r.$$

We define the function:

$$\beta^n_{T,S}(t,x) = \begin{cases} \dfrac{\psi^n\left(x, \partial_x u^n_{T+S}(t,x)\sigma(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}^n_{T+S}(t,x)\sigma(x)\right)}{\left|\partial_x w^n_{T+S}(t,x)\sigma(x)\right|^2} \left(\partial_x w^n_{T+S}(t,x)\sigma(x)\right)^*, & \\ & \text{if } t < S \text{ and } \partial_x w^n_{T+S}(t,x) \neq 0, \\ \dfrac{\psi^n\left(x, \partial_x u^n_{T+S}(S,x)\sigma(x)\right) - \psi^n\left(x, \partial_x \widetilde{u}^n_{T+S}(S,x)\sigma(x)\right)}{\left|\partial_x w^n_{T+S}(S,x)\sigma(x)\right|^2} \left(\partial_x w^n_{T+S}(S,x)\sigma(x)\right)^*, & \\ & \text{if } t \geqslant S \text{ and } \partial_x w^n_{T+S}(S,x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

This function is bounded by $K_z \left\|\sigma^{-1}\right\|_\infty$; $\widetilde{W}_t = W_t - \int_0^t \beta^n_{T,s}\left(r, X^x_r\right) \mathrm{d}r$ is a Brownian motion on $[0,S]$ under the probability $\mathbb{Q}^{T,S,n,x}$. This way, we can write:

$$w^n_{T+S}(0,x) = \mathbb{E}^{\mathbb{Q}^{T,S,n,x}}\left[Y^n_S - \widetilde{Y}^n_S\right] = \mathbb{E}^{\mathbb{Q}^{T,S,n,x}}\left[w^n_{T+S}\left(S, X^x_S\right)\right] = \mathbb{E}^{\mathbb{Q}^{T,S,n,x}}\left[w^n_T\left(0, X^x_S\right)\right] = \mathcal{P}^n_S\left[w^n_T(0,\bullet)\right](x),$$

where $\mathcal{P}^n$ is the Kolmogorov semigroup of the SDE: $\mathrm{d}U_t = \Xi\left(U_t\right)\mathrm{d}t + \sigma\left(U_t\right)\left(\mathrm{d}W_t + \beta^n_{T,S}\left(t, U_t\right)\mathrm{d}t\right)$. Without any loss of generality, we can suppose that $T_i > T_i'$, for every $i \in \mathbb{N}$. We take $T = T_i'$ and $S = T_i - T_i'$. This way:

$$\forall x \in \mathbb{R}^d,\ \mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) = w^n_{T_i}(0,x) \underset{n\to\infty}{\longrightarrow} w_{T_i}(0,x) \underset{i\to\infty}{\longrightarrow} L.$$

Now, to prove that $L = L_K$, it suffices to show that $\lim\limits_{i\to\infty} \limsup\limits_{n\to\infty} \mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) = L_K$. We have:

$$\left|\mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - L_K\right| \leqslant \left|\mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - w^n_{T_i'}(0,x)\right| + \left|w^n_{T_i'}(0,x) - w_{T_i'}(0,x)\right| + \left|w_{T_i'}(0,x) - L_K\right|,$$

$$\limsup_{n\to\infty}\left|\mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - L_K\right| \leqslant \limsup_{n\to\infty}\left|\mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - w^n_{T_i'}(0,x)\right| + \left|w_{T_i'}(0,x) - L_K\right|.$$

Thanks to Lemma 15, $\beta^n_{T,S}$ is the limit of $\mathcal{C}^1$-functions, uniformly bounded, with bounded derivatives $\left(\beta^{n,m}_{T,S}\right)_{m\geqslant 0}$. We define $U^{m,x}$ as the solution of the SDE: $\mathrm{d}U^{m,x}_t = \Xi\left(U^{m,x}_t\right)\mathrm{d}t + \sigma\left(U^{m,x}_t\right)\left(\mathrm{d}W_t + \beta^{n,m}_{T,S}\left(t, U^{m,x}_t\right)\mathrm{d}t\right)$. Thus,

$$\begin{aligned} \left|\mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - w^n_{T_i'}(0,x)\right| &= \left|\lim_{m\to\infty}\mathbb{E}\left[w^n_{T_i'}\left(0, U^{m,x}_{T_i - T_i'}\right)\right] - w^n_{T_i'}(0,x)\right| \\ &\leqslant \limsup_{m\to\infty}\mathbb{E}\left[\left|w^n_{T_i'}\left(0, U^{m,x}_{T_i - T_i'}\right) - w^n_{T_i'}(0,x)\right|\right] \\ &\leqslant \widehat{c}\left(1 + \limsup_{m\to\infty}\mathbb{E}\left[\left|U^{m,x}_{T_i - T_i'}\right|^\mu\right] + |x|^\mu\right)\mathrm{e}^{-\widehat{\nu}T_i'}, \end{aligned}$$

and the constants $\widehat{c}$ and $\widehat{\nu}$ are independent of n, m and T_i'. Thanks to proposition 5, we have $\mathbb{E}\left[\left|U^{m,x}_t\right|^\mu\right] \leqslant C\left(1 + |x|^\mu\right)$, with C independent of n, m and t. So,

$$\left|\mathcal{P}^n_{T_i - T_i'}\left[w^n_{T_i'}(0,\bullet)\right](x) - w^n_{T_i'}(0,x)\right| \leqslant \widetilde{C}\left(1 + |x|^\mu\right)\mathrm{e}^{-\nu T_i'},$$

with $\widetilde{C}$ and ν independent of n and T_i'. And we can conclude that $L = L_K$.

Step 8: Speed of convergence. Let $x \in \mathbb{R}^d$ and $T > 0$. We have:

$$\begin{aligned}|w_T(0,x) - L| &= \lim_{V\to\infty} |w_T(0,x) - w_V(0,x)| = \lim_{V\to\infty}\lim_{n\to\infty} |w_T^n(0,x) - w_V^n(0,x)| \\ &= \lim_{V\to\infty}\lim_{n\to\infty} \left|w_T^n(0,x) - \mathcal{P}_{V-T}^n\left[w_T^n(0,\bullet)\right](x)\right|,\end{aligned}$$

where $\mathcal{P}^n$ is the Kolmogorov semigroup of the SDE: $\mathrm{d}U_t = \Xi\left(U_t\right)\,\mathrm{d}t + \sigma\left(U_t\right)\left(\mathrm{d}W_t + \beta_{T,V-T}^n\left(t, U_t\right)\,\mathrm{d}t\right)$. Like before, we use Lemma 15 and proposition 5, and we get, for δ small enough:

$$\begin{aligned}|w_T(0,x) - L| &= \lim_{V\to\infty}\lim_{n\to\infty}\left|w_T^n(0,x) - \lim_{m\to\infty}\mathbb{E}\left[w_T^n\left(0, U_{V-T}^{m,x}\right)\right]\right| \\ &\leqslant \limsup_{V\to\infty}\limsup_{n\to\infty}\limsup_{m\to\infty}\mathbb{E}\left[\left|w_T^n(0,x) - w_T^n\left(0, U_{V-T}^{m,x}\right)\right|\right] \leqslant \widetilde{C}\left(1+|x|^\mu\right)\mathrm{e}^{-\nu T}.\end{aligned}$$

■

6 Application to Optimal ergodic problem and HJB equation

6.1 Optimal ergodic problem

In this section, we apply our results to an ergodic control problems. The proofs of the following results are so similar to the ones of [HMR15] that we omit them. As before, we consider a process X^x satisfying:

$$X_t^x = x + \int_0^t \Xi\left(X_s^x\right)\,\mathrm{d}s + \int_0^t \sigma\left(X_s^x\right)\,\mathrm{d}W_s,$$

where Ξ and σ are Lipschitz continuous, Ξ is weakly dissipative with $\langle \Xi(x), x\rangle \leqslant \eta_1 - \eta_2|x|^2$, the function $x \mapsto \sigma(x)^{-1}$ is bounded and $|\sigma(x)|_F^2 \leqslant r_1 + r_2|x|^2$. Moreover, we pick $\mu \geqslant 2$ such that $\frac{\mu-1}{2}r_2 < \eta_2$. Let U be a separable metric space; we call "control" any progressively measurable U-valued process. We consider some measurable functions satisfying the following assumptions:

- $R : U \to \mathbb{R}^d$ is bounded;
- $L : \mathbb{R}^d \times U \to \mathbb{R}$ is uniformly Lipschitz continuous in $x \in \mathbb{R}^d$ w.r.t. $a \in U$;
- $g : \mathbb{R}^d \to \mathbb{R}$ is continuous, has polynomial growth and is locally Lipschitz continuous: $|g(x)| \leqslant C\left(1+|x|^\mu\right)$ and $|g(x) - g(x')| \leqslant C\left(1+|x|^\mu + |x'|^\mu\right)|x - x'|$.

Let X^x be the solution of the SDE (6); for any control a and horizon $T > 0$, we set:

$$\rho_T^{x,a} = \exp\left(\int_0^T \sigma\left(X_t^x\right)^{-1} R\left(a_t\right)\,\mathrm{d}W_t - \frac{1}{2}\int_0^T \left|\sigma\left(X_t^x\right)^{-1} R\left(a_t\right)\right|^2\,\mathrm{d}t\right) \text{ and } \mathbb{P}_T^{x,a} = \rho_T^{x,a}\mathbb{P} \text{ over } \mathcal{F}_T.$$

We define the finite horizon cost as: $J^T(x,a) = \mathbb{E}_T^{x,a}\left[\int_0^T L\left(X_t^x, a_t\right)\,\mathrm{d}t\right] + \mathbb{E}_T^{x,a}\left[g\left(X_t^x\right)\right]$, and the associated optimal control problem is to minimise $J^T(x,a)$ over the set of controls $a^T : \Omega \times [0,T] \to U$. We also define an other cost, called "ergodic cost": $J(x,a) = \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}_T^{x,a}\left[\int_0^T L\left(X_t^x, a_t\right)\,\mathrm{d}t\right]$, and the associated optimal control problem is to minimise $J(x,a)$ over the set of controls $a : \Omega \times \mathbb{R}_+ \to U$. Due to Girsanov's theorem, $W_t^{a,x} = W_t - \int_0^t \sigma\left(X_s^x\right)^{-1} R\left(a_s\right)\,\mathrm{d}s$ is a Brownian motion on $[0,T]$ under $\mathbb{P}_T^{x,a}$ and

$$\mathrm{d}X_t^x = \left[\Xi\left(X_t^x\right) + R\left(a_t\right)\right]\,\mathrm{d}t + \sigma\left(X_t^x\right)\,\mathrm{d}W_t^{x,a}.$$

We define the Hamiltonian in the following way:

$$\psi(x,z) = \inf_{a\in U}\left\{L(x,a) + z\sigma(x)^{-1}R(a)\right\}. \tag{15}$$

We recall that, if this infimum is attained for every x and z, by Filippov's theorem (see [MW67]), there exists a measurable function $\gamma : \mathbb{R}^d \times \left(\mathbb{R}^d\right)^* \to U$ such that: $\psi(x,z) = L(x,\gamma(x,z)) + z\sigma(x)^{-1}R(\gamma(x,z))$.

Lemma 25

The Hamiltonian ψ_0 satisfies:

- $\forall x \in \mathbb{R}^d$, $|\psi_0(x,0)| \leqslant \left(\|L\|_{\text{lip}} + |L(0,0)|\right)(1+|x|)$;
- $\forall x, x' \in \mathbb{R}^d$, $\forall z, z' \in \left(\mathbb{R}^d\right)^*$, $|\psi_0(x,z) - \psi_0(x',z')| \leqslant \|L\|_{\text{lip}}|x-x'| + \|R\|_\infty \left|z\sigma(x)^{-1} - z'\sigma(x')^{-1}\right|$.

Lemma 26

For every control a, we have: $J^T(x,a) \geqslant Y_0^{T,x}$, where $Y^{T,x}$ is part of the solution of the finite horizon BSDE:

$$Y_t^{T,x} = g\left(X_T^x\right) + \int_t^T \psi\left(X_s^x, Z_s^{T,x}\right) \mathrm{d}s - \int_t^T Z_s^{T,x} \mathrm{d}W_s, \quad \forall t \in [0,T]. \tag{16}$$

Moreover, if the infimum is attained for every x and z in equation (15), we have $J\left(x, \overline{a}^T\right) = Y_0^{T,x}$, where we set $\overline{a}_t^T = \gamma\left(X_t^{x,\overline{a}^T}, \nabla u\left(t, X_t^{x,\overline{a}^T}\right)\sigma\left(X_t^{x,\overline{a}^T}\right)\right)$.

Lemma 27

For every control a, we have: $J(x,a) \geqslant \lambda$, where λ is part of the solution of the EBSDE:

$$Y_t^x = Y_T^x + \int_t^T \left\{\psi\left(X_s^x, Z_s^x\right) - \lambda\right\} \mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \quad \forall 0 \leqslant t \leqslant T < \infty.$$

Moreover, if the infimum is attained for every x and z in equation (15), we have $J\left(x, \overline{a}^T\right) = \lambda$, where we set $\overline{a}_t = \gamma\left(X_t^{x,\overline{a}}, \nabla v\left(X_t^{x,\overline{a}}\right)\sigma\left(X_t^{x,\overline{a}}\right)\right)$.

Theorem 28

For every control a, we have:

$$\liminf_{T\to\infty} \frac{J^T(x,a)}{T} \geqslant \lambda.$$

Moreover, if the infimum is attained for every x and z in equation (15), we have

$$\left|J^T\left(x, \overline{a}^T\right) - J\left(x, \overline{a}\right)T + Y_0^x + L\right| \leqslant C\left(1+|x|^2\right)\mathrm{e}^{-\nu T}.$$

Proof :

This is a straightforward consequence of the previous lemmas and Theorem 22. ■

Remark 29. All the results of this subsection can be rephrased in terms of viscosity solution of PDEs (17) and (18).

6.2 Large time behaviour of viscosity solution of HJB equation

We consider the ergodic PDE:

$$\mathcal{L}v(x) + \psi\left(x, \nabla v(x)\sigma(x)\right) - \lambda = 0, \tag{17}$$

where $\mathcal{L}$ is the generator of the Kolmogorov semigroup of X^x, solution of (2). We recall that the couple (v, λ) is a viscosity subsolution (resp. supersolution) if:

- $v : \mathbb{R}^d \to \mathbb{R}$ is a continuous function with polynomial growth;
- for any function $\phi \in \mathcal{C}^2\left(\mathbb{R}^d, \mathbb{R}\right)$, for every $x \in \mathbb{R}^d$ of local maximum (resp. minimum) of $v - \phi$:

$$\mathcal{L}\phi(x) + \psi(x, \nabla\phi(x)\sigma(x)) - \lambda \geqslant 0 \quad (\text{resp.} \leqslant 0).$$

Proposition 30 (*Existence of ergodic viscosity solution*)

Under Assumption 10, the couple $(\overline{v}, \overline{\lambda})$ obtained with the solution given in Theorem 18 is a viscosity solution of equation (17).

Proof :

Note that we already know that $\overline{v}$ is continuous and has quadratic growth. The proof of this result is classical and can easily be adapted from Theorem 4.3 of [Par96]. ■

Proposition 31 (*Uniqueness of ergodic viscosity solution*)

Let $p > 0$; we suppose that $\sqrt{r_2} K_z \left\|\sigma^{-1}\right\|_\infty + [(p \vee 2) - 1]\dfrac{r_2}{2} < \eta_2$ and that Assumption 10 holds true.
Then uniqueness holds for viscosity solutions (v, λ) of equation (17) in the class of viscosity solutions such that $\exists a \in \mathbb{R}^d, v(a) = \widetilde{v}(a)$ and v and $\widetilde{v}$ have polynomial growth of at most degree p.

Proof :

The proof is quite the same as in Lemma 3.18 of [HM16]. Let (v, λ) and $(\widetilde{v}, \widetilde{\lambda})$ be two viscosity solutions of (17). We fix $T > 0$, and we consider the solution $\left(Y^{T,x}, Z^{T,x}\right)$ of the BSDE

$$Y_t^{T,x} = v\left(X_T^x\right) + \int_t^T \left\{\psi\left(X_s^x, Z_s^{T,x}\right) - \lambda\right\} \mathrm{d}s - \int_t^T Z_s^{T,x} \mathrm{d}W_s, \quad t \in [0, T].$$

Because this BSDE has a unique solution, we can claim that $v(x) = Y_0^{T,x}$. We define a couple $\left(\widetilde{Y}^{T,x}, \widetilde{Z}^{T,x}\right)$ by replacing in the previous equation (v, λ) by $(\widetilde{v}, \widetilde{\lambda})$. Then, for any $T > 0$ and $x \in \mathbb{R}^d$:

$$(v - \widetilde{v})(x) = (v - \widetilde{v})\left(X_T^x\right) + \int_0^T \left\{\psi\left(X_s^x, Z_s^{T,x}\right) - \psi\left(X_s^x, \widetilde{Z}_s^{T,x}\right)\right\} \mathrm{d}s - \left(\lambda - \widetilde{\lambda}\right) T - \int_0^T \left\{Z_s^{T,x} - \widetilde{Z}_s^{T,x}\right\} \mathrm{d}W_s.$$

We set

$$\beta_s = \begin{cases} \dfrac{\psi\left(X_s^x, Z_s^{T,x}\right) - \psi\left(X_s^x, \widetilde{Z}_s^{T,x}\right)}{\left|Z_s^{T,x} - \widetilde{Z}_s^{T,x}\right|^2} \left(Z_s^{T,x} - \widetilde{Z}_s^{T,x}\right)^*, & \text{if } Z_s^{T,x} \neq \widetilde{Z}_s^{T,x}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the process β is bounded by $K_z \left\|\sigma^{-1}\right\|_\infty$, by Girsanov's theorem, there exists a new probability measure $\mathbb{Q}^T$ equivalent to $\mathbb{P}$ and under which $W - \int_0^\bullet \beta_s \, \mathrm{d}s$ is a Brownian motion. Then:

$$\frac{(v - \widetilde{v})(x)}{T} = \frac{\mathbb{E}^{\mathbb{Q}^T}\left[(v - \widetilde{v})\left(X_T^x\right)\right]}{T} - \left(\lambda - \widetilde{\lambda}\right).$$

Thanks to proposition 5 and the polynomial growth of v and $\widetilde{v}$, letting $T \to \infty$ gives us $\lambda = \widetilde{\lambda}$. Applying the same argument as that in Theorem 20, we deduce the uniqueness claimed. ■

We recall the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + \psi\left(x, \nabla u(t,x)\sigma(x)\right) = 0 & \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(T,x) = g(x) & \forall x \in \mathbb{R}^d, \end{cases} \tag{18}$$

whose viscosity solution is linked to the BSDE (16) via $Y_t^{T,x} = u\left(T - t, X_t^x\right)$. We can rephrase Theorem 22 as:

Theorem 32

We consider the equations (17) and (18). We suppose that Assumption 10 holds true; moreover, we assume that $g : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $\forall x, x' \in \mathbb{R}^d$, $|g(x)| \leqslant C\left(1 + |x|^\mu\right)$ and $|g(x) - g(x')| \leqslant C\left(1 + |x|^\mu + |x'|^\mu\right)|x - x'|$ with $\sqrt{r_2} K_z \left\|\sigma^{-1}\right\|_\infty + \dfrac{\mu - 1}{2} r_2 < \eta_2$ and $\mu \geqslant 2$.

Then, there exists $L \in \mathbb{R}$, such that: $\forall x \in \mathbb{R}^d$, $u(T,x) - \lambda T - v(x) \underset{T\to\infty}{\longrightarrow} L$. Furthermore,

$$\forall x \in \mathbb{R}^d,\ \forall T > 0,\ |u(T,x) - \lambda T - v(x) - L| \leqslant C\,(1 + |x|^{\mu})\,\mathrm{e}^{-\nu T}.$$

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A Appendix about the Lipschitz continuity of v^α

We consider the forward-backward system, where X is a vector in $\mathbb{R}^d$, Y a real, Z a row with d real coefficients, and W a Brownian motion in $\mathbb{R}^d$:

$$X_s^{t,x} = x + \int_t^s b\left(X_r^{t,x}\right)\,\mathrm{d}r + \int_t^s \sigma\left(X_r^{t,x}\right)\,\mathrm{d}W_r, \tag{19}$$

$$Y_s^{t,x} = g\left(X_T^{t,x}\right) + \int_s^T f\left(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\,\mathrm{d}r - \int_s^T Z_r^{t,x}\,\mathrm{d}W_r. \tag{20}$$

The first result is an adaptation to our case of the Theorem 3.1 of [MZ02].

Proposition 33

We make the following assumptions:

- $\sigma \in \mathcal{C}^1\left(\mathbb{R}^d, \mathrm{GL}_d(\mathbb{R})\right)$ and $b \in \mathcal{C}^1\left(\mathbb{R}^d, \mathbb{R}^d\right)$ have bounded derivatives;
- the function $\sigma(\bullet)^{-1}$ is bounded;
- $g \in \mathcal{C}^1\left(\mathbb{R}^d, \mathbb{R}\right)$ has bounded derivatives;
- if we define $h : (x, y, p) \mapsto f(x, y, p\sigma(x))$, then $h \in \mathcal{C}^1\left(\mathbb{R}^d \times \mathbb{R} \times \left(\mathbb{R}^d\right)^*, \mathbb{R}\right)$ has bounded derivatives (the bounds of the derivatives of h will be denoted H_x, H_y and H_p w.r.t. x, y and p).

Under those assumptions, we already know that the forward-backward system (19)-(20) admits a unique solution $\left(Y^{t,x}, Z^{t,x}\right)$.
Moreover, we assume:

- $\forall x \in \mathbb{R}^d$, $|f(x,0,0)| \leqslant M_f\,(1+|x|^\mu)$;
- $\forall p \geqslant 2$, $\forall x \in \mathbb{R}^d$, $\sup\limits_{|\varepsilon|\leqslant 1} \mathbb{E}\left[\left|Y^{t,x+\varepsilon}\right|_{t,T}^{*,p}\right] < \infty$.

Then, the function $u : (t,x) \mapsto Y_t^{t,x}$ is of class $\mathcal{C}^1$ w.r.t. x and

$$\forall s \in [t,T],\ Z_s^{t,x} = \partial_x u\left(s, X_s^{t,x}\right)\sigma\left(X_s^{t,x}\right)\ \mathbb{P}\text{-a.s.}$$

Proof : We will prove the proposition in the case $d = 1$, to simplify presentation.

Step 1: we have, for all $p \geqslant 1$:

$$\mathbb{E}\left[\left(\int_t^T \left|Z_r^{t,x}\right|^2 \mathrm{d}r\right)^p\right] \leqslant C e^{2ap(T-t)}\left\{\mathbb{E}\left[\left|Y^{t,x}\right|_{t,T}^{*,2p}\right] + C_T\left(1+|x|^{2p\mu}\right)\right\}, \tag{21}$$

where C comes from the Lemma 3.1 of [BDH$^+$03] and C_T depends only on p, μ, T, $T-t$, r_1, $\|\sigma\|_{\text{lip}}$, $|b(0)|$, $\|b\|_{\text{lip}}$ and M_f. The last assumption of our proposition allows us to bound $\mathbb{E}\left[\left(\int_t^T \left|Z_r^{t,x+\varepsilon}\right|^2 \mathrm{d}r\right)^p\right]$ independently of ε.

Step 2: we will show that: $\forall p \geqslant 2$, $\forall x \in \mathbb{R}^d$, $\mathbb{E}\left[\left|Y^{t,x+\varepsilon} - Y^{t,x}\right|_{t,T}^{*,p} + \left(\int_t^T \left|Z_s^{t,x+\varepsilon} - Z_s^{t,x}\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Let $(t,x) \in [0,T]\times\mathbb{R}$ be fixed. We define, for any $\varepsilon \neq 0$,

$$\nabla X^\varepsilon = \frac{X^{t,x+\varepsilon} - X^{t,x}}{\varepsilon},\ \nabla Y^\varepsilon = \frac{Y^{t,x+\varepsilon} - Y^{t,x}}{\varepsilon},\ \nabla Z^\varepsilon = \frac{Z^{t,x+\varepsilon} - Z^{t,x}}{\varepsilon}.$$

We also define the solution of the variational system:

$$\nabla X_s = 1 + \int_t^s \partial_x b\left(X_r^{t,x}\right)\nabla X_r\,\mathrm{d}r + \int_t^s \partial_x \sigma\left(X_r^{t,x}\right)\nabla X_r\,\mathrm{d}W_r; \tag{22}$$

$$\nabla Y_s = \partial_x g\left(X_T^{t,x}\right)\nabla X_T + \int_s^T \left[\partial_x f\left(\Theta_r^{t,x}\right)\nabla X_r + \partial_y f\left(\Theta_r^{t,x}\right)\nabla Y_r + \partial_z f\left(\Theta_r^{t,x}\right)\nabla Z_r\right]\mathrm{d}r - \int_s^T \nabla Z_r\,\mathrm{d}W_r, \tag{23}$$

where Θ stands for (X,Y,Z). For every $s \in [t,T]$, we have:

$$\nabla Y_s^\varepsilon = g_x^\varepsilon \nabla X_T^\varepsilon + \int_s^T \left[f_x^\varepsilon(r)\nabla X_r^\varepsilon + f_y^\varepsilon(r)\nabla Y_r^\varepsilon + f_z^\varepsilon(r)\nabla Z_r^\varepsilon\right]\mathrm{d}r - \int_s^T \nabla Z_r^\varepsilon\,\mathrm{d}W_r, \tag{24}$$

where $g_x^\varepsilon = \int_0^1 \partial_x g\left(X_T^{t,x} + w\varepsilon\nabla X_T^\varepsilon\right)\mathrm{d}w$, $f_x^\varepsilon(r) = \int_0^1 \partial_x f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w$,

$f_y^\varepsilon(r) = \int_0^1 \partial_y f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w$ and $f_z^\varepsilon(r) = \int_0^1 \partial_z f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w$.

In order to apply the proposition 3.2 of [BDH$^+$03] later; we set

$$i(r,y,z) := \int_0^1 \partial_x f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w\,\nabla X_r^\varepsilon + \int_0^1 \partial_y f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w\,y + \int_0^1 \partial_z f\left(\Theta_r^{t,x} + w\varepsilon\nabla\Theta_r^\varepsilon\right)\mathrm{d}w\,z.$$

This way, we can check that:

$$|i(r,y,z)| \leqslant \underbrace{\left(H_x + H_p\left(\left|Z_r^{t,x}\right| + \left|Z_r^{t,x+\varepsilon}\right|\right)\left\|\sigma^{-1}\right\|_{\text{lip}}\right)\left|\nabla X_r^\varepsilon\right|}_{=i_r} + H_y|y| + H_p\left\|\sigma^{-1}\right\|_\infty |z|,$$

$$I := \int_t^T i_r\,\mathrm{d}r \leqslant C\left(1 + \int_t^T \left|Z_r^{t,x}\right|\mathrm{d}r + \int_t^T \left|Z_r^{t,x+\varepsilon}\right|\mathrm{d}r\right)\left|\nabla X^\varepsilon\right|_{t,T}^*,$$

$$\mathbb{E}\left[|I|^p\right] \leqslant C\left(1+\mathbb{E}\left[\left(\int_t^T \left|Z_r^{t,x}\right|^2 \mathrm{d}r\right)^p\right]^{\frac{1}{2}}+\mathbb{E}\left[\left(\int_t^T \left|Z_r^{t,x+\varepsilon}\right|^2 \mathrm{d}r\right)^p\right]^{\frac{1}{2}}\right)\mathbb{E}\left[\left|\nabla X^\varepsilon\right|_{t,T}^{*,2p}\right]^{\frac{1}{2}}.$$

By the step 1 and the converence of ∇X^ε in $\mathcal{S}^p$, $\mathbb{E}\left[|I|^p\right]$ can clearly be bounded independently of ε. Using the proposition 3.2 of [BDH$^+$03], we are now able to say that, for a great enough,

$$\mathbb{E}\left[\left|\nabla Y^\varepsilon\right|_{t,T}^{*,p}+\left(\int_t^T \left|\nabla Z_r^\varepsilon\right|^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \leqslant C\mathrm{e}^{ap(T-t)}\left(\|\partial_x g\|_\infty \mathbb{E}\left[\left|\nabla X_T^\varepsilon\right|^p\right]+\mathbb{E}\left[I^p\right]\right).$$

Step 3: some dominated convergences. Thanks to the previous step, for all $p \geqslant 2$,

$$\mathbb{E}\left[\left|g_x^\varepsilon-g_x^0\right|^p\right] \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad \mathbb{E}\left[\left(\int_t^T \left|f_x^\varepsilon(r)-f_x^0(r)\right|^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

$$\mathbb{E}\left[\int_t^T \left|f_y^\varepsilon(r)-f_y^0(r)\right|^p \mathrm{d}r\right] \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad \mathbb{E}\left[\int_t^T \left|f_z^\varepsilon(r)-f_z^0(r)\right|^p \mathrm{d}r\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The first, third and fourth convergences are easy to prove since f have bounded derivatives. Also, the inequality $\left|\partial_x f\left(\Theta_r^{t,x}+w\varepsilon\nabla\Theta_r^\varepsilon\right)-\partial_x f\left(\Theta_r^{t,x}\right)\right| \leqslant 2H_x+H_p\left\|\sigma^{-1}\right\|_{\mathrm{lip}}\left(2\left|Z_r^{t,x}\right|+\left|Z_r^{t,x+\varepsilon}\right|\right)$ let us use dominated convergence.

Step 4: we will show that: $\forall x \in \mathbb{R}^d$, $\mathbb{E}\left[\left|\nabla Y^\varepsilon-\nabla Y\right|_{t,T}^{*,2}+\left(\int_t^T \left|\nabla Z_s^\varepsilon-\nabla Z_s\right|^2 \mathrm{d}s\right)\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$.
Combining equations (23) and (24), we get:

$$\Delta Y_s^\varepsilon = g_x^\varepsilon \Delta X_T^\varepsilon+\left(g_x^\varepsilon-g_x^0\right)\nabla X_T+\int_s^T\left[f_x^\varepsilon(r)\nabla X_r^\varepsilon-f_x^0(r)\nabla X_r+f_y^\varepsilon(r)\Delta Y_r^\varepsilon+f_z^\varepsilon(r)\Delta Z_r^\varepsilon+\beta^\varepsilon(r)\right]\mathrm{d}r$$
$$-\int_s^T \Delta Z_r^\varepsilon \,\mathrm{d}W_r,$$

where we set $\beta^\varepsilon(s)=\left[f_y^\varepsilon(r)-f_y^0(r)\right]\nabla Y_r+\left[f_z^\varepsilon(r)-f_z^0(r)\right]\nabla Z_r$, $\Delta X_s^\varepsilon=\nabla X_s^\varepsilon-\nabla X_s$, $\Delta Y_s^\varepsilon=\nabla Y_s^\varepsilon-\nabla Y_s$ and $\Delta Z_s^\varepsilon=\nabla Z_s^\varepsilon-\nabla Z_s$. The Lemma 2.2 of [MZ02] gives:

$$\mathbb{E}\left[\left|\Delta Y^\varepsilon\right|_{t,T}^{*,2}+\int_t^T \left|\Delta Z_r^\varepsilon\right|^2 \mathrm{d}r\right] \leqslant C\mathbb{E}\left[\left|g_x^\varepsilon\Delta X_T^\varepsilon+\left(g_x^\varepsilon-g_x^0\right)\nabla X_T\right|^2+\int_t^T\left|f_x^\varepsilon(r)\nabla X_r^\varepsilon-f_x^0(r)\nabla X_r+\beta^\varepsilon(r)\right|^2 \mathrm{d}r\right].$$

First, $\mathbb{E}\left[\left|g_x^\varepsilon\Delta X_T^\varepsilon+\left(g_x^\varepsilon-g_x^0\right)\nabla X_T\right|^2\right] \leqslant 2\|\partial_x g\|_\infty \mathbb{E}\left[\left|\Delta X_T^\varepsilon\right|^2\right]+2\mathbb{E}\left[\left|g_x^\varepsilon-g_x^0\right|^4\right]^{\frac{1}{2}}\mathbb{E}\left[\left|\nabla X_T\right|^4\right]^{\frac{1}{2}} \underset{\varepsilon\to 0}{\longrightarrow}$ 0, by Lemma 2.1 of [MZ02]. By dominated convergence, we get $\mathbb{E}\left[\int_t^T \left|\beta^\varepsilon(r)\right|^2 \mathrm{d}r\right] \underset{\varepsilon\to 0}{\longrightarrow} 0$, and then

$$\mathbb{E}\left[\int_t^T\left|f_x^\varepsilon(r)\nabla X_r^\varepsilon-f_x^0(r)\nabla X_r\right|^2 \mathrm{d}r\right] \leqslant 2\mathbb{E}\left[\left|\Delta X^\varepsilon\right|_{t,T}^{*,4}\right]^{\frac{1}{2}}\mathbb{E}\left[\left(\int_t^T\left|f_x^0(r)\right|^2 \mathrm{d}r\right)^2\right]^{\frac{1}{2}}$$
$$+2\mathbb{E}\left[\left|\nabla X^\varepsilon\right|_{t,T}^{*,4}\right]^{\frac{1}{2}}\mathbb{E}\left[\left(\int_t^T\left|f_x^\varepsilon(r)-f_x^0(r)\right|^2 \mathrm{d}r\right)^2\right]^{\frac{1}{2}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Step 5: we conclude that $\partial_x u$ exists and that $\partial_x u(t,x)=\nabla Y_t^{t,x}$, for all (t,x). See Theorem 3.1 of [MZ02].

Step 6: we will show that $\partial_x u$ is continuous. Let $(t_i,x_i) \in [0,T]\times\mathbb{R}^d$, $i=1,2$, with $t_1<t_2$. To simplify, we write:

$$\Theta^i=\left(X^i,Y^i,Z^i\right)=\left(X^{t_i,x_i},Y^{t_i,x_i},Z^{t_i,x_i}\right),\ f_x^i(r)=\partial_x f\left(\Theta_r^i\right),\ f_y^i(r)=\partial_y f\left(\Theta_r^i\right),\ f_z^i(r)=\partial_z f\left(\Theta_r^i\right),$$

$$g_x^i=\partial_x g\left(X_T^i\right),\ b_x^i(r)=\partial_x b\left(X_r^i\right) \text{ and } \sigma_x^i(r)=\partial_x\sigma\left(X_r^i\right).$$

We set $\widetilde{\Delta}X_r = \nabla X_r^1 - \nabla X_r^2$, $\widetilde{\Delta}Y_r = \nabla Y_r^1 - \nabla Y_r^2$, $\widetilde{\Delta}Z_r = \nabla Z_r^1 - \nabla Z_r^2$ and for any function φ, $\widetilde{\Delta}_{12}[\varphi] = \varphi^1 - \varphi^2$.

$$
\begin{aligned}
&|\partial_x u(t_1,x_1) - \partial_x u(t_2,x_2)| \\
&= \Bigg| \mathbb{E}\left[g_x^1 \nabla X_T^1 + \int_{t_1}^T \left\{ f_x^1(r)\nabla X_r^1 + f_y^1(r)\nabla Y_r^1 + f_z^1(r)\nabla Z_r^1 \right\} \mathrm{d}r \right] \\
&\qquad\qquad - \mathbb{E}\left[g_x^2 \nabla X_T^2 + \int_{t_2}^T \left\{ f_x^2(r)\nabla X_r^2 + f_y^2(r)\nabla Y_r^2 + f_z^2(r)\nabla Z_r^2 \right\} \mathrm{d}r \right] \Bigg| \\
&\leqslant \mathbb{E}\left[\left| g_x^1 \nabla X_T^1 - g_x^2 \nabla X_T^2 \right| \right] + \mathbb{E}\left[\int_{t_1}^{t_2} \left\{ \left| f_x^1(r) \right| \left| \nabla X_r^1 \right| + \left| f_y^1(r) \right| \left| \nabla Y_r^1 \right| + \left| f_z^1(r) \right| \left| \nabla Z_r^1 \right| \right\} \mathrm{d}r \right] \\
&\qquad + \mathbb{E}\Bigg[\int_{t_2}^T \Big\{ \left| f_x^1(r)\widetilde{\Delta}X_r \right| + \left| \widetilde{\Delta}_{12}[f_x](r) \right| \left| \nabla X_r^2 \right| + \|\partial_y f\|_\infty \left| \widetilde{\Delta}Y_r \right| + \left| \widetilde{\Delta}_{12}[f_y](r) \right| \left| \nabla Y_r^2 \right| \\
&\qquad\qquad + \|\partial_z f\|_\infty \left| \widetilde{\Delta}Z_r \right| + \left| \widetilde{\Delta}_{12}[f_z](r) \right| \left| \nabla Z_r^2 \right| \Big\} \mathrm{d}r \Bigg]
\end{aligned}
$$

First of all, $\mathbb{E}\left[\int_{t_1}^{t_2} \left\{ \left| f_x^1(r) \right| \left| \nabla X_r^1 \right| + \|\partial_y f\|_\infty \left| \nabla Y_r^1 \right| + \|\partial_z f\|_\infty \left| \nabla Z_r^1 \right| \right\} \mathrm{d}r \right] \underset{t_1 \to t_2}{\longrightarrow} 0$, by dominated convergence, because $\mathbb{E}\left[\left| \nabla X^1 \right|_{t,T}^{*,2} + \left| \nabla Y^1 \right|_{t,T}^{*,2} + \int_t^T \left| \nabla Z_s^1 \right|^2 \mathrm{d}s + \int_t^T \left| Z_s^1 \right|^2 \mathrm{d}s \right] < \infty$. Using Lemma 2.2 of [MZ02],

$$
\begin{aligned}
&\mathbb{E}\left[\left| \widetilde{\Delta}Y \right|_{t_2,T}^{*,2} + \int_{t_2}^T \left| \widetilde{\Delta}Z_s \right|^2 \mathrm{d}s \right] \leqslant C \Bigg\{ \|\partial_x g\|_\infty^2 \, \mathbb{E}\left[\left| \widetilde{\Delta}X_T \right|^2 \right] + \mathbb{E}\left[\left| \widetilde{\Delta}_{12}[g_x] \right|^2 \left| \nabla X_T^2 \right|^2 \right] \\
&\quad + \mathbb{E}\left[\int_{t_2}^T \left(\left| f_x^1(r) \right|^2 \left| \widetilde{\Delta}X_r \right|^2 + \left| \widetilde{\Delta}_{12}[f_x](r) \right|^2 \left| \nabla X_r^2 \right|^2 + \left| \widetilde{\Delta}_{12}[f_y](r) \right|^2 \left| \nabla Y_r^2 \right|^2 + \left| \widetilde{\Delta}_{12}[f_z](r) \right|^2 \left| \nabla Z_r^2 \right|^2 \right) \mathrm{d}r \right] \Bigg\}.
\end{aligned}
$$

We can adapt the step 2, we replace (t,x) by (t_2,x_2) and $x+h$ by $X_{t_2}^{t_1,x_1}$ and get:

$$
\forall p \geqslant 2,\ \mathbb{E}\left[\left| X^1 - X^2 \right|_{t_2,T}^{*,p} + \left| Y^1 - Y^2 \right|_{t_2,T}^{*,p} + \left(\int_{t_2}^T \left| Z_s^1 - Z_s^2 \right|^2 \mathrm{d}s \right)^{\frac{p}{2}} \right] \longrightarrow 0.
$$

By dominated convergence, for φ bounded, $\mathbb{E}\left[\left| \widetilde{\Delta}_{12}[\varphi] \right|_{t_2,T}^{*,p} \right] \longrightarrow 0$. Since $\mathbb{E}\left[\left(\int_{t_2}^T \left| Z_r^i \right|^2 \mathrm{d}r \right)^3 \right] < \infty$, we have $\mathbb{E}\left[\int_{t_2}^T \left\{ \left| f_x^1(r)\widetilde{\Delta}X_r \right|^2 + \left| \widetilde{\Delta}_{12}[f_x](r)\nabla X_r^2 \right|^2 \right\} \mathrm{d}r \right] \longrightarrow 0$. To sum up, we have shown that $|\partial_x u(t_1,x_1) - \partial_x u(t_2,x_2)| \longrightarrow 0$ when $t_1 \to t_2$ and $x_1 \to x_2$. We can prove it when $t_2 \to t_1$ and $x_2 \to x_1$: $\partial_x u$ is continuous on $[0,T] \times \mathbb{R}^d$.

Step 7: to show the relation $\forall s \in [t,T]$, $Z_s^{t,x} = \partial_x u\left(s, X_s^{t,x}\right) \sigma\left(X_s^{t,x}\right)$ $\mathbb{P}$-a.s., we can do the same as in the Theorem 3.1 of [MZ02]. We approximate b, σ, h and g by functions b^ε, σ^ε, h^ε and g^ε which are of class $\mathcal{C}^\infty$ with bounded derivatives (and the bound is independent of ε), and converge uniformly. Then, $(\sigma^\varepsilon)^{-1}$ converges uniformly to σ^{-1}. We define a function f^ε (we want it to approximate f and to be smooth) by $f^\varepsilon(x,y,z) = h^\varepsilon\left(x,y,z\sigma^\varepsilon(x)^{-1}\right)$ and it satisfies $|f^\varepsilon(x,y,z) - f(x,y,z)| \leqslant \|h^\varepsilon - h\|_\infty + H_p|z| \left\| (\sigma^\varepsilon)^{-1} - \sigma^{-1} \right\|_\infty$. Like before, results from [BDH+03] are useful to adapt the end of the proof in [MZ02]. ■

Lemma 34

Under the same assumptions, for every $T > t$, we have the following inequality:

$$
|\partial_x u(t,x)| \leqslant \mathcal{C}\left(1 + |x|^\mu + \mathbb{E}\left[\left| g\left(X_T^{t,x}\right) \right|^4 \right]^{\frac{1}{4}} \right),
$$

where $\mathcal{C}$ depends on T, $T-t$, H_x, H_y, H_p, M_f, $\left\|\sigma^{-1}\right\|_\infty$, $|b(0)|$, $|\sigma(0)|$, $\|b\|_{\mathrm{lip}}$, $\|\sigma\|_{\mathrm{lip}}$ and μ.

Proof :

Our proof is based on the ideas developed in Theorem 3.3 of [Ric11]. In the following, $\Theta_r^{t,x} = \left(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)$. We can rewrite the equation verified by $Y^{t,x}$ and $Z^{t,x}$, under differential form:

$$\mathrm{d}Y_s^{t,x} = -f\left(\Theta_s^{t,x}\right)\,\mathrm{d}s + \sum_{j=1}^{d}\left[Z_s^{t,x}\right]^{(j)}\,\mathrm{d}W_s^j, \tag{25}$$

where $\left[Z_s^{t,x}\right]^{(j)}$ stands for the jth coefficient of the row $Z_s^{t,x}$. By differentiation of (25) w.r.t. x and discounting method,

$$\begin{aligned}&\mathrm{d}\left(\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_s^{t,x}\right)\\ &\qquad = \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\left(-\partial_x f\left(\Theta_s^{t,x}\right)\nabla X_s^{t,x}\,\mathrm{d}s - \sum_{j=1}^{d}\partial_{z_j} f\left(\Theta_s^{t,x}\right)\nabla\left[Z_s^{t,x}\right]^{(j)}\,\mathrm{d}s + \sum_{j=1}^{d}\nabla\left[Z_s^{t,x}\right]^{(j)}\,\mathrm{d}W_s^j\right).\end{aligned}$$

We set, for every $j \in [\![1,d]\!]$, $\mathrm{d}\widetilde{W}_s^j = \mathrm{d}W_s^j - \partial_{z_j} f\left(\Theta_s^{t,x}\right)\mathrm{d}s$; $\widetilde{W}$ is a Brownian motion under a new probability denoted $\mathbb{Q}$. Between s and T (for $s \in [t,T]$), the integral form of this equation is:

$$\begin{aligned}&\mathrm{e}^{\int_t^T \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_T^{t,x} - \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_s^{t,x}\\ &\qquad = -\int_s^T \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\partial_x f\left(\Theta_r^{t,x}\right)\nabla X_r^{t,x}\,\mathrm{d}r + \sum_{j=1}^{d}\int_s^T \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\nabla\left[Z_r^{t,x}\right]^{(j)}\,\mathrm{d}\widetilde{W}_r^j.\end{aligned} \tag{26}$$

We define $F_s^{t,x} = \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla Y_s^{t,x} + \int_t^s \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\partial_x f\left(\Theta_r^{t,x}\right)\nabla X_r^{t,x}\,\mathrm{d}r$, and then equation (26) becomes:

$$F_s^{t,x} = F_T^{t,x} - \sum_{j=1}^{d}\int_s^T \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\nabla\left[Z_r^{t,x}\right]^{(j)}\,\mathrm{d}\widetilde{W}_r^j, \tag{27}$$

and it tells us that $F^{t,x}$ is a $\mathbb{Q}$-martingale. We recall, for every $s \in [t,T]$, that $Y_s^{t,x} = u\left(s, X_s^{t,x}\right)$, $\nabla Y_s^{t,x} = \partial_x u\left(s, X_s^{t,x}\right)\nabla X_s^{t,x}$ and $Z_s^{t,x} = \partial_x u\left(s, X_s^{t,x}\right)\sigma\left(X_s^{t,x}\right) = \nabla Y_s^{t,x}\left(\nabla X_s^{t,x}\right)^{-1}\sigma\left(X_s^{t,x}\right)$. Indeed, $\nabla X_s^{t,x}$ is invertible. We can show that $\nabla X^{t,x}$ is the solution of the linear SDE (because b and σ are $\mathcal{C}^1$ with bounded derivatives):

$$\mathrm{d}\nabla X_s^{t,x} = \partial_x b\left(X_s^{t,x}\right)\nabla X_s^{t,x}\,\mathrm{d}s + \sum_{j=1}^{d}\partial_x^j\sigma\left(X_s^{t,x}\right)\nabla X_s^{t,x}\,\mathrm{d}W_s^j,$$

whose solution is $\nabla X_s^{t,x} = \exp\left(\int_s^t\left\{\partial_x b\left(X_r^{t,x}\right) - \frac{1}{2}\sum_{j=1}^{d}\partial_x^j\sigma\left(X_r^{t,x}\right)\left[\partial_x^j\sigma\left(X_r^{t,x}\right)\right]^*\right\}\mathrm{d}r + \sum_{j=1}^{d}\int_t^s\partial_x^j\sigma\left(X_r^{t,x}\right)\,\mathrm{d}W_r^j\right)$,

where the notation $\partial_x^j\sigma\left(X_s^{t,x}\right)$ stands for the $(d\times d)$-matrix $\begin{pmatrix}\partial_x\left[\sigma\left(X_s^{t,x}\right)\right]^{(1,j)}\\ \hline \vdots \\ \hline \partial_x\left[\sigma\left(X_s^{t,x}\right)\right]^{(d,j)}\end{pmatrix}$, and where $\left[\sigma\left(X_s^{t,x}\right)\right]^{(i,j)}$ is the coefficient at line i and column j in the matrix $\sigma\left(X_s^{t,x}\right)$. This way, we see that $\left(\nabla X_s^{t,x}\right)^{-1}$ is solution of the following linear SDE:

$$\mathrm{d}\left(\nabla X_s^{t,x}\right)^{-1} = \left(\nabla X_s^{t,x}\right)^{-1}\left\{-\partial_x b\left(X_s^{t,x}\right) + \frac{1}{2}\sum_{j=1}^{d}\partial_x^j\sigma\left(X_r^{t,x}\right)\left[\partial_x^j\sigma\left(X_r^{t,x}\right)\right]^*\right\}\mathrm{d}s - \left(\nabla X_s^{t,x}\right)^{-1}\sum_{j=1}^{d}\partial_x^j\sigma\left(X_s^{t,x}\right)\,\mathrm{d}W_s^j. \tag{28}$$

In the following, for every $s \in [t,T]$, we set $R_s^{t,x} = Z_s^{t,x}\sigma\left(X_s^{t,x}\right)^{-1} = \nabla Y_s^{t,x}\left(\nabla X_s^{t,x}\right)^{-1} = \partial_x u\left(s, X_s^{t,x}\right)$, $\beta_s^{t,x} = \int_t^s \mathrm{e}^{\int_t^r \partial_y f\left(\Theta_u^{t,x}\right)\mathrm{d}u}\partial_x f\left(\Theta_r^{t,x}\right)\nabla X_r^{t,x}\,\mathrm{d}r\left(\nabla X_s^{t,x}\right)^{-1}$, $\widetilde{R}_s^{t,x} = F_s^{t,x}\left(\nabla X_s^{t,x}\right)^{-1} = \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}R_s^{t,x} + \beta_s^{t,x}$.

Using the integration by parts formula, combining (27) and (28), one gets:

$$\begin{aligned}
\mathrm{d}\widetilde{R}_s^{t,x} = \widetilde{R}_s^{t,x} &\left\{ -\partial_x b\left(X_s^{t,x}\right) + \sum_{j=1}^d \partial_x^j \sigma\left(X_r^{t,x}\right)\left[\partial_x^j \sigma\left(X_r^{t,x}\right)\right]^* - \sum_{j=1}^d \partial_{z_j} f\left(\Theta_s^{t,x}\right)\partial_x^j \sigma\left(X_s^{t,x}\right)\right\} \mathrm{d}s \\
&- \sum_{j=1}^d \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r} \nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1}\partial_x^j \sigma\left(X_s^{t,x}\right)\mathrm{d}s \\
&+ \sum_{j=1}^d \left\{\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r} \nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\mathrm{d}\widetilde{W}_s^j. \qquad (29)
\end{aligned}$$

Then, let us take a new parameter $\lambda \in \mathbb{R}$; we have:

$$\begin{aligned}
\mathrm{d}\left|\mathrm{e}^{\lambda s}\widetilde{R}_s^{t,x}\right|^2 = \mathrm{e}^{2\lambda s}\widetilde{R}_s^{t,x} &\left\{2\lambda \mathrm{I}_d - 2\partial_x b\left(X_s^{t,x}\right) + 2\sum_{j=1}^d \partial_x^j \sigma\left(X_r^{t,x}\right)\left[\partial_x^j \sigma\left(X_r^{t,x}\right)\right]^* - 2\sum_{j=1}^d \partial_{z_j} f\left(\Theta_s^{t,x}\right)\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^* \mathrm{d}s \\
&-2\mathrm{e}^{2\lambda s}\sum_{j=1}^d \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1}\partial_x^j \sigma\left(X_s^{t,x}\right)\left(\widetilde{R}_s^{t,x}\right)^* \mathrm{d}s \\
&+2\mathrm{e}^{2\lambda s}\sum_{j=1}^d \left\{\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^* \mathrm{d}\widetilde{W}_s^j \\
&+\mathrm{e}^{2\lambda s}\sum_{j=1}^d \left\{\left[\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right]\right. \\
&\left.\times\left[\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right]^*\right\}\mathrm{d}s. \qquad (30)
\end{aligned}$$

If we denote $\gamma = \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1}$ and $\delta = \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)$, we remark that we have the inequality $-2\gamma\delta^* + |\gamma-\delta|^2 = |\gamma - 2\delta|^2 - 3|\delta|^2$. Thus,

$$\begin{aligned}
\mathrm{d}\left|\mathrm{e}^{\lambda s}\widetilde{R}_s^{t,x}\right|^2 = \mathrm{e}^{2\lambda s}\widetilde{R}_s^{t,x} &\left\{2\lambda \mathrm{I}_d - 2\partial_x b\left(X_s^{t,x}\right) - \sum_{j=1}^d \partial_x^j \sigma\left(X_r^{t,x}\right)\left[\partial_x^j \sigma\left(X_r^{t,x}\right)\right]^* - 2\sum_{j=1}^d \partial_{z_j} f\left(\Theta_s^{t,x}\right)\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^* \mathrm{d}s \\
&+2\mathrm{e}^{2\lambda s}\sum_{j=1}^d \left\{\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right\}\left(\widetilde{R}_s^{t,x}\right)^* \mathrm{d}\widetilde{W}_s^j \\
&+\mathrm{e}^{2\lambda s}\sum_{j=1}^d \left|\mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}\nabla\left[Z_s^{t,x}\right]^{(j)}\left(\nabla X_s^{t,x}\right)^{-1} - \widetilde{R}_s^{t,x}\partial_x^j \sigma\left(X_s^{t,x}\right)\right|^2 \mathrm{d}s. \qquad (31)
\end{aligned}$$

This way, we can see that, for λ great enough (that is to say bigger than something depending only on H_p, $\left\|\sigma^{-1}\right\|_\infty$ and on bounds over the derivatives of b and σ), the process $\left(\left|\mathrm{e}^{\lambda s}\widetilde{R}_s^{t,x}\right|^2\right)_{s\in[t,T]}$ is a $\mathbb{Q}$-submartingale. So, we get:

$$\left|R_t^{t,x}\right|^2(T-t) \leqslant \mathrm{e}^{-2\lambda t}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{2\lambda s}\left|\widetilde{R}_s^{t,x}\right|^2 \mathrm{d}s\right] \leqslant \mathrm{e}^{2\lambda(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|\widetilde{R}_s^{t,x}\right|^2 \mathrm{d}s\right].$$

But $\widetilde{R}_s^{t,x} = \mathrm{e}^{\int_t^s \partial_y f\left(\Theta_r^{t,x}\right)\mathrm{d}r}R_s^{t,x} + \beta_s^{t,x}$ and $\partial_y f$ is bounded by a constant H_y, so

$$\left|R_t^{t,x}\right|^2(T-t) \leqslant \mathrm{e}^{2(\lambda+H_y)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|R_s^{t,x}\right|^2 \mathrm{d}s\right] + \mathrm{e}^{2\lambda(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|\beta_s^{t,x}\right|^2 \mathrm{d}s\right].$$

On the one hand, we have:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|R_s^{t,x}\right|^2 \mathrm{d}s\right] &\leqslant \left\|\sigma^{-1}\right\|_\infty^2 \mathbb{E}\left[\mathrm{e}^{\int_t^T \partial_z f\left(\Theta_s^{t,x}\right)\mathrm{d}W_s - \frac{1}{2}\int_t^T \left|\partial_z f\left(\Theta_s^{t,x}\right)\right|^2 \mathrm{d}s}\int_t^T \left|Z_s^{t,x}\right|^2 \mathrm{d}s\right] \\
&\leqslant \left\|\sigma^{-1}\right\|_\infty^2 \mathrm{e}^{\frac{1}{2}H_p^2\left\|\sigma^{-1}\right\|_\infty^2(T-t)}\mathbb{E}\left[\left(\int_t^T \left|Z_s^{t,x}\right|^2 \mathrm{d}s\right)^2\right]^{\frac{1}{2}} \qquad (32)
\end{aligned}$$

We recall equation (21): $\mathbb{E}\left[\left(\int_t^T \left|Z_r^{t,x}\right|^2 \mathrm{d}r\right)^2\right] \leqslant Ce^{4a(T-t)}\left\{\mathbb{E}\left[\left|Y^{t,x}\right|_{t,T}^{*,4}\right] + C_T\left(1+|x|^{4\mu}\right)\right\}$. By using the Lemma 2.2 of [MZ02], we have, where C is a constant:

$$\mathbb{E}\left[\left|Y^{t,x}\right|_{t,T}^{*,4}\right] \leqslant C\mathbb{E}\left[\left|g\left(X_T^{t,x}\right)\right|^4 + \int_t^T \left|f\left(X_r^{t,x},0,0\right)\right|^4 \mathrm{d}r\right].$$

But, using proposition 4 and the assumption made on $f(\bullet,0,0)$, we get:

$$\mathbb{E}\left[\int_t^T \left|f\left(X_r^{t,x},0,0\right)\right|^4 \mathrm{d}r\right] \leqslant C\left(1+|x|^{4\mu}\right),$$

where C only depends on μ, M_f, $T-t$, T, $|\sigma(0)|$, $\|\sigma\|_{\mathrm{lip}}$, $|b(0)|$ and $\|b\|_{\mathrm{lip}}$. On the other hand, using the Lemma 2.1 of [MZ02], we can show that $\nabla X^{t,x}$ and $\left(\nabla X^{t,x}\right)^{-1}$ are in $\mathcal{S}^p$, for all $p<\infty$. Also, we have $\left|\partial_x f\left(\Theta_r^{t,x}\right)\right| \leqslant H_x + H_p\left|Z_r^{t,x}\right|\left\|\partial_x\sigma^{-1}\right\|_\infty$. Then, with Cauchy-Schwarz inequality and a priori estimates of [BDH+03], we get that $\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|\beta_s^{t,x}\right|^2 \mathrm{d}s\right]$ is bounded. We recall that $R_t^{t,x} = \partial_x u(t,x)$. We have the conclusion:

$$|\partial_x u(t,x)|^2 \leqslant \mathcal{C}\left(1+|x|^{2\mu} + \mathbb{E}\left[\left|g\left(X_T^{t,x}\right)\right|^4\right]^{\frac{1}{2}}\right).$$

■

Theorem 35

We make the following assumptions:

- σ and b are Lipschitz continuous;
- the function $\sigma(\bullet)^{-1}$ is bounded;
- $f \in \mathcal{C}^0\left(\mathbb{R}^d \times \mathbb{R} \times \left(\mathbb{R}^d\right)^*, \mathbb{R}\right)$ and $\forall x \in \mathbb{R}^d$, $|f(x,0,0)| \leqslant M_f\left(1+|x|^\mu\right)$ and $|g(x)| \leqslant M_g\left(1+|x|^\nu\right)$;
- we can write $f(x,y,z) = \psi(x,z) - \alpha y$, with $\alpha \in (0,1]$ and if we define $h : (x,p) \mapsto \psi(x,p\sigma(x))$, then h is Lipschitz continuous (with bounds denoted H_x and H_p w.r.t. x and p).

Under those assumptions, the function $u : (t,x) \mapsto Y_t^{t,x}$ satisfies the following inequality, for every $T>t$:

$$\left|u(t,x) - u(t,x')\right| \leqslant \mathcal{C}\left(1+|x|^{\mu\vee\nu} + |x'|^{\mu\vee\nu}\right)|x-x'|,$$

where $\mathcal{C}$ depends on T, $T-t$, H_x, H_p, M_f, M_g, $\left\|\sigma^{-1}\right\|_\infty$, $|b(0)|$, $|\sigma(0)|$, $\|b\|_{\mathrm{lip}}$, $\|\sigma\|_{\mathrm{lip}}$ and μ.

Proof :

We approximate the functions h, g, b and σ by some uniformly converging sequences $(h^\varepsilon)_{\varepsilon>0}$, $(g^\varepsilon)_{\varepsilon>0}$, $(b^\varepsilon)_{\varepsilon>0}$ and $(\sigma^\varepsilon)_{\varepsilon>0}$ of functions of class $\mathcal{C}^1$ with bounded derivatives. We can assume that there exists a bound, independent of ε, of all the derivatives of h^ε, b^ε and σ^ε, for every $\varepsilon>0$. We define $f^\varepsilon(x,y,z) = h^\varepsilon\left(x, z\sigma(x)^{-1}\right) - \alpha y$ and we can assume that $f^\varepsilon(\bullet,0,0) = h^\varepsilon(\bullet,0,0)$ and g^ε have polynomial growth independently of ε, with constants M_f and M_g and exponents μ and ν. For every $(t,x) \in [0,T]\times\mathbb{R}^d$, $|\partial_x u^\varepsilon(t,x)| \leqslant \mathcal{C}\left(1+|x|^{\mu\vee\nu}\right)$, where $\mathcal{C}$ does not depend on ε or x. Using the mean-value theorem, it comes that :

$$\forall t \in [0,T), \forall x,x' \in \mathbb{R}^d,\ \left|u^\varepsilon(t,x) - u^\varepsilon(t,x')\right| \leqslant \mathcal{C}\left(1+|x|^{\mu\vee\nu} + |x'|^{\mu\vee\nu}\right)|x-x'|. \tag{33}$$

Then, for every $T>1$, the set $\{u^\varepsilon(t,\bullet)|\varepsilon>0, t\in[0,T-1]\}$ is equicontinuous; also it is pointwise bounded: we can show it by writing the equation verified by $Y^{\varepsilon,t,x}$ and using the Lemma 2.2 of [MZ02]. Let K be a compact subset of $\mathbb{R}^d$. By the Arzelà-Ascoli theorem, we get, for a sequence $\left(\varepsilon_n^{(K)}\right)_{n\geqslant 0}$ which has 0 as limit:

$$\forall t \in [0,T-1],\ u^{\varepsilon_n^{(K)}}(t,\bullet)\big|_K \xrightarrow[n\to\infty]{\|.\|_\infty} u_K(t,\bullet).$$

Let $x \in K$; we have: $u^{\varepsilon_n^{(K)}}(t,x) \xrightarrow[n\to\infty]{} u_K(t,x)$ and $u^{\varepsilon_n^{(K)}}(t,x) = Y_t^{\varepsilon_n^{(K)},t,x} \xrightarrow[n\to\infty]{} Y_t^{t,x}$. So the limit $u_K(t,x) = Y_t^{t,x}$ does not depend on K; and the convergence of $u^{\varepsilon_n^{(K)}}(t,\bullet)$ is uniform on K. The following triangle

inequality gives the result claimed after taking the limit as n goes to infinity (because $\mathcal{C}$ is independent of K): for every $x, x' \in K$ and $t \in [0, T-1]$,

$$\forall n \in \mathbb{N},\ |u(t,x) - u(t,x')| \leqslant \left| u(t,x) - u^{\varepsilon_n^{(K)}}(t,x) \right| + \mathcal{C}\left(1 + |x|^{\mu \vee \nu} + |x'|^{\mu \vee \nu}\right)|x - x'| + \left| u^{\varepsilon_n^{(K)}}(t,x') - u(t,x') \right| . \blacksquare$$